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APPLICATIONS OF CIRCUIT ANALYSIS COMPUTER PROGRAMS TO ESTUARINE POLLUTION PROBLEMS

BY

HALBERT HOMER BYBEE, JR.

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ABSTRACT

APPLICATIONS OF CIRCUIT ANALYSIS COMPUTER PROGRAMS
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This research establishes a method of analyzing the equations that describe pollution problems employing a digital computer program previously in existence, the IBM Electronic Circuit Analysis Program (ECAP). In this analysis, a waterway is approximated by an electrical network which adapts the partial differential equation normally used as a mathematical model for fluid flow, to solution by ECAP.

This modeling method is investigated by analyses made in the frequency and time domains for specific estuarine pollution problems and by checking the results against the analytic solution of the partial differential equation. The application of ECAP is further expanded by calculations of the sensitivities of the various solutions to changes in the parameters of the river, using Leeds' method.

This research provides an accurate method of investigating pollution problems on a digital computer without requiring any special programming for this problem.

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H. H. B., Jr.

Houston, Texas

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I. INTRODUCTION

Digital computer programs which calculate the solutions to electrical network problems with a minimum of effort on the part of the investigator are enjoying wider and wider use in the field of electrical engineering. One might reasonably ask if these programs could not be used with equal ease to analyze problems in other fields. This thesis reports the successful application of one of these programs to the analysis of one type of water pollution problems, that of estuarine pollution.

As the population of the world has increased man has been forced to become more efficient in all of his endeavors, even in waste disposal. In the past man has just dumped his sewage into the nearest river; when it was polluted, he moved on to cleaner areas. Today, however, nearly all the streams and rivers of the world are in use, both as sources of fresh water and as receptacles for man's sewage. Technology has therefore been called upon to help man repurify his water, before and after use. The problem is already out of hand, and there is an urgent need for a more efficient way of insuring a constant supply of clean water, not just on the basis of the demands of today, but also for the increasing long-range use of water.

To obtain the purification efficiency required, man must be able to analyze the results of his efforts and choose the better methods: mathematical analysis is often a more efficient approach than empirical methodology. For example, to decide where a sewerage plant should be located, it would be foolishly expensive to build plants at several different locations in order to determine the best site. Since the processes involved in the water-use cycle are too complex for direct analysis, researchers must get the answers to their questions from simplified models of the system, and if progress is to be made, the models used must predict the proposed systems with good accuracy.

One type of model used in research on the water-use cycle is simply a reduced-scale construction of an actual waterway, so that flow, flooding, and pollution may be observed and studied at close range and under controlled conditions. The scale model of a section of the Delaware River and its tributaries, built by the United States Army Corps of Engineers at Vicksburg, Mississippi,[1] exemplifies this type of model.

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[1]Discussed by D. J. O'Connor et al., in The Analog Computer and Estuarine Pollution Problems.

A second type is a mathematical model, in which a set of simultaneous differential equations is used to approximate the pollution action in a river. One model of this type was developed by D. J. O'Connor and solved on an analog computer.[2]

In the research reported in this thesis, a third type of model was developed. An electrical-network model was realized from a set of simultaneous differential equations. These equations were solved as a circuit by a digital computer program of previously established accuracy, rather than on an analog computer. This procedure allowed greater flexibility and speed in processing data on river pollution.

In order to apply the existing digital computer program to the study of pollution problems, it was necessary to transform the river configuration data into the form of an electrical network which the digital program could analyze. To do this, the river was first modeled in terms of a partial differential equation which expressed pollution concentration as a function of time, location, and characteristics of the river. An approximation of

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[2]Discussed in O'Connor, et al, Mathematical Analysis of Estaurine Pollution.

this equation produced the set of simultaneous differential equations which could then be represented by the electrical-network model for the computer program to handle, with no need for further approximations.

This network model can accurately approximate the behavior of five different types of waterways. In increasing order of complexity, these are (1) a stagnant body of water, (2) a river flowing steadily in one direction, (3) an estuary, or river whose flow is one-directional but perturbed by the action of the tide as it nears the sea, (4) a lake in which the gross transport of water is in more than one direction but where there is no tidal perturbation, and (5) a bay in which the gross water transport is multidirectional and which is also affected by the action of the tide.

The third type of waterway, an estuary, was chosen for the application of this digital program because the estuary level is complex enough to demonstrate the extent of the program's capabilities and because sufficient data were available on typical estuaries. Data from the Vicksburg hydraulic model were used to verify the coefficients in the network model. The model was checked further against analytic solutions to the partial differential equation model.

Calculations of the sensitivity, or effect of changes in the coefficients on the results obtained from the network model, revealed a number of methods new to the pollution modeling field. The sensitivity information revealed a method of determining the accuracy required in the measurement of general estuary characteristics, to produce a highly accurate model of that particular estuary. Also, the effect of changing estuary parameters upon pollution could be predicted from this information.

The development of this electrical-network model, which enables the investigator to utilize an existing digital computer program to analyze pollution problems, has shown the applicability of the reported modeling technique to practical problems in water pollution research. Casting the model in the form of an electrical network makes possible the investigation of pollution problems on a digital computer without requiring special programming, as well as facilitating the application of Leeds' network method of sensitivity calculations to the study of water pollution models.

II. DEVELOPMENT OF THE ELECTRICAL-NETWORK MODEL

The Partial Differential Equation

The development of the network model begins with the partial differential continuity equation used by O'Connor in his research on Delaware River pollution, in which he established a mathematical model to be analyzed on an analog computer. This equation describes an approximation of a real river, in which a line represents the one-dimensional model of the river. (Only the one-dimensional form of the equation was used because it produced results of sufficient accuracy for the available data, without requiring as much effort in its solution as would the two- or three-dimensional forms. Variations across the profile of the river are of course averaged out of the calculations by the use of the one-dimensional form.)

At any point on the line, where

c = the pollution concentration at this point;

t = the time variable;

x = the distance variable, measured positively in the downstream direction and negatively in the upstream direction;

A = the cross-sectional area of the river, perpendicular to the flow, at this point;

E = the coefficient of eddy diffusivity (eddy diffusivity is defined as the net effect of mixing due to turbulence and molecular diffusion, resulting in the reduction of the gradients of concentration);

U = the velocity of the river at this point;

K = the rate at which the pollutant decays at this point (it may be zero for certain pollutants which do not break down);

and S = the rate at which the pollutant is added to the system at this point (it will be zero unless there is an external source at this point),

the equation is

$$\frac{\partial c}{\partial t} = \frac{1}{A}\frac{\partial}{\partial x}\left(EA\frac{\partial c}{\partial x}\right) - \frac{1}{A}\frac{\partial}{\partial x}(AUc) - Kc + S.$$

As expressed by the left-hand side of the equation, each of the terms on the right-hand side contributes to the net rate of change per unit time of the amount of pollutant in a unit volume of river.

The first right-hand term

$$\frac{1}{A}\frac{\partial}{\partial x}\left(EA\frac{\partial c}{\partial x}\right)$$

is derived from Fick's second law, and gives the time rate of concentration due to the net diffusion of pollutant <u>into</u>

the point from points of higher concentration and <u>out</u> to points of lower concentration.

The second right-hand term

$$- \frac{1}{A} \frac{\partial}{\partial x} (AUc)$$

comes from fluid dynamics, and gives the time rate of change of concentration due to the accumulation of pollutant being carried in and out by the flow of the river.

The third term, - Kc, is the rate of decay of the pollutant in the unit volume and is assumed directly proportional to its concentration.

The final term, + S, is the contribution to the net rate of change of pollutant due to external sources.

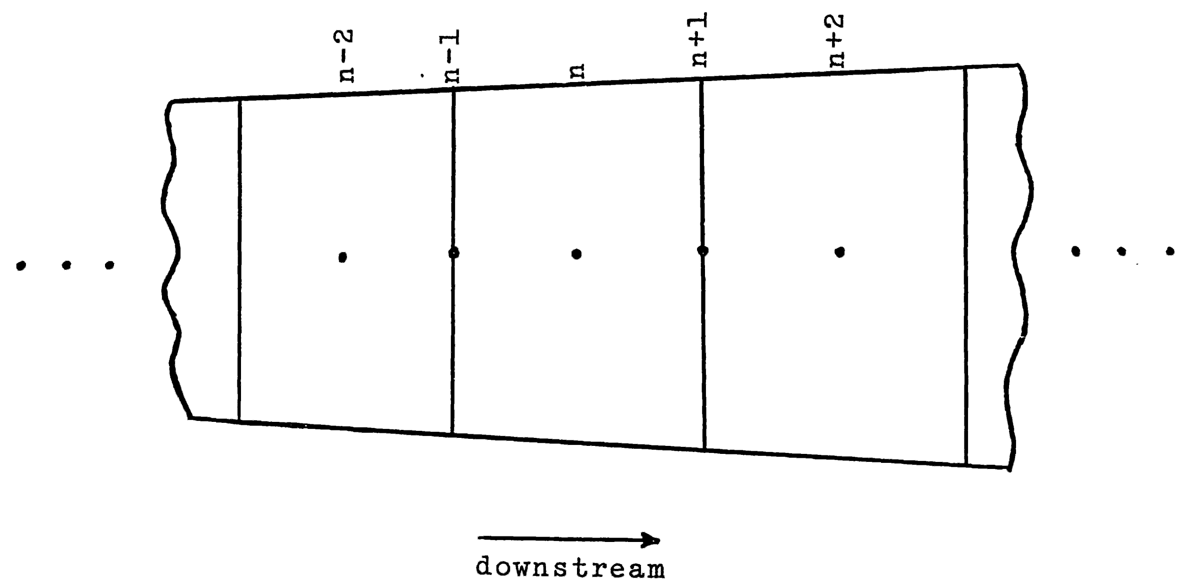
The Set of Simultaneous Differential Equations

Obtaining the set of simultaneous ordinary differential equations by performing the operation of finite differencing on the partial differential equation is the second important step in the development of the electrical-network model for the digital computer to analyze. Up to the process of finite differencing the partial differential equation, this research followed very closely that done by O'Connor in establishing the mathematical model for the analog computer to solve, but after this step the methods are radically different. In both this research and O'Connor's research, it was necessary to reduce the partial differential equation, whose derivatives are with respect to two variables, to a set of equations whose derivatives are with respect to only one variable. Before this finite differencing could be done, however, three simplifying assumptions were required to reduce the complexity of the partial differential equation.

The first of these assumptions is that "time-smoothing," as used by O'Connor, can be performed on the equation without impairing the accuracy of the results. By the operation of time-smoothing, the time-varying coefficients (i.e., those which are a function of time, as velocity, eddy diffusivity, and cross-sectional area) are replaced by time-invariant average coefficients. In the time-smoothing for this estuarine pollution analysis, the period of time-averaging was chosen as one complete tide-cycle. Of course, this eliminated any possibility of obtaining accurate solutions to a time scale as small as one tide-cycle; but the resulting simplification of the equation justifies the limitations on the time range.

The second simplifying assumption is that K, the rate at which the pollutant decays at a point in the model, can be considered constant; i.e., that K is a function of neither time nor distance.

Third, the finite-difference form of the equation was written assuming that the coefficients had no distance dependence. The line model was reduced to a chain of finite length sections so that the average concentration in each section, rather than at each point on the line, was considered in the finite differencing. In the investigations of the final network model, however, the distance variation of the coefficients could be reinserted, using different constants in each section depending on its location along the river. O'Connor used this "variable-constant" scheme in his analog computer solutions of the equation, notably in matching the cross-sectional area coefficients to the area of the river for each section, which established the changes in the velocity coefficients to maintain a constant volume flow-rate down the river. O'Connor, however, did not give the values of the cross-sectional area as a function of distance in his papers, and the absence of these data led to the use of a model with distance-invariant coefficients in this research; i.e., model properties were investigated rather than actual river characteristics.



O'Connor's River Model, in which five points are taken to obtain concentration for a section.

FIGURE 1

In calculating his set of simultaneous differential equations to use as the analog computer model of the diffusion equation, O'Connor used five points in his model to obtain the concentration for a section. The points n-2, n, and n+2 were taken at the centers of adjacent sections, and the points n-1 and n+1 were taken at the boundaries between sections.

The finite-difference form of the original partial differential equation resulting from this sectioning scheme and the assumptions of time-smoothed and sectionwise, distance-invariant coefficients is

$$\frac{dc_n}{dt} = \frac{1}{(x_{n+1} - x_{n-1})A_n}\left(\frac{E_n A_{n-1}(c_{n-2} - c_n)}{x_n - x_{n-2}} - \frac{E_{n+2}A_{n+1}(c_n - c_{n+2})}{x_{n+2} - x_n}\right) + \frac{U_{n-1}A_{n-1}c_{n-1} - U_{n+1}A_{n+1}c_{n+1}}{A_n(x_{n+1} - x_{n-1})} - Kc_n + S_n .$$

O'Connor chose not to use the full accuracy of this set of equations, but to set the concentration at the points n-1 and n+1 to values interpolated from the adjacent points, obtaining the following:

$$c_{n+1} = c_n + (c_{n+2} - c_n)\frac{x_{n+1} - x_n}{x_{n+2} - x_n} ,$$

$$c_{n-1} = c_{n-2} + (c_n - c_{n-2})\frac{x_{n-1} - x_{n-2}}{x_n - x_{n-2}} .$$

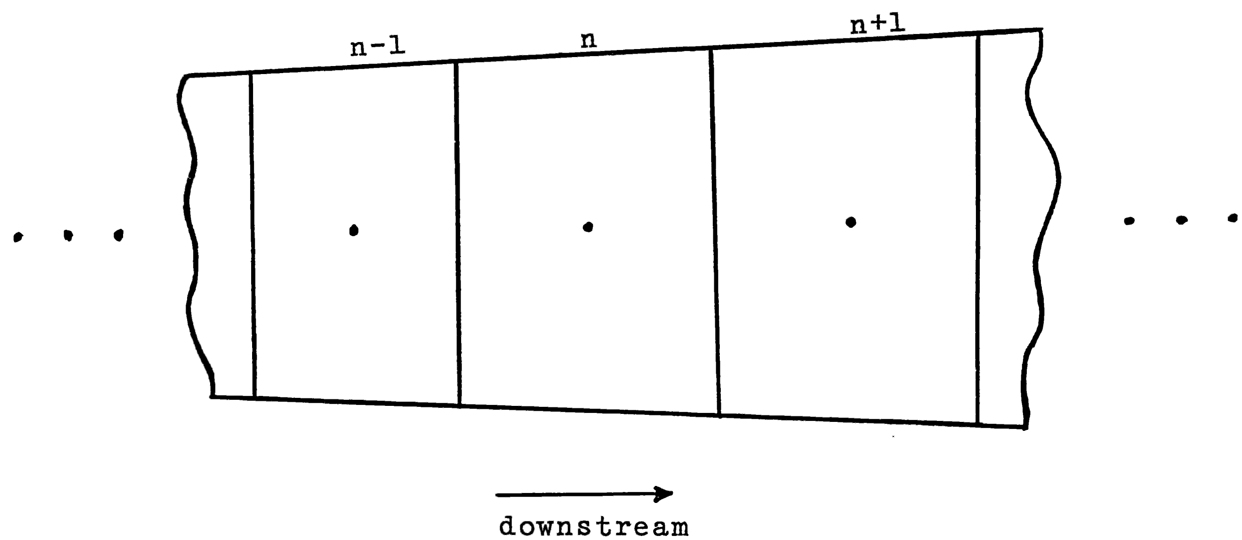
Contrary to O'Connor's method of differencing, in this work only three points were used in the equation for the concentration in a section. The points n-1, n, and n+1 were taken at the centers of adjacent sections. (See Figure 2.)

This sectioning scheme and the assumption of strictly invariant coefficients gave the following finite-difference equation, with L for the section length:

$$\frac{dc_n}{dt} = \frac{E(c_{n+1} - c_n)}{L^2} - \frac{E(c_n - c_{n-1})}{L^2} - \frac{U(c_{n+1} - c_{n-1})}{2L} - Kc_n + S_n .$$

A useful rearrangement of this equation, for network realization, was seen to be

$$\frac{dc_n}{dt} = \left(\frac{E}{L^2} + \frac{U}{2L}\right)\left(c_{n-1} - c_n\right) - \left(\frac{E}{L^2} + \frac{U}{2L}\right)\left(c_n - c_{n+1}\right) + \frac{U}{L}(c_n - c_{n-1}) - Kc_n + S_n .$$



River model for this research, in which three points were taken to obtain concentration in a section.

FIGURE 2

In this modification of the equation, only nearest neighbor differences were explicit. The network sections realized from this form of the equation were much easier to concatenate than the cross-linked sections which would have resulted from the first form of the finite-difference equation. The set of simultaneous ordinary differential equations from which the electrical-network model is derived is then obtained by evaluating n for each section of the river model.

The Electrical-Network Model

To realize an electrical-network model of the equation

$$\frac{dc_n}{dt} = \left(\frac{E}{L^2} + \frac{U}{2L}\right)\left(c_{n-1} - c_n\right) - \left(\frac{E}{L^2} + \frac{U}{2L}\right)\left(c_n - c_{n+1}\right)$$

$$+ \frac{U}{L}\,(c_n - c_{n-1}) - Kc_n + S_n \;,$$

the concentration of the pollutant was associated with the voltage in the network. The model equation could then be approximated by a ladder of similar network sections, each of which represents a region along the river. The

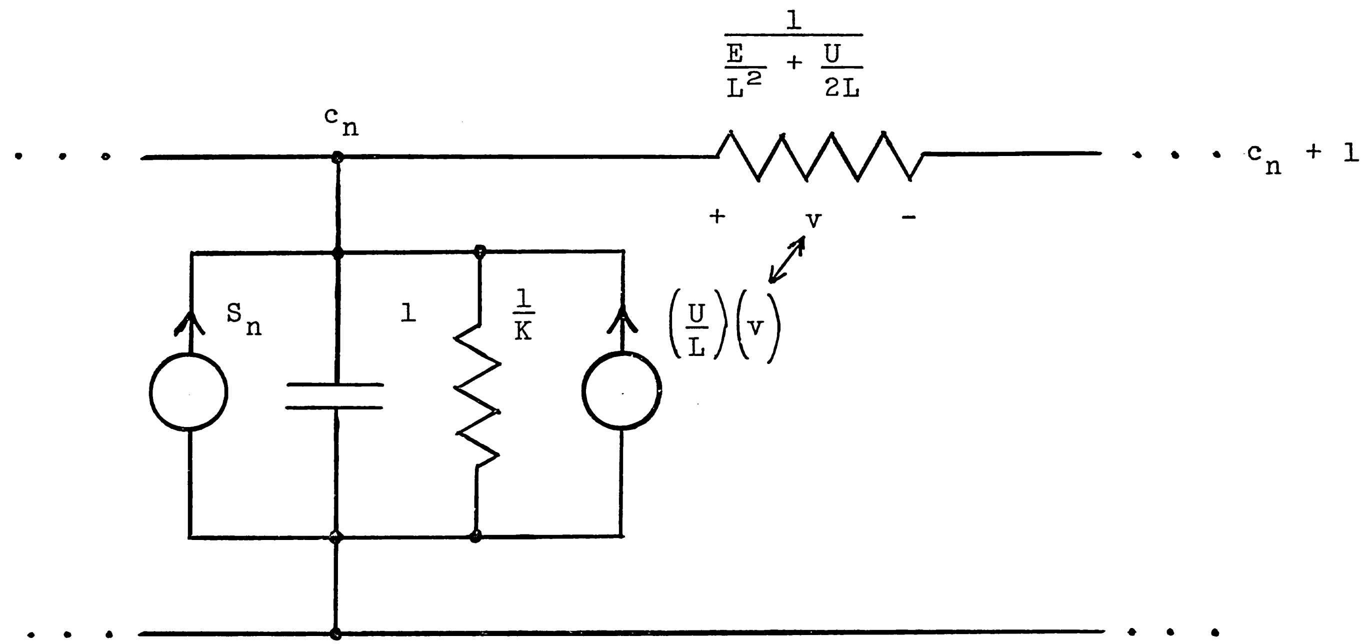


Diagram of a section of the electrical-network model

FIGURE 3

derived network section is shown in schematic form in Figure 3.

Under this identification of concentration with voltage, the equation was viewed as Kirchoff's current law, i.e., that the summation of the currents flowing into any node in an electrical circuit is identically zero. To calculate the component values for each of the sections, the following associations were made:

(1) The left-hand side of the equation

$$\frac{dc_n}{dt}$$

is of the form of a current flowing into a capacitor, and this is represented by the one-Farad capacitor in the network's shunt arm.

(2) The first two terms of the right-hand side,

$$\left(\frac{E}{L^2} + \frac{U}{2L}\right)\left(c_{n-1} - c_n\right) \quad \text{and} \quad -\left(\frac{E}{L^2} + \frac{U}{2L}\right)\left(c_n - c_{n+1}\right)$$

are represented by the series arms of the network ladder which connects nearest neighbor points. (With the first form of the equation, these resistances would have been augmented by others connecting next-nearest neighbor points.)

(3) The third term on the right-hand side of the equation

$$\frac{U}{L} (c_n - c_{n-1})$$

is represented by a voltage-controlled current source in the shunt arm. This dependent current source senses the voltage in the downstream series arm to determine its current value.

(4) The decay term, $- Kc_n$, and the source term, $+ S_n$, are, respectively, a resistor and a current source in the shunt arm.

Thus, the finite-difference equation is translated into an electrical-network model to be analyzed by a digital computer program.

III. THE DIGITAL COMPUTER PROGRAM

The digital computer program for which this research establishes a method of analyzing water pollution problems is the Electronic Circuit Analysis Program (ECAP), written by the International Business Machines Corporation. This program had been in use at Rice University since January 1965 and was updated as version 3-2 through corrections outlined in IBM newsletters since that time. Test circuit problems had been run and checked, showing that the program was calculating the correct solution in each of its parts. As it was thus unnecessary to question the accuracy of the program, attention could be focused on the problem statement whenever unexpected results were obtained.

ECAP can solve electrical-network problems when given only the topology and element values of the network and the required forcing functions. As well as obtaining standard response solutions to a network, it can calculate the sensitivity of any network voltage or current to a change in any of the network parameters at the same time it calculates the response of the network to its forcing functions.[3]

Like all computer programs, ECAP solves an electrical circuit problem in three steps: input, computation, and output. The circuit to be solved must be initially approximated by elements of ECAP's input language, e.g., by resistors, capacitors, inductors, independent sources, current sources dependent upon the voltage of another branch, and current controlled sources. Then the program establishes a set of up to fifty nodal admittance equations which it solves for the direct current solution, the alternating current solution, or the transient response to given forcing functions. Output is relayed in the form of tables of all the node voltages, all branch voltages, and/or all currents, even where only one "input" current and one "output" voltage are of interest to the investigator.

To better utilize the output of ECAP, the output section of the program was modified to plot the frequency response of the transient response of any voltages or

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[3]This method of sensitivity calculation was developed by Dr. J. V. Leeds, Jr., of the Department of Electrical Engineering at Rice University.

currents in the circuit. In keeping with the ECAP idea of a user-oriented input language, the plotting section needs to be told only which quantities to plot and whether magnitude, phase, or log-magnitude graphs are desired. For this research, factors were internally calculated to provide labelled, full-scale, 8 × 10 inch graphs.

Another modification, less widely useful but nevertheless significant, permitted the storage on magnetic tape of complete sets of ECAP output for possible subsequent processing. Programs were written to read this tape and to extract desired values from successive sets of data for alternate graphs or listings. With this modification, graphs of voltage versus node number (concentration as a function of distance in the pollution model) could be obtained. The graphical output features proved very useful in the evaluation of solutions of the various pollution problems examined.

Typical run times on the computer were taken for a network of fifty nodes and no more than 200 connecting branches, as this is the limit of ECAP's capacity. To calculate and print out the transient response for fifty output intervals with ten calculation time steps per output interval, the Rice IBM 7040 digital computer required a few seconds for loading from disk files and nine minutes of computation time. The equivalent calculation on an IBM 7094 computer used a main frame time of one minute for loading and one minute for execution.

IV. INVESTIGATIONS

A number of investigations were run on the ECAP network model, primarily to check the accuracy of the model. The coefficient values to be used throughout the investigations were first determined by calculating the impulse response; then investigations were run on the step response, the frequency response, and the sensitivies in the network model. In all the investigations, the results of the ECAP calculations were checked against the analytic solutions to the model and against published data showing the results obtained by O'Connor on an analog computer. In discussing the analysis of the network model, terminology more closely related to network analysis than to pollution control will be used.

Impulse Response: Coefficient Setting

One of the sections in the center of the ladder of the network model is designated as the source section. For each of the responses run on the model, the exciting source is placed in this section. It is the only section with either nonzero initial conditions or a nonzero current source.

The first investigation run on ECAP was the calculation of the _impulse response_, for the purpose of determining the coefficient values to be used in the network model. The impulse response of the network is the set of the voltages as functions of time for each section when the source section is excited with an initially charged capacitor. The resulting concentration-proportional voltages were observed in each section at the end of each tide cycle.

The initial coefficient values used in this investigation were taken from a typical set of O'Connor's values, based on data from the Delaware River hydraulic model at Vicksburg. Where O'Connor's data were incomplete, values from the table of Delaware River data in the Coast and Geodetic Survey Tide Tables[4] were used. One constant set of parameters was used for all sections. The results of the ECAP runs were checked against the Vicksburg hydraulic model data, which were also used by O'Connor to check the results of his analog computer runs. Since the Vicksburg data, with which the ECAP values were to be compared,

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[4]_Tidal Current Tables 1965_, U.S. Department of Commerce (_Coast and Geodetic Survey_).

showed impulse response in the form of dye concentrations along the model as functions of time and distance for a single injection of dye, the impulse response of the network model was calculated by running a transient response with the only forcing function being an initial charge on the capacitor in the source section.

The ranges and typical values of O'Connor's coefficients are given in the table below, along with the final set of coefficients obtained for this research. In these data, one tide cycle was taken as twelve hours.

	Coefficient ranges (O'Connor)	Typical values (O'Connor)	Final Coefficients (this research)
eddy diffusivity E (ft.^{2}/tide cycle)	$(0 \text{ to } 28) \times 10^7$	5.6×10^7	4.2×10^7
stream velocity U (ft./tide cycle)	-	-	3.3×10^3
decay rate K (tide cycle^{-1})	0 to 0.1	2.2×10^{-2}	2.2×10^{-2}
section length (ft.)	$(0 \text{ to } 2.5) \times 10^4$	$(1 \text{ to } 2) \times 10^4$	1.1×10^4
area A (ft.^{2})	$(5 \text{ to } 50) \times 10^4$	-	9.1×10^4
section volume (ft.^{3})	$(0.07 \text{ to } 10) \times 10^9$	-	1.0×10^9
flow rate (ft.^{3}/sec.)	$(3.4 \text{ to } 18) \times 10^3$	7×10^3	7.0×10^3

For the first ECAP runs, a velocity of 6×10^3 feet per cycle and a section length of 2×10^4 feet were estimated from the two sources of data. The solution obtained from this approximation disagreed with the Vicksburg data in two ways: at points above the pollution injection point, the solution contained a damped oscillation as a function of distance, instead of the exponential decay verified by O'Connor; at points downstream from the source, the concentration peak arrived too soon in comparison to O'Connor's data. To correct these errors, the component values of the network were changed in an attempt to reduce the section length and velocity.

The computations were repeated several times with different coefficients, in an attempt to get a better fit to the Vicksburg data. When the shapes of the two sets of data looked fairly similar, the river parameters, as given above, were calculated from the values of the network components. These component values were used for the rest of the investigations. An exact reproduction of the Vicksburg data was not sought, however, since the present model intrinsically introduces error due to the distance invariance.

It should be noted that O'Connor was able to produce good agreement with the data from the hydraulic model at Vicksburg. The best agreement was obtained with coefficients related to measurements between high tide slack times, where the largest cross-sectional areas were present. O'Connor also noted that the product of area and measured diffusivity was almost a constant independent of position along the river. In his papers, however, he did not give the values of the cross-sectional area as a function of distance, and the lack of these data led to the use of a model with position-invariant coefficients in this research. In other words, model properties were investigated instead of actual river characteristics.

Step Response

The step response was run over fifty tide-cycle periods in order to determine delays and steady-state section gains. The step response of the network is the set of section voltages as functions of time, where the forcing function in the source section is a constant-valued current source turned on at time equal to zero. The voltages of each section were observed at integral tide-cycle periods. The steady-state value of the step response is defined by the set of constant voltages attained by the system after all time variation has died out. The section gain is the ratio of the "output" voltage of a section to its "input" voltage at a given instant or for a given frequency. For upstream gains (up river from the source section), "input" is at the downstream end of the section (that closest to the source) and "output" is at the upstream end. For downstream gain (at sections down river from the source section), "output" is at the downstream end (that farthest from the source) and "input" is at the upstream end. Thus a one-section downstream gain would be the ratio of the voltage at point n+1 to that at point n.

The source section reached its steady-state value after about thirty cycles; other sections reached steady-state values with a time lag of about four cycles per section. At the end of fifty cycles, the eleven central sections were at steady-state values. These center sections showed gains of 0.934 per section downstream for sections below the source and 0.37 per section upstream for sections above the source. Runs were not made for more than fifty cycles, as this was considered the point of diminishing returns for information obtained.

The step response for three different points in the model is shown in Graph 1 (page 45). Graph 2 (page 46) shows concentration voltage as a function of distance in the network model at the end of the fiftieth tide cycle. Because of the rapid rate of change with distance of the concentration in the upstream half of the model, the linear approximation used to generate the network began to break down, and the steady-state section gain was about ten percent higher than that predicted by the analytic solution of the partial differential equation, as we shall see later.

A further minor error was introduced by shorting the end sections of the network to ground. The resulting boundary conditions were thus for a river terminating in an infinite pollution sink. To approximate termination in a channel of the same configuration, the characteristic impedance* of a section was calculated to terminate the downstream end of the model. Using the Laplace transform of the partial-differential equation, the following equation resulted for the characteristic impedance, z:

$$z = \frac{.676 - s \pm \sqrt{s^2 + .648s + .501}}{s + .022},$$

where the symbol s is used for the frequency in radians

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*The characteristic impedance of a two-port device is defined as the input impedance of the device when it is terminated in that same impedance.

per tide-cycle period times the square root of minus one; i.e., the complex frequency.

The first approximation to this impedance is a sixty-three Ohm resistor in parallel with a one-Farad capacitor. The use of this new termination gave errors in only the two end sections instead of in three as without it. Higher order approximations were not attempted, as it was felt that the number of useable nodes would not be increased.

To check the accuracy of the network model, the steady-state response to the step input was compared with the analytic solutions of the partial differential equation being modeled. The analytic solutions of the continuity equation used for the estuary model were derived from some of O'Connor's preliminary work[5] in which the constant, time-smoothed coefficients were used. For the steady state problem, the time derivative of the concentration is zero, and the solution of the equation is

$$c = \frac{S}{AU\sqrt{1 + \frac{4KE}{U^2}}} \; e^{\left[\frac{U}{2E}\left(1 \pm \sqrt{1 + \frac{4KE}{U^2}}\right)x\right]} .$$

The plus sign in the exponent is for the portion of the

[5]O'Connor, The Analog Computer ...

model upstream from the source, and the minus sign is for the portion downstream from the source. The source is located at the origin of the x-axial. For ease in checking with the ECAP data, a distance equal to one section length was used.

The effects of sectioning were examined by repeating the problem with different sectioning lengths and comparing the solutions with the analytic solution. In these calculations, the error due to finite differencing of the model appeared. Five different section length runs were made for the step response. The lengths were in the following ratios to the standard section length: one-fourth, one-half, one, two, and four. The steady-state gains per section for each length are given below with the ratios between ECAP and analytic values.

length	downstream calc.	downstream ECAP	(ECAP / calc.)	upstream calc.	upstream ECAP	(ECAP / calc.)
1/4	0.983	0.983	1.00	0.800	0.791	0.99
1/2	0.966	0.966	1.00	0.640	0.622	0.97
1	0.934	0.934	1.00	0.410	0.370	0.90
2	0.872	0.870	1.00	0.168	0.062	0.37
4	0.760	0.752	0.99	0.028	-0.18	-

The last entry shows the damped oscillations similar to those of the very first run of the model. The errors shown in this table would be of different magnitudes for the same section lengths but with the other coefficients different; i.e., in a different river or in a different part of the same river.

To reduce the error from the upstream gain to the size of that in the downstream gain, the section length would have to be reduced by a factor of ten. This became evident in examination of the exponents in the analytic solution. The exponent for the upstream form of the equation is about ten times the exponent for the other choice of signs. Reducing the section length would reduce the size of the exponential concentration change that the linear model was trying to approximate to a value that it could handle. Since the downstream portion was of the most interest in this research, the error associated with the eleven-thousand-foot sectioning was tolerated.

Frequency Response

In order to determine the validity of the model for high-resolution time responses, it was necessary to examine the frequency response of the model. The frequency response of the network is the set of all section voltages as a function of the frequency of the sinusoidal current source exciting the source section. Both the phase relative to the source and the magnitude of the sine wave of the section voltages are required to make up the frequency response. Under a sinusoidal current source excitation, the section voltages were observed at a number of different frequencies. The magnitude of the resulting sine wave and its phase relationship to the sinusoidal forcing function were calculated by ECAP at each point.

As expected from the configuration of the network, the model had low pass characteristics. The roll-off was approximately six decibels per octave per section, with a cutoff frequency of about 0.064 forcing function cycles per tide cycle period, or approximately eight days per cycle. The cutoff frequency is defined to be that frequency at which the section gain is reduced to 0.707 of the zero frequency section gain.

The zero frequency gain was 0.934 downstream for sections below the source and 0.37 upstream for sections above the source. Again the finite-differencing error for the upstream portion of the model was ten percent per section at zero frequency. (The analytic solution calculated later shows other errors in the network at high frequency.)

Graph 3 (page 47) shows the magnitude frequency response of the first and fourth downstream sections on a full log graph, and Graph 4 (page 48) gives the corresponding phase information.

The frequency response obtained by ECAP was compared with the analytic solutions of the partial differential equation as a check on the accuracy of the network model, in much the same way that the step response was compared with the analytic solutions. The Laplace transform of the differential equation was taken, and the resulting complex frequency term was combined with the decay term, yielding another homogeneous equation whose solution is

$$c = \frac{S(s)}{AU\sqrt{1 + \frac{4E(K + s)}{U^2}}} \cdot e^{\left[\frac{U}{2E}\left(1 \pm \sqrt{1 + \frac{4E(K + s)}{U^2}}\right)x\right]}$$

where s is the complex frequency. In the exponent, the plus sign is for the portion of the model upstream from the source and the minus sign is for the portion downstream from the source. A distance equal to one section length was used. As shown by the exponent in the equation, for any multiple m of the section length, the section gain is the original section gain raised to the m^{th} power, and the phase shift is the original shift times m.

The frequency response was calculated analytically for five frequencies. As in the step response, the errors introduced by the finite-differenced model could be seen from these calculations. The upstream section gain was always in error. The downstream section gain developed a five percent error by the time the frequency attained approximately the cutoff frequency. In the phase shift per section, which is the same for upstream as for downstream, an error developed in the second digit for frequencies above the cutoff.

Even though they are not independent, the phase shift of the model was investigated along with the magnitude response, since the phase is often the first quantity to show an error when the model begins to fail. For example, in this model the phase error would always exist no matter how fine the sectioning became, because, as the analytic solution shows, the phase increases as the square root of the frequency for high frequencies. The individual sections, however, are limited to a phase shift of ninety degrees by their configurations.

The table below gives both the calculated analytical solutions and the ECAP solutions for section gain and phase shift.

frequency cycles/ tide cycle	downstream gain		upstream gain		phase shift (degrees per sec.)	
	calc.	ECAP	calc.	ECAP	calc.	ECAP
0.0	0.934	0.934	0.410	0.370	0	0
0.032	0.794	0.787	0.335	0.310	24.8	24.7
0.064	0.663	0.638	0.280	0.252	38.8	37.7
0.361	0.252	0.207	0.106	0.082	99.5	73.0
4.081	0.0036	0.020	0.0015	0.0077	346.	88.4

There is a reversal in the trend of the errors as the frequency becomes very high and the wavelength becomes small with regard to the section length. The model was previously known not to be valid in this region because of the initial time smoothing of the coefficients. In the frequency region where the model is applicable, the error would again be lessened by reducing the section length by a factor of ten, as in the step response investigations.

Sensitivity

Sensitivities of the network model were obtained by employing Leeds' method of sensitivity calculation, which makes possible the study of the effects of changes in stream flow, pollutant diffusivity, and decay rate.

The sensitivity values show how sensitive a given voltage is to the value (or to the perturbation in the value) of a given coefficient somewhere in the network; i.e., the partial derivative of the former with respect to the latter. When the percent change in the coefficient value is multiplied by the sensitivity of the given voltage with respect to the coefficient value, the product is the resulting percent change in the voltage. In all cases in this investigation, the sensitivity of the concentration voltage was taken with respect to a change in a particular parameter of the model. The sensitivities in the network model are less than one, which means that a change of a given percentage in any parameter will produce a change of a smaller percentage in the voltage. A sensitivity value of + 0.01 for a concentration in section n with respect to a coefficient in section m means that a + 5 percent (or + 1 percent) change in the coefficient in section m would produce a concentration change of + 0.05 percent (or + 0.01 percent) in section n.

Sensitivities with respect to diffusivity coefficients, velocity coefficients, and decay coefficients were calculated. Graph 5 (page 49) shows the sensitivity of the concentration at the source with respect to a diffusivity coefficient two sections (four miles) downstream, as a function of time for a step response. The maximum value never exceeded 0.01.

The steady-state value of the sensitivity is not dependent on the location of the source, except when the source is between the two sections of interest; e.g., the final value of the sensitivity for the concentration in the second section downstream with respect to the diffusivity coefficient in the fourth section downstream would be the same as the final value shown in Graph 5, where the concentration section contains the source. Its peak, however, would be delayed by the time required for the first pollution to reach the second section. The peaking portion would also be distorted by the fact that the second section was receiving only a rounded approximation of a step excitation, due to the attenuation of high frequency components by the network.

The sensitivities in the network were all fairly small. That of the concentration in the source section to that section's diffusivity coefficient never exceeded 0.11. The velocity sensitivity curve was shaped like that of the diffusivity sensitivity. Its maximum value across one section never exceeded 0.08. The sensitivity curve from the decay coefficient was shaped like the concentration curves. It had a maximum value across one section of 0.02. The fact that the sensitivities were so low meant that, near the source, the model was not very dependent upon coefficient values in other parts of the model. This gave some validity to the constant-coefficient model as a predictor of real phenomena.

These sensitivity calculations, which O'Connor did not attempt, give some economic importance to the methods used in this research. Once the desired accuracy of the model has been established, one can calculate the accuracy required in the actual field measurements of the parameters, with only a rough initial guess at the coefficients. Much more time could thus be spent on the measurements of the parameters which actually contribute to the accuracy of the model.

Another interesting sensitivity calculation might be for total flow. With an accurate mode, the sensitivity of concentration at each point with respect to flow changes in all sections could be calculated in one application of the Leeds sensitivity method, thus predicting pollution changes as a function of stream flow as controlled from a dam, without requiring another special computer program. The sensitivity data could be useful in many other investigations as well.

V. CONCLUSIONS

The most important conclusion to be drawn from this research is that the existing digital computer program, ECAP, can be successfully applied to analysis of problems of water pollution. This was determined by the investigations run on the program once the electrical-network model had been developed from the partial differential equation normally used as a mathematical model of fluid flow.

Exciting the network model with a "pollution" source at its middle section and observing the resulting "concentration" in all of the up- and downstream sections enabled the investigator to solve for a number of characteristic properties of the electrical network, each of which corresponds directly to a property of the waterway being modeled and studied.

By solving for the impulse response of the network, the values of the coefficients for use in investigations on that waterway were accurately determined.

The accuracy of the network was further verified by solving for the step response and the frequency response and then by comparing the results of the ECAP calculations to solutions obtained from analytic methods.

Sensitivity calculations on the network provide a method of determining tolerances required of the coefficients for any given waterway to produce a desired accuracy of simulation. The sensitivity information also enabled new investigations to be effected on the sensitivities of the river to various characteristics of its configuration.

Other investigations may be extended from the foundations set by this research. By eliminating the assumption of constant coefficients, a model with variable area or variable diffusivity could be obtained, permitting, for example, a more accurate method of calculating sensitivity with respect to area. This is of particular interest, as change in cross-sectional area is the most important factor in the flooding of a river. Closer approximation of the diffusivity in the partial differential equation is another inherent advantage of a model with variable coefficients.

The accuracy of the upstream gain in the model could be increased by a method of finite differencing more suited to rapid exponential changes.

A third expansion of the model might be to permit the solution of bay problems by using a two-dimensional, rather than a one-dimensional model. The basic sections could be connected into a mesh instead of a ladder. Even though the molecular component of diffusivity in this case would be independent of direction, it would be somewhat more involved to determine the first approximations to the coefficients, because fluid flow due to tidal motion in the bay could cause different degrees of mixing in the longitudinal direction and the lateral direction; thus the time-smoothed effective diffusion term at one point could have different longitudinal and lateral components. The velocity terms could not be predicted by a model as simple as the free-plane turbulent jet[6] which approximates the turbulent action of water coming from a plane surface into a semi-infinite body, because the large convection currents coming in from the sides would require flow across the side boundaries of the bay, violating its boundary conditions. Instead, the model of divergent and convergent boundary layers,[7] which can be shaped to fit the conditions of the bay, would probably be better suited to velocity calculations of this type of waterway. Flow data from navigational charts might, however, suffice for calculating sensitivities and laying out field campaigns.

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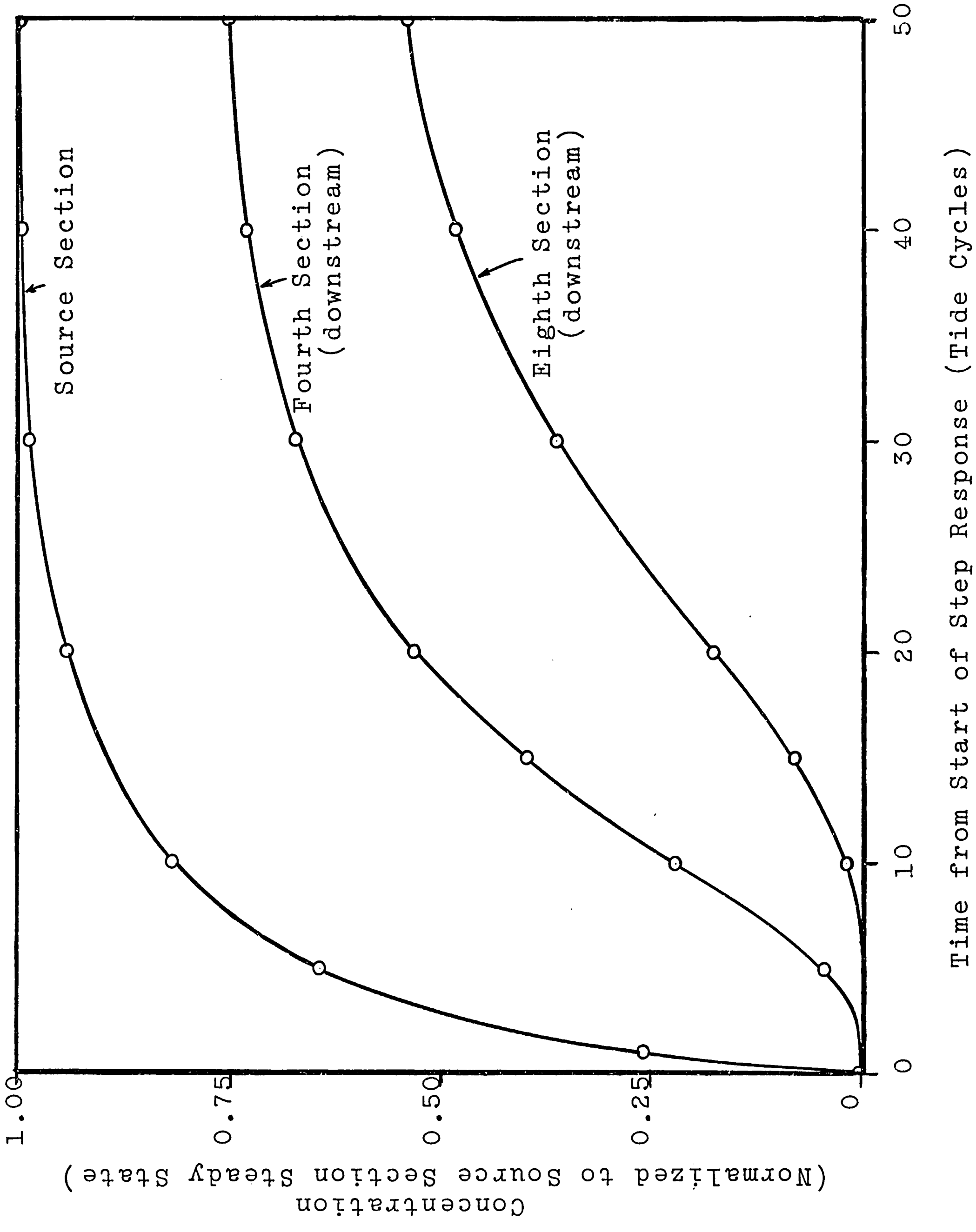
[6]Bird, et al., Transport Phenomena, p. 178.

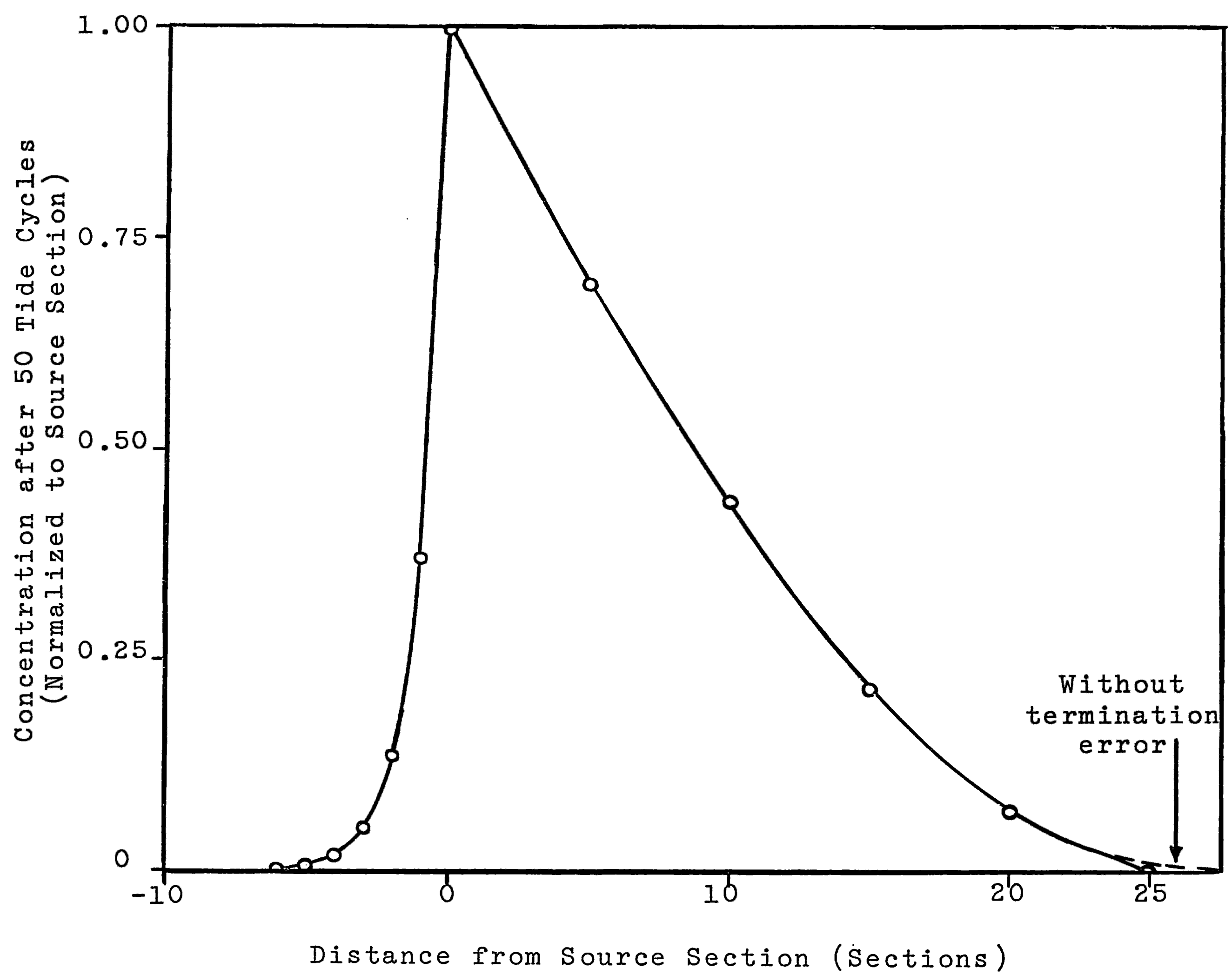
[7]Schlichting, Boundary Layer Theory, p. 586.

The method of developing an electrical-network model for digital computer program analysis need not be limited to the equation used in research on water pollution. It could be applied as well in other fields where mathematical models could be adapted for solution by a digital program. For example, given the stress-strain equations for a beam, a network model could be developed to compute certain structural specifications in architecture.

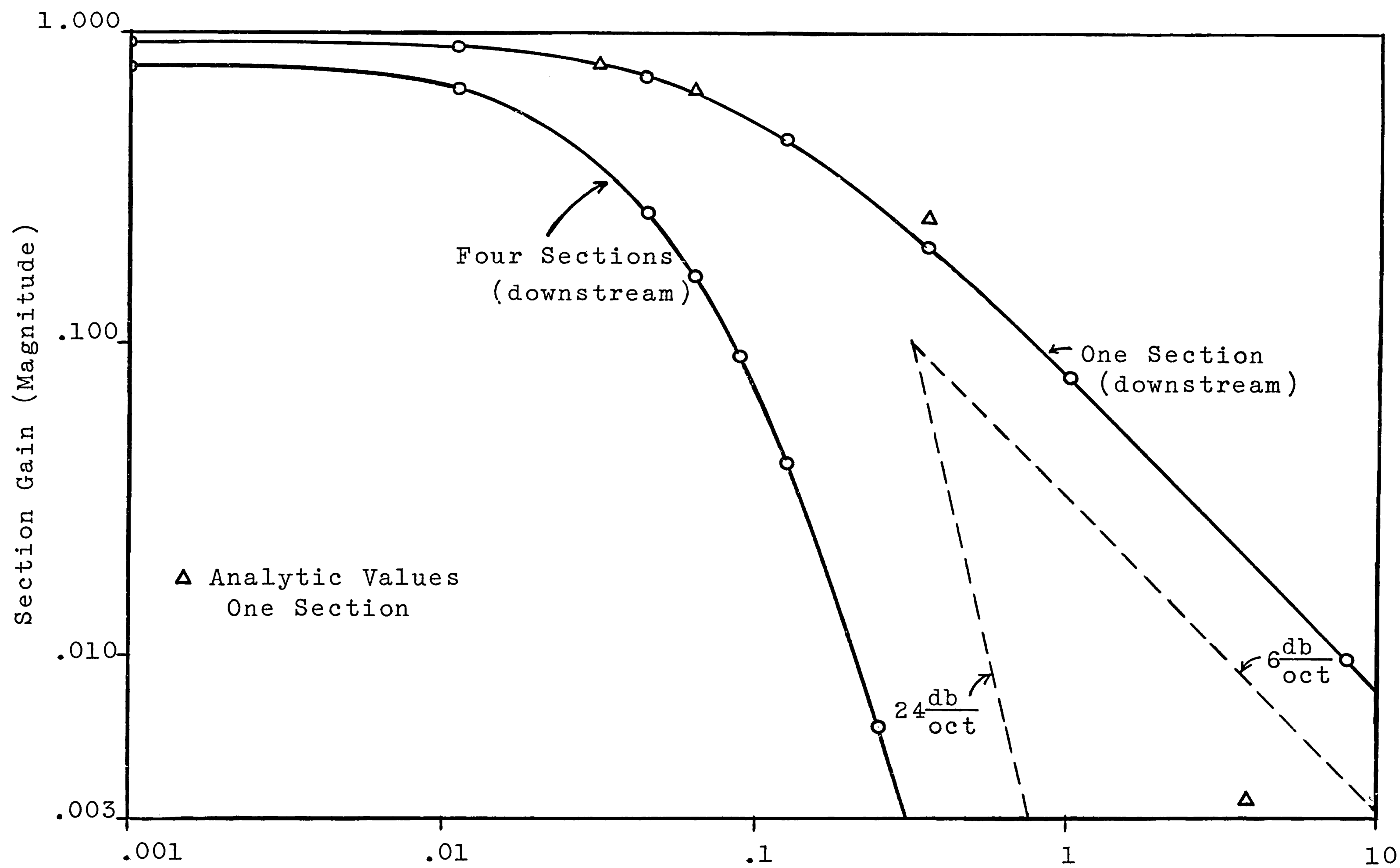
Thus this research has established a network model for the analysis of water pollution problems by ECAP, a method of demonstrated accuracy which is easier to apply and less expensive than other methods of pollution analysis, such as direct observation of an actual river, a scale model of a waterway, or computer programs specifically written for each problem. Most important, it is not limited strictly to analysis, but applies to prediction and synthesis as well. This method of easier analysis lays the groundwork for further research, both on the problems of water pollution and in other diverse fields.


Source Section
Fourth Section (downstream)
Eighth Section (downstream)
Concentration (Normalized to Source Section Steady State)
1.00
0.75
0.50
0.25
0
Time from Start of Step Response (Tide Cycles)
0
10
20
30
40
50

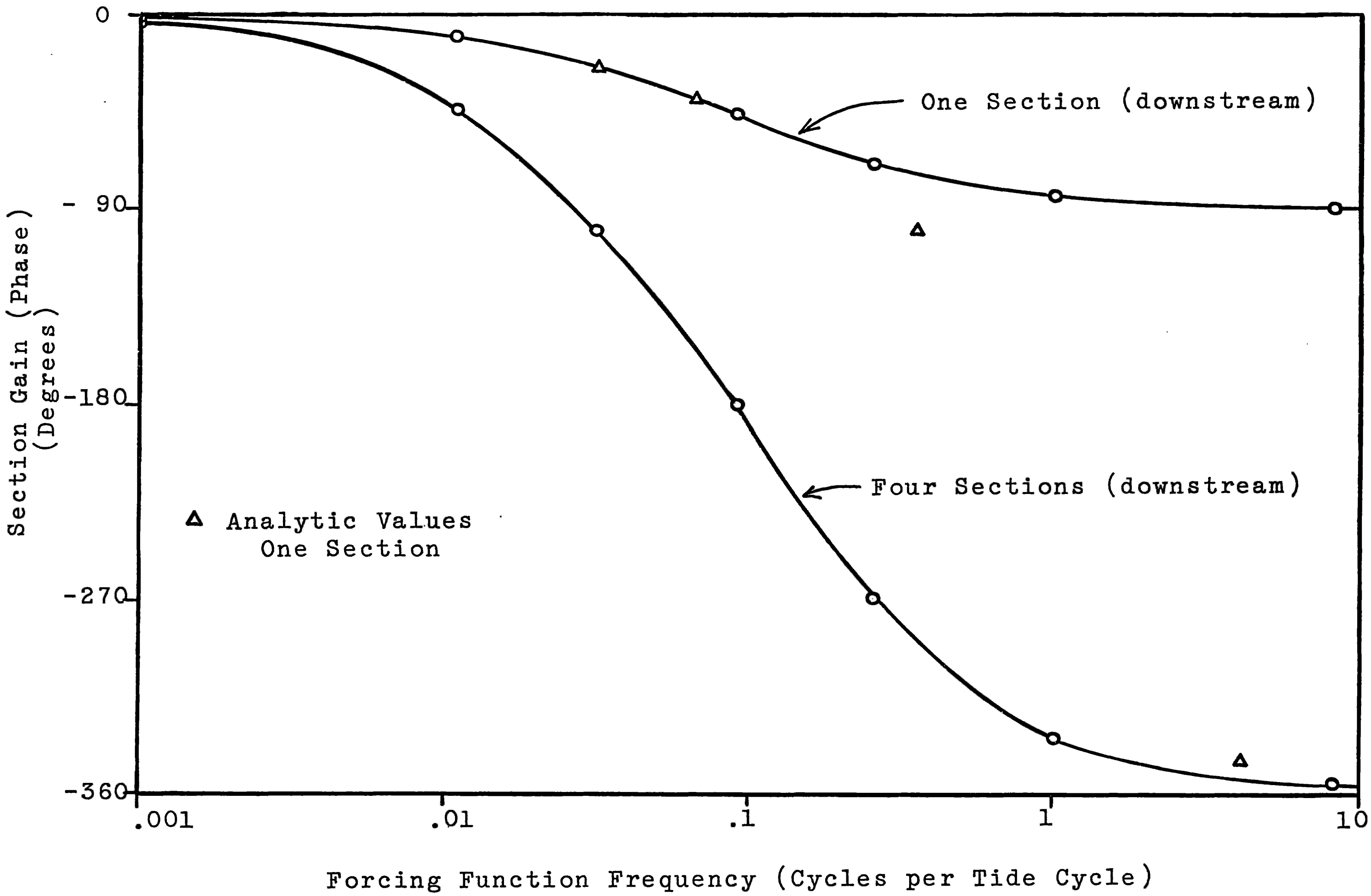
GRAPH 1


1.00
0.75
0.50
0.25
0
Concentration after 50 Tide Cycles
(Normalized to Source Section)
-10
0
10
20
25
Without
termination
error
Distance from Source Section (Sections)

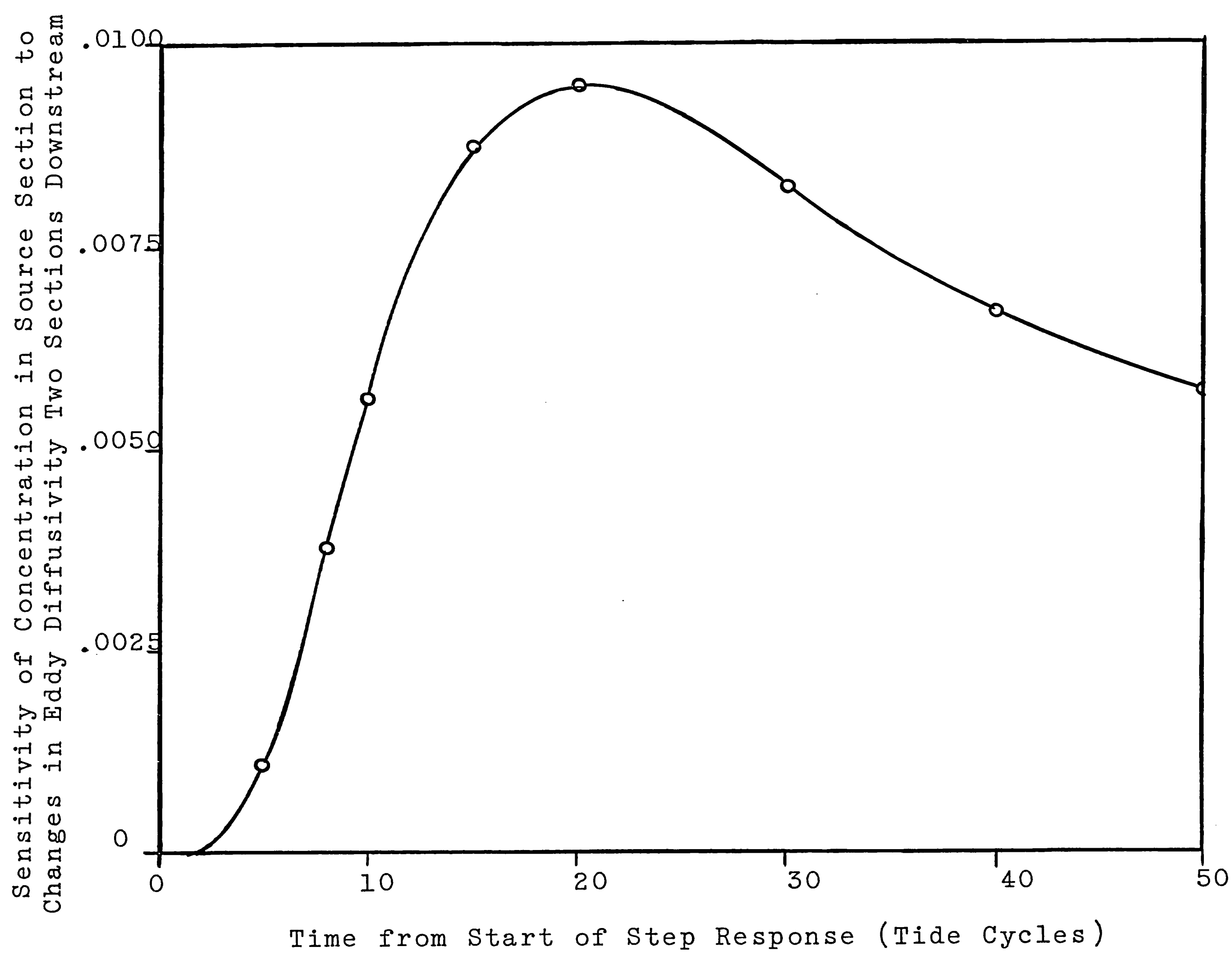
GRAPH 2


1.000
.100
.010
.003
Section Gain (Magnitude)
.001
.01
.1
1
10
Forcing Function Frequency (Cycles per Tide Cycle)
Four Sections
(downstream)
One Section
(downstream)
Analytic Values
One Section
24 db/oct
6 db/oct

GRAPH 3


0
- 90
-180
-270
-360
Section Gain (Phase)
(Degrees)
One Section (downstream)
Four Sections (downstream)
Analytic Values
One Section
.001
.01
.1
1
10
Forcing Function Frequency (Cycles per Tide Cycle)

GRAPH 4


Sensitivity of Concentration in Source Section to Changes in Eddy Diffusivity Two Sections Downstream
.0100
.0075
.0050
.0025
0
0
10
20
30
40
50
Time from Start of Step Response (Tide Cycles)

GRAPH 5

VII. BIBLIOGRAPHY

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