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Mixed Finite Element Methods for Flow in Porous Media

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Abstract

Mixed finite element discretizations for problems arising in flow in porous medium applications are considered. We first study second order elliptic equations which model single phase flow. We consider the recently introduced expanded mixed method. Combined with global mapping techniques, the method is suitable for full conductivity tensors and general geometry domains. In the case of the lowest order Raviart-Thomas spaces, quadrature rules reduce the method to cell-centered finite differences, making it very efficient computationally.

We consider problems with discontinuous coefficients on multiblock domains. To obtain accurate approximations, we enhance the scheme by introducing Lagrange multiplier pressures along subdomain boundaries and coefficient discontinuities. This modification comes at no extra computational cost, if the method is implemented in parallel, using non-overlapping domain decomposition algorithms. Moreover, for regular solutions, it provides optimal convergence and discrete superconvergence for both pressure and velocity.

We next consider the standard mixed finite element method on non-matching grids. We introduce mortar pressures along the non-matching interfaces. The mortar space is chosen to have higher approximability than the normal trace of the velocity spaces. The method is shown to be optimally convergent for all variables. Superconvergence for the subdomain pressures and, if the tensor coefficient is diagonal, for the velocities and the mortar pressures is also proven.

We also consider the expanded mixed method on general geometry multiblock domains with non-matching grids. We analyze the resulting finite difference scheme and show superconvergence for all variables. Efficiency is not sacrificed by adding the mortar pressures. The computational complexity is shown to be comparable to the one on matching grids. Numerical results are presented, that verify the theory.

We finally consider mixed finite element discretizations for the nonlinear multiphase flow system. The system is reformulated as a pressure and a saturation equation. The methods described above are directly applied to the elliptic or parabolic pressure equation. We present an analysis of a mixed method on non-matching grids for the saturation equation of degenerate parabolic type.

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Chapter 1

Introduction

Numerical modeling of physical processes has played an increasing role in solving engineering problems in recent years. In many applications coupled systems of non-linear partial differential equations describe the mathematical models. The need for accurate solutions to these equations challenges numerical analysts to design new methods. The finite element methods have been the preferred tool for both mathematicians and engineers due to their simple physical interpretation, robustness, and elegant mathematical form. The mixed finite element methods [67, 22] gained even greater popularity in the last two decades. The reason is twofold. First, they provide very accurate approximations of two physical quantities - the primary unknown and its flux. Second, they conserve mass locally (on any element).

This work concentrates on applications such as ground-water contaminant transport and reservoir simulation which require modeling of fluid flow in a porous medium. Conservation of mass is a very important property in these cases and motivates the choice of the mixed finite element method as the main technique for numerical approximation. Russell and Wheeler [70] showed that special quadrature rules reduce a certain mixed method on rectangles to cell-centered finite differences, which have been used by the petroleum engineers for many years.

We first consider a single phase, non-gravitational flow model for the pressure p and the Darcy velocity $\mathbf{u}$:

$$\nabla \cdot \mathbf{u} = q$$
$$\mathbf{u} = -\frac{K}{\mu}\nabla p,$$

where K is the absolute permeability or hydraulic conductivity tensor, μ is the viscosity, and q is the source term. Even in this relatively simple model, some of the intrinsic features of the differential problem impose serious difficulties on the formulation and the analysis of the numerical method.

- The conductivity K is a full tensor. Most of the existing results for the mixed methods assume diagonal tensors.
- The medium can be highly heterogeneous with discontinuous conductivity.
- The domain can have general geometry. For very irregular domains a multiblock structure may be desirable to describe the geometry.
- In the presence of faults, non-matching grids in different parts of the domain may be needed.
- Accurate approximation of local phenomena (e.g., high gradients around the wells) may require locally refined grids.

The governing equations for multiphase fluid motion through a porous medium are mass conservation of phases [12, 24]. We consider an immiscible two-phase model

$$\frac{\partial(\phi s_i \rho_i)}{\partial t} + \nabla \cdot \rho_i \mathbf{u}_i = q_i,$$

where $i = w$ (wetting), n (non-wetting) denotes the phase, s_i is the phase saturation, ρ_i is the phase density, ϕ is the porosity, q_i is the source term, and

$$\mathbf{u}_i = -\frac{k_i(s_i)K}{\mu_i}(\nabla p_i - \rho_i g \nabla D)$$

is the Darcy velocity. Here p_i is the phase pressure, $k_i(s_i)$ is the phase relative permeability, μ_i is the phase viscosity, g is the gravitational constant, and D is the depth. The two equations are coupled via the volume balance equation

$$s_w + s_n = 1$$

and the capillary pressure relation

$$p_c(s_w) = p_n - p_w.$$

In addition to the above mentioned difficulties, in this case we have to deal with a coupled system of highly nonlinear transient partial differential equations. The equations are degenerate - the relative permeability $k_i(s_i)$ vanishes at $s_i = 0$, causing lack of regularity for the solution; therefore, the standard techniques in the analysis of the parabolic and elliptic equations cannot be applied directly.

A common approach has been to reformulate the two-phase system as a pressure equation of parabolic or elliptic type and a saturation equation of advection - diffusion type [24]. Therefore, we first concentrate on the mixed methods for elliptic and parabolic equations.

The mixed methods for second order elliptic equations have been extensively studied. Several mixed finite element spaces on triangles, rectangles, and prisms have been developed [72, 67, 62, 21, 19, 20, 26]. Their convergence and superconvergence properties are well understood [72, 67, 34, 61, 74, 37, 41]. The quasi-linear case has been studied in [60, 33, 36], and mixed methods for parabolic equations have been analyzed in [51, 39].

Most of the above works assume a diagonal tensor coefficient, and while the convergence results and the pressure superconvergence results can be easily generalized for a full tensor, this is not the case for the superconvergence in the velocity. Recently, a variant of the mixed finite element method, which we call "the expanded mixed method", has been introduced [75, 52, 25, 9]. In [9], full tensor coefficients are handled efficiently on rectangles by deriving an equivalent cell-centered finite difference scheme for the pressure, which generalizes the result of Russell and Wheeler [70] for diagonal tensors. Moreover, superconvergence for the velocity is obtained at certain discrete points.

Serious problems arise in the analysis and the implementation of the mixed methods on domains with irregular geometry. Partitions of triangles or tetrahedra lead to unstructured data and much larger problems to solve, compared to logically rectangular meshes of quadrilateral-type elements. Also, the computed velocities are not superconvergent on triangles, even for diagonal tensors. This motivates the choice of logically rectangular grids. Thomas [72] showed optimal convergence for both pressure and velocity on quadrilaterals. In [5], a modification of the expanded mixed method is considered for handling full tensors and general domains, leading to cell-centered finite difference schemes on logically rectangular and triangular grids. All computations are performed on a regular grid after mapping the problem to a reference computational domain. Under assumptions of smoothness of the mapping and the coefficients, optimal convergence for the pressure and the velocity is obtained. Moreover, superconvergence for both variables is obtained on quadrilateral-type elements. Unfortunately, the theory breaks down if the assumption of smoothness is omitted and no convergence can be obtained near discontinuities. This is confirmed by numerical experiments.

As we noted above, in many applications the medium is heterogeneous and the permeability tensor is discontinuous. Also, in describing the geometry of a highly irregular domain with a logically rectangular grid, a multiblock structure has to be used. The true domain is then mapped to a union of rectangles (parallelepipeds) and the grid may be non-smooth across the interfaces. It turns out, however, that this leads to a discontinuous tensor on the computational domain. Therefore both problems can be attacked with similar techniques.

The macro-hybrid form of the mixed method (see [11]) has to be used for handling discontinuities. In this formulation, pressure Lagrange multipliers are added along the interfaces of discontinuities. This allows the pressure gradient to be approximated by a discontinuous function. Numerical results in [5] indicate that convergence is regained in this case. In Chapter 2 we present theoretical analysis for the macro-hybrid form of the expanded mixed method for problems with piece-wise smooth coefficients.

The presence of faults imposes another very interesting problem for the numerical method. These natural discontinuities in the geological structure are very difficult to follow by continuous grids. Therefore non-matching grids in different parts of the domain are needed. Non-matching grids also allow for using a different coordinate system in each block, leading to a greater flexibility in describing irregular geometries. To the author's knowledge, no analysis has been done for mixed finite element methods on non-matching grids.

In our applications, it is important to construct schemes that conserve mass across the faults. To achieve this we employ "the mortar finite element method" (see [16] for such work with Galerkin finite elements). The idea is to introduce an interface finite element space (called a mortar space) and use it to impose continuity of fluxes in a weak sense. In the mixed method this is the pressure Lagrange multiplier space. In the case of matching grids, we may use the standard choice of the normal trace of the velocity space on the interface. Therefore we recover the macro-hybrid form of the mixed method on matching grids. If the grids on the two sides do not match, a different mortar space is needed. We show theoretically and numerically that, in order to preserve optimal convergence in this case, the mortar space must consist of piece-wise polynomials of one degree higher then the degree of the normal trace of the velocity spaces. Moreover, we are able to maintain the known pressure and velocity superconvergence properties of the mixed methods.

The mortar finite element method provides a new way to handle locally refined grids. In fact, grids with local refinement are just a special case of non-matching grids. In the previous research (see, e.g., [18, 40, 23, 59, 43]) authors use the notion of "slave" nodes, forcing fluxes to be continuous across the coarse-fine interface. This however, forces all fine interface velocities within one coarse cell to be the same, which leads to large numerical errors on the interface. A special choice of the mortar space recovers the known schemes. However, we can relax the strict continuity of fluxes by taking coarser mortar space. As numerical examples indicate, this reduces the size of the velocity error.

The above formulations are also computationally attractive when combined with domain decomposition solution techniques. The Glowinski-Wheeler algorithm [48], later generalized in [29], and the balancing domain decomposition [57, 58, 28] exploit similar hybridization techniques. These methods were originally implemented for the standard mixed method by Cowsar in the simulator ParFlow1, which was later modified by San Soucie and the author to handle the expanded mixed formulation. The code was later generalized by the author to handle general geometry domains with non-matching logically rectangular grids in different subdomains.

Recently some authors showed that the Lagrange multiplier formulation allows optimal order substructuring preconditioners on non-matching grids to be constructed [1, 53].

We next apply the above methods to the coupled system of multiphase flow equations. As we mentioned above, a common approach is to rewrite the system as a pressure and a saturation equation. The latter is a degenerate advection - diffusion equation with a diffusion term vanishing at $s = 0, 1$.

Many authors have addressed the issues of analysis and approximation of solutions to degenerate parabolic equations. Existence and uniqueness of weak solutions have been studied in [65, 46, 54, 3], and low regularity for the solutions have been shown.

A widely used technique is to regularize the original problem, and then approximate the regularized problem (see, e.g., [69, 50, 63, 45, 44]). A suitable choice of the regularization parameter allows error estimates to be derived.

Another common tool is the Kirchoff transformation. For the saturation equation or its simplified version - the porous medium equation, it allows one to simplify the elliptic term, treating the degeneracy analytically [69, 45, 44, 10]. The obtained form is very close to the form of the two phase Stefan problem, making the analysis of the two problems similar [50, 63, 64].

The only previous work that uses mixed methods for degenerate parabolic equations is [10]. No regularization is used there and via the Kirchoff transformation the authors bound the discretization error in approximation theory terms.

Very limited work has been done on analysis of the coupled system of equations. Existence of weak solutions to the incompressible model has been shown in [2, 4]. The few studies of numerical approximation to the problem treat only the nondegenerate case [32, 27].

In this work we formulate and study mixed finite element schemes for the numerical solution of the multiphase flow system. We consider the expanded mixed method applied to pressure-saturation formulation. The method is shown to conserve mass of both phases locally. It is also extended to handle discontinuous permeability tensors, multiblock domains, and non-matching and locally refined grids. Analysis of a mixed method with mortars for the degenerate saturation equation on non-matching grids is presented.

The rest of the thesis is organized as follows. In Chapter 2 we recall the expanded mixed method for linear elliptic problems with tensor coefficients on general geometry and present theoretical and numerical results for the macro-hybrid formulation of the method on multiblock domains and discontinuous coefficients. The subject of Chapter 3 is the standard mixed methods with mortars on non-matching and locally refined grids. In Chapter 4 we present and analyze the expanded mixed methods for elliptic problems on irregular multiblock domains with non-matching grids. Mixed finite element discretizations for the coupled system of multiphase flow equations are discussed in Chapter 5.

Chapter 2

The expanded mixed method on general geometry

We consider a second order elliptic problem on a geometrically general domain $\Omega \subseteq \mathbf{R}^d, d = 2$ or 3, with a Lipschitz boundary $\partial\Omega$. In its mixed form we seek $(\mathbf{u}, p)$ such that

$$\mathbf{u} = -K\nabla p \qquad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = f \qquad \text{in } \Omega, \tag{2.2}$$

$$p = g^D \qquad \text{on } \Gamma^D, \tag{2.3}$$

$$\mathbf{u} \cdot \nu = g^N \qquad \text{on } \Gamma^N, \tag{2.4}$$

where K is a symmetric positive definite tensor with $L^\infty(\Omega)$ components, and ν is the outward unit normal vector on $\partial\Omega$, which is decomposed into Γ^D and Γ^N. In flow in porous medium applications p is the pressure, $\mathbf{u}$ is the velocity vector, and K is the conductivity tensor divided by the viscosity.

2.1 Some notation and formulation of the method

For a bounded domain $S \subset \mathbf{R}^d$, let $(\cdot,\cdot)_S$ denote the $L^2(S)$ inner product or duality pairing. We may omit S if $S = \Omega$. The $L^2(S)$ norm is given by

$$\|\phi\|_{0,S} = (\phi, \phi)_S^{1/2}.$$

We may also omit the subscript 0. Let $\langle\cdot,\cdot\rangle_{\partial S}$ denote the $L^2(\partial S)$ inner product or duality pairing. Define

$$H(\text{div}; S) = \{\mathbf{v} \in (L^2(S))^d : \nabla \cdot \mathbf{v} \in L^2(S)\},$$

with a norm

$$\|\mathbf{v}\|_{H(\text{div};S)} = \left\{ \int_S \left(|\mathbf{v}|^2 + |\nabla \cdot \mathbf{v}|^2\right) dx \right\}^{1/2}.$$

Let $\|\cdot\|_{m,p,S}$ denote the norm of $W^{m,p}(S)$, the Sobolev space of m-times differentiable functions in $L^p(S)$. We will mainly use the Hilbert space $H^m(S) = W^{m,2}(S)$ with a norm

$$\|u\|_{m,S} = \left(\sum_{|\alpha|\le m} \int_S |\partial^\alpha u|^2\, dx \right)^{1/2}.$$

For a non-integer r, $m < r < m+1$, $H^r(S)$ is defined by interpolation of $H^m(S)$ and $H^{m+1}(S)$ (see [56]). Let, for any real r,

$$H_0^r(S) = \text{ closure of } C_0^\infty(S) \text{ in } H^r(S)$$

We denote by $H^{-r}(S)$ the dual space of $H_0^r(S)$ with a norm $\|\cdot\|_{-r,S}$. The Sobolev spaces on ∂S are defined in a similar fashion. For $0 < r < 1$, $H^r(\partial S)$ is equipped with the norm

$$\|u\|_{r,\partial S} = \left\{ \|u\|_{0,S}^2 + |u|_{r,S}^2 \right\}^{1/2}, \quad |u|_{r,S}^2 = \int_S \int_S \frac{|u(t_1) - u(t_2)|^2}{|t_1 - t_2|^{d-1+2r}} ds_{t_1} ds_{t_2}.$$

We recall two important trace theorems needed in the analysis. The proof of the first result can be found in [56].

Theorem 2.1 (Trace) Let $\partial\Omega$ be of class C^k and $1/2 < s \le k$. Then the trace map γ, with $\gamma u = u|_{\partial\Omega}$ for smooth functions u, can be extended as a continuous map

$$\gamma : H^s(\Omega) \overset{\text{onto}}{\rightarrow} H^{s-1/2}(\partial\Omega).$$

The next theorem is proven in [67].

Theorem 2.2 (Normal trace) For a Lipschitz domain Ω, there exists a continuous map

$$\gamma_\nu : H(\text{div}; \Omega) \overset{\text{onto}}{\rightarrow} H^{-1/2}(\partial\Omega),$$

such that $\gamma_\nu \mathbf{u} = \mathbf{u} \cdot \nu|_{\partial\Omega}$ for smooth vector functions $\mathbf{u}$.

We now continue with the formulation of the expanded mixed method. Following [9, 5] we introduce the adjusted pressure gradient

$$\tilde{\mathbf{u}} = -M^{-1}\nabla p,$$

where M is some symmetric positive definite tensor related to the geometry of Ω. Then

$$\mathbf{u} = KM\tilde{\mathbf{u}}.$$

Let

$$\mathbf{V} = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}\cdot\nu \in L^2(\Gamma^N)\},\ \tilde{\mathbf{V}} = (L^2(\Omega))^d,\ W = L^2(\Omega),\ \Lambda = L^2(\Gamma^N).$$

We have the following expanded mixed variational formulation. A weak solution of (2.1)–(2.4) is $(\mathbf{u}, \tilde{\mathbf{u}}, p, \lambda) \in \mathbf{V} \times \tilde{\mathbf{V}} \times W \times \Lambda$ such that

$$(M\mathbf{u}, \tilde{\mathbf{v}}) = (MKM\tilde{\mathbf{u}}, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}, \tag{2.5}$$

$$(M\tilde{\mathbf{u}}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) \tag{2.6}$$

$$= -\langle g^D, \mathbf{v}\cdot\nu\rangle_{\Gamma^D} - \langle \lambda, \mathbf{v}\cdot\nu\rangle_{\Gamma^N}, \qquad \mathbf{v} \in \mathbf{V},$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \qquad w \in W, \tag{2.7}$$

$$\langle \mathbf{u}\cdot\nu, \mu\rangle_{\Gamma^N} = \langle g^N, \mu\rangle_{\Gamma^N}, \qquad \mu \in \Lambda. \tag{2.8}$$

Our main requirement is that there exists a smooth (at least C^2) invertible map

$$F : \mathbf{R}^d \to \mathbf{R}^d, \quad F(\hat{\Omega}) = \Omega,$$

where $\hat{\Omega}$ is a computational reference domain with a standard shape. Let $\hat{\mathcal{T}}_h$ be a quasi-uniform family of partitions of $\hat{\Omega}$ into standard shaped elements. Then F defines a curved element partition $\mathcal{T}_h$ of Ω. Denote by $\hat{\mathbf{V}}_h \times \hat{W}_h \subset H(\mathrm{div};\hat{\Omega}) \times L^2(\hat{\Omega})$ any of the known mixed finite element spaces defined on $\hat{\mathcal{T}}_h$ [72, 67, 62, 21, 19, 20, 26]. Let $\hat{\tilde{\mathbf{V}}}_h$ be a finite element subspace of $(L^2(\hat{\Omega}))^d$ such that $\hat{\mathbf{V}}_h \subseteq \hat{\tilde{\mathbf{V}}}_h$. Let $\hat{\Lambda}_h^N \subset L^2(\hat{\Gamma}^N)$ be the Lagrange multiplier space on $\hat{\Gamma}^N$, corresponding to the normal trace of $\hat{\mathbf{V}}_h$.

The spaces $\mathbf{V}_h$, W_h, $\tilde{\mathbf{V}}_h$, and Λ_h^N on the partition $\mathcal{T}_h$ of Ω are defined by the Piola transformation for the vector spaces and by the standard isomorphism for the scalar spaces as follows (see also [72, 5]). For each $\hat{\mathbf{v}} \in \hat{\mathbf{V}}_h$ ($\hat{\tilde{\mathbf{V}}}_h$), $\hat{w} \in \hat{W}_h$, and $\hat{\mu} \in \hat{\Lambda}_h^N$, define $\mathbf{v} \in \mathbf{V}_h$ ($\tilde{\mathbf{V}}_h$), $w \in W_h$, and $\mu \in \Lambda_h^N$ by

$$\mathbf{v}(x) = \left(\frac{1}{J} DF\,\hat{\mathbf{v}}\right) \circ F^{-1}(x), \tag{2.9}$$

$$w(x) = \hat{w} \circ F^{-1}(x), \tag{2.10}$$

$$\mu(x) = \hat{\mu} \circ F^{-1}(x), \tag{2.11}$$

where DF is the Jacobian matrix of F and

$$J = |\det(DF)|.$$

In the expanded mixed method on general geometry we solve for $\mathbf{u}_h \in \mathbf{V}_h$, $\tilde{\mathbf{u}}_h \in \tilde{\mathbf{V}}_h$, $p_h \in W_h$, and $\lambda_h \in \Lambda_h^N$ satisfying

$$(M\mathbf{u}_h, \tilde{\mathbf{v}}) = (MKM\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_h, \tag{2.12}$$

$$(M\tilde{\mathbf{u}}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) \tag{2.13}$$

$$= -\langle g^D, \mathbf{v} \cdot \nu \rangle_{\Gamma^D} - \langle \lambda_h, \mathbf{v} \cdot \nu \rangle_{\Gamma^N}, \qquad \mathbf{v} \in \mathbf{V}_h,$$

$$(\nabla \cdot \mathbf{u}_h, w) = (f, w), \qquad w \in W_h, \tag{2.14}$$

$$\langle \mathbf{u}_h \cdot \nu, \mu \rangle_{\Gamma^N} = \langle g^N, \mu \rangle_{\Gamma^N}, \qquad \mu \in \Lambda_h^N. \tag{2.15}$$

Optimal convergence for $\|p - p_h\|_0$, $\|\lambda - \lambda_h\|_{-1/2,\Gamma^N}$, $\|\mathbf{u} - \mathbf{u}_h\|_0$, $\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0$, and $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_0$ has been shown in [9, 5] under the assumption of smoothness of K and M.

2.2 A Cell-centered finite difference scheme on logically rectangular grids

The main motivation to study the expanded mixed method is that it can be implemented as cell-centered finite differences for the pressure, even for problems with full tensor coefficients on general geometry domains. Special choice of M simplifies the interactions of the vector basis functions in (2.12) and (2.13). Define

$$M(F(\hat{x})) = \left(J(DF^{-1})^T DF^{-1}\right)(\hat{x}). \tag{2.16}$$

A transformation of (2.12)-(2.15) through (2.9)-(2.11) leads to the following problem on the reference domain $\hat{\Omega}$. Find $\hat{\mathbf{u}}_h \in \hat{\mathbf{V}}_h$, $\hat{\tilde{\mathbf{u}}}_h \in \hat{\tilde{\mathbf{V}}}_h$, $\hat{p}_h \in \hat{W}_h$, and $\hat{\lambda}_h \in \hat{\Lambda}_h^N$ such that

$$(\hat{\mathbf{u}}_h, \hat{\tilde{\mathbf{v}}})_{\hat{\Omega}} = \left(JDF^{-1}K(DF^{-1})^T\hat{\tilde{\mathbf{u}}}_h, \hat{\tilde{\mathbf{v}}}\right)_{\hat{\Omega}}, \qquad \hat{\tilde{\mathbf{v}}} \in \hat{\tilde{\mathbf{V}}}_h, \tag{2.17}$$

$$(\hat{\tilde{\mathbf{u}}}_h, \hat{\mathbf{v}})_{\hat{\Omega}} - (\hat{p}_h, \hat{\nabla} \cdot \hat{\mathbf{v}})_{\hat{\Omega}} \tag{2.18}$$

$$= -\langle \hat{g}^D, \hat{\mathbf{v}} \cdot \hat{\nu} \rangle_{\hat{\Gamma}^D} - \langle \hat{\lambda}_h, \hat{\mathbf{v}} \cdot \hat{\nu} \rangle_{\hat{\Gamma}^N}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h,$$

$$(\hat{\nabla} \cdot \hat{\mathbf{u}}_h, \hat{w})_{\hat{\Omega}} = (J\hat{f}, \hat{w})_{\hat{\Omega}}, \qquad \hat{w} \in \hat{W}_h, \tag{2.19}$$

$$\langle \hat{\mathbf{u}}_h \cdot \hat{\nu}, \hat{\mu} \rangle_{\hat{\Gamma}^N} = \langle J_{\hat{\nu}} \hat{g}^N, \hat{\mu} \rangle_{\hat{\Gamma}^N}, \qquad \hat{\mu} \in \hat{\Lambda}_h^N, \tag{2.20}$$

where

$$J_{\hat{\nu}}(\hat{x}) = J(\hat{x})|(DF^{-1})^T \hat{\nu}|.$$

To obtain cell-centered finite differences on logically rectangular grids, we consider the lowest order Raviart-Thomas spaces [67] and take $\tilde{\mathbf{V}}_h = \mathbf{V}_h$. We employ quadrature rules to approximate the vector integrals in (2.17), (2.18). The two equations are replaced by

$$(\hat{\mathbf{u}}_h, \hat{\hat{\mathbf{v}}})_{\mathrm{TM},\hat{\Omega}} = \left(JDF^{-1}K(DF^{-1})^T\hat{\hat{\mathbf{u}}}_h, \hat{\hat{\mathbf{v}}}\right)_{\mathrm{T},\hat{\Omega}}, \qquad \hat{\hat{\mathbf{v}}} \in \hat{\hat{\mathbf{V}}}_h, \tag{2.21}$$

$$\begin{aligned}(\hat{\hat{\mathbf{u}}}_h, \hat{\mathbf{v}})_{\mathrm{TM},\hat{\Omega}} - (\hat{p}_h, \hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}} \\ = -\langle \hat{g}^D, \hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}^D} - \langle \hat{\lambda}_h, \hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}^N}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h,\end{aligned} \tag{2.22}$$

where $(\cdot,\cdot)_{\mathrm{M},S}$, $(\cdot,\cdot)_{\mathrm{T},S}$ denote an application of the midpoint and trapezoidal rule, respectively, to the $L^2(S)$ inner product with respect to $\hat{\mathcal{T}}_h$, and $(\cdot,\cdot)_{\mathrm{TM},S}$ is the trapezoidal-midpoint rule, defined for vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ as

$$(\mathbf{v},\mathbf{q})_{\mathrm{TM},S} = (v_1, q_1)_{\mathrm{T}\times\mathrm{M}\times\mathrm{M},S} + (v_2, q_2)_{\mathrm{M}\times\mathrm{T}\times\mathrm{M},S} + (v_3, q_3)_{\mathrm{M}\times\mathrm{M}\times\mathrm{T},S}. \tag{2.23}$$

This choice of quadrature rules gives diagonal coefficient matrices for $\hat{\mathbf{u}}_h$ in (2.21) and for $\hat{\hat{\mathbf{u}}}_h$ in (2.22), since the TM rule uses exactly the nodal velocity points. Therefore the vector unknowns can be eliminated, leading to cell-centered finite differences for the pressure with a 9 point stencil in two dimensions and a 19 point stencil in three dimensions.

The convergence properties of this scheme have been analyzed in [9, 5] under the assumption of smoothness of K and F. There, superconvergence for the pressure, velocity, and its divergence at the midpoints of the elements is shown.

2.3 The expanded mixed method for problems with discontinuous coefficients on multiblock domains

The results of the previous section required the conductivity K and the mapping F to be smooth functions. As we mentioned in the introduction, this is often not the case. The conductivity of heterogeneous media can be discontinuous. On the other hand, very irregular domains are difficult to map to a regular shape domain via smooth mappings. In those cases, multiblock structures may have to be used, with different mappings for the different blocks. Therefore the global map is piecewise smooth and could be non-differentiable across the interfaces.

Computational results in [6, 5] indicate loss of convergence along interfaces of map or coefficient discontinuities. Note that the transformed computational tensor is

$$\mathcal{K} = JDF^{-1}K(DF^{-1})^T, \tag{2.24}$$

so a nonsmooth mapping leads to a discontinuous coefficient on the computational domain. This implies discontinuous pressure gradient. In the scheme, however, $\hat{\tilde{\mathbf{u}}}_h = \hat{\nabla}\hat{p}_h$ is continuous in the normal direction. This inconsistency causes the loss of convergence along the discontinuities.

To correct for the above problem, we need to relax the continuity of $\tilde{\mathbf{V}}_h$ across the interfaces. However, we would like to keep the matrices arising from vector inner products of type $(\mathbf{v}, \tilde{\mathbf{v}})$ square. In this case we can eliminate both vector unknowns to obtain finite differences for the pressure. Therefore we also relax the continuity of $\mathbf{V}_h$ across the interfaces and then impose it weakly. We do so by adding pressure Lagrange multipliers along the interfaces, obtaining a partially hybridized mixed formulation.

Assume that Ω can be decomposed into a set of non-overlapping subdomains such that the restrictions of K and F to any subdomain are smooth functions. This can be achieved by aligning large discontinuities in K (like interfaces between rock strata) with subdomain boundaries. If K varies on a smaller scale, homogenization can be used to obtain conductivity that is smooth within a subdomain (see [15, 17, 38] and references therein).

Let

$$\Omega = \cup_{i=1}^n \Omega_i, \quad \Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j, \quad \Gamma_i = \partial\Omega_i \backslash \Gamma^D, \quad \Gamma = \cup_{i=1}^n \Gamma_i.$$

Let

$$\mathbf{V}_i = \left\{ \mathbf{v} \in H(\mathrm{div}; \Omega_i) : \mathbf{v} \cdot \nu \in L^2(\Gamma_i) \right\}, \qquad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

$$\tilde{\mathbf{V}}_i = (L^2(\Omega_i))^d, \qquad \tilde{\mathbf{V}} = \bigoplus_{i=1}^n \tilde{\mathbf{V}}_i = (L^2(\Omega))^d,$$

$$W_i = L^2(\Omega_i), \qquad W = \bigoplus_{i=1}^n W_i = L^2(\Omega),$$

and

$$\Lambda = L^2(\Gamma).$$

The following weak form for $(\mathbf{u}, \tilde{\mathbf{u}}, p, \lambda) \in \mathbf{V} \times \tilde{\mathbf{V}} \times W \times \Lambda$ can be obtained by integrating the original equations over each Ω_i and summing.

$$(M\mathbf{u}, \tilde{\mathbf{v}}) = (MKM\tilde{\mathbf{u}}, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}, \quad (2.25)$$

$$(M\tilde{\mathbf{u}}, \mathbf{v}) = \sum_{i=1}^n \Big((p, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} \Big) - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\Gamma^D}, \qquad \mathbf{v} \in \mathbf{V}, \quad (2.26)$$

$$\sum_{i=1}^{n}(\nabla\cdot\mathbf{u},w)_{\Omega_i} = (f,w), \qquad w\in W, \quad (2.27)$$

$$\sum_{i=1}^{n}\langle\mathbf{u}\cdot\nu,\mu\rangle_{\Gamma_i} = \langle g^N,\mu\rangle_{\Gamma^N}, \qquad \mu\in\Lambda. \quad (2.28)$$

The flux continuity equation (2.28) implies that $\mathbf{u}\in H(\mathrm{div};\Omega)$, and we recover the weak form (2.5)–(2.8).

Let F maps a union of reference subdomains $\hat{\Omega}=\cup_{i=1}^{n}\hat{\Omega}_i$ onto Ω, $F(\hat{\Omega}_i)=\Omega_i$. Let $\mathcal{T}_{h,i}$ be a curved element partition of Ω_i, $1\le i\le n$, defined by mapping a standard shape element partition $\hat{\mathcal{T}}_{h,i}$ of $\hat{\Omega}_i$, and let $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ match on $\Gamma_{i,j}$. Let

$$\mathbf{V}_{h,i}\times W_{h,i}\subset\mathbf{V}_i\times W_i$$

be any of the known mixed spaces on Ω_i. Let

$$\mathbf{V}_h=\bigoplus_{i=1}^{n}\mathbf{V}_{h,i}, \quad \tilde{\mathbf{V}}_h=\bigoplus_{i=1}^{n}\tilde{\mathbf{V}}_{h,i}, \quad W_h=\bigoplus_{i=1}^{n}W_{h,i},$$

and let Λ_h denote the Lagrange multiplier space on Γ associated with $\mathbf{V}_h\times W_h$. For convinienience we assume that Λ_h is extended by zero on Γ^D. We then solve for $\mathbf{u}_h\in\mathbf{V}_h$, $\tilde{\mathbf{u}}_h\in\tilde{\mathbf{V}}_h$, $p_h\in W_h$, and $\lambda_h\in\Lambda_h$ satisfying

$$(M\mathbf{u}_h,\tilde{\mathbf{v}}) = (MKM\tilde{\mathbf{u}}_h,\tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}}\in\tilde{\mathbf{V}}_h, \quad (2.29)$$

$$(M\tilde{\mathbf{u}}_h,\mathbf{v}) = \sum_{i=1}^{n}\Big((p_h,\nabla\cdot\mathbf{v})_{\Omega_i}-\langle\lambda_h,\mathbf{v}\cdot\nu\rangle_{\Gamma_i}\Big)-\langle g^D,\mathbf{v}\cdot\nu\rangle_{\Gamma^D}, \qquad \mathbf{v}\in\mathbf{V}_h, \quad (2.30)$$

$$\sum_{i=1}^{n}(\nabla\cdot\mathbf{u}_h,w)_{\Omega_i} = (f,w), \qquad w\in W_h, \quad (2.31)$$

$$\sum_{i=1}^{n}\langle\mathbf{u}_h\cdot\nu,\mu\rangle_{\Gamma_i} = \langle g^N,\mu\rangle_{\Gamma^N}, \qquad \mu\in\Lambda_h. \quad (2.32)$$

Following [5] we refer to (2.29)–(2.32) as the enhanced method. Note that the flux-matching condition (2.32) guarantees that $\mathbf{u}_h\in H(\mathrm{div};\Omega)$, while $\tilde{\mathbf{u}}_h$ is discontinuous across the subdomain interfaces.

With M defined in (2.16), (2.29)–(2.32) is transformed into the following problem on the reference domain $\hat{\Omega}$. Find $\hat{\mathbf{u}}_h\in\hat{\mathbf{V}}_h$, $\hat{\tilde{\mathbf{u}}}_h\in\hat{\tilde{\mathbf{V}}}_h$, $\hat{p}_h\in\hat{W}_h$, and $\hat{\lambda}_h\in\hat{\Lambda}_h$ such that

$$(\hat{\mathbf{u}}_h,\hat{\tilde{\mathbf{v}}})_{\hat{\Omega}} = (\mathcal{K}\hat{\tilde{\mathbf{u}}}_h,\hat{\tilde{\mathbf{v}}})_{\hat{\Omega}}, \qquad \hat{\tilde{\mathbf{v}}}\in\hat{\tilde{\mathbf{V}}}_h, \quad (2.33)$$

$$(\hat{\tilde{\mathbf{u}}}_h,\hat{\mathbf{v}})_{\hat{\Omega}} = \sum_{i=1}^{n}\Big((\hat{p}_h,\hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}_i}-\langle\hat{\lambda}_h,\hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}_i}\Big)-\langle\hat{g}^D,\hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}^D}, \qquad \hat{\mathbf{v}}\in\hat{\mathbf{V}}_h, \quad (2.34)$$

$$\sum_{i=1}^{n}(\hat{\nabla}\cdot\hat{\mathbf{u}}_h,\hat{w})_{\hat{\Omega}_i} = (J\hat{f},\hat{w})_{\hat{\Omega}}, \qquad \hat{w}\in\hat{W}_h, \tag{2.35}$$

$$\sum_{i=1}^{n}\langle\hat{\mathbf{u}}_h\cdot\hat{\nu},\hat{\mu}\rangle_{\hat{\Gamma}_i} = \langle J_\nu\hat{g}^N,\hat{\mu}\rangle_{\hat{\Gamma}^N}, \qquad \hat{\mu}\in\hat{\Lambda}_h. \tag{2.36}$$

Now take $\mathbf{V}_h^i \times W_h^i$ to be the RT_0 spaces on a curved logically rectangular grid $\mathcal{T}_{h,i}$ on Ω_i. As in Section 2.2, quadrature rules are employed to approximate the vector inner products. Equations (2.33) and (2.34) are replaced with

$$(\hat{\mathbf{u}}_h,\hat{\hat{\mathbf{v}}})_{\mathrm{TM},\hat{\Omega}} = (\mathcal{K}\hat{\hat{\mathbf{u}}}_h,\hat{\hat{\mathbf{v}}})_{\mathrm{T},\hat{\Omega}}, \qquad \hat{\hat{\mathbf{v}}}\in\hat{\hat{\mathbf{V}}}_h, \tag{2.37}$$

$$(\hat{\hat{\mathbf{u}}}_h,\hat{\mathbf{v}})_{\mathrm{TM},\hat{\Omega}} = \sum_{i=1}^{n}\left((\hat{p}_h,\hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}_i} - \langle\hat{\lambda}_h,\hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}_i}\right) - \langle\hat{g}^D,\hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}^D}, \quad \hat{\mathbf{v}}\in\hat{\mathbf{V}}_h. \tag{2.38}$$

Elimination of the velocities leads to a sparse symmetric positive definite system for p_h and λ_h. The stencil is slightly modified only near the interfaces where Lagrange multipliers are added. Therefore the computational cost for the enhanced method is compatible with the one for the expanded method. Moreover, we use the Glowinski-Wheeler domain decomposition algorithm [48, 29] to solve the linear system. There, Lagrange multipliers are introduced on subdomain interfaces for the purpose of parallelism, so the solution to the enhanced method comes at no extra cost.

Computational results in [6, 5] show that convergence along the discontinuities is regained for the enhanced method. Using techniques similar to those used in [9, 5] we are able to confirm these results theoretically.

2.3.1 A regularity theorem

For the error analysis we need a regularity result for the solution of the transformed problem. To avoid technical complications, we assume for the rest of the chapter that the union of $\hat{\Omega}_i$ forms a rectangular block and $\Gamma_D = \partial\Omega$.

We need some preliminary results. Let e_i be the unit coordinate vector in the x_i direction. For any function u on Ω, define the *difference quotient* in the direction e_i by

$$\delta_i^h u(x) = \frac{u(x+he_i)-u(x)}{h}, \quad h\neq 0.$$

The following two results can be found in [47], Lemma 7.23 and Lemma 7.24.

Lemma 2.1 If u in $H^1(\Omega)$, then $\delta_i^h u \in L^2(\Omega')$, for any $\Omega' \subset\subset \Omega$ such that $|h| < \text{dist}(\Omega', \partial\Omega)$. Moreover,

$$\|\delta_i^h u\|_{L^2(\Omega')} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}.$$

Lemma 2.2 Let $u \in L^2(\Omega)$ and suppose there exists a constant M such that $\delta_i^h u \in L^2(\Omega')$ and $\|\delta_i^h u\|_{L^2(\Omega')} \leq M$ for all $h \neq 0$ and $\Omega' \subset\subset \Omega$ satisfying $|h| < \text{dist}(\Omega', \partial\Omega)$. Then the weak derivative $\partial u/\partial x_i$ exists and satisfies $\|\partial u/\partial x_i\|_{L^2(\Omega)} \leq M$.

We now state the regularity result.

Theorem 2.3 Let p be the solution to

$$-\nabla \cdot K\nabla p = f \qquad \text{in } \Omega, \tag{2.39}$$

$$p = g \qquad \text{on } \partial\Omega, \tag{2.40}$$

where $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, is a rectangular domain decomposed into a union of rectangular blocks Ω_i, $1 \leq i \leq n$, K is a symmetric positive definite matrix with components that are uniformly Lipschitz continuous on Ω_i, $f \in L^2(\Omega)$, and $g \in H^2(\Omega_i) \cap H^1(\Omega)$. Let $\Omega_i^\varepsilon = \Omega_i \backslash (\cup B_\varepsilon(\xi))$, where ξ is any corner of Ω_i that is not a corner of Ω if $d = 2$, or any point on an edge of Ω_i that is not an edge of Ω if $d = 3$, and $B_\varepsilon(\xi)$ is a ball with a center ξ and a radius ε. Then $p \in H^2(\Omega_i^\varepsilon)$ and there exists a positive constant $C = C(K, \Omega, \varepsilon)$ such that

$$\sum_{i=1}^{n} \|p\|_{2,\Omega_i^\varepsilon} \leq C(\|f\|_0 + \|g\|_1 + \sum_{i=1}^{n} \|g\|_{2,\Omega_i}).$$

Proof Cover the compact set $\overline{\Omega}$ by open sets $U_1, ..., U_N$, where $U_i \subset\subset \Omega_i$, $1 \leq i \leq n$, and the rest are balls $B_\varepsilon(\xi)$ with centers ξ on $\Gamma \cup \partial\Omega$ and a small radius ε. Assume that, for any corner ξ of Ω_i, a ball with a center ξ is an element of the covering set. If $d = 3$, assume also that the balls that cover the subdomain edges have centers on the edges. Next, find a partition of unity $\phi_1, ..., \phi_N$ subordinate to this cover, i.e.,

$$\sum_{j=1}^{N} \phi_j(x) = 1, \quad x \in \Omega, \qquad \text{and } \phi_j \in C_0^\infty(U_j), \quad 1 \leq j \leq N.$$

Since $\|p\| \le \sum_{j=1}^{N} \|\phi_j p\|$, it is enough to consider each $\phi_j p$. Let $Lp = -\nabla \cdot K\nabla p$ and note that

$$L(\phi p) = \phi Lp + pL\phi - 2K\nabla p \cdot \nabla\phi. \tag{2.41}$$

First consider the open sets $U_j \subset\subset \Omega_j$, $1 \le j \le n$. With K Lipschitz on Ω_j, we have by elliptic regularity (see, e.g., [47], Theorem 8.12, Corollary 8.7) and (2.41)

$$\|\phi_j p\|_{2,U_j} \le C\|L(\phi_j p)\|_{0,U_j} \le C(\|f\|_{0,U_j} + \|p\|_{1,U_j}) \le C(\|f\|_0 + \|g\|_1), \quad 1 \le j \le n. \tag{2.42}$$

Next consider any ball $U_j = U$ on an interior interface away from the corners (and the edges if $d = 3$). Let the interface divide U into two semi-balls U^- and U^+. To simplify the notation, let $u = \phi_j p$. Since $u = 0$ on ∂U, using Poincare's inequality we have

$$\|u\|_{1,U}^2 \le C_0\|\nabla u\|_{0,U}^2 \le C_1(K\nabla u, \nabla u)_U = C_1(Lu, u)_U \le C_1\|Lu\|_{-1,U}\|u\|_{1,U};$$

therefore,

$$\|u\|_{1,U} \le C_1\|Lu\|_{-1,U}. \tag{2.43}$$

Let $U' \subset\subset U$ be the smallest ball that contains $\operatorname{supp}(u)$. For any h such that $0 < 2|h| < \operatorname{dist}(U', \partial U)$, define $U_h = \{x \in U : \operatorname{dist}(x, \partial U) > |h|\}$. Now consider $\delta^h u = \delta_i^h u \in H_0^1(U_h)$ for any direction x_i tangential to the interface. Let

$$u_h(x) = u(x + he_i).$$

A bound for $\delta^h u$ on U_h, similar to (2.43), gives

$$\|\delta^h u\|_{1,U_h} \le C\|L(\delta^h u)\|_{-1,U_h} \le C(\|\delta^h(Lu)\|_{-1,U_h} + \|(\delta^h L)u_h\|_{-1,U_h}), \tag{2.44}$$

using

$$\delta^h(Lu) = (\delta^h L)u_h + L(\delta^h u)$$

for the last inequality. We claim that the right hand side of (2.44) is uniformly bounded as $|h| \to 0$. For the first term, take any $\varphi \in H_0^1(U_h)$ and write

$$(\delta^h(Lu), \varphi)_{U_h} = -(Lu, \delta^{-h}\varphi)_{U_h} \le \|Lu\|_{0,U_h}\|\delta^{-h}\varphi\|_{0,U_h} \le C\|Lu\|_{0,U_h}\|\varphi\|_{1,U_h},$$

using Lemma 2.1 for the last inequality. Therefore, with an estimate similar to (2.42),

$$\|\delta^h(Lu)\|_{-1,U_h} \le C(\|f\|_0 + \|g\|_1). \tag{2.45}$$

For the second term, since L has Lipschitz coefficients on Ω_i, $\delta^h L$ is a second order operator with coefficients bounded uniformly in h. Hence,

$$\|(\delta^h L)u_h\|_{-1,U_h} \leq C\|u_h\|_{1,U_h} = C\|u\|_{1,U_h} \leq C(\|f\|_0 + \|g\|_1),$$

which, combined with (2.44) and (2.45), implies

$$\|\delta^h u\|_{1,U_h} \leq C(\|f\|_0 + \|g\|_1)$$

uniformly in h. Lemma 2.2 now implies that all second order derivatives of u are in $L^2(U)$, except $\partial^2 u/\partial x_k^2$, where x_k is the direction orthogonal to the interface. We now use the equation $Lu = L(\phi_j p) = \phi_j f + pL\phi_j - 2K\nabla p \cdot \nabla \phi_j$ to solve for $\partial^2 u/\partial x_k^2$ in terms of the other derivatives. Thus we conclude that $\partial^2 u/\partial x_k^2 \in L^2(U^-)$ and $\partial^2 u/\partial x_k^2 \in L^2(U^+)$, and

$$\|\phi_j p\|_{2,U^-} + \|\phi_j p\|_{2,U^+} \leq C(\|f\|_0 + \|g\|_1).$$

Now consider any ball $U_j = U$ with a center on $\partial\Omega \cap \partial\Omega_i$ away from the corners (and the edges if $d = 3$). Let $U^+ = U \cap \Omega_i$. A similar argument as above gives

$$\|\phi_j p\|_{2,U^+} \leq C(\|f\|_0 + \|g\|_1 + \|g\|_{2,\Omega_i}).$$

It remains to show that the solution is regular near the corners (edges if $d = 3$) of Ω. Consider any such corner (edge) and assume it is shared by Ω_i. Choose a sequence Ω_i^m of convex domains with C^2 boundaries Γ_i^m such that $\Omega_i^m \subseteq \Omega_i$ and $\Gamma_i^m = \Gamma_i^* \cup \gamma_i^m$, where Γ_i^* is fixed away from the corners (edges) of Ω_i and $\mathrm{dist}(\gamma_i^m, \partial\Omega) \to 0$ as $m \to \infty$. By the Trace theorem and the above analysis, the trace of p on Γ_i^* is in $H^{3/2}(\Gamma_i^*)$ and

$$\|p\|_{3/2,\Gamma_i^*} \leq C(\|f\|_0 + \|g\|_1 + \|g\|_{2,\Omega_i}). \tag{2.46}$$

Now consider the sequence of problems

$$Lp_m = f \qquad \text{in } \Omega_i^m, \tag{2.47}$$

$$p_m = p \qquad \text{on } \Gamma_i^*, \tag{2.48}$$

$$p_m = g \qquad \text{on } \gamma_i^m = \partial\Omega_i^m \backslash \Gamma_i^*, \tag{2.49}$$

For the regular problems (2.47)–(2.49) we have

$$\|p_m\|_{2,\Omega_i^m} \leq C(\|f\|_{0,\Omega_i^m} + \|p\|_{3/2,\Gamma_i^*} + \|g\|_{3/2,\gamma_i^m}) \leq C(\|f\|_0 + \|g\|_1 + \|g\|_{2,\Omega_i}),$$

using (2.46) and the Trace theorem for the last inequality. An argument by Grisvard [49], Theorem 3.2.1.2, implies that $p_m \to p$ weakly in H^2 and

$$\|p\|_{2,\Omega_i^*} \le C(\|f\|_0 + \|g\|_1 + \|g\|_{2,\Omega_i}),$$

where $\Omega_i^m \to \Omega_i^*$ as $m \to \infty$. This completes the proof of the theorem. □

> **Remark 2.1** The above estimate is sharp. A two dimensional example of a problem with a piese-wise smooth coefficient in [71], Section 8.1 shows that the solution may have a regularity $H^{1+\beta}$, $0 < \beta < 1$, in a neighborhood of an interior subdomain corner.

For the analysis, we assume that the solution is also regular near the corners (edges).

(H0) The solution p of (2.39)–(2.40) is in $H^2(\Omega_i)$ and

$$\sum_{i=1}^{n} \|p\|_{2,\Omega_i} \le C(\|f\|_0 + \|g\|_1 + \sum_{i=1}^{n} \|g\|_{2,\Omega_i}).$$

2.3.2 Error analysis of the finite difference scheme

We start with some relatively standard finite difference notation, given here in two dimensions for simplicity. Denote the grid points on $\hat{\Omega}_k$, $1 \le k \le n$, by

$$(\hat{x}_{i+1/2}, \hat{y}_{j+1/2}), \quad i = 0, ..., N_{\hat{x}}^k,\ j = 0, ..., N_{\hat{y}}^k,$$

and then define

$$\begin{aligned}
\hat{x}_i &= \frac{1}{2}(\hat{x}_{i+1/2} + \hat{x}_{i-1/2}), & i &= 1, N_{\hat{x}}^k,\\
\hat{y}_j &= \frac{1}{2}(\hat{y}_{j+1/2} + \hat{y}_{j-1/2}), & j &= 1, N_{\hat{y}}^k,\\
\hat{h}_i^x &= \hat{x}_{i+1/2} - \hat{x}_{i-1/2}, & i &= 1, N_{\hat{x}}^k,\\
\hat{h}_j^y &= \hat{y}_{j+1/2} - \hat{y}_{j-1/2}, & j &= 1, N_{\hat{y}}^k.
\end{aligned}$$

These points are mapped to points in Ω_k defined by the corresponding symbol without the caret. We write $\mathbf{v} = (v^x, v^y)$ for $\mathbf{v} \in \mathbf{R}^2$, and for any function $\hat{\psi}(\hat{x}, \hat{y})$, let $\hat{\psi}_{ij}$ denote $\hat{\psi}(\hat{x}_i, \hat{y}_j)$, let $\hat{\psi}_{i+1/2,j}$ denote $\hat{\psi}(\hat{x}_{i+1/2}, \hat{y}_j)$, etc., with a similar definition for functions and points without carets.

We need the following definition.

Definition 2.1 An asymptotic family of rectangular grids is said to be generated by a C^2 map if each grid is the image by a fixed map of a grid that is uniform in each coordinate direction, where each component of the map is strictly monotone and in $C^2(\bar{\hat{\Omega}})$.

Denote this map by $\mathbf{G}(\hat{x},\hat{y}) = \big(G^x(\hat{x}), G^y(\hat{y})\big)$ and note that in this case

$$\hat{h}^x_{i+1} - \hat{h}^x_i = G^x_{i+3/2} - 2G^x_{i+1/2} + G^x_{i-1/2} = \frac{d^2G^x(\bar{x})}{d\hat{x}^2}(\bar{h}^x)^2,$$

where $\bar{x}$ is between $\hat{x}_{i-1/2}$ and $\hat{x}_{i+3/2}$, and $\bar{h}^x$ is the uniform grid spacing. This, together with the smoothness of $\mathbf{G}$, implies

$$|\hat{h}^x_{i+1} - \hat{h}^x_i| \le C\hat{h}^2 \quad \text{and, similarly,} \quad |\hat{h}^y_{j+1} - \hat{h}^y_j| \le C\hat{h}^2. \tag{2.50}$$

In the analysis we will need several projection operators related to the mixed finite element spaces on the reference grids. For $1 \le i \le n$, let $\mathcal{P}_{\hat{h}}$ denote the L^2-projection of $\hat{W}_i$ onto $\hat{W}_{h,i}$. For $\varphi \in \hat{W}_i$, $\mathcal{P}_{\hat{h}}\varphi \in \hat{W}_{h,i}$ is defined by

$$(\mathcal{P}_{\hat{h}}\varphi - \varphi, w)_{\hat{\Omega}_i} = 0, \quad w \in \hat{W}_{h,i}.$$

For $\varphi \in H^1(\hat{\Omega}_i)$,

$$\|\mathcal{P}_{\hat{h}}\varphi - \varphi\|_{0,\hat{\Omega}_i} \le C\|\varphi\|_{1,\hat{\Omega}_i}\hat{h}. \tag{2.51}$$

Similarly, let $\mathcal{Q}_{\hat{h}}$ denote the $L^2(\partial\hat{\Omega}_i)$-projection onto $\hat{\mathbf{V}}_{h,i}\cdot\nu|_{\partial\hat{\Omega}_i}$. For $\psi \in H^1(\partial\hat{\Omega}_i)$,

$$\|\mathcal{Q}_{\hat{h}}\psi - \psi\|_{0,\partial\hat{\Omega}_i} \le C\|\psi\|_{1,\partial\hat{\Omega}_i}\hat{h}. \tag{2.52}$$

We also need the standard mixed projection operator $\hat{\Pi} : (H^1(\hat{\Omega}_i))^d \to \hat{\mathbf{V}}_{h,i}$ satisfying, for $\mathbf{q} \in (H^1(\hat{\Omega}_i))^d$,

$$(\nabla\cdot(\mathbf{q} - \hat{\Pi}\mathbf{q}), w)_{\hat{\Omega}_i} = 0, \qquad w \in \hat{W}_{h,i}, \tag{2.53}$$

$$\langle(\mathbf{q} - \hat{\Pi}\mathbf{q})\cdot\hat{\nu}, \mathbf{v}\cdot\nu\rangle_{\partial\hat{\Omega}_i} = 0, \qquad \mathbf{v} \in \hat{\mathbf{V}}_{h,i}. \tag{2.54}$$

For $\mathbf{q} \in (H^1(\hat{\Omega}_i))^d$,

$$\|\mathbf{q} - \hat{\Pi}\mathbf{q}\|_{0,\hat{\Omega}_i} \le C\|\mathbf{q}\|_{1,\hat{\Omega}_i}\hat{h}. \tag{2.55}$$

Equations (2.53) and (2.54) imply that $\nabla\cdot\hat{\Pi}\mathbf{q} = \mathcal{P}_{\hat{h}}\nabla\cdot\mathbf{q}$ and $\hat{\Pi}\mathbf{q}\cdot\hat{\nu} = \mathcal{Q}_{\hat{h}}\mathbf{q}\cdot\hat{\nu}$; therefore, with (2.51) and (2.52),

$$\|\nabla\cdot(\mathbf{q} - \hat{\Pi}\mathbf{q})\|_{0,\hat{\Omega}_i} \le C\|\nabla\cdot\mathbf{q}\|_{1,\hat{\Omega}_i}\hat{h}, \tag{2.56}$$

$$\|(\mathbf{q} - \hat{\Pi}\mathbf{q})\cdot\hat{\nu}\|_{0,\partial\hat{\Omega}_i} \le C\|\mathbf{q}\cdot\hat{\nu}\|_{1,\partial\hat{\Omega}_i}\hat{h}. \tag{2.57}$$

We next note that the quadrature rules $(\cdot,\cdot)_{\mathrm{M}}$, $(\cdot,\cdot)_{\mathrm{T}}$, and $(\cdot,\cdot)_{\mathrm{TM}}$ introduced in (2.21), (2.22) on $\hat{\mathcal{T}}_h$, are naturally defined on $\mathcal{T}_h$ via the mapping F. For $w \in W \cap C^0(\bar{\Omega})$, $\mathbf{v} \in \tilde{\mathbf{V}} \cap \left(C^0(\bar{\Omega})\right)^d$, and h implicitly fixed, let

$$\|w\|^2_{\mathrm{M},S} = (w,w)_{\mathrm{M},S}, \quad \|\mathbf{v}\|^2_{\mathrm{TM},S} = (\mathbf{v},\mathbf{v})_{\mathrm{TM},S}, \quad \text{and} \quad \|\mathbf{v}\|^2_{\mathrm{T},S} = (\mathbf{v},\mathbf{v})_{\mathrm{T},S},$$

where again we omit S if $S = \Omega$; these can also be defined on W_h or $\mathbf{V}_h$, where they are norms squared. Clearly for $\hat{\mathbf{v}} \in \hat{\mathbf{V}}_h$,

$$\frac{1}{C}\|\hat{\mathbf{v}}\|_0 \le \|\hat{\mathbf{v}}\|_{\mathrm{TM}} = \|\hat{\mathbf{v}}\|_{\mathrm{T}} \le C\|\hat{\mathbf{v}}\|_0,$$

that is, these three norms are equivalent on the reference velocity space.

We are now ready to state the main result of this section.

Theorem 2.4 Assume that (H0) hold. For the cell-centered finite difference approximation of the enhanced mixed method (2.37), (2.38), (2.35), (2.36) on a logically rectangular grid, if the corresponding computational grids are generated by a C^2 map, and if $p \in C^{3,1}(\bar{\Omega}_i) \bigcap C^0(\bar{\Omega})$, $\mathbf{u} \in \left(C^1(\bar{\Omega}_i) \bigcap W^{2,\infty}(\Omega_i)\right)^d \bigcap H(\mathrm{div};\Omega)$, and $K \in \left(C^1(\bar{\Omega}_i) \bigcap W^{2,\infty}(\Omega_i)\right)^{d\times d}$, then there exists a constant C, independent of h but dependent on the solution and K as indicated , and on $\|F\|_{3,\infty,\Omega_i}$, $\|F^{-1}\|_{3,\infty,\Omega_i}$, $\|DF\|_{0,\infty}$, and $\|DF^{-1}\|_{0,\infty}$, such that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{M}} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathrm{M}} \le Ch^{3/2}, \tag{2.58}$$

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{\mathrm{M}} \le Ch^2, \tag{2.59}$$

$$\|p - p_h\|_{\mathrm{M}} \le Ch^2. \tag{2.60}$$

For the proof of Theorem 2.4 we need several auxiliary estimates.

Lemma 2.3 Under the assumptions of Theorem 2.4, there exist $\hat{\mathbf{U}} \in \hat{\mathbf{V}}_h$, $\hat{\hat{\mathbf{U}}} \in \hat{\mathbf{V}}_h$, $\hat{P} \in \hat{W}_h$, and $\hat{\lambda}^* \in \hat{\Lambda}_h$ such that

$$(\hat{\hat{\mathbf{U}}}, \hat{\mathbf{v}})_{\mathrm{TM}} = \sum_{i=1}^{n} \left((\hat{P}, \hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}_i} - \langle \hat{\lambda}^*, \hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}_i} \right) - \langle \hat{g}^D, \hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}^D}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h, \tag{2.61}$$

$$(\hat{\mathbf{U}}, \hat{\hat{\mathbf{v}}})_{\mathrm{TM}} = (\mathcal{K}\hat{\hat{\mathbf{U}}}, \hat{\hat{\mathbf{v}}})_{\mathrm{T}}, \qquad \hat{\hat{\mathbf{v}}} \in \hat{\mathbf{V}}_h, \tag{2.62}$$

and there exists a constant C, independent of h, but dependent on the solution, K, and F as indicated in Theorem 2.4, such that, for all i, j,

$$|\hat{P}_{ij} - \hat{p}_{ij}| \leq C\hat{h}^2, \tag{2.63}$$

$$|\hat{\hat{U}}^x_{i+1/2,j} - \hat{\hat{u}}^x_{i+1/2,j}| + |\hat{\hat{U}}^y_{i,j+1/2} - \hat{\hat{u}}^y_{i,j+1/2}| \leq C\hat{h}^{\tilde{r}}, \tag{2.64}$$

$$|\hat{U}^x_{i+1/2,j} - \hat{u}^x_{i+1/2,j}| + |\hat{U}^y_{i,j+1/2} - \hat{u}^y_{i,j+1/2}| \leq C\hat{h}^r, \tag{2.65}$$

where $\tilde{r} = 2$ for all points not on $\hat{\Gamma}^D$ and not on $\cup_{i,j=1}^n \hat{\Gamma}_{i,j}$, and $\tilde{r} = 1$ otherwise, $r = 2$ for points strictly in the interior of Ω_i that lie on an edge or face e such that $\bar{e} \cap \bar{\hat{\Gamma}}^D = \emptyset$ and $\bar{e} \cap \bar{\hat{\Gamma}}_{i,j} = \emptyset$, and $r = 1$ otherwise.

Remark 2.2 The velocity space $\hat{\mathbf{V}}_h$ is discontinuous acros interior interfaces. For such points, (2.64) and (2.65) hold for the values on both sides of the interface, i.e.,

$$|\hat{\hat{U}}^{x,+}_{i+1/2,j} - \hat{\hat{u}}^{x,+}_{i+1/2,j}| + |\hat{\hat{U}}^{y,+}_{i,j+1/2} - \hat{\hat{u}}^{y,+}_{i,j+1/2}| \leq Ch, \tag{2.66}$$

$$|\hat{\hat{U}}^{x,-}_{i+1/2,j} - \hat{\hat{u}}^{x,-}_{i+1/2,j}| + |\hat{\hat{U}}^{y,-}_{i,j+1/2} - \hat{\hat{u}}^{y,-}_{i,j+1/2}| \leq Ch, \tag{2.67}$$

where

$$\hat{\hat{u}}^{x,+}_{i+1/2,j} = \lim_{\hat{x} \to \hat{x}^+_{i+1/2}} \hat{\hat{u}}^x(x, y_j), \quad \hat{\hat{u}}^{x,-}_{i+1/2,j} = \lim_{\hat{x} \to \hat{x}^-_{i+1/2}} \hat{\hat{u}}^x(x, y_j),$$

with a similar expression for (2.65).

Remark 2.3 To avoid confusion, we note that i and j used for indexing the subdomains differ from i and j used for indexing the grid points.

Proof (Lemma 2.3) On a subdomain $\hat{\Omega}_i$, we apply a construction due to Weiser and Wheeler (see [74], Lemma 4.1 and appendix) to $(\hat{\mathbf{u}}, \hat{p})$, that gives a $\hat{P}$ satisfying (2.63), and a $\hat{\hat{\mathbf{U}}}$, satisfying (2.61) and (2.64) with $\tilde{r} = 2$ strictly in the interior. Note that the constants in this construction depend on the Lipschitz constant of $\hat{\partial}^3 \hat{p}$ on $\hat{\Omega}_i$, and, through (2.10), on the Lipschitz constant of $\partial^3 p$ on Ω_i and on $\|F\|_{3,\infty,\Omega_i}$. On $\partial\hat{\Omega}_i \cap \hat{\Gamma}_N$ we define $\hat{\lambda}^*$ to give

$$\hat{\hat{U}}^x_{i+1/2,j} = \hat{\hat{u}}^x_{i+1/2,j} \quad \text{and} \quad \hat{\hat{U}}^y_{i,j+1/2} = \hat{\hat{u}}^y_{i,j+1/2},$$

therefore (2.64) holds with $\tilde{r} = 2$ on $\hat{\Gamma}^N$. Clearly (2.64) holds with $\tilde{r} = 1$ on $\hat{\Gamma}^D$. Finally, let

$$\hat{\lambda}^*_{i+1/2,j} = \hat{\lambda}_{i+1/2,j}, \quad \hat{\lambda}^*_{i,j+1/2} = \hat{\lambda}_{i,j+1/2}$$

on any interior interface $\hat{\Gamma}_{i,j}$. Then, with (2.61) and (2.63),

$$\hat{\tilde{U}}^{x,-}_{i+1/2,j} = \frac{\hat{P}_{ij} - \hat{\lambda}^*_{i+1/2,j}}{\hat{h}^x_i/2} = \frac{\hat{p}_{ij} - \hat{\lambda}_{i+1/2,j}}{\hat{h}^x_i/2} + O(h) = \hat{\tilde{u}}^{x,-}_{i+1/2,j} + O(h).$$

Similar estimates for $\hat{\tilde{U}}^{x,+}$, $\hat{\tilde{U}}^{y,-}$, and $\hat{\tilde{U}}^{y,+}$ imply (2.66) and (2.67).

To show (2.65), we choose $\hat{\tilde{\mathbf{v}}}$ in (2.62) to be the basis function associated with node $(i+1/2,j)$. For the interior of $\hat{\Omega}_i$ we have

$$\begin{aligned}\hat{U}^x_{i+1/2,j} = {} & \frac{1}{2}\left[(\mathcal{K}_{11})_{i+1/2,j-1/2} + (\mathcal{K}_{11})_{i+1/2,j+1/2}\right]\hat{\tilde{U}}^x_{i+1/2,j} \\ & + \frac{1}{2(\hat{h}^x_i + \hat{h}^x_{i+1})}\Big\{ \\ & \left[(\mathcal{K}_{12})_{i+1/2,j-1/2}\hat{\tilde{U}}^y_{i+1,j-1/2} + (\mathcal{K}_{12})_{i+1/2,j+1/2}\hat{\tilde{U}}^y_{i+1,j+1/2}\right]\hat{h}^x_{i+1} \\ & + \left[(\mathcal{K}_{12})_{i+1/2,j-1/2}\tilde{U}^y_{i,j-1/2} + (\mathcal{K}_{12})_{i+1/2,j+1/2}\tilde{U}^y_{i,j+1/2}\right]\hat{h}^x_i\Big\},\end{aligned} \tag{2.68}$$

and on $\partial\hat{\Omega}_i$ we have

$$\begin{aligned}\hat{U}^x_{i+1/2,j} = {} & \frac{1}{2}\Big\{\left[(\mathcal{K}_{11})_{i+1/2,j-1/2} + (\mathcal{K}_{11})_{i+1/2,j+1/2}\right]\hat{\tilde{U}}^x_{i+1/2,j} \\ & + \left[(\mathcal{K}_{12})_{i+1/2,j-1/2}\hat{\tilde{U}}^y_{i+1,j-1/2} + (\mathcal{K}_{12})_{i+1/2,j+1/2}\hat{\tilde{U}}^y_{i+1,j+1/2}\right]\Big\},\end{aligned} \tag{2.69}$$

where $i = \hat{\imath} = 0$ for the left boundary and $i = \hat{\imath} + 1 = N^i_{\hat{x}}$ for the right boundary. Taylor's theorem for $\hat{\mathbf{u}} = \mathcal{K}\hat{\tilde{\mathbf{u}}}$ gives

$$\begin{aligned}\hat{u}^x_{i+1/2,j} = {} & \frac{1}{2}\left[(\mathcal{K}_{11})_{i+1/2,j-1/2} + (\mathcal{K}_{11})_{i+1/2,j+1/2}\right]\hat{\tilde{u}}^x_{i+1/2,j} \\ & + \frac{1}{2(\hat{h}^x_i + \hat{h}^x_{i+1})}\Big\{ \\ & \left[(\mathcal{K}_{12})_{i+1/2,j-1/2}\hat{\tilde{u}}^y_{i+1,j-1/2} + (\mathcal{K}_{12})_{i+1/2,j+1/2}\hat{\tilde{u}}^y_{i+1,j+1/2}\right]\hat{h}^x_i \\ & + \left[(\mathcal{K}_{12})_{i+1/2,j-1/2}\hat{\tilde{u}}^y_{i,j-1/2} + (\mathcal{K}_{12})_{i+1/2,j+1/2}\hat{\tilde{u}}^y_{i,j+1/2}\right]\hat{h}^x_{i+1}\Big\} \\ & + O(\hat{h}^2),\end{aligned} \tag{2.70}$$

$$\begin{aligned}\hat{u}^x_{i+1/2,j} = {} & \frac{1}{2}\Big\{\left[(\mathcal{K}_{11})_{i+1/2,j-1/2} + (\mathcal{K}_{11})_{i+1/2,j+1/2}\right]\hat{\tilde{u}}^x_{i+1/2,j} \\ & + \left[(\mathcal{K}_{12})_{i+1/2,j-1/2}\hat{\tilde{u}}^y_{i+1,j-1/2} + (\mathcal{K}_{12})_{i+1/2,j+1/2}\hat{\tilde{u}}^y_{i+1,j+1/2}\right]\Big\} \\ & + O(\hat{h}), \quad i = \hat{\imath} = 0 \text{ or } i = \hat{\imath} + 1 = N^i_{\hat{x}},\end{aligned} \tag{2.71}$$

with constants depending on $\mathbf{u}$ and K as indicated in the assumptions, and on $\|F\|_{3,\infty,\Omega_i}$ and $\|F^{-1}\|_{3,\infty,\Omega_i}$, because of (2.9) and (2.24). Note that the coefficients in (2.68) and (2.70) differ only in the weights $\hat{h}_i^x$ and $\hat{h}_{i+1}^x$. Adding and subtracting $\hat{h}_{i+1}^x$ and $\hat{h}_i^x$ to the weights of the second and third term on the right side of (2.70), respectively, and subtracting from (2.68), we have

$$\begin{aligned}
&|\hat{U}^x_{i+1/2,j} - \hat{u}^x_{i+1/2,j}| \\
&\quad\le C\Big\{\hat{h}^2 + \Big|\hat{\hat{U}}^x_{i+1/2,j} - \hat{\hat{u}}^x_{i+1/2,j}\Big| \\
&\qquad + \Big|\hat{\hat{U}}^y_{i+1,j-1/2} - \hat{\hat{u}}^y_{i+1,j-1/2}\Big| + \Big|\hat{\hat{U}}^y_{i,j-1/2} - \hat{\hat{U}}^y_{i,j-1/2}\Big| \\
&\qquad + \Big|\hat{\hat{U}}^y_{i+1,j+1/2} - \hat{\hat{u}}^y_{i+1,j+1/2}\Big| + \Big|\hat{\hat{U}}^y_{i,j+1/2} - \hat{\hat{U}}^y_{i,j+1/2}\Big|\Big\} \\
&\qquad + \frac{1}{4}\Bigg\{\bigg|(\mathcal{K}_{12})_{i+1/2,j-1/2}\frac{\partial \hat{\hat{u}}^y}{\partial \hat{x}}(\bar{x}', y_{j-1/2})\bigg| \\
&\qquad\quad + \bigg|(\mathcal{K}_{12})_{i+1/2,j+1/2}\frac{\partial \hat{\hat{u}}^y}{\partial \hat{x}}(\bar{x}'', y_{j+1/2})\bigg|\Bigg\}|\hat{h}^x_{i+1} - \hat{h}^x_i|,
\end{aligned} \tag{2.72}$$

where $\bar{x}'$ and $\bar{x}''$ are points between $\hat{x}_i$ and $\hat{x}_{i+1}$. Estimate (2.65) follows in the interior from (2.72), (2.64), and (2.50); and on $\partial\hat{\Omega}_i$ from (2.69) and (2.71). $\square$

Since $\cup_{i=1}^n \partial\hat{\Omega}_i$ is a set of dimension $d-1$, the following corollary holds.

Corollary 2.1 For the functions in Lemma 2.3, there exists a constant C, independent of $\hat{h}$, such that

$$\|\hat{\hat{\mathbf{U}}} - \hat{\hat{\mathbf{u}}}\|_{\mathrm{TM},\hat{\Omega}} + \|\hat{\mathbf{U}} - \hat{\mathbf{u}}\|_{\mathrm{TM},\hat{\Omega}} \le C\hat{h}^{3/2}.$$

Lemma 2.4 Under the assumptions of Theorem 2.4,there exists a constant C, independent of $\hat{h}$, such that

$$\|\hat{\hat{\mathbf{u}}} - \hat{\hat{\mathbf{u}}}_h\|_{\mathrm{TM},\hat{\Omega}} + \|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{\mathrm{TM},\hat{\Omega}} \le C\hat{h}^{3/2}.$$

Proof (Lemma 2.4) Subtracting (2.33)–(2.36) from (2.62), (2.61), the transformed (2.27) and (2.28), respectively, we obtain the error equations

$$(\hat{\mathbf{U}} - \hat{\mathbf{u}}_h, \hat{\mathbf{v}})_{\mathrm{TM},\hat{\Omega}} = (\mathcal{K}(\hat{\hat{\mathbf{U}}} - \hat{\hat{\mathbf{u}}}_h), \hat{\mathbf{v}})_{\mathrm{T},\hat{\Omega}}, \qquad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h, \tag{2.73}$$

$$(\hat{\hat{\mathbf{U}}} - \hat{\hat{\mathbf{u}}}_h, \hat{\mathbf{v}})_{\mathrm{TM},\hat{\Omega}} = \sum_{i=1}^n \Big((\hat{P} - \hat{p}_h, \hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}_i} - \langle\hat{\lambda}^* - \hat{\lambda}_h, \hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}_i}\Big), \quad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h, \tag{2.74}$$

$$\sum_{i=1}^{n}(\hat{\nabla}\cdot(\hat{\Pi}\hat{\mathbf{u}}-\hat{\mathbf{u}}_h),\hat{w})_{\hat{\Omega}_i}=0, \qquad \hat{w}\in\hat{W}_h, \quad (2.75)$$

$$\sum_{i=1}^{n}\langle(\hat{\Pi}\hat{\mathbf{u}}-\hat{\mathbf{u}}_h)\cdot\hat{\nu},\hat{\mu}\rangle_{\hat{\Gamma}_i}=0, \qquad \hat{\mu}\in\hat{\Lambda}_h. \quad (2.76)$$

We take $\hat{\hat{\mathbf{v}}}=\hat{\hat{\mathbf{U}}}-\hat{\hat{\mathbf{u}}}_h$, $\hat{\mathbf{v}}=\hat{\Pi}\hat{\mathbf{u}}-\hat{\mathbf{u}}_h$, $\hat{w}=\hat{P}-\hat{p}_h$, and $\hat{\mu}=\hat{\lambda}^*-\hat{\lambda}_h$ to obtain

$$(\mathcal{K}(\hat{\hat{\mathbf{U}}}-\hat{\hat{\mathbf{u}}}_h),\hat{\hat{\mathbf{U}}}-\hat{\hat{\mathbf{u}}}_h)_{\mathrm{T},\hat{\Omega}}=(\hat{\hat{\mathbf{U}}}-\hat{\hat{\mathbf{u}}}_h,\hat{\mathbf{U}}-\hat{\Pi}\hat{\mathbf{u}})_{\mathrm{TM},\hat{\Omega}}.$$

An application of Schwarz inequality gives

$$\|\hat{\hat{\mathbf{U}}}-\hat{\hat{\mathbf{u}}}_h\|_{\mathrm{TM},\hat{\Omega}}\le C\|\hat{\mathbf{U}}-\hat{\Pi}\hat{\mathbf{u}}\|_{\mathrm{TM},\hat{\Omega}}.$$

Take $\hat{\hat{\mathbf{v}}}=\hat{\mathbf{U}}-\hat{\mathbf{u}}_h$ in (2.73) to get

$$\|\hat{\mathbf{U}}-\hat{\mathbf{u}}_h\|_{\mathrm{TM},\hat{\Omega}}\le C\|\hat{\hat{\mathbf{U}}}-\hat{\hat{\mathbf{u}}}_h\|_{\mathrm{TM},\hat{\Omega}}.$$

The Lemma now follows from Corollary 2.1 and the known estimate [37]

$$\|\hat{\Pi}\hat{\mathbf{u}}-\hat{\mathbf{u}}\|_{\mathrm{TM},\hat{\Omega}}\le C\hat{h}^2.$$

□

Finally, we need a bound on $\hat{p}-\hat{p}_h$. From (2.33) – (2.36) and the transformed (2.25) – (2.28) we get the error equations

$$(\hat{\mathbf{u}}-\hat{\mathbf{u}}_h,\hat{\hat{\mathbf{v}}})_{\hat{\Omega}}=(\mathcal{K}(\hat{\hat{\mathbf{u}}}-\hat{\hat{\mathbf{u}}}_h),\hat{\hat{\mathbf{v}}})_{\hat{\Omega}}-E_{\mathrm{TM}}(\hat{\mathbf{u}}_h,\hat{\hat{\mathbf{v}}}) + E_{\mathrm{T}}(\mathcal{K}\hat{\hat{\mathbf{u}}}_h,\hat{\hat{\mathbf{v}}}), \qquad \hat{\hat{\mathbf{v}}}\in\hat{\mathbf{V}}_h, \quad (2.77)$$

$$(\hat{\hat{\mathbf{u}}}-\hat{\hat{\mathbf{u}}}_h,\hat{\mathbf{v}})_{\hat{\Omega}}=\sum_{i=1}^{n}\left((\mathcal{P}_{\hat{h}}\hat{p}-\hat{p}_h,\hat{\nabla}\cdot\hat{\mathbf{v}})_{\hat{\Omega}_i}-\langle\mathcal{Q}_{\hat{h}}\hat{\lambda}-\hat{\lambda}_h,\hat{\mathbf{v}}\cdot\hat{\nu}\rangle_{\hat{\Gamma}_i}\right) - E_{\mathrm{TM}}(\hat{\hat{\mathbf{u}}}_h,\hat{\mathbf{v}}), \qquad \hat{\mathbf{v}}\in\hat{\mathbf{V}}_h, \quad (2.78)$$

$$\sum_{i=1}^{n}(\hat{\nabla}\cdot(\hat{\mathbf{u}}-\hat{\mathbf{u}}_h),\hat{w})_{\hat{\Omega}_i}=0, \qquad \hat{w}\in\hat{W}_h, \quad (2.79)$$

$$\sum_{i=1}^{n}\langle(\hat{\mathbf{u}}-\hat{\mathbf{u}}_h)\cdot\hat{\nu},\hat{\mu}\rangle_{\hat{\Gamma}_i}=0, \qquad \hat{\mu}\in\hat{\Lambda}_h, \quad (2.80)$$

where

$$E_{\mathrm{Q}}(\mathbf{q},\mathbf{v})=(\mathbf{q},\mathbf{v})-(\mathbf{q},\mathbf{v})_{\mathrm{Q}}, \quad \mathrm{Q}=\mathrm{TM}\text{ or }\mathrm{T}.$$

It is well known [30] that the error in approximating an integral by either of these rules is of order $\hat{h}^2$:

$$|E_{\mathrm{Q}}(\mathbf{q},\mathbf{v})| \le C \sum_{\hat{E}} \sum_{|\alpha|=2} \left\| \frac{\partial^\alpha}{\partial \hat{x}^\alpha}(\mathbf{q}\cdot\mathbf{v}) \right\|_{0,1,\hat{E}} \hat{h}^2. \tag{2.81}$$

The following lemma is proven in [9]. The proof is included for completeness.

Lemma 2.5 For the lowest order RTN spaces on rectangles, for any $\mathbf{q} = (q^x, q^y) \in H^1(\hat{\Omega}_i)$ and $\hat{E} \in \hat{\mathcal{T}}_h$,

$$\left\| \frac{\partial}{\partial \hat{x}}(\hat{\Pi}\mathbf{q})^x \right\|_{0,\hat{E}} \le \left\| \frac{\partial q^x}{\partial \hat{x}} \right\|_{0,\hat{E}} \quad \text{and} \quad \left\| \frac{\partial}{\partial \hat{y}}(\hat{\Pi}\mathbf{q})^y \right\|_{0,\hat{E}} \le \left\| \frac{\partial q^y}{\partial \hat{y}} \right\|_{0,\hat{E}}.$$

Proof Without loss of generality, assume that $\hat{E}$ is the unit square. By definition, $\hat{\Pi}$ satisfies on each edge $\hat{e}$ of $\hat{E}$

$$\int_{\hat{e}} (\mathbf{q}\cdot\hat{\nu} - \hat{\Pi}\mathbf{q}\cdot\hat{\nu})\, d\hat{s} = 0.$$

Writing this for the two vertical edges, we have

$$\int_0^1 [q^x(1,\hat{y}) - (\hat{\Pi}\mathbf{q})^x(1,\hat{y})]\, d\hat{y} = 0 \quad \text{and} \quad \int_0^1 [q^x(0,\hat{y}) - (\hat{\Pi}\mathbf{q})^x(0,\hat{y})]\, d\hat{y} = 0.$$

Subtraction of the above equations and the fundamental theorem of calculus imply

$$\int_0^1 \int_0^1 \left[\frac{\partial}{\partial \hat{x}} q^x(\hat{x},\hat{y}) - \frac{\partial}{\partial \hat{x}}(\hat{\Pi}\mathbf{q})^x(\hat{x},\hat{y}) \right] \frac{\partial v^x}{\partial \hat{x}}\, d\hat{x}\, d\hat{y} = 0, \quad \mathbf{v} \in \hat{\mathbf{V}}_h.$$

Therefore $(\hat{\Pi}\mathbf{q})^x$ is the H_0^1-projection of q^x in the $\hat{x}$ direction. Similarly, $(\hat{\Pi}\mathbf{q})^y$ is the H_0^1-projection of q^y in the $\hat{y}$ direction, which proves the lemma. □

We now present a bound on $\hat{p} - \hat{p}_h$.

Lemma 2.6 Under the assumption of Lemma 2.3,there exists a constant C, independent of $\hat{h}$, such that

$$\|\mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h\|_{\hat{\Omega}} \le C\hat{h}^2.$$

Proof To estimate the pressure error, we use a duality argument. Let φ solve

$$-\hat{\nabla}\cdot\mathcal{K}\hat{\nabla}\varphi = \mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h \qquad \text{in } \hat{\Omega}, \tag{2.82}$$

$$\varphi = 0 \qquad \text{on } \partial\hat{\Omega}. \tag{2.83}$$

With (H0) we have

$$\sum_{i=1}^{n} \|\varphi\|_{2,\hat{\Omega}_i} \leq C\|\mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h\|_{\hat{\Omega}}. \tag{2.84}$$

Take $\hat{\mathbf{v}} = -\hat{\Pi}\mathcal{K}\hat{\nabla}\varphi$ in (2.78) and get

$$\begin{aligned}
&\|\mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h\|_{\hat{\Omega}}^2 \\
&\quad = -\sum_{i=1}^{n} \Big((\hat{\tilde{\mathbf{u}}} - \hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\mathcal{K}\hat{\nabla}\varphi - \mathcal{K}\hat{\nabla}\varphi)_{\hat{\Omega}_i} - (\mathcal{K}(\hat{\tilde{\mathbf{u}}} - \hat{\tilde{\mathbf{u}}}_h), \hat{\nabla}\varphi - \hat{\Pi}\hat{\nabla}\varphi)_{\hat{\Omega}_i} \\
&\qquad -(\mathcal{K}(\hat{\tilde{\mathbf{u}}} - \hat{\tilde{\mathbf{u}}}_h), \hat{\Pi}\hat{\nabla}\varphi)_{\hat{\Omega}_i}\Big) + E_{\mathrm{TM}}(\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\mathcal{K}\hat{\nabla}\varphi),
\end{aligned} \tag{2.85}$$

and, using (2.77), (2.79), and (2.80), the third term on the right is

$$\begin{aligned}
&-\sum_{i=1}^{n} (\mathcal{K}(\hat{\tilde{\mathbf{u}}} - \hat{\tilde{\mathbf{u}}}_h), \hat{\Pi}\hat{\nabla}\varphi)_{\hat{\Omega}_i} \\
&\quad = -\sum_{i=1}^{n} \Big((\hat{\mathbf{u}} - \hat{\mathbf{u}}_h, \hat{\Pi}\hat{\nabla}\varphi - \hat{\nabla}\varphi)_{\hat{\Omega}_i} - (\hat{\mathbf{u}} - \hat{\mathbf{u}}_h, \hat{\nabla}\varphi)_{\hat{\Omega}_i}\Big) \\
&\qquad -E_{\mathrm{TM}}(\hat{\mathbf{u}}_h, \hat{\Pi}\hat{\nabla}\varphi) + E_{\mathrm{T}}(\mathcal{K}\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\hat{\nabla}\varphi) \\
&\quad = -\sum_{i=1}^{n} \Big((\hat{\mathbf{u}} - \hat{\mathbf{u}}_h, \hat{\Pi}\hat{\nabla}\varphi - \hat{\nabla}\varphi)_{\hat{\Omega}_i} + (\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h), \varphi - \mathcal{P}_{\hat{h}}\varphi)_{\hat{\Omega}_i} \\
&\qquad -\langle(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\cdot\hat{\nu}, \varphi - \mathcal{Q}_{\hat{h}}\varphi\rangle_{\partial\hat{\Omega}_i}\Big) - E_{\mathrm{TM}}(\hat{\mathbf{u}}_h, \hat{\Pi}\hat{\nabla}\varphi) + E_{\mathrm{T}}(\mathcal{K}\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\hat{\nabla}\varphi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h\|_{\hat{\Omega}}^2 \\
&\quad \leq C\Big\{\|\hat{\tilde{\mathbf{u}}} - \hat{\tilde{\mathbf{u}}}_h\|_{0,\hat{\Omega}} + \|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{0,\hat{\Omega}} + \sum_{i=1}^{n} \|\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\|_{0,\hat{\Omega}_i} \\
&\qquad + \sum_{i=1}^{n} \|(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\cdot\hat{\nu}\|_{0,\partial\hat{\Omega}_i}\Big\} \sum_{i=1}^{n} \|\varphi\|_{2,\hat{\Omega}_i}\hat{h} \\
&\qquad + \Big|E_{\mathrm{TM}}(\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\mathcal{K}\hat{\nabla}\varphi)\Big| + \Big|E_{\mathrm{TM}}(\hat{\mathbf{u}}_h, \hat{\Pi}\hat{\nabla}\varphi)\Big| + \Big|E_{\mathrm{T}}(\mathcal{K}\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\hat{\nabla}\varphi)\Big|,
\end{aligned} \tag{2.86}$$

using (2.55), (2.51), and (2.52).

Using (2.81) and the fact that the functions are in the discrete space, we have

$$\begin{aligned}
|E_{\mathrm{TM}}(\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\mathcal{K}\hat{\nabla}\varphi)| \leq C\sum_{\hat{E}} \bigg\{ &\bigg\|\frac{\partial \hat{\tilde{u}}_h^{\hat{x}}}{\partial \hat{x}}\bigg\|_{0,\hat{E}} \bigg\|\frac{\partial}{\partial \hat{x}}(\hat{\Pi}\mathcal{K}\hat{\nabla}\varphi)^{\hat{x}}\bigg\|_{0,\hat{E}} \\
&+ \bigg\|\frac{\partial \hat{\tilde{u}}_h^{\hat{y}}}{\partial \hat{y}}\bigg\|_{0,\hat{E}} \bigg\|\frac{\partial}{\partial \hat{y}}(\hat{\Pi}\mathcal{K}\hat{\nabla}\varphi)^{\hat{y}}\bigg\|_{0,\hat{E}} \bigg\}\hat{h}^2.
\end{aligned} \tag{2.87}$$

By the inverse inequality (valid for quasi-uniform grids) and Lemma 2.5,

$$\left\|\frac{\partial \hat{\tilde{u}}_h^{\hat{x}}}{\partial \hat{x}}\right\|_{0,\hat{E}} = \left\|\frac{\partial}{\partial \hat{x}}(\tilde{u}_h^{\hat{x}} - (\hat{\Pi}\hat{\mathbf{u}})^{\hat{x}})\right\|_{0,\hat{E}} + \left\|\frac{\partial}{\partial \hat{x}}(\hat{\Pi}\hat{\mathbf{u}})^{\hat{x}}\right\|_{0,\hat{E}}$$
$$\leq C\|\hat{\tilde{u}}_h^{\hat{x}} - (\hat{\Pi}\hat{\mathbf{u}})^{\hat{x}}\|_{0,\hat{E}}\hat{h}^{-1} + \left\|\frac{\partial \hat{\tilde{u}}^{\hat{x}}}{\partial \hat{x}}\right\|_{0,\hat{E}}$$
$$\leq C\|\hat{\tilde{\mathbf{u}}}_h - \hat{\Pi}\hat{\tilde{\mathbf{u}}}\|_{0,\hat{E}}\hat{h}^{-1} + \|\hat{\tilde{\mathbf{u}}}\|_{1,\hat{E}}.$$

A similar expression holds for the $\hat{y}$-direction. Now, with (2.87) and Lemma 2.5,

$$\begin{aligned}
&|E_{\mathrm{TM}}(\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\mathcal{K}\hat{\nabla}\varphi)| \\
&\quad \leq C\sum_{\hat{E}}\left\{\|\hat{\tilde{\mathbf{u}}}_h - \hat{\Pi}\hat{\tilde{\mathbf{u}}}\|_{0,\hat{E}}\hat{h}^{-1} + \|\hat{\tilde{\mathbf{u}}}\|_{1,\hat{E}}\right\}\|\mathcal{K}\hat{\nabla}\varphi\|_{1,\hat{E}}\hat{h}^2 \\
&\quad \leq C\sum_{\hat{E}}\left\{\|\hat{\tilde{\mathbf{u}}}_h - \hat{\Pi}\hat{\tilde{\mathbf{u}}}\|_{0,\hat{E}} + \|\hat{\tilde{\mathbf{u}}}\|_{1,\hat{E}}\,\hat{h}\right\}\|\varphi\|_{2,\hat{E}}\,\hat{h} \\
&\quad \leq C\Big\{\|\hat{\tilde{\mathbf{u}}}_h - \hat{\Pi}\hat{\tilde{\mathbf{u}}}\|_{0,\hat{\Omega}} + \sum_{i=1}^{n}\|\hat{\tilde{\mathbf{u}}}\|_{1,\hat{\Omega}_i}\,\hat{h}\Big\}\sum_{i=1}^{n}\|\varphi\|_{2,\hat{\Omega}_i}\,\hat{h}.
\end{aligned} \tag{2.88}$$

We bound the other two quadrature error terms similarly:

$$|E_{\mathrm{TM}}(\hat{\mathbf{u}}_h, \hat{\Pi}\hat{\nabla}\varphi)| \leq C\Big\{\|\hat{\mathbf{u}}_h - \hat{\Pi}\hat{\mathbf{u}}\|_{0,\hat{\Omega}} + \sum_{i=1}^{n}\|\hat{\mathbf{u}}\|_{1,\hat{\Omega}_i}\,\hat{h}\Big\}\sum_{i=1}^{n}\|\varphi\|_{2,\hat{\Omega}_i}\,\hat{h},$$
$$|E_{\mathrm{T}}(\mathcal{K}\hat{\tilde{\mathbf{u}}}_h, \hat{\Pi}\hat{\nabla}\varphi)| \leq C\Big\{\|\hat{\tilde{\mathbf{u}}}_h - \hat{\Pi}\hat{\tilde{\mathbf{u}}}\|_{0,\hat{\Omega}} + \sum_{i=1}^{n}\|\hat{\tilde{\mathbf{u}}}\|_{1,\hat{\Omega}_i}\,\hat{h}\Big\}\sum_{i=1}^{n}\|\varphi\|_{2,\hat{\Omega}_i}\,\hat{h},$$

noting that the constants depend on $\sum_{i=1}^{n}\|\mathcal{K}\|_{2,\infty,\hat{\Omega}_i}$. From (2.86), then,

$$\begin{aligned}
&\|\mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h\|_{\hat{\Omega}}^2 \\
&\quad \leq C\Big\{\|\hat{\tilde{\mathbf{u}}} - \hat{\tilde{\mathbf{u}}}_h\|_{0,\hat{\Omega}} + \|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{0,\hat{\Omega}} + \|\hat{\tilde{\mathbf{u}}}_h - \hat{\Pi}\hat{\tilde{\mathbf{u}}}\|_{0,\hat{\Omega}} + \|\hat{\mathbf{u}}_h - \hat{\Pi}\hat{\mathbf{u}}\|_{0,\hat{\Omega}} \\
&\qquad + \sum_{i=1}^{n}\|\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\|_{0,\hat{\Omega}_i} + \sum_{i=1}^{n}\|(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\cdot\hat{\nu}\|_{0,\partial\hat{\Omega}_i} + \hat{h}\Big\}\sum_{i=1}^{n}\|\varphi\|_{2,\hat{\Omega}_i}\,\hat{h}.
\end{aligned} \tag{2.89}$$

The first four terms on the right are bounded by Lemma 2.4 and (2.55). To bound the fifth term, we note that (2.79) and (2.53) imply $\hat{\nabla}\cdot(\hat{\Pi}\hat{\mathbf{u}} - \hat{\mathbf{u}}_h) = 0$; therefore, with (2.56),

$$\|\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\|_{0,\hat{\Omega}_i} = \|\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}})\|_{0,\hat{\Omega}_i} \leq C\hat{h}.$$

For the sixth term, using the inverse inequality, (2.57), the Normal trace theorem, and Lemma 2.4, we have

$$\|(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\cdot\hat{\nu}\|_{0,\partial\hat{\Omega}_i} = \|(\hat{\mathbf{u}} - \hat{\Pi}\hat{\mathbf{u}})\cdot\hat{\nu}\|_{0,\partial\hat{\Omega}_i} + \|(\hat{\Pi}\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\cdot\hat{\nu}\|_{0,\partial\hat{\Omega}_i}$$

$$
\begin{aligned}
&\leq C(\hat{h} + \hat{h}^{-1/2}\|(\hat{\Pi}\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\cdot\hat{\nu}\|_{-1/2,\partial\hat{\Omega}_i}) \\
&\leq C(\hat{h} + \hat{h}^{-1/2}\|\hat{\Pi}\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{H(\mathrm{div};\hat{\Omega}_i)}) \\
&\leq C(\hat{h} + \hat{h}^{-1/2}\|\hat{\Pi}\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{\mathrm{TM},\hat{\Omega}_i} \\
&\leq C\hat{h}
\end{aligned}
$$

With the above bounds, (2.89) gives

$$
\|\mathcal{P}_{\hat{h}}\hat{p} - \hat{p}_h\|^2_{\hat{\Omega}} \leq C\sum_{i=1}^{n}\|\varphi\|_{2,\hat{\Omega}_i}\,\hat{h}^2,
$$

which, combined with (2.84), proves the result. □

We are now ready to prove the main theorem.

Proof (Theorem 2.4) To show (2.58), we write

$$
\|\mathbf{u} - \mathbf{u}_h\|^2_{\mathrm{M}} = \left\|J^{-1}DF(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\right\|^2_{\mathrm{M}} \leq C\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|^2_{\mathrm{M}}.
$$

Now

$$
\begin{aligned}
\|\hat{u}^x - \hat{u}^x_h\|^2_{\mathrm{M}} &= \sum_{\hat{E}}(\hat{u}^x - \hat{u}^x_h)^2_{ij}|\hat{E}| \\
&\leq \sum_{\hat{E}}\left\{\frac{1}{2}\Big[(\hat{u}^x - \hat{u}^x_h)_{i-1/2,j} + (\hat{u}^x - \hat{u}^x_h)_{i+1/2,j}\Big] + C\hat{h}^2\right\}^2|\hat{E}| \\
&\leq C\Big(\|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|^2_{\mathrm{TM}} + \|\hat{\mathbf{u}} - \hat{\mathbf{u}}_h\|_{\mathrm{TM}}\hat{h}^2 + \hat{h}^4\Big) \leq Ch^3, \qquad (2.90)
\end{aligned}
$$

using Lemma 2.4 for the last inequality. A similar estimate for $\|\hat{u}^y - \hat{u}^y_h\|_{\mathrm{M}}$ completes the proof of (2.58). To prove (2.59), we observe that $\hat{\nabla}\cdot\hat{\mathbf{u}}_h = \mathcal{P}_{\hat{h}}\hat{\nabla}\cdot\hat{\mathbf{u}}$ implies

$$
\|\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\|_{\mathrm{M}} \leq C\sum_{i=1}^{n}\|\hat{\nabla}\cdot\hat{\mathbf{u}}\|_{2,\hat{\Omega}_i}\,\hat{h}^2,
$$

and, since $\widehat{\nabla\cdot\mathbf{u}} = J^{-1}\hat{\nabla}\cdot\hat{\mathbf{u}}$,

$$
\|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_{\mathrm{M}} \leq C\|\hat{\nabla}\cdot(\hat{\mathbf{u}} - \hat{\mathbf{u}}_h)\|_{\mathrm{M}} \leq Ch^2.
$$

Finally, (2.60) follows from Lemma 2.6. □

> **Remark 2.4** Theorem 2.4 implies super-convergence of the computed pressure, velocity and its divergence at the midpoints of the elements. The loss of half a power of h for the velocity is due to the $O(h)$ approximation on the interfaces and on $\partial\Omega$.

2.4 Numerical experiments

We present computational results from the 3D parallel single phase flow simulator ParFlow1. To illustrate the theoretical results, we solve a 2D problem with a known analytic solution and mapping

$$p(x,y) = \begin{cases} xy & \text{for } x \le 1/2, \\ xy + (x-1/2)(y+1/2) & \text{for } x > 1/2, \end{cases}$$

$$K(x,y) = \begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{for } x < 1/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } x > 1/2, \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} + \frac{1}{10}\sin(6\hat{x}) \end{pmatrix}.$$

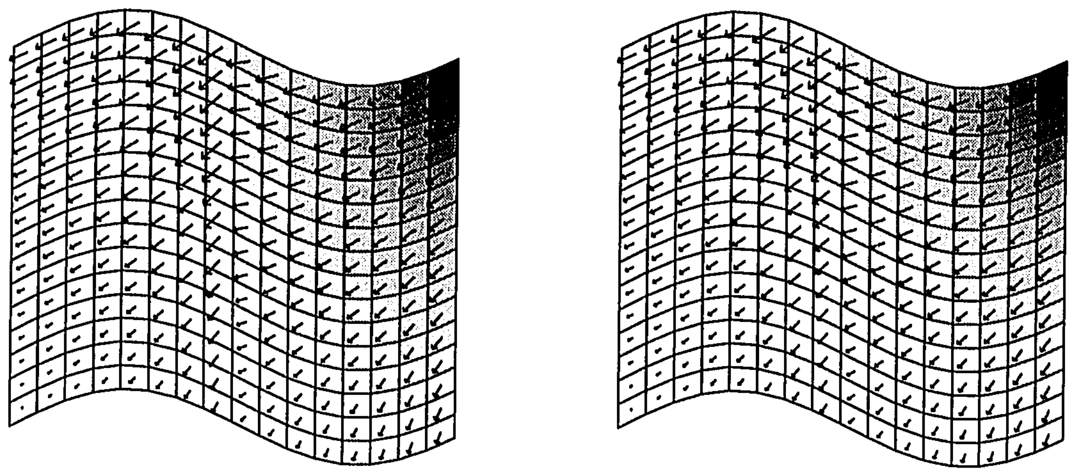
The computational domain is the unit square. Neumann boundary conditions are imposed on the boundary. We compare the results from solvimg the above problem with and without Lagrange multipliers on the interface of the discontinuity. In the first case the domain is divided into two subdomains with an interface along the $x = 1/2$ line. The discrete norm errors and convergence rates are given in Table 2.1. The results show loss of convergence for the expanded mixed method. Superconvergence for pressure and velocity is regained by adding Lagrange multipliers. Note that this is achieved at no extra computational cost, since the Glowinski-Wheeler substructuring algorithm, used in the code, is in fact a macro-hybrid formulation over subdomains.

	No Lagrange multipliers		With Lagrange multipliers		
$1/h$	$\|\|p-p_h\|\|_M$	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_{TM}$	$\|\|p-p_h\|\|_M$	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_{TM}$	$\|\|\lambda-\lambda_h\|\|_M$
8	1.23E-2	1.19E-1	5.11E-3	3.62E-2	6.90E-3
16	3.78E-3	1.01E-1	1.73E-3	1.57E-2	2.10E-3
32	1.83E-3	7.49E-2	4.99E-4	5.48E-3	5.83E-4
64	9.58E-4	5.39E-2	1.33E-4	1.85E-3	1.53E-4
128	—	—	3.42E-5	6.34E-4	3.93E-5
levels 1-5	$O(h^{1.21})$	$O(h^{0.39})$	$O(h^{1.82})$	$O(h^{1.48})$	$O(h^{1.87})$
levels 4-5	—	—	$O(h^{1.96})$	$O(h^{1.55})$	$O(h^{1.96})$

Table 2.1 Discrete norm errors and convergence rates.

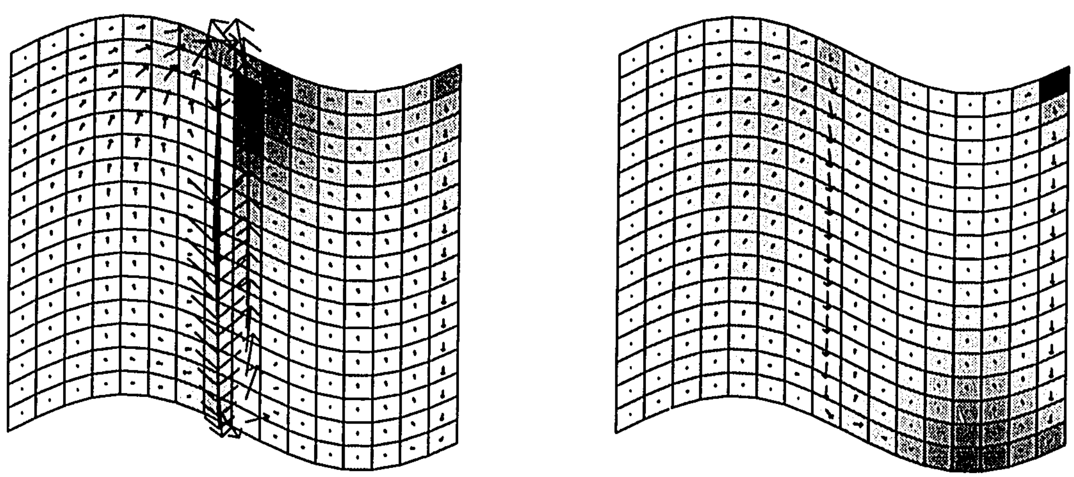
The slight degradation from the theoretical convergence rates is due to only approximate computation of the derivatives of the map and the cell centers of the true cells, where the error is computed. The relative importance of this approximation becomes negligible for fine enough grids and the theoretical rates are reached asymptotically.

The computed solution in each case on a 16×16 grid is shown on Figure 2.1. Although both solutions look the same, the errors from Table 2.1 indicate that they actually differ. This can also be seen on Figure 2.2, where the magnified error is shown.



A. No Lagrange multipliers B. With Lagrange multipliers

Figure 2.1 Computed pressure (shade) and velocity (arrows).



A. No Lagrange multipliers B. With Lagrange multipliers

Figure 2.2 Pressure (shade) and velocity (arrows) error.

Chapter 3

Mixed finite element methods on non-matching and locally refined grids

In many applications the geometry of the domain or the behavior of the solution may require using different grids in different parts of the domain, possibly non-matching on the interface. Typical examples are reservoirs with faults or locally refined grids for accurate approximation of local phenomena such as high gradients around wells. This chapter is devoted to mixed finite element approximations on non-matching grids.

A number of papers deal with the analysis and the implementation of the mixed methods for elliptic problems on regular grids (see, e.g., [72, 67, 62, 21, 19, 20, 26, 34, 61, 74, 37, 41, 9, 5]). Mixed methods on nested locally refined grids are considered in [40, 43]. These works apply the notion of "slave" nodes to force continuity of fluxes across the interfaces. The results rely heavily on the fact that the grids are nested and cannot be extended to non-matching grids.

In the present work we employ a partially hybridized form of the mixed methods to obtain accurate approximations on non-matching grids. We assume that Ω is a rectangular domain decomposed into a union of non-overlapping rectangular blocks, each of them covered by a rectangular grid. Pressure Lagrange multipliers are introduced on the interblock boundaries (see [11, 22]). Since the grids are different on the two sides of the interface, the Lagrange multiplier space can no longer be the normal trace of the velocity space. A different boundary space is needed, which we call a mortar finite element space, using the terminology from previous works on Galerkin and spectral methods (see [16] and references therein). As we show later in the analysis, the boundary space has to possess higher approximability than the velocity space. In the case of the lowest order Raviart-Thomas spaces considered in the analysis, we take continuous or discontinuous piecewise linears for the Lagrange multiplier space. Both choices provide optimal convergence rates and superconvergence at certain discrete points. The discontinuous mortars, however, have better local mass conservation properties across the interfaces, because they force the fluxes to match on every element of the boundary grid.

The considered scheme is also computationally efficient when implemented in parallel using non-overlapping domain decomposition algorithms. In particular, we modify the Glowinski-Wheeler algorithm [48, 29] to handle non-matching grids. Since this algorithm uses Lagrange multipliers on the interface, the only additional cost is computing projections of piecewise multilinear functions onto the normal trace of the local velocity spaces and vice-versa.

3.1 Formulation of the method

For simplicity of the presentation we assume that $\Gamma^D = \partial\Omega$. A weak solution of (2.1)–(2.3) is a pair $\mathbf{u} \in H(\text{div};\Omega)$, $p \in L^2(\Omega)$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega}, \qquad \mathbf{v} \in H(\text{div};\Omega), \tag{3.1}$$

$$(\nabla \cdot \mathbf{u}, w) = (f, w), \qquad w \in L^2(\Omega), \tag{3.2}$$

It is well known (see, e.g., [22, 68]) that (3.1)–(3.2) has an unique solution.

We now introduce another weak formulation related to the mixed method with continuous or discontinuous mortar elements.

Let $\Omega = \cup_{i=1}^n \Omega_i$, $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{i,j=1}^n \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma$. Let

$$\mathbf{V}_i = \{\mathbf{v} \in H(\text{div};\Omega_i) : \mathbf{v} \cdot \nu \in L^2(\Gamma_i)\}, \qquad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

$$W_i = L^2(\Omega_i), \qquad W = \bigoplus_{i=1}^n W_i = L^2(\Omega), \qquad \Lambda = L^2(\Gamma).$$

Let $\hat{\mathbf{u}} \in \mathbf{V}$, $\hat{p} \in W$, $\hat{\lambda} \in \Lambda$ satisfy, for $1 \le i \le n$,

$$(K^{-1}\hat{\mathbf{u}}, \mathbf{v})_{\Omega_i} = (\hat{p}, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \hat{\lambda}, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_i, \tag{3.3}$$

$$(\nabla \cdot \hat{\mathbf{u}}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_i, \tag{3.4}$$

$$\sum_{i=1}^n \langle \hat{\mathbf{u}} \cdot \nu, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda, \tag{3.5}$$

We first observe that (3.3)–(3.5) has at most one solution. The proof of this fact is essentially the same as the uniqueness proof in the finite dimensional case, Lemma 3.1.

Next, under the assumption that the solution $(\mathbf{u}, p)$ of (3.1)–(3.2) belongs to $(H(\mathrm{div};\Omega)\cap\mathbf{V})\times H^1(\Omega)$, it is easy to see that $(\mathbf{u}, p, p|_\Gamma)$ satisfy (3.3)–(3.5). Therefore

$$\mathbf{u} = \hat{\mathbf{u}}, \quad p = \hat{p}, \quad p|_\Gamma = \lambda.$$

For the purpose of the analysis, we introduce a reduced problem involving only the function λ. This reduced problem arose naturally in the work of Glowinski and Wheeler [48] on substructuring domain decomposition methods for mixed finite elements and is closely related to the inter-element multiplier formulation of Arnold and Brezzi [11]. Define a bilinear form $d : \Lambda \times \Lambda \to \mathbf{R}$ by

$$d(\lambda, \mu) = \sum_{i=1}^{n} d_i(\lambda, \mu) = -\sum_{i=1}^{n} \langle \mathbf{u}^*(\lambda)\cdot\nu, \mu\rangle_{\Gamma_i},$$

where $(\mathbf{u}^*(\lambda), p^*(\lambda)) \in \mathbf{V}\times W$ satisfy, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u}^*(\lambda), \mathbf{v})_{\Omega_i} = (p^*(\lambda), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}, \qquad \mathbf{v}\in\mathbf{V}_i, \tag{3.6}$$

$$(\nabla\cdot\mathbf{u}^*(\lambda), w)_{\Omega_i} = 0, \qquad w \in W_i. \tag{3.7}$$

Define a linear functional $g : \Lambda \to \mathbf{R}$ by

$$g(\mu) = \sum_{i=1}^{n} g_i(\mu) = \sum_{i=1}^{n} \langle \bar{\mathbf{u}}\cdot\nu, \mu\rangle_{\Gamma_i},$$

where for $1 \le i \le n$, $(\bar{\mathbf{u}}, \bar{p}) \in \mathbf{V}\times W$ solve

$$(K^{-1}\bar{\mathbf{u}}, \mathbf{v})_{\Omega_i} = (\bar{p}, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle g^D, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\setminus\Gamma}, \qquad \mathbf{v}\in\mathbf{V}_i, \tag{3.8}$$

$$(\nabla\cdot\bar{\mathbf{u}}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w\in W_i. \tag{3.9}$$

Then, as shown in [48], the solution $(\mathbf{u}, p, \lambda)$ of (3.3)–(3.5) satisfies

$$d(\lambda, \mu) = g(\mu), \quad \mu\in\Lambda, \tag{3.10}$$

$$\mathbf{u} = \mathbf{u}^*(\lambda) + \bar{\mathbf{u}}, \qquad p = p^*(\lambda) + \bar{p}. \tag{3.11}$$

3.1.1 Mixed finite element approximation

Let $\mathcal{T}_{h,i}$ be a quasi-uniform rectangular partition of Ω_i, $1 \le i \le n$, allowing for the possibility that $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not align on Γ_{ij}. Let

$$\mathbf{V}_{h,i}\times W_{h,i} \subset \mathbf{V}_i \times W_i$$

be the lowest order Raviart-Thomas-Nedelec (RTN) spaces on $\mathcal{T}_{h,i}$ (see, [67, 62, 68, 22]). Let

$$\mathbf{V}_h = \bigoplus_{i=1}^{n} \mathbf{V}_{h,i}, \qquad W_h = \bigoplus_{i=1}^{n} W_{h,i}.$$

Recall that

$$\nabla \cdot \mathbf{V}_{h,i} = W_{h,i},$$

and there exists a projection Π onto $\mathbf{V}_{h,i}$, satisfying amongst other properties that for any $\mathbf{q} \in (H^1(\Omega_i))^d$

$$(\nabla \cdot (\Pi \mathbf{q} - \mathbf{q}), w)_{\Omega_i} = 0, \qquad w \in W_{h,i} \tag{3.12}$$

$$\langle (\mathbf{q} - \Pi \mathbf{q}) \cdot \nu, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i} = 0, \qquad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{3.13}$$

Let $\mathcal{T}_h^{\Gamma_{i,j}}$ be a quasi-uniform rectangular partition of $\Gamma_{i,j}$. Denote by $\Lambda_{h,i,j} \subset L^2(\Gamma_{i,j})$ the space of either continuous or discontinuous piecewise multilinears on $\mathcal{T}_h^{\Gamma_{i,j}}$. Let

$$\Lambda_h = \bigoplus_{1 \le i < j \le n} \Lambda_{h,i,j}.$$

In the following we treat any function $\mu \in \Lambda_h$ as extended by zero on $\partial\Omega$. An additional assumption on the space Λ_h and hence $\mathcal{T}_h^{\Gamma_{i,j}}$ will be made below.

> **Remark 3.1** We limit our presentation to the case of the lowest order RTN spaces on rectangles. However, many of the results can be generalized in a straightforward way to any of the known mixed spaces of higher order and to simplex-type elements.

In the mixed finite element approximation of (3.3)–(3.5), we seek $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, $\lambda_h \in \Lambda_h$ such that, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{3.14}$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_{h,i}, \tag{3.15}$$

$$\sum_{i=1}^{n} \langle \mathbf{u}_h \cdot \nu, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h. \tag{3.16}$$

For each subdomain Ω_i, define a projection $\mathcal{Q}_{h,i} : L^2(\partial\Omega_i) \to \mathbf{V}_{h,i} \cdot \nu|_{\partial\Omega_i}$ such that, for any $\phi \in L^2(\partial\Omega_i)$,

$$\langle \phi - \mathcal{Q}_{h,i}\phi, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{3.17}$$

Lemma 3.1 Assume that for any $\phi \in \Lambda_h$,

$$\mathcal{Q}_{h,i}\phi = 0, \ 1 \le i \le n, \text{ implies that } \phi = 0. \tag{3.18}$$

Then there exists a unique solution of (3.14)-(3.16).

Proof Since (3.14)-(3.16) is a square system, it is enough to show uniqueness. Let $f = 0$, $g^D = 0$. Setting $\mathbf{v} = \mathbf{u}_h$, $w = p_h$, and $\mu = \lambda_h$, adding (3.14)-(3.16) together, and summing over $1 \le i \le n$, we arrive at

$$\sum_{i=1}^{n}(K^{-1}\mathbf{u}_h, \mathbf{u}_h)_{\Omega_i} = 0.$$

Hence, $\mathbf{u}_h = 0$. Denote, for $1 \le i \le n$,

$$\overline{p_{h,i}} = \frac{1}{|\Omega_i|}\int_{\Omega_i} p_h \, dx, \qquad \overline{\mathcal{Q}_{h,i}\lambda_h} = \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i} \mathcal{Q}_{h,i}\lambda_h \, ds$$

and consider the auxiliary problem

$$\begin{aligned} -\nabla \cdot K\nabla\varphi_i &= p_h - \overline{p_{h,i}} && \text{in } \Omega_i, \\ -K\nabla\varphi_i \cdot \nu &= -\left(\mathcal{Q}_{h,i}\lambda_h - \overline{\mathcal{Q}_{h,i}\lambda_h}\right) && \text{on } \partial\Omega_i. \end{aligned}$$

Note that the problem is well posed with φ_i determined up to a constant. Setting $\mathbf{v} = -\Pi K\nabla\varphi_i$ in (3.14), we have

$$(p_h, p_h - \overline{p_{h,i}})_{\Omega_i} + \left\langle \mathcal{Q}_{h,i}\lambda_h, \mathcal{Q}_{h,i}\lambda_h - \overline{\mathcal{Q}_{h,i}\lambda_h}\right\rangle_{\partial\Omega_i} = 0,$$

implying

$$p_h|_{\Omega_i} = \overline{p_{h,i}} \equiv p_i, \quad \mathcal{Q}_{h,i}\lambda_h = \overline{\mathcal{Q}_{h,i}\lambda_h}.$$

Since

$$p_i(1, \nabla\cdot\mathbf{v})_{\Omega_i} - \mathcal{Q}_{h,i}\lambda_h\langle 1, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i} = 0,$$

the divergence theorem implies $p_i = \mathcal{Q}_{h,i}\lambda_h$.

Note that $\lambda_h = 0$ on $\partial\Omega$, therefore $p_i = \mathcal{Q}_{h,i}\lambda_h = 0$ for those domains with $\partial\Omega_i \cap \partial\Omega \neq \emptyset$. For any i and j such that $\partial\Omega_i \cap \partial\Omega_j = \Gamma_{ij} \neq \emptyset$, (3.17) implies that

$$\mathcal{Q}_{h,i}\lambda_h|_{\Gamma_{ij}} = \mathcal{Q}_{h,j}\lambda_h|_{\Gamma_{ij}} = \frac{1}{|\Gamma_{ij}|}\int_{\Gamma_{ij}} \lambda_h \, ds.$$

We conclude that $\mathcal{Q}_{h,i}\lambda_h = 0$ for all $1 \le i \le n$; hence, $\lambda_h = 0$ by the hypothesis of the lemma. $\square$

Analogous to the continuous variational formulation, we consider the following equivalent reduced mixed finite element formulation. Define a bilinear form d_h : $\Lambda \times \Lambda \to \mathbf{R}$ by

$$d_h(\lambda, \mu) = \sum_{i=1}^{n} d_{h,i}(\lambda, \mu) = -\sum_{i=1}^{n} \langle \mathbf{u}_h^*(\lambda) \cdot \nu, \mu \rangle_{\Gamma_i},$$

where for $\lambda \in \Lambda$, $(\mathbf{u}_h^*(\lambda), p_h^*(\lambda)) \in \mathbf{V}_h \times W_h$ solve, for $1 \leq i \leq n$,

$$(K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{v})_{\Omega_i} = (p_h^*(\lambda), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu \rangle_{\Gamma_i}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{3.19}$$

$$(\nabla \cdot \mathbf{u}_h^*(\lambda), w)_{\Omega_i} = 0, \qquad w \in W_{h,i}. \tag{3.20}$$

Note that, with $\mathbf{v} = \mathbf{u}_h^*(\mu)$ for some $\mu \in \Lambda$, we have

$$d_{h,i}(\mu, \lambda) = -\langle \lambda, \mathbf{u}_h^*(\mu) \cdot \nu \rangle_{\Gamma_i} = (K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{u}_h^*(\mu))_{\Omega_i} = d_{h,i}(\lambda, \mu) \tag{3.21}$$

Define a linear functional $g_h : \Lambda \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i=1}^{n} g_{h,i}(\mu) = \sum_{i=1}^{n} \langle \bar{\mathbf{u}}_h \cdot \nu, \mu \rangle_{\Gamma_i},$$

where $(\bar{\mathbf{u}}_h, \bar{p}_h) \in \mathbf{V}_h \times W_h$ solve, for $1 \leq i \leq n$,

$$(K^{-1}\bar{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} = (\bar{p}_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{3.22}$$

$$(\nabla \cdot \bar{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_{h,i}. \tag{3.23}$$

It is straightforward to show (see [48]) that the solution $(\mathbf{u}_h, p_h, \lambda_h)$ of (3.14)–(3.16) satisfies

$$d_h(\lambda_h, \mu) = g_h(\mu), \quad \mu \in \Lambda_h, \tag{3.24}$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \qquad p_h = p_h^*(\lambda_h) + \bar{p}_h. \tag{3.25}$$

3.2 Error analysis

For the error analysis we assume that

$$\text{(H1)} \qquad \partial\Omega_i \cap \partial\Omega \neq \emptyset, \quad 1 \leq i \leq n.$$

This hypothesis is needed for the proof of Lemma 3.2 and may be omitted if a lower order term with a positive coefficient is added to the equation.

In the analysis we will use several projection operators. Let $\mathcal{P}_h$ be the L^2 projection onto Λ_h satisfying for any $\psi \in \Lambda$

$$\langle \psi - \mathcal{P}_h\psi, \mu\rangle_\Gamma = 0, \quad \mu \in \Lambda_h.$$

Let , for any $\varphi \in L^2(\Omega)$, $\hat{\varphi} \in W_h$ be its L^2 projection satisfying

$$(\varphi - \hat{\varphi}, w) = 0, \quad w \in W_h.$$

These operators, along with the defined earlier projections Π and $\mathcal{Q}_{h,i}$, have the following approximation properties:

$$\sum_{\tau \in \mathcal{T}_h^{\Gamma_{i,j}}} \|\psi - \mathcal{P}_h\psi\|_{s,\tau} \le C\|\psi\|_{l,\Gamma_{i,j}} h^{l-s}, \quad 0 \le l \le 2,\ 0 \le s \le l, \tag{3.26}$$

$$\|\varphi - \hat{\varphi}\| \le C\|\varphi\|_1 h, \tag{3.27}$$

$$\|\mathbf{q} - \Pi\mathbf{q}\|_{\Omega_i} \le C\|\mathbf{q}\|_{1,\Omega_i} h, \tag{3.28}$$

$$\|\nabla\cdot(\mathbf{q} - \Pi\mathbf{q})\|_{\Omega_i} \le C\|\nabla\cdot\mathbf{q}\|_{1,\Omega_i} h, \tag{3.29}$$

$$\|\psi - \mathcal{Q}_{h,i}\psi\|_{-s,\Gamma_i} \le C\|\psi\|_{l,\Gamma_i} h^{l+s}, \quad 0 \le l \le 1,\ 0 \le s \le 1. \tag{3.30}$$

$$\|(\mathbf{q} - \Pi\mathbf{q})\cdot\nu\|_{-s,\Gamma_i} \le C\|\mathbf{q}\|_{l,\Gamma_i} h^{l+s}, \quad 0 \le l \le 1,\ 0 \le s \le 1. \tag{3.31}$$

Let

$$(\cdot,\cdot)_{S,\mathrm{M}}, \quad \langle\cdot,\cdot\rangle_{\partial S,\mathrm{M}}$$

denote an application of the midpoint rule associated with $\mathcal{T}_{h,i}$ for computing the L^2 inner product on S or ∂S, respectively. Let

$$\|\cdot\|_{S,\mathrm{M}}, \quad \|\cdot\|_{\partial S,\mathrm{M}}$$

be the induced seminorms. The error in approximating integrals by the midpoint rule is bounded by $O(h^2)$ (see [30]):

$$|E_{\mathrm{M}}(\varphi,\psi)| \equiv |(\varphi,\psi) - (\varphi,\psi)_{\mathrm{M}}| \le C\sum_E \sum_{|\alpha|=2} \left\| \frac{\partial^\alpha}{\partial x^\alpha}(\varphi\psi) \right\|_{0,1,E} h^2. \tag{3.32}$$

Let $\hat{\mathcal{T}}_{h,i}^{\partial\Omega_i}$ be a refinement of $\mathcal{T}_{h,i}|_{\partial\Omega_i}$ with vertices at the centers of the elements of $\mathcal{T}_{h,i}|_{\partial\Omega_i}$ and the vertices of $\mathcal{T}_{h,i}|_{\partial\Omega_i}$. Let $\hat{U}_{h,i}$ be the space of continuous piece-wise linears on $\hat{\mathcal{T}}_{h,i}^{\partial\Omega_i}$. Let, for any $\phi \in \mathbf{V}_{h,i}\cdot\nu$, $\mathcal{I}^{\partial\Omega_i}\phi \in \hat{U}_{h,i}|_{\partial\Omega_i}$ be such that

$$\mathcal{I}^{\partial\Omega_i}\phi = \phi, \quad \text{at the midpoints,}$$

and

$$\mathcal{I}^{\partial\Omega_i}\phi = \text{ the average of the adjacent midpoint values, at the other nodes.}$$

Recall that two quadratic forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are equivalent on $\mathcal{D}$, if there exist positive constants c_1 and c_2, such that

$$c_1\mathcal{Q}_1(\phi,\phi) \le \mathcal{Q}_2(\phi,\phi) \le c_2\mathcal{Q}_1(\phi,\phi), \quad \forall\phi \in \mathcal{D}.$$

When the constants c_1 and c_2 are independent of h, we will denote this equivalence by $\mathcal{Q}_1 \simeq \mathcal{Q}_2$.

For the error analysis we need the following lemma.

Lemma 3.2 Assume that (H1) holds. Then for any $\phi \in \bigoplus_{i=1}^n \mathcal{Q}_{h,i}\Lambda_h$,

$$d_h(\phi,\phi) \simeq \sum_{i=1}^n \|\mathcal{I}^{\partial\Omega_i}\phi\|_{1/2,\partial\Omega_i}^2.$$

Proof It is proven in [28] that

$$d_h(\phi,\phi) \simeq \sum_{i=1}^n |\mathcal{I}^{\partial\Omega_i}\phi|_{1/2,\partial\Omega_i}^2.$$

The discrete bilinear form $d_h(\cdot,\cdot)$ in [28] is defined via the full hybrid form of the mixed method, which is equivalent to our macro-hybrid formulation [11]. Since $\phi = 0$ on $\partial\Omega$, by Poincare's inequality,

$$|\mathcal{I}^{\partial\Omega_i}\phi|_{1/2,\partial\Omega_i} \simeq \|\mathcal{I}^{\partial\Omega_i}\phi\|_{1/2,\partial\Omega_i}, \quad 1 \le i \le n. \tag{3.33}$$

□

Let us introduce, for any $\mu \in \Lambda_h$, the norm

$$|||\mu|||_{1/2,\Gamma_{i,j}} = \left(\sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}} \|\mu\|_{1/2,\tau}^2 \right)^{1/2}$$

We need the following hypothesis.

(H2) $|||\mu|||_{1/2,\Gamma_{i,j}} \le C(\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\Gamma_{i,j}} + \|\mathcal{I}^{\partial\Omega_j}\mathcal{Q}_{h,j}\mu\|_{1/2,\Gamma_{i,j}}), \quad \forall\mu \in \Lambda_h.$

Note that, because of the solvability assumption (3.18), both sides in (H2) represent norms on Λ_h. Therefore (H2) holds with a constant possibly dependent on h. Since the highest derivatives involved in both norms are of order 1/2, we should expect that the constant is independent of h. The following lemma shows that this is indeed the case for a particular choice of Λ_h and indicates why (H2) is reasonable in general for any choice of Λ_h satisfying (3.18). Note also that the left norm requires only local smoothness (on an element), so the inequality in the other direction does not hold.

Lemma 3.3 Let $\mathcal{T}_{h,i}|_{\Gamma_{i,j}}$ be a refinement by two of $\mathcal{T}_h^{\Gamma_{i,j}}$. Then, for any $\mu \in \Lambda_h$,

$$|||\mu|||_{1/2,\Gamma_{i,j}} \leq C \|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu\|_{1/2,\Gamma_{i,j}}.$$

Proof Denote $\hat{s} = \mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu$. It is easy to see that, for any element τ of $\mathcal{T}_h^{\Gamma_{i,j}}$, and for any element $\hat{\tau}$ of $\hat{\mathcal{T}}_{h,i}^{\Gamma_{i,j}}$,

$$|\mu|_{1,\tau}^2 \simeq |\tau|^{1-2/d} \sum_{\substack{\text{vertices}\\ v_l, v_k \in \tau}} (\mu(v_l) - \mu(v_k))^2, \tag{3.34}$$

$$|\hat{s}|_{1,\hat{\tau}}^2 \simeq |\hat{\tau}|^{1-2/d} \sum_{\substack{\text{vertices}\\ \hat{v}_l, \hat{v}_k \in \hat{\tau}}} (\hat{s}(\hat{v}_l) - \hat{s}(\hat{v}_k))^2, \tag{3.35}$$

$$|\mu|_{0,\tau}^2 \simeq |\tau| \sum_{\substack{\text{vertices}\\ v_l \in \tau}} \mu(v_l)^2, \tag{3.36}$$

$$|\hat{s}|_{0,\hat{\tau}}^2 \simeq |\hat{\tau}| \sum_{\substack{\text{vertices}\\ \hat{v}_l \in \hat{\tau}}} \hat{s}(\hat{v}_l)^2. \tag{3.37}$$

Denote

$$\mu_k = \mu(v_k).$$

Now consider an element τ of $\mathcal{T}_h^{\Gamma_{i,j}}$ in the case $d = 3$. We note that the following argument works also for $d = 2$. Denote the vertices of τ by v_{11}, v_{12}, v_{21}, v_{22}. Let x and y be the boundary variables and

$$\mu(x, y) = \varphi(x)\psi(y).$$

We have that

$$\mu_{lk} = \mu(v_{lk}) = \varphi_l \psi_k, \quad l, k = 1, 2.$$

Let

$$\tau = \bigcup_{l,k=1,2} \tau_{lk}, \quad \tau_{lk} \in \mathcal{T}_{h,i}|_{\Gamma_{i,j}}.$$

Let

$$q_{lk} = \mathcal{Q}_{h,i}\mu|_{\tau_{lk}}, \quad l, k = 1, 2. \tag{3.38}$$

Assume for simplicity that the grids are uniform. The generalization to non-uniform grids is straightforward. Now we have

$$\begin{aligned} q_{11} &= \frac{1}{4}\Big(\varphi_1 + \frac{\varphi_1 + \varphi_2}{2}\Big)\Big(\psi_1 + \frac{\psi_1 + \psi_2}{2}\Big), \\ q_{21} &= \frac{1}{4}\Big(\varphi_2 + \frac{\varphi_1 + \varphi_2}{2}\Big)\Big(\psi_1 + \frac{\psi_1 + \psi_2}{2}\Big), \end{aligned}$$

with similar expressions for q_{12} and q_{22}. Denote by q the value of $\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu$ at the center of τ. We have that

$$\begin{aligned} 2\Big[(q - q_{11})^2 + (q - q_{21})^2\Big] &\geq (q_{11} - q_{21})^2 \\ &= \frac{1}{16}(\varphi_1 - \varphi_2)^2\Big(\psi_1 + \frac{\psi_1 + \psi_2}{2}\Big)^2. \end{aligned}$$

Similarly,

$$\begin{aligned} 2\Big[(q - q_{12})^2 + (q - q_{22})^2\Big] &\geq (q_{12} - q_{22})^2 \\ &= \frac{1}{16}(\varphi_1 - \varphi_2)^2\Big(\psi_2 + \frac{\psi_1 + \psi_2}{2}\Big)^2. \end{aligned}$$

Therefore

$$\begin{aligned} &(q - q_{11})^2 + (q - q_{21})^2 + (q - q_{12})^2 + (q - q_{22})^2 \\ &\geq C(\varphi_1 - \varphi_2)^2(\psi_1^2 + \psi_2^2) \\ &= C\Big[(\mu_{11} - \mu_{21})^2 + (\mu_{12} - \mu_{22})^2\Big]. \end{aligned} \tag{3.39}$$

A similar argument gives

$$\begin{aligned} &(q - q_{11})^2 + (q - q_{21})^2 + (q - q_{12})^2 + (q - q_{22})^2 \\ &\geq C\Big[(\mu_{11} - \mu_{12})^2 + (\mu_{21} - \mu_{22})^2\Big]. \end{aligned} \tag{3.40}$$

Combining (3.39), (3.40), (3.34), and (3.35) we conclude that

$$|\mu|_{1,\tau} \leq C|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu|_{1,\tau}. \tag{3.41}$$

Similar argument as above implies

$$q_{11}^2 + q_{21}^2 \geq \frac{1}{16}(\varphi_1^2+\varphi_2^2)\Big(\psi_1 + \frac{\psi_1+\psi_2}{2}\Big)^2,$$

and

$$q_{12}^2 + q_{22}^2 \geq \frac{1}{16}(\varphi_1^2+\varphi_2^2)\Big(\psi_2 + \frac{\psi_1+\psi_2}{2}\Big)^2.$$

Therefore,

$$\begin{aligned} q_{11}^2 + q_{21}^2 + q_{12}^2 + q_{22}^2 &\geq \frac{1}{16}(\varphi_1^2+\varphi_2^2)(\psi_1^2+\psi_2^2) \\ &= \frac{1}{16}(\mu_{11}^2+\mu_{21}^2+\mu_{12}^2+\mu_{22}^2), \end{aligned}$$

which, combined with (3.36) and (3.37), implies

$$|\mu|_{0,\tau} \leq C|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu|_{0,\tau}. \tag{3.42}$$

The interpolation theory of Sobolev spaces [56] and bounds (3.41) and (3.42) imply the statement of the lemma. □

In the analysis we also need the following Lemma.

Lemma 3.4 For any function $\mathbf{v} \in \mathbf{V}_{h,i}$,

$$\|\mathbf{v}\cdot\nu\|_{0,\partial\Omega_i} \leq Ch^{-1/2}\|\mathbf{v}\|_{0,\Omega_i}.$$

Proof Since $\mathbf{v}\cdot\nu$ is piece-wise constant on $\partial\Omega_i$, we have

$$\|\mathbf{v}\cdot\nu\|_{0,\partial\Omega_i}^2 = \sum_{e\in\mathcal{T}_{h,i}|_{\partial\Omega_i}} |e|v(e)^2, \tag{3.43}$$

where $v(e)$ is the value of $\mathbf{v}\cdot\nu$ on e. Since the first component v_1 of $\mathbf{v}$ is piece-wise linear in the first direction and piece-wise constant in the other directions, we have

$$\|v_1\|_{0,\Omega_i}^2 = \sum_{\tau\in\mathcal{T}_{h,i}} \frac{1}{2}|\tau|(v_1(\tau,1)^2 + v_1(\tau,2)^2), \tag{3.44}$$

where $v_1(\tau,1)$ and $v_1(\tau,2)$ are the values of v_1 on the left and the right face (edge) of τ, respectively. Similar expressions hold for v_2 and v_3. The lemma now follows from (3.43), (3.44), and the fact that

$$|\tau| \simeq h|e|.$$

□

We continue with the error analysis by deriving a bound on $\lambda - \lambda_h$. Subtracting (3.24) from (3.10) we get

$$d_h(\lambda - \lambda_h, \mu) = d_h(\lambda, \mu) - d(\lambda, \mu) + g(\mu) - g_h(\mu), \quad \mu \in \Lambda_h. \tag{3.45}$$

We have that

$$\begin{aligned} &d_h(\lambda - \lambda_h, \mu) \\ &\quad = \sum_{i=1}^n \Big(- \langle \mathbf{u}_h^*(\lambda) \cdot \nu, \mu \rangle_{\Gamma_i} - \langle \bar{\mathbf{u}}_h \cdot \nu, \mu \rangle_{\Gamma_i} + \langle \mathbf{u}^*(\lambda) \cdot \nu, \mu \rangle_{\Gamma_i} + \langle \bar{\mathbf{u}} \cdot \nu, \mu \rangle_{\Gamma_i} \Big). \end{aligned}$$

Define

$$\mathbf{u}_h(\lambda) = \mathbf{u}_h^*(\lambda) + \bar{\mathbf{u}}_h, \quad p_h(\lambda) = p_h^*(\lambda) + \bar{p}_h,$$

and note that $(\mathbf{u}_h(\lambda), p_h(\lambda)) \in \mathbf{V}_h \times W_h$ satisfy for $1 \le i \le n$

$$\begin{aligned} (K^{-1}\mathbf{u}_h(\lambda), \mathbf{v})_{\Omega_i} &= (p_h(\lambda), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} \\ &\qquad - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \qquad (3.46) \\ (\nabla \cdot \mathbf{u}_h(\lambda), w)_{\Omega_i} &= (f, w)_{\Omega_i}, \qquad w \in W_{h,i}. \qquad (3.47) \end{aligned}$$

We now have

$$\begin{aligned} d_h(\lambda - \lambda_h, \mu) &= \sum_{i=1}^n \Big(- \langle \mathbf{u}_h(\lambda) \cdot \nu, \mu \rangle_{\Gamma_i} + \langle \mathbf{u} \cdot \nu, \mu \rangle_{\Gamma_i} \Big) \\ &= \sum_{i=1}^n \Big(- \langle \mathbf{u}_h(\lambda) \cdot \nu, \mu \rangle_{\Gamma_i} + \langle \Pi\mathbf{u} \cdot \nu, \mu \rangle_{\Gamma_i} - \langle (\Pi\mathbf{u} - \mathbf{u}) \cdot \nu, \mu \rangle_{\Gamma_i} \Big) \\ &= \sum_{i=1}^n \Big(- \langle (\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu, \mu \rangle_{\Gamma_i} - \langle (\Pi\mathbf{u} - \mathbf{u}) \cdot \nu, \mu \rangle_{\Gamma_i} \Big). \qquad (3.48) \end{aligned}$$

Since $\mu = 0$ on $\partial\Omega$, for the first term on the right we have

$$\begin{aligned} &\langle (\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu, \mu \rangle_{\Gamma_i} \\ &\quad = \langle (\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu, \mathcal{Q}_{h,i}\mu \rangle_{\partial\Omega_i, M} \\ &\quad = \langle (\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu, \mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu \rangle_{\partial\Omega_i, M} - \langle (\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu, \mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu \rangle_{\partial\Omega_i} \\ &\qquad\qquad + \langle (\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu, \mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu \rangle_{\partial\Omega_i} \\ &\quad \le C \sum_{\tau \in \mathcal{T}_{h,i}|_{\Gamma_i}} \|(\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu\|_{0,\tau} \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1,\tau} h \\ &\qquad\qquad + \|(\mathbf{u}_h(\lambda) - \Pi\mathbf{u}) \cdot \nu\|_{-1/2,\partial\Omega_i} \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i} \\ &\quad \le C\Big(h^{-1/2} \|\mathbf{u}_h(\lambda) - \Pi\mathbf{u}\|_{0,\Omega_i} h^{-1/2} \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i} h \\ &\qquad\qquad + \|\mathbf{u}_h(\lambda) - \Pi\mathbf{u}\|_{H(\mathrm{div};\Omega_i)} \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i} \Big) \\ &\quad \le C \|\mathbf{u}\|_{1,\Omega_i} h \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}, \qquad (3.49) \end{aligned}$$

where we used (3.32) for the first inequality, Lemma 3.4 and the Normal Trace Theorem for the second inequality, and the standard mixed method estimates for (3.3)–(3.4) and (3.46)–(3.47) (see [72, 67, 34]) for the last inequality.

For the second term on the right in (3.48), using the fact that $\Pi\mathbf{u}\cdot\nu$ is the L^2-projection of $\mathbf{u}\cdot\nu$ onto $\mathbf{V}_{h,i}\cdot\nu$ and (H2), we have

$$\begin{aligned}
&\langle(\Pi\mathbf{u}-\mathbf{u})\cdot\nu,\mu\rangle_{\Gamma_{i,j}} \\
&\leq C\sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}}\|(\Pi\mathbf{u}-\mathbf{u})\cdot\nu\|_{-1/2,\tau}\|\mu\|_{1/2,\tau} \\
&\leq C\|\mathbf{u}\|_{1,\Omega_i}h\Big(\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}+\|\mathcal{I}^{\partial\Omega_j}\mathcal{Q}_{h,j}\mu\|_{1/2,\partial\Omega_j}\Big).
\end{aligned}\tag{3.50}$$

Combining (3.48), (3.49), and (3.50) we obtain

$$d_h(\lambda-\lambda_h,\mu)\leq C\sum_{i=1}^{n}\|\mathbf{u}\|_{1,\Omega_i}h\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}.\tag{3.51}$$

Now take $\mu=\mathcal{P}_h\lambda-\lambda_h$ and get

$$\begin{aligned}
&d_h(\mathcal{P}_h\lambda-\lambda_h,\mathcal{P}_h\lambda-\lambda_h) \\
&\leq|d_h(\mathcal{P}_h\lambda-\lambda,\mathcal{P}_h\lambda-\lambda_h)|+C\sum_{i=1}^{n}\|\mathbf{u}\|_{1,\Omega_i}h\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}(\mathcal{P}_h\lambda-\lambda_h)\|_{1/2,\partial\Omega_i}.
\end{aligned}\tag{3.52}$$

Since, for any $\phi,\psi\in\Lambda$,

$$d_{h,i}(\phi,\psi)=d_{h,i}(\mathcal{Q}_{h,i}\phi,\mathcal{Q}_{h,i}\psi),$$

an application of Lemma 3.2 and Schwartz inequality for $d_{h,i}(\cdot,\cdot)$ gives

$$\sum_{i=1}^{n}\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}(\mathcal{P}_h\lambda-\lambda_h)\|_{1/2,\partial\Omega_i}^2\leq C\sum_{i=1}^{n}\Big(d_{h,i}(\mathcal{P}_h\lambda-\lambda,\mathcal{P}_h\lambda-\lambda)+\|\mathbf{u}\|_{1,\Omega_i}^2h^2\Big)\tag{3.53}$$

To bound the first term on the left, in (3.19)–(3.20), we replace λ by $\mathcal{P}_h\lambda-\lambda$, and take $\mathbf{v}=\mathbf{u}_h^*(\mathcal{P}_h\lambda-\lambda)$ and $w=p_h^*(\mathcal{P}_h\lambda-\lambda)$ to obtain

$$\begin{aligned}
(K^{-1}\mathbf{u}_h^*(\mathcal{P}_h\lambda-\lambda),\mathbf{u}_h^*(\mathcal{P}_h\lambda-\lambda))_{\Omega_i}&=-\langle\mathcal{P}_h\lambda-\lambda,\mathbf{u}_h^*(\mathcal{P}_h\lambda-\lambda)\cdot\nu\rangle_{\Gamma_i} \\
&\leq\|\mathcal{P}_h\lambda-\lambda\|_{0,\Gamma_i}\|\mathbf{u}_h^*(\mathcal{P}_h\lambda-\lambda)\cdot\nu\|_{0,\Gamma_i} \\
&\leq C\|\lambda\|_{3/2,\Gamma_i}h^{3/2}\|\mathbf{u}_h^*(\mathcal{P}_h\lambda-\lambda)\|_{0,\Omega_i}h^{-1/2},
\end{aligned}\tag{3.54}$$

using (3.26) and Lemma 3.4 for the last inequality. Therefore, with (3.21),

$$d_{h,i}(\mathcal{P}_h\lambda-\lambda,\mathcal{P}_h\lambda-\lambda)\leq C\|p\|_{2,\Omega_i}^2h^2,$$

implying, together with (3.53),

$$\left(\sum_{i=1}^{n} \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}(\mathcal{P}_h\lambda - \lambda_h)\|^2_{1/2,\partial\Omega_i}\right)^{1/2} \leq C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i} + \|\mathbf{u}\|_{1,\Omega_i})h.$$

Now, with (H2) we have

$$|||\mathcal{P}_h\lambda - \lambda_h|||_{1/2,\Gamma} \equiv \left(\sum_{1\leq i<j\leq n} |||\mathcal{P}_h\lambda - \lambda_h|||^2_{1/2,\Gamma_{i,j}}\right)^{1/2} \leq C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i} + \|\mathbf{u}\|_{1,\Omega_i})h, \tag{3.55}$$

and, with (3.26),

$$|||\lambda - \lambda_h|||_{1/2,\Gamma} \leq C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i} + \|\mathbf{u}\|_{1,\Omega_i})h. \tag{3.56}$$

We next derive estimates for $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$. Subtracting (3.14)–(3.16) from (3.3)–(3.5) we get the error equations

$$(K^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v})_{\Omega_i} = (p - p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda - \lambda_h, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{3.57}$$

$$(\nabla\cdot(\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} = 0, \qquad w \in W_{h,i}, \tag{3.58}$$

$$\sum_{i=1}^{n}\langle(\mathbf{u} - \mathbf{u}_h)\cdot\nu, \mu\rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h. \tag{3.59}$$

Note that (3.58) implies

$$\nabla\cdot(\Pi\mathbf{u} - \mathbf{u}_h) = 0 \text{ on } \Omega_i. \tag{3.60}$$

Take $\mathbf{v} = \Pi\mathbf{u} - \mathbf{u}_h$, $w = \hat{p} - p_h$, and $\mu = \mathcal{P}_h\lambda - \lambda_h$ to get

$$\begin{aligned}
&\sum_{i=1}^{n}(K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi\mathbf{u} - \mathbf{u}_h)_{\Omega_i} \\
&\quad = \sum_{i=1}^{n}\Big(-\langle\lambda - \mathcal{P}_h\lambda, (\Pi\mathbf{u} - \mathbf{u}_h)\cdot\nu\rangle_{\Gamma_i} + \langle(\mathbf{u} - \Pi\mathbf{u})\cdot\nu, \mathcal{P}_h\lambda - \lambda_h\rangle_{\Gamma_i}\Big) \\
&\quad \leq \sum_{i=1}^{n}\|\lambda - \mathcal{P}_h\lambda\|_{0,\Gamma_i}\|(\Pi\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\partial\Omega_i} \\
&\qquad\qquad + \sum_{i,j=1}^{n}\sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}}\|(\mathbf{u} - \Pi\mathbf{u})\cdot\nu\|_{-1/2,\tau}\|\mathcal{P}_h\lambda - \lambda_h\|_{1/2,\tau} \\
&\quad \leq C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i}h\|\Pi\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} + \|\mathbf{u}\|_{1,\Omega_i}h|||\mathcal{P}_h\lambda - \lambda_h|||_{1/2,\Gamma_i},
\end{aligned} \tag{3.61}$$

using Lemma 3.4, (3.26) and (3.31) for the last inequality. Therefore, with (3.28) and (3.55),

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i}+\|\mathbf{u}\|_{1,\Omega_i})h. \tag{3.62}$$

The estimate

$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_i} \le C\|\nabla\cdot\mathbf{u}\|_{1,\Omega_i}h \tag{3.63}$$

follows from (3.60) and (3.29).

We next use a duality argument to derive a superconvergence estimate for $\hat{p}-p_h$. Let φ be the solution of

$$\begin{aligned} -\nabla\cdot K\nabla\varphi &= -(\hat{p}-p_h) && \text{in } \Omega,\\ \varphi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Assuming that K restricted to any subdomain is smooth, we have by (H0)

$$\sum_{i=1}^{n}\|\varphi\|_{2,\Omega_i} \le C\|\hat{p}-p_h\|_0. \tag{3.64}$$

Take $\mathbf{v}=\Pi K\nabla\varphi$ in (3.57) and get

$$\begin{aligned} \|\hat{p}-p_h\|_0^2 &= \sum_{i=1}^{n}(\hat{p}-p_h,\nabla\cdot\Pi K\nabla\varphi)_{\Omega_i}\\ &= \sum_{i=1}^{n}\Big((K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi K\nabla\varphi)_{\Omega_i}+\langle\lambda-\lambda_h,\Pi K\nabla\varphi\cdot\nu\rangle_{\Gamma_i}\Big). \end{aligned} \tag{3.65}$$

The first term on the right can be manipulated as

$$\begin{aligned} &\sum_{i=1}^{n}(K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi K\nabla\varphi)_{\Omega_i}\\ &\quad= \sum_{i=1}^{n}\Big((K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi K\nabla\varphi-K\nabla\varphi)_{\Omega_i}+(\mathbf{u}-\mathbf{u}_h,\nabla\varphi)_{\Omega_i}\Big)\\ &\quad= \sum_{i=1}^{n}\Big((K^{-1}(\mathbf{u}-\mathbf{u}_h),\Pi K\nabla\varphi-K\nabla\varphi)_{\Omega_i}\\ &\qquad\qquad+(\nabla\cdot(\mathbf{u}-\mathbf{u}_h),\varphi-\hat{\varphi})_{\Omega_i}-\langle(\mathbf{u}-\mathbf{u}_h)\cdot\nu,\varphi-\mathcal{P}_h\varphi\rangle_{\partial\Omega_i}\Big)\\ &\quad\le C\sum_{i=1}^{n}\Big(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega_i}+\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_i}+\|(\mathbf{u}-\mathbf{u}_h)\cdot\nu\|_{-1/2,\partial\Omega_i}\|\Big)h\sum_{i=1}^{n}\|\varphi\|_{2,\Omega_i}, \end{aligned} \tag{3.66}$$

using (3.28), (3.27), and (3.26) for the last inequality with $C=C(\max_i\|K\|_{1,\infty,\Omega_i})$.

For the second term on the right in (3.65) we have

$$
\begin{aligned}
\sum_{i=1}^{n}&\langle\lambda-\lambda_h,\Pi K\nabla\varphi\cdot\nu\rangle_{\Gamma_i}\\
&=\sum_{i=1}^{n}\langle\mathcal{Q}_{h,i}(\lambda-\lambda_h),K\nabla\varphi\cdot\nu\rangle_{\Gamma_i}\\
&=\sum_{i=1}^{n}\langle\mathcal{Q}_{h,i}(\lambda-\lambda_h)-(\lambda-\lambda_h),K\nabla\varphi\cdot\nu\rangle_{\Gamma_i}\\
&\le C\sum_{1\le i<j\le n}\sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}}\Big(\|\mathcal{Q}_{h,i}(\lambda-\lambda_h)-(\lambda-\lambda_h)\|_{-1/2,\tau}\\
&\qquad\qquad+\|\mathcal{Q}_{h,j}(\lambda-\lambda_h)-(\lambda-\lambda_h)\|_{-1/2,\tau}\Big)\sum_{i=1}^{n}\|\varphi\|_{2,\Omega_i}\\
&\le Ch|||\lambda-\lambda_h|||_{1/2,\Gamma}\sum_{i=1}^{n}\|\varphi\|_{2,\Omega_i},
\end{aligned}\tag{3.67}
$$

using (3.30) for the last inequality.

Combining estimates (3.65), (3.66), and (3.67) with elliptic regularity (3.64) we obtain

$$
\|\hat{p}-p_h\|_0\le C\Big(\|\mathbf{u}-\mathbf{u}_h\|_0+\sum_{i=1}^{n}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_i}+|||\lambda-\lambda_h|||_{1/2,\Gamma}\Big)h. \tag{3.68}
$$

With (3.56), (3.62), (3.63), (3.68), and (3.27) we have shown following theorem.

Theorem 3.1 For the mixed method on non-matching grids (3.14)-(3.16), if (H0)-(H2) hold, then there exists a positive constant C dependent on $\max_i\|K\|_{1,\infty,\Omega_i}$, but independent of h, such that

$$
\|\mathbf{u}-\mathbf{u}_h\|_0\le C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i}+\|\mathbf{u}\|_{1,\Omega_i})h, \tag{3.69}
$$

$$
\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega_i}\le C\|\nabla\cdot\mathbf{u}\|_{1,\Omega_i}h, \tag{3.70}
$$

$$
|||\lambda-\lambda_h|||_{1/2,\Gamma}\le C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i}+\|\mathbf{u}\|_{1,\Omega_i})h, \tag{3.71}
$$

$$
\|p-p_h\|_0\le C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i}+\|\mathbf{u}\|_{1,\Omega_i}+\|\nabla\cdot\mathbf{u}\|_{1,\Omega_i})h, \tag{3.72}
$$

$$
\|\hat{p}-p_h\|_0\le C\sum_{i=1}^{n}(\|p\|_{2,\Omega_i}+\|\mathbf{u}\|_{1,\Omega_i}+\|\nabla\cdot\mathbf{u}\|_{1,\Omega_i})h^2. \tag{3.73}
$$

3.2.1 Superconvergence for the velocity

To obtain a velocity superconvergence estimate, we need a better bound on the Lagrange multiplier error. For the rest of the section we assume that the tensor K is diagonal. We now modify the last inequality in (3.49) to get

$$\langle(\mathbf{u}_h(\lambda) - \Pi\mathbf{u})\cdot\nu, \mu\rangle_{\Gamma_i} \le C\|\mathbf{u}\|_{2,\Omega_i} h^2 \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}, \tag{3.74}$$

using a superconvergence estimate for the standard mixed method derived in [37] (see also [61, 41]). Estimate (3.50) can be modified as

$$\begin{aligned}
&\langle(\Pi\mathbf{u} - \mathbf{u})\cdot\nu, \mu\rangle_{\Gamma_{i,j}} \\
&\quad\le C \sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}} \|(\Pi\mathbf{u} - \mathbf{u})\cdot\nu\|_{-1/2,\tau}\|\mu\|_{1/2,\tau} \\
&\quad\le C\|\mathbf{u}\|_{3/2,\Omega_i} h^{3/2}\Big(\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i} + \|\mathcal{I}^{\partial\Omega_j}\mathcal{Q}_{h,j}\mu\|_{1/2,\partial\Omega_j}\Big),
\end{aligned} \tag{3.75}$$

using (3.31). Combining (3.74) and (3.75) with (3.48) we obtain

$$d_h(\lambda - \lambda_h, \mu) \le C\sum_{i=1}^{n} \|\mathbf{u}\|_{2,\Omega_i} h^{3/2}\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}. \tag{3.76}$$

Take $\mu = \mathcal{P}_h\lambda - \lambda_h$ and continue the argument as in (3.52)–(3.53), to obtain

$$\sum_{i=1}^{n} \|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}(\mathcal{P}_h\lambda - \lambda_h)\|^2_{1/2,\partial\Omega_i} \le C\sum_{i=1}^{n}\left(d_{h,i}(\mathcal{P}_h\lambda - \lambda, \mathcal{P}_h\lambda - \lambda) + \|\mathbf{u}\|^2_{2,\Omega_i}h^3\right) \tag{3.77}$$

To bound the first term on the left, we proceed as in (3.54), using a sharper bound

$$\|\mathcal{P}_h\lambda - \lambda\|_{0,\Gamma_i} \le C\|\lambda\|_{2,\Gamma_i}h^2$$

from (3.26) for the last inequality of (3.54), to conclude that

$$d_{h,i}(\mathcal{P}_h\lambda - \lambda, \mathcal{P}_h\lambda - \lambda) \le C\|p\|^2_{5/2,\Omega_i}h^3. \tag{3.78}$$

Combining (3.77), (3.78), and using (H2) and (3.26), we obtain

$$|||\lambda - \lambda_h|||_{1/2,\Gamma} \le C\sum_{i=1}^{n}(\|p\|_{5/2,\Omega_i} + \|\mathbf{u}\|_{2,\Omega_i})h^{3/2}. \tag{3.79}$$

To obtain a superconvergence estimate for the velocity, we modify the last inequality in (3.61), using (3.26) and (3.31):

$$\begin{aligned}
&\sum_{i=1}^{n}(K^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi\mathbf{u} - \mathbf{u}_h)_{\Omega_i} \\
&\quad\le C\sum_{i=1}^{n}\left(\|p\|_{5/2,\Omega_i}h^{3/2}\|\Pi\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i} + \|\mathbf{u}\|_{3/2,\Omega_i}h^{3/2}|||\mathcal{P}_h\lambda - \lambda_h|||_{1/2,\Gamma_i}\right).
\end{aligned} \tag{3.80}$$

Now, a combination of (3.80), (3.79), and an estimate by Duran ([37], Theorem 3.1)

$$(K^{-1}(\Pi\mathbf{u} - \mathbf{u}), \Pi\mathbf{u} - \mathbf{u}_h)_{\Omega_i} \leq Ch^2\|\mathbf{u}\|_{2,\Omega_i}\|\Pi\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_i},$$

gives that

$$\|\Pi\mathbf{u} - \mathbf{u}_h\|_0 \leq C\sum_{i=1}^{n}(\|\mathbf{u}\|_{2,\Omega_i} + \|p\|_{5/2,\Omega_i})h^{3/2}. \tag{3.81}$$

This estimate implies superconvergence of the velocity at Gaussian points. Recall that $\|\cdot\|_{\mathrm{TM}}$ is a seminorm in $\mathbf{V}$, induced by the quadrature inner product $(\cdot,\cdot)_{\mathrm{TM}}$ (see (2.23)). Note that this seminorm only involves function values at the Gaussian points and is equal to the L^2-norm on $\mathbf{V}_h$.

Now (3.81) and the bound (see [37])

$$\|\mathbf{u} - \Pi\mathbf{u}\|_{\mathrm{TM},\Omega_i} \leq C\|\mathbf{u}\|_{2,\Omega_i}h^2$$

imply that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}} \leq C\sum_{i=1}^{n}(\|\mathbf{u}\|_{2,\Omega_i} + \|p\|_{5/2,\Omega_i})h^{3/2}. \tag{3.82}$$

We have shown the following theorem.

Theorem 3.2 For the mixed method on non-matching grids (3.14)–(3.16), if (H1) and (H2) hold, and K is diagonal, then there exists a positive constant C dependent on $\max_i \|K\|_{1,\infty,\Omega_i}$, but independent of h, such that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}} + |||\lambda - \lambda_h|||_{1/2,\Gamma} \leq C\sum_{i=1}^{n}(\|\mathbf{u}\|_{2,\Omega_i} + \|p\|_{5/2,\Omega_i})h^{3/2}.$$

3.3 Numerical experiments

The method described above has been implemented in the parallel simulator ParFlow1. The domain decomposition algorithm by Glowinski and Wheeler [48] has been modified to handle non-matching grids. The algorithm solves the interface problem (3.24) using the conjugate gradient method. On every iteration Dirichlet subdomain problems (3.19)–(3.20) has to be solved. Because of the property

$$d_{h,i}(\lambda,\mu) = d_{h,i}(\mathcal{Q}_{h,i}\lambda, \mathcal{Q}_{h,i}\mu),$$

the conjugate gradient is performed in the space $\bigoplus_{i=1}^{n}\mathcal{Q}_{h,i}\Lambda_h$ and the subdomain solves use Dirichlet data $\mathcal{Q}_{h,i}\lambda_h$. Therefore the local solves are the same as the solves in the case of matching grids and standard Lagrange multipliers. The only additional steps are projecting the local boundary flux onto Λ_h using $\mathcal{Q}_{h,i}^T$ and then projecting the jump in the flux back to $\mathbf{V}_{h,i} \cdot \nu$, using $\mathcal{Q}_{h,i}$. This makes the computational effort for non-matching grids comparable to that for matching grids.

We present some numerical tests that exhibit the theoretical convergence rates. In the first example we solve a problem on the unit square with a known analytic solution

$$p(x, y) = x^3y^2 + \sin(xy)$$

and conductivity tensor

$$K = \begin{pmatrix} 10 + 5\cos(xy) & 0 \\ 0 & 1 \end{pmatrix}.$$

The boundary conditions are Dirichlet on the left and right edge and Neumann on the rest of the boundary. The domain is divided into four subdomains with interfaces along the $x = 1/2$ and $y = 1/2$ lines. The domains are numbered starting from the lower left corner and first increasing x. The initial non-matching grids are shown on Figure 3.1 We test both continuous and discontinuous mortars. The initial mortar grids are chosen as shown in Table 3.1. Convergence rates for the test case are given in Table 3.2. The rates were established by running the test case for 5 levels of grid refinement and computing a least squares fit to the error. We observe numerically convergence rates corresponding to those predicted by the theory.

In the second example we would like to compare the mortar element mixed method on locally refined grids to the more common "slave" nodes local refinement technique [40, 43]. In the latter, the fine grid interface fluxes within a coarse cell are forced to be equal to the coarse grid flux. We note that this scheme can be recovered as a special case of the mortar element method with discontinuous mortars, if the trace of

	Continuous mortars		Discontinuous mortars	
interface	# elements	# d.o.f.	# elements	# d.o.f.
Γ_{12}	4	5	2	4
Γ_{13}	3	4	1	2
Γ_{24}	3	4	1	2
Γ_{34}	4	5	2	4

Table 3.1 Initial mortar grids.

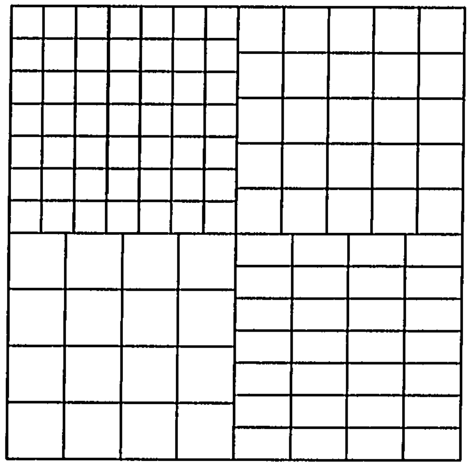


Figure 3.1 Initial non-matching grids.

	Continuous mortars			Discontinuous mortars		
$1/h$	$\|p - p_h\|_{\mathrm{M}}$	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}}$	$\|\lambda - \lambda_h\|_{\mathrm{M}}$	$\|p - p_h\|_{\mathrm{M}}$	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}}$	$\|\lambda - \lambda_h\|_{\mathrm{M}}$
8	1.45E-3	5.37E-2	2.75E-3	1.71E-3	9.36E-2	9.76E-3
16	3.56E-4	1.66E-2	7.80E-4	3.97E-4	3.32E-2	1.47E-3
32	8.72E-5	4.99E-3	1.96E-4	8.90E-5	7.16E-3	2.71E-4
64	2.17E-5	1.40E-3	4.87E-5	2.20E-5	2.52E-3	6.74E-5
128	5.42E-5	3.90E-4	1.21E-5	5.46E-6	8.76E-4	1.66E-5
rate	$O(h^{2.02})$	$O(h^{1.78})$	$O(h^{1.96})$	$O(h^{2.08})$	$O(h^{1.72})$	$O(h^{2.28})$

Table 3.2 Discrete norm errors and convergence rates for the first example.

the fine grid is a refinement by two of the interface grid. Indeed, in this case the flux matching condition (3.16) becomes a local condition over two (four if $d = 3$) fine grid boundary elements and forces all fine grid fluxes to be equal to the coarse grid flux. Our theory also recovers the convergence and superconvergence results derived by Ewing and Wang [43]. In the mortar method however, the flux continuity condition can be relaxed by choosing a coarser mortar space. In this case the fine grid fluxes are not forced to be equal and approximate the solution better. Our numerical experience shows that choosing the mortar grid to have one element less in each direction than the coarse grid, generally reduces the flux error on the interface by a factor of two.

We solve the problem of the first example on locally refined grids. The initial grids are 4×4 on Ω_1–Ω_3 and 16×16 on Ω_4. We use discontinuous piese-wise linear mortars on the non-matching interface. We report the numericall error on four levels of refinement for two cases. If the coarse grid is $n \times n$, we take a mortar grid with $n-1$ elements in the first case and $2n$ elements in the second case, which is equivalent to the "slave" nodes method. The results are summarized in Table 3.3. The pressure and velocity error on the second level of refinement are shown on Figure 3.2 and Figure 3.3.

	Disontinuous mortars			"Slave" nodes		
$1/h_c$	$\|\|p-p_h\|\|_{\mathrm{M}}$	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_{\mathrm{TM}}$	$\|\|\lambda-\lambda_h\|\|_{\mathrm{M}}$	$\|\|p-p_h\|\|_{\mathrm{M}}$	$\|\|\mathbf{u}-\mathbf{u}_h\|\|_{\mathrm{TM}}$	$\|\|\lambda-\lambda_h\|\|_{\mathrm{M}}$
8	1.12E-3	6.70E-2	3.80E-3	1.30E-3	1.45E-1	5.74E-3
16	2.67E-4	2.48E-2	1.03E-3	2.90E-4	5.00E-2	1.39E-3
32	6.57E-5	9.77E-3	2.72E-4	6.86E-5	1.74E-2	3.41E-4
64	1.64E-5	3.62E-3	6.93E-5	1.66E-5	6.09E-3	8.42E-5
rate	$O(h^{2.03})$	$O(h^{1.40})$	$O(h^{1.93})$	$O(h^{2.09})$	$O(h^{1.52})$	$O(h^{2.03})$

Table 3.3 Discrete norm errors and convergence rates on locally refined grids.

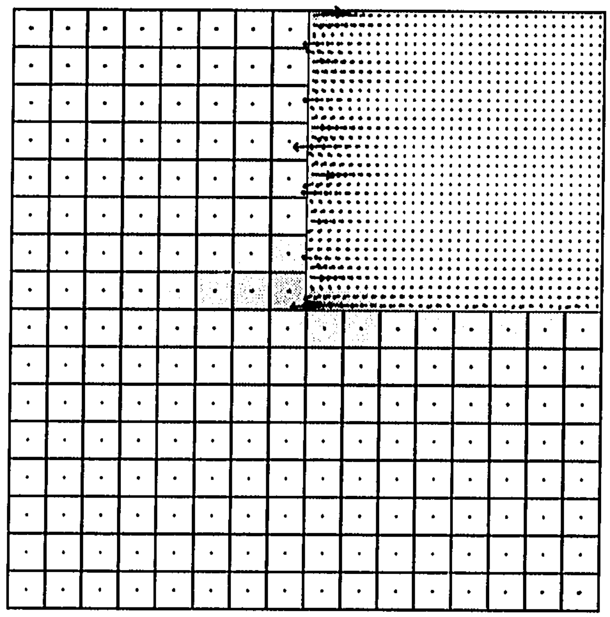


Figure 3.2 Pressure (shade) and velocity (arrows) error on locally refined grids with discontinuous mortars.

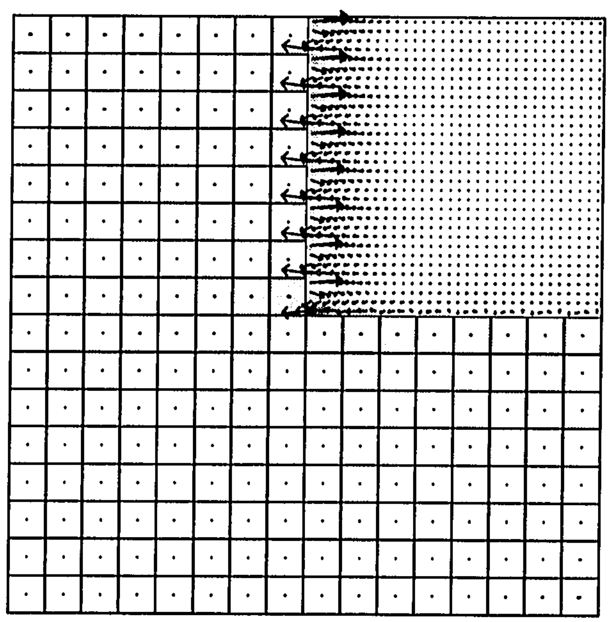


Figure 3.3 Pressure (shade) and velocity (arrows) error on locally refined grids with "slave" nodes.

Chapter 4

The expanded mixed method on non-matching grids and general geometry

In this chapter we combine techniques from the previous two chapters to formulate and analyze efficient and accurate mixed methods for elliptic problems on multiblock, irregularly shaped domains with non-matching grids on the interface. By introducing piece-wise linear mortar pressures on the subdomain boundaries we recover the superconvergence properties of the expanded mixed method on matching grids. We do not sacrifice the ease of the implementation. The mixed method is reduced to finite differences for the pressures and all computations are performed on a union of reference rectangular blocks.

4.1 Formulation of the method

We again consider problem (2.1)–(2.3), assuming $\Gamma^D = \partial\Omega$ for simplicity. Here Ω is a union of non-overlapping irregularly shaped blocks Ω_i, $1 \leq i \leq n$. We assume that there exists a continuous piece-wise smooth (at least C^2) map F from a union of rectangular blocks $\hat{\Omega} = \cup_{i=1}^n \hat{\Omega}_i$ onto Ω, such that

$$F(\hat{\Omega}_i) = \Omega_i.$$

Recall that in the expanded mixed formulation $\tilde{\mathbf{u}} = -M^{-1}\nabla p$ is the adjusted pressure gradient, for some symmetric positive definite matrix M. Using the notation from Section 3.1, we have the following variational formulation. Find $\mathbf{u} \in \mathbf{V}$, $\tilde{\mathbf{u}} \in \tilde{\mathbf{V}}$, $p \in W$, and $\lambda \in \Lambda$, such that, for $1 \leq i \leq n$,

$$(M\mathbf{u}, \tilde{\mathbf{v}})_{\Omega_i} = (MKM\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_{\Omega_i}, \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_i, \qquad (4.1)$$

$$(M\tilde{\mathbf{u}}, \mathbf{v})_{\Omega_i} = (p, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_i, \qquad (4.2)$$

$$(\nabla \cdot \mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_i, \qquad (4.3)$$

$$\sum_{i=1}^{n} \langle \mathbf{u} \cdot \nu, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda, \qquad (4.4)$$

where $\tilde{\mathbf{V}}_i = (L^2(\Omega_i))^d$ and $\tilde{\mathbf{V}} = \bigoplus_{i=1}^n \tilde{\mathbf{V}}_i$. The argument that (4.1)–(4.4) is equivalent to the more standard variational formulation (2.5)–(2.8) is similar to the one from Section 3.1.1.

To formulate the expanded mixed method, we first define the reference mixed finite element spaces, as in Section 3.1.1. Let $\hat{\mathcal{T}}_{h,i}$ be a quasi-uniform rectangular partition of $\hat{\Omega}_i$, $1 \leq i \leq n$, allowing for the possibility that $\hat{\mathcal{T}}_{h,i}$ and $\hat{\mathcal{T}}_{h,j}$ need not align on $\hat{\Gamma}_{ij}$. Let

$$\hat{\mathbf{V}}_{h,i} \times \hat{W}_{h,i} \subset \hat{\mathbf{V}}_i \times \hat{W}_i$$

be the lowest order RTN spaces on $\hat{\mathcal{T}}_{h,i}$. Let

$$\hat{\mathbf{V}}_h = \bigoplus_{i=1}^n \hat{\mathbf{V}}_{h,i}, \qquad \hat{W}_h = \bigoplus_{i=1}^n W_{h,i}.$$

Let $\hat{\tilde{\mathbf{V}}}_{h,i}$ be a finite element subspace of $(L^2(\hat{\Omega}_i))^d$ such that $\hat{\mathbf{V}}_{h,i} \subseteq \hat{\tilde{\mathbf{V}}}_{h,i}$ and let $\hat{\tilde{\mathbf{V}}}_h = \bigoplus_{i=1}^n \hat{\tilde{\mathbf{V}}}_{h,i}$. Let $\hat{\mathcal{T}}_h^{\hat{\Gamma}_{i,j}}$ be a quasi-uniform rectangular partition of $\hat{\Gamma}_{i,j}$. Denote by $\hat{\Lambda}_{h,i,j} \subset \hat{\Lambda}_{i,j}$ the space of either continuous or discontinuous piecewise multilinears on $\hat{\mathcal{T}}_h^{\hat{\Gamma}_{i,j}}$. Let

$$\hat{\Lambda}_h = \bigoplus_{1 \leq i < j \leq n} \hat{\Lambda}_{h,i,j}.$$

Given the reference grids, the mapping F defines logically rectangular grids $\mathcal{T}_{h,i}$ on Ω_i and $\mathcal{T}_h^{\Gamma_{i,j}}$ on $\Gamma_{i,j}$. The finite element spaces on these curvilinear grids are defined via (2.9) for $\mathbf{V}_h$ and $\tilde{\mathbf{V}}_h$, (2.10) for W_h, and (2.11) for Λ_h.

In the expanded mixed method for (4.1)–(4.4) we solve for $\mathbf{u}_h \in \mathbf{V}_h$, $\tilde{\mathbf{u}}_h \in \tilde{\mathbf{V}}_h$, $p_h \in W_h$, and $\lambda_h \in \Lambda_h$ such that, for $1 \leq i \leq n$,

$$(M\mathbf{u}_h, \tilde{\mathbf{v}})_{\Omega_i} = (MKM\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\Omega_i}, \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_{h,i}, \quad (4.5)$$

$$(M\tilde{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (4.6)$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_{h,i}, \quad (4.7)$$

$$\sum_{i=1}^n \langle \mathbf{u}_h \cdot \nu, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h. \quad (4.8)$$

The choice $M(F(\hat{x})) = \Big(J(DF^{-1})^T DF^{-1}\Big)(\hat{x})$ leads to a significantly simplified problem after a transformation to $\hat{\Omega}$. In Theorem 2.4 we showed that, if F is piecewise smooth, the analysis of the original problem is reduced to the analysis of the transformed problem. We therefore concentrate on solving the following variational problem on the reference domain $\hat{\Omega}$, wherein we omit the hats and the Jacobian factors that appear after the transformation. Find $\mathbf{u} \in \mathbf{V}$, $\tilde{\mathbf{u}} \in \tilde{\mathbf{V}}$, $p \in W$, and $\lambda \in \Lambda$, such that, for $1 \le i \le n$,

$$(\mathbf{u}, \tilde{\mathbf{v}})_{\Omega_i} = (K\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_{\Omega_i}, \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_i, \tag{4.9}$$

$$(\tilde{\mathbf{u}}, \mathbf{v})_{\Omega_i} = (p, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_i, \tag{4.10}$$

$$(\nabla \cdot \mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_i, \tag{4.11}$$

$$\sum_{i=1}^{n} \langle \mathbf{u} \cdot \nu, \mu \rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda, \tag{4.12}$$

For the purpose of the analysis, we consider the following equivalent variational formulation. Define a bilinear form $d : \Lambda \times \Lambda \to \mathbf{R}$ by

$$d(\lambda, \mu) = \sum_{i=1}^{n} d_i(\lambda, \mu) = -\sum_{i=1}^{n} \langle \mathbf{u}^*(\lambda) \cdot \nu, \mu \rangle_{\Gamma_i},$$

where $(\mathbf{u}^*(\lambda), \tilde{\mathbf{u}}^*(\lambda), p^*(\lambda)) \in \mathbf{V} \times \tilde{\mathbf{V}} \times W$ satisfy, for $1 \le i \le n$,

$$(\mathbf{u}^*(\lambda), \tilde{\mathbf{v}})_{\Omega_i} = (K\tilde{\mathbf{u}}^*(\lambda), \tilde{\mathbf{v}})_{\Omega_i}, \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_i, \tag{4.13}$$

$$(\tilde{\mathbf{u}}^*(\lambda), \mathbf{v})_{\Omega_i} = (p^*(\lambda), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu \rangle_{\Gamma_i}, \qquad \mathbf{v} \in \mathbf{V}_i, \tag{4.14}$$

$$(\nabla \cdot \mathbf{u}^*(\lambda), w)_{\Omega_i} = 0, \qquad w \in W_i. \tag{4.15}$$

Define a linear functional $g : \Lambda \to \mathbf{R}$ by

$$g(\mu) = \sum_{i=1}^{n} g_i(\mu) = \sum_{i=1}^{n} \langle \bar{\mathbf{u}} \cdot \nu, \mu \rangle_{\Gamma_i},$$

where, for $1 \le i \le n$, $(\bar{\mathbf{u}}, \bar{\tilde{\mathbf{u}}}, \bar{p}) \in \mathbf{V} \times \tilde{\mathbf{V}} \times W$ solve

$$(\bar{\mathbf{u}}, \tilde{\mathbf{v}})_{\Omega_i} = (K\bar{\tilde{\mathbf{u}}}, \tilde{\mathbf{v}})_{\Omega_i}, \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_i, \tag{4.16}$$

$$(\bar{\tilde{\mathbf{u}}}, \mathbf{v})_{\Omega_i} = (\bar{p}, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle g^D, \mathbf{v} \cdot \nu \rangle_{\partial\Omega_i \setminus \Gamma}, \qquad \mathbf{v} \in \mathbf{V}_i, \tag{4.17}$$

$$(\nabla \cdot \bar{\mathbf{u}}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_i. \tag{4.18}$$

It is easy to see that the solution $(\mathbf{u}, \tilde{\mathbf{u}}, p, \lambda)$ of (4.9)–(4.12) satisfies

$$d(\lambda, \mu) = g(\mu), \quad \mu \in \Lambda, \tag{4.19}$$

$$\mathbf{u} = \mathbf{u}^*(\lambda) + \bar{\mathbf{u}}, \quad \tilde{\mathbf{u}} = \tilde{\mathbf{u}}^*(\lambda) + \bar{\tilde{\mathbf{u}}}, \quad p = p^*(\lambda) + \bar{p}. \tag{4.20}$$

We now formulate the expanded mixed method on non-matching grids with quadrature for approximating (4.9)–(4.12). We take $\tilde{\mathbf{V}}_{h,i} = \mathbf{V}_{h,i}$ to be the lowest order Raviart-Thomas spaces on Ω_i and seek for $\mathbf{u}_h \in \mathbf{V}_h$, $\tilde{\mathbf{u}}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_h \in \Lambda_h$ such that, for $1 \le i \le n$,

$$(\mathbf{u}_h, \tilde{\mathbf{v}})_{\mathrm{TM},\Omega_i} = (K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\mathrm{T},\Omega_i}, \qquad \tilde{\mathbf{v}} \in \mathbf{V}_{h,i}, \tag{4.21}$$

$$(\tilde{\mathbf{u}}_h, \mathbf{v})_{\mathrm{TM},\Omega_i} = (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v}\cdot\nu\rangle_{\Gamma_i} - \langle g^D, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\setminus\Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{4.22}$$

$$(\nabla\cdot\mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_{h,i}, \tag{4.23}$$

$$\sum_{i=1}^{n}\langle \mathbf{u}_h\cdot\nu, \mu\rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h, \tag{4.24}$$

The proof of the following lemma is a trivial modification of the proof of Lemma 3.1.

Lemma 4.1 Assume that for any $\phi \in \Lambda_h$, $\mathcal{Q}_{h,i}\phi = 0$, $1 \le i \le n$, implies that $\phi = 0$. Then there exists a unique solution of (4.21)–(4.24).

Analogous to the continuous variational formulation, we consider the following equivalent reduced mixed finite element formulation. Define a bilinear form d_h : $\Lambda_h \times \Lambda_h \to \mathbf{R}$ by

$$d_h(\lambda,\mu) = \sum_{i=1}^{n} d_{h,i}(\lambda,\mu) = -\sum_{i=1}^{n}\langle \mathbf{u}_h^*(\lambda)\cdot\nu, \mu\rangle_{\Gamma_i},$$

where $(\mathbf{u}_h^*(\lambda), \tilde{\mathbf{u}}_h^*(\lambda), p_h^*(\lambda)) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h$ satisfy, for $1 \le i \le n$,

$$(\mathbf{u}_h^*(\lambda), \tilde{\mathbf{v}})_{\mathrm{TM},\Omega_i} = (K\tilde{\mathbf{u}}_h^*(\lambda), \tilde{\mathbf{v}})_{\mathrm{T},\Omega_i}, \qquad \tilde{\mathbf{v}} \in \mathbf{V}_{h,i}, \tag{4.25}$$

$$(\tilde{\mathbf{u}}_h^*(\lambda), \mathbf{v})_{\mathrm{TM},\Omega_i} = (p_h^*(\lambda), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{4.26}$$

$$(\nabla\cdot\mathbf{u}_h^*(\lambda), w)_{\Omega_i} = 0, \qquad w \in W_{h,i}. \tag{4.27}$$

Define a linear functional $g_h : \Lambda_h \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i=1}^{n} g_{h,i}(\mu) = \sum_{i=1}^{n}\langle \bar{\mathbf{u}}_h\cdot\nu, \mu\rangle_{\Gamma_i},$$

where, for $1 \le i \le n$, $(\bar{\mathbf{u}}_h, \bar{\tilde{\mathbf{u}}}_h, \bar{p}_h) \in \mathbf{V}_h \times \mathbf{V}_h \times W_h$ solve

$$(\bar{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\mathrm{TM},\Omega_i} = (K\bar{\tilde{\mathbf{u}}}_h, \tilde{\mathbf{v}})_{\mathrm{T},\Omega_i}, \qquad \tilde{\mathbf{v}} \in \mathbf{V}_{h,i}, \tag{4.28}$$

$$(\bar{\tilde{\mathbf{u}}}_h, \mathbf{v})_{\mathrm{TM},\Omega_i} = (\bar{p}_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle g^D, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\setminus\Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{4.29}$$

$$(\nabla\cdot\bar{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \qquad w \in W_{h,i}. \tag{4.30}$$

It is easy to see that the solution $(\mathbf{u}_h, \tilde{\mathbf{u}}_h, p_h, \lambda_h)$ of (4.21)-(4.24) satisfies

$$d_h(\lambda_h, \mu) = g_h(\mu), \quad \mu \in \Lambda_h, \tag{4.31}$$

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad \tilde{\mathbf{u}}_h = \tilde{\mathbf{u}}_h^*(\lambda) + \bar{\tilde{\mathbf{u}}}_h, \quad p_h = p_h^*(\lambda) + \bar{p}_h. \tag{4.32}$$

We finish the section with a characterization of $d_h(\cdot,\cdot)$, needed later in the analysis. Let $\Lambda_h^{\Omega_i}$ be the standard space of interelement Lagrange multipliers on Ω_i, i.e.,

$$\Lambda_h^{\Omega_i} = \{\mu \in L^2(\cup e) : \mu = \text{ constant on any face (edge) } e\}.$$

Given $\lambda \in \Lambda$, define $\lambda_h^*(\lambda) \in \Lambda_h^{\Omega_i}$ as follows. For any internal face (edge) e, let $\lambda_h^*(\lambda)|_e$ be the average of the values of $p_h^*(\lambda)$ on the two adjacent elements. Let $\lambda_h^*(\lambda) = \mathcal{Q}_{h,i}\lambda$ on $\partial\Omega_i$. For any element $\tau \in \mathcal{T}_{h,i}$, let the set of nodes on τ be the center of τ and the centers of the faces (edges) of τ.

Lemma 4.2 Let $q = [p_h^*(\lambda), \lambda_h^*(\lambda)] \in W_{h,i} \times \Lambda_h^{\Omega_i}$. Then

$$d_{h,i}(\lambda,\lambda) \simeq \sum_{\tau\in\mathcal{T}_{h,i}} |\tau|^{1-2/d} \sum_{\substack{\text{nodes} \\ n_l, n_k \in \tau}} (q(n_l) - q(n_k))^2$$

with constants dependent on the maximum and minimum eigenvalue of K, but independent of h.

Proof For any $\mu \in \Lambda_h$, take $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}_h^*(\mu)$, $\mathbf{v} = \mathbf{u}_h^*(\mu)$, and $w = p_h^*(\mu)$ in (4.25)-(4.27). We have

$$d_{h,i}(\mu,\lambda) = -\langle \lambda, \mathbf{u}_h^*(\mu)\cdot\nu\rangle_{\Gamma_i} = (K\tilde{\mathbf{u}}_h^*(\lambda), \tilde{\mathbf{u}}_h^*(\mu))_{\mathrm{T},\Omega_i} = d_{h,i}(\lambda,\mu). \tag{4.33}$$

Therefore

$$d_{h,i}(\lambda,\lambda) \simeq \sum_{\tau\in\mathcal{T}_{h,i}} |\tau| \sum_{\substack{\text{edges} \\ e\subset\partial\tau}} (\tilde{\mathbf{u}}_h^*(\lambda)\cdot\nu|_e)^2 \simeq \sum_{\tau\in\mathcal{T}_{h,i}} |\tau|^{1-2/d} \sum_{\substack{\text{nodes} \\ n_l, n_k \in \tau}} (q(n_l) - q(n_k))^2,$$

where the last equivalence follows from (4.26) and the definition of $\lambda_h^*(\lambda)$. □

Remark 4.1 We introduce the space $\Lambda_h^{\Omega_i}$ only for the purpose of the analysis. It is not involved in the actual computation. The above result is a counterpart of Theorem 4.3 in [28] and allows us to employ the theory developed there.

4.2 Error analysis of the finite differences scheme

Let $\mathcal{I}^{\partial\Omega_i}$ be the interpolation operator defined in Section 3.2. Because of Lemma 4.2, the theory of [28] applies to give

$$d_h(\phi,\phi) \simeq \sum_{i=1}^{n} |\mathcal{I}^{\partial\Omega_i}\phi|^2_{1/2,\partial\Omega_i}, \quad \forall\phi \in \bigoplus \mathbf{V}_{h,i}\cdot\nu.$$

The proof of the following lemma is analogous to the proof of Lemma 3.2.

Lemma 4.3 Assume that (H1) holds. Then for any $\phi \in \bigoplus_{i=1}^{n} \mathcal{Q}_{h,i}\Lambda_h$,

$$d_h(\phi,\phi) \simeq \sum_{i=1}^{n} \|\mathcal{I}^{\partial\Omega_i}\phi\|^2_{1/2,\partial\Omega_i}.$$

Throughout the analysis we assume that the subdomain grids are generated by a C^2 map (see Definition 2.1). We proceed by deriving a bound on $\lambda - \lambda_h$. Following the argument (3.45)–(3.48), we have, for any $\mu \in \Lambda_h$,

$$d_h(\lambda-\lambda_h,\mu) = \sum_{i=1}^{n}\Big(-\langle(\mathbf{u}_h(\lambda)-\Pi\mathbf{u})\cdot\nu,\mu\rangle_{\Gamma_i} - \langle(\Pi\mathbf{u}-\mathbf{u})\cdot\nu,\mu\rangle_{\Gamma_i}\Big), \tag{4.34}$$

where $\mathbf{u}_h(\lambda) = \mathbf{u}_h^*(\lambda) + \bar{\mathbf{u}}_h$, $\tilde{\mathbf{u}}_h(\lambda) = \tilde{\mathbf{u}}_h^*(\lambda) + \bar{\tilde{\mathbf{u}}}_h$, and $p_h(\lambda) = p_h^*(\lambda) + \bar{p}_h$ satisfy, for $1 \le i \le n$,

$$(\mathbf{u}_h(\lambda),\tilde{\mathbf{v}})_{\mathrm{TM},\Omega_i} = (K\tilde{\mathbf{u}}_h(\lambda),\tilde{\mathbf{v}})_{\mathrm{T},\Omega_i}, \qquad \tilde{\mathbf{v}} \in \mathbf{V}_{h,i}, \tag{4.35}$$

$$(\tilde{\mathbf{u}}_h(\lambda),\mathbf{v})_{\mathrm{TM},\Omega_i} = (p_h(\lambda),\nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda,\mathbf{v}\cdot\nu\rangle_{\Gamma_i} - \langle g^D,\mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\setminus\Gamma}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{4.36}$$

$$(\nabla\cdot\mathbf{u}_h(\lambda),w)_{\Omega_i} = (f,w)_{\Omega_i}, \qquad w \in W_{h,i}. \tag{4.37}$$

Now, as in (3.49), we have for the first term on the right in (4.34)

$$\begin{aligned}\langle(\mathbf{u}_h(\lambda)-\Pi\mathbf{u})\cdot\nu,\mu\rangle_{\Gamma_i} &\le C\|\mathbf{u}_h(\lambda)-\Pi\mathbf{u}\|_{0,\Omega_i}\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}\\ &\le Ch^{3/2}\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i},\end{aligned} \tag{4.38}$$

where the last inequality follows from the analysis of the finite difference scheme (4.35)–(4.37) in [9], Theorem 5.6, with a constant C depending on $p \in C^{3,1}(\bar{\Omega}_i)$, $\mathbf{u} \in \left(C^1(\bar{\Omega}_i)\bigcap W^{2,\infty}(\Omega_i)\right)^d$, and $K \in \left(C^1(\bar{\Omega}_i)\bigcap W^{2,\infty}(\Omega_i)\right)^{d\times d}$. For the rest of the section C will be a generic positive constant that may depend on the above quantities.

With (4.34), (4.38), and (3.75), we have

$$d_h(\lambda - \lambda_h, \mu) \le C \sum_{i=1}^{n} h^{3/2} \|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu\|_{1/2,\partial\Omega_i}. \tag{4.39}$$

Now take $\mu = \mathcal{P}_h\lambda - \lambda_h$ and get

$$\begin{aligned} & d_h(\mathcal{P}_h\lambda - \lambda_h, \mathcal{P}_h\lambda - \lambda_h) \\ & \quad \le |d_h(\mathcal{P}_h\lambda - \lambda, \mathcal{P}_h\lambda - \lambda_h)| + C \sum_{i=1}^{n} h^{3/2} \|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}(\mathcal{P}_h\lambda - \lambda_h)\|_{1/2,\partial\Omega_i}. \end{aligned} \tag{4.40}$$

Since, for any $\phi, \psi \in \Lambda$,

$$d_{h,i}(\phi, \psi) = d_{h,i}(\mathcal{Q}_{h,i}\phi, \mathcal{Q}_{h,i}\psi),$$

an application of Lemma 4.3 and Schwartz inequality for $d_{h,i}(\cdot,\cdot)$ gives

$$\sum_{i=1}^{n} \|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}(\mathcal{P}_h\lambda - \lambda_h)\|_{1/2,\partial\Omega_i}^2 \le C\Big(\sum_{i=1}^{n} d_{h,i}(\mathcal{P}_h\lambda - \lambda, \mathcal{P}_h\lambda - \lambda) + h^3 \Big) \tag{4.41}$$

To bound the first term on the left, we replace in (4.25)–(4.27) λ by $\mathcal{P}_h\lambda - \lambda$, and take $\mathbf{v} = \mathbf{u}_h^*(\mathcal{P}_h\lambda - \lambda)$, $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}_h^*(\mathcal{P}_h\lambda - \lambda)$, and $w = p_h^*(\mathcal{P}_h\lambda - \lambda)$ to obtain

$$\begin{aligned} (K\tilde{\mathbf{u}}_h^*(\mathcal{P}_h\lambda - \lambda), \tilde{\mathbf{u}}_h^*(\mathcal{P}_h\lambda - \lambda))_{\mathrm{T},\Omega_i} &= -\langle \mathcal{P}_h\lambda - \lambda, \mathbf{u}_h^*(\mathcal{P}_h\lambda - \lambda)\cdot\nu\rangle_{\Gamma_i} \\ &\le \|\mathcal{P}_h\lambda - \lambda\|_{0,\Gamma_i} \|\mathbf{u}_h^*(\mathcal{P}_h\lambda - \lambda)\|_{0,\Gamma_i} \\ &\le C\|\lambda\|_{2,\Gamma_i} h^2 \|\mathbf{u}_h^*(\mathcal{P}_h\lambda - \lambda)\|_{0,\Omega_i} h^{-1/2}, \end{aligned} \tag{4.42}$$

using (3.26) and Lemma 3.4 for the last inequality. With $\tilde{\mathbf{v}} = \mathbf{u}_h^*(\mathcal{P}_h\lambda - \lambda)$ in (4.25), we have

$$\|\mathbf{u}_h^*(\mathcal{P}_h\lambda - \lambda)\|_{\mathrm{TM},\Omega_i} \le C\|\tilde{\mathbf{u}}_h^*(\mathcal{P}_h\lambda - \lambda)\|_{\mathrm{T},\Omega_i},$$

and, since the norms $\|\cdot\|_{\mathrm{TM},\Omega_i}$, $\|\cdot\|_{\mathrm{T},\Omega_i}$, and $\|\cdot\|_{0,\Omega_i}$ are equivalent on $\mathbf{V}_{h,i}$, we conclude from (4.42) that

$$\|\tilde{\mathbf{u}}_h^*(\mathcal{P}_h\lambda - \lambda)\|_{0,\Omega_i} \le C\|\lambda\|_{2,\Gamma_i} h^{3/2}.$$

Therefore, with (4.33),

$$d_{h,i}(\mathcal{P}_h\lambda - \lambda, \mathcal{P}_h\lambda - \lambda) \le C\|p\|_{5/2,\Omega_i}^2 h^3,$$

implying, together with (4.41),

$$\left(\sum_{i=1}^{n} \|\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}(\mathcal{P}_h\lambda - \lambda_h)\|_{1/2,\partial\Omega_i}^2 \right)^{1/2} \le Ch^{3/2}.$$

Now, with (H2) we have

$$|||\mathcal{P}_h\lambda - \lambda_h|||_{1/2,\Gamma} \le Ch^{3/2}, \tag{4.43}$$

and, with (3.26),

$$|||\lambda - \lambda_h|||_{1/2,\Gamma} \le Ch^{3/2}. \tag{4.44}$$

Next, we derive an estimate on $\mathbf{u} - \mathbf{u}_h$. In the following we use the 2D finite difference notation from Section 2.3.2. We start with the following auxiliary result.

Lemma 4.4 If $p \in C^{3,1}(\bar{\Omega}_k) \bigcap C^0(\bar{\Omega})$, $\mathbf{u} \in \left(C^1(\bar{\Omega}_k) \bigcap W^{2,\infty}(\Omega_k)\right)^d \bigcap H(\mathrm{div};\Omega)$, and $K \in \left(C^1(\bar{\Omega}_k) \bigcap W^{2,\infty}(\Omega_k)\right)^{d\times d}$, then there exist $\mathbf{U} \in \mathbf{V}_h$, $\tilde{\mathbf{U}} \in \mathbf{V}_h$, $P \in W_h$, and $\lambda^* \in \Lambda_h$ such that, for $1 \le k \le n$,

$$\begin{aligned}
(\tilde{\mathbf{U}}, \mathbf{v})_{\mathrm{TM},\Omega_k} &= (P, \nabla\cdot\mathbf{v})_{\Omega_k} - \langle\lambda^*, \mathbf{v}\cdot\nu\rangle_{\Gamma_k} \\
&\quad - \langle g^D, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_k\setminus\Gamma}, && \mathbf{v} \in \mathbf{V}_{h,k}, \quad (4.45)\\
(\mathbf{U}, \tilde{\mathbf{v}})_{\mathrm{TM},\Omega_k} &= (K\tilde{\mathbf{U}}, \tilde{\mathbf{v}})_{\mathrm{T},\Omega_k}, && \tilde{\mathbf{v}} \in \mathbf{V}_{h,k}, \quad (4.46)
\end{aligned}$$

and there exists a constant C, independent of h, but dependent on the solution, and K as indicated, such that, for all i, j on a subdomain Ω_k,

$$\begin{aligned}
&|P_{ij} - p_{ij}| \le Ch^2, && (4.47)\\
&|\tilde{U}^x_{i+1/2,j} - \tilde{u}^x_{i+1/2,j}| + |\tilde{U}^y_{i,j+1/2} - \tilde{u}^y_{i,j+1/2}| \le Ch^{\tilde{r}}, && (4.48)\\
&|U^x_{i+1/2,j} - u^x_{i+1/2,j}| + |U^y_{i,j+1/2} - u^y_{i,j+1/2}| \le Ch^{r}, && (4.49)\\
&|||\lambda^* - \lambda|||_{1/2,\Gamma_k} \le Ch^{3/2}, && (4.50)
\end{aligned}$$

where $\tilde{r} = 2$ for all points not on $\Gamma \cup \partial\Omega$, and $\tilde{r} = 1$ otherwise, $r = 2$ for points strictly in the interior of Ω_k that lie on an edge or face e such that $\bar{e} \cap \partial\bar{\Omega}_k = \emptyset$, and $r = 1$ otherwise.

Proof On a subdomain Ω_k, we apply a construction due to Weiser and Wheeler (see [74], Lemma 4.1 and appendix) to $(\tilde{\mathbf{u}}, p)$, that gives a P satisfying (4.47), and a $\tilde{\mathbf{U}}$, satisfying (4.45) and (4.48) with $\tilde{r} = 2$ strictly in the interior and $\tilde{r} = 1$ on $\partial\Omega_k \cap \partial\Omega$. Note that the constants in this construction depend on the Lipschitz constant of $\partial^3 p$ on Ω_k. Let

$$\lambda^* = \mathcal{P}_h\lambda \text{ on } \Gamma_k;$$

therefore, with (3.26), (4.50) holds. For any point $(x_{i+1/2}, y_j)$ on the left edge of Γ_k, (4.45) gives

$$\tilde{U}^x_{i+1/2,j} = \frac{2}{h^x_{i+1}}\left(\frac{1}{|e|}\int_e \lambda^* - P_{i+1,j}\right),$$

where e is the element edge with a center $(x_{i+1/2}, y_j)$, and by Taylor theorem,

$$\tilde{u}^x_{i+1/2,j} = \frac{2}{h^x_{i+1}}(\lambda_{i+1/2,j} - p_{i+1,j}) + O(h).$$

Then, with (4.47),

$$\tilde{U}^x_{i+1/2,j} - \tilde{u}^x_{i+1/2,j} = \frac{2}{h^x_{i+1}}\frac{1}{|e|}\int_e(\lambda^* - \lambda) + O(h) = O(h),$$

where the last equality follows from the approximation properties of $\mathcal{P}_h$. Treating similarly the rest of Γ_k, we conclude that (4.48) holds on Γ_k with $\tilde{r} = 1$.

The proof of (4.49) is analogous to the argument (2.68)-(2.72) from Lemma 2.3. □

Since $\cup_{i=1}^n \partial\Omega_i$ is a set of dimension $d-1$, the following corollary holds.

Corollary 4.1 For the functions in Lemma 4.4, there exists a constant C, independent of h, such that

$$\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}\|_{\mathrm{TM}} + \|\mathbf{U} - \mathbf{u}\|_{\mathrm{TM}} \le Ch^{3/2}.$$

Subtracting (4.21)-(4.24) from (4.46), (4.45), (4.11), and (4.12), respectively, we obtain the error equations

$$(\mathbf{U} - \mathbf{u}_h, \tilde{\mathbf{v}})_{\mathrm{TM},\Omega_i} = (K(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h), \tilde{\mathbf{v}})_{\mathrm{T},\Omega_i}, \qquad \tilde{\mathbf{v}} \in \mathbf{V}_{h,i}, \quad (4.51)$$

$$(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h, \mathbf{v})_{\mathrm{TM},\Omega_i} = (P - p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda^* - \lambda_h, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (4.52)$$

$$(\nabla\cdot(\Pi\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} = 0, \qquad w \in W_{h,i}, \quad (4.53)$$

$$\sum_{i=1}^n \langle(\mathbf{u} - \mathbf{u}_h)\cdot\nu, \mu\rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h, \quad (4.54)$$

With $\mathbf{v} = \Pi\mathbf{u} - \mathbf{u}_h$, $w = P - p_h$, and $\mu = \lambda^* - \lambda_h$, we obtain

$$\begin{aligned}(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h, \Pi\mathbf{u} - \mathbf{u}_h)_{\mathrm{TM}} &= \sum_{i=1}^n \langle(\mathbf{u} - \Pi\mathbf{u})\cdot\nu, \lambda^* - \lambda_h\rangle_{\Gamma_i} \\ &\le \sum_{i,j=1}^n \sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}} \|(\mathbf{u} - \Pi\mathbf{u})\cdot\nu\|_{-1/2,\tau}\|\lambda^* - \lambda_h\|_{1/2,\tau} \\ &\le C(h^3 + |||\lambda^* - \lambda_h|||^2_{1/2,\Gamma}) \le Ch^3, \end{aligned} \quad (4.55)$$

using the triangle inequality, (4.44), and (4.50) for the last inequality. Now, with $\tilde{\mathbf{v}} = \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h$ in (4.51),

$$(K(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h), \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)_{\mathrm{T}} = (\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h, \mathbf{U} - \Pi\mathbf{u})_{\mathrm{TM}} + O(h^3).$$

An application of Schwarz inequality gives

$$\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM}} \leq C(\|\mathbf{U} - \Pi\mathbf{u}\|_{\mathrm{TM}} + h^{3/2}) \leq Ch^{3/2},$$

using Corollary 4.1 and the bound [37]

$$\|\mathbf{u} - \Pi\mathbf{u}\|_{\mathrm{TM}} \leq Ch^2 \tag{4.56}$$

for the last inequality. Taking $\tilde{\mathbf{v}} = \mathbf{U} - \mathbf{u}_h$ in (4.51) implies

$$\|\mathbf{U} - \mathbf{u}_h\|_{\mathrm{TM}} \leq C\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM}} \leq Ch^{3/2}.$$

An application of Corollary 4.1 now gives the following bound on the velocity error.

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{TM}} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathrm{TM}} \leq Ch^{3/2}. \tag{4.57}$$

To bound $\nabla \cdot (\mathbf{u} - \mathbf{u}_h)$, we observe that (4.53) implies $\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h) = 0$; therefore, with (3.29),

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_i} = \|\nabla \cdot (\mathbf{u} - \Pi\mathbf{u})\|_{0,\Omega_i} \leq C\|\nabla \cdot \mathbf{u}\|_{1,\Omega_i} h. \tag{4.58}$$

We next use a duality argument to derive a superconvergence bound for $\hat{p} - p_h$. From (4.21)-(4.24) and (4.9)-(4.12) we get the error equations

$$\begin{aligned}
(\mathbf{u} - \mathbf{u}_h, \tilde{\mathbf{v}})_{\Omega_i} &= (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \tilde{\mathbf{v}})_{\Omega_i} - E_{\mathrm{TM}}(\mathbf{u}_h, \tilde{\mathbf{v}})_{\Omega_i} \\
&\quad + E_{\mathrm{T}}(K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\Omega_i}, && \tilde{\mathbf{v}} \in \mathbf{V}_{h,i}, && (4.59) \\
(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} &= (\hat{p} - p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda - \lambda_h, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} \\
&\quad - E_{\mathrm{TM}}(\tilde{\mathbf{u}}_h, \mathbf{v})_{\Omega_i}, && \mathbf{v} \in \mathbf{V}_{h,i}, && (4.60) \\
(\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w)_{\Omega_i} &= 0, && w \in W_{h,i}, && (4.61) \\
\sum_{i=1}^{n} \langle (\mathbf{u} - \mathbf{u}_h) \cdot \nu, \mu \rangle_{\Gamma_i} &= 0, && \mu \in \Lambda_h, && (4.62)
\end{aligned}$$

where

$$E_{\mathrm{Q}}(\mathbf{q}, \mathbf{v})_{\Omega_i} = (\mathbf{q}, \mathbf{v})_{\Omega_i} - (\mathbf{q}, \mathbf{v})_{\mathrm{Q},\Omega_i}, \quad \mathrm{Q} = \mathrm{TM} \text{ or } \mathrm{T}.$$

Let φ solve

$$-\nabla\cdot K\nabla\varphi = \hat{p} - p_h \qquad \text{in } \Omega, \tag{4.63}$$

$$\varphi = 0 \qquad \text{on } \partial\Omega. \tag{4.64}$$

With (H0) we have

$$\sum_{i=1}^{n} \|\varphi\|_{2,\Omega_i} \le C\|\hat{p} - p_h\|. \tag{4.65}$$

Take $\mathbf{v} = -\Pi K\nabla\varphi$ in (4.60) and get

$$\begin{aligned}
&\|\hat{p} - p_h\|_0^2 \\
&\quad = -\sum_{i=1}^{n}\Big((\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \Pi K\nabla\varphi - K\nabla\varphi)_{\Omega_i} + (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \nabla\varphi - \Pi\nabla\varphi)_{\Omega_i} \\
&\qquad + (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \Pi\nabla\varphi)_{\Omega_i} + \langle\lambda - \lambda_h, \Pi K\nabla\varphi\cdot\nu\rangle_{\Gamma_i}\Big) + E_{\mathrm{TM}}(\tilde{\mathbf{u}}_h, \Pi K\nabla\varphi),
\end{aligned} \tag{4.66}$$

and, using (4.59), (4.61), and (4.62), the third term on the right is

$$\begin{aligned}
&-\sum_{i=1}^{n}(K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \Pi\nabla\varphi)_{\Omega_i} \\
&\quad = -\sum_{i=1}^{n}\Big((\mathbf{u} - \mathbf{u}_h, \Pi\nabla\varphi - \nabla\varphi)_{\Omega_i} + (\mathbf{u} - \mathbf{u}_h, \nabla\varphi)_{\Omega_i}\Big) \\
&\qquad - E_{\mathrm{TM}}(\mathbf{u}_h, \Pi\nabla\varphi) + E_{\mathrm{T}}(K\tilde{\mathbf{u}}_h, \Pi\nabla\varphi) \\
&\quad = -\sum_{i=1}^{n}\Big((\mathbf{u} - \mathbf{u}_h, \Pi\nabla\varphi - \nabla\varphi)_{\Omega_i} - (\nabla\cdot(\mathbf{u} - \mathbf{u}_h), \varphi - \hat{\varphi})_{\Omega_i} \\
&\qquad + \langle(\mathbf{u} - \mathbf{u}_h)\cdot\nu, \varphi - \mathcal{P}_h\varphi\rangle_{\partial\Omega_i}\Big) - E_{\mathrm{TM}}(\mathbf{u}_h, \Pi\nabla\varphi) + E_{\mathrm{T}}(K\tilde{\mathbf{u}}_h, \Pi\nabla\varphi).
\end{aligned}$$

The fourth term on the right in (4.66) is

$$\begin{aligned}
&\sum_{i=1}^{n}\langle\lambda - \lambda_h, \Pi K\nabla\varphi\cdot\nu\rangle_{\Gamma_i} \\
&\qquad = \sum_{i=1}^{n}\langle\mathcal{Q}_{h,i}(\lambda - \lambda_h), K\nabla\varphi\cdot\nu\rangle_{\Gamma_i} \\
&\qquad = \sum_{i=1}^{n}\langle\mathcal{Q}_{h,i}(\lambda - \lambda_h) - (\lambda - \lambda_h), K\nabla\varphi\cdot\nu\rangle_{\Gamma_i} \\
&\qquad \le C\sum_{1\le i<j\le n}\ \sum_{\tau\in\mathcal{T}_h^{\Gamma_{i,j}}}\Big(\|\mathcal{Q}_{h,i}(\lambda - \lambda_h) - (\lambda - \lambda_h)\|_{-1/2,\tau} \\
&\qquad\qquad + \|\mathcal{Q}_{h,j}(\lambda - \lambda_h) - (\lambda - \lambda_h)\|_{-1/2,\tau}\Big)\sum_{i=1}^{n}\|\varphi\|_{2,\Omega_i} \\
&\qquad \le Ch|||\lambda - \lambda_h|||_{1/2,\Gamma}\sum_{i=1}^{n}\|\varphi\|_{2,\Omega_i},
\end{aligned} \tag{4.67}$$

using (3.30) for the last inequality.

Therefore, using (3.28), (3.27), and (3.26),

$$
\begin{aligned}
&\|\hat{p} - p_h\|_0^2 \\
&\quad \leq C\Big\{\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 + \sum_{i=1}^{n} \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_i} \\
&\qquad + \sum_{i=1}^{n} \|(\mathbf{u} - \mathbf{u}_h) \cdot \nu\|_{-1/2,\partial\Omega_i} + |||\lambda - \lambda_h|||_{1/2,\Gamma}\Big\} \sum_{i=1}^{n} \|\varphi\|_{2,\Omega_i} h \\
&\qquad + \Big|E_{\mathrm{TM}}(\tilde{\mathbf{u}}_h, \Pi K \nabla\varphi)\Big| + \Big|E_{\mathrm{TM}}(\mathbf{u}_h, \Pi\nabla\varphi)\Big| + \Big|E_{\mathrm{T}}(K\tilde{\mathbf{u}}_h, \Pi\nabla\varphi)\Big|. \qquad (4.68)
\end{aligned}
$$

The quadrature error terms are bounded as in the proof of Lemma 2.6 (2.87)–(2.88):

$$
\begin{aligned}
|E_{\mathrm{TM}}(\tilde{\mathbf{u}}_h, \Pi K\nabla\varphi)| &\leq C\Big\{\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_{0,\Omega} + \sum_{i=1}^{n} \|\tilde{\mathbf{u}}\|_{1,\Omega_i} h\Big\} \sum_{i=1}^{n} \|\varphi\|_{2,\Omega_i} h \\
|E_{\mathrm{TM}}(\mathbf{u}_h, \Pi\nabla\varphi)| &\leq C\Big\{\|\mathbf{u}_h - \Pi\mathbf{u}\|_{0,\Omega} + \sum_{i=1}^{n} \|\mathbf{u}\|_{1,\Omega_i} h\Big\} \sum_{i=1}^{n} \|\varphi\|_{2,\Omega_i} h, \\
|E_{\mathrm{T}}(K\tilde{\mathbf{u}}_h, \Pi\nabla\varphi)| &\leq C\Big\{\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_{0,\Omega} + \sum_{i=1}^{n} \|\tilde{\mathbf{u}}\|_{1,\Omega_i} h\Big\} \sum_{i=1}^{n} \|\varphi\|_{2,\Omega_i} h,
\end{aligned}
$$

with constants depending on $\sum_{i=1}^{n} \|K\|_{2,\infty,\Omega_i}$.

From (4.68), then,

$$
\begin{aligned}
&\|\hat{p} - p_h\|_0^2 \\
&\quad \leq C\Big\{\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_0 + \|\mathbf{u}_h - \Pi\mathbf{u}\|_0 \\
&\qquad + \sum_{i=1}^{n} \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_i} + \sum_{i=1}^{n} \|(\mathbf{u} - \mathbf{u}_h) \cdot \nu\|_{-1/2,\partial\Omega_i} \\
&\qquad + |||\lambda - \lambda_h|||_{1/2,\Gamma} + h\Big\} \sum_{i=1}^{n} \|\varphi\|_{2,\Omega_i} h. \qquad (4.69)
\end{aligned}
$$

The first four terms on the right are bounded in (4.57) and (4.56); the fifth term is bounded in (4.58)

For the sixth term, using the Normal trace theorem, we have

$$
\|(\mathbf{u} - \mathbf{u}_h) \cdot \nu\|_{-1/2,\partial\Omega_i} \leq C\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div};\Omega_i)}.
$$

With (4.44), (4.65), and the above bounds, (4.69) gives

$$
\|\hat{p} - p_h\|_0 \leq Ch^2. \qquad (4.70)
$$

With (4.44), (4.57), (4.58), and (4.70), we have shown the following theorem.

Theorem 4.1 For the cell-centered finite difference approximation of the expanded mixed method on non-matching grids (4.21)-(4.24), if (H0)-(H2) hold, and if $p \in C^{3,1}(\bar{\Omega}_i)\bigcap C^0(\bar{\Omega})$, $\mathbf{u} \in \left(C^1(\bar{\Omega}_i)\bigcap W^{2,\infty}(\Omega_i)\right)^d \bigcap H(\text{div};\Omega)$, and $K \in \left(C^1(\bar{\Omega}_i)\bigcap W^{2,\infty}(\Omega_i)\right)^{d\times d}$, then there exists a constant C, independent of h but dependent on the solution and K as indicated , such that

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{TM}} + \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{\mathrm{TM}} \le Ch^{3/2}, \tag{4.71}$$

$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 \le Ch, \tag{4.72}$$

$$\|p-p_h\|_{\mathrm{M}} \le Ch^2, \tag{4.73}$$

$$|||\lambda-\lambda_h|||_{1/2,\Gamma} \le Ch^{3/2}. \tag{4.74}$$

4.3 Numerical experiments

The parallel simulator ParFlow1, developed originally as a rectangular code, was modified to handle logically rectangular grids. A preprocessor was added to modify the coefficients of the problem as described in Section 2.2, and a postprocessor was added to transform the reference solution $(\hat{p}_h, \hat{\mathbf{u}}_h, \hat{\lambda}_h)$ to the solution $(p_h, \mathbf{u}_h, \lambda_h)$ on the physical domain.

Our first example exhibits the theoretical convergence rates. We solve a problem with a known analytic solution and mapping

$$p(x,y) = \begin{cases} xy & \text{for } x \le 1/2, \\ xy + (x-1/2)(y+1/2) & \text{for } x > 1/2, \end{cases}$$

$$K(x,y) = \begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{for } x < 1/2, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } x > 1/2, \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} + \frac{1}{10}\sin(6\hat{x}) \end{pmatrix}.$$

The computational domain is the unit square. The boundary conditions are Dirichlet on the left edge and Neumann on the rest of the boundary. The domain is divided into two sub-domains with an interface along the $x = 1/2$ line. The non-matching grids are initially 4×8 on the left and 4×11 on the right. Continuous mortars on a grid of 7 elements with 8 degrees of freedom or discontinuous mortars on a grid of 3 elements with 6 degrees of freedom are introduced on the interface. Convergence rates for the test case are given in Table 4.1. As in the example from Section 2.4, we observe slight degradation from the theoretical convergence rates due to approximate computation of the derivatives of the map and the cell centers of the true cells. The relative importance of this approximation becomes negligible for fine enough grids and the theoretical rates are reached asymptotically.

	Continuous mortars			Discontinuous mortars		
$1/h$	$\lVert p-p_h\rVert_M$	$\lVert \mathbf{u}-\mathbf{u}_h\rVert_M$	$\lVert \lambda-\lambda_h\rVert_M$	$\lVert p-p_h\rVert_M$	$\lVert \mathbf{u}-\mathbf{u}_h\rVert_M$	$\lVert \lambda-\lambda_h\rVert_M$
8	5.97E-3	3.62E-2	7.80E-3	5.97E-3	3.62E-2	7.78E-3
16	2.07E-3	1.58E-2	2.29E-3	2.07E-3	1.58E-2	2.28E-3
32	6.11E-4	5.50E-3	6.08E-4	6.11E-4	5.51E-3	6.09E-4
64	1.65E-5	1.86E-3	1.55E-4	1.65E-5	1.87E-3	1.56E-4
128	4.26E-5	6.34E-4	3.91E-5	4.26E-5	6.39E-4	3.93E-5
levels 1-5	$O(h^{1.80})$	$O(h^{1.48})$	$O(h^{1.92})$	$O(h^{1.80})$	$O(h^{1.47})$	$O(h^{1.91})$
levels 4-5	$O(h^{1.95})$	$O(h^{1.55})$	$O(h^{1.99})$	$O(h^{1.95})$	$O(h^{1.55})$	$O(h^{1.99})$

Table 4.1 Discrete norm errors and convergence rates for the example with a known analytic solution.

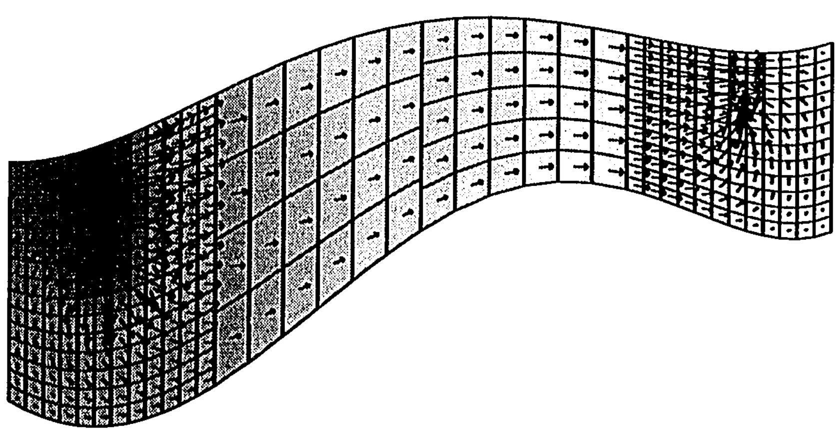


Figure 4.1 Computed pressure (shade) and velocity field (arrows) for the more practical example.

Our second example shows a more practical application. We model flow through a three dimensional aquifer with a vertical fault cutting the domain near its middle. A vertical cross-section, perpendicular to the fault, of the computed pressure and velocity field is shown in Fig. 4.1. The injection well on the left and the production well on the right penetrate through half the aquifer depth; no flow is specified on the boundary and gravity is neglected. The aquifer is divided into four sub-domains. The fault coincide with two sub-domain boundaries, and the grid is refined around the wells for a better approximation of the velocities.

Chapter 5

Mixed finite element methods for multiphase flow

In this chapter we discuss mixed finite element discretizations of the coupled system of two-phase flow equations. The original system is reduced to a pressure equation of elliptic type and a saturation of degenerate parabolic type. The mixed methods developed in the previous chapters are then directly applied to the pressure equation. For the saturation equation, following [10], we employ the Kirchoff transformation to handle the degenerate diffusion.

5.1 Fractional flow formulation

We consider two phase immiscible flow in an irregular, heterogeneous reservoir Ω. It is modeled by the system of conservation equations [12, 24]

$$\frac{\partial(\phi s_i \rho_i)}{\partial t} + \nabla \cdot \rho_i \mathbf{u}_i = q_i \qquad \text{in } \Omega \times (0,T], \tag{5.1}$$

$$\mathbf{u}_i = -\frac{k_i(s_i)K}{\mu_i}(\nabla p_i - \rho_i g \nabla D) \qquad \text{in } \Omega \times (0,T], \tag{5.2}$$

$i = w$ (wetting), n (non-wetting), coupled with

$$s_w + s_n = 1, \tag{5.3}$$

$$p_c(s_w) = p_n - p_w, \tag{5.4}$$

where s_i is the phase saturation, ρ_i is the phase density, ϕ is the porosity, q_i is the source term, $\mathbf{u}_i$ is the Darcy velocity, p_i is the phase pressure, K is the absolute permeability tensor, $k_i(s_i)$ is the phase relative permeability, μ_i is the phase viscosity, g is the gravitational constant, and D is the depth. For simplicity we assume that no flow boundary conditions are imposed, although more general boundary conditions can also be treated. Initial wetting phase saturation $s_0(x)$ is specified on Ω.

We start by reformulating the problem in a fractional flow form (pressure and saturation equation). Later we show that the proposed numerical scheme is equivalent to direct discretization of the conservation equations. Let

$$\lambda_i = \frac{k_i}{\mu_i}, \quad i = w, n,$$

denote the phase mobilities, and let

$$\lambda = \lambda_w + \lambda_n$$

be the total mobility. Let

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}_n$$

be the total velocity. For simplicity of the presentation, assume incompressible flow and medium (constant ρ_i and ϕ), and neglect gravity effects. Multiplying (5.1) by $1/\rho_i$ and adding them together, we get

$$\nabla \cdot \mathbf{u} = q, \tag{5.5}$$

where $q = q_w/\rho_w + q_n/\rho_n$. Let $s = s_w$, and define the global pressure [24] to be

$$p = p_w + \int_0^{p_c(s)} \Big(\frac{\lambda_n}{\lambda}\Big)\big(p_c^{-1}(\zeta)\big)\, d\zeta.$$

Thus

$$\mathbf{u} = -\lambda K \nabla p. \tag{5.6}$$

Equation (5.5) is referred to as the pressure equation. Since $\lambda > 0$ and K is a symmetric positive definite tensor, this is an elliptic equation. For compressible flow the pressure equation is parabolic.

To derive the saturation equation, we first observe that

$$\frac{\lambda_w}{\lambda}\mathbf{u} = \mathbf{u}_w - \frac{\lambda_w \lambda_n}{\lambda} K \nabla p_c(s).$$

Substituting this expression into the water conservation equation (5.1), we get the saturation equation

$$\phi \frac{\partial s}{\partial t} + \nabla \cdot \Big(\beta(s)\mathbf{u} + \alpha(s) K \nabla p_c(s)\Big) = \tilde{q}_w, \tag{5.7}$$

where $\beta(s) = \lambda_w/\lambda$, $\alpha(s) = \lambda_w\lambda_n/\lambda$, and $\tilde{q}_w = q_w/\rho_w$. Note that $p_c(s)$ is a strictly monotone decreasing function. We can write the last term on the left in (5.7) as

$$\alpha(s) K \frac{\partial p_c}{\partial s} \nabla s \equiv -\sigma(s) K \nabla s.$$

Therefore (5.7) is an advection-diffusion equation. The diffusion term vanishes at $s = 0, 1$ - the minimum and maximum values of the saturation. This is due to the behavior of the relative permeability and the capillary pressure functions (see [13]).

This double degeneracy is the main source of difficulties in the numerical approximation. The solutions of degenerate parabolic equations have very low regularity. It has been shown that (see [65, 46, 54, 3, 2, 4]

$$s \in L^\infty(0,T;L_1(\Omega)), \tag{5.8}$$

$$s_t \in L^2(0,T;H^{-1}(\Omega)). \tag{5.9}$$

The solutions can have compact support, thus behaving very differently form the solutions to non-degenerate parabolic problems. Many authors introduce a regularized problem and then approximate it [69, 50, 63, 45, 44]. The numerical error is then a sum of the discretization error for the regularized problem and the difference between the solutions to the regularized and the original problems.

Another difficulty in the analysis comes from the treatment of the degenerate diffusion term. Because of the degeneracy, certain error terms cannot be directly bounded. A common technique is to handle the degeneracy analytically via the Kirchoff transformation [69, 45, 44, 10]. Let

$$D(s) = \int_0^s \sigma(\zeta)\, d\zeta.$$

Then

$$\nabla D(s) = \sigma(s)\nabla s,$$

and (5.7) becomes

$$\phi\frac{\partial s}{\partial t} + \nabla \cdot \Big(\beta(s)\mathbf{u} - K\nabla D(s)\Big) = \tilde{q}_w. \tag{5.10}$$

5.2 The expanded mixed method for multiphase flow

In the following we omit the porosity ϕ, assuming that a linear change of variables $s' = \phi s$ has been made. To obtain the expanded mixed variational formulation of (5.1)-(5.7) we introduce the variables

$$\tilde{\mathbf{u}} = -\nabla p, \tag{5.11}$$

$$\tilde{\psi} = -\nabla D(s), \tag{5.12}$$

$$\psi = \beta(s)\mathbf{u} + K\tilde{\psi}. \tag{5.13}$$

Because of (5.9), s_t is not in $L^2(\Omega)$ for a given t, thus we integrate (5.7) in time to obtain the equivalent equation

$$s + \nabla \cdot \int_0^t \psi \, d\tau = \int_0^t \tilde{q}_w \, d\tau + s_0. \tag{5.14}$$

Let

$$\mathbf{V} = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v} \cdot \nu = 0 \text{ on } \partial\Omega\}$$

We now have, for every time $t \in [0,T]$, the variational form for $\mathbf{u}(\cdot,t) \in \mathbf{V}$, $\tilde{\mathbf{u}}(\cdot,t) \in (L^2(\Omega))^d$, and $p(\cdot,t) \in L^2(\Omega)$ as

$$(\mathbf{u}, \tilde{\mathbf{v}}) = (\lambda(s)K\tilde{\mathbf{u}}, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in (L^2(\Omega))^d, \tag{5.15}$$

$$(\tilde{\mathbf{u}}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0, \qquad \mathbf{v} \in \mathbf{V}, \tag{5.16}$$

$$(\nabla \cdot \mathbf{u}, w) = (q(s), w), \qquad w \in L^2(\Omega), \tag{5.17}$$

and the variational form for $\psi(\cdot,t) \in \mathbf{V}$, $\tilde{\psi}(\cdot,t) \in (L^2(\Omega))^d$, and $s(\cdot,t) \in L^2(\Omega)$ as

$$(\psi, \tilde{\mathbf{v}}) = (\beta(s)\mathbf{u}, \tilde{\mathbf{v}}) + (K\tilde{\psi}, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in (L^2(\Omega))^d, \tag{5.18}$$

$$(\tilde{\psi}, \mathbf{v}) - (D(s), \nabla \cdot \mathbf{v}) = 0, \qquad \mathbf{v} \in \mathbf{V}, \tag{5.19}$$

$$(s, w) + \left(\nabla \cdot \int_0^t \psi \, d\tau, w\right) = \left(\int_0^t \tilde{q}_w(s) \, d\tau, w\right) + (s_0, w), \qquad w \in L^2(\Omega). \tag{5.20}$$

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times L^2(\Omega)$ be any mixed finite element spaces on a partition $\mathcal{T}_h$ of Ω, and let $\tilde{\mathbf{V}}_h$ be a finite element subspace of $(L^2(\Omega))^d$ such that $\mathbf{V}_h \subseteq \tilde{\mathbf{V}}_h$. We then have the following semidiscrete expanded mixed finite element approximation to the system (5.15)–(5.20). For each $t \in [0,T]$, let $(\mathbf{u}_h(\cdot,t), \tilde{\mathbf{u}}_h(\cdot,t), p_h(\cdot,t)) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h \times W_h$ be the approximation of $(\mathbf{u}(\cdot,t), \tilde{\mathbf{u}}(\cdot,t), p(\cdot,t))$ such that

$$(\mathbf{u}_h, \tilde{\mathbf{v}}) = (\lambda(s_h)K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_h, \tag{5.21}$$

$$(\tilde{\mathbf{u}}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) = 0, \qquad \mathbf{v} \in \mathbf{V}_h, \tag{5.22}$$

$$(\nabla \cdot \mathbf{u}_h, w) = (q(s_h), w), \qquad w \in W_h, \tag{5.23}$$

and $(\psi_h(\cdot,t), \tilde{\psi}_h(\cdot,t), s_h(\cdot,t)) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h \times W_h$ be the approximation of $(\psi(\cdot,t), \tilde{\psi}(\cdot,t), s(\cdot,t))$ such that

$$(\psi_h, \tilde{\mathbf{v}}) = (\beta(s_h)\mathbf{u}_h, \tilde{\mathbf{v}}) + (K\tilde{\psi}_h, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_h, \tag{5.24}$$

$$(\tilde{\psi}_h, \mathbf{v}) - (D(s_h), \nabla \cdot \mathbf{v}) = 0, \qquad \mathbf{v} \in \mathbf{V}_h, \tag{5.25}$$

$$(s_h, w) + \left(\nabla \cdot \int_0^t \psi_h \, d\tau, w\right) = \left(\int_0^t \tilde{q}_w(s_h) \, d\tau, w\right) + (s_{0,h}, w), \qquad w \in W_h, \tag{5.26}$$

where $s_{0,h}$ is an approximation of s_0.

We next consider a backward Euler time discretization. Let $\{t_n\}_{n=0}^N$ be a monotone partition of $[0,T]$ with $t_0 = 0$ and $t_N = T$, let $\Delta t^n = t_n - t_{n-1}$, and let $f^n = f(t_n)$.

In the fully discrete mixed method we seek, for any $0 \le n \le N$, $(\mathbf{u}_h^n, \tilde{\mathbf{u}}_h^n, p_h^n) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h \times W_h$ such that

$$(\mathbf{u}_h^n, \tilde{\mathbf{v}}) = (\lambda(s_h^n) K \tilde{\mathbf{u}}_h^n, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_h, \tag{5.27}$$

$$(\tilde{\mathbf{u}}_h^n, \mathbf{v}) - (p_h^n, \nabla \cdot \mathbf{v}) = 0, \qquad \mathbf{v} \in \mathbf{V}_h, \tag{5.28}$$

$$(\nabla \cdot \mathbf{u}_h^n, w) = (q(s_h^n), w), \qquad w \in W_h, \tag{5.29}$$

and $(\psi_h^n, \tilde{\psi}_h^n, s_h^n) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h \times W_h$ such that

$$(\psi_h^n, \tilde{\mathbf{v}}) = (\beta(s_h^n)\mathbf{u}_h^n, \tilde{\mathbf{v}}) + (K\tilde{\psi}_h^n, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_h, \tag{5.30}$$

$$(\tilde{\psi}_h^n, \mathbf{v}) - (D(s_h^n), \nabla \cdot \mathbf{v}) = 0, \qquad \mathbf{v} \in \mathbf{V}_h, \tag{5.31}$$

$$\begin{aligned}(s_h^n, w) &+ \left(\nabla \cdot \sum_{j=1}^n \psi_h^j \Delta t^j, w\right) \\ &= \left(\sum_{j=1}^n \tilde{q}_w(s_h^j)\Delta t^j, w\right) + (s_{0,h}, w), \qquad w \in W_h. \end{aligned} \tag{5.32}$$

Equation (5.32) can be rewritten in the usual backward Euler form

$$\left(\frac{s_h^n - s_h^{n-1}}{\Delta t^n}, w\right) + (\nabla \cdot \psi_h^n, w) = (\tilde{q}_w(s_h^n), w), \qquad w \in W_h, \tag{5.33}$$

$$(s_h^0, w) = (s_{0,h}, w), \qquad w \in W_h. \tag{5.34}$$

Remark 5.1 The choice of fully implicit time discretization is motivated from the fact that all explicit or semi-implicit schemes suffer from severe stability limitations when advection processes are dominating.

Remark 5.2 Equation (5.33) is a locally mass conservative approximation of the wetting phase conservation equation (note that by construction $\psi = \mathbf{u}_w$). Subtracting (5.33) from (5.29) we get

$$-\left(\frac{s_h^n - s_h^{n-1}}{\Delta t^n}, w\right) + (\nabla \cdot (\mathbf{u}_h^n - \psi_h^n), w) = (\tilde{q}_n(s_h^n), w), \quad w \in W_h, \tag{5.35}$$

which is an element by element approximation to the non-wetting phase conservation equation, so the scheme conserves the mass of both phases locally. Therefo-re the mixed method for the pressure-saturation formulation preserves the conservation properties of the cell-centered finite difference scheme applied directly to the phase conservation equations, a method commonly used by the petroleum engineers [66, 12].

In the case of RT_0 mixed spaces, the use of quadrature rules for approximating the vector inner products in (5.27)–(5.32) (see Section 2.2, (2.21)–(2.22)) allows the vector unknowns to be trivially eliminated, if $\tilde{\mathbf{V}}_h = \mathbf{V}_h$. Thus, a coupled cell-centered finite difference system for the pressure and the saturation is obtained. The stencil for the pressure or the saturation is 5 points for $d = 2$ and 7 points for $d = 3$ if K is a diagonal tensor, 9 points for $d = 3$ and 19 points for $d = 3$ if K is a full tensor.

5.2.1 Extensions to general geometry domains, discontinuous coefficients, and non-matching grids

An advantage of the pressure-saturation formulation is that the methods discussed in the previous chapters can be applied to handle irregularly shaped domains, discontinuous coefficients, and non-matching grids. For the case of general geometry, we only need to modify definitions (5.11)–(5.13) by

$$\tilde{\mathbf{u}} = -M^{-1}\nabla p, \tag{5.36}$$

$$\tilde{\psi} = -M^{-1}\nabla D(s), \tag{5.37}$$

$$\psi = \beta(s)\mathbf{u} + KM\tilde{\psi}, \tag{5.38}$$

where M is defined in (2.16). Now, the procedure from Section 2.2 can be applied to give cell-centered finite differences for p and s on logically rectangular grids.

If the permeability tensor is discontinuous, or the domain has a multiblock structure with non-smooth grids across the interfaces, the macro-hybrid form of the expanded mixed method has to be used. It introduces Lagrange multipliers for p and s along the discontinuities. If the grids match, we may use the usual normal trace of the vector space. If the grids do not match, a higher order mortar finite element space, as described in Chapter 3 and Chapter 4, must be used on the interface.

The expanded mixed method for the pressure equation has been analyzed in the previous chapters. In the next section we concentrate on the saturation equation.

5.3 A mixed method for the saturation equation on non-matching grids

Let Ω be a union of n non-overlapping subdomain blocks Ω_i, $1 \le i \le n$. The saturation $s(x,t)$ satisfies

$$\frac{\partial s}{\partial t} + \nabla \cdot \Big(\beta(s)\mathbf{u} - K\nabla D(s)\Big) = \tilde{q}_w(s) \qquad \text{in } \Omega \times (0,T], \tag{5.39}$$

$$\Big(\beta(s)\mathbf{u} - K\nabla D(s)\Big) \cdot \nu = 0 \qquad \text{on } \partial\Omega \times [0,T], \tag{5.40}$$

$$s(x,0) = s_0(x) \qquad \text{on } \Omega. \tag{5.41}$$

We need several assumptions on the coefficients of the above equation. Recall that $D(s) = \int_0^s \sigma(\zeta)\, d\zeta$, and assume that

$$\sigma(s) \ge \begin{cases} \beta_1 |s|^{\nu_1}, & 0 \le s \le \alpha_1, \\ \beta_2, & \alpha_1 \le s \le \alpha_2, \\ \beta_3 |1-s|^{\nu_2}, & \alpha_2 \le s \le 1, \end{cases} \tag{5.42}$$

where β_i, $1 \le i \le 3$, are positive constants, and α_i and ν_i, $i = 1,2$, satisfy

$$0 < \alpha_1 < 1/2 < \alpha_2 < 1, \quad 0 < \nu_i \le 2.$$

Note that (5.42) controls the rate of degeneracy of the diffusion. It is based on the physical behavior of the relative permeabilities and the capillary pressure (see, e.g., [13, 55]). We also have that there exists a positive constant C such that

$$\|D(s_1) - D(s_2)\|_0^2 \le C(D(s_1) - D(s_2), s_1 - s_2), \quad \text{for } s_1, s_2 \in L^2(\Omega). \tag{5.43}$$

A sufficient condition for (5.43) is

$$0 \le \frac{\partial D}{\partial s}(x,t;s) \le C \quad \text{for } (x,t) \in \Omega \times [0,T],\ 0 \le s \le 1,$$

which again follows from the behavior of the relative permeabilities and the capillary pressure (see also [10]). Finally, we assume that, for $0 \le s_1, s_2 \le 1$,

$$|\beta(s_1) - \beta(s_2)|^2 \le C(D(s_1) - D(s_2))(s_1 - s_2), \tag{5.44}$$

$$|\tilde{q}_w(s_1) - \tilde{q}_w(s_2)|^2 \le C(D(s_1) - D(s_2))(s_1 - s_2). \tag{5.45}$$

Bounds (5.44) and (5.45) are justified by the following lemma, proven in [45].

Lemma 5.1 Suppose σ satisfies (5.42). If $f \in C^1[0,1]$ and $f'(0) = f'(1) = 0$ with f' Lipschitz at 0 and 1, then there exists a positive constant C such that

$$|f(a) - f(b)|^2 \le C(D(a) - D(b))(a-b), \quad \text{for } 0 \le a, b \le 1.$$

Note the fractional flow function $\beta(s)$ satisfies the conditions of Lemma 5.1. The well term $\tilde{q}_w(s)$ satisfies the conditions of Lemma 5.1 at the injection wells. At the production wells, $\tilde{q}_w(s) \sim k_w(s)$, so $\tilde{q}_w'(0) = 0$. Therefore (5.45) holds, if $s \le s^* < 1$ at the production well, which covers all cases of physical interest.

Remark 5.3 The fractional flow function $\beta(s)$ and the integrated diffusion function $D(s)$ are both S-shaped with zero derivatives at the end points. Bound (5.44) relates the rates of degeneracy of the derivatives of the two functions and indicates, in a sense, that the diffusion dominates the advection.

Let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{i,j=1}^n \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma$. Let

$$\mathbf{V}_i = \{\mathbf{v} \in H(\mathrm{div};\Omega_i) : \mathbf{v}\cdot\nu \in L^2(\partial\Omega_i) \text{ and } \mathbf{v}\cdot\nu = 0 \text{ on } \partial\Omega\}, \qquad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

$$W_i = L^2(\Omega_i), \qquad W = \bigoplus_{i=1}^n W_i = L^2(\Omega), \qquad \Lambda = L^2(\Gamma).$$

With $a = K^{-1}$, $\psi = \beta(s)\mathbf{u} - K\nabla D(s)$, and $\gamma =$ the trace of $D(s)$ on Γ, we have the following variational formulation. For every time $t \in [0,T]$, $\psi(\cdot,t) \in \mathbf{V}$, $s(\cdot,t) \in W$, and $\gamma(\cdot,t) \in \Lambda$ satisfy, for $1 \le i \le n$,

$$(a\psi, \mathbf{v})_{\Omega_i} = (D(s), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle\gamma, \mathbf{v}\cdot\nu\rangle_{\Gamma_i} + (a\beta(s)\mathbf{u}, \mathbf{v})_{\Omega_i}, \qquad \mathbf{v} \in \mathbf{V}_i, \tag{5.46}$$

$$(s,w)_{\Omega_i} + \left(\nabla\cdot\int_0^t \psi\,d\tau, w\right)_{\Omega_i} = \left(\int_0^t \tilde{q}_w(s)\,d\tau, w\right)_{\Omega_i} + (s_0, w)_{\Omega_i}, \quad w \in W_i, \tag{5.47}$$

$$\sum_{i=1}^n \left\langle \int_0^t \psi\cdot\nu\,d\tau, \mu\right\rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda. \tag{5.48}$$

Let $\mathcal{T}_{h,i}$ be a finite element partition of Ω_i with maximal element diameter h. We allow for the possibility that $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not match on $\Gamma_{i,j}$. Let $\mathbf{V}_{h,i} \times W_{h,i}$ be the RT_0 spaces on $\mathcal{T}_{h,i}$. Let $\mathcal{T}_h^{\Gamma_{i,j}}$ be a finite element partition of $\Gamma_{i,j}$ with maximal element diameter h. Let $\Lambda_{h,i,j} \subset \Lambda_{i,j}$ be the space of continuous or discontinuous piece-wise multi-linears on $\mathcal{T}_h^{\Gamma_{i,j}}$. An additional assumption on $\Lambda_{h,i,j}$ and hence on $\mathcal{T}_h^{\Gamma_{i,j}}$ will be made later.

In the continuous time mixed finite element method for approximating (5.46)-(5.48) we seek, for each $t \in [0,T]$, $\psi_h(\cdot,t) \in \mathbf{V}_h$, $s_h(\cdot,t) \in W_h$, and $\gamma_h(\cdot,t) \in \Lambda_h$ such that, for $1 \le i \le n$,

$$(a\psi_h, \mathbf{v})_{\Omega_i} = (D(s_h), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \gamma_h, \mathbf{v}\cdot\nu\rangle_{\Gamma_i} + (a\beta(s_h)\mathbf{u}, \mathbf{v})_{\Omega_i}, \qquad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{5.49}$$

$$(s_h, w)_{\Omega_i} + \left(\nabla\cdot\int_0^t \psi_h\, d\tau, w\right)_{\Omega_i} = \left(\int_0^t \tilde{q}_w(s_h)\, d\tau, w\right)_{\Omega_i} + (s_{0,h}, w)_{\Omega_i}, \qquad w \in W_{h,i}, \tag{5.50}$$

$$\sum_{i=1}^n \left\langle \int_0^t \psi_h\cdot\nu\, d\tau, \mu\right\rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h, \tag{5.51}$$

where $s_{0,h} \in W_h$ is an approximation of s_0.

5.3.1 Error analysis of the semidiscrete scheme

We combine ideas from [10], where the mixed method for the saturation equation on a single block is analyzed, with techniques from the previous chapters, to analyze the mixed method for the saturation equation on a multiblock domain with non-matching grids.

We need the following projections onto the finite element spaces. The standard mixed projection operator $\Pi : (H^1(\Omega_i))^d \to \mathbf{V}_{h,i}$ satisfies, for $\mathbf{q} \in (H^1(\Omega_i))^d$,

$$(\nabla\cdot(\mathbf{q} - \Pi\mathbf{q}), w)_{\Omega_i} = 0, \qquad w \in W_{h,i}, \tag{5.52}$$

$$\langle(\mathbf{q} - \Pi\mathbf{q})\cdot\nu, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i} = 0, \qquad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{5.53}$$

In the analysis we apply Π to $\int_0^t \psi$, which is justified, since $\int_0^t \psi \in (H^1(\Omega_i \times (0,T]))^d$, as shown in [10].

Let, for any $\varphi \in W$, $\hat{\varphi} \in W_h$ be its L^2-projection, satisfying

$$(\varphi - \hat{\varphi}, w) = 0, \quad w \in W_h.$$

In a similar way we define the L^2-projections $\mathcal{P}_h : \Lambda \to \Lambda_h$, and $\mathcal{Q}_{h,i} : \Lambda \to \mathbf{V}_{h,i}\cdot\nu$. For smooth enough functions, these operators have optimal order approximation properties:

$$\|\mathbf{q} - \Pi\mathbf{q}\|_{\Omega_i} \le C\|\mathbf{q}\|_{1,\Omega_i} h, \tag{5.54}$$

$$\|\varphi - \hat{\varphi}\| \le C\|\varphi\|_l h^l, \quad 0 \le l \le 1, \tag{5.55}$$

$$\|\psi - \mathcal{P}_h\psi\|_{0,\Gamma} \le C\|\psi\|_{l,\Gamma} h^l, \quad 0 \le l \le 2, \tag{5.56}$$

$$\|\psi - \mathcal{Q}_{h,i}\psi\|_{0,\Gamma_i} \le C\|\psi\|_{l,\Gamma_i} h^l, \quad 0 \le l \le 1. \tag{5.57}$$

We make explicit the following assumption on the mortar space Λ_h.

$$\text{(H3)} \qquad \|\mu\|_{0,\Gamma_{i,j}} \le C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}}), \quad \forall \mu \in \Lambda_h,$$

which is a weaker version of hypothesis (H2) from Section 3.2 and is justified by Lemma 3.3.

We now proceed with the error analysis. Subtracting (5.49)–(5.51) from (5.46)–(5.48), we obtain the error equations

$$\begin{aligned}
(a(\psi - \psi_h), \mathbf{v})_{\Omega_i} &= (D(s) - D(s_h), \nabla \cdot \mathbf{v})_{\Omega_i} \\
&\quad - \langle \gamma - \gamma_h, \mathbf{v} \cdot \nu \rangle_{\Gamma_i} + (a(\beta(s) - \beta(s_h))\mathbf{u}, \mathbf{v})_{\Omega_i}, \qquad \mathbf{v} \in \mathbf{V}_{h,i},
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
(s - s_h, w)_{\Omega_i} &+ \left(\nabla \cdot \int_0^t (\psi - \psi_h)\, d\tau, w\right)_{\Omega_i} \\
&= \left(\int_0^t (\tilde{q}_w(s) - \tilde{q}_w(s_h))\, d\tau, w\right)_{\Omega_i} + (s_0 - s_{0,h}, w)_{\Omega_i}, \qquad w \in W_{h,i},
\end{aligned} \tag{5.59}$$

$$\sum_{i=1}^{n} \left\langle \int_0^t (\psi - \psi_h) \cdot \nu\, d\tau, \mu \right\rangle_{\Gamma_i} = 0, \qquad \mu \in \Lambda_h, \tag{5.60}$$

To simplify notations, let $\overline{\Phi}(t) = \Pi \int_0^t (\psi - \psi_h)\, d\tau$. We choose $s_{0,h} = \hat{s}_0$ and take $\mathbf{v} = \overline{\Phi}$, $w = \widehat{D(s)} - \widehat{D(s_h)}$, and $\mu = \mathcal{P}_h\gamma - \gamma_h$ in (5.58)–(5.60). We then have

$$\begin{aligned}
(s - s_h, \widehat{D(s)} - \widehat{D(s_h)}) &+ (a(\psi - \psi_h), \overline{\Phi}) \\
&= \left(\int_0^t (\tilde{q}_w(s) - \tilde{q}_w(s_h))\, d\tau, \widehat{D(s)} - \widehat{D(s_h)}\right) \\
&\quad + (a(\beta(s) - \beta(s_h))\mathbf{u}, \overline{\Phi}) - \sum_{i=1}^{n} \langle \gamma - \mathcal{P}_h\gamma, \overline{\Phi} \cdot \nu \rangle_{\Gamma_i} \\
&\quad + \sum_{i=1}^{n} \left\langle \left(\int_0^t \psi\, d\tau - \Pi \int_0^t \psi\, d\tau\right) \cdot \nu, \mathcal{P}_h\gamma - \gamma_h \right\rangle_{\Gamma_i}
\end{aligned} \tag{5.61}$$

We integrate (5.61) in time form 0 to t. The first term on the left becomes

$$\int_0^t (s - s_h, \widehat{D(s)} - \widehat{D(s_h)})\, d\tau = \int_0^t (s - s_h, D(s) - D(s_h))\, d\tau + T_1, \tag{5.62}$$

where

$$T_1 = \int_0^t (\hat{s} - s, D(s) - D(s_h))\, d\tau.$$

The second term on the left-hand side of (5.61) becomes

$$\int_0^t (a(\psi - \psi_h), \overline{\Phi})\, d\tau = \frac{1}{2}\Big\|a^{1/2}\int_0^t (\psi - \psi_h)\, d\tau\Big\|_0^2 + T_2, \tag{5.63}$$

where

$$T_2 = \int_0^t \Big(a(\psi - \psi_h), \Pi\int_0^\tau \psi\, d\xi - \int_0^\tau \psi\, d\xi\Big)\, d\tau.$$

Combining (5.61)–(5.63), we obtain

$$\int_0^t (s - s_h, D(s) - D(s_h))\, d\tau + \frac{1}{2}\Big\|a^{1/2}\int_0^t (\psi - \psi_h)\Big\|_0^2 = \sum_{k=1}^{6} T_k, \tag{5.64}$$

where

$$T_3 = \int_0^t \left(\int_0^\tau (\tilde{q}_w(s) - \tilde{q}_w(s_h))\, d\xi, \widehat{D(s)} - \widehat{D(s_h)}\right) d\tau,$$

$$T_4 = \int_0^t (a(\beta(s) - \beta(s_h))\mathbf{u}, \overline{\Phi})\, d\tau,$$

$$T_5 = -\sum_{i=1}^{n} \int_0^t \langle \gamma - \mathcal{P}_h\gamma, \overline{\Phi}\cdot\nu\rangle_{\Gamma_i}\, d\tau,$$

$$T_6 = \sum_{i=1}^{n} \int_0^t \Big\langle\Big(\int_0^\tau \psi\, d\xi - \Pi\int_0^\tau \psi\, d\xi\Big)\cdot\nu\, d\xi, \mathcal{P}_h\gamma - \gamma_h\Big\rangle_{\Gamma_i}\, d\tau.$$

We now bound each T_k, $k = 1, \ldots, 6$. For any $\varepsilon > 0$, we have

$$|T_1| \le C\int_0^t \|\hat{s} - s\|_0^2\, d\tau + \varepsilon\int_0^t \|D(s) - D(s_h)\|_0^2\, d\tau,$$

$$|T_3| \le C\int_0^t\int_0^\tau (s - s_h, D(s) - D(s_h))\, d\xi\, d\tau + \varepsilon\int_0^t \|D(s) - D(s_h)\|_0^2\, d\tau,$$

$$|T_4| \le \varepsilon\int_0^t (s - s_h, D(s) - D(s_h))\, d\tau + C\int_0^t \|\overline{\Phi}\|_0^2\, d\tau,$$

$$|T_5| \le C\Big\{h^{-1}\int_0^t \|\gamma - \mathcal{P}_h\gamma\|_{0,\Gamma}^2\, d\tau + \int_0^t \|\overline{\Phi}\|_0^2\, d\tau\Big\},$$

using (5.45) for the bound of T_3, (5.44) for the bound of T_4, and Lemma 3.4 for the bound of T_5. To estimate T_2 we integrate by parts in time:

$$T_2 = -\int_0^t \left(\int_0^\tau a(\psi - \psi_h)\, d\xi, \frac{\partial}{\partial\tau}\Big(\Pi\int_0^\tau \psi\, d\xi - \int_0^\tau \psi\, d\xi\Big)\right) d\tau$$
$$+\left(\int_0^t a(\psi - \psi_h)\, d\tau, \Pi\int_0^t \psi\, d\tau - \int_0^t \psi\, d\tau\right);$$

therefore,

$$|T_2| \le C\Big\{ \int_0^t \Big\| \int_0^\tau (\psi - \psi_h)\, d\xi \Big\|_0^2 d\tau + \int_0^t \Big\| \frac{\partial}{\partial\tau}\big(\Pi \int_0^\tau \psi\, d\xi - \int_0^\tau \psi\, d\xi\big) \Big\|_0^2 d\tau$$
$$+ \Big\|\Pi \int_0^t \psi\, d\tau - \int_0^t \psi\, d\tau\Big\|_0^2 \Big\} + \varepsilon \Big\| \int_0^t (\psi - \psi_h)\, d\tau \Big\|_0^2$$

Integration by parts in time for T_6 gives

$$T_6 = -\sum_{i=1}^n \int_0^t \Big\langle \frac{\partial}{\partial\tau}\Big(\int_0^\tau \psi\, d\xi - \Pi \int_0^\tau \psi\, d\xi\Big) \cdot \nu, \int_0^\tau (\mathcal{P}_h\gamma - \gamma_h)\, d\xi \Big\rangle_{\Gamma_i} d\tau$$
$$+ \sum_{i=1}^n \Big\langle \Big(\int_0^t \psi\, d\tau - \Pi \int_0^t \psi\, d\tau\Big) \cdot \nu\, d\tau, \int_0^t (\mathcal{P}_h\gamma - \gamma_h)\, d\tau \Big\rangle_{\Gamma_i}.$$

Therefore,

$$|T_6| \le C\Big\{ \int_0^t \Big\| \frac{\partial}{\partial\tau}\Big(\int_0^\tau \psi\, d\xi - \Pi \int_0^\tau \psi\, d\xi\Big) \cdot \nu \Big\|_{0,\Gamma}^2 d\tau$$
$$+ \Big\| \Big(\int_0^t \psi\, d\tau - \Pi \int_0^t \psi\, d\tau\Big) \cdot \nu \Big\|_{0,\Gamma}^2 d\tau \Big\} + \varepsilon \int_0^t \Big\| \int_0^\tau (\mathcal{P}_h\gamma - \gamma_h)\, d\xi \Big\|_{0,\Gamma}^2 d\tau$$
$$+ \varepsilon \Big\| \int_0^t (\mathcal{P}_h\gamma - \gamma_h)\, d\tau \Big\|_{0,\Gamma}^2.$$

To bound the last two terms, we consider, for $1 \le i \le n$ and any fixed $t \in (0,T]$, the auxiliary problem

$$\varphi - \Delta\varphi = 0, \qquad \text{in } \Omega_i, \tag{5.65}$$

$$\nabla\varphi \cdot \nu = \int_0^t \mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\, d\tau, \qquad \text{on } \partial\Omega_i, \tag{5.66}$$

where $\mathcal{I}^{\partial\Omega_i}$ is the interpolation operator, defined in Section 3.2. By elliptic regularity,

$$\|\varphi\|_{m,\Omega_i} \le C \Big\| \int_0^t \mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\, d\tau \Big\|_{m-3/2,\partial\Omega_i}, \quad m = 1, 2. \tag{5.67}$$

We now integrate in time from 0 to t and take $\mathbf{v} = \Pi\nabla\varphi$ in (5.58) to obtain

$$\Big\langle \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\, d\tau, \int_0^t \mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\, d\tau \Big\rangle_{\partial\Omega_i}$$
$$= -\Big(\int_0^t a(\psi - \psi_h)\, d\tau, \Pi\nabla\varphi \Big)_{\Omega_i} + \Big(\int_0^t (\widehat{D(s)} - \widehat{D(s_h)})\, d\tau, \varphi \Big)_{\Omega_i}$$
$$+ \Big(\int_0^t a(\beta(s) - \beta(s_h))\mathbf{u}\, d\tau, \Pi\nabla\varphi \Big)_{\Omega_i} + \Big\langle \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma)\, d\tau, \nabla\varphi \cdot \nu \Big\rangle_{\partial\Omega_i}$$
$$= T_7 + T_8 + T_9 + T_{10}. \tag{5.68}$$

It is easy to see, from the definition of $\mathcal{I}^{\partial\Omega_i}$, that

$$\Big\langle \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau, \int_0^t \mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\rangle_{\partial\Omega_i} \geq C\Big\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i}^2 \tag{5.69}$$

and

$$\Big\| \int_0^t \mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i} \leq C\Big\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i}. \tag{5.70}$$

We bound the terms on the right-hand side of (5.68) as follows. For any $\delta > 0$,

$$\begin{aligned} |T_7| &\leq C\Big\{ \Big\| \int_0^t \psi\,d\tau - \Pi \int_0^t \psi\,d\tau \Big\|_{0,\Omega_i}^2 + \|\overline{\Phi}(t)\|_{0,\Omega_i}^2 \Big\} + \delta(\|\Pi\nabla\varphi - \nabla\varphi\|_{0,\Omega_i}^2 + \|\nabla\varphi\|_{0,\Omega_i}^2) \\ &\leq C\Big\{ \Big\| \int_0^t \psi\,d\tau - \Pi \int_0^t \psi\,d\tau \Big\|_{0,\Omega_i}^2 + \|\overline{\Phi}(t)\|_{0,\Omega_i}^2 \Big\} + \delta\Big\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i}^2, \end{aligned}$$

using (5.54), (5.67), the inverse inequality, and (5.70) for the last bound. Similarly,

$$\begin{aligned} |T_8| &\leq C\int_0^t \|D(s) - D(s_h)\|_{0,\Omega_i}^2\,d\tau + \delta\Big\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i}^2, \\ |T_9| &\leq C\int_0^t (s - s_h, D(s) - D(s_h))_{\Omega_i}\,d\tau + \delta\Big\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i}^2, \\ |T_{10}| &\leq C\int_0^t \|\mathcal{P}_h\gamma - \gamma\|_{0,\partial\Omega_i}^2\,d\tau + \delta\Big\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\partial\Omega_i}^2. \end{aligned}$$

Combining together (5.68), (5.69), the bounds on T_7–T_{10}, and (H3), we obtain

$$\begin{aligned} &\Big\| \int_0^t (\mathcal{P}_h\gamma - \gamma_h)\,d\tau \Big\|_{0,\Gamma}^2 \\ &\quad\leq C\Big\{ \Big\| \int_0^t \psi\,d\tau - \Pi \int_0^t \psi\,d\tau \Big\|_0^2 + \|\overline{\Phi}(t)\|_0^2 + \int_0^t \|D(s) - D(s_h)\|_0^2\,d\tau \\ &\qquad + \int_0^t (s - s_h, D(s) - D(s_h))\,d\tau + h^{-1}\int_0^t \|\mathcal{P}_h\gamma - \gamma\|_{0,\Gamma}^2\,d\tau \Big\} \end{aligned} \tag{5.71}$$

Combining (5.64), the bounds on T_1–T_6, (5.71), and using (5.43) and Gronwall's inequality, we arrive at the following result.

> **Theorem 5.1** Assume that (5.42)–(5.45) and (H3) hold. For the semi-discrete mixed finite element approximation (5.49)–(5.51) of problem (5.39)–(5.41), there exists a positive constant C such that, for every

$t \in [0, T]$,

$$\int_0^t (s - s_h, D(s) - D(s_h))\, d\tau + \Big\| \int_0^t (\psi - \psi_h)\, d\tau \Big\|_0^2$$
$$\leq C\Big\{ \int_0^t \|\hat{s} - s\|_0^2\, d\tau + \Big\| \Pi \int_0^t \psi\, d\tau - \int_0^t \psi\, d\tau \Big\|_0^2$$
$$+ h^{-1} \int_0^t \|\gamma - \mathcal{P}_h \gamma\|_{0,\Gamma}^2\, d\tau + \Big\| \Big(\int_0^t \psi\, d\tau - \Pi \int_0^t \psi\, d\tau \Big) \cdot \nu \Big\|_{0,\Gamma}^2$$
$$+ \int_0^t \Big\| \frac{\partial}{\partial \tau} \Big(\int_0^\tau \psi\, d\xi - \Pi \int_0^\tau \psi\, d\xi \Big) \Big\|_0^2\, d\tau$$
$$+ \int_0^t \Big\| \frac{\partial}{\partial \tau} \Big(\int_0^\tau \psi\, d\xi - \Pi \int_0^\tau \psi\, d\xi \Big) \cdot \nu \Big\|_{0,\Gamma}^2\, d\tau \Big\}.$$

Theorem 5.1 bounds the size of $\|D(s) - D(s_h)\|_0$ by (5.43). It also allows us to derive a bound on $\|s - s_h\|_{-1}$.

Theorem 5.2 Assume that (5.42)-(5.45) and (H3) hold. For the semi-discrete mixed finite element approximation (5.49)-(5.51) of problem (5.39)-(5.41), there exists a positive constant C such that, for every $t \in [0, T]$,

$$\int_0^t (s - s_h, D(s) - D(s_h))\, d\tau + \Big\| \int_0^t (\psi - \psi_h)\, d\tau \Big\|_0^2$$
$$\leq C\Big\{ \int_0^t \|\hat{s} - s\|_0^2\, d\tau + \Big\| \Pi \int_0^t \psi\, d\tau - \int_0^t \psi\, d\tau \Big\|_0^2$$
$$+ h^{-1} \int_0^t \|\gamma - \mathcal{P}_h \gamma\|_{0,\Gamma}^2\, d\tau + \Big\| \Big(\int_0^t \psi\, d\tau - \Pi \int_0^t \psi\, d\tau \Big) \cdot \nu \Big\|_{0,\Gamma}^2$$
$$+ \int_0^t \Big\| \frac{\partial}{\partial \tau} \Big(\int_0^\tau \psi\, d\xi - \Pi \int_0^\tau \psi\, d\xi \Big) \Big\|_0^2\, d\tau$$
$$+ \int_0^t \Big\| \frac{\partial}{\partial \tau} \Big(\int_0^\tau \psi\, d\xi - \Pi \int_0^\tau \psi\, d\xi \Big) \cdot \nu \Big\|_{0,\Gamma}^2\, d\tau \Big\}.$$

Proof For any $\varphi \in H_0^1(\Omega)$, we have

$$(s - s_h, \varphi) = (s - s_h, \varphi - \hat{\varphi}) + (s - s_h, \hat{\varphi}) = (s - \hat{s}, \varphi - \hat{\varphi}) + (s - s_h, \hat{\varphi}).$$

By (5.59) we have that

$$(s - s_h, \hat{\varphi}) = -\sum_{i=1}^n \Big(\nabla \cdot \Big(\Pi \int_0^t \psi\, d\tau - \int_0^t \psi_h\, d\tau \Big), \hat{\varphi} \Big)_{\Omega_i} + \Big(\int_0^t (\tilde{q}_w(s) - \tilde{q}_w(s_h))\, d\tau, \hat{\varphi} \Big).$$

For the first term on the right we write

$$
\begin{aligned}
-\sum_{i=1}^{n}\Big(\nabla\cdot\big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi_h\,d\tau\big),\hat{\varphi}\Big)_{\Omega_i} &= -\sum_{i=1}^{n}\Big(\nabla\cdot\big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi_h\,d\tau\big),\varphi\Big)_{\Omega_i}\\
&= \sum_{i=1}^{n}\Big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi_h\,d\tau,\nabla\varphi\Big)_{\Omega_i} - \sum_{i=1}^{n}\Big\langle\big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi_h\,d\tau\big)\cdot\nu,\varphi\Big\rangle_{\Gamma_i}\\
&= \sum_{i=1}^{n}\Big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi_h\,d\tau,\nabla\varphi\Big)_{\Omega_i} - \sum_{i=1}^{n}\Big\langle\big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi\,d\tau\big)\cdot\nu,\varphi\Big\rangle_{\Gamma_i}\\
&\quad -\sum_{i=1}^{n}\Big\langle\int_0^t (\psi-\psi_h)\cdot\nu\,d\tau,\varphi-\mathcal{P}_h\varphi\Big\rangle_{\Gamma_i},
\end{aligned}
$$

using (5.60) for the last equality. Therefore

$$
\begin{aligned}
&|(s-s_h,\varphi)|\\
&\quad\le C\Big\{h\|s-\hat{s}\|_0+\Big\|\Pi\int_0^t \psi\,d\tau-\int_0^t \psi_h\,d\tau\|_0+\Big\|\big(\Pi\int_0^t \psi\,d\tau-\int_0^t \psi\,d\tau\big)\cdot\nu\Big\|_{0,\Gamma}\\
&\qquad+\Big\|\int_0^t(\psi-\psi_h)\,d\tau\Big\|_0+\int_0^t(s-s_h,D(s)-D(s_h))\,d\tau\Big\}\|\varphi\|_1,
\end{aligned}
$$

using (5.55) for the first term on the right, Lemma 3.4 and (5.56) for the fourth term, and (5.43) for the last term. An application of Theorem 5.1 completes the proof. □

We end the section with some remarks on the approximation of the advective term in the saturation equation. The issues here are time step limitations because of stability problems and numerical diffusion that smears the sharp fronts. The method that is commonly used by the petroleum engineers is upstream-weighting [73]. It can be shown to be unconditionally stable when used in implicit schemes. However, it introduces an excessive amount of artificial diffusion. Explicit Godunov schemes [14] have time step stability limitations. The modified method of characteristics-Galerkin [35, 42, 31] does not conserve mass locally. A characteristics-mixed method for linear transport problems developed by Arbogast and Wheeler [7] is locally mass conservative and allows larger time steps. The trace-back integrals needed in the method can be difficult to evaluate. This method, in conjunction with the expanded mixed method, has been successfully used for advection dominated transport problems [8].

The schemes described earlier in the chapter do not treat the advective term in any special way. They can be improved by incorporating some of the above mentioned transport methods to allow for larger time steps without adding too much numerical diffusion.

Bibliography

[1] Y. ACHDOU, Y. A. KUZNETSOV, AND O. PIRONNEAU, *Substructuring preconditioners for the Q_1 mortar element method*, Numer. Math., 71 (1995), pp. 419–450.

[2] H. W. ALT AND E. DIBENEDETTO, *Nonsteady flow of water and oil through inhomogeneous porous media*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 12 (1985), pp. 335–392.

[3] H. W. ALT AND S. LUCKHAUS, *Quasilinear elliptic-parabolic differential equations*, Math. Z., 183 (1983), pp. 311–341.

[4] T. ARBOGAST, *The existence of weak solutions to single porosity and simple dual-porosity models of two-phase incompressible flow*, Nonlinear Analysis, Theory, Methods and Applications, 19 (1992), pp. 1009–1031.

[5] T. ARBOGAST, C. N. DAWSON, P. T. KEENAN, M. F. WHEELER, AND I. YOTOV, *Enhanced cell-centered finite differences for elliptic equations on general geometry*. To appear in SIAM J. Sci. Comp.

[6] T. ARBOGAST, P. T. KEENAN, M. F. WHEELER, AND I. YOTOV, *Logically rectangular mixed methods for Darcy flow on general geometry*, in Thirteenth SPE Symposium on Reservoir Simulation, San Antonio, Texas, Society of Petroleum Engineers, Feb. 1995, pp. 51–59. SPE 29099.

[7] T. ARBOGAST AND M. F. WHEELER, *A characteristics-mixed finite element method for advection-dominated transport problems*, SIAM J. Numer. Anal., 32 (1992), pp. 404–424.

[8] T. ARBOGAST, M. F. WHEELER, AND I. YOTOV, *Logically rectangular mixed methods for groundwater flow and transport on general geometry*, in Computational Methods in Water Resources X, A. Peters et al., eds., Dordrecht,The Netherlands, 1994, Kluwer Academic Publishers, pp. 149–156.

[9] ——, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997). To appear.

[10] T. Arbogast, M. F. Wheeler, and N. Zhang, *A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media*, SIAM J. Numer. Anal., 33 (1996). To appear.

[11] D. N. Arnold and F. Brezzi, *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Modèl. Math. Anal. Numèr., 19 (1985), pp. 7–32.

[12] K. Aziz and A. Settari, *Petroleum reservoir simulation*, Applied Science Publishers, London and New York, 1979.

[13] J. Bear, *Dynamics of fluids in porous media*, Dover, New York, 1972.

[14] J. B. Bell, C. N. Dawson, and G. R. Shubin, *An unsplit higher-order godunov scheme for scalar conservation laws in two dimensions*, J. Comp. Physics, 74 (1988), pp. 1–24.

[15] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North Holland, Amsterdam, 1978.

[16] C. Bernardi, Y. Maday, and A. T. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications, H. Brezis and J. L. Lions, eds., Longman Scientific & Technical, UK, 1994.

[17] A. Bourgeat, *Homogenized behavior of two-phase flows in naturally fractured reservoirs with uniform fractures distribution*, Comput. Meth. Appl. Mech. Eng., 47 (1984), pp. 205–216.

[18] J. H. Bramble, R. E. Ewing, J. E. Pasciak, and A. H. Schatz, *A preconditioning technique for the efficient solution of problems with local grid refinement*, Comput. Meth. Appl. Mech. Eng., 67 (1988), pp. 149–159.

[19] F. Brezzi, J. Douglas, Jr., R. Duràn, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.

[20] F. BREZZI, J. DOUGLAS, JR., M. FORTIN, AND L. D. MARINI, *Efficient rectangular mixed finite elements in two and three space variables*, RAIRO Modèl. Math. Anal. Numèr., 21 (1987), pp. 581–604.

[21] F. BREZZI, J. DOUGLAS, JR., AND L. D. MARINI, *Two families of mixed elements for second order elliptic problems*, Numer. Math., 88 (1985), pp. 217–235.

[22] F. BREZZI AND M. FORTIN, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[23] Z. CAI AND S. F. MCCORMICK, *On the accuracy of the finite volume method for diffusion equations on composite grids*, SIAM J. Numer. Anal., 27 (1990), pp. 636–655.

[24] G. CHAVENT AND J. JAFFRE, *Mathematical models and finite elements for reservoir simulation*, North-Holland, Amsterdam, 1986.

[25] Z. CHEN, *Expanded mixed finite element methods for linear second order elliptic problems I*, in IMA preprint series # 1219, Institute for Mathematics and its Applications, University of Minnesota, 1994.

[26] Z. CHEN AND J. DOUGLAS, JR., *Prismatic mixed finite elements for second order elliptic problems*, Calcolo, 26 (1989), pp. 135–148.

[27] Z. CHEN, M. ESPEDAL, AND R. E. EWING, *Finite element analysis of multiphase flow in groundwater hydrology*, Tech. Rep. ISC-94-11-MATH, Inst. Sci. Comp., Texas A&M University, 1994.

[28] L. C. COWSAR, J. MANDEL, AND M. F. WHEELER, *Balancing domain decomposition for mixed finite elements*, Math. Comp., 64 (1995), pp. 989–1015.

[29] L. C. COWSAR AND M. F. WHEELER, *Parallel domain decomposition method for mixed finite elements for elliptic partial differential equations*, in Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. Kuznetsov, G. Meurant, J. Periaux, and O. Widlund, eds., Philadelphia, 1991, SIAM.

[30] P. J. DAVIS, *Interpolation and Approximation*, Dover Publications, New York, 1975.

[31] C. N. DAWSON, T. F. RUSSELL, AND M. F. WHEELER, *Some improved error estimates for the modified method of characteristics*, SIAM J. Numer. Anal., 26 (1989), pp. 1487–1512.

[32] J. DOUGLAS, JR., *Finite difference methods for two-phase incompressible flow in porous media*, SIAM J. Numer. Anal., 20 (1983), pp. 681–696.

[33] J. DOUGLAS, JR. AND F. A. MILNER, *Interior and superconvergence estimates for mixed methods for second order elliptic problems*, RAIRO Modèl. Math. Anal. Numèr., 19 (1985), pp. 397–428.

[34] J. DOUGLAS, JR. AND J. E. ROBERTS, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.

[35] J. DOUGLAS, JR. AND T. F. RUSSELL, *Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures*, SIAM J. Numer. Anal., 19 (1982), pp. 871–885.

[36] R. DURÁN, *Error analysis in $L^p, 1 \leq p \leq \infty$, for mixed finite element methods for linear and quasi-linear elliptic problems.* Preprint.

[37] ——, *Superconvergence for rectangular mixed finite elements*, Numer. Math., 58 (1990), pp. 287–298.

[38] L. J. DURLOFSKY, *Numerical calculation of equivalent grid block permeability tensors for heterogeneous porous media*, Water Resources Research, 27 (1991), pp. 699–708.

[39] R. E. EWING AND R. D. LAZAROV, *Superconvergence of the mixed finite element approximations of parabolic problems using rectangular finite elements*, East-West J. Numer. Math., 1 (1993), pp. 199–212.

[40] R. E. EWING, R. D. LAZAROV, T. F. RUSSELL, AND P. S. VASSILEVSKI, *Analysis of the mixed finite element method for rectangular Raviart-Thomas elements with local refinement*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, R. Glowinski, J. Periaux, and O. B. Widlund, eds., Philadelphia, 1990, SIAM, pp. 98–114.

[41] R. E. Ewing, R. D. Lazarov, and J. Wang, *Superconvergence of the velocity along the Gauss lines in mixed finite element methods*, SIAM J. Numer. Anal., 28 (1991), pp. 1015–1029.

[42] R. E. Ewing, T. F. Russell, and M. F. Wheeler, *Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics*, Comp. Meth. Appl. Mech. Eng., 47 (1984), pp. 73–92.

[43] R. E. Ewing and J. Wang, *Analysis of mixed finite element methods on locally refined grids*, Numer. Math., 63 (1992), pp. 183–194.

[44] K. Fadimba and R. Sharpley, *Galerkin finite element method for a class of porous medium equations*. Preprint.

[45] ——, *A priori estimates and regularization for a class of porous medium equations*. Submitted.

[46] A. Friedman, *The Stefan problem in several space variables*, Trans. Amer. Math. Soc., 133 (1968), pp. 51–87.

[47] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[48] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds., Philadelphia, 1988, SIAM, pp. 144–172.

[49] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, 1985.

[50] J. Jerome and M. Rose, *Error estimates for the multidimensional two-phase stefan problem*, Math. Comp., 39 (1982), pp. 377–414.

[51] C. Johnson and V. Thomée, *Error estimates for some mixed finite element methods for parabolic type problems*, RAIRO Anal. Numèr, 15 (1981), pp. 1015–1029.

[52] J. KOEBBE, *A computationally efficient modification of mixed finite element methods for flow problems with full transmissivity tensors*, Numer. Meth. for PDE's, 9 (1993), pp. 339–355.

[53] Y. A. KUZNETSOV AND M. F. WHEELER, *Optimal order substructuring preconditioners for mixed finite element methods on non-matching grids*, East-West J. Numer. Math., 3 (1995), pp. 127–143.

[54] O. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, vol. 23 of Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1968.

[55] L. W. LAKE, *Fundamentals of enhanced oil recovery*, Prentice Hall, New Jersey, 1989.

[56] J. L. LIONS AND E. MAGENES, *Non-homogeneous boundary value problems and applications*, vol. 1, Springer-Verlag, 1972.

[57] J. MANDEL, *Balancing domain decomposition.* To appear in Comm. Appl. Numer. Meth.

[58] J. MANDEL AND M. BREZINA, *Balancing domain decomposition: theory and performance in two and three dimensions.* Submitted.

[59] T. P. MATHEW, *Domain decomposition and iterative refinement methods for mixed finite element discretizations of elliptic problems*, PhD thesis, New York University, 1989.

[60] F. A. MILNER, *Mixed finite element methods for quasilinear secon-order elliptic problems*, Math. Comp., 44 (1985), pp. 303–320.

[61] M. NAKATA, A. WEISER, AND M. F. WHEELER, *Some superconvergence results for mixed finite element methods for elliptic problems on rectangular domains*, in The Mathematics of Finite Elements and Applications V, J. R. Whiteman, ed., Academic Press, London, 1985, pp. 367–389.

[62] J. C. NEDELEC, *Mixed finite elements in* $\mathbf{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[63] R. N. NOCHETTO, *Error estimates for the multidimensional singular parabolic problems*, Japan J. Appl. Math., 4 (1987), pp. 111–138.

[64] R. N. NOCHETTO AND C. VERDI, *Approximation of degenerate parabolic problems using numerical integration*, SIAM J. Numer. Anal., 25 (1988), pp. 784–814.

[65] O. A. OLEINIK, A. S. KALASHNIKOV, AND C. YUI-LIN, *The Cauchy problem and boundary value problems for equations of type of nonstationary filtration*, Izv. Akad. Nauk SSSR Ser Mat, 22 (1958), pp. 667–704. In russian.

[66] D. W. PEACEMAN, *Fundamentals of numerical reservoir simulation*, Elsevier, Amsterdam, 1977.

[67] R. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, vol. 606, Springer-Verlag, New York, 1977, pp. 292–315.

[68] J. E. ROBERTRS AND J.-M. THOMAS, *Mixed and hybrid methods*, in Handbook of Numerical Analysis, Vol. II, P. G. Ciarlet and J. Lions, eds., Elsevier Science Publishers B.V., 1991, pp. 523–639.

[69] M. ROSE, *Numerical methods for flows through porous media. I*, Math. Comp., 40 (1983), pp. 435–467.

[70] T. F. RUSSELL AND M. F. WHEELER, *Finite element and finite difference methods for continuous flows in porous media*, in The Mathematics of Reservoir Simulation, R. E. Ewing, ed., SIAM, Philadelphia, 1983, pp. 35–106.

[71] W. G. STRANG AND G. J. FIX, *An analysis of the finite element method*, Prentice-Hall, New Jersey, 1973.

[72] J. M. THOMAS, *These de Doctorat d'etat*, 'a l'Universite Pierre et Marie Curie, 1977.

[73] M. R. TODD, P. M. O'DELL, AND G. J. HIRASAKI, *Methods for increased accuracy in numerical reservoir simulators*, Soc. Petrol. Eng. J., 12 (1972), pp. 515–530.

[74] A. WEISER AND M. F. WHEELER, *On convergence of block-centered finite-differences for elliptic problems*, SIAM J. Numer. Anal., 25 (1988), pp. 351–375.

[75] M. F. WHEELER, K. R. ROBERSON, AND A. CHILAKAPATI, *Three-dimensional bioremediation modeling in heterogeneous porous media*, in Computational Methods in Water Resources IX, Vol. 2: Mathematical Modeling in Water Resources, T. F. Russell, R. E. Ewing, C. A. Brebbia, W. G. Gray, and G. F. Pindar, eds., Southampton, U. K., 1992, Computational Mechanics Publications, pp. 299–315.