IV

APPLICATIONS TO THE MUTUAL INFLUENCE OF TWO PENDULUMS

35. Let us consider the problem of two ideal pendulums, the bobs being treated as particles of masses μ, μ', and the rods as weightless and having lengths b, b'. They are hung from points attached to a bar of mass M capable of horizontal motion only: all frictional effects are neglected.

We shall first suppose that the bar is not acted on by any external force. If each body in the system has zero velocity initially, the horizontal component of the momentum is zero; also the total energy remains unchanged throughout the

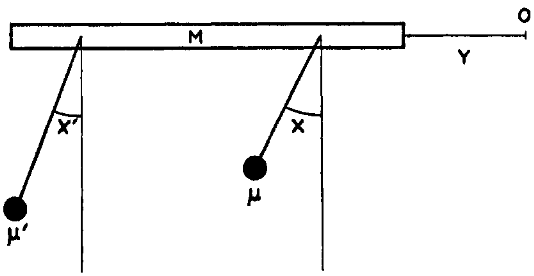

FIG. 2

motion. Let x, x' be the angles which the rods make with the vertical at time t and y the horizontal distance of the center of mass of M from a fixed point O at the same instant. The assumed zero value of the total horizontal momentum gives

(35.1) $My+\mu(y+b \sin x)+\mu'(y+b' \sin x')=\text{const.}$

The forces and accelerations resolved perpendicularly to the rods give, after division by the masses,

$$\frac{d^2y}{dt^2}\cos x + b\,\frac{d^2x}{dt^2} = -g\sin x, \tag{35.2}$$

$$\frac{d^2y}{dt^2}\cos x' + b'\,\frac{d^2x'}{dt^2} = -g\sin x'. \tag{35.3}$$

The substitution of (35.1) in (35.2) furnishes an equation which may be written

$$b\,\frac{d^2x}{dt^2} + g\sin x - \frac{\mu b}{M+\mu+\mu'}\cos x\,\frac{d^2}{dt^2}(\sin x) = \frac{\mu' b'}{M+\mu+\mu'}\cos x\,\frac{d^2}{dt^2}(\sin x'). \tag{35.4}$$

If the right-hand member of (35.4) be neglected, the equation becomes integrable on multiplication by dx/dt. Since $\cos x \cdot dx/dt = d(\sin x)/dt$, this integral is

$$\tfrac{1}{2}b\left(\frac{dx}{dt}\right)^2 - g\cos x - \tfrac{1}{2}\,\frac{\mu b}{M+\mu+\mu'}\left(\cos x\,\frac{dx}{dt}\right)^2 = \text{const.} = \tfrac{1}{2}C,$$

or

$$\left(\frac{ax}{dt}\right)^2 = (C+2g\cos x) \div b\left(1 - \frac{\mu}{M+\mu+\mu'}\cos^2 x\right) \equiv -2f(x).$$

On differentiating this result we obtain

$$\frac{d^2x}{dt^2} + f'(x) = 0, \tag{35.5}$$

and $f(x)$ may be written in the form

$$f(x) = -\frac{M+\mu+\mu'}{2b}\cdot\frac{C+2g\cos x}{M+\mu'+\mu\sin^2 x}. \tag{35.6}$$

Since $f(x)$ lies between finite limits and has no singularity, equation (35.6) is of the type considered in section 20, and the analysis of that and of the succeeding sections may be applied. But the calculations may be much simplified by making approximations which can be shown to be sufficient to investigate the resonance phenomena when the latter are present.

36. The solutions we seek are those in which x, x' are oscillating through angles which are never very great, and in which μ/M, μ'/M are small. The approximations will be made on the assumption that powers and products of x^2, x'^2, μ/M, μ'/M beyond the first can be neglected in $f(x)$: this involves the neglect of products of x^3, x'^3 by μ/M, μ'/M in $f'(x)$.

With these limitations $f(x)$ reduces to

$$f(x) = \text{const.} - \frac{M+\mu+\mu'}{b(M+\mu')} g \cos x \,,$$

so that (35.5) becomes the equation of motion of a simple pendulum of length $b(M+\mu') \div (M+\mu+\mu')$. Further, in the right-hand member of (35.4) we can put $\cos x = 1$, $\sin x' = x'$. If then the notation

$$\kappa^2 = \frac{g}{b}\,\frac{M+\mu+\mu'}{M+\mu'}, \quad \kappa'^2 = \frac{g}{b}\,\frac{M+\mu+\mu'}{M+\mu}\,,$$

$$m = \frac{\mu'}{M+\mu}, \quad m' = \frac{\mu}{M+\mu'}\,,$$

be adopted, (35.4) and the similar equation for x' may be written

$$\text{(36.1)} \qquad \frac{d^2x}{dt^2} + \kappa^2 \sin x = m\frac{\kappa^2}{\kappa'^2}\,\frac{d^2x'}{dt^2}\,,$$

$$\text{(36.2)} \qquad \frac{d^2x'}{dt^2} + \kappa'^2 \sin x' = m'\frac{\kappa'^2}{\kappa^2}\,\frac{d^2x}{dt^2}\,.$$

Finally, the neglect of the product mm' enables us to substitute for d^2x'/dt^2 in (36.1), its value derived from (36.2) when the right-hand member of the latter is neglected: equation (36.2) may be similarly treated, and we may put $\sin x = x$, $\sin x' = x'$ in the right-hand members. We then obtain

$$\text{(36.3)} \qquad \frac{d^2x}{dt^2} + \kappa^2 \sin x = -m\kappa^2 x' = -m\kappa^2 \frac{\partial}{\partial x}(xx'),$$

$$(36.4)\qquad \frac{d^2x'}{dt^2}+\kappa'^2 \sin x' = -m'\kappa'^2 x = -m'\kappa'^2 \frac{\partial}{\partial x'}(xx').$$

Each of these equations now has the form ready for the application of the methods developed above.

37. When $m=m'=0$, each of the equations (36.3), (36.4) reduces to that for the motion of a simple pendulum. The variable x is then changed to the variables c, l as in the previous example, and x' to similar variables c', l'. Also, as in the previous example, it is sufficient for a first approximation to confine the solution (36.3) when $m=0$ to

$$(37.1)\qquad x=c\sin l,\quad n=\kappa\left(1-\frac{c^2}{16}\right),\quad K=\kappa c;$$

and the solution of (36.4) when $m'=0$ to

$$(37.2)\qquad x'=c'\sin l',\quad n'=k'\left(1-\frac{c'^2}{16}\right),\quad K'=\kappa'c'.$$

If, in the general formulae, we replace m by $m\kappa^2$, and by $m'\kappa'^2$ for the respective equations, we shall then have

$$\phi=-xx'=-cc'\sin l\sin l'$$
$$(37.3)\qquad =-\tfrac{1}{2}cc'\cos(l-l')+\tfrac{1}{2}cc'\cos(l+l').$$

The resonance case corresponds to that in which $dl/dt - dl'/dt$ is nearly zero. The first term of (37.3) is therefore to be used and we can pass immediately to (26.3) with n given by (37.1) and $a_i=\tfrac{1}{2}cc'$. We obtain

$$\frac{d^2l}{dt^2}+\frac{m\kappa^2}{\kappa c}\left\{-\frac{1}{2}cc'\left(-\frac{1}{8}\kappa c\right)+\frac{1}{2}(n-n')c'\right\}\sin(l-l')=0.$$

The first approximation is obtained by putting $c=c_0$, $c'=c'_0$. When there is resonance we have $n_0=n'_0$. The equation therefore reduces to

$$(37.4)\qquad \frac{d^2l}{dt^2}+\frac{m\kappa^2}{16}c_0c'_0\sin(l-l')=0.$$

Similarly

$$\frac{d^2l'}{dt^2}+\frac{m'\kappa'^2}{16}\,c_o\,c'_o\sin\,(l'-l)=0. \tag{37.5}$$

From these we deduce

$$m'\kappa'^2\frac{d^2l}{dt^2}+m\kappa^2\;\frac{d^2l'}{dt^2}=0. \tag{37.6}$$

$$\frac{d^2}{dt^2}(l-l')+\frac{m\kappa^2+m'\kappa'^2}{16}\,c_o\,c'_o\sin\,(l-l')=0. \tag{37.7}$$

The last equation has the standard resonance form.

38. Since c_0, c'_0 are positive by definition, equation (37.7) shows that in resonance $l-l'$ oscillates about the value zero and that $l-l'=\pi$ is the limiting case between resonance and non-resonance. Hence with the relation $n_0=n'_0$ we must also have $\epsilon_0=\epsilon'_0$. Thus

The stable resonance configuration of two pendulums attached to a massive block free to move horizontally is that in which the rods are always approximately parallel to one another.

If a slight disturbance be given to the system, an oscillation (libration) defined by

$$l-l'=\lambda\sin\,(pt+\lambda_o),\; p^2=\frac{m\kappa^2+m'\kappa'^2}{16}c_o\,c'_o, \tag{38.1}$$

will be present. The combination of (38.1) with (37.6) shows that the librations of the two pendulums will have opposite phases.

Suppose the pendulums had been started from opposite sides of the vertical so that $n_0=n'_0$, but $l-l'=\pi$. Evidently M will in this case be at rest. But any small disturbance of the system will ultimately compel $l-l'$ to pass through nearly all values between π and $-\pi$, with consequent oscillation of M. Or else resonance, as defined here, will not be present, but "beats" at very long intervals will occur, according to the nature of the disturbance. Thus the motion will be highly sensitive to small disturbing forces and considerable irregularities will occur if the pendulums be used for accurate measures of time. Near the stable case, on the other hand, the irregularities will be small, but the pendulums become in effect one unit for time measurement instead of being two separate units, as in the non-resonance case, each giving its own measure.

The condition $n_0 = n'_0$, demands that

$$\kappa\left(1-\frac{c_0^2}{16}\right)=\kappa'\left(1-\frac{c_0'^2}{16}\right).$$

Since c_0, c'_0 were assumed to be small, this condition demands that κ, κ', and therefore by the definitions in section 36, b, b' shall be nearly the same. It is to be noticed that small differences from equality in b, b' can be compensated in resonance by the arcs through which the pendulums swing. The masses may be quite different provided they are both small compared with M.

The variation of c is given by

$$\frac{dc}{dt}=\frac{m\kappa^2}{K}\frac{\partial\phi}{\partial l}=\tfrac{1}{2}m\kappa c'\sin(l-l').$$

Substituting for sin $(l-l')$ from (37.8) and integrating, we obtain

$$c=c_0-\frac{8m\kappa}{c_0(m\kappa^2+m'\kappa'^2)}\frac{d}{dt}(l-l'),$$

which, with the use of (38.1), gives

$$\begin{aligned}c&=c_0-\frac{8m\kappa}{m\kappa^2+m'\kappa'^2}\frac{p\lambda}{c_0}\cos\,(pt+\lambda_0)\\&=c_0-\frac{2m\kappa}{(m\kappa^2+m'\kappa'^2)^{\frac{1}{2}}}\left(\frac{c'_0}{c_0}\right)^{\frac{1}{2}}\lambda\cos(pt+\lambda_0).\end{aligned}$$

A similar expression gives the variation of c'. The maximum amplitudes of c and of c' we obtain from (29.1).

It is evident that neither c_0 nor c'_0 can be very small if the approximations are to be valid. If $\kappa=\kappa'$ we have $c_0=c'_0$ and the difficulty disappears. If however the difference between κ, κ' and the starting conditions were such that c_0 or c'_0 were zero it would be necessary to reconstruct the analysis, possibly in a manner similar to that of section 33.

39. When $b=b'$, $\mu=\mu'$, it is possible to explain without transforming the equations of motion, why the unstable case of resonance is that in which the pendulums are started from rest with equal angles on opposite sides of the vertical.

Here equation (35.4) and that obtained by interchanging x, x', can be written

$$\frac{d^2x}{dt^2}+\kappa^2\sin x=m\kappa^2\cos x\frac{d^2}{dt^2}(\sin x+\sin x'), \tag{39.1}$$

$$\frac{d^2x'}{dt^2}+\kappa'^2\sin x'=m\kappa'^2\cos x'\frac{d^2}{dt^2}(\sin x+\sin x'). \tag{39.2}$$

A particular solution of these equations when $\kappa=\kappa^1$; is

$$x=-x',\quad \frac{d^2x}{dt^2}+\kappa^2\sin x=0,$$

the case in which the phases are opposite.

Suppose that a small disturbance be given to the system. The right-hand members of (39.1), (39.2) are then of the order m times disturbance, and so extremely small. The motion of one pendulum will affect the other very little, and with different amplitudes, their periods will be different and the phase difference will tend to increase until the right-hand members become large enough to affect it.

On the other hand, with the particular solution

$$x=x',\quad \frac{d^2x}{dt^2}+\kappa^2\sin x=2m\kappa^2\cos x\frac{d^2}{dt^2}(\sin x),$$

the case in which they start on the same side of the vertical, a disturbance affects both pendulums. But the difference between (39.1), (39.2), having the factor $\cos x - \cos x'$, which is of the order $(x-x')(x+x')$, is always very small.

The distinction between the two cases consists in the fact that though the effect of one pendulum on the other is very small in the unstable case, it tends to accumulate; while in the stable case, the limit of accumulation is sooner reached and the effect is then reversed.

40. Suppose that (35.4) and the similar equation for x' had been reduced to the linear form at the outset by putting $\cos x = 1$, $\cos x' = 1$, $\sin x = x$, $\sin x' = x'$. They would have become

$$b\,\frac{d^2x}{dt^2}\left(1-\frac{\mu}{M+\mu+\mu'}\right)+gx-\frac{\mu' b'}{M+\mu+\mu'}\,\frac{d^2x'}{dt^2}=0,$$

$$b\,\frac{d^2x}{dt^2}\,1-\left(\frac{\mu'}{M+\mu+\mu'}\right)+gx'-\frac{\mu b}{M+\mu+\mu'}\,\frac{d^2x}{dt^2}=0.$$

When $b=b'$, $\mu=\mu'$, these, by subtraction and addition, may be written

$$b\,\frac{d^2}{dt^2}(x-x')+g(x-x')=0,$$

$$b\left(1-\frac{2\mu}{M+2\mu}\right)\frac{d^2}{dt^2}(x+x')+g(x+x')=0,$$

giving to the oscillations of $x-x'$, $x+x'$, the frequencies

$$\left(\frac{g}{b}\right)^{\frac{1}{2}},\quad \left(\frac{g}{b}\,\frac{M+2\mu}{M}\right)^{\frac{1}{2}}.$$

These show the possible existence of resonance but give no information as to the nature of the motion under such a condition.

41. *Pendulums Mounted on a Massive Pier.* Let us now suppose that the bar M, instead of being free, is confined in its motion by stiff springs, so that its natural free period of oscillation is very short compared with those of the pen-

dulums. If N be its natural frequency, the equation for the motion of M will be

$$M\left(\frac{d^2y}{dt^2}+N^2y\right)=T\sin x+T'\sin x',$$

where T, T' are the tensions of the pendulum rods.

With the earlier hypotheses concerning the magnitudes of the masses and the angles we may put $T=\mu g$, $T'=\mu' g$, $\sin x=x$, $\sin x'=x'$, in this equation so that it reduces to

$$\frac{d^2y}{dt^2}+N^2y=\frac{g}{M}(\mu x+\mu' x').$$

For oscillations of x, x' with periods very long compared with $2\pi/N$, we have N^2y large compared with d^2y/dt^2. Hence, approximately,

$$y=\frac{g}{MN^2}(\mu x+\mu' x').$$

This equation is similar to (35.1) when the latter is reduced by putting $\sin x=x$, $\sin x'=x'$, and the previous developments can be utilized by proper choices of m, m'.

The difference of chief importance is due to the fact that m, m' now have signs opposite to those of the earlier problem. Hence a reference to (37.8) shows that $l-l'$ now oscillates about the value π instead of about zero.

The present problem is substantially the same as that of two free pendulums attached to a massive pier. Such a pier will, in general, have natural periods of oscillation very short compared with those of the pendulums. Hence

The stable resonance case of two pendulums attached to a massive pier capable of oscillation is that in which the phases are opposite so that the pier does not sensibly vibrate. A small disturbance from this configuration will produce only small differences of phase.

The case separating this from the earlier problem is evidently that in which the period of oscillation of M is nearly the same as those of the pendulums. Such a case would give rise to resonances between three vibrating systems—a complicated problem which I have not attempted to attack.