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A class of symmetric two dimensional logarithmic potentials

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A Class of Symmetric Two Dimensional Logarithmic Potentials

by

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Abstract

This thesis discusses the solution of an elliptical conductor problem in two dimensions. This problem can be easily solved in higher dimensions, but in two dimensions, the potential function is logarithmic near infinity. This thesis also discusses some particular cases of the solution obtained by solving the above problem.

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To Ma and Bai-Dada.

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Chapter 1

The Conductor Problem for an Ellipse

We first consider the problem of finding the distribution of a charge in equilibrium on an elliptical conductor subject to the conditions:

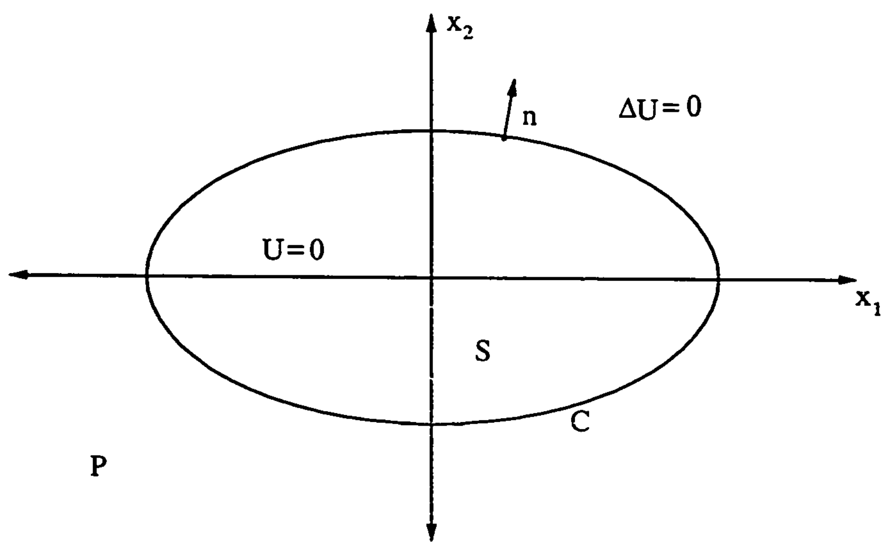

Figure 1.1 Elliptical Conductor C: P is the exterior of C and S is its interior.

(a) $U = 0$ in S,

(b) $\Delta U = 0$ in P,

(c) U continuous on $\mathbb{R}^2$,

(d) near ∞, $U \sim \log|x|$, i.e., $U - \log|x| = O(1)$.

"Potentials" like these are usually determined up to an additive constant, but we have fixed this constant by condition (a). Further this solution is unique if it exists:

Uniqueness Theorem. *The conductor problem has a unique solution if it exists.*

Proof. Let U, V be two solutions of the above problem. Let $W = U - V$. Then W is continuous on $\mathbb{R}^2$ and is bounded at infinity (by Condition (d)). Consider the function

$$\frac{\log \frac{|\mathrm{x}|}{\epsilon}}{\log \frac{a}{\epsilon}}$$

defined on the annular region between the circles of radii ϵ and a. Since W is bounded at infinity, there exists a constant c such that $|W| \leq c$ on $\mathbb{R}^2$. Hence if ϵ is small enough,

$$|W| \leq \frac{c \log \frac{|\mathrm{x}|}{\epsilon}}{\log \frac{a}{\epsilon}}$$

for $|\mathrm{x}| = \epsilon$ and $|\mathrm{x}| = a$ (the boundary of the region as in Fig (1.2)). The right-hand

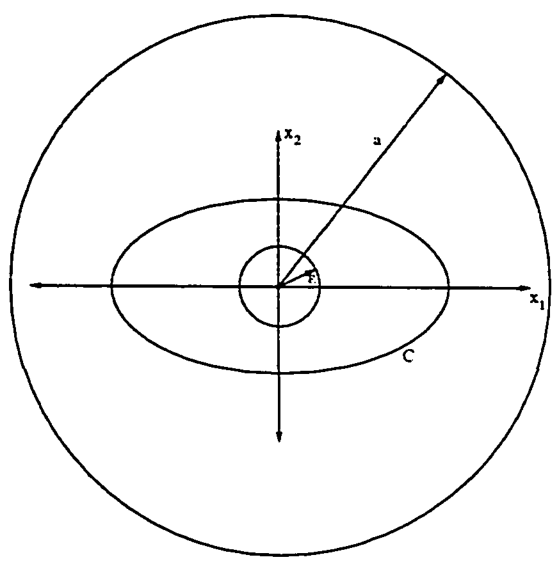

Figure 1.2

side is harmonic on the region, W is subharmonic. Hence the maximum principle implies

$$|W| \leq \frac{c \log \frac{|x|}{\epsilon}}{\log \frac{a}{\epsilon}}$$

holds on the entire region. This is true for any a greater than ϵ. Hence, as $a \to \infty$, we get $|W| \leq 0$, which implies $W = 0$. QED.

We shall discover that U is also of class C^1 in S $\cup$ C and P $\cup$ C. Then we shall define a continuous function σ on C such that

$$\frac{\partial U}{\partial n_-} - \frac{\partial U}{\partial n_+} = -\sigma,$$

where $\frac{\partial U}{\partial n_-}$ and $\frac{\partial U}{\partial n_+}$ are the directional derivatives of U in the direction n of the unit outer normal to C (see Fig 1.1), computed using $U|_S$ and $U|_P$, respectively. The σ mentioned here is thought of as the charge density on C.

1.1 Elliptical Coordinates

Consider a basic ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \; .$$

We assume $0 < b < a$. Also consider the functions

$$\begin{aligned} Q(x, s) &= \frac{x_1^2}{a^2 + s} + \frac{x_2^2}{b^2 + s} - 1, \\ \Phi(s) &= (a^2 + s)(b^2 + s). \end{aligned}$$

The equation $Q(x, s) = 0$, when s has any fixed value not a root of $\Phi(s)$, represents a quadratic curve. For $s > -b^2$ it is an ellipse. For large values of s it is an ellipse nearly in circular form. For $-a^2 < s < -b^2$ it is a hyperbola. For $s = 0$ it is our basic ellipse.

As s approaches $(-b^2)_+$, the semi-axes of the ellipse approach $\sqrt{a^2 - b^2}$ and 0 respectively. Therefore for given (x_1, x_2) not on the coordinate axes, we find the curves (ellipse and hyperbola) that intersect at (x_1, x_2) by the equation $Q(x, \cdot) = 0$ subject to $\Phi(s) \neq 0$. $Q(x, s)\Phi(s) = 0$ is a quadratic equation which in general has two roots.

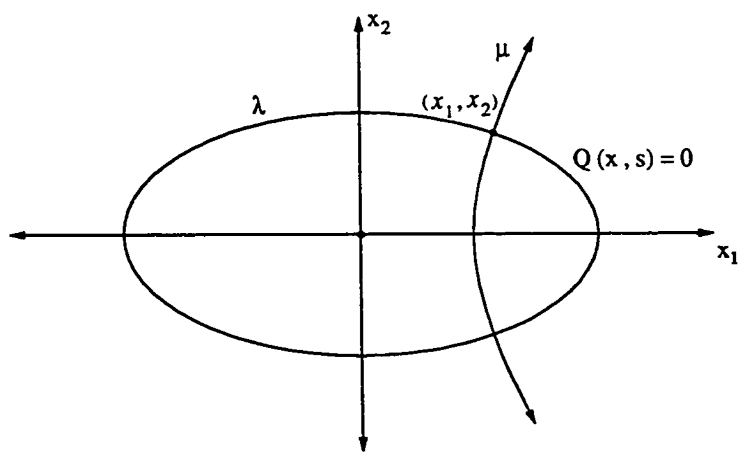

Figure 1.3 Through every point (x_1, x_2), a hyperbola and an ellipse pass. λ and μ are roots of Q(x, s) = 0.

It is geometrically evident (see Fig (1.3)) that the roots λ and μ of this quadratic equation satisfy $-a^2 < \mu < -b^2 < \lambda$. Let us denote by μ-curve and λ-curve, the curves corresponding to $\mu =$ constant and $\lambda =$ constant which represent a hyperbola and an ellipse respectively. Given the μ-curve and λ-curve, we can find a point that is represented by them, i.e., the point of intersection of the two. Factoring the quadratic function of s,

$$Q(s)\Phi(s) = x_1^2(b^2 + s) + x_2^2(a^2 + s) - (a^2 + s)(b^2 + s) \ = -(s - \lambda)(s - \mu) \ .$$

Substituting $s = -a^2$, we get

$$x_1^2(b^2 - a^2) = -(-a^2 - \lambda)(-a^2 - \mu),$$

so

$$\left.\begin{array}{rcl} x_1^2 & = & \frac{(a^2+\lambda)(a^2+\mu)}{a^2-b^2} \; . \\ \text{Similarly,} \;\; s=-b^2 & & \text{gives} \\ x_2^2 & = & -\frac{(b^2+\lambda)(b^2+\mu)}{a^2-b^2} \; . \end{array}\right\} \tag{1.1}$$

Therefore we get four points lying on the λ, μ curves which are symmetric about the coordinate axes and since we will be dealing with symmetric functions that do not distinguish between any two symmetric points, we can choose points (x_1, x_2) using any convention.

We will now show that the λ, μ coordinate system is orthogonal, i.e., we show that $\nabla\lambda \cdot \nabla\mu = 0$.

Consider

$$\begin{array}{rcl} Q(x,\lambda) & = & 0 \; , \\ \text{i.e.,} \;\; \frac{x_1^2}{a^2+\lambda} + \frac{x_2^2}{b^2+\lambda} - 1 & = & 0 \; . \end{array}$$

Differentiating with respect to x_1 we get

$$\frac{\partial}{\partial x_1} Q(x,\lambda) = \frac{\partial Q}{\partial \lambda}\frac{\partial \lambda}{\partial x_1} + \frac{\partial Q}{\partial x_1} = 0 \, .$$

Hence

$$\begin{array}{rcl} \frac{\partial \lambda}{\partial x_1} & = & -\frac{\partial Q/\partial x_1}{Q'(\lambda)} \, , \qquad \text{where } Q'(\lambda) = \frac{\partial Q}{\partial \lambda} \, , \\ & = & \frac{-2x_1/(a^2+\lambda)}{Q'(\lambda)} \, . \end{array}$$

Similarly,

$$\frac{\partial \lambda}{\partial x_2} = -\frac{\partial Q/\partial x_2}{Q'(\lambda)} = \frac{-2x_2/(b^2+\lambda)}{Q'(\lambda)} \, .$$

Therefore

$$\nabla\lambda = \frac{1}{Q'(\lambda)}\left(\frac{-2x_1}{a^2+\lambda}, \frac{-2x_2}{b^2+\lambda}\right).$$

Notice that

$$\frac{\partial Q}{\partial \lambda} \neq 0 \, , \text{ since } \frac{\partial Q}{\partial \lambda} = -\frac{x_1^2}{(a^2+\lambda)^2} - \frac{x_2^2}{(b^2+\lambda)^2} < 0 \, .$$

Similarly,

$$\nabla\mu = \frac{1}{Q'(\mu)}\left(\frac{-2x_1}{a^2+\mu}, \frac{-2x_2}{b^2+\mu}\right).$$

Thus,

$$\begin{aligned}\nabla\lambda\cdot\nabla\mu &= \frac{4}{Q'(\lambda)Q'(\mu)}\left(\frac{x_1^2}{(a^2+\lambda)(a^2+\mu)}+\frac{x_2^2}{(b^2+\lambda)(b^2+\mu)}\right)\\ &= 0\\ (\text{since } 0 &= Q(\lambda)-Q(\mu)\\ &= (\lambda-\mu)\left\{\frac{x_1^2}{(a^2+\lambda)(a^2+\mu)}+\frac{x_2^2}{(b^2+\lambda)(b^2+\mu)}\right\},\quad \text{and } \lambda-\mu\neq 0).\end{aligned}$$

Further,

$$(\nabla\lambda)^2 = \frac{1}{Q'(\lambda)^2}\left\{\frac{4x_1^2}{(a^2+\lambda)^2}+\frac{4x_2^2}{(b^2+\lambda)^2}\right\}\neq 0.$$

Thus the $\lambda-\mu$ system of curves is orthogonal.Therefore we can use λ,μ as an orthogonal coordinate system.And with our convention we have a unique representation of a point (x_1,x_2) in terms of (λ,μ).

1.2 ξ,η Coordinate System

$$ds^2 = dx_1^2+dx_2^2 .$$

$$x_1^2 = \frac{(a^2+\lambda)(a^2+\mu)}{a^2-b^2} \Rightarrow 2x_1\frac{\partial x_1}{\partial\lambda} = \frac{a^2+\mu}{a^2-b^2} \Rightarrow \frac{\partial x_1}{\partial\lambda} = \frac{a^2+\mu}{2x_1(a^2-b^2)} .$$

$$\text{Similarly } \frac{\partial x_1}{\partial\mu} = \frac{a^2+\lambda}{2x_1(a^2-b^2)} .$$

$$\begin{aligned}dx_1 &= \frac{\partial x_1}{\partial\lambda}d\lambda+\frac{\partial x_1}{\partial\mu}d\mu\\ &= \frac{a^2+\mu}{2x_1(a^2-b^2)}d\lambda+\frac{a^2+\lambda}{2x_1(a^2-b^2)}d\mu .\\ \text{Similarly } dx_2 &= -\frac{b^2+\mu}{2x_2(a^2-b^2)}d\lambda-\frac{b^2+\lambda}{2x_2(a^2-b^2)}d\mu .\end{aligned}$$

Therefore

$$dx_1^2 + dx_2^2 = \left[\frac{(a^2+\mu)}{2x_1(a^2-b^2)}d\lambda + \frac{(a^2+\lambda)}{2x_1(a^2-b^2)}d\mu\right]^2$$

$$+\left[-\frac{(b^2+\mu)}{2x_2(a^2-b^2)}d\lambda - \frac{(b^2+\lambda)}{2x_2(a^2-b^2)}d\mu\right]^2$$

$$= \left[\frac{(a^2+\mu)^2}{4x_1^2(a^2-b^2)^2} + \frac{(b^2+\mu)^2}{4x_2^2(a^2-b^2)^2}\right] d\lambda^2 + \left[\frac{(a^2+\lambda)^2}{4x_1^2(a^2-b^2)^2} + \frac{(b^2+\lambda)^2}{4x_2^2(a^2-b^2)^2}\right] d\mu^2$$

$$+2\left[\frac{\partial x_1}{\partial\lambda}\frac{\partial x_1}{\partial\mu} + \frac{\partial x_2}{\partial\lambda}\frac{\partial x_2}{\partial\mu}\right] d\lambda d\mu.$$

Since λ and μ are orthogonal, the expression in the last bracket vanishes. Thus, from (1.1)

$$ds^2 = \frac{1}{4(a^2-b^2)}\left\{\left[\frac{a^2+\mu}{a^2+\lambda} - \frac{b^2+\mu}{b^2+\lambda}\right] d\lambda^2 + \left[\frac{a^2+\lambda}{a^2+\mu} - \frac{b^2+\lambda}{b^2+\mu}\right] d\mu^2\right\}$$

$$= \frac{(\lambda-\mu)}{4}\left[\frac{d\lambda^2}{(a^2+\lambda)(b^2+\lambda)} - \frac{d\mu^2}{(a^2+\mu)(b^2+\mu)}\right]$$

$$= \frac{(\lambda-\mu)}{4}\left\{\left[\frac{d\lambda}{\sqrt{\Phi(\lambda)}}\right]^2 + \left[\frac{d\mu}{\sqrt{-\Phi(\mu)}}\right]^2\right\}.$$

Let $d\xi = d\lambda/\sqrt{\Phi(\lambda)}$ be a function of λ, and $d\eta = d\mu/\sqrt{-\Phi(\mu)}$, a function of μ. Then ξ, η system of curves is orthogonal and hence can also be taken as a coordinate system of curves. Specifically define

$$\xi = \int_0^\lambda \frac{d(s)}{\sqrt{\Phi(s)}}, \qquad -b^2 < \lambda < \infty,$$

$$\eta = \int_{-a^2}^\mu \frac{ds}{\sqrt{-\Phi(s)}} \qquad -a^2 < \mu < -b^2.$$

1.3 Laplace's Equation $\Delta U(x_1, x_2) = 0$ in ξ, η Coordinates

$$\Delta U = \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2}.$$

Let $U(x_1, x_2) = u(\xi, \eta)$. We have (refer to [2, 1])

$$ds^2 = \frac{(\lambda - \mu)}{4}(d\xi^2 + d\eta^2) = Q_1\ d\xi^2 + Q_2\ d\eta^2 + Q_{12}\ d\xi\ d\eta$$

where $Q_1 = Q_2 = \frac{\lambda-\mu}{4}$ and $Q_{12} = 0$. Therefore

$$\Delta U = \frac{1}{\sqrt{Q_1 Q_2}}\left(\frac{\partial}{\partial \xi}\frac{\partial u}{\partial \xi} + \frac{\partial}{\partial \eta}\frac{\partial u}{\partial \eta}\right)$$
$$= \frac{4}{\lambda - \mu}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right).$$

Therefore $\Delta U = \frac{4}{\lambda-\mu}\Delta u$ and $\Delta U = 0 \Leftrightarrow \Delta u = 0$.

1.4 Solution of Laplace's Problem as a Function of λ

As we saw in the previous section, $\Delta U = 0 \Leftrightarrow \Delta u = 0$.

Therefore we are led to search for a solution of a conductor problem in terms of a function of λ alone. Hence the conditions of our elliptical PDE problem, (a) $\leftrightarrow$ (d) become

(a) $u = 0$ for $-b^2 < \lambda \leq 0$,

(b) $\Delta u = 0$ for $\lambda > 0$,

(c) u continuous on $[-b^2, \infty)$,

(d) near ∞, $u \sim \frac{1}{2}\log\lambda$ (since $\lambda \sim |\mathrm{x}|^2$ at ∞) .

As u is supposed to be independent of η, we have

$$\frac{d^2u}{d\xi^2} = 0\ .$$

Therefore,

$$u = A\xi + B$$

$$= A\int_0^\lambda \frac{ds}{\sqrt{\Phi(s)}} + B\ , \quad \text{defined for } -b^2 \leq \lambda < \infty\ ,$$

$$= A\left[\log\left(\lambda + \frac{a^2+b^2}{2} + \sqrt{\Phi(\lambda)}\right) - \log\frac{(a+b)^2}{2}\right] + B \qquad \text{by Appendix A .}$$

Now $\xi(0) = 0$. Therefore,

$$A\xi(0) + B = 0 \Rightarrow B = 0 \text{ and } A\xi \sim \log|\mathrm{x}| \text{ near } \infty \Rightarrow A = \frac{1}{2}$$

for

$$\lambda + \mu = x_1^2 + x_2^2 - (a^2+b^2) = |\mathrm{x}|^2 - (a^2+b^2) \Rightarrow \lambda \sim |x|^2\ .$$

Therefore,

$$\begin{aligned} u = & \ \tfrac{1}{2}\textstyle\int_0^\lambda \frac{ds}{\sqrt{\Phi(s)}} && \text{for } \lambda \geq 0 \\ = & \ 0 && \text{for } \lambda < 0. \end{aligned}$$

That is,

$$u = \frac{1}{2}\int_0^{\max\{\lambda,0\}} \frac{ds}{\sqrt{\Phi(s)}}\ .$$

1.5 Calculation of σ

Using the condition (d), now we will find the density σ.

As u is constant inside the interior of C,

$$\frac{\partial u}{\partial n_-} = 0$$

and we have

$$\frac{\partial u}{\partial n_+} = \sigma,$$

or

$$\sigma = \frac{\partial u}{\partial n_+} = \frac{1}{2}\frac{1}{\sqrt{\Phi(\lambda)}}\frac{\partial \lambda}{\partial n_+}\ , \qquad \text{where } \lambda = 0\ .$$

We had

$$ds^2 = \frac{(\lambda-\mu)}{4\Phi(\lambda)}d\lambda^2 + \frac{(\mu-\lambda)}{4\Phi(\mu)}d\mu^2$$

$$= dn^2 + dt^2.$$

Since the outward normal points in the direction of a λ-curve, we have

$$dn^2 = \frac{(\lambda-\mu)}{4\Phi(\lambda)}d\lambda^2 \Rightarrow \frac{d\lambda}{dn_+} = \frac{2\sqrt{\Phi(\lambda)}}{\sqrt{(-\mu)}} = \frac{2ab}{\sqrt{-\mu}} \quad (\text{since on C, } \lambda = 0) \ .$$

Hence

$$\sigma = \frac{1}{2}\frac{1}{ab}\frac{2ab}{\sqrt{(-\mu)}}$$
$$= \frac{1}{\sqrt{(-\mu)}} \ .$$

$Q(s)\Phi(s) = 0$ gives $x_1^2(b^2+s) + x_2^2(a^2+s) - (a^2+s)(b^2+s) = 0$.

Therefore,

$$s^2 + (a^2+b^2-x_1^2-x_2^2)s - x_1^2b^2 - a^2x_2^2 + a^2b^2 = 0 \ .$$

Therefore,

$$\lambda + \mu = x_1^2 + x_2^2 - a^2 - b^2 \ ,$$

and hence, on C,

$$-\mu = a^2 + b^2 - x_1^2 - x_2^2 \ .$$

Since on C, $x_1 = a\cos\theta$ and $x_2 = b\sin\theta$ are valid parametric equations,

$$\sigma = \frac{1}{\sqrt{a^2+b^2-a^2\cos^2\theta - b^2\sin^2\theta}} = \frac{1}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

The equation of the line tangent to the ellipse (see Fig (1.4)) $\lambda = 0$ at (x_1, x_2) is

$$(\mathrm{X}_1 - x_1)\frac{x_1}{a^2} + (\mathrm{X}_2 - x_2)\frac{x_2}{b^2} = 0 \ .$$

Distance of this line from the origin is

$$p = \left|\frac{-\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}}{\sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}}\right| = \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4}}} \qquad \left(\text{since } \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \text{ on C}\right)$$
$$= \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} = ab\sigma \ .$$

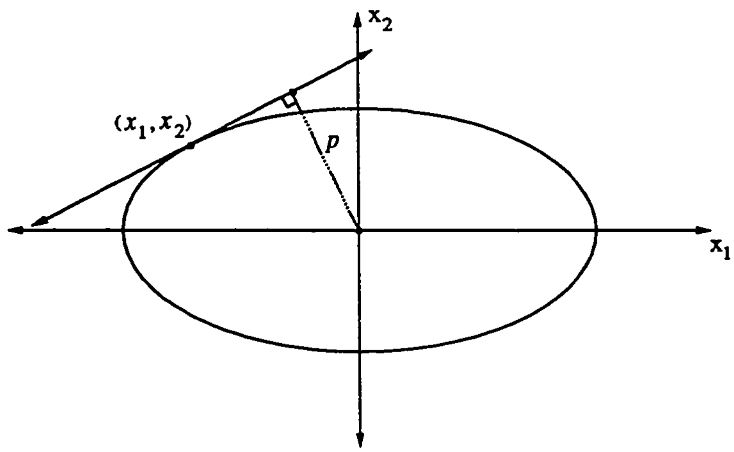

Figure 1.4

Thus the charge density at any point of the ellipse is proportional to the distance from the center to the tangent line at that point.

1.6 Computation of $\Delta U \in \mathcal{D}'(\mathbb{R}^2)$

Let ϕ be a test function. That is, $\phi \in C^\infty(\mathbb{R}^2)$ and supp ϕ is compact.

$$\begin{aligned}
\langle \Delta U, \phi \rangle &= \left\langle \nabla^2 U, \phi \right\rangle \\
&= -\langle \nabla U, \nabla \phi \rangle \\
&= -\int_P \nabla U \cdot \nabla \phi \, dx \qquad \text{(since } U = 0 \text{ in the interior of C)} \\
&= -\int_P \left[\nabla U \cdot \nabla \phi + (\nabla^2 U)\phi \right] \, dx \qquad \text{(since } \nabla^2 U = 0 \text{ in P)} \\
&= -\int_P \nabla \cdot (\phi \nabla U) \, dx \\
&= \int_C \phi \nabla U \cdot n_+ \, ds \qquad \text{(by the divergence theorem)}. \\
\text{Hence } \langle \Delta U, \phi \rangle &= \int \phi \frac{\partial U}{\partial n_+} \, ds. \\
\text{Hence } \Delta U &= \frac{\partial U}{\partial n_+} \, ds \quad \text{on C}
\end{aligned}$$

$$= \sigma \, ds$$
$$= \frac{1}{ab\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}}\, ds, \qquad \text{supported on C.}$$

1.7 Logarithmic Potential

Define $V = \frac{\log|\mathrm{x}|}{2\pi} * \frac{1}{ab\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}} ds$ $\qquad x \in \mathbb{R}^2.$ Convolution defined in the sense of "Distributions"

$$\Delta V = \frac{1}{ab\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}}\, ds = \Delta U \,. \qquad \begin{pmatrix} \text{Since } \Delta \log|\mathrm{x}| = 2\pi\delta_0, \\ \Delta(\log|\mathrm{x}| * F) = 2\pi F. \end{pmatrix}$$

Hence $\Delta(V-U) = 0.$

Therefore $V-U$ is harmonic on $\mathbb{R}^2$.

$$V(\mathrm{y}) = \frac{1}{2\pi ab}\int_C \log|\mathrm{y}-\mathrm{x}| \frac{ds(\mathrm{x})}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}} \qquad \text{where } \mathrm{y}=(y_1,y_2)\in\mathbb{R}^2,\ \mathrm{x}=(x_1,x_2)\in C\,.$$

Near ∞ $\log|\mathrm{y}-\mathrm{x}| \sim \log|\mathrm{y}|$. Hence

$$V(\mathrm{y}) \sim \frac{1}{2\pi ab}\log|\mathrm{y}|\int_C \frac{ds(x)}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}} = \frac{1}{2\pi ab}\log|\mathrm{y}|\int_0^{2\pi} ab\, d\theta$$

Hence

$$V(y) \sim \log|\mathrm{y}|$$

and

$$\mathrm{u}(\lambda) = \frac{1}{2}\log\left(\lambda + \frac{a^2+b^2}{2} + \sqrt{(a^2+\lambda)(b^2+\lambda)}\right) - \frac{1}{2}\log\frac{(a+b)^2}{2}.$$

Therefore

$$\begin{aligned}\lim_{\mathrm{y}\to\infty}(V-\mathrm{u}) &= \lim_{\mathrm{y}\to\infty}\log|\mathrm{y}| - \frac{1}{2}\log 2\lambda + O(\frac{1}{\lambda}) + \frac{1}{2}\log\frac{(a+b)^2}{2} \\ &= -\frac{1}{2}\log 2 + \frac{1}{2}\log\frac{(a+b)^2}{2} \\ &= \log\frac{(a+b)}{2}\,.\end{aligned}$$

Hence by the Liouville's principle V - u is constant on $\mathbb{R}^2$. Hence we conclude that V(y) = A(λ), a function of λ, where

$$A(\lambda) = u(\lambda) + \log\frac{(a+b)}{2}, \qquad -b^2 \leq \lambda.$$

Summary: The unique solution U of the conductor problem given in Section 1 is represented either in the form

$$U(y) = \frac{1}{2}\int_0^{\max\{\lambda(y),0\}} \frac{ds}{\sqrt{\Phi(s)}},$$

or in the form

$$U(y) = \frac{1}{2\pi ab}\int_C \log|y-x|\frac{ds(x)}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}} - \log\frac{a+b}{2}.$$

The asymptotic relationship is precisely

$$\lim_{|x|\to\infty} U(x) - \log|x| = -\log\frac{a+b}{2}.$$

1.8 Logarithmic Potential at a Point Due to a Certain Density Distribution on $\mathbb{R}^2$

Let charge be distributed on $\mathbb{R}^2$ as a function $f(\omega^2)$ where,

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} = \omega^2$$

for points $(z_1, z_2) \in \mathbb{R}^2$, that is, the density at a point (z_1, z_2) is $f(z_1^2/a^2 + z_2^2/b^2)$. We assume the function f has compact support. By definition, f is constant on each ellipse concentric and similar to C. Then the logarithmic potential at y due to this distribution is given by

$$I = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|y-z|\; f(\frac{z_1^2}{a^2}+\frac{z_2^2}{b^2})\, dz_1\, dz_2 .$$

Employ $z_1 = a\omega\cos\theta,\ z_2 = b\omega\sin\theta$, and $dz_1 dz_2 = ab\ \omega d\omega d\theta$. Hence,

$$I = \frac{1}{2\pi}\int_0^\infty \int_0^{2\pi} \log|\mathrm{y}-\mathrm{z}|\, f(\omega^2) ab\,\omega\, d\omega\, d\theta$$
$$= \frac{1}{2\pi}\int_0^\infty f(\omega^2)\omega d\omega \int_0^{2\pi} \log|\mathrm{y}-\mathrm{z}|\ ab\, d\theta.$$

Let $\mathrm{z} = \omega\mathrm{x},\ \mathrm{x} \in C$. Then,

$$I = \frac{1}{2\pi}\int_0^\infty f(\omega^2)\omega\, d\omega \int_0^{2\pi} \{\log|\frac{\mathrm{y}}{\omega}-\mathrm{x}| + \log\omega\} ab\, d\theta$$
$$= \frac{1}{2\pi}\int_0^\infty f(\omega^2)\omega\, d\omega \left[\int_0^{2\pi} \log|\frac{\mathrm{y}}{\omega}-\mathrm{x}|\ ab\, d\theta + 2\pi ab\log\omega\right]$$
$$= ab\int_0^\infty f(\omega^2)\omega\log\omega\, d\omega + \frac{1}{2\pi}\int_0^\infty f(\omega^2)\omega d\omega \int_0^{2\pi} \log|\frac{\mathrm{y}}{\omega}-\mathrm{x}|\ ab d\theta.$$

We had

$$V = \frac{1}{2\pi ab}\int_C \log|\mathrm{y}-\mathrm{x}|\ \frac{ds(\mathrm{x})}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}}$$

where $x_1 = a\cos\theta, x_2 = b\sin\theta$, and

$$ds(\mathrm{x}) = \sqrt{(\frac{dx_1}{d\theta})^2 + (\frac{dx_2}{d\theta})^2}\, d\theta = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta.$$

Therefore,

$$V = \frac{1}{2\pi ab}\int_0^{2\pi} \log|\mathrm{y}-\mathrm{x}(\theta)|\ ab\, d\theta = A(\lambda).$$

It follows that

$$I = ab\int_0^\infty f(\omega^2)\omega\log\omega\, d\omega + ab\int_0^\infty f(\omega^2)\omega A(\lambda(\frac{\mathrm{y}}{\omega}))\, d\omega$$
$$= c_1 + ab\int_0^\infty f(\omega^2)\omega A(\lambda(\frac{\mathrm{y}}{\omega}))\, d\omega$$

with $|c_1| < \infty$, since f has compact support. Hence,

$$I = c_1 + ab\int_0^\infty f(\omega^2)\omega\left[u(\lambda(\frac{\mathrm{y}}{\omega})) + \log\frac{a+b}{2}\right] d\omega$$
$$= c_1 + c_2 + ab\int_0^\infty f(\omega^2)\omega\frac{1}{2}\int_0^{\max\{\lambda(\frac{\mathrm{y}}{\omega}),0\}} \frac{ds}{\sqrt{\Phi(s)}}\, d\omega$$

$$= k + \frac{ab}{2}\int_0^\infty f(\omega^2)\omega \int_0^{\max(\lambda(\frac{y}{\omega}),0)} \frac{ds}{\sqrt{\Phi(s)}}\, d\omega$$

where k is a new constant. Since

$$\frac{\frac{y_1^2}{\omega^2}}{a^2+\lambda(\frac{y}{\omega})} + \frac{\frac{y_2^2}{\omega^2}}{b^2+\lambda(\frac{y}{\omega})} = 1,$$

we have

$$s \le \lambda(\frac{y}{\omega}) \Longleftrightarrow \frac{y_1^2}{a^2+s} + \frac{y_2^2}{b^2+s} \ge \omega^2 \Longleftrightarrow P(y,s) \ge \omega^2$$

with $P(y,s) = \frac{y_1^2}{a^2+s} + \frac{y_2^2}{b^2+s}$. This gives us, with the order of integration changed,

$$I = k + \frac{ab}{2}\int_0^\infty \frac{ds}{\sqrt{\Phi(s)}} \int_0^{\sqrt{P(y,s)}} f(\omega^2)\omega\, d\omega.$$

Now letting $\omega^2 = \zeta$, we have $2\omega d\omega = d\zeta$, and

$$I = k + \frac{ab}{4}\int_0^\infty \frac{ds}{\sqrt{\Phi(s)}} \int_0^{P(y,s)} f(\zeta)\, d\zeta\ .$$

Letting $F(v) = \int_0^v f$, we get

$$I = k + \frac{ab}{4}\int_0^\infty F(\frac{y_1^2}{a^2+s} + \frac{y_2^2}{b^2+s})\frac{ds}{\sqrt{\Phi(s)}}\ .$$

1.9 Logarithmic Potential for the Constant Density Case

In particular, if

$$f = \begin{cases} 1, & \omega \le 1\ , \\ 0, & \text{if } \omega > 1\ , \end{cases}$$

then

$$F(v) = \begin{cases} \int_0^v f = v, & \text{for } v \le 1\ , \\ 1, & \text{else.} \end{cases}$$

We then have

$$I = k + \frac{ab}{4}\int_0^{\max\{\lambda(y),0\}} \frac{ds}{\sqrt{\Phi(s)}} + \frac{ab}{4}\int_{\max\{\lambda(y),0\}}^\infty P(y,s)\frac{ds}{\sqrt{\Phi(s)}}\ .$$

Expanding the integral, we get for all y $\in S$

$$I = k + \frac{ab}{4}\left[\int_0^\infty \frac{y_1^2}{(a^2+s)\sqrt{(a^2+s)(b^2+s)}}\,ds + \int_0^\infty \frac{y_2^2}{(b^2+s)\sqrt{(a^2+s)(b^2+s)}}\,ds\right].$$

By simple calculations (Appendix B), we get

$$\begin{aligned} I &= k + \frac{ab}{4}\left\{\frac{2y_1^2}{a^2-b^2}\sqrt{\frac{s+b^2}{s+a^2}} - \frac{2y_2^2}{a^2-b^2}\sqrt{\frac{s+a^2}{s+b^2}}\right\}\Bigg|_0^\infty \\ &= k + \frac{ab}{2}\left\{\frac{y_1^2}{a^2-b^2}\left(1-\frac{b}{a}\right) - \frac{y_2^2}{a^2-b^2}\left(1-\frac{a}{b}\right)\right\} \\ &= k + \frac{ab}{2(a^2-b^2)}\left\{y_1^2 - y_2^2 - ab\left(\frac{y_1^2}{a^2}-\frac{y_2^2}{b^2}\right)\right\} \\ &= k + \frac{by_1^2+ay_2^2}{2(a+b)}. \end{aligned}$$

1.10 Calculation of k

$$\begin{aligned} c_1 &= \int_0^1 \omega \log\omega \; d\omega \\ &= \left[\frac{\omega^2}{2}\log\omega - \frac{\omega^2}{4}\right]\Bigg|_0^1 \\ &= -\frac{1}{4}\,. \\ c_2 &= \log\frac{(a+b)}{2}\int_0^1 \omega d\omega \\ &= \frac{1}{2}\log\frac{(a+b)}{2}\,. \end{aligned}$$

Hence

$$k = ab\left(-\frac{1}{4}+\frac{1}{2}\log\frac{(a+b)}{2}\right)\,.$$

Therefore

$$\mathrm{I} = -\frac{ab}{4}\left(1-2\log\frac{(a+b)}{2}\right) + \frac{ab}{4(a^2-b^2)}\left\{y_1^2 - y_2^2 - ab\left(\frac{y_1^2}{a^2}-\frac{y_2^2}{b^2}\right)\right\}.$$

When y $= 0$,

$$\mathrm{I} = -\frac{ab}{4}\left(1-2\log\frac{(a+b)}{2}\right)\,.$$

1.11

It is an interesting exercise to verify independently of our result for the formula for U(y) does actually give 0 when y = 0. That is, that

$$\int_C \log|\mathrm{x}| \frac{ds(\mathrm{x})}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}} = 2\pi ab \log\frac{a+b}{2}.$$

That is,

$$\int_0^{2\pi} \log\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\ \ d\theta = 2\pi\log\frac{a+b}{2}.$$

$$V(y_1,y_2) = \frac{1}{2\pi ab}\int_C \log|\mathrm{y}-\mathrm{x}| \frac{ds(\mathrm{x})}{\sqrt{\frac{x_1^2}{a^4}+\frac{x_2^2}{b^4}}}\ .$$

Hence

$$\begin{aligned}
V(0,0) &= \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\ \ d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{(a^2-b^2)\cos^2\theta + b^2}\ \ d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} \left[\log b^2 + \log\left(1+\frac{a^2-b^2}{b^2}\cos^2\theta\right)\right] d\theta \\
&= \log b \ + \ \frac{1}{4\pi}\int_0^{2\pi} \log\left(1+\frac{a^2-b^2}{b^2}\cos^2\theta\right) d\theta \\
&= \log b \ + \ \frac{1}{4\pi}\int_0^{2\pi} \log\left(1+\frac{a^2-b^2}{2b^2}+\frac{a^2-b^2}{2b^2}\cos 2\theta\right) d\theta \\
&= \log b \ + \ \frac{1}{8\pi}\int_0^{4\pi} \log\left(\frac{(a^2+b^2)}{2b^2}+\frac{a^2-b^2}{2b^2}\cos\theta\right) d\theta \\
&= \frac{1}{2}\log\frac{(a^2+b^2)}{2} + \frac{1}{4\pi}\int_0^{2\pi} \log\left(1+\frac{a^2-b^2}{(a^2+b^2)}\cos\theta\right) d\theta\ .
\end{aligned}$$

1.12 Computation of $\int_0^{2\pi} \log\left(1+k^2\cos\theta\right) d\theta$

The function $\log|\mathrm{x}-l|$ is harmonic on the circle of radius 1 (see Fig (1.5))

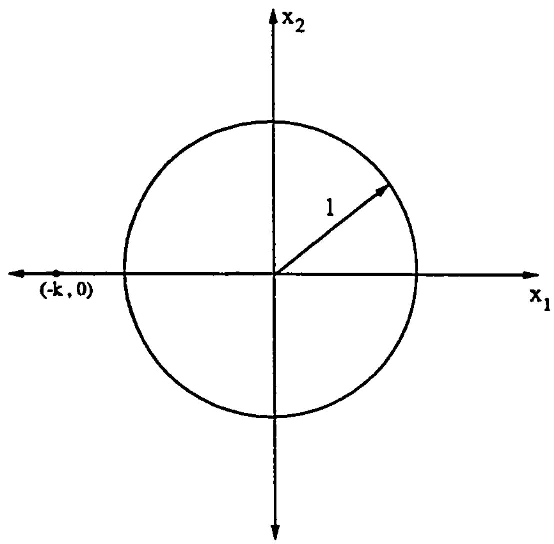

Figure 1.5

with center at the origin, for any l not in this circle. Therefore if we choose $l = (k, 0)$, the mean value property $\Rightarrow$

$$\log |k|^2 = \frac{1}{2\pi}\int_0^{2\pi} \log[(\cos\theta + k)^2 + \sin^2\theta]\, d\theta \qquad k > 1$$

i.e.,

$$2\pi \log k^2 = \int_0^{2\pi} \log\left[(1+k^2) + 2k\cos\theta\right] d\theta$$

$$= 2\pi\log(1+k^2) + \int_0^{2\pi} \log\left[1 + \frac{2k}{1+k^2}\cos\theta\right] d\theta \, .$$

Hence

$$2\pi \log\frac{k^2}{1+k^2} = \int_0^{2\pi} \log\left[1 + \frac{2k}{1+k^2}\cos\theta\right] d\theta \, .$$

By choosing $t = 2k/(1+k^2)$, we obtain

$$\int_0^{2\pi} \log(1 + t\cos\theta)\, d\theta = 2\pi\log\frac{1+\sqrt{1-t^2}}{2} \, .$$

Therefore

$$\int_0^{2\pi} \log\left(1 + \frac{a^2-b^2}{(a^2+b^2)}\cos\theta\right) d\theta = 2\pi\log\frac{(a+b)^2}{2(a^2+b^2)} \, .$$

Hence

$$V(0,0) = \log \frac{a+b}{2} \ .$$

Although A is defined for $\lambda \geq -b^2$ (which takes care of all points except those on the x-axis between $(-\sqrt{a^2-b^2}, 0)$ and $(\sqrt{a^2-b^2}, 0)$, U = 0 here. We can verify that $V = \log \frac{(a+b)}{2}$ at the origin from the observation that V–u $= \log \frac{(a+b)}{2}$.

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Appendix A

$$\int_0^\lambda \frac{ds}{\sqrt{\Phi(s)}} = \int_0^\lambda \frac{ds}{\sqrt{(a^2+s)(b^2+s)}} = \int_0^\lambda \frac{ds}{\sqrt{(s+\frac{a^2+b^2}{2})^2 - (\frac{a^2-b^2}{2})^2}}$$

$$= \log\,(s + \frac{a^2+b^2}{2}) + \sqrt{(s+\frac{a^2+b^2}{2})^2 + (\frac{a^2-b^2}{2})^2}\,\Big|_0^\lambda \text{ (from integral tables).}$$

$$= \log\,(\lambda + \frac{a^2+b^2}{2} + \sqrt{(a^2+\lambda)(b^2+\lambda)}) - \log\frac{(a+b)^2}{2}.$$

Appendix B

$$I = \int \frac{ds}{(a^2+s)\sqrt{(a^2+s)(b^2+s)}}.$$

Let $a^2 + s = t^2$. Then,

$$I = \int \frac{2\,dt}{t^2\sqrt{t^2-(a^2-b^2)}}.$$

Substituting $t^2 = (a^2 - b^2)\sec^2\theta$ we get

$$I = \frac{2}{a^2-b^2}\int \cos\theta\, d\theta = \frac{2}{a^2-b^2}\sin\theta = \frac{2}{a^2-b^2}\sqrt{\frac{s+b^2}{s+a^2}}.$$

Similarly,

$$\int \frac{ds}{(b^2+s)\sqrt{(a^2+s)(b^2+s)}} = \frac{2}{b^2-a^2}\sqrt{\frac{s+a^2}{s+b^2}}.$$