Remembering John Napier and His Logarithms

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Abstract

My decision to first read and study *The Construction of the Wonderful Canon of Logarithms* was not motivated by an interest in logarithms. I was trying to learn more about the origin of the natural exponential function. I had known Leonhard Euler $(1707 - 1783)$ probably did the most to engender wide spread understanding and acceptance of modern exponential functions (including the natural exponential function), but John Napier had over one hundred years earlier written a couple of papers creating something he called logarithms, which through hindsight, we have learned have a very intimate relationship with the natural exponential function. I wanted to understand this relationship with the hope that it would help me understand the context in which the natural exponential function was born.

I also discovered something unexpected. Napier's *Canon* is a marvelous example of how engineering problems are solved in practice. Often times engineers are prohibited from directly applying textbook solutions because of the complexity of the problem. They, instead, make simplifying assumptions and estimate quantities of interest. Napier faced a similar situation. He *defined* logarithms, yet he could not directly compute them. So, he computed their estimates. His ability to do so, I believe, shows the real ingenuity and inspiration of John Napier.

Throughout this paper, I have tried to preserve most of Napier's original proofs and arguments, updating the language and mathematics where necessary. At the end, I added a section hopefully answering some nagging questions a modern reader might have about a 400 year old paper. Such as, why did Napier choose the word logarithm and how do Napierian logarithms relate to ordinary modern logarithms? I enjoy placing mathematics into historical context and I hope the reader gains a new understanding and appreciation of logarithms which goes beyond what is usually gleamed from modern texts.

Remembering John Napier and His Logarithms

I. Introduction

John Napier $(1550 - 1617)$ was a laird of the Merichston estate near Edinburgh, Scotland. He was not employed as a professional mathematician, although he is now most remembered as one of the inventors of logarithms. From what is known about his life, Napier spent a considerable amount of time studying mathematics searching for easier and more efficient ways of multiplying numbers [1, 2]. During the late sixteenth and early seventeenth centuries, multiplication as well as, division and the extraction of roots were in general slow and tedious calculations. The invention of logarithms almost certainly, came as a long awaited relief to the labor of these calculations.

The sum of Napier's work on logarithms is found in two treatises, *The Description of the Wonderful Canon of Logarithms* and *The Construction of the Wonderful Canon of Logarithms*. The *Description* was published in 1614 and the *Construction*, although written before the *Description*, was only published posthumously by his son Robert Napier in 1619. This paper focuses on the ideas and arguments presented by Napier in the *Construction*.

Napier devised an ingenious mathematical tool without the advantages of modern mathematics. Differential and integral calculus had not been invented nor had exponential notation (terms like base and exponent were not routinely used until much later). In fact, most of our contemporary mathematical language did not exist, so Napier could not even express his thoughts as we would today. Consequently, he initially described logarithms through geometry and not as the inverse of the exponential function. It was in Napier's lifetime, decimal notation began to be widely accepted and Johann Kepler $(1571 - 1630)$ derived his laws of planetary motion.

Similar to modern mathematical texts, the *Construction*, is written in a more or less axiomatic format. Napier begins with a few basic definitions and then progressively builds on them. The logarithmic function he describes is not the natural logarithmic function known today, but forms the basis and very essence of modern logarithms. As the name implies, the bulk of the *Construction* explains how to tabulate values for this function.

Napier did not develop an explicit mathematical expression for the logarithm of a number. He estimated them by finding a number whose logarithm possesses upper and lower bounds that differ by an "insignificant amount". He then reasoned the average of these bounds was a good estimate of the actual logarithm. Through these types of estimates and by taking advantage of certain properties of logarithms, Napier built his entire Table.

II. Arithmetic and Geometric Progressions

The first portion of the *Construction* is introductory, but nevertheless, presents some ideas which are seen throughout the treatise and which are crucial to understanding central concepts. Arithmetic and geometric progressions (sequences) are two such topics worth highlighting. An arithmetical progression "proceeds by equal intervals" (Article 2) such that succeeding terms differ by a constant. A geometrical progression advances by "unequal but proportionally increasing or decreasing intervals". That is, the ratio of succeeding terms is constant. As examples, Napier offers

Arithmetical progressions:	$1, 2, 3, 4, 5, 6, 7, \ldots$
	$2, 4, 6, 8, 10, 12, 14, 16, \ldots$
Geometrical progressions:	$1, 2, 4, 8, 16, 32, 64, \ldots$
	$243, 81, 27, 9, 3, 1, \ldots$

Despite the fact that the terms in these examples are integers, Napier required them to be real numbers in his definition. (This fact is important to remember but easily forgotten when one delves into the construction of the Table.) If Napier had modern notation, he might have described arithmetic and geometric motion by the functions

Arithmetic motion:	$f(t) = ct + b$
Geometric motion:	$f(t) = ca^t$

where in both cases $f(\cdot)$ is a real valued function of a real variable, and $a, b, c \in \mathbb{R}$.

III. Definition

Today, the natural logarithmic function is usually defined as the inverse of the natural exponential function (e.g. [3]), or through the integral equation

$$\ln x = \int_1^x \frac{1}{t}\, dt, \quad x > 0$$

(e.g. [4]). Napier offered a more qualitative definition. Consider two points α, β moving along the lines shown in Figure 1. Let α move arithmetically from left to right along the first line such that in equal time increments T it moves equal distances. In other words, let α travel with constant velocity. Set the length of the second line equal to 10^7 and let β travel geometrically from left to right so that the distance

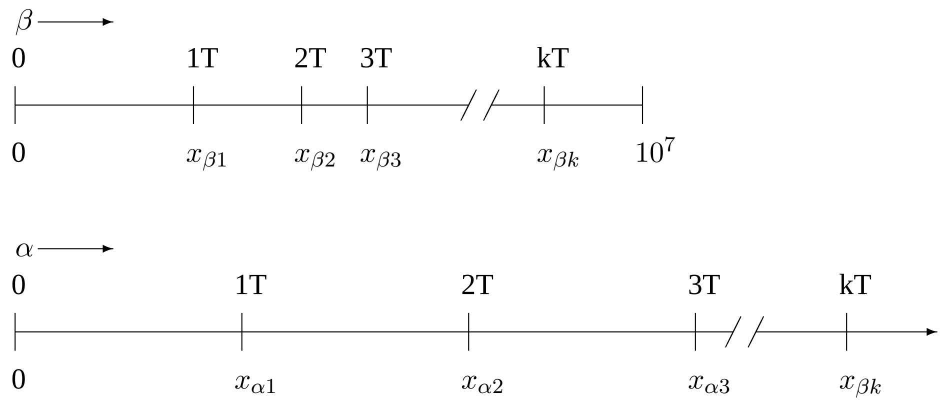

Fig. 1. β moves such that its speed geometrically decreases while α travels at a constant velocity. The position of β or α at each moment in time kT is denoted by $x_{\beta k}$ and $x_{\alpha k}$ respectively. The logarithm of the distance $10^7 - x_{\beta k}$ is defined as $x_{\alpha k}$.

travelled in each time increment equals a constant fractional proportion of the remaining length. (Napier's choice of 10^7 is discussed below.) Since these lengths decrease, the velocity of β decreases in time.

Set the magnitude of the velocity of α equal to 10^7 and require the velocity of β at any moment in time be equal (in magnitude) to the remaining distance at the same time. Thus at time 0, the velocity of β equals 10^7, at time T the velocity would equal $10^7 - x_{\beta 1}$, and at time $2T$, it would equal $10^7 - x_{\beta 2}$. "Hence, whatever be the proportion of the distances [10^7, $10^7 - x_{\beta 1}$, $10^7 - x_{\beta 2}$, $10^7 - x_{\beta 3}$], etc. to each other, that of the velocities of [β] at the points [0,$x_{\beta 1}$, $x_{\beta 2}$, $x_{\beta 3}$], etc. to one another, will be the same." (Article 25)

Finally, let β move geometrically as described above from 0 to $x_{\beta k}$ in time kT and let α move arithmetically (with constant velocity equal to 10^7) for the same amount of time from 0 to $x_{\alpha k}$. The distance $x_{\alpha k}$ is called the *logarithm* of the distance $10^7 - x_{\beta k}$. (Article 26)

To the modern reader this definition may seem imprecise, nevertheless, it explicitly expresses the relationship between a number and its logarithm. In particular, setting the velocity of β equal to the lengths of the remaining distances forced the base of the exponential function describing the motion of β to be e^{-1}. (Refer to section V for a more complete explanation.)

Thus, knowing this base, Napier's definition can easily be restated in modern terminology. Let the functions

$$x_\alpha(t) = 10^7 t \tag{1}$$

$$x_\beta(t) = 10^7 - 10^7 e^{-t} \tag{2}$$

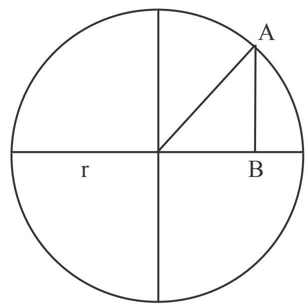

Fig. 2. In the sixteenth century the sine of an angle θ was defined as the half chord AB of a circle of radius r.

describe the motion of α and β respectively. Then the function

$$z_\beta(t) = 10^7 e^{-t} \tag{3}$$

describes the remaining distance from β to the end of the second line as a function of time.

Thus for a given time kT, the Napierian logarithm of $z(kT)$ is defined as

$$\text{Nap } \log\left(z(kT)\right) = x_\alpha(kT). \tag{4}$$

Napier chose to tabulate the logarithms of sines of angles because many intensive computational problems of the day involved trigonometry. Note that in the sixteenth century the sine of an angle was not normalized to a circle of radius one. Instead, it depended on the radius of the circle of interest. Specifically, the sine of an angle θ was defined as the half chord AB of a circle of radius r. Refer to Figure III. The length of the second line represents the radius of Napier's circle.

He chose 10^7, in particular, because he wanted to guarantee accuracy and ease of computation. (Article 3) Multiplying everything by 10^7 produced large integral numbers and achieved this goal.

Unfortunately, requiring the length of the second line to equal 10^7 instead of one, convolutes the computational advantages normally seen in modern logarithms. Napier himself was aware of the improvements which would be gained by rescaling and he, in fact, proposed a new and "better kind" of logarithm in the appendix of the *Construction*.

IV. Construction of the Table

Besides reflecting his approach, the arrangement of Napier's Table and his method of constructing it reflect the challenge of computing logarithms. As said before, his definition did not provide an explicit expression. Calculating the logarithm of a number wasn't simply a matter of plugging numbers into an equation. Napier cleverly built his Table in a specific way so that he could take advantage of certain

First Table	Second Table
10000000.0000000	10000000.0000000
9999999.0000000	9999900.0000000
9999998.0000001	9999800.0010000
9999997.0000003	9999700.0030000
9999996.0000006	9999600.0060000
9999995.0000010	9999500.0099999
9999994.0000015	9999400.0149998
9999993.0000021	9999300.0209996
9999992.0000028	9999200.0279994
9999991.0000036	9999100.0359992
⋮	⋮
9999900.0004950	9995001.224804[1]

TABLE I

NAPIER'S FIRST AND SECOND TABLES OF DECREASING GEOMETRIC PROGRESSIONS. THE FIRST PROCEEDS WITH A PROPORTION OF ONE ONE MILLIONTH AND THE SECOND WITH ONE ONE HUNDRED THOUSANDTH.

properties and arrive at estimates for his logarithms. He began by creating a decreasing geometric sequence which was easy to compute. Starting with the number $a_0 = 10^7$, he subtracted one one millionth from it to obtain the second number in the sequence. That is, he subtracted 1 from 10^7 to obtain 9999999.00000. Similarly, he subtracted one one millionth of 9999999.00000 from 9999999.00000 to obtain the third number, i.e. $a_2 = a_1 - a_1(.0000001) = 9999998.0000001$. In general, $a_n = a_{n-1} - a_{n-1}(.000001)$. Napier continued the sequence for 100 iterations and called it the First Table (Table I).

Next, he created a second decreasing geometric progression starting again with the number 10^7 but using a different proportion. This time in order to make everything fit together, he looked at the first and last elements in the First Table and noted that their difference was roughly 100 which is one one hundred thousandth of 10^7. He used this proportion to build his Second Table. It continued in the same manner as the First Table, but only for 50 iterations ending with the number 9995001.224804 (Table I). He did not use the exact proportion existing between the first and last elements of the First Table for the very simple reason that using a round number like 100 made the computations easier.

Building on these two tables, Napier created a larger Third Table. See Table II. This table had 21 rows

[1]This number mistakenly appeared in the *Construction* as 9995001.222927. Napier most likely made a computational error when constructing the Second Table. This error impacted later calculations, the effects of which are seen in Napier's final logarithm table. [1]

1st Column	2nd Column	3rd Column	$\cdots$	69th Column
10000000.0000000	9900000.0000000	9801000.0000000		5048858.8878707
9995000.0000000	9895050.0000000	9796099.5000000		5046334.4584268
9990002.5000000	9890102.4750000	9791201.4502500		5043811.2911976
9985007.4987500	9885157.4237625	9786305.8495249		5041289.3855520
9980014.9950006	9880214.8450506	9781412.6966001		5038768.7408592
$\vdots$	$\vdots$	$\vdots$		$\vdots$
9900473.5780233	9801468.8422431	9703454.1538206	$\cdots$	4998609.4018532

TABLE II

NAPIER'S THIRD TABLE. THE PROPORTION IN EACH COLUMN IS ONE TWO THOUSANDTH AND ONE ONE HUNDREDTH IN EACH ROW.

and 69 columns and each row and column was a decreasing geometric sequence. The proportion in each column was one two thousandth; in each row one one hundredth. The first element in the first column was 10^7 like before and the last element in the 69th column was 4998609.4018532.

Napier began with these decreasing geometric progressions because his goal was to build a table which listed the logarithms of sines between zero and ninety degrees. This meant, given the archaic definition of the sine of an angle (and a radius of 10^7), he was ultimately searching for the logarithms of 0 through 10^7. He did not tabulate progressions with a proportion of e^{-1} since he could not directly compute the terms. He instead created progressions which were easy to compute and which yielded estimates of their logarithms.

It follows directly from Napier's definition that the logarithm of 10^7 is zero since the distance α has traveled at time 0 is 0. Napier estimated the logarithm of 9999999.0000000 by introducing the following bounds.

Consider the abstraction of Figure 1 shown in Figure 3. Denote the endpoints of the line along which β travels as A and B and fix a point C arbitrarily between them. Extend this line to the left to the point D such that the length of the line segment DB is in the same proportion to AB as AB is to CB, i.e. let

$$\frac{DB}{AB} = \frac{AB}{CB}.$$

Allow β to move geometrically to the right beginning at D such that it moves from D to A and from A to C in equal times (the speed of β at D is equal to the length of the line segment DB which is greater than 10^7). Allow α to move arithmetically from E to F in the same amount of time.

The logarithm of the length of the segment CB is, by definition, the length of EF. From the figure

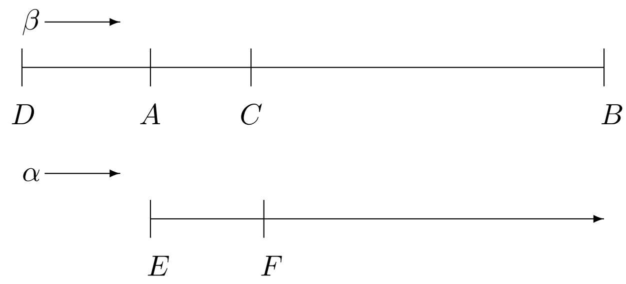

Fig. 3. The logarithm of the line segment CB has an upper bound equal to the length of DA and a lower bound equal to the length of AC.

we also see that the logarithm of CB is greater than the length of the segment AC because α moves at a constant speed as it travels from E to F while β continuously slows down as it moves from A to C. Thus, the length of AC is a lower bound of the length of EF. Similarly, the length of DA is an upper bound since β moves at higher speeds between D and A than α does between E and F.

Expressions for these bounds are easily found. Note that the length of AC equals the length of AB minus CB, that is 10^7 minus the number with which you are taking the logarithm. An expression for the upper bound is found by noticing that the segment DA is in the same proportion to AB as AC is to CB. This yields

$$DA = \frac{(AB)(AC)}{CB}.$$

With these bounds Napier returned to the problem of computing the the logarithm of 9999999.0000000. If the length of CB equals 9999999.0000000, the lengths of AC and DA equal 1.0000000 and 1.0000001, respectively. Napier reasoned that these bounds differed “insensibly” (Article 31) and that any number between the bounds sufficed as an estimate of the logarithm. He decided to take the average of the bounds and set the logarithm of 999999.0000000 equal to 1.00000005.

If Napier had started with a different proportion in his progressions, his bounds could have very easily yielded an inaccurate estimate. If, for example, he used one tenth instead of one ten millionth in his First Table, he would have computed the bounds of 9000000.0000000 as 1000000.00 and 11111111.11!

It is evident from Napier's definition that as the remaining lengths decrease geometrically, their logarithms increase arithmetically. For a given time interval T, β will move from 0 to $x_{\beta 1}$ (Figure 4) and by definition, the logarithm of $10^7 - x_{\beta 1}$ is $x_{\alpha 1}$. At time $2T$, the remaining length becomes $10^7 - x_{\beta 2}$ and the logarithm of this distance is simply twice the logarithm of $10^7 - x_{\beta 1}$ since $x_{\alpha 2}$ is twice $x_{\alpha 1}$. Likewise, the logarithm of $10^7 - x_{\beta 3}$ equals three times $x_{\alpha 1}$.

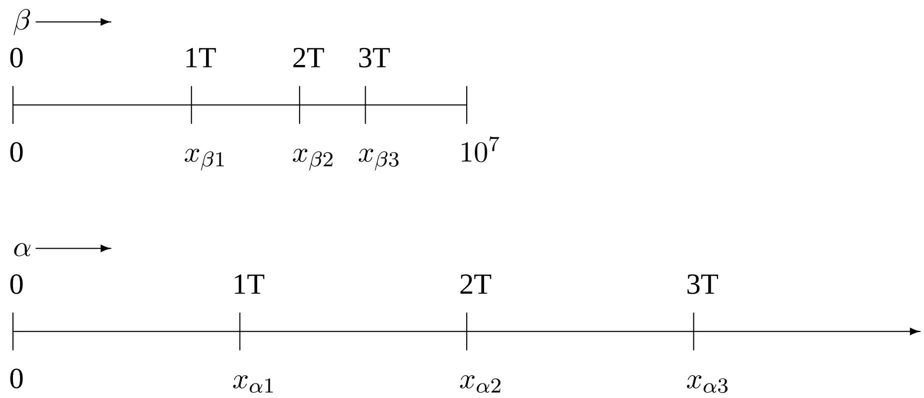

Fig. 4. As the length of the remaining distances ($10^7 - x_{\beta 1}$, $10^7 - x_{\beta 2}$, etc.) decrease geometrically, their logarithms increase arithmetically.

Using this logic, Napier took his estimate for the logarithm of 999999.0000000 and doubled, tripled, quadrupled, etc. it and computed all the logarithms of the First Table (Table III).

Number	**Logarithm**
10000000.0000000	0.0000000
9999999.0000000	1.0000000
9999998.0000001	2.0000001
9999997.0000003	3.0000002
9999996.0000006	4.0000002
9999995.0000010	5.0000002
9999994.0000015	6.0000003
9999993.0000021	7.0000004
9999992.0000028	8.0000004
9999991.0000036	9.0000004
9999990.0000045	10.0000005
⋮	⋮
9999900.0004950	100.0000050

TABLE III
COMPLETED FIRST TABLE.

Napier now tackled the Second Table. The first value is 10^7 which we already know has a logarithm equal to zero. To find logarithm of 9999900.0000000, Napier proceeded essentially as he did before seeking an upper and a lower bound which differed by an insensible amount. This time, though, the process becomes more complicated. It involves three more properties.

1) The difference of the logarithm of a given number and the logarithm of 10^7 is just the logarithm of the given number (Article 34).

   Proof: This is self-evident since the logarithm of 10^7 is zero. $\square$

2) The logarithms of similarly proportioned numbers are equidifferent (Article 36).

   Proof: This property follows directly from Napier's definition since the point β will travel for equal time increments between any two numbers that are similarly proportioned. *Example:* From Table III, the proportion of the numbers 9999993.0000021 and 9999991.0000036 is $1:5000000$, and the difference of their logarithms is 2.0. The numbers 9999992.0000028 and 9999990.0000045 are similarly proportioned and the difference of their logarithms $(10.0000005 - 8.0000004)$ is also 2.0. $\square$

3) The difference of two logarithms is bounded (Article 39).

   Proof: Refer to Figure 5. Let the line segment AB have length 10^7 and let the segments CB and DB represent the numbers whose logarithms are of interest. Define the points E and F such that
   $$\frac{EA}{AB} = \frac{CD}{DB} \tag{5}$$
   and
   $$\frac{AF}{AB} = \frac{CD}{CB}. \tag{6}$$
   Solving equation 5 for EA, we see that $EA = (AB)(CD)/DB$ and thus
   $$\frac{EB}{AB} = \frac{\frac{(AB)(CD)}{DB} + AB}{AB} = \frac{AB\left(\frac{CD}{DB}+1\right)}{AB} = \frac{CD+DB}{DB} = \frac{CB}{DB}.$$
   Therefore by the second property, the logarithms of CB and DB differ by the same amount as the logarithms of AB and FB since they are similarly proportional. By the first property, this difference is simply equal to the logarithm of FB, and as before, is bounded by EA and AF.

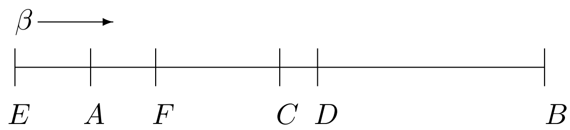

Fig. 5. The lengths of the line segments EA and AF are upper and lower bounds for the difference of the logarithms of CB and DB

Therefore, Napier concluded that the difference of two logarithms is bounded.

$$\frac{(AB)(CD)}{CB} = AF < \text{Nap } \log{(CB)} - \text{Nap } \log{(DB)} < EA = \frac{(AB)(CD)}{DB} \qquad \square$$

This last property is the key to finding the bounds of the logarithm of 9999900.0000000 and to determining all of the logarithms of the Second Table. Specifically, Napier found the number closest to 9999900.0000000 in the First Table (9999900.0004950) and noted its logarithm (100.00000050) and bounds (100.0000100 and 100.0000000). Using property 3, he determined that $.0004950$ should be added to the bounds of 9999900.0004950 to yield the bounds of 9999900.0000000. The result being 100.0005050 and 100.0004950. Finally, as before, Napier reasoned these bounds differed by an insensible amount so either bound or any number between them could be taken to be the true logarithm of 9999900.0000000. Thus, Napier completed the Second Table by repeating what he had done for the First. He doubled, tripled, quadrupled, etc. the logarithm of 9999900.0000000 as he stepped through the progression. See Table IV.

Napier was now ready to find the logarithms of the Third Table. The problem boiled down to finding the logarithms of two numbers. If he could find the logarithm of 9995000.0000000 (first element, first column), he could calculate all of the logarithms in the first column since they are in a particular geometric progression. Likewise, if could calculate the logarithm of 9900000.0000000 (first element, second column), he could fill in all of the remaining columns since all of the these elements are geometrically related. Using the same method as before, Napier determined the upper and lower bounds for the logarithm of 9995000.0000000. The closest number in the Second Table to 9995000.0000000 is 9995001.224804. The bounds for this logarithm as calculated before are 5000.02525 and 5000.02475. Seeking the bounds for the difference of the logarithms results in 1.2254167 and 1.2254165 for the upper and lower bound respectively. Adding these bounds to 5000.02525 and 5000.02475 yields the bounds for the logarithm of 9995000.0000000. These are 5001.2506667 and 5001.2501665. The true logarithm of 9995000.0000000 was taken to be the average of these bounds or 5001.2504166. Following the same

Number	Logarithm
10000000.0000000	0.0000000
9999900.0000000	100.0005000
9999800.0010000	200.0010000
9999700.0030000	300.0015000
9999600.0060000	400.0020000
9999500.0099999	500.0025000
9999400.0149998	600.0030000
9999300.0209996	700.0035000
9999200.0279994	800.0040000
9999100.0359992	900.0045000
⋮	⋮
9995001.2248040	5000.0250000

TABLE IV
COMPLETED SECOND TABLE.

process, Napier determined the logarithm of 9900000.000000 to be 100503.35853072.

Taking 5001.2504166 to be the logarithm of 9995000.0000000, Napier once again doubled, tripled, quadrupled, etc. it to determine the logarithms for all of the numbers in the first column of the Third Table. Then starting again with 5001.2504166, he successively added 100503.3585307 to it to obtain all of the logarithms of the first row. To fill in the remainder of the Table, one can either take the first element in each column and successively add to it 5001.2504166 to obtain the logarithms for each column, or take the first element in each row and successively add to it 100503.3585307 to obtain the logarithms for each row.

Having found all of the logarithms in the Third Table, Napier instructed the reader to arrange it as seen in Table V so that in his own words, it "may be made complete and perfect." (Article 47) From this point forward, Napier referred to the Third Table as The Radical Table and stated that it be used to construct the Logarithmic Table of sines. The only work left undone was to determine how to calculate the logarithms of all the numbers in between the numbers tabulated in the Radical Table and to calculate the logarithms from zero to 5000000.

Fortunately, Napier already had the answers. For any number greater than 9996700, the logarithm can be determined by simply computing its lower bound (subtract given number from 10^7) as was done in estimating the logarithms of the First Table. To find the logarithm of any number embraced within the Radical Table, the bounds should be computed as was done in estimating the logarithms of the Second and Third Tables. To find the logarithms of numbers less than 5000000, Napier further exploited the property that the logarithms of similarly proportioned numbers are equidifferent. Given a number, one can multiple it by any convenient proportional factor and obtain a number which will lie within the limits of the Radical Table. This number is, of course, in proportion to the original number and its logarithm will differ from the logarithm of the original number by an amount dependent upon the proportional factor. The logarithm of the proportional number can be determined as before and all one has to do to obtain the logarithm of the original number is to add to it the difference of the logarithms.

Napier concluded by describing how his Logarithmic Table should be arranged. "Prepare forty-five pages" (Article 59) each with seven columns. Each page was devoted to two degrees. The first and the last columns each listed every minute within the two degrees in such a way that they were complements of each other. Wherefore, the first column began with 0 degrees, 0 minutes and ended with 0 degrees, 60 minutes, and the last column began with 89 degrees, 60 minutes and ended with 89 degrees, 0 minutes. In the second and sixth columns the corresponding sines were listed next to the angle with which it was associated and likewise, the logarithms of the sines were tabulated in the third and fifth columns. Lastly, the difference of the two logarithms from each row appeared in the fourth column. This was Napier's Logarithmic Table, the culmination of over twenty years work and a milestone in the history of mathematics.

V. Remaining Questions

Why Napier's logarithms are essentially to the base e^{-1}?

Napier required the velocity of β to be the same as the remaining distances, hence he necessarily, if unknowingly, specified the base of its geometric motion. To see this, suppose

$$x_\beta(t) = 10^7 - 10^7(1-p)^t \tag{7}$$

describes the motion of β, where p is an arbitrary proportional factor.

Thus,

$$z_\beta(t) = 10^7(1-p)^t \tag{8}$$

describes the lengths of the remaining distances.

The velocity of β is the derivative of equation 7

$$v_\beta(t) = \frac{d}{dt}\left[10^7 - 10^7(1-p)^t\right] = -10^7(1-p)^t \ln(1-p). \tag{9}$$

According to the definition, equations 8 and 9 must be equal. But equality holds *only* if $-\ln(1-p) = 1$, or if $1-p = 1/e$. Therefore, by definition, the base of the exponential function describing the motion of β is e^{-1}.

This fact has direct bearing on determining the base of Napier's logarithms. *If the total length of the second line equaled* 1, *Napier's logarithms would exactly be to the base* e^{-1}. However, since Napier choose 10^7, he multiplied his numbers and logarithms by that amount. For example, the logarithm of 9811277.6670907 as listed in the Radical Table is 190525.8660295. But the (modern) logarithm to the base e^{-1} of $190525.8660295/10^7$ is 0.01905258660379. Multiplying this result by 10^7 yields an answer which agrees with Napier's tabulated value up to eleven significant figures! The majority of the small error derives from the method of estimation for the logarithm of 9999999.

If the factor 10^7 is strictly taken into account, it can be argued that the true base is e^{-10^7} (shown below), however it is my opinion, that describing Napier's logarithms as having base e^{-1} best preserves their true nature.

Relation between Napierian and modern logarithms

Solve equation 3 for kT to obtain

$$kT = -\ln\left(\frac{z_\beta(kT)}{10^7}\right).$$

Substituting this result into equation 1 yields

$$x_\alpha(kT) = -10^7 \ln\left(\frac{z_\beta(kT)}{10^7}\right).$$

Thus equation 4 becomes

$$\text{Nap } \log z_\beta(kT) = -10^7 \ln\left(\frac{z_\beta(kT)}{10^7}\right) = -10^7 \ln z_\beta(kT) + 10^7 \ln 10^7.$$

Changing the base of the natural logarithm we can rewrite $-10^7 \ln z_\beta(kT)$ as $\log_b z_\beta(kT)$ where $b = e^{-1/10^7}$. Therefore, Napierian logarithms can also be thought of as logarithms to the base b shifted by $10^7 \ln 10^7$.

$$\text{Nap } \log z_\beta(kT) = \log_b z_\beta(kT) + 10^7 \ln 10^7$$

Why the word logarithm?

Napier coined the Latin word *logarithmus* which derives its meaning from two Greek words: *logos* meaning a principle relationship between numbers or ratio, and *arithmos* meaning number. Napier did not explain his view of the literal meaning of logarithimus, but seems appropriate since the concept of proportion is central to the idea of logarithms.

Utility

Modern logarithms derive most of their computational utility from the following three properties:

$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^n) = n\log_a x$$

Since Napier used 10^7 instead of 1, his logarithms do not strictly possess the first and third properties. However, if this factor is properly taken into account, his logarithms behave and can be used like modern logarithms. Certainly, Napier recognized the advantages early on, for he writes, "by it [his logarithmic table] all multiplications, divisions, and the more difficult extraction of roots are avoided." (Article 1)

VI. Conclusion

As a closing note, it should be remembered that Napier did not exist in a vacuum. The full development of logarithms sprang from the efforts of many individuals. Without Burgi, Briggs, Vlacq and many others logarithms would not have reached the mathematical maturity nor the usefulness they have. Napier himself welcomed collaboration and encouraged further development. For he said, "Nothing is perfect at birth."

VII. Acknowledgement

I thank Anthony F. Lexa, my father, for his comments and encouragement.

References

[1] John Napier, "The construction of the wonderful cannon of logarithms," 1619, 1889 English translation by William MacDonald.

[2] Carl B. Boyer, *A History of Mathematics*, John Wiley & Sons, Inc., New York, 1991.

[3] Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, Inc., New York, 3 edition, 1964.

[4] Watson Fulks, *Advanced Calculus: An Introduction to Analysis*, John Wiley & Sons, New York, 3 edition, 1978.

1st Column		2nd Column		3rd Column			69th Column	
Number	Logarithm	Number	Logarithm	Number	Logarithm		Number	Logarithm
10^7.0000000	0.0000000	9900000.0000000	100503.3585307	9801000.0000000	201006.7170614		5048858.8878707	6834228.3800876
9995000.0000000	5001.2504166	9895050.0000000	105504.6089473	9796099.5000000	206007.9674780		5046334.4584268	6839229.6305042
9990002.5000000	10002.5008332	9890102.4750000	110505.8593639	9791201.4502500	211009.2178946		5043811.2911976	6844230.8809208
9985007.4987500	15003.7512498	9885157.4237625	115507.1097805	9786305.8495249	216010.4683112		5041289.3855520	6849232.1313374
9980014.9950006	20005.0016664	9880214.8450506	120508.3601971	9781412.6966001	221011.7187278		5038768.7408592	6854233.3817540
9975024.9875031	25006.2520830	9875274.7376281	125509.6106137	9776521.9902518	226012.9691444		5036249.3564887	6859234.6321706
9970037.4750094	30007.5024996	9870337.1002593	130510.8610303	9771633.7292567	231014.2195610		5033731.2318105	6864235.8825872
9965052.4562719	35008.7529162	9865401.9317092	135512.1114469	9766747.9123921	236015.4699776		5031214.3661946	6869237.1330038
9960069.9300437	40010.0033328	9860469.2307433	140513.3618635	9761864.5384359	241016.7203942		5028698.7590115	6874238.3834204
9955089.8950787	45011.2537494	9855538.9961279	145514.6122801	9756983.6061666	246017.9708108		5026184.4096320	6879239.6338370
9950112.3501312	50012.5041660	9850611.2266299	150515.8626967	9752105.1143636	251019.2212274		5023671.3174272	6884240.8842536
9945137.2939561	55013.7545826	9845685.9210165	155517.1131133	9747229.0618064	256020.4716440		5021159.4817685	6889242.1346702
9940164.7253091	60015.0049992	9840763.0780560	160518.3635299	9742355.4472755	261021.7220606		5018648.9020276	6894243.3850868
9935194.6429465	65016.2554158	9835842.6965170	165519.6139465	9737484.2695518	266022.9724772		5016139.5775766	6899244.6355034
9930227.0456250	70017.5058324	9830924.7751688	170520.8643631	9732615.5274171	271024.2228938		5013631.5077878	6904245.8859200
9925261.9321022	75018.7562490	9826009.3127812	175522.1147797	9727749.2196534	276025.4733104		5011124.6920339	6909247.1363366
9920299.3011361	80020.0066656	9821096.3081248	180523.3651963	9722885.3450435	281026.7237270		5008619.1296879	6914248.3867532
9915339.1514856	85021.2570822	9816185.7599707	185524.6156129	9718023.9023710	286027.9741436		5006114.8201230	6919249.6371698
9910381.4819098	90022.5074988	9811277.6670907	190525.8660295	9713164.8904198	291029.2245602		5003611.7627130	6924250.8875864
9905426.2911689	95023.7579154	9806372.0282572	195527.1164461	9708308.3079746	296030.4749768		5001109.9568316	6929252.1380030
9900473.5780233	100025.0083320	9801468.8422431	200528.3668627	9703454.1538206	301031.7253934		4998609.4018532	6934253.3884196

TABLE V

RADICAL TABLE