

XII

SERIES HAVING THE CIRCLE OF CONVERGENCE AS A CUT

Theorem 1 of Chapter V is sometimes called the Hadamard-Fabry theorem. This chapter has for its object the consideration of related theorems. We shall obtain less restrictive conditions on the $\{\lambda_n\}$ in order that the circle of convergence shall be a cut. We begin by developing some ideas which, although elementary, are important in themselves as well as in their bearing on the theorems in question.

Let $f(x)$ denote the function defined by $\sum a_n x^n$ within the unit circle. If $|\beta| < 1$, we have

$$f(\beta) = a_0 + a_1\beta + \cdots,$$
$$f^{(m)}(\beta) = \sum_{q=0}^{\infty} \frac{(m+q)!}{q!} a_{m+q} \beta^q.$$

For the expansion of $f(x)$ about the point $x = \beta$, we may write

$$f(x) = \sum_{m=0}^{\infty} c_m (x - \beta)^m, \quad \beta = be^{i\phi}.$$

Let C_1, R_1 denote, respectively, the circle and the radius of convergence of this series. We have

$$c_m = \frac{f^{(m)}(\beta)}{m!} = \sum_{q=0}^{\infty} \frac{(m+q)!}{m!q!} a_{m+q} \beta^q. \quad (1)$$

It is clear that $R_1 \geq 1 - b$, since $f(x)$ has no singularities within the circle C_2 , center at β , radius $R_2 = 1 - b$. Hence

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C_2 is tangent to the unit circle at the point $e^{i\phi}$. If $R_1 = R_2$, then $e^{i\phi}$ is singular, for all other points on C_2 are interior to the unit circle. Conversely, if $e^{i\phi}$ is singular, we must have $R_1 = R_2 = 1 - b$. Hence the requirement $R_1 = 1 - b$ is necessary and sufficient in order that $e^{i\phi}$ be singular. Similarly, the necessary and sufficient condition that $e^{i\phi}$ be regular is $R_1 > 1 - b$.

Expressing R_1 in terms of the c_n , we have

(A) *If the following inequality holds:*

$$R_1 = \frac{1}{\lim \sqrt[n]{|c_n|}} > 1 - b, \quad \lim \sqrt[n]{|c_n|} < \frac{1}{1 - b}, \quad (2)$$

then $e^{i\phi}$ is regular;

(B) *If*

$$R_1 = \frac{1}{\lim \sqrt[n]{|c_n|}} = 1 - b, \quad \lim \sqrt[n]{|c_n|} = \frac{1}{1 - b}, \quad (3)$$

then $e^{i\phi}$ is singular.

If, in (A) and (B), the c_n are replaced by their values given by (1), we may say that the point $e^{i\phi}$ is singular if for an arbitrary $\epsilon > 0$ we have, for an infinity of values of m ,

$$\begin{aligned} |c_m| &= |a_m + (m + 1)a_{m+1}be^{i\phi} + \dots + C_{m+q}^q a_{m+q}b^q e^{qi\phi} + \dots| \\ &> \left(\frac{1 - \epsilon}{1 - b}\right)^m, \end{aligned} \quad (5)$$

and $e^{i\phi}$ is regular if there exists an $\epsilon > 0$ so small that

$$|c_m| < \left(\frac{1 - \epsilon}{1 - b}\right)^m \quad (6)$$

for $m > m_0$.

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In what follows we shall use the fact that *the left hand side of (5) and (6) may be replaced by a finite sum of terms occurring in this expression, e.g., by*

$$| a_m + (m + 1)a_{m+1}be^{i\phi} + \dots + C_{m+q}^q a_{m+q} b^q e^{qi\phi} |,$$

where q is arbitrary, but $q \geq p = p(m)$, where this last number depends only on m . This principle, due to Hadamard,¹ has been employed by Fabry, Leau and others in obtaining important results.

54. We shall prove the existence of the number $p = p(m)$. Suppose for simplicity that $\phi = 0$. Since the quantity $C_{m+q}^q m^m q^q$ is a term of the expansion of $(m + q)^{m+q}$ we have

$$C_{m+q}^q m^m q^q < (m + q)^{m+q} \quad (7)$$

Moreover, since this term is the largest in the expansion,² and since there are $m + q + 1$ terms, we have

$$C_{m+q}^q m^m q^q > \frac{(m + q)^{m+q}}{m + q + 1}. \quad (8)$$

From (7) and (8),

$$\begin{aligned} \lim_{m \rightarrow \infty} \sqrt[m]{\frac{(m + q)^{m+q}}{m^m q^q} b^q} &\geq \lim_{m \rightarrow \infty} \sqrt[m]{C_{m+q}^q b^q} \\ &\geq \lim_{m \rightarrow \infty} \sqrt[m]{\frac{(m + q)^{m+q}}{(m + q + 1)m^m q^q} b^q}, \end{aligned} \quad (9)$$

where q may be supposed to vary with m , and where the existence of the limits is assumed.

¹ *loc. cit.*

² Write $(m + q)^{m+q} = \alpha_0 + \alpha_1 + \dots + \alpha_{m+q}$, where

$$\alpha_k = \frac{(m + q)!}{(m + q - k)! k!} m^{m+q-k} q^k.$$

Then

$$\alpha_k = \frac{m + q - k + 1}{k} \frac{q}{m} \alpha_{k-1},$$

so that if $k \leq q$, then $\alpha_k > \alpha_{k-1}$; whereas if $k > q$, $\alpha_k < \alpha_{k-1}$. The maximum term in the expansion of $(m + q)^{m+q}$ is therefore

$$\alpha_q = C_{m+q}^q m^m q^q.$$

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From (7), and from the fact that the radius of convergence is 1 ($|a_{m+q}| < (1 + \eta)^{m+q}$ for $\eta > 0$, arbitrary but fixed, m sufficiently large), we have

$$\begin{aligned} \sqrt[m]{C_{m+q}^a a_{m+q} b^q} &< \left(1 + \frac{q}{m}\right)^{\frac{m+q}{m}} m^{\frac{q}{m}} q^{-\frac{q}{m}} b^{\frac{q}{m}} (1 + \eta)^{\frac{m+q}{m}} \\ &= \left(1 + \frac{q}{m}\right)^{1 + \frac{q}{m}} (1 + \eta)^{1 + \frac{q}{m}} \left(\frac{bm}{q}\right)^{\frac{q}{m}} \\ &= \left(1 + \frac{q}{m}\right) (1 + \eta) \left[(1 + \eta) b \left(1 + \frac{m}{q}\right) \right]^{\frac{q}{m}}. \end{aligned} \quad (10)$$

Let l be a number satisfying the inequality

$$b(1 + \eta) < l < 1.$$

To each integer m corresponds a number $p = p(m)$ such that for $q \geq p$ the following inequalities hold:

$$(1 + \eta) b \left(1 + \frac{m}{q}\right) < l, \quad (11)$$

$$\left(1 + \frac{q}{m}\right) (1 + \eta) l^{\frac{q}{m}} < \frac{l}{1 - b}. \quad (12)$$

It is possible to choose b and l in such a way that (11) and (12) are surely satisfied for $\frac{p}{m} > \lambda'$, where λ' is an arbitrary fixed number greater than 1. In fact, on the one hand,

$$\lim_{b \rightarrow 0} (1 + \eta) b \left(1 + \frac{1}{\lambda'}\right) = 0;$$

the inequality

$$(1 + \eta) b \left(1 + \frac{1}{\lambda'}\right) < l \quad (11)$$

is therefore satisfied for a pair of numbers b_1, l_1 , both arbitrarily small, b_1 being chosen small enough with reference to l_1 .

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On the other hand,

$$\lim_{l \rightarrow 0} (1 + \lambda')(1 + \eta)l^{\lambda'-1} = 0,$$

since $\lambda' > 1$; and

$$\lim_{b \rightarrow 0} \frac{1}{1-b} = 1.$$

Accordingly, for $l = l_1$, $b = b_1$, these numbers being sufficiently small, we have

$$(1 + \lambda')(1 + \eta)l^{\lambda'-1} < \frac{1}{1-b},$$

or

$$(1 + \lambda')(1 + \eta)l^{\lambda'} < \frac{l}{1-b}. \quad (12')$$

Hence there exists a pair of numbers l_1 , b_1 , arbitrarily small, such that (11') and (12') are verified. We may take $l_1 < \frac{1}{e}$.

Under the same conditions, the two quantities

$$(1 + \eta)b_1 \left(1 + \frac{1}{x}\right), \quad (1 + \eta)(1 + x)l_1^{x-1},$$

$l_1 \left(< \frac{1}{e}\right)$ and b_1 being fixed, decrease as x increases. The truth of this statement is obvious for the first expression; as for the second, its derivative with respect to x is negative when $l_1 < \frac{1}{e}$. Since (11') and (12') hold for $l = l_1 < \frac{1}{e}$, $b = b_1$, and $\lambda' > 1$, it follows from what we have just seen that these inequalities are verified *a fortiori* if we replace λ' by $\frac{q}{m} > \lambda'$.

It is therefore always possible to satisfy (11) and (12) by a suitable choice of l and b , and by letting p be arbitrary but greater than $m\lambda'$, where $\lambda' > 1$ is fixed. Hereafter it will be assumed that b , l and p have been so chosen.

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From (11),

$$\left[(1 + \eta)b \left(1 + \frac{m}{q} \right) \right]^{\frac{q}{m}} < \frac{q}{l^m}, \quad q \geq p; \quad (13)$$

consequently, from (10) and (12),

$$|C_{m+q}^q a_{m+q} b^q| < \left(\frac{l}{1-b} \right)^m, \quad q \geq p. \quad (14)$$

Now, from the inequality

$$|a_{m+q}| < (1 + \eta)^{m+q},$$

we have

$$\begin{aligned} & |C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + C_{m+q+2}^{q+2} a_{m+q+2} b^{q+2} + \dots| \\ & < C_{m+q+1}^{q+1} (1 + \eta)^{m+q+1} b^{q+1} + C_{m+q+2}^{q+2} (1 + \eta)^{m+q+2} b^{q+2} \\ & \qquad \qquad \qquad + \dots, \quad q \geq p. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{C_{m+r+1}^{r+1} (1 + \eta)^{m+r+1} b^{r+1}}{C_{m+r}^r (1 + \eta)^{m+r} b^r} &= b (1 + \eta)^{\frac{m+r+1}{r+1}} \\ &< b \left(1 + \frac{m}{r} \right) (1 + \eta), \end{aligned} \quad (15)$$

and from (11),

$$\frac{C_{m+r+1}^{r+1} (1 + \eta)^{m+r+1} b^{r+1}}{C_{m+r}^r (1 + \eta)^{m+r} b^r} < l. \quad (16)$$

Combining this inequality with (14), we have

$$\begin{aligned} & |C_{m+q}^q a_{m+q} b^q + C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + \dots| \\ & < |C_{m+q}^q a_{m+q} b^q| (1 + l + l^2 + \dots) \\ & < \left(\frac{l}{1-b} \right)^m \frac{1}{1-l}, \end{aligned}$$

and therefore

$$\begin{aligned} \lim_{m \rightarrow \infty} \sqrt[m]{|C_{m+q}^q a_{m+q} b^q + C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + \dots|} \\ < \frac{1}{1-b} \quad \text{if } q \geq p. \end{aligned}$$

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But if $\{A_n\}$, $\{B_n\}$ are two sequences such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|A_n|} = A,$$

and

$$\lim_{n \rightarrow \infty} \sqrt[n]{|B_n|} = B < A,$$

then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|A_n + B_n|} = A, \quad (18)$$

which is equivalent to the statement that if the radius of convergence of $\Sigma B_n x^n$ is greater than that of $\Sigma A_n x^n$, then the series $\Sigma(A_n + B_n)x^n$ has the same radius of convergence as $\Sigma A_n x^n$.

Let

$$\left. \begin{aligned} A_m &= a_m + (m+1)a_{m+1}b + \dots + C_{m+q}^q a_{m+q} b^q, \\ B_m &= C_{m+q+1}^{q+1} a_{m+q+1} b^{q+1} + \dots \end{aligned} \right\} \quad (19)$$

Then the point 1 is singular if

$$\begin{aligned} & \sqrt[n]{|a_m + C_{m+1}^1 a_{m+1} b + \dots + C_{m+q}^q a_{m+q} b^q|} \\ &= \sqrt[n]{|A_m + B_m|} > \frac{1 - \epsilon}{1 - b} \end{aligned}$$

for an infinity of m , and for every $\epsilon > 0$. The point 1 is regular if for an arbitrary ϵ ,

$$\sqrt[m]{|a_m + C_{m+1}^1 a_{m+1} b + \dots + C_{m+q}^q a_{m+q} b^q|} < \frac{1 - \epsilon}{1 - b}$$

for m sufficiently large, and for b sufficiently small, depending on ϵ .

In general, the point $e^{i\phi}$ is regular if

$$\lim_{m \rightarrow \infty} \sqrt[m]{|a_m + C_{m+1}^1 a_{m+1} b e^{i\phi} + \dots + C_{m+q}^q a_{m+q} b^q e^{qi\phi}|} < \frac{1}{1 - b},$$

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and singular if

$$\lim_{m \rightarrow \infty} \sqrt[m]{|a_m + C_{m+1}^1 a_{m+1} b e^{i\phi} + \dots + C_{m+q}^q a_{m+q} b^q e^{iq\phi}|} = \frac{1}{1-b}, \quad (5')$$

for $q \geq p$, and for b sufficiently small.

55. THEOREM 1: *The circle of convergence of the series*

$$\sum a_{\lambda_n} x^{\lambda_n} = \sum k_m x^m, \quad (20)$$

where

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_n} > \lambda > 1, \quad n = 1, 2, \dots, \quad (20')$$

λ fixed, is a cut.¹

We assume that the radius of convergence is 1. Corresponding to each m , choose a number q' such that

$$\lim_{m \rightarrow \infty} \frac{q'}{m} = \frac{b}{1-b}, \quad (21)$$

where b is given as in the previous section. Then

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sqrt[m]{\frac{(m+q')^{m+q'}}{m^m q'^{q'}} b^{q'}} \\ &= \lim_{m \rightarrow \infty} \left(1 + \frac{q'}{m}\right)^{1 + \frac{q'}{m}} m^{\frac{q'}{m}} m^{\frac{q'}{m}} q'^{-\frac{q'}{m}} b^{\frac{q'}{m}} \\ &= \left(\frac{1}{1-b}\right)^{\frac{1}{1-b}} \left(\frac{1}{1-b}\right)^{-\frac{b}{1-b}} = \frac{1}{1-b}. \end{aligned} \quad (22)$$

On the other hand,

$$\lim_{m \rightarrow \infty} \sqrt[m]{m + q' + 1} = 1.$$

Hence, from (22),

$$\lim_{m \rightarrow \infty} \sqrt[m]{\frac{(m+q')^{m+q'}}{(m+q'+1)m^m q'^{q'}} b^{q'}} = \frac{1}{1-b},$$

¹ See footnote, p. 351

and from (9),

$$\lim_{m \rightarrow \infty} \sqrt[m]{C_{m+q}^{q'} b^{q'}} = \frac{1}{1-b}. \quad (23)$$

We choose b so small that

$$\frac{b}{1-b} < 1. \quad (24)$$

Let λ' be a number satisfying the inequality

$$1 < \lambda' < \lambda, \quad (25)$$

and define the partial sequence $\{m_n\}$ by

$$m_n = E[\lambda_n(1-b)].^1 \quad (26)$$

We have

$$\lim_{n \rightarrow \infty} \frac{\lambda_n - m_n}{m_n} = \frac{b}{1-b}. \quad (26')$$

In (5') let m take on the values m_n , placing

$$q = \lambda_{n+1} - m_n - 1. \quad (26'')$$

Since, from (20'), we have

$$\lambda_{n-1} < \frac{\lambda_n}{1+\lambda},$$

where $\lambda > \frac{b}{1-b}$, it follows that $\lambda_{n-1} < \lambda_n(1-b)$, hence, from (26),

$$m_n > \lambda_{n-1}.$$

Noting that $\lambda_n > m_n$, we obtain from (26''),

$$\frac{q}{m_n} = \frac{\lambda_{n+1}}{m_n} - 1 - \frac{1}{m_n} > \frac{\lambda_{n+1}}{\lambda_n} - 1 - \frac{1}{m_n}.$$

Hence, for n sufficiently large,

$$\frac{q}{m_n} > \lambda',$$

¹ $E[x]$ denotes the smallest integer $\geq x$.

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since $\lim_{n \rightarrow \infty} \frac{1}{m_n} = 0$. But we have seen (page 343) that, under these conditions, the inequalities (11) and (12) hold, provided further that l and b are suitably chosen, b sufficiently small with reference to l , which is arbitrarily small. If then q is chosen so as to satisfy (26''), *i.e.*, if

$$m_n + q = \lambda_{n+1} - 1, \tag{27}$$

then q may be substituted in (5') and (6') in order to obtain the criterion for determining whether or not a given point on the circle of convergence is singular. From (27), all the coefficients k_m from k_{m_n} to k_{m_n+q} vanish except k_{λ_n} , hence

$$\begin{aligned} k_{m_n} + C_{m+1}^1 k_{m_n+1} b e^{i\phi} + \cdots + C_{m_n+q}^q k_{m_n+q} b^q e^{qi\phi} \\ = C_{\lambda_n}^{\lambda_n - m_n} k_{\lambda_n} b^{\lambda_n - m_n} e^{(\lambda_n - m_n)i\phi} \\ = C_{\lambda_n}^{q'} k_{\lambda_n} b^{q'} e^{q'i\phi}, \end{aligned} \tag{28}$$

where $q' = \lambda_n - m_n$. By (26), the number q' satisfies the requirement (21) if m takes on the values m_n .

From (23),

$$\lim_{n \rightarrow \infty} \sqrt[m_n]{|C_{\lambda_n}^{q'} b^{q'} e^{q'i\phi}|} = \frac{1}{1 - b}.$$

Now there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that

$$\lim_{i \rightarrow \infty} \sqrt[\lambda_{n_i}]{|k_{\lambda_{n_i}}|} = 1.$$

But from (26),

$$\lim_{i \rightarrow \infty} \frac{\lambda_{n_i}}{m_{n_i}} = \frac{1}{1 - b},$$

and therefore

$$\lim_{i \rightarrow \infty} \sqrt[m_{n_i}]{|k_{\lambda_{n_i}}|} = \left[\lim_{i \rightarrow \infty} \sqrt[\lambda_{n_i}]{|k_{\lambda_{n_i}}|} \right]^{\frac{\lambda_{n_i}}{m_{n_i}}} = 1.$$

Hence

$$\lim_{n \rightarrow \infty} \sqrt[m_n]{|k_{\lambda_n}|} = 1. \tag{30}$$

We obtain from (29) and (30),

$$\lim_{n \rightarrow \infty} \sqrt[m_n]{|C_{\lambda_n}^{q'} b^{q'} e^{q' i \phi} k_{\lambda_n}|} = \frac{1}{1 - b},$$

and from (28),

$$\lim_{n \rightarrow \infty} \sqrt[m_n]{|k_{m_n} + C_{m_n+1}^1 k_{m_n+1} b e^{i \phi} + \dots + C_{m_n+q}^q k_{m_n+q} b^q e^{q i \phi}|} = \frac{1}{1 - b}.$$

Comparison with (5') shows that every point $e^{i \phi}$ is singular. Theorem 1 is therefore proved.

56. THEOREM 2: *The circle of convergence of the series*

$$\sum a_{\lambda_n} x^{\lambda_n}$$

is a cut provided that there exists a number $\sigma < 1$ such that the series

$$\sum \frac{1}{\lambda_n^\sigma}$$

converges.

Let $\Sigma a_{\lambda_n} x^{\lambda_n} = \Sigma k'_m x^m$. From the sequence $\{\lambda_n\}$ we may extract a subsequence $\{\mu_m\}$ satisfying the conditions

$$\frac{\mu_{m+1} - \mu_m}{\mu_m} > \lambda > 1, \tag{31}$$

λ being fixed; and

$$\lim_{m \rightarrow \infty} \sqrt[\mu_m]{|a_{\mu_m}|} = \lim_{m \rightarrow \infty} \sqrt[\mu_m]{|k'_{\mu_m}|} = 1. \tag{32}$$

Denote by $\{\eta_k\}$ the sequence of integers contained in $\{\lambda_n\}$ and having no element in common with $\{\mu_m\}$. The sequence

$\{\eta_k\}$ clearly has the property that the series $\sum \frac{1}{\eta_k^\sigma}$ converges.

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We may form an integral function

$$g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\eta_k} \right)$$

having the points η_k and no others as zeros. By Theorem 5, Chapter V, the function defined by $\Sigma g(n)x^n$ has just one singularity in the entire plane.

On the other hand, it follows from the work of Poincaré¹ that the function $g(z)$ has the property that

$$|g(z)| > e^{-r^{\sigma+\epsilon}}, \quad r = |z|, \quad \epsilon > 0,$$

for $r > r_0$, the point z being such that within the ring formed by the circles with centers at the origin, radii $|z| - 1$, $|z| + 1$, respectively, the function $g(z)$ does not vanish.

But the number μ_m has the property required for z , from the way in which $g(z)$ was formed; hence, for k sufficiently large,

$$|g(\mu_k)| > e^{-\mu_k^{\sigma+\epsilon}}, \tag{33}$$

for $\epsilon > 0$ arbitrary.

Now the series

$$\Sigma g(n)k'_n x^n = \Sigma d_n x^n \tag{34}$$

has only the d_{μ_m} as non-zero coefficients, since for the indices n different from μ_m we have either $g(n) = 0$ or $k'_n = 0$. Hence we may write

$$\Sigma g(n)k'_n x^n = \Sigma d_{\mu_m} x^{\mu_m}.$$

But we have, from (33),

$$\begin{aligned} \lim_{m \rightarrow \infty} \sqrt[\mu_m]{|g(\mu_m)|} &\geq \lim_{m \rightarrow \infty} e^{-\mu_m^{\sigma+\epsilon-1}} \\ &= 1, \text{ since } 0 < \sigma + \epsilon < 1. \end{aligned}$$

¹ Bull. de la Soc. Math. de France, t. 11 (1883), p. 136.

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Since $\Sigma g(n)x^n$ converges within the unit circle, we have

$$\overline{\lim}_{m \rightarrow \infty} \sqrt[m]{|g(\mu_m)|} \leq 1.$$

Hence

$$\overline{\lim}_{m \rightarrow \infty} \sqrt[m]{|g(\mu_m)|} = 1.$$

Hence, from (32) and (34),

$$\overline{\lim}_{m \rightarrow \infty} \sqrt[m]{|d_{\mu_m}|} = 1,$$

that is, the series $\Sigma d_{\mu_m} x^{\mu_m}$ has its radius of convergence equal to 1. The function defined by this series accordingly has the unit circle as a cut, since the sequence $\{\mu_m\}$ satisfies (31).

Now

$$\Sigma d_{\mu_m} x^{\mu_m} = \Sigma d_n x^n = \Sigma g(n) k_n x^n,$$

and since $\Sigma g(n)x^n$ defines a function having the point 1 as its only singularity, it follows from Hadamard's theorem on the multiplication of singularities that $\Sigma k_n x^n$ likewise has its circle of convergence as a cut. For otherwise the circle of convergence of $\Sigma d_n x^n$ would not be a cut. The proof is therefore complete.

57. THEOREM 3: *The circle of convergence of the series $\Sigma a_{\lambda_n} x^{\lambda_n}$ is a cut provided that*

$$\frac{\lambda_{n+1} - \lambda_n}{\lambda_n^\alpha} > h,$$

where α, h are arbitrary but fixed positive numbers.¹

¹ When $\alpha = 1$, the theorem reduces to Theorem 1, Ch. V. Taking $\alpha = \frac{1}{2}$, we have the theorem of Borel, Jour. de Math., t. ii (1896), p. 441. Finally, the most general theorem, viz., that the circle of convergence is a cut if $\lim_{n \rightarrow \infty} (\lambda_{n+1} - \lambda_n) = \infty$, is due to Fabry, Ann. de l'École Norm. Sup., t. xiii (1896), p. 367.

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We have, by hypothesis,

$$\lambda_{n+p} - \lambda_{n+p-1} > h\lambda_{n+p-1}^\alpha.$$

Obviously $\lambda_{n+k} > k + 1$ for k sufficiently large. We may therefore assume that this inequality holds for all k . We have, then,

$$\lambda_{n+p} - \lambda_{n+p-1} > hp^\alpha.$$

Hence,

$$\begin{aligned} \lambda_{n+p} - \lambda_n &> h(1^\alpha + 2^\alpha + \cdots + p^\alpha), \\ \lambda_{n+p} &> h(1^\alpha + 2^\alpha + \cdots + p^\alpha) \\ &> h \int_0^p x^\alpha dx \\ &= \frac{h}{\alpha + 1} p^{\alpha+1} \end{aligned}$$

We may choose a number r , $0 < r < 1$, such that

$$r(\alpha + 1) > 1 + \delta, \quad \delta > 0,$$

provided δ is small enough. Consequently

$$\begin{aligned} \lambda_{n+p}^r &> \left(\frac{h}{\alpha + 1}\right)^r p^{r(\alpha+1)} \\ &> \left(\frac{h}{\alpha + 1}\right)^r p^{1+\delta} \end{aligned}$$

The series $\sum \frac{1}{\lambda_n^r}$ is therefore convergent. Hence, by Theorem 2, the circle of convergence of the given series is a cut.

This transition from Theorem 1 to Theorem 3 by means of Theorem 2 is due to Faber.¹ Starting with Theorem 1 he also proves by an analogous process the theorem of Fabry referred to in the footnote on p. 351. The latter theorem includes as particular cases the theorems of Hadamard and Borel.

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¹ Sitzungsber. der Math.-Phys. Classe der Münicher Acad. (1904), p. 62.



