

RICE UNIVERSITY

Essays on Fair Division and Social Choice

by

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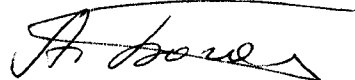
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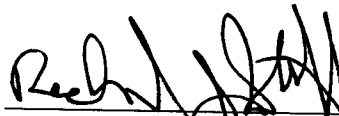
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ABSTRACT

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Sinan Ertemel

In my dissertation, I studied Social Choice and Fair Division problems under uncertainty. In the first chapter, I defined welfare egalitarianism in the form of certainty equivalence where the individuals are endowed with state contingent consumption bundles and provided an axiomatic characterization of this ordering by efficiency, equity and monotonicity axioms. In the second chapter, I introduced two natural extensions of the proportional rules on the rationing problem with state contingent claims and provided the characterization of those two rules by No Advantageous Reallocation, i.e. no group of agents can benefit from reallocating their claims amongst each other, which is defined across states or individuals, combined with some standard axioms in the literature. And finally in the last

chapter, I consider a class of resolute social choice correspondences and characterize the strong Nash equilibrium outcomes of their voting games in terms of a generalization of the Condorcet principle.

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Contents

ABSTRACT	ii
Acknowledgements	iv
List of Figures	viii
Chapter 1. Welfare Egalitarianism under Uncertainty	1
1.1. Introduction	1
1.2. Preliminaries	5
1.3. The Results	10
1.4. Conclusion	17
Chapter 2. Proportional Allocation Rules under Uncertainty	21
2.1. Introduction	21
2.2. Preliminaries	26
2.3. Characterizations	28
2.4. Conclusion	39
2.5. Appendix	40
Chapter 3. Voting Games of Resolute Social Choice Correspondences	42

3.1. Introduction	42
3.2. The Model	44
3.3. Characterizations	51
3.4. Sets as generalized Condorcet winners	57
3.5. Conclusion	69
References	71

List of Figures

Figure 1.1. CE Leximin Ordering.....	8
Figure 1.2. Proof of Lemma 6.....	13
Figure 1.3. Proof of Lemma 7.....	14
Figure 1.4. Proof of Theorem 8	16
Figure 2.1. Ex-ante vs. Ex-post Proportional Rule	28
Figure 2.2. The sum of the claims is constant.....	29
Figure 2.3. The ratio of the claims is constant.....	29
Figure 2.4. Mean preserving spread.....	37

CHAPTER 1

Welfare Egalitarianism under Uncertainty

1.1. Introduction

Consider an environment where individuals are endowed with state contingent consumption bundles. Our main motivation is to come up with an intuitive and fair method of aggregating individual preferences into a social preference in this risky environment. Harsanyi (1955)'s aggregation theorem shows that if individuals and social planners has expected utility consistent preferences, then the Pareto principle forces the social welfare to be affine with respect to individual utilities. This utilitarian form of social welfare is indifferent to the distribution of welfare which is a huge drawback in terms of social justice. To accommodate egalitarianism, one either takes ex-ante approach by relaxing rationality, i.e. Diamond (1967) or by taking ex-post approach by relaxing Pareto principle, i.e. Hammond (1983). In this chapter, by employing ordinal and noncomparable individual preferences, following Fair Social Choice Theory introduced by Fleurbaey and Maniquet (1996), we characterize an egalitarian social welfare ordering, that is, giving the priority to the worse-off.

Fair Social Choice Theory seeks Social Welfare Orderings for all possible allocations, not only efficient but also satisfying some fairness properties. It provides a crucial link between Social Choice and Fair Allocation Theory. It evaluates allocation of the resources by constructing social preferences from Social Choice Theory and borrows equity axioms from the Fair Allocation literature.¹ Arrovian Social Choice Theory is after defining social choice functions which gives a complete ranking over all the feasible allocations. On the other hand, fair allocation theory provides rules which give the optimal allocations, that is, it gives a two-tier social ordering, optimal and non-optimal ones. Fair Social Choice Theory takes social choice approach in the sense that it gives fine grained rankings. This approach has clear advantages if one is interested in the implementation problems, that is, sometimes policy maker has to choose among the non-optimal allocations due incentive constraints coming from asymmetric information, or status quo problems (for example, linear taxation).²

Arrovian Social Choice Theory showed the Independence of the Irrelevance axiom is quite incompatible with Pareto axioms. Eventhough Independence axiom brings informational simplicity, combined with Pareto axioms, it gives nondesirable (dictatorial) outcomes. For example Bordes and Le Breton (1989) showed that

¹For a more detailed treatment of fair allocation rules one can see Moulin and Thomson (1997) and Thomson (2013)

²One can see Maniquet and Sprumont (2006,2007 and 2011) for this second best approach in the optimal taxation problem.

under supersaturating preference domain Independence and Weak Pareto results in dictatorial outcomes. Fair Social Choice Theory aims to weaken the Independence axiom by replacing with equity axioms inspired by Fair Allocation Rules and comes up with the possibility results, mostly in the egalitarian sense.

Fair Social Choice Theory can also be considered as a welfarist approach, it provides a social welfare ordering from given individual welfare indices. The welfarist approach uses exogenous interpersonally comparable utility functions.³ Instead of taking exogenous welfare indices, Fair Social Choice Theory takes ordinal preferences and obtains interpersonal comparisons drawn by preferences over resources. This follows the idea by Rawls (1971), and Sen (1992) saying that utility comparisons involve value judgments and therefore it cannot be compared across individuals. And interpersonal comparisons should be based on resource metric. Furthermore fair social orderings literature differs from other models in the sense that it allows heterogeneous preferences. However mostly egalitarian aggregation methods are possible through this approach.

Fair Social Choice Theory provides a hierarchy in the normative criteria which is also followed in this chapter. Efficiency is seen as the first and foremost condition to be satisfied. Then various criteria of fairness are introduced. There is an efficiency-equality conflict in the sense that reducing inequalities in the resource

³Bossert and Weymark (2004) and d'Aspremont and Gevers (2002) are excellent surveys for characterizations of cardinal preferences.

does not necessarily lead to efficient outcomes.⁴Equity axioms are weakened until they capture some basic form of efficiency. Next, the robustness conditions are introduced. A robust allocation implies that social preference is independent of changes of some irrelevant parameters of the model. Efficiency and relevant equity conditions, combined with the robustness conditions, give us a set of acceptable social orderings.

In one public and one private good model, Moulin (1987a) defined egalitarian equivalent allocations by finding highest level of public good that is consumed for free which yields feasible utility distribution) by Pareto Optimality, Cost Monotonicity, Individual Rationality and No Private Transfers (no agent receives positive amount of private good). Maniquet and Sprumont (2004) used an alternative approach, that is, fair social orderings. They defined welfare egalitarianism in the economies with one private good and one partially excludable nonrival good. First they define an individual's welfare as the amount of nonrival good which leaves him indifferent to his initial consumption bundle. They then ranked these bundles by the leximin criterion and characterized the maximin ordering by Unanimous Indifference, Responsiveness, and Free Lunch Aversion axioms. Maximizing this social ordering with respect to technological constraints gives exactly the same public good level proposed by Moulin (1987a). This chapter can be regarded as an extension of Maniquet and Sprumont (2004) to economies with state contingent

⁴On the full domain, no social choice function satisfies Pigou-Dalton principle and weak Pareto. See Fleurbaey and Maniquet (2011).

endowment vectors. The natural way of defining welfare in this framework is the "riskless" allocation, e.g. certainty equivalent allocation. The main contribution of this chapter can be seen as defining an equity criterion ensuring some form of aversion to income inequality where inequality is defined as two individuals being affected from an event in opposite directions. One can find this axiom quite compelling for some catastrophic events, such as natural disasters (earthquake, hurricane, etc.), where it is socially undesirable for some individuals to benefit from that event at the expense of others. This axiom, combined with efficiency and robustness conditions, leads to a social ordering with an infinite aversion to inequality – a maximin ordering.

The rest of the chapter is organized as follows. In Section 2 the axioms and the model are introduced. The results are stated in Section 3. Section 4 concludes with possible directions for future research.

1.2. Preliminaries

Consider a finite set of individuals N with $|N| \geq 2$. S is a finite set of distinct states of nature, with $|S| \geq 2$. $\Omega \in (\mathbb{R}_+^S)^N$ denotes the *social endowment* of the state contingent goods. Consumption of individual $i \in N$ at state $s \in S$ is denoted as $z_{is} \in \mathbb{R}_+$. $R_i \in \mathcal{R}$ is ex-ante and state independent preference of individual $i \in N$ which is a binary relation over state contingent goods, that is complete, transitive,

convex, continuous, and strictly increasing in each state contingent good. *Social preference profile* is denoted as $R = (R_i)_{i \in N} \in \mathcal{R}^N$. An *economy* is defined as a quadruple $E = (N, S, \Omega, R) \in \mathcal{E}$. An *allocation* is a vector of $z_N = (z_i)_{i \in N} \in (\mathbb{R}_+^S)^N$. An allocation is feasible if $\sum z_i \leq \Omega$. The set of feasible allocations is denoted as $Z(E)$. *Upper contour set* of R_i at z_i is denoted as $B(R_i, z_i) = \{z'_i \in \mathbb{R}_+^S \mid z'_i R_i z_i\}$. *Social ordering function* \mathbf{R} assigns a binary and transitive ranking for all $E \in \mathcal{E}$, e.g. $z_N \mathbf{R}(E) z'_N$ means allocation z_N is socially preferred to z'_N . $\mathbf{I}(E)$ and $\mathbf{P}(E)$ are defined as counterparts for social indifference and social strict preference respectively.

Next, we will define the notion of Certainty Equivalent Egalitarianism. Individual welfare levels are measured on the certainty ray, that is the sure allocation that leaves an individual indifferent to his original allocation. For the sake of exposition, throughout the chapter, we will provide our results for two states.⁵ State contingent endowment of individual i is denoted as $z_i = (x_i, y_i) \in \mathbb{R}_+^2$ where x_i denotes individual i 's endowment for state 1 and y_i denotes individual i 's endowment for state 2. Certainty Equivalent welfare level of agent $i \in N$ with a preference relation R_i at the allocation z_i is given as $c_i \in \mathbb{R}_{++}$ where $z_i I_i(c_i, c_i)$. Then, social preference is found by applying leximin ordering to the individual welfare levels. We will provide three axioms that would provide a characterization of this particular maximin ordering. First, Unanimous Indifference condition says

⁵This is by no means a simplification as the results follow for any S as any $S - 1$ states can be represented as a projection to one state.

that two allocations that leave all the individuals indifferent should be deemed socially equivalent. This is a weaker condition than Pareto, and it is clearly satisfied by Certainty Equivalent Leximin ordering. The Responsiveness condition ensures that social ordering is preserved if better sets for all individuals shrink for the better allocation, and they expand for the worse allocation. And finally, Aversion to Attendant Gains is the equity condition requiring a transfer between two agents as a social improvement, as long as they have the same endowment under one event and the transfer is done under the event in which the endowment of two agents lie on the opposite sides of the certainty equivalent line provided that their orientation with respect to certainty ray does not change after transfer. Figure 1.1 illustrates how Certainty Equivalent Leximin ordering satisfies the Aversion to Attendant Gains condition. By Unanimous Indifference, one can move along the indifference curve such that $(z_1, z_2)\mathbf{I}(E)(\bar{z}_1, \bar{z}_2)$. And by Aversion to Attendant Gains, we have $(z'_1, z'_2)\mathbf{R}(E)(\bar{z}_1, \bar{z}_2)$ as $\min(c'_i, c'_j) = c'_i > c_i = \min(c_i, c_j)$.

Now, we will turn to the formal model. The first axiom captures the minimum efficiency condition. Unanimous Indifference requires social preferences to agree with individual preferences, e.g. if all agents are indifferent to two different bundles then social preference agrees with it. This axiom is weaker than the Pareto principle. In the next section, we will show that this axiom, combined with the Responsiveness and Aversion to Attendant Gains axioms, will give Unanimous Preference and Unanimous Strict Preference.

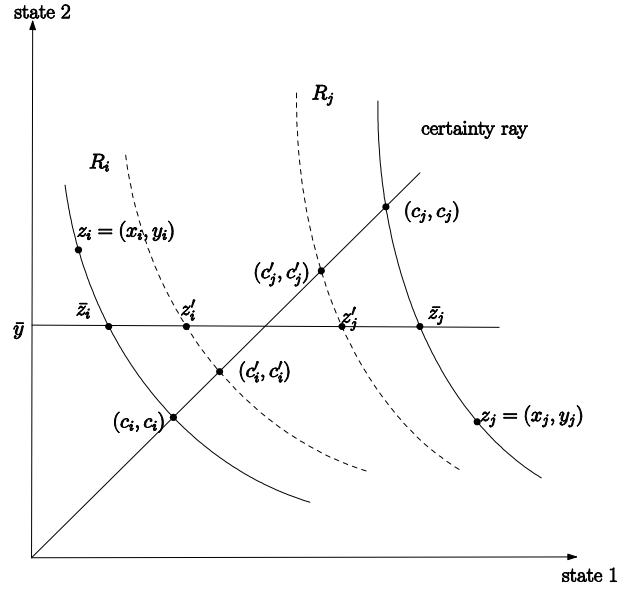


Figure 1.1. CE Leximin ordering satisfies AAG.

Definition 1. Unanimous Indifference (UI): Let $E = (N, S, \Omega, R) \in \mathcal{E}$ be given. Let $z_N, z'_N \in Z(E)$, If $z_i I_i z'_i$, for all $i \in N$, then $z_N \mathbf{I}(E) z'_N$

Now, we will define an equity criterion relevant to our framework which is inspired by Free Lunch Aversion Axiom introduced by Maniquet and Sprumont (2004). It is a fairly minimal inequality aversion condition whose ethical justification was presented in the introduction. Aversion to Attendant Gains condition says that if two individuals face the risk of one unexpected event in opposite directions, then reducing the gap of that risk by transfer improves social welfare, provided that the orientation with respect to certainty ray would not change after transfer.

This axiom is clearly weaker than Pigou-Dalton transfer which contradicts with the efficiency.⁶

Definition 2. Aversion to the Attendant Gains (AAG) with respect to state s : Let $E = (N, S, \Omega, R) \in \mathcal{E}$ be given. Let $z_N, z'_N \in Z(E)$ such that there exist $s \in S$ and $i, j \in N$ with $z_{is} = z_{js}$ and there exist $t \in S$ and $\Delta > 0$ such that $z_{it} < z_{it} + \Delta = z'_{it} < z_{is} = z_{js} < z'_{jt} = z_{jt} - \Delta < z_{jt}$ and $z_{ks} = z'_{ks}$ for all $k \neq i, j$ and for all $s \in S$. Then $z'_N \mathbf{P}(E) z_N$.

The third axiom presents the robustness condition which can also be seen as an independence axiom. It is borrowed from Fleurbaey and Maniquet (1996). Say an allocation z_N is socially preferred to another allocation z'_N . The Responsiveness condition ensures that social preference is preserved if better sets of all the individuals shrink for the "better" allocation and they shrink for the "worse" allocation.

Definition 3. Responsiveness (R): Let $E = (N, A, \Omega, R) \in \mathcal{E}$ and $E' = (N, A, \Omega, R') \in \mathcal{E}$ be given. Let $z_N, z'_N \in Z(E)$. Let $B(R'_i, z_i) \subseteq B(R_i, z_i)$ and $B(R'_i, z'_i) \supseteq B(R_i, z'_i)$ for all $i \in N$, then $\{z_N \mathbf{R}(E) z'_N\} \Rightarrow \{z_N \mathbf{R}(E') z'_N\}$ and $\{z_N \mathbf{P}(E) z'_N\} \Rightarrow \{z_N \mathbf{P}(E') z'_N\}$

⁶See Theorem 2.1. Fleurbaey and Maniquet (2011).

1.3. The Results

Before stating our results, we will formally define Certainty Equivalent Welfare Ordering. For each $R_i \in \mathcal{R}$ and for each $z_i \in \mathbb{R}_+^S$, there is a unique level of $c(R_i, z_i) \in \mathbb{R}_+$ such that $z_i I_i c(R_i, z_i) \mathbf{1}_s$ where $\mathbf{1}_s = (1, \dots, 1) \in \mathbb{R}_+^S$. Certainty equivalent welfare level of individual i with preference profile R at z_i is denoted by $c(R_i, z_i)$. A social ordering is in the form of certainty equivalent maximin, if the ordering of two social allocations are obtained according to the maximin ordering of certainty equivalent welfare levels. That is, for any $R \in \mathcal{R}^N$ and for any $z_N, z'_N \in (\mathbb{R}_+^S)^N$

$$\min_{i \in N} c(R_i, z_i) > \min_{i \in N} c(R_i, z'_i) \implies z_N \mathbf{P}(E) z'_N$$

Leximin ordering is the eminent example of the maximin ordering. Let \succsim_{lex} denote the usual leximin ordering⁷ on $(\mathbb{R}_+^S)^N$. Certainty Equivalent Welfare Leximin Ordering \mathbf{R}^L ranks the vectors of certainty equivalent welfare levels by applying leximin ordering. For any $R \in \mathcal{R}^N$ and for any $z_N, z'_N \in (\mathbb{R}_+^S)^N$

$$z_N \mathbf{R}^L(E) z'_N \iff (c(R_i, z_i))_{i \in N} \succsim_{lex} (c(R_i, z'_i))_{i \in N}$$

⁷For two vectors $u_N, v_N \in \mathbb{R}_+^N$, we have $u_N \succsim_{lex} v_N$ if the smallest component of u_N is larger than v_N . If they are equal the next smallest component is compared, and so on.

Before going into our characterization theorem, we will state two lemmas. It is important to note that Unanimous Indifference is a fairly minimal condition of efficiency. The next two lemmas show that stronger efficiency criteria, such as Unanimous Preference and Unanimous Strict Preference, could be obtained by adding Responsiveness and Aversion to the Attendant Gains conditions.

Definition 4. Unanimous Preference (UP): *Let $E = (N, S, \Omega, R) \in \mathcal{E}$ be given. Let $z_N, z'_N \in Z(E)$. If $z_i R_i z'_i$, for all $i \in N$, then $z_N \mathbf{R}(E) z'_N$.*

Definition 5. Unanimous Strict Preference (USP): *Let $E = (N, S, \Omega, R) \in \mathcal{E}$ be given. Let $z_N, z'_N \in Z(E)$. If $z_i P_i z'_i$, for all $i \in N$, then $z_N \mathbf{P}(E) z'_N$.*

The following results are mostly adapted from Maniquet and Sprumont (2004).

Lemma 6. *If a social ordering satisfies Unanimous Indifference and Responsiveness, then it satisfies Unanimous Preference.*

Proof. Suppose \mathbf{R} satisfies Unanimous Indifference and Responsiveness. To get a contradiction, assume that \mathbf{R} fails Unanimous Preference. That is, there exist $R \in \mathcal{R}^N$ and two social allocations $z_N^1, z_N^2 \in Z(E)$ with $z_N^1 \mathbf{P}(E) z_N^2$ and there

exists $M \subseteq N$ such that $z_i^2 P_i z_i^1$, for all $i \in M$ and $z_j^2 I_j z_j^1$, for all $j \in N \setminus M$. Without loss of generality assume that $M = \{i\}$.⁸

As shown in Figure 1.2, choose z_i^3 such that $z_i^3 I_i z_i^1$ and $y_i^3 > y_i^1, y_i^2$. Let C be the convex hull of $\{(x_i, y_i) \in B(R_i, z_i^1) \mid y_i^1 \geq y_i^3\} \cup B(R_i, z_i^2)$ and let $\partial C = \{(x_i, y_i) \in C \mid ((x'_i, y'_i) = (x_i, y_i), \text{ for all } (x_i, y_i) \in C \text{ such that } x'_i \leq x_i \text{ and } y'_i \leq y_i)\}$. So, there exists $z_i^4 \in \partial C$ such that $z_i^4 I_i z_i^2$. By Unanimous Indifference, $(z_i^3, z_{-i}^1) \mathbf{P}(E)(z_i^4, z_{-i}^2)$. Now we can construct $R'_i \in \mathfrak{R}$ such that $B(R'_i, z_i^3) = C$. By continuity and strict monotonicity of the preferences there exists $z_i^4 \in \partial C$ such that $z_i^4 I'_i z_i^3$. Since $B(R'_i, z_i^3) \subseteq B(R_i, z_i^3)$ and $B(R'_i, z_i^4) \supseteq B(R_i, z_i^4)$, by Responsiveness we get $(z_i^3, z_{-i}^1) \mathbf{P}(E')(z_i^4, z_{-i}^2)$, which contradicts with the Unanimous Indifference. □

Lemma 7. *If a social ordering satisfies Unanimous Preference and Aversion to the Attendant Gains, then it satisfies Unanimous Strict Preference.*

Proof. Suppose \mathbf{R} satisfies Unanimous Preference and Aversion to the Attendant Gains. To get a contradiction, assume that \mathbf{R} fails Unanimous Strict Preference. That is, there exist $R \in \mathcal{R}^N$ and two social allocations $z_N, \tilde{z}_N \in Z(E)$ with $z_N \mathbf{R}(E) \tilde{z}_N$ such that $\tilde{z}_i P_i z_i$ for all $i \in N$. Without loss of generality, assume that

⁸For $|M| \geq 2$, construct a sequence of $\{z(t)\}_{t=0}^{t=|N|}$ where $z_j(t) = z_j^2$ for $j \leq t$ and z_j^1 otherwise. Because R is transitive, there exists some $t \in \{1, \dots, |N|\}$ such that $z(t-1) \mathbf{P}(R) z(t)$.

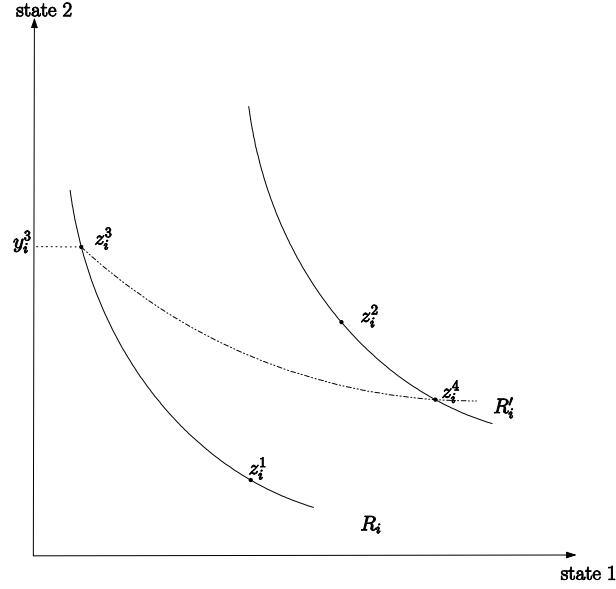


Figure 1.2. UI and R implies UP.

$c(R_1, z_1) \geq c(R_i, z_i)$, for all $i \in N$. Therefore $c(R_1, \tilde{z}_1) \geq c(R_i, z_i)$, for all $i \in N$. As shown in Figure 1.3, we can choose $\hat{z}_1 = (\hat{x}_1, \bar{y})$ and $\hat{z}_2 = (\hat{x}_2, \bar{y})$.

Then there exists $\Delta > 0$ such that $\hat{x}_2 + \Delta \leq y \leq \hat{x}_1 - \Delta$ and $(\hat{x}_1, y)P_1(\hat{x}_1 - \Delta, y)$ and $(\hat{x}_2 + \Delta, y)P_2(\hat{x}_2, y)$.

By Aversion to the Attendant Gains, $((\hat{x}_1 - \Delta, y), (\hat{x}_2 + \Delta, y), z_{-12})\mathbf{P}(E)((\hat{x}_1, y), (\hat{x}_2, y), z_{-12})$.

By Unanimous Indifference, $((\hat{x}_1, y), (\hat{x}_2, y), z_{-12})\mathbf{I}(E)(\tilde{z}_1, z_2, z_{-12})$.

And by Unanimous Preference $(\tilde{z}_1, z_2, z_{-12})\mathbf{R}(E)(z_1, z_2, z_{-12})$.

Since $z_N\mathbf{R}(E)\tilde{z}_N$ we get $((\hat{x}_1 - \Delta, y), (\hat{x}_2 + \Delta, y), z_{-12})\mathbf{P}(E)(\tilde{z}_1, \tilde{z}_2, \tilde{z}_{-12})$, which contradicts with the Unanimous Preference. \square

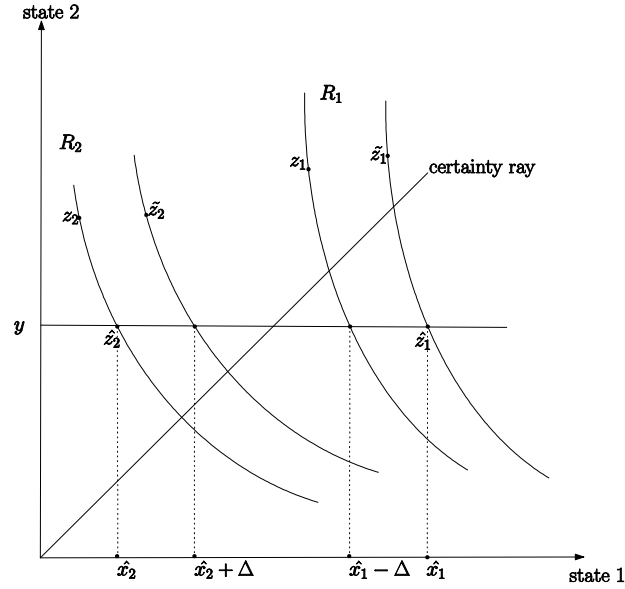


Figure 1.3. UR and AAG implies USP.

The previous two lemmas show that social preferences follow, not only for indifference of individual preferences, but also follow for weak and strict preferences. Now we are ready to state our main characterization theorem.

Theorem 8. *The Certainty Equivalent Leximin ordering \mathbf{R}^L satisfies Unanimous Indifference, Responsiveness and Aversion to Attendant Gains. Conversely, every social ordering \mathbf{R} satisfying Unanimous Indifference, Responsiveness and Aversion to Attendant Gains is in the form of certainty equivalent maximin.*

Proof. First we will show that Certainty Equivalent Leximin ordering \mathbf{R}^L satisfies Unanimous Indifference, Responsiveness and Aversion to the Attendant Gains.

Let $R \in \mathcal{R}^N$ and $z_N, z'_N \in Z(E)$ such that $z_i I_i z'_i$ for all $i \in N$. So $c(R_i, z_i) = c(R_i, z'_i)$ for all $i \in N$. Therefore $z_N \mathbf{I}(E) z'_N$. So Unanimous Indifference holds.

To show that Responsiveness is satisfied assume that $z_N \mathbf{R}(E) z'_N$ with $B(R'_i, z_i) \subseteq B(R_i, z_i)$ and $B(R'_i, z'_i) \supseteq B(R_i, z'_i)$ for all $i \in N$. Then $c(R'_i, z_i) \geq c(R_i, z_i)$ and $c(R'_i, z'_i) \leq c(R_i, z'_i)$, for all $i \in N$. So $z_N \mathbf{R}(E') z'_N$. Hence Responsiveness holds.

And to check Aversion to the Attendant Gains, let $i, j \in N$ and assume that $z_i = (x_i, y)$; $z_j = (x_j, y)$ where $x_i > y$ and $x_j < y$ and $x_j < x'_j = x_j + \Delta \leq y \leq x_i - \Delta = x'_i < x_i$. Further assume that $z_{-ij} = z'_{-ij}$.

Then $c(R_i, (x'_i, y)) < c(R_i, z_i)$ and $c(R_j, (x'_j, y)) > c(R_j, z_j)$

So $(c(R_i, z'_i))_{i \in N} \succ_{lex} (c(R_i, z_i))_{i \in N}$ which implies $z' \mathbf{P}(E) z$. Thus Aversion to the Attendant Gains holds as well.

Now we will prove that a social ordering satisfying Unanimous Indifference, Responsiveness and Aversion to the Attendant Gains has to be in the form of certainty equivalent maximin.

To get a contradiction, suppose that there exists $R \in \mathcal{R}^N$ and $z_N, z'_N \in Z(E)$ such that $\min_{i \in N} c(R_i, z_i) < \min_{i \in N} c(R_i, z'_i)$ yet $z_N \mathbf{R}(E) z'_N$.

So $c(R_i, z_i) \leq \min_{k \in N} c(R_k, z'_k) \leq c(R_j, z_j)$ for all $i \in M$ and for all $j \in N \setminus M$.

Since $z_N \mathbf{R}(E) z'_N$ we have $|M| > 0$. And we have $|M| < |N|$ as $|M| = |N|$ contradicts with the Unanimous Strict Preference. Take $|M'| = |M| + 1$ and construct $R' \in \mathcal{R}^N$ such that $c(R'_i, q_i) < \min_{k \in N} c(R'_k, q_k) \leq c(R'_j, q_j)$ for all $i \in M$ and for all $j \in N \setminus M'$ and $q_N \mathbf{R}(E) q'_N$.

By repeating this construction $|N| - |M|$ times, we get a contradiction with the Unanimous Strict Preference.

Without loss of generality, we will take $1 \in M$, $2 \in N \setminus M$ and assume that $c(R_1, z_1) < c(R_2, z'_2) = \min_{k \in N} c(R_k, z'_k) < c(R_1, z'_1) < c(R_2, z_2)$.

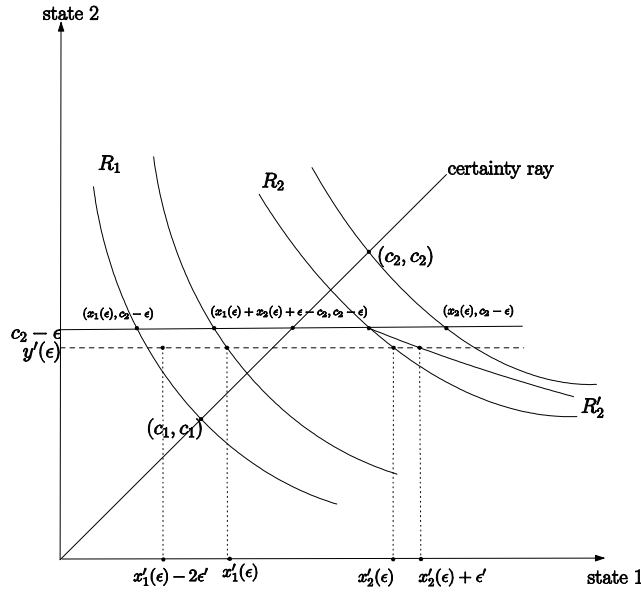


Figure 1.4. UI, R, and AAG forces CE Maximin ordering

So $((c_1, c_1), (c_2, c_2), z_{-12}) \mathbf{R}(E) ((c'_1, c'_1), (c'_2, c'_2), z'_{-12})$. As shown in Figure 1.4, by continuity and strict monotonicity, there exists $\epsilon > 0$ such that $x_1(\epsilon) < c_2 - \epsilon$ and $x_2(\epsilon) > c_2 - \epsilon$ which ensures $(x_1(\epsilon), c_2 - \epsilon) I_1(c_1, c_1)$ and $(x_2(\epsilon), c_2 - \epsilon) I_2(c_2, c_2)$ and $x_1(\epsilon) + x_2(\epsilon) < c_2 - \epsilon$. Then, there exist $y'(\epsilon) > y(\epsilon)$ and $x'_1(\epsilon) < y'(\epsilon)$ and $x'_2(\epsilon) > y'(\epsilon)$ which implies $(x'_1(\epsilon), y'(\epsilon)) I_1(x_1(\epsilon) + x_2(\epsilon) + \epsilon - c_2, c_2 - \epsilon)$ and $(x'_2(\epsilon), y'(\epsilon)) I_2(c'_2, c'_2)$ and $c_1 < y(\epsilon) < y'(\epsilon) < c'_2$

Now, we will choose $\varepsilon' > 0$ small enough to ensure that $(c_2, c_2)P_2(x'_2(\varepsilon) + \varepsilon', y'(\varepsilon))$. Construct a preference $R'_2 \in \mathcal{R}$ such that $B(R'_2, (c'_2, c'_2)) = B(R_2, (c'_2, c'_2), (x'_2(\varepsilon) + \varepsilon', y'(\varepsilon)))I'_2(c_2 - \varepsilon, c_2 - \varepsilon), (x_2(\varepsilon), c_2 - \varepsilon)I'_2(c_2, c_2)$.

Let $R'_i = R_i$, for all $i \in N \setminus \{2\}$ and let $q = ((x'_1(\varepsilon) + 2\varepsilon', y'(\varepsilon)), (x'_2(\varepsilon) - \varepsilon', y'(\varepsilon)), z_{-12})$.

$$q_N \mathbf{P}(E')((x'_1(\varepsilon) + y'(\varepsilon)), (x'_2(\varepsilon) + \varepsilon', y'(\varepsilon)), z_{-12})$$

$$\mathbf{I}(E')((x_1(\varepsilon) + x_2(\varepsilon) + c_2 - \varepsilon, c_2 - \varepsilon), (c_2 - \varepsilon, c_2 - \varepsilon), z_{-12})$$

$$\mathbf{P}(E')((x_1(\varepsilon), c_2 - \varepsilon), (x_2(\varepsilon), c_2 - \varepsilon), z_{-12})$$

$$\mathbf{I}(E')((c_1, c_1), (c_2, c_2), z_{-12})$$

$\mathbf{R}(E')((c'_1, c'_1), (c'_2, c'_2), z_{-12}) = q'_N$ by applying Aversion to Attendant Gains, Unanimous Indifference, Aversion to Attendant Gains, Unanimous Indifference and Responsiveness respectively.

Now, take $M' = M \cup \{2\}$ and repeat these steps until you get contradiction. \square

1.4. Conclusion

In this chapter, we provide an axiomatic characterization of welfare egalitarianism defined by the certainty equivalence form. The equity condition formulated by the Aversion to the Attendant Gains axiom, which is a fairly minimal condition combined with Unanimous Indifference and Responsiveness, leads to an ordering which gives absolute priority to the worse off, that is, infinite aversion to inequality. By making use of ordinal and noncomparable preferences, and providing social

orderings for all the possible preference profiles, this model is quite rich for policy analysis which seeks to recommend second best allocations. For problems in which the policy maker has imperfect information on the individuals who are bounded by incentive constraints, the efficient allocations might not be implementable. Social welfare ordering defined in this chapter can give the second best allocations by maximizing this ordering, subject to the relative constraints defined by that particular problem, e.g. status quo, incentive constraints, etc. One can take any other reference bundle than the certainty ray. For example in the standard model, total endowment vector is meaningful with the fairness criterion like equal-split.

Certainty Equivalent Leximin ordering defined in this chapter can also be seen as a contribution to the welfarist approach. It differs from the classical characterizations which are defined for cardinal and comparable preferences. Those models define indices of the welfare exogenously. On the other hand, Certainty Equivalent Leximin ordering utilizes ordinal and noncomparable preferences and defines the welfare by a fairness condition specific to the model itself.

There are various resource equality axioms in the fair allocations literature such as Equal Split Transfer, Proportional Allocations Transfer, Equal Split Allocation, Transfer among Equals, and Nested Contour Transfer. One can clearly see that Certainty Equivalent Leximin Ordering satisfies all of these axioms. One axiom

stands out here in the state contingent endowment framework: Proportional Allocations transfer in which proportionality is defined on the certainty ray. This axiom is clearly weaker than the Aversion to the Attendant Gains axiom. It is an interesting problem to study other robustness conditions weaker than Responsiveness, so that it forces social ordering to be in leximin form combined with Unanimous Indifference and Proportional Allocations transfer.

Here we studied the full domain of preferences. In decision theory, it is very practical to restrict the domain to additively separable preferences, i.e. expected utility consistent preferences. Moreover in this restricted domain the certainty equivalence becomes a stronger benchmark as all redistributions of wealth even the risky transfers satisfy Pareto efficiency. However Responsiveness axiom loses much of its bite in this domain because knowing indifference curves of expected utility maximizers does not provide much information for the rest of the indifference map. One can conjecture that by introducing stronger Responsiveness condition or introducing another transfer axiom, i.e. certainty transfer, one can extend our characterization to this restricted domain, as shown in Fleurbaey and Maniquet (2011) with a different set of axioms.

Social ordering in the leximin form can be seen as strongly egalitarian, i.e. giving absolute priority to the worse off. There are other social ordering functions in the literature relaxing this strong form of egalitarianism. One example is the

Nash-product social welfare function instead of the leximin criterion. This social ordering satisfies Pareto in the strong sense and the Proportional Allocations Transfer, but not the rest of the aforementioned transfer axioms. For future research, one can study possible characterization of Nash-product maximin ordering with appropriate robustness conditions.

CHAPTER 2

Proportional Allocation Rules under Uncertainty

2.1. Introduction

Rationing problem is arguably the simplest model of the distributive justice. The problem involves a resource to be divided among a number of agents who submit claims for the resource. Rationing is required when the sum of the claims is larger than the resource, with typical examples being bankruptcy, taxation, inheritance, etc. Perhaps the problem of rationing is as old as the history of civilization itself; and one can find documentation of such problems in ancient texts such as Talmud, Aristotle's books, etc. The very first formal analysis to the rationing problem was presented by O'Neill (1982) where he interprets the resource as "inheritance". Aumann and Maschler (1985) provides a rule from Talmud in the "bankruptcy" context. Young (1987a) characterizes a class of parametric rules in the "taxation" problem.

The problem of rationing concerns more of ethical or normative issues since market or traditional institutions can not convincingly provide a way out. For this reason adopting axiomatic approach has been the focus of the literature on rationing. Probably the most natural rule in this context arises from Aristotle's

maxim, "Equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences" from *Nicomachean Ethics*. Proportional rule gives shares in proportion to claims. There are various normative treatments of the proportional rule, such as O'Neill (1982), Moulin (1987b), Chun (1988), Young (1988), and Ju et al. (2007), etc. Two other rules central in the literature exhibit some form of egalitarianism. Uniform Gains rule equalizes the shares such as the shares do not exceed the claims. And Uniform Losses rule equalizes the losses (difference between claim and share) as much as possible. One can refer to some axiomatic characterizations of egalitarian rules in Dagan (1996), Herrero and Villar (2001), Sprumont (1991), etc. Furthermore Young (1987b) introduced another interesting family of rules called the "equal sacrifice" rules. One can see Moulin (2002) and Thomson (2003) for a survey of axiomatic characterization of the rationing rules.

We consider the rationing problems where the claims are state contingent. In stage one, each agent submits a claim for every possible state of the world. The realization of state happens in stage two. A rule must distribute the resources in the stage one i.e., before the realization of the state of the world. Such a situation may arise, for instance, in the allocation of fiscal budget of a country. Different departments of the government may require different amounts based on the state of the world to be realized in the coming fiscal year. For example, the Department of Defense may have different requirements depending on its relations

with the neighbouring countries in the following year. Department of Agriculture have requirements based on factors like the rainfall in the following year. The Department of Health may have requirements that depend on factors like incidence of epidemic, weather, etc. However, the federal budget must be allocated at the beginning of the fiscal year. Another example of our setting is the distribution of research funds (or travel grants) among the graduate students of a department in a university who expect travels or research expenses based on the state of the world (e.g., expenses based on the results of her research, travels based on the conferences accepting her paper, etc.). A situation like our setting also arises in the allocation of university funds among different departments based on their performance/need, or NSF funds to researchers from various universities, etc.

This natural framework of rationing problem has not been given much consideration in the literature. A fairly close setting called multi-issue allocations (MIA), introduced in Calleja et al. (2005), has been studied in the literature. Bergantiños et al. (2011), and Lorenzo et al. (2009) provide several axiomatic characterizations of Uniform Gains and Uniform Losses rules in MIA whereas Moreno-Ternero (2009), and Bergantiños et al. (2010) provide axiomatic characterizations of Proportional rule in MIA. The MIA framework does not consider uncertainty. A similar framework to ours that considers uncertainty has been studied by Habis and Herings (2013). They are interested in checking the stability¹ of the stochas-

tic extensions of various rationing rules and show that the only stable rule is the stochastic extension of the Uniform Gains.

In our two-stage framework where the agents submit their claims in the first stage and uncertainty is resolved in the second stage, the resource must be allocated in the first stage. Two particularly natural approaches in such situations arise. The first one is to apply a rationing rule on the expectation of the claims, which we call ex-ante rationing rule. The other approach is to consider the expectation of the allocations (by a rule) corresponding to the various state contingent claims, which we call ex-post rationing rule. In this chapter, we will focus our attention to proportional rules and characterize both ex-ante and ex-post proportional rules. Our axiomatic characterizations are based on No Advantageous Reallocation axiom introduced by Moulin (1985). This axiom states that no group of agents can benefit from reallocating their claims amongst them. We extend this concept to our state contingent framework and introduce two nonmanipulability conditions. The first extension which we call No Advantageous Reallocation across Individuals (*NARAI*) requires that no group of agents benefits if transfers are allowed within a state. The next extension considers transfers across states which we call No Advantageous Reallocation across States (*NARAS*). We also use the axioms of Anonymity (*AN*), Symmetry (*SYM*), Continuity (*CONT*), No Award for Null

¹They used a notion that they call "Weak Sequential Core" as the stability criterion.

(*NAN*), and Independence (*IND*). *AN* says that the rule should not distinguish based on the names of the agents and *SYM* requires that the names of the states do not matter. *CONT* states that the rule should be a continuous function in its arguments and *NAN* says that agents with zero claims in all states should be allocated zero amount. *IND* says that if we mix two lotteries² with a third one, then the allocation rule associated with these two mixed lotteries does not depend on the third lottery used. We show that ex-ante proportional rule is the only rule satisfying *AN*, *SYM*, *CONT*, *NAN*, *NARAI*, and *NARAS* whereas ex-post proportional rule is characterized by *AN*, *SYM*, *CONT*, *NAN*, *NARAI*, and *IND*.

Another interesting aspect of this problem is to compare the shares allocated by the ex-ante and ex-post proportional rules. In the appendix, we do the comparison for the ex-ante and ex-post allocations by the proportional rules for various distributions of claims and find sufficient conditions under which a particular agent will be favoured by one approach compared to the other. Section 2 provides the preliminaries. Section 3 gives the characterization results, and section 4 concludes with some directions for future research.

²By lottery we mean probability distribution over states of the world to be realized in the stage two.

2.2. Preliminaries

In the state contingent claims' framework, a rationing problem is a tuple (N, S, x, p, t) where N is a finite set of agents and S is a finite set of the states of the world.³ The state contingent claim matrix $x \in \mathbb{R}_+^{N \times S}$ represents the claims of agents in various states, where x_{is} denotes the claim of agent i in state s . The probabilities of states is denoted by $p \in \Delta^{|S|}$ and $t \geq 0$ is the resource to be shared among the agents.⁴ It is assumed that $\sum_{i \in N} x_{is} \geq t$ for all $s \in S$. Throughout the chapter, we consider a fixed population N and a fixed set S of states. For the sake of brevity, we denote our problem (x, p, t) . A nonempty set of problems is called a domain and is denoted by \mathcal{D} .⁵ A rationing rule $\varphi : \mathcal{D} \rightarrow \mathbb{R}_+^N$ gives a vector of shares such that $\sum_{i \in N} \varphi_i(x, p, t) = t$. We restrict our attention to rich domains which is defined as follows:

Definition 9. *A domain \mathcal{D} is rich if for all $x, x' \in \mathbb{R}_+^{N \times S}$ for all $p \in \Delta^{|S|}$ for all $t \geq 0$ with $x_{Ns} = x'_{Ns}$ ⁶ for all $s \in S$, then $\{(x, p, t) \in \mathcal{D} \Rightarrow (x', p, t) \in \mathcal{D}\}$.*

³The standard rationing problem is defined as (N, x, t) where N is a finite set of agents, x is a claim vector $x = (x_i)_{i \in N} \geq 0$ and $t \geq 0$ is the resource to be shared among the agents. A rationing rule φ , assigns a vector of shares $\varphi(N, x, t) \in \mathbb{R}_+^N$ to every rationing problem such that $\sum_{i \in N} \varphi_i(N, x, t) = t$.

⁴ $\Delta^{|S|}$ denotes an $|S|$ dimensional simplex.

⁵More precisely this is a restricted domain of problems where N and S are fixed so a better notation would be $\mathcal{D}(N, S)$. However, for notational simplicity we use \mathcal{D} since it does not raise any confusion.

⁶We use the notation $x_{Ts} := \sum_{i \in T} x_{is}$, where $T \subseteq N$.

Now we will define two rationing rules which involve proportionality idea. Since our rules are based on proportionality idea, let us recall the standard proportional rule when there is only one (certain) state of the world s , i.e., $S = \{s\}$.

The proportional rule is defined as

$$pr_i(x, p, t) = \frac{x_{is}}{x_{Ns}} t \text{ for all } i \in N$$

Ex-ante proportional rule is defined as applying proportional rule to the expectation of the state contingent claims.

$$\overline{pr}_i(x, p, t) := pr_i(E_p[x], t) = \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{j \in N} \sum_{s \in S} (p_s x_{js})} t = \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{s \in S} (p_s x_{Ns})} t \text{ for all } i \in N$$

Ex-post proportional rule is defined as expectation of the shares found by the proportional rule on the state contingent claims.

$$\tilde{pr}_i(x, p, t) := E_p[pr_i(x, t)] = \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) t \text{ for all } i \in N$$

The illustration of the ex-ante and ex-post proportional rules for a simple economy with two people and two states is presented in Figure 2.1.

As it is shown in the Figure 2.1, the ex-ante and ex-post proportional rules do not necessarily coincide. For $|N| = 2$, aforementioned rules give identical shares

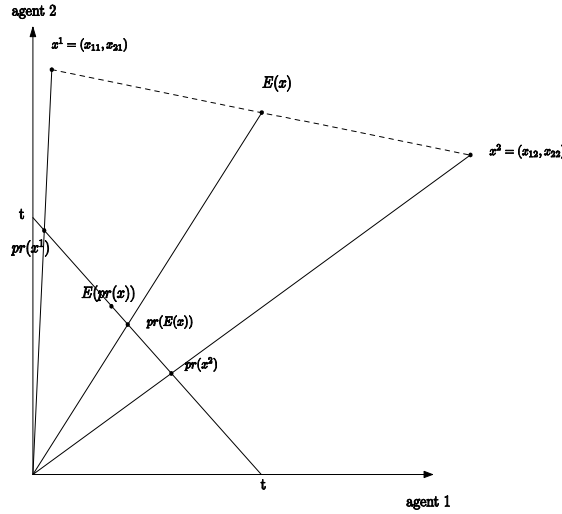


Figure 2.1. Ex-ante vs Ex-Post Proportional Rule (for $p_1 = p_2 = \frac{1}{2}$)

when the sum of the claims is equal for each state (Figure 2.2) or the ratio of the claims for each state is equal (Figure 2.3). The difference of the shares for the ex-ante and ex-post proportional rules for general economy with finite number of people and states are given in the Appendix.

2.3. Characterizations

Now by using natural axioms we will characterize the rules we introduced above.

Continuity (CONT): For all $(x, p, t) \in \mathcal{D}$ and for all sequences $(x^k, p^k, t^k) \in \mathcal{D}$, if $(x^k, p^k, t^k) \rightarrow (x, p, t)$, then $\varphi(x^k, p^k, t^k) \rightarrow \varphi(x, p, t)$.

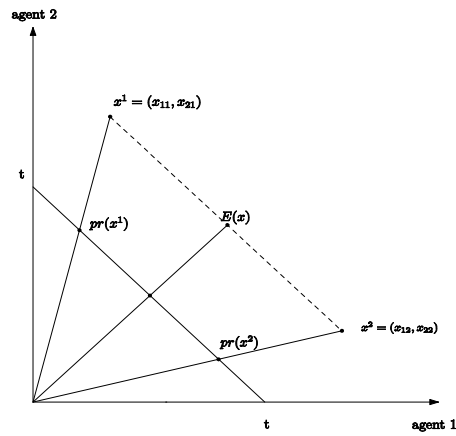


Figure 2.2. The sum of the claims for each state is equal: $x_{N1} = x_{N2}$

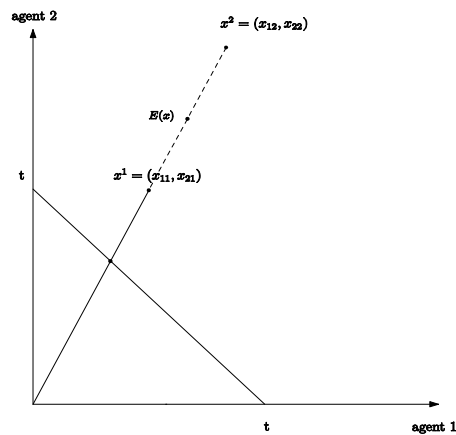


Figure 2.3. The ratio of the claims for each state is equal: $\frac{x_{11}}{x_{21}} = \frac{x_{12}}{x_{22}}$

Continuity tells us that small changes in the parameters of the problem do not bring big jumps in the allocations. Continuity is desirable because we do not want small errors (e.g., measurement errors) to lead to big changes in the allocations.

Anonymity (AN): For all $(x, p, t) \in \mathcal{D}$, for all permutations $\sigma : N \rightarrow N$, and for all $i \in N$, $\varphi_i(x, p, t) = \varphi_{\sigma(i)}(x_\sigma, p, t)$, where $x_\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(|N|)})$.

Anonymity says that the names of the agents do not matter. This is very natural axiom and is central to the literature on fairness.

Symmetry (SYM): For all $(x, p, t) \in \mathcal{D}$, for all permutations $\rho : S \rightarrow S$, and for all $i \in N$, $\varphi_i(x, p, t) = \varphi_i(x_\rho, p_\rho, t)$, where $p_\rho = (p_{\rho(1)}, p_{\rho(2)}, \dots, p_{\rho(|S|)})$ and $x_\rho = (x_{\rho(1)}, x_{\rho(2)}, \dots, x_{\rho(|S|)})$.

Symmetry is similar to the the Anonymity axiom with the role of agents substituted by states. It says that the names of the states do not matter.

No Award for Null (NAN): For all $(x, p, t) \in \mathcal{D}$ and for all $i \in N$, if $x_{is} = 0$ for all $s \in S$, then $\varphi_i(x, p, t) = 0$.

No Award for Null axiom says that an agent with zero claim for each state should get zero share. This axiom is also called dummy axiom in the literature.

Moulin (1985) defined Non-Advantageous Reallocation axiom to characterize the egalitarian and utilitarian solutions in quasi-linear social choice problems.

We will define two axioms on invariance to reallocation in a similar manner where transfers are made either across individuals or across states.

Non-advantageous Reallocation across Individuals (NARAI): For all $(x, p, t), (x', p, t) \in \mathcal{D}$ and for all $i \in N$, if $\sum_{j \in N \setminus \{i\}} x_{js} = \sum_{j \in N \setminus \{i\}} x'_{js}$ and $x_{is} = x'_{is}$ for all $s \in S$, then $\varphi_i(x, p, t) = \varphi_i(x', p, t)$.

NARAI states that the share of agent i depends on the sum of the total claim of the agents other than him. In other words, individuals other than i cannot affect the share of i by reallocating their claims among themselves, i.e. the share of individual i is a function of $x_i, x_{N \setminus i}, p$, and t .

Non-advantageous Reallocation across States (NARAS): For all $(x, p, t), (x', p, t) \in \mathcal{D}$ and for all $i \in N$, if $\sum_{s \in S} (p_s x_{is}) = \sum_{s \in S} (p_s x'_{is})$ and $x_{js} = x'_{js}$ for all $j \in N \setminus \{i\}$ and for all $s \in S$, then $\varphi_j(x, p, t) = \varphi_j(x', p, t)$ for all $j \in N \setminus \{i\}$.

NARAS implies that if agent i reallocates his claim across all the states given his expected claim is constant then the share of the other individuals would not change.

Now we will characterize the class of sharing rules satisfying NARAI, Anonymity, and Continuity.

Theorem 10. *Let $|N| \geq 3$. Let (x, p, t) and $(x', p, t) \in \mathcal{D}$. A sharing rule φ satisfies NARAI, AN, and CONT if there exists a continuous $W_s : \mathbb{R}^S \times \Delta^{|S|} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ for all $s \in S$ such that for all $i \in N$ we have*

$$(2.1) \quad \varphi_i(x, p, t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N, p, t) \right]$$

And conversely every rule satisfying NARAI, AN, and CONT must be in the form of [2.1].

Proof. The first statement is obvious. We will prove the second statement. Let $(x, p, t) \in \mathcal{D}$. Let φ be a rationing rule satisfying NARAI, AN, and CONT.

Let $x' = (x_1 + x_2, 0, x_3, \dots)$. Apply NARAI for the coalition $\{1, 2\}$, we get

$$(2.2) \quad \varphi_1(x, p, t) + \varphi_2(x, p, t) = \varphi_1(x', p, t) + \varphi_2(x', p, t).$$

Let $x'' = (x_1, x_{N \setminus \{1\}}, 0, 0, \dots)$. Now we'll apply NARAI for the coalition $N \setminus \{1\}$.

This implies

$$(2.3) \quad \varphi_{N \setminus \{1\}}(x, p, t) = \varphi_{N \setminus \{1\}}(x'', p, t).$$

Thus we have $t - \varphi_{N \setminus \{1\}}(x, p, t) = \varphi_1(x, p, t) = \varphi_1(x'', p, t) = t - \varphi_{N \setminus \{1\}}(x'', p, t)$.

$\varphi_2(x, p, t) = \varphi_1(x_2, x_{N \setminus \{2\}}, 0, \dots, 0, p, t)$ by [2.3] and AN.

$\varphi_1(x', p, t) = \varphi_1(x_1 + x_2, x_{N \setminus \{1,2\}}, 0, \dots, 0, p, t)$ by [2.3].

$\varphi_2(x', p, t) = \varphi_1(0, x_N, 0, \dots, 0, p, t)$ by [2.3] and AN.

Let us plug these back into [2.2].

$$\varphi_1(x_1, x_{N \setminus \{1\}}, 0, \dots, 0, p, t) + \varphi_1(x_2, x_{N \setminus \{2\}}, 0, \dots, 0, p, t) = \varphi_1(x_1 + x_2, x_{N \setminus \{1,2\}}, 0, \dots, 0, p, t) + \varphi_1(0, x_N, 0, \dots, 0, p, t)$$

Let us define $f : \mathbb{R}^S \times \mathbb{R}^S \times \Delta^{|S|} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$f(x_i, x_N, p, t) = \varphi_1(x_i, x_{N \setminus \{i\}}, 0, \dots, 0, p, t) - \varphi_1(0, x_N, 0, \dots, 0, p, t)$$

and define $g : \mathbb{R}^S \times \Delta^{|S|} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $g(x_N, p, t) = \varphi_1(0, x_N, 0, \dots, 0, p, t)$.

Thus we get

$$f(x_1, x_N, p, t) + f(x_2, x_N, p, t) = f(x_1 + x_2, x_N, p, t).$$

f is additive in the first term and by definition, f is continuous (φ is continuous). Fix (x_N, p, t) . So by invoking Eichhorn (1978) - Cor 3.1.9, p.51, we deduce that f is linear in the first term, that is, there exists a continuous $W : \mathbb{R}^S \times \Delta^{|S|} \times \mathbb{R}_+ \rightarrow \mathbb{R}^S$ such that

$$f(x_i, x_N, p, t) = \sum_{s \in S} [W_s(x_N, p, t) x_{is}].$$

$$\text{So } \varphi_i(x, p, t) = \sum_{s \in S} [(W_s(x_N, p, t) x_{is})] + g(x_N, p, t).$$

Summing over $i \in N$ we get

$$\sum_{i \in N} \varphi_i(x, p, t) = \sum_{s \in S} [(W_s(x_N, p, t) x_{Ns})] + |N|g(x_N, p, t) = t.$$

$$\text{So } g(x_N, p, t) = \frac{t - \sum_{s \in S} [(W_s(x_N, p, t) x_{Ns})]}{|N|}.$$

Hence we get the desired functional form.

$$\varphi_i(x, p, t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N, p, t) \right] \text{ for all } i \in N. \quad \square$$

Example 11. *Note that proportional and egalitarian rules are members of the family of the rules characterized in the previous theorem.*

- We have equal split rule, $\varphi_i(x, p, t) = \frac{t}{|N|}$ when $W_s(x_N, p, t) = 0$.
- $W_s(x_N, p, t)$ satisfying $\sum_{s \in S} [W_s(x_N, p, t)x_{Ns}] = t$ gives the family of proportional rules, e.g. $\varphi_i(x, p, t) = \sum_{s \in S} [W_s(x_N, p, t)x_{is}]$.
- If the weight functions are uniform with respect to states, that is, $W_s(x_N, p, t) = \frac{t}{\sum_{s \in S} x_{Ns}}$ for all s then $\varphi_i(x, p, t) = \frac{\sum_{s \in S} x_{is}}{\sum_{s \in S} x_{Ns}} t$.
- We have ex-ante proportional rule, $\varphi_i(x, p, t) = \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{s \in S} (p_s x_{Ns})} t$ when $W_s(x_N, p, t) = \frac{p_s t}{\sum_{s \in S} (p_s x_{Ns})}$.
- We get ex-post proportional rule, $\varphi_i(x, p, t) = \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) t$ when $W_s(x_N, p, t) = \frac{p_s t}{x_{Ns}}$.
- The family contains nonsymmetric proportional rules with respect to states as well, e.g. $\varphi_i(x, p, t) = \frac{x_{i1}}{x_{N1}} t$ when $W_1(x_N, p, t) = \frac{t}{x_{N1}}, W_2(x_N, p, t) = W_3(x_N, p, t) = \dots = 0$ (all the weight is given to state 1).

Theorem 12. *Let $|N| \geq 3$. A sharing rule φ satisfies NARAI, NARAS, AN, SYM, CONT, and NAN if and only if φ is ex-ante proportional rule.*

Proof. "If" part is obvious. We will prove the "only if" part. Let $(x, p, t) \in \mathcal{D}$. Let φ be a rationing rule satisfying NARAI, NARAS, AN, CONT, and NAN. Given that φ satisfies the premises of Theorem 1, we have $\varphi_i(x, p, t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N, p, t) \right]$. Fix $i \in N$ and let $x_{is} = 0$ for all $s \in S$. NAN

implies that $\varphi_i(x, p, t) = \frac{t}{|N|} + \sum_{s \in S} \left[\left(x_{is} - \frac{x_{Ns}}{|N|} \right) W_s(x_N, p, t) \right] = 0$. So we get $\sum_{s \in S} [W_s(x_N, p, t)x_{Ns}] = t$. Hence $\varphi_i(x, p, t) = \sum_{s \in S} [(W_s(x_N, p, t)x_{is})]$.

Let $\sum_{s \in S} (p_s x_{is}) = \sum_{s \in S} (p_s x'_{is})$ and $x_{js} = x'_{js}$ for all $j \in N \setminus \{i\}$ and for all $s \in S$.

So we get

$$(2.4) \quad \sum_{s \in S} [p_s (x_{is} - x'_{is})] = 0 \text{ for all } i \in N.$$

By NARAS, we have $\varphi_i(x, p, t) = \varphi_i(x', p, t)$ for all $i \in N$. Then

$$(2.5) \quad \sum_{s \in S} [(W_s(x_N, p, t)x_{is})] = \sum_{s \in S} [(W_s(x'_N, p, t)x'_{is})] \text{ for all } i \in N.$$

Fix $j \in N \setminus \{i\}$, by [2.5] we have $\sum_{s \in S} [(W_s(x_N, p, t)x_{js})] = \sum_{s \in S} [(W_s(x'_N, p, t)x'_{js})] = \sum_{s \in S} [(W_s(x'_N, p, t)x_{js})]$.

By the richness of \mathcal{D} , we have $W_s(x_N, p, t) = W_s(x'_N, p, t)$ for all $s \in S$. By using [2.5] we get

$$(2.6) \quad \sum_{s \in S} [W_s(x_N, p, t) (x_{is} - x'_{is})] = 0 \text{ for all } i \in N.$$

By [2.4] and [2.6], we deduce that p and W are colinear. So for all $s \in S$ there exists $h_s : \mathbb{R}^S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $W_s(x_N, p, t) = h_s(x_N, t)p_s$ for all $s \in S$. By SYM, we have $h_s = h_t$ for all $t \in S \setminus \{s\}$. So we can write it as $h(x_N, t)$.

Summing over $i \in N$, $\sum_{i \in N} \sum_{s \in S} [(W_s(x_N, p, t)x_{is})] = \sum_{i \in N} \sum_{s \in S} [h(x_N, t)p_s x_{is}] = h(x_N, t) \sum_{s \in S} (p_s x_{Ns}) = t$.

So $h(x_N, t) = \frac{t}{\sum_{s \in S} (p_s x_{Ns})}$. And $\varphi_i(x, p, t) = \sum_{s \in S} [h(x_N, t)p_s x_{is}] = \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{s \in S} (p_s x_{Ns})} t$ for all $i \in N$. \square

Before characterizing ex-post proportional rule we will show that NARAS axiom is not necessarily satisfied by ex-post proportional rules. Here we provide an example. Suppose that each agent can reallocate their claims across states, that is, their expected claim is constant, e.g. $\sum_{s \in S} (p_s x_{is}) = c_i \in \mathbb{R}_+$. Let us find the optimal strategy of agent i when all the other agents have a deterministic claim, e.g. $x_{js} = c_j \in \mathbb{R}_+$ for all $j \neq i$ and for all $s \in S$.

$$\begin{aligned} & \max \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) \\ & \text{s.t. } \sum_{s \in S} (p_s x_{is}) = c_i \\ & \quad x_{is} \geq 0 \text{ for all } s \in S \\ & L = \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) - \lambda \left[\sum_{s \in S} (p_s x_{is}) - c_i \right] \end{aligned}$$

$$\text{FOC: } \frac{\partial L}{\partial x_{is}} = p_s \frac{x_{(N \setminus \{i\})s}}{(x_{Ns})^2} - \lambda p_s \leq 0 \text{ for all } s \in S$$

By symmetry, $x_{is} = x_s > 0$ for all $i \in N$

$$\text{So } \lambda = \frac{(n-1)x_s}{(nx_s)^2}. \text{ Hence } x_s = \frac{n-1}{n^2 \lambda}.$$

$$\sum_{s \in S} (p_s x_s) = c_i \implies \sum_{s \in S} \left(p_s \frac{n-1}{n^2 \lambda} \right) = c_i \implies \lambda = \frac{n-1}{c_i n^2} \implies x_s = \frac{n-1}{n^2 \lambda} = \frac{n-1}{n^2 \frac{n-1}{c_i n^2}} = c_i$$

We showed that there is a symmetric Nash equilibrium where every agent i claims c_i independent of states. Therefore a claim vector satisfying mean preserving spread makes one worse off. Hence NARAS is not satisfied. Figure 2.4 below illustrates this fact, i.e. agent 1 is weakly better off having a deterministic claim provided that agent 2 is having a deterministic claim.

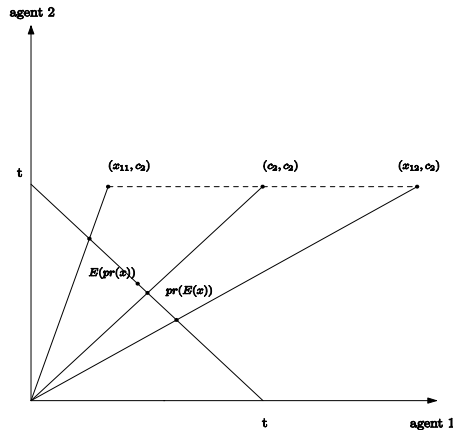


Figure 2.4. Mean preserving spread makes agent 1 worse off.

Now we will characterize ex-post proportional rule. The functional form of ex-post proportional rule exhibits additively separable preferences with respect to the states. Hence we are going to utilize a version of Expected Utility Theorem. To arrive there we are going to utilize an independence axiom in the spirit of von Neumann - Morgenstern.

Independence (IND): For all $(x, p, t), (x, q, t), (x, r, t) \in \mathcal{D}$, for all $i \in N$, and for all $\lambda \in (0, 1)$, we have $\varphi_i(x, p, t) \geq \varphi_i(x, q, t)$ if and only if $\varphi_i(x, \lambda p + (1 - \lambda)r, t) \geq \varphi_i(x, \lambda q + (1 - \lambda)r, t)$.

Theorem 13. *Let $|N| \geq 3$. A sharing rule φ satisfies NARAI, AN, SYM, IND, CONT, and NAN if and only if φ is ex-post proportional rule.*

Proof. "If" part is obvious. We will prove the "only if" part. Let $(x, p, t) \in \mathcal{D}$. Let φ be a rationing rule satisfying NARAI, AN, SYM, IND, CONT, and NAN. By Theorem 1 and NAN, we have

$$(2.7) \quad \varphi_i(x, p, t) = \sum_{s \in S} [(W_s(x_N, p, t)x_{is})]$$

By CONT and IND we can invoke celebrated Expected Utility Characterization of von Neumann and Morgenstern (1947) and deduce that φ_i should be additively separable with respect to probabilities. That is, for all $x \in \mathbb{R}^{N \times S}$ for all $p \in \Delta^{|S|}$ for all $i \in N$ and for all $s \in S$ there exists $u_{is} : \mathbb{R}^{N \times S} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$(2.8) \quad \varphi_i(x, p, t) = \sum_{s \in S} [p_s u_{is}(x, t)]$$

By [2.7] and [2.8] we deduce that for all $s \in S$ there exists $v_s : \mathbb{R}^S \times \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$(2.9) \quad \varphi_i(x, p, t) = \sum_{s \in S} [p_s x_{is} v_s(x_N, t)]$$

Consider a degenerate lottery δ_s , that is, fix $s \in S$ and let $p_s = 1$.

So $\varphi_i(x, \delta_s, t) = x_{is} v_s(x_N, t)$.

Summing over $i \in N$, we get $\sum_{i \in N} \varphi_i(x, \delta_s, t) = \sum_{i \in N} [x_{is} v_s(x_N, t)] = v_s(x_N, t) x_{Ns} = t$. So $v_s(x_N, t) = \frac{t}{x_{Ns}}$. Hence we get the ex-post proportional rule.

$$\varphi_i(x, p, t) = \sum_{s \in S} [p_s x_{is} v_s(x_N, t)] = \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) t. \quad \square$$

2.4. Conclusion

We studied a resource allocation problem where the claims are state contingent. We consider proportional rules and introduce two extensions of the proportional rules in our framework – ex-ante and ex-post proportional rules. Applying the proportional rule to the expected claim gives the ex-ante proportional rule. Ex-post proportional rule is defined as the expectation of the shares given by the proportional rule for various states. To characterize these rules we propose two extensions of No Advantageous Reallocation (introduced by Moulin(1985)). The first extension, *NARAI*, requires that no group of agents benefits if transfers are allowed across individuals for each state whereas the second extension, *NARAS*, considers transfers across states. We characterize ex-ante proportional rule by *NARAI*

and *NARAS* combined with Anonymity, Symmetry, Continuity, and No Award for Null. To characterize ex-post proportional rule, we introduce an Independence axiom similar to Independence axiom used in Expected Utility Theory. This axiom says that by mixing two lotteries with a third one, the allocation rule remains unaffected by the choice of the third lottery. Replacing *NARAS* with the aforementioned Independence axiom gives the characterization of ex-post proportional rule.

This chapter leads us to two particularly important questions to be considered in future research. The first question is to find interesting characterizations of the extensions of other important rules, such as, Uniform Gains, Uniform Losses, etc. It will also be interesting to extend our framework to situations like, (a) where the resource itself is state contingent, (b) the individuals are taking subjective probabilities, that is, the probabilities for the states are not uniform across the individuals.

2.5. Appendix

Proposition 14. *Let $(x, p, t) \in \mathcal{D}$ be given.*

$$\bar{p}r_i(x, p, t) - \tilde{p}r_i(x, p, t) = \frac{\sum_{s \in S} \left[p_s x_{is} \left(1 - p_s - \sum_{j \in N} \frac{x_{js}}{x_{is}} \sum_{r \neq s} \frac{p_r x_{ir}}{x_{Nr}} \right) \right]}{\sum_{s \in S} (p_s x_{Ns})} t$$

Proof. Define $\alpha_{js} = \frac{x_{js}}{x_{is}}$ for all $j \in N$ and for all $s \in S$.

So ex-ante proportional rule for agent i is given by

$$\bar{p}r_i(x, p, t) = \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{s \in S} (p_s x_{Ns})} t = \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{s \in S} (p_s x_{is} \alpha_{Ns})} t$$

And ex-post proportional rule for agent i is given by

$$\tilde{p}r_i(x, p, t) = \sum_{s \in S} \left(p_s \frac{x_{is}}{x_{Ns}} \right) t = \sum_{s \in S} \frac{p_s}{\alpha_{Ns}} t$$

So the difference between ex-ante and ex-post proportional rule is

$$\begin{aligned} \bar{p}r_i(x, p, t) - \tilde{p}r_i(x, p, t) &= \frac{\sum_{s \in S} (p_s x_{is})}{\sum_{s \in S} (p_s x_{is} \alpha_{Ns})} t - \sum_{s \in S} \frac{p_s}{\alpha_{Ns}} t = \\ &= \frac{\sum_{s \in S} (p_s x_{is}) \prod_{s \in S} \alpha_{Ns} - \sum_{s \in S} (p_s x_{is} \alpha_{Ns}) \prod_{s \in S} \alpha_{Ns} \sum_{s \in S} \frac{p_s}{\alpha_{Ns}}}{\prod_{s \in S} \alpha_{Ns} \sum_{s \in S} (p_s x_{is} \alpha_{Ns})} t = \\ &= \frac{\prod_{s \in S} \alpha_{Ns} \left[\sum_{s \in S} (p_s x_{is}) - \sum_{s \in S} [p_s x_{is} \alpha_{Ns}] \sum_{s \in S} \frac{p_s}{\alpha_{Ns}} \right]}{\prod_{s \in S} \alpha_{Ns} \sum_{s \in S} (p_s x_{is} \alpha_{Ns})} t = \frac{\sum_{s \in S} [(p_s - p_s^2) x_{is}] - \sum_{s \in S} (p_s x_{is} \alpha_{Ns}) \sum_{r \neq s} \frac{p_r}{\alpha_{Nr}}}{\sum_{s \in S} (p_s x_{is} \alpha_{Ns})} t = \\ &= \frac{\sum_{s \in S} \left[p_s x_{is} \left(1 - p_s - \sum_{j \in N} \alpha_{js} \sum_{r \neq s} \frac{p_r}{\alpha_{Nr}} \right) \right]}{\sum_{s \in S} (p_s x_{is} \alpha_{Ns})} t = \frac{\sum_{s \in S} \left[p_s x_{is} \left(1 - p_s - \sum_{j \in N} \frac{x_{js}}{x_{is}} \sum_{r \neq s} \frac{p_r x_{ir}}{x_{Nr}} \right) \right]}{\sum_{s \in S} (p_s x_{Ns})} t. \quad \square \end{aligned}$$

CHAPTER 3

Voting Games of Resolute Social Choice Correspondences

3.1. Introduction

Gibbard (1973) and Satterthwaite (1975) establish a major difficulty in implementing social choice rules by asking voters their preferences: Except for trivial or particular cases, voters' incentives to misrepresent their preferences cannot be ruled out. The pretended preferences of voters lead to a social outcome which Hurwicz and Sertel (1999) call the *performance* of the social choice rule. There is no a priori reason to believe that the performance of a social choice rule coincides with the social choice rule itself.¹ Besides, the divergence between a social choice rule and its performance manifests the difference between the aimed and reached social outcome.

We aim to determine the performance of voting rules. This requires a modelling of the preference declaration process in terms of a *voting game* whose equilibria are computed. Such an analysis depends on the structure of the voting game, as well as the assumed game-theoretic solution concept. Although alternative models exist, it is quite common to conceive the voting game as a normal form *preference pretension*

¹In fact, Otani and Sicilian (1982), Thomson (1984), Tadenuma and Thomson (1995), Sertel and Sanver (1999), Sanver (2002, 2005), Sertel and Sanver (2004) exemplify cases of divergence between a social choice rule and its performance, in a variety of environments.

game where voters are allowed to pretend any logically possible preference while the outcome is determined by applying the prevailing social choice rule to the pretended preference profile. On the other hand, the choice of the game-theoretic solution concept exhibits a broad variety.²

Our analysis is closely related to Sertel and Sanver (2004) who characterize strong equilibrium outcomes of preference pretension games in terms of what they call “ (n, q) -Condorcet winners” - a concept which generalizes the usual notion of a Condorcet winner. Although their environment covers a quite wide class of social choice rules, it is restricted to *social choice functions* where the outcome is a single alternative at every preference profile. However, many interesting social choice rules are *social choice correspondences* which allow sets of alternatives as outcomes. These sets can be *irresolute* outcomes arising from the inevitability of ties when both voters and alternatives are equally treated (see Moulin (1983)) but also *resolute* ones when the social choice problem is concerned with the determination a list of mutually compatible alternatives, such as the members of a committee. In the latter case, the social choice rule is called a *resolute social choice correspondence*.³

We ask the extent to which the characterization results of Sertel and Sanver (2004) subsume resolute social choice correspondences. In Section 2, we introduce

²For example, we see the use of Nash equilibrium by Farquharson (1969); dominance solvability by Moulin (1979); coalition proof Nash equilibrium by Keiding and Peleg (2002); strong Nash equilibrium by Sertel and Sanver (2004), Barbera and Coelho (2009).

³For a more detailed treatment of resolute social choice correspondences, one can see Ozyurt and Sanver (2008) and Reffgen (2008).

the model. Although we essentially borrow the model of Sertel and Sanver (2004), our analysis handles the additional complication of determining individuals' preferences over sets. In Section 3, we characterize strong equilibrium outcomes of voting games induced by resolute social choice correspondences in terms of sets which are (n, q) -Condorcet winners according to preferences over sets. Under very mild axioms to extend a ranking over a set to its power set, our characterizations directly generalize those of Sertel and Sanver (2004). However, unlike their counterpart in Sertel and Sanver (2004), they do not fully describe the strong equilibrium outcomes we are after. For, the property of being an (n, q) -Condorcet winner according to preferences over sets needs to be translated into a property defined according to preferences over alternatives. We address this problem in Section 4 and solve it by using the techniques introduced by Kaymak and Sanver (2003) and further elaborated by Barbera and Coelho (2008), hence being able to reach the characterization we are after. We conclude in Section 5.

3.2. The Model

3.2.1. Basic Concepts

Consider a society $N = \{1, \dots, n\}$ with $n \geq 2$, confronting a finite set of alternatives A with $\#A = m \geq 2$. Writing Π for the set of complete, transitive and antisymmetric binary relations over A , we interpret $\rho_i \in \Pi$ as the *preference* of

individual $i \in N$ over A . We write ρ_i^* for the strict counterpart of ρ_i .⁴ A *preference profile* over A is an n -tuple $\rho_N = (\rho_1, \dots, \rho_n) \in \Pi^N$ of individual preferences over A . Throughout the chapter we fix some $k \in \{1, \dots, m-1\}$ and write $\mathcal{A}_k = \{X \subset A : \#X = k\}$ for the set of k -element subsets of A . A (*resolute*) *social choice correspondence* (SCC) is a mapping $F : \Pi^N \rightarrow \mathcal{A}_k$.

Let \mathfrak{R} be the set of reflexive and transitive binary relations over $\mathcal{A} = 2^A \setminus \{\emptyset\}$ such that for any $R \in \mathfrak{R}$ and for any $X, Y \in \mathcal{A}$ we have

- (i) $X R Y$ or $Y R X$ whenever $\#X = \#Y$;
- (ii) $\sim X R Y$ and $\sim Y R X$ whenever $\#X \neq \#Y$.

So any two sets of equal cardinality are compared by R while sets of different cardinality are incomparable. The strict counterpart of R is denoted P . We interpret $R_i \in \mathfrak{R}$ as the preference of $i \in N$ over \mathcal{A} while $R_N = (R_1, \dots, R_n) \in \mathfrak{R}^N$ is a preference profile over \mathcal{A} . We assume that the preference of an individual over \mathcal{A} exhibits a consistency with his preference over A . This is expressed through an *extension map* α which assigns to each $\rho \in \Pi$ a non-empty subset $\alpha(\rho)$ of \mathfrak{R} such that for every $R \in \alpha(\rho)$ we have $\{x\}R\{y\} \iff x \rho y \forall x, y \in A$. The elements of $\alpha(\rho)$ are interpreted as the preferences over \mathcal{A} that are admissible under ρ . Remark that $\alpha(\rho) \cap \alpha(\rho') = \emptyset$ for distinct $\rho, \rho' \in \Pi$. We consider extension maps which satisfy the following *Condition Γ* : Let $\max_k(\rho) \in \mathcal{A}_k$ be the set consisting

⁴So for any $x, y \in A$, we have $x \rho^* y$ iff $x \rho y$ holds but $y \rho x$ does not. As ρ is antisymmetric, when x and y are distinct, we have either $x \rho^* y$ or $y \rho^* x$.

of the best k elements in A with respect to $\rho \in \Pi$, i.e., $x\rho y$ holds $\forall x \in \max_k(\rho)$, $\forall y \in A \setminus \max_k(\rho)$. Given any $\rho \in \Pi$ and any $R \in \alpha(\rho)$, we have $\max_k(\rho) P Y$ for all $Y \in \mathcal{A}_k \setminus \{\max_k(\rho)\}$. So Condition Γ requires the best k element set to be preferred to any other k element set. From now on, we pick an extension map α satisfying Condition Γ and use it throughout the chapter. By a slight abuse of notation, we write $\alpha(\rho_N) = \{R_N \in \mathfrak{R}^N : R_i \in \alpha(\rho_i) \text{ for each } i \in N\}$ for the set of preference profiles over \mathcal{A} that are admissible under $\rho_N \in \Pi^N$.

The first extension map we consider is a separability condition which Roth and Sotomayor (1990) call “responsiveness”.⁵ We call this the RS extension and define it as follows: For any $\rho \in \Pi$ and any $R \in \mathfrak{R}$, we have $R \in \alpha(\rho) \iff X P Y \forall X, Y \in \mathcal{A}_k$ with $X = (Y \setminus \{y\}) \cup \{x\}$ for some $y \in Y$ and $x \in A \setminus Y$ satisfying $x \rho y$.

The next extension map we consider is a modified version of the extension map introduced by Kelly (1977): We say that α is the *Kelly extension* iff given any $\rho \in \Pi$ and any $R \in \mathfrak{R}$ we have $R \in \alpha(\rho)$ if and only if

(i) $X P Y$ for all distinct $X, Y \in \mathcal{A}_k$ with $x \rho^* y \forall x \in X, \forall y \in Y$

and

(ii) $\max_k(\rho) P Y \forall Y \in \mathcal{A}_k \setminus \{\max_k(\rho)\}$.

⁵Nevertheless, Erdamar and Sanver (2008) argue for the inappropriateness of this axiom when sets are non-resolute outcomes.

Example 15. *In order to illustrate our extension maps consider the preference $\rho = (a, b, c, d)$ where the leftmost alternative is preferred the most.*

Based on the RS extension map for two-element sets the preferences over sets that are consistent with ρ are

$$R_1 = (\{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}, \{c, d\})$$

$$\text{and } R_2 = (\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}).$$

Similarly, based on the Kelly extension map for the two-element sets the preferences that are consistent with ρ are so that $\{a, b\}$ is listed as the top preference and the rest of the rankings can be anything.

We close this section by introducing a generalization of the concept of a Condorcet winner, defined over elements of \mathcal{A}_k . Given any $q \in \{0, 1, \dots, n + 1\}$, any $R_N \in \mathfrak{R}^N$ and any $X, Y \in \mathcal{A}_k$, we say that X (n, q) -dominates Y (according to R_N) iff $\#\{i \in N : X P_i Y\} \geq q$. We write $X D(R_N; n, q) Y$ when X (n, q) -dominates Y . The set $C(R_N; n, q) = \{X \in \mathcal{A}_k : \text{there exists no } Y \in \mathcal{A}_k \setminus \{X\} \text{ with } Y D(R_N; n, q) X\}$ contains the elements of \mathcal{A}_k which are undominated according to $D(R_N; n, q)$ or simply the (n, q) -Condorcet winners at R_N . Our definition of an (n, q) -Condorcet winner generalizes the standard notion. In particular, when n is even, letting $\mu^* = \min\{k \in \{0, 1, \dots, n\} : k > n - k\}$ and $\mu = \mu^* - 1$, we have $C(R_N; n, \mu^*)$ as the set of weak Condorcet winners at R_N and $C(R_N; n, \mu)$ as the set of strong Condorcet winners at R_N . When n is odd,

$C(R_N; n, \mu^*)$ is the set of Condorcet winners at R_N while the distinction between “strong” and “weak” vanishes.

Remark that (n, q) –Condorcet winners are defined with respect to preference profiles over \mathcal{A} . Following Kaymak and Sanver (2003), we extend the concept so that it is defined with respect to preference profiles over A . We say that $X \in \mathcal{A}_k$ is an (n, q) –Condorcet winner at $\rho_N \in \Pi^N$ iff $X \in C(R_N; n, q)$ for some $R_N \in \alpha(\rho_N)$. We call $X \in \mathcal{A}_k$ a *universal* (n, q) –Condorcet winner at $\rho_N \in \Pi^N$ iff $X \in C(R_N; n, q)$ for all $R_N \in \alpha(\rho_N)$. We denote the set of (n, q) –Condorcet winners at ρ_N as $C(\rho_N; n, q)$ while $C^*(\rho_N; n, q)$ stands for the set of universal (n, q) –Condorcet winners at ρ_N .⁶

3.2.2. Voting games

Pick some SCC $F : \Pi^N \rightarrow \mathcal{A}_k$. At each $\rho_N \in \Pi^N$, take some $R_N \in \alpha(\rho_N)$ and consider the (normal form) voting game $G(R_N) = \{(S_i, R_i)\}_{i \in N}$ where $S_i = \Pi$ is the strategy space of $i \in N$ while $s_N R_i s'_N \iff F(s_N) R_i F(s'_N)$ holds for all $s_N, s'_N \in S = \prod_{i \in N} S_i$. We say that $s = \{s_i\}_{i \in N} \in S$ is a *strong (Nash) equilibrium* of $G(R_N)$ iff given any $K \subseteq N$ there exists no $s'_N = \{s'_i\}_{i \in N} \in S$ with $s'_j = s_j$ for all $j \in N \setminus K$ such that $s'_N P_i s_N$ for each $i \in K$. We denote the set of

⁶Qualifying $X \in \mathcal{A}_k$ as a (universal) (n, q) –Condorcet winner at $\rho_N \in \Pi^N$ is not informative about the properties of X with respect to individual preferences at ρ_N . We address this issue in Section 4.

strong equilibria of $G(R_N)$ as $\sigma[G(R_N)]$. Whenever $s_N \in \sigma[G(R_N)]$, we call $F(s_N)$ a *strong equilibrium outcome* of $G(R_N)$.

Let $\Gamma_F(\rho_N) = \{G(R_N)\}_{R_N \in \alpha(\rho_N)}$ be the class of voting games induced by F at $\rho_N \in \Pi^N$. We say that $X \in \mathcal{A}_k$ is a *strong equilibrium outcome* of F at $\rho_N \in \Pi^N$ iff X is a strong equilibrium outcome of some $G \in \Gamma_F(\rho_N)$. We call $X \in \mathcal{A}_k$ a *universally strong equilibrium outcome* of F at $\rho_N \in \Pi^N$ iff X is a strong equilibrium outcome of every $G \in \Gamma_F(\rho_N)$. We denote the set of strong equilibrium outcomes of F at $\rho_N \in \Pi^N$ as $F_\sigma(\rho_N)$ while $F_\sigma^*(\rho_N)$ stands for the universally strong equilibrium outcomes of F at $\rho_N \in \Pi^N$. Of course $F_\sigma(\rho_N)$, a fortiori $F_\sigma^*(\rho_N)$ may be empty.

Let $\mathcal{D} = \bigcup_{\rho \in \Pi} \alpha(\rho)$ be the set of all admissible individual preferences over \mathcal{A} under the extension map α . We define the function $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ which maps every $R_N \in \mathcal{D}^N$ to $f(R_N) = F(\rho_N)$ where $R_N \in \alpha(\rho_N)$. As $\alpha(\rho_N) \cap \alpha(\rho'_N) = \emptyset$ for distinct $\rho_N, \rho'_N \in \Pi^N$, f is well-defined. We call f the *equivalent social choice hyperfunction (SCH)* of F .

When the SCH $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is instituted as the social choice rule, at each $R_N \in \mathcal{D}^N$, it induces a voting game $\Gamma_f(R_N) = \{(S_i, R_i)\}_{i \in N}$ where $S_i = \mathcal{D}$ is the strategy space of $i \in N$ while $s_N R_i s'_N \iff f(s_N) R_i f(s'_N)$ holds for all $s_N, s'_N \in S = \prod_{i \in N} S_i$. We say that $s_N = \{s_i\}_{i \in N} \in S$ is a strong (Nash) equilibrium of $\Gamma_f(R_N)$ iff given any $K \subseteq N$ there exists no $s'_N = \{s'_i\}_{i \in N} \in S$ with $s'_j = s_j$ for all $j \in N \setminus K$ such that $s'_N P_i s_N$ for each $i \in K$. We denote the set of strong equilibria of $\Gamma_f(R_N)$ as $\sigma[\Gamma_f(R_N)]$. Whenever $s_N \in \sigma[\Gamma_f(R_N)]$, we

call $f(s_N)$ a strong equilibrium outcome of $\Gamma_f(R_N)$. Let $f_\sigma(R_N)$ be the (possibly empty) set of strong equilibrium outcomes of $\Gamma_f(R_N)$.

We are after characterizing the (universal) strong equilibrium outcomes of F , expressed through the correspondences F_σ and F_σ^* . As the proposition below states, these are related to the strong equilibrium outcomes of its equivalent SCH f , expressed through the correspondence f_σ .

Proposition 16. *Given any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$, we have*

(i) $X \in F_\sigma(\rho_N) \iff X \in f_\sigma(R_N)$ for some $R_N \in \alpha(\rho_N)$.

(ii) $X \in F_\sigma^*(\rho_N) \iff X \in f_\sigma(R_N)$ for all $R_N \in \alpha(\rho_N)$.

Proof. Take any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$. We first prove (i). To show the “if” part, let $X \in f_\sigma(R_N)$ for some $R_N \in \alpha(\rho_N)$. So there exists $s_N \in \mathcal{D}^N$ with $s_N \in \sigma(\Gamma_f(R_N))$ and $f(s_N) = X$. Let $r_N \in \Pi^N$ be such that $s_N \in \alpha(r_N)$. So $F(r_N) = X$. We consider $G(R_N) \in \Gamma_F(\rho_N)$ and claim $r_N \in \sigma(G(R_N))$, hence $X \in F_\sigma(\rho_N)$. Suppose, for a contradiction, $r_N \notin \sigma(G(R_N))$. So there exist $K \subseteq N$ and $r'_N \in \Pi^N$ with $r'_j = r_j$ for all $j \in N \setminus K$ such that $F(r'_N) \not P_i F(r_N) = X$ for each $i \in K$. Pick some $s'_N \in \mathcal{D}^N$ with $s'_j = s_j$ for all $j \in N \setminus K$ while $s'_N \in \alpha(r'_N)$. As $f(s'_N) = F(r'_N)$ and $F(r_N) = f(s_N)$, we have $f(s'_N) \not P_i f(s_N)$ for each $i \in K$, hence $s_N \notin \sigma(\Gamma_f(R_N))$, giving the desired contradiction. To show the “only if” part, let $X \in F_\sigma(\rho_N)$. So there exists $r_N \in \Pi^N$ such that $r_N \in \sigma(G(R_N))$ for some $G(R_N) \in \Gamma_F(\rho_N)$ while $F(r_N) = X$. Take some $s_N \in \mathcal{D}^N$ with $s_N \in \alpha(r_N)$. So $f(s_N) = X$. We claim $s_N \in \sigma(\Gamma_f(R_N))$, hence $X \in f_\sigma(R_N)$ for some $R_N \in \alpha(\rho_N)$. Suppose,

for a contradiction, $s_N \notin \sigma(\Gamma_f(R_N))$. So there exist $K \subseteq N$ and $s'_N \in \mathcal{D}^N$ with $s'_j = s_j$ for all $j \in N \setminus K$ such that $f(s'_N) P_i f(s_N)$ for each $i \in K$. Pick some $r'_N \in \Pi^N$ with $r'_j = r_j$ for all $j \in N \setminus K$ while $s'_N \in \alpha(r'_N)$. As $f(s'_N) = F(r'_N)$ and $F(r_N) = f(s_N)$, we have $F(r'_N) P_i F(r_N)$ for each $i \in K$, hence $r_N \notin \sigma(G(R_N))$, giving the desired contradiction. This establishes (i).

We now prove (ii). To show the “if” part, let $X \in f_\sigma(R_N)$ for every $R_N \in \alpha(\rho_N)$. Now take any $G(R_N) \in \Gamma_F(\rho_N)$. Consider $s_N \in \mathcal{D}^N$ with $s_N \in \sigma(\Gamma_f(R_N))$ and $f(s_N) = X$. Take any $r_N \in \Pi^N$ with $s_N \in \alpha(r_N)$, hence $F(r_N) = X$. The arguments used in proving the “if” part of (i), mutatis mutandis, establish $r_N \in \sigma(G(R_N))$, hence $X \in F_\sigma^*(\rho_N)$. To show the “only if” part, let $X \in F_\sigma^*(\rho_N)$. Take any $R_N \in \alpha(\rho_N)$. As $X \in F_\sigma^*(\rho_N)$, there exists $r_N \in \Pi^N$ with $F(r_N) = X$ such that $r_N \in \sigma(G(R_N))$ where $G(R_N) \in \Gamma_F(\rho_N)$. Pick any $s_N \in \mathcal{D}^N$ with $s_N \in \alpha(r_N)$, hence $f(s_N) = X$. The arguments used in proving the “only if” part of (i), mutatis mutandis, establish that $s_N \in \sigma(\Gamma_f(R_N))$, hence $X \in f_\sigma(R_N)$, completing the proof. \square

We devote the next section to characterizations of f_σ , F_σ , and F_σ^* .

3.3. Characterizations

In characterizing strong equilibrium outcomes of F in terms of (n, q) –Condorcet winners, the value that q gets depends on the power distribution that the equivalent SCH f induces among coalitions. Following Moulin and Peleg (1982), Moulin

(1983), Peleg (1984), and Ichiishi (1986), we express this power distribution through the “effectivity” of a coalition. Given the SCH $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$, a coalition $K \subseteq N$ of individuals is β -effective for $X \in \mathcal{A}_k$ iff for all $R_{N \setminus K} \in \mathcal{D}^{N \setminus K}$, there exists $R_K \in \mathcal{D}^K$ such that $f(R_K, R_{N \setminus K}) = X$. Let $\beta_f^+(X)$ be the set of coalitions which are β -effective for X under f . A coalition $K \subseteq N$ of individuals is *weakly* β -effective for $X \in \mathcal{A}_k$ iff there exists $Y \in \mathcal{A}_k \setminus \{X\}$ such that for some $R_{N \setminus K} \in \mathcal{D}^{N \setminus K}$ with $\{Y\} = \arg \max_{\mathcal{A}_k} R_i$ for all $i \in N \setminus K$, there exists $R_K \in \mathcal{D}^K$ such that $X = f(R_K, R_{N \setminus K})$. Let $\beta_f^-(X)$ be the set of coalitions which are weakly β -effective for X under f . We define $b_f^+(X) = \min\{\#K\}_{K \in \beta_f^+(X)}$ if $\beta_f^+(X) \neq \emptyset$ and $b_f^-(X) = \min\{\#K\}_{K \in \beta_f^-(X)}$ if $\beta_f^-(X) \neq \emptyset$. So in case $\beta_f^+(X)$ (resp., $\beta_f^-(X)$) is non-empty, $b_f^+(X)$ (resp., $b_f^-(X)$) is the cardinality of a minimal cardinality coalition belonging to $\beta_f^+(X)$ (resp., $\beta_f^-(X)$). By convention, we set $b_f^+(X) = n + 1$ if $\beta_f^+(X) = \emptyset$ and $b_f^-(X) = n + 1$ if $\beta_f^-(X) = \emptyset$. Finally, let $b_f^+ = \max_{X \in \mathcal{A}_k} b_f^+(X)$ and $b_f^- = \min_{X \in \mathcal{A}_k} b_f^-(X)$. Note that as $\beta_f^+(X) \subseteq \beta_f^-(X) \forall X \in \mathcal{A}_k$ holds by definition, we have $b_f^- \leq b_f^+$.

We start by characterizing f_σ .⁷ We say that $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is *anonymous* iff given any permutation $p : N \rightarrow N$ of individuals and any $(R_i)_{i \in N} \in \mathcal{D}^N$, we have $f((R_i)_{i \in N}) = f((R_{p(i)})_{i \in N})$.

⁷By conceiving each set as an independent alternative, a SCH can be conceived as a resolute social choice rule. Hence the environment of Sertel and Sanver (2004) includes the voting game induced by $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$. As a result, Proposition 2, Proposition 3, and Proposition 4 below indeed follow from their Theorem 4.3, Theorem 4.1, and Theorem 4.7 respectively.

Proposition 17. *If $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is anonymous, then $f_\sigma(R_N) \subseteq C(R_N; n, b_f^+)$ $\forall R_N \in \mathfrak{R}^N$.*

Proof. Let $f : \mathcal{D}^N \rightarrow \mathcal{A}$ be an arbitrary anonymous SCH. Take any $R_N \in \mathcal{D}^N$, any $X \in f_\sigma(R_N)$ and any $Y \in \mathcal{A}_k \setminus \{X\}$. As $X \in f_\sigma(R_N)$, there exists $s_N \in \sigma[\Gamma_f(R_N)]$ with $f(s_N) = X$. Thus, $\{i \in N : Y P_i X\} \notin \beta_f^+(Y)$. Moreover, as f is anonymous, there exists an integer k such that $\beta_f^+(Y) = \{K \subseteq N : \#K \geq k\}$. Hence $\#\{i \in N : Y P_i X\} < b_f^+(Y) \leq b_f^+$, establishing the failure of $Y D (R_N; n, b_f^+) X$, which shows $X \in C (R_N; n, b_f^+)$. \square

We say that $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is *top-unanimous* iff for any $X \in \mathcal{A}_k$ and any $R_N \in \mathcal{D}$ with $\#\{i \in N : X P_i Y \forall Y \in \mathcal{A}_k \setminus \{X\}\} = n$, we have $f(R_N) = X$.

Proposition 18. *If $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is top-unanimous, then $C(R_N; n, b_F^-) \subseteq f_\sigma(R_N) \forall R_N \in \mathfrak{R}^N$.*

Proof. Let $f : \mathcal{D}^N \rightarrow \mathcal{A}$ be an arbitrary top-unanimous SCH. Take any $R_N \in \mathcal{D}^N$ and any $X \in C(R_N; n, b_F^-)$. Consider the game $\Gamma_f(R_N)$ and pick some $s_N \in \mathcal{D}^N$ with $X P_i Y \forall Y \in \mathcal{A}_k \setminus \{X\}, \forall i \in N$. As f is top-unanimous, we have $f(s_N) = X$. We complete the proof by establishing $s_N \in \sigma[\Gamma_f(R_N)]$, hence $X \in f_\sigma(R_N)$. Take any $Y \in \mathcal{A}_k \setminus \{X\}$. Since $X \in C(R_N; n, b_F^-)$, we have $\#\{i \in N : Y P_i X\} < b_F^- \leq b_F^-(Y)$, which implies $\{i \in N : Y P_i X\} \notin \beta_f^-(Y)$. So there exists no

$s'_N \in \mathcal{D}^N$ with $s'_j = s_j \forall j \in N \setminus \{i \in N : Y P_i X\}$ while $f(s'_N) = Y$, which shows $s_N \in \sigma[\Gamma_f(R_N)]$. \square

We say that $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is *top-majoritarian* iff for any $X \in \mathcal{A}_k$ and any $R_N \in \mathcal{D}$ with $\#\{i \in N : X P_i Y \forall Y \in \mathcal{A}_k \setminus \{X\}\} \geq \mu^*$, we have $f(R_N) = X$. Top-majoritarianism implies top-unanimity. Moreover, for a top-majoritarian SCH f , we have $b_f^- = b_f^+ = \mu^*$ when n is odd and $\{b_f^-, b_f^+\} \subseteq \{\mu, \mu^*\}$ when n is even. These observations, together with Proposition 17 and Proposition 18, lead to the following corollary:

Proposition 19. *Let $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ be anonymous and top-majoritarian. For any $R_N \in \mathfrak{R}^N$, we have*

- (i) $f_\sigma(R_N) = C(R_N; n, \mu^*)$ if n is odd.
- (ii) $C(R_N; n, \mu) \subseteq f_\sigma(R_N) \subseteq C(R_N; n, \mu^*)$ if n is even.

We now characterize F_σ and F_σ^* . A SCC $F : \Pi^N \rightarrow \mathcal{A}_k$ is *anonymous* iff given any permutation $p : N \rightarrow N$ of individuals and any $(\rho_i)_{i \in N} \in \Pi^N$, we have $F((\rho_i)_{i \in N}) = F((\rho_{p(i)})_{i \in N})$. We first show that under anonymous SCCs, (universal) equilibrium outcomes are sets that are (universal) (n, b_F^+) -Condorcet winners with respect to preferences over sets.

Theorem 20. *If $F : \Pi^N \rightarrow \mathcal{A}_k$ is anonymous, then $F_\sigma(\rho_N) \subseteq C(\rho_N; n, b_F^+)$ and $F_\sigma^*(\rho_N) \subseteq C^*(\rho_N; n, b_F^+)$ hold for every $\rho_N \in \Pi^N$.*

Proof. Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be anonymous. Take any $\rho_N \in \Pi^N$. We first show $F_\sigma(\rho_N) \subseteq C(\rho_N; n, b_F^+)$. Take any $X \in F_\sigma(\rho_N)$. By Proposition 16, there exists $R_N \in \alpha(\rho_N)$ such that $X \in f_\sigma(R_N)$. The anonymity of F implies the anonymity of its equivalent SCH $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$. So by Proposition 17, $X \in C(R_N; n, b_f^+)$, implying $X \in C(\rho_N; n, b_F^+)$. These arguments, mutatis mutandis, establish $F_\sigma^*(\rho_N) \subseteq C^*(\rho_N; n, b_F^+)$. \square

A SCC $F : \Pi^N \rightarrow \mathcal{A}_k$ is *top-unanimous* iff given any $X \in \mathcal{A}_k$ and any $\rho_N \in \Pi^N$ with $\max_k(\rho_i) = X \forall i \in N$, we have $F(\rho_N) = X$. We now show that under top-unanimous SCCs, sets that are (universal) (n, b_F^-) -Condorcet winners with respect to preferences over sets are (universal) equilibrium outcomes.

Theorem 21. *If $F : \Pi^N \rightarrow \mathcal{A}_k$ is top-unanimous, then $C(\rho_N; n, b_F^-) \subseteq F_\sigma(\rho_N)$ and $C^*(\rho_N; n, b_F^-) \subseteq F_\sigma^*(\rho_N)$ hold for every $\rho_N \in \Pi^N$.*

Proof. Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be top-unanimous. Take any $\rho_N \in \Pi^N$. We first show $C(\rho_N; n, b_F^-) \subseteq F_\sigma(\rho_N)$. Take any $X \in C(\rho_N; n, b_F^-)$. So $X \in C(R_N; n, b_F^-)$ for some $R_N \in \alpha(\rho_N)$. Since the extension map α satisfies Condition Γ , the top-unanimity of F implies the top-unanimity of its equivalent SCH $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$. So by Proposition 18, $X \in f_\sigma(R_N)$ and by Proposition 16, $X \in F_\sigma(\rho_N)$. These arguments, mutatis mutandis, establish $C^*(\rho_N; n, b_F^-) \subseteq F_\sigma^*(\rho_N)$. \square

A SCC $F : \Pi^N \rightarrow \mathcal{A}_k$ is *top-majoritarian* iff given any $X \in \mathcal{A}_k$ and any $\rho_N \in \Pi^N$ with $\#\{i \in N : \max_k(\rho_i) = X\} \geq \mu^*$, we have $F(\rho_N) = X$. Under top-majoritarian and anonymous SCCs, (universal) equilibrium outcomes coincide with Condorcet winners in the usual sense, as we state and show below.

Theorem 22. *Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be anonymous and top-majoritarian. For any $\rho_N \in \Pi^N$, we have*

(i) $F_\sigma(\rho_N) = C(\rho_N; n, \mu^*)$ and $F_\sigma^*(\rho_N) = C^*(\rho_N; n, \mu^*)$ if n is odd.

(ii) $C(\rho_N; n, \mu) \subseteq F_\sigma(\rho_N) \subseteq C(\rho_N; n, \mu^*)$ and $C^*(\rho_N; n, \mu) \subseteq F_\sigma^*(\rho_N) \subseteq C^*(\rho_N; n, \mu^*)$ if n is even.

Proof. Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be anonymous and top-majoritarian. Take any $\rho_N \in \Pi^N$. We prove (i). Let n be odd. We first show $F_\sigma(\rho_N) = C(\rho_N; n, \mu^*)$. Take any $X \in F_\sigma(\rho_N)$. By Proposition 16, $X \in f_\sigma(R_N)$ for some $R_N \in \alpha(\rho_N)$ where $f : \mathcal{D}^N \rightarrow \mathcal{A}_k$ is the equivalent SCH of F . The anonymity of F implies the anonymity of f . Moreover, as F is top-majoritarian and α satisfies Condition Γ , f is also top-majoritarian. So by Proposition 19, $X \in C(R_N; n, \mu^*)$, hence $X \in C(\rho_N; n, \mu^*)$. Now take any $X \in C(\rho_N; n, \mu^*)$. So $X \in C(R_N; n, \mu^*)$ for some $R_N \in \alpha(\rho_N)$. As f is anonymous and top-majoritarian, we have $X \in f_\sigma(R_N)$ by Proposition 19 and $X \in F_\sigma(\rho_N)$ by Proposition 16. Thus, $F_\sigma(\rho_N) = C(\rho_N; n, \mu^*)$. These arguments, mutatis mutandis, establish $F_\sigma^*(\rho_N) = C^*(\rho_N; n, \mu^*)$.

We now prove (ii). Let n be even. Recall that f is anonymous and top-majoritarian. Take any $X \in C(\rho_N; n, \mu)$. So $X \in C(R_N; n, \mu)$ for some R_N

$\in \alpha(\rho_N)$ and by Proposition 19, $X \in f_\sigma(R_N)$, which, by Proposition 16, implies $X \in F_\sigma(\rho_N)$. Now take any $X \in F_\sigma(\rho_N)$. By Proposition 16, $X \in f_\sigma(R_N)$ for some $R_N \in \alpha(\rho_N)$ and by Proposition 19, $X \in C(R_N; n, \mu^*)$, hence $X \in C(\rho_N; n, \mu^*)$, establishing $C(\rho_N; n, \mu) \subseteq F_\sigma(\rho_N) \subseteq C(\rho_N; n, \mu^*)$. These arguments, mutatis mutandis, establish $C^*(\rho_N; n, \mu) \subseteq F_\sigma^*(\rho_N) \subseteq C^*(\rho_N; n, \mu^*)$. \square

Theorems 20, 21, and 22 characterize $F_\sigma(\rho_N)$ and $F_\sigma^*(\rho_N)$ in terms of sets which are generalized Condorcet winners with respect to preferences over sets. However, this is not informative about the properties of equilibrium outcomes with respect to individual preferences over alternatives. We devote the next section to a clarification of this issue.

3.4. Sets as generalized Condorcet winners

Let $U_\alpha(X; \rho) = \{Y \in \mathcal{A}_k : Y P X \forall R \in \alpha(\rho)\}$ be the *upper contour set* of $X \in \mathcal{A}_k$ at $\rho \in \Pi$ with respect to α . So $U_\alpha(X; \rho)$ contains the k -element sets which are ranked above X at every ranking of sets admissible under ρ . Note that by Condition Γ , $\max_k(\rho) \in U_\alpha(X; \rho)$ for all $X \in \mathcal{A}_k \setminus \{\max_k(\rho)\}$. Clearly, any $R \in \alpha(\rho)$ does not contradict U_α , i.e., if $R \in \alpha(\rho)$ then $X R Y \implies Y \notin U_\alpha(X; \rho) \forall X, Y \in \mathcal{A}_k$. On the other hand, not necessarily every $R \in \mathfrak{R}$ which does not contradict U_α is in $\alpha(\rho)$. We introduce a richness condition which ensures the latter implication. We qualify α as *rich* iff given any $\rho \in \Pi$ and any $R \in \mathfrak{R}$ with X

$R Y \implies Y \notin U_\alpha(X; \rho) \forall X, Y \in \mathcal{A}_k$, we have $R \in \alpha(\rho)$. Throughout this section, we consider rich extension maps.⁸

Pick some $q \in \{0, 1, \dots, n+1\}$. At each $\rho_N \in \Pi^N$, we induce through $U_\alpha(X; \rho)$, a binary relation $\delta_\alpha(\rho_N; q)$ over \mathcal{A}_k as follows: Given any $X, Y \in \mathcal{A}_k$, we have $Y \delta_\alpha(\rho_N; q) X \iff \#\{i \in N : Y \in U_\alpha(X; \rho_i)\} \geq q$.

Proposition 23. *Given any $q \in \{0, 1, \dots, n+1\}$, any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$, we have*

(i) $X \in C(\rho_N; n, q) \iff X$ is undominated with respect to $\delta_\alpha(\rho_N; q)$.

(ii) $X \in C^*(\rho_N; n, q) \iff X$ is dominant with respect to $\delta_\alpha(\rho_N; n - q + 1)$.

Proof. Take any $q \in \{0, 1, \dots, n+1\}$, any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$.

We prove (i). To show the “if” part, let X be undominated with respect to $\delta_\alpha(\rho_N; q)$. So for any $Y \in \mathcal{A}_k$ we have $\#\{i \in N : Y \in U_\alpha(X; \rho_i)\} < q$. Pick some $R_N \in \alpha(\rho_N)$ such that for any $i \in N$ we have $X R_i Y$ for all $Y \notin U_\alpha(X; \rho_i)$. The existence of R_N is ensured by the richness of α . By construction of R_N , we have $\#\{i \in N : Y P_i X\} < q$ for all $Y \in \mathcal{A}_k$. Thus $X \in C(R_N; n, q)$, which in turn implies $X \in C(\rho_N; n, q)$. To show the “only if” part, suppose $Y \delta_\alpha(\rho_N; q) X$ for some $Y \in \mathcal{A}_k$, which implies $\#\{i \in N : Y \in U_\alpha(X; \rho_i)\} \geq q$. Hence, $X \in C(R_N; n, q)$ fails for every $R_N \in \alpha(\rho_N)$, establishing $X \notin C(\rho_N; n, q)$.

We prove (ii). To show the “if” part, let $X \notin C^*(\rho_N; n, q)$. So there exists $Y \in \mathcal{A}_k$ such that $\#\{i \in N : Y P_i X\} \geq q$ for some $R_N \in \alpha(\rho_N)$. Thus,

⁸Consider the preference relation given in the Example 15. Under RS extension, α is not rich either R_1 or R_2 is not in α .

$\#\{i \in N : X \in U_\alpha(Y; \rho_i)\} \leq n - q$, implying the failure of $X \delta_\alpha(\rho_N; n - q + 1) Y$. Hence, X is not dominant with respect to $\delta_\alpha(\rho_N; n - q + 1)$. To show the “only if” part, let $X \delta_\alpha(\rho_N; n - q + 1) Y$ fail for some $Y \in \mathcal{A}_k \setminus \{X\}$. So $\#\{i \in N : X \in U_\alpha(Y; \rho_i)\} < n - q + 1$. Hence, there exists $R_N \in \alpha(\rho_N)$ with $\#\{i \in N : Y P_i X\} \geq q$, which implies $X \notin C^*(\rho_N; n, q)$. \square

Clearly, the structure of $U_\alpha(X; \rho)$ and $\delta_\alpha(\rho_N; q)$, hence of $C^*(\rho_N; n, q)$ and $C(\rho_N; n, q)$ depends on the particularities of α . We consider two well-known extension maps. The first extension map we consider is a separability condition which Roth and Sotomayor (1990) call “responsiveness”.⁹ We call this the RS extension and define it as follows: For any $\rho \in \Pi$ and any $R \in \mathfrak{R}$, we have $R \in \alpha(\rho) \iff X P Y \forall X, Y \in A_k$ with $X = (Y \setminus \{y\}) \cup \{x\}$ for some $y \in Y$ and $x \in A \setminus Y$ satisfying $x \rho y$. Note that the RS extension is rich.

Given any $\rho \in \Pi$, let, without loss of generality, $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ such that $x_j \rho x_{j+1}$ and $y_j \rho y_{j+1}$ for all $j \in \{1, \dots, k - 1\}$. Given any $\rho \in \Pi$, we introduce the following componentwise dominance relation $\gamma(\rho)$ over A_k : For any $X, Y \in A_k$, $Y \gamma(\rho) X \iff y_j \rho x_j$ for all $j \in \{1, \dots, k\}$.

Proposition 24. *Let α be the RS extension. For any $\rho \in \Pi$ and any distinct $X, Y \in \mathcal{A}_k$, we have $Y \in U_\alpha(X; \rho) \iff Y \gamma(\rho) X$.*

⁹Nevertheless, Erdamar and Sanver (2008) argue for the inappropriateness of this axiom when sets are non-resolute outcomes.

Proof. Let α be the RS extension. Take any $\rho \in \Pi$ and any $X, Y \in \mathcal{A}_k$. To show the “if” part let $Y \gamma(\rho) X$. Write, without loss of generality, $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$ such that $x_j \rho x_{j+1}$ and $y_j \rho y_{j+1}$ for all $j \in \{1, 2, \dots, k-1\}$. Since $Y \gamma(\rho) X$, we have $y_j \rho x_j$ for all $j \in \{1, 2, \dots, k-1\}$. Let $Z_0 = Y$, $Z_k = X$, and $Z_s = \{x_1, \dots, x_s, y_{s+1}, \dots, y_k\}$ for $s \in \{1, 2, \dots, k-1\}$. As α is the RS extension, for any $R \in \alpha(\rho)$ we have $Z_t R Z_{t+1}$ for each $t \in \{0, 1, 2, \dots, k-1\}$ and $Z_t P Z_{t+1}$ for some $t \in \{0, 1, 2, \dots, k-1\}$. Hence, by transitivity of every $R \in \alpha(\rho)$, we have $Y \in U_\alpha(X; \rho)$. To show the “only if” part, let $Y \in U_\alpha(X; \rho)$. Suppose, for a contradiction, that $Y \gamma(\rho) X$ fails. So $x_j \rho^* y_j$ for some $j \in \{1, 2, \dots, k\}$. Note that if the RS extension does not enforce a ranking between X and Y , then by richness any ranking between X and Y is possible for some $R \in \alpha(\rho)$. The RS extension enforces $Y P X$ for any $R \in \alpha(\rho)$, by transitivity, only if there is a sequence of sets $\{Z_t\}$ such that $Z_0 = Y$, $Z_n = X$, $Z_t = (Z_{t-1} \setminus \{z_{t-1}\}) \cup \{z_t\}$ with $z_{t-1} \rho z_t$ for $t \in \{1, 2, \dots, n\}$. But since $x_j \rho^* y_j$ we can not find such a sequence. Hence, we have $Y \notin U_\alpha(X; \rho)$. This is a contradiction. \square

Proposition 24 establishes, under the RS extension, the relationship between $\gamma(\rho)$ and $U_\alpha(\cdot; \rho)$. The general relationship between $U_\alpha(X; \rho)$ and $\delta_\alpha(\rho_N; q)$ is given by definition. Hence we have the relationship between $\gamma(\rho)$ and $\delta_\alpha(\rho_N; q)$ under the RS extension, which we state in the following proposition.

Proposition 25. *Let α be the RS extension. For any $q \in \{0, 1, \dots, n+1\}$, any $\rho_N \in \Pi^N$, and any $X \in \mathcal{A}_k$ we have*

(i) *X is undominated with respect to $\delta_\alpha(\rho_N; q) \iff$ there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \gamma(\rho_i) X\} \geq q$.*

(ii) *X is dominant with respect to $\delta_\alpha(\rho_N; q) \iff \#\{i \in N : X \gamma(\rho_i) Y\} \geq q$ for any $Y \in \mathcal{A}_k \setminus \{X\}$.*

The three theorems below follow as a corollary to Theorems 20-22, Proposition 23, and Proposition 25.

Theorem 26. *Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be anonymous and α be the RS extension. For any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$, we have*

(i) *$X \in F_\sigma(\rho_N) \implies$ there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \gamma(\rho_i) X\} \geq b_F^+$.*

(ii) *$X \in F_\sigma^*(\rho_N) \implies \#\{i \in N : X \gamma(\rho_i) Y\} \geq n - b_F^+ + 1$ for all $Y \in \mathcal{A}_k \setminus \{X\}$.*

Theorem 26 announces for anonymous SCCs and under the RS extension that a set X is a strong equilibrium outcome only if for any other set Y , the number of individuals i for whom Y dominates X with respect to the binary relation $\gamma(\rho_i)$ is less than b_F^+ ; and X is a universal strong equilibrium outcome only if for any other set Y , the number of individuals i for whom X dominates Y with respect to the binary relation $\gamma(\rho_i)$ is at least $n - b_F^+ + 1$.

Theorem 27. *Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be top-unanimous and α be the RS extension.*

For any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$, we have

(i) There exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \gamma(\rho_i) X\} \geq b_F^- \implies X \in F_\sigma(\rho_N)$.

(ii) $\#\{i \in N : X \gamma(\rho_i) Y\} \geq n - b_F^- + 1$ for all $Y \in \mathcal{A}_k \setminus \{X\} \implies X \in F_\sigma^(\rho_N)$.*

Theorem 27 announces for top-unanimous SCCs and under the RS extension that a set X is a strong equilibrium outcome if for any other set Y , the number of individuals i for whom Y dominates X with respect to the binary relation $\gamma(\rho_i)$ is less than b_F^- ; and X is a universal strong equilibrium outcome if for any other set Y , the number of individuals i for whom X dominates Y with respect to the binary relation $\gamma(\rho_i)$ is at least $n - b_F^- + 1$.

By conjoining the two partial characterizations established by Theorem 26 and Theorem 27, we obtain a full characterization for anonymous and top-majoritarian (hence top-unanimous) SCCs, where generalized Condorcet winners coincide with the standard definition of the concept. This is formally stated by the following theorem, whose proof is left to the reader:

Theorem 28. *Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be anonymous and top-majoritarian while α is the RS extension. For any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$, we have*

(a) For n being odd,

(i) $X \in F_\sigma(\rho_N) \iff$ there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \gamma(\rho_i) X\} \geq \mu^*$.

(ii) $X \in F_\sigma^*(\rho_N) \iff \#\{i \in N : X \gamma(\rho_i) Y\} \geq n - \mu^* + 1$ for all $Y \in \mathcal{A}_k \setminus \{X\}$.

(b) For n being even,

(i) There exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \gamma(\rho_i) X\} \geq \mu \implies X \in F_\sigma(\rho_N) \implies$ there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \gamma(\rho_i) X\} \geq \mu^*$.

(ii) $\#\{i \in N : X \gamma(\rho_i) Y\} \geq n - \mu + 1$ for all $Y \in \mathcal{A}_k \setminus \{X\} \implies X \in F_\sigma^*(\rho_N) \implies \#\{i \in N : X \gamma(\rho_i) Y\} \geq n - \mu^* + 1$ for all $Y \in \mathcal{A}_k \setminus \{X\}$.

The next extension map we consider is a modified version of the extension map introduced by Kelly (1977): We say that α is the Kelly extension iff given any $\rho \in \Pi$ and any $R \in \mathfrak{R}$ we have $R \in \alpha(\rho)$ if and only if

(i) $X P Y$ for all distinct $X, Y \in A_k$ with $x \rho^* y \forall x \in X, \forall y \in Y$

and

(ii) $\max_k(\rho) P Y \forall Y \in A_k \setminus \{\max_k(\rho)\}$.

Note that the Kelly extension is rich. Also remark that for any $X, Y \in A_k$ $Y \in U_\alpha(X; \rho)$ implies $X \cap Y = \emptyset$ or $Y = \max_k(\rho)$. We write $\max(X; \rho) \in A$ for the best element in $X \in A_k$ with respect to $\rho \in \Pi$, i.e., $\max(X; \rho) \rho x$ holds for all $x \in X$. Similarly, $\min(X; \rho) \in A$ is the worst element in $X \in A_k$ with respect to $\rho \in \Pi$, i.e., $x \rho \min(X; \rho)$ holds for all $x \in X$. Given any $\rho \in \Pi$, we introduce the following absolute dominance relation $\kappa(\rho)$ over A_k : For any $X, Y \in A_k$, $Y \kappa(\rho) X$

$\iff \min(Y; \rho) \rho^* \max(X; \rho)$ or $Y = \max_k(\rho)$. Note that under the Kelly extension, given any $\rho \in \Pi$ and any distinct $X, Y \in A_k$, we have $Y \in U_\alpha(X; \rho) \iff Y \kappa(\rho) X$. This relationship between $\kappa(\rho)$ and $U_\alpha(\cdot; \rho)$ which exists under the Kelly extension conjoined with the general relationship between $U_\alpha(X; \rho)$ and $\delta_\alpha(\rho_N; q)$ establishes the relationship between $\kappa(\rho)$ and $\delta_\alpha(\rho_N; q)$ under the Kelly extension, which we state in the following proposition.

Proposition 29. *Let α be the Kelly extension. Take any $q \in \{0, 1, \dots, n + 1\}$, any $\rho_N \in \Pi^N$, and any $X \in A_k$.*

(i) *X is undominated with respect to $\delta_\alpha(\rho_N; q) \iff$ there exists no $Y \in A_k \setminus \{X\}$ with $\#\{i \in N : Y \kappa(\rho_i) X\} \geq q$.*

(ii) *X is dominant with respect to $\delta_\alpha(\rho_N; q) \iff \#\{i \in N : X \kappa(\rho_i) Y\} \geq q$ for all $Y \in A_k \setminus \{X\}$.*

The three theorems below follow as a corollary to Theorems 20-22, Proposition 23, and Proposition 29.

Theorem 30. *Let $F : \Pi^N \rightarrow A_k$ be anonymous and α be the Kelly extension.*

For any $\rho_N \in \Pi^N$ and any $X \in A_k$, we have

(i) *$X \in F_\sigma(\rho_N) \implies$ there exists no $Y \in A_k \setminus \{X\}$ with $\#\{i \in N : Y \kappa(\rho_i) X\} \geq b_F^+$.*

(ii) *$X \in F_\sigma^*(\rho_N) \implies \#\{i \in N : X \kappa(\rho_i) Y\} \geq n - b_F^+ + 1$ for all $Y \in A_k \setminus \{X\}$.*

Theorem 30 announces for anonymous SCCs and under the Kelly extension that a set X is a strong equilibrium outcome only if for any other set Y , the number of individuals, for whom the worst element of Y is better than the best element of X or Y is the set of best k alternatives, is less than b_F^+ . On the other hand, X is a universal strong equilibrium outcome only if for any other set Y , the number of individuals, for whom the worst element of X is better than the best element of Y or X is the set of best k alternatives, is at least $n - b_F^+ + 1$.

Theorem 31. *Let $F : \Pi^N \rightarrow A_k$ be top-unanimous and α be the Kelly extension. For any $\rho_N \in \Pi^N$ and any $X \in A_k$, we have*

(i) *There exists no $Y \in A_k \setminus \{X\}$ with $\#\{i \in N : Y \kappa(\rho_i) X\} \geq b_F^- \implies X \in F_\sigma(\rho_N)$.*

(ii) *$\#\{i \in N : X \kappa(\rho_i) Y\} \geq n - b_F^- + 1$ for all $Y \in A_k \setminus \{X\} \implies X \in F_\sigma^*(\rho_N)$.*

Theorem 31 announces for top-unanimous SCCs and under the Kelly extension that a set X is a strong equilibrium outcome if for any other set Y , the number of individuals, for whom the worst element of Y is better than the best element of X or Y is the set of best k alternatives, is less than b_F^- . Moreover, X is a universal strong equilibrium outcome only if for any other set Y , the number of individuals, for whom the worst element of X is better than the best element of Y or X is the set of best k alternatives, is at least $n - b_F^- + 1$.

By conjoining the two partial characterizations established by Theorem 30 and Theorem 31, we obtain a full characterization for anonymous and top-majoritarian (hence top-unanimous) SCCs, where generalized Condorcet winners coincide with the standard definition of the concept. This is formally stated by the following theorem, whose proof is left to the reader:

Theorem 32. *Let $F : \Pi^N \rightarrow \mathcal{A}_k$ be anonymous and top-majoritarian while α is the Kelly extension. For any $\rho_N \in \Pi^N$ and any $X \in \mathcal{A}_k$, we have*

(a) *For n being odd,*

(i) $X \in F_\sigma(\rho_N) \iff$ *there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \kappa(\rho_i) X\} \geq \mu^*$.*

(ii) $X \in F_\sigma^*(\rho_N) \iff \#\{i \in N : X \kappa(\rho_i) Y\} \geq n - \mu^* + 1$ *for all $Y \in \mathcal{A}_k \setminus \{X\}$.*

(b) *For n being even,*

(i) *There exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \kappa(\rho_i) X\} \geq \mu \implies X \in F_\sigma(\rho_N) \implies$ there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ with $\#\{i \in N : Y \kappa(\rho_i) X\} \geq \mu^*$.*

(ii) $\#\{i \in N : X \kappa(\rho_i) Y\} \geq n - \mu + 1$ *for all $Y \in \mathcal{A}_k \setminus \{X\} \implies X \in F_\sigma^*(\rho_N) \implies \#\{i \in N : X \kappa(\rho_i) Y\} \geq n - \mu^* + 1$ for all $Y \in \mathcal{A}_k \setminus \{X\}$.*

Now, we explore the existence of (n, q) -Condorcet winners and universal (n, q) -Condorcet winners. Our approach is similar to that of Sertel and Sanver (2004). We find a lower bound for q which ensures the existence of (n, q) -Condorcet winner and an upper bound for q at which higher values q implies

non-existence of the universal (n, q) -Condorcet winner. Hence, for the universal Condorcet winner case we provide a condition for non-existence rather than existence. For this purpose we define $q(n, c) = \lfloor n(c-1)/c \rfloor + 1$ where $c = \frac{m!}{k!(m-k)!}$ is the number of k combinations from set of alternatives and $\lfloor \cdot \rfloor$ denotes the floor function. The following proposition shows the acyclicity of the binary relation $\delta_\alpha(\rho_N, q(n, c))$.

Proposition 33. *For any $\rho_N \in \Pi^N$ and $j \in \{2, 3, \dots, c\}$, we have $X_i \delta_\alpha(\rho_N, q(n, c)) X_{i+1}$ for all $i \in \{1, 2, \dots, j-1\} \Rightarrow \sim X_j \delta_\alpha(\rho_N, q(n, c)) X_1$ where $X_i \in \mathcal{A}_k$ for all $i \in \{1, 2, \dots, j-1\}$.¹⁰*

Proof. Take any $j \in \{2, 3, \dots, c\}$, any $\rho_N \in \Pi^N$ and assume that $X_i \delta_\alpha(\rho_N, q(n, c)) X_{i+1}$ for all $i \in \{1, 2, \dots, j-1\}$. Denote $K_i = \{t \in N : X_i \in U_\alpha(X_{i+1}; \rho_t)\}$ for all $i \in \{1, 2, \dots, j-1\}$. Note that since $X_i \in U_\alpha(X_{i+1}; \rho_t)$ and $X_{i+1} \in U_\alpha(X_{i+2}; \rho_t)$ implies $X_i \in U_\alpha(X_{i+2}; \rho_t)$, we have $\{t \in N : X_1 \in U_\alpha(X_j; \rho_t)\} \supseteq \bigcap_{i=1}^{j-1} K_i$. Note also that $\#\left(\bigcap_{i=1}^{j-1} K_i\right)^C = \#\bigcup_{i=1}^{j-1} K_i^C \leq \sum_{i=1}^{j-1} \#K_i^C \leq (j-1)(n-q(n, c))$. Hence, $\#\bigcap_{i=1}^{j-1} K_i \geq n - (j-1)(n-q(n, c))$ implying that $\#\{t \in N : X_j \in U_\alpha(X_1; \rho_t)\} \leq (j-1)(n-q(n, c))$. Thus it is enough to have $(c-1)(n-q(n, c)) < q(n, c)$ in order to get acyclicity. Hence, $\frac{c-1}{c}n < q(n, c)$ or $q(n, c) = \lfloor n(c-1)/c \rfloor + 1$ is a sufficient condition for acyclicity □

¹⁰Essentially our proposition is the same as Nakamura's (1979) result in a slightly different framework. See also Moulin (1981) for a generalization of Nakamura's result in the context of effectivity functions.

Proposition 34. *For any $\rho_N \in \Pi^N$ and $q \in \{0, 1, \dots, n + 1\}$, if we have X is undominated with respect to $\delta_\alpha(\rho_N, q)$, then $q^* > n - q + 1$ assures that X is not dominant with respect to $\delta_\alpha(\rho_N, q^*)$.*

Proof. Assume that X is undominated with respect to $\delta_\alpha(\rho_N, q)$. Then, there exists no $Y \in \mathcal{A}_k \setminus \{X\}$ such that $\{i \in N : Y \in U_\alpha(X; \rho_i)\} \geq q$. Hence, for any $Y \in \mathcal{A}_k \setminus \{X\}$ we have $\{i \in N : Y \in U_\alpha(X; \rho_i)\} < q$. Then, it is possible that $q - 1$ individuals can rank Y above X for some $R_N \in \alpha(\rho_N)$. Therefore, X cannot be dominant with respect to $\delta_\alpha(\rho_N, q^*)$ if $q^* > n - q + 1$. \square

By Propositions 33 and 34 we conclude that X is undominated with respect to $\delta_\alpha(\rho_N, q(n, c))$ and X cannot be dominant with respect to $\delta_\alpha(\rho_N, q^*)$ when $q^* > n - q(n, c) + 1$. So although we do not know the minimal q assuring the existence of universal (n, q) -Condorcet winner, we know that in order to assure existence we have to restrict ourselves to a subset of $\{0, 1, 2, \dots, n - q(n, c) + 1\}$. The following theorem follows from Propositions 23, 33, and 34.

Theorem 35. *For every $\rho_N \in \Pi^N$ we have $C(\rho_N; n, q) \neq \emptyset$ for all $q \geq q(n, c)$ and $C^*(\rho_N; n, q^*) = \emptyset$ for all $q^* < q(n, c)$.*

3.5. Conclusion

We characterize equilibrium outcomes of voting games induced by a fairly large class of resolute social choice correspondences. We show that under a reasonable extension map,

- For anonymous SCCs, (universal) strong equilibrium outcomes are sets that are (universal) generalized Condorcet winners with respect to preferences over sets.
- For top-unanimous SCCs, sets that are (universal) generalized Condorcet winners with respect to preferences over sets are (universal) strong equilibrium outcomes.
- The parameter q which generalizes the concept of a Condorcet winner depends on the power distribution induced by the instituted social choice correspondence. For top-majoritarian social choice correspondences, q becomes equal to the usual majority, hence (n, q) -Condorcet winners coincide with the standard definition of a Condorcet winner.

These results sound as transferring the findings of Sertel and Sanver (2004) to our more general environment.¹¹ However, they can be further elaborated by revealing the meaning of “being a generalized Condorcet winner with respect to preferences over sets” in terms of preferences over alternatives. We do this for

¹¹When $k = 1$, the definition of “being a universal strong equilibrium outcome” and of “being a strong equilibrium outcome” are equivalent. Our findings -naturally- comply with this equivalence.

two well-known extension maps of the literature, namely the RS extension and the Kelly extension. For each of these, we are able to express (universal) strong equilibrium outcomes in terms of their dominance/undominance with respect to a binary relation derived from individual preferences over alternatives. Moreover, through this binary relation we provide an existence/non-existence condition for (universal) generalized Condorcet winners.

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