III

GRAVITATION AND THE ELECTRON[1]

In order to describe physical quantities by numerical data we must refer them to a Cartesian system of axes in space which consists of three mutually perpendicular vectors e of length 1. Transition to another allowable set of axes is accomplished by an orthogonal transformation or rotation

$$e'_\alpha = \Sigma_\beta O_{\alpha\beta} e_\beta. \tag{1}$$

Vectors and tensors are quantities which are determined relative to an axis system by a set of numbers, their components, in such a way that these components are transformed in a definite way on transition to another axis system. More precisely, the linear transformation of the components is related to the arbitrary rotation (1) by a certain law in such a way that to composition of two rotations corresponds the composition of the two associated transformations. Such a correspondence between the elements of a group—here the group of rotations—and linear transformations is called a representation of the group. *Each kind of quantity is characterized by a definite representation of the rotation group.* However the familiar vectors and tensors are not the only quantities of this kind; a possibility which had not cropped up previously in physics is necessary for the description of the electron spin in wave mechanics.

A rotation is a transformation of the unit sphere into

[1] For similar treatments see H. Weyl, *Gravitation and the Electron*, Proc. Nat. Acad. Sciences, V, 15, No. 4, pp. 323–34, 1929, and *Elektron und Gravitation I*, Zeit. für Phy., 56 Bd., 5 & 6 Hft., p. 330, 1929. Full references are included in these papers.

itself. The sphere can be projected stereographically onto its equatorial plane, which is to be considered as described by the complex variable $\zeta = x + iy$. We then replace the one complex coordinate ζ on the sphere by two homogeneous coordinates ψ_1, ψ_2 by writing $\zeta = \psi_2/\psi_1$ (in order to include in the representation the center of projection on the sphere). The formulas for the stereographic projection are then (x_α being the Cartesian coordinates in space)

$$x_1 : x_2 : x_3 : 1 =$$

$$\psi_2\bar{\psi}_1 + \psi_1\bar{\psi}_2 : \frac{1}{i}(\psi_2\bar{\psi}_1 - \psi_1\bar{\psi}_2) : \psi_1\bar{\psi}_1 - \psi_2\bar{\psi}_2 : \psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2 \tag{2}$$

Each rotation s of the sphere can now be represented by a unitary transformation of the coordinates ψ_1, ψ_2, i.e., by a linear transformation (s) which leaves $\psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_2$ invariant (Cayley and Helmholtz). Of course this transformation (s) is only definite in the homogeneous sense, so that transformations which are obtained from each other by multiplying all coefficients by a number $e^{i\lambda}$ of absolute value 1 are to be considered as the same. We can avoid this arbitrariness by a normalization, requiring that the determinant of the transformation shall be $+1$; it is still double-valued, however, since multiplication of all coefficients by -1 does not disturb the normalization. (2) are Hermitean forms in ψ_1, ψ_2 with the coefficient matrices

$$S_1 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \quad S_2 = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix}, \quad S_3 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}. \tag{3}$$

Denoting by ψ the matrix with the one *column* ψ_1, ψ_2 and by $\bar{\psi}$ the matrix with one *row* consisting of the conjugate complex quantities $\bar{\psi}_1$, $\bar{\psi}_2$, the expressions (2) for the rectangular components become

$$x_\alpha = \bar{\psi} S_\alpha \psi.$$

The law of transformation of the ψ is consequently determined by the requirement that the three quantities $\bar{\psi} S_\alpha \psi$

transform like the components of a vector. A quantity having two components with this law of transformation describes the wave field of an electron in Pauli's theory of the electron spin; both components of ψ are here of course functions of position (and time).

It follows immediately from the above law that

$$\Sigma_\alpha S_\alpha \frac{\partial}{\partial x_\alpha} = \nabla$$

is a differential operator independent of coordinate system which transforms ψ into quantity $\nabla\psi$ of the same kind. Dirac remarked that the reiterated operator $\nabla\nabla$ is the Poisson operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

He therefore replaced the de Broglie-Schrödinger wave equation of second order for the scalar quantity ψ, which cannot take account of the spin, by two first order equations in the two components ψ_1, ψ_2 in which this differential operator ∇ plays the same rôle as the Poisson operator in the scalar theory, and was in this way able to give a most satisfactory explanation of the anomolous Zeeman effect in all its details.

But he did still more. We must really operate in the four-dimensional world—instead of in three-dimensional space—which has, in accordance with the special theory of relativity a geometry (Minkowski's geometry) analogous to that of three-dimensional Euclidean space. A Cartesian system of axes there consists of three real space-like vectors $e(1)$, $e(2)$, $e(3)$ and a pure imaginary vector $e(0)$; we expressly demand that the real time-like vector $e(0)/i$ point toward the future. Unfortunately the quantity ψ has then four components ψ_1^+, ψ_2^+; ψ_1^-, ψ_2^- instead of two. $S(\alpha)$ are now linear transformations of the four components under which the two $+$-components and the two $-$-components transform among themselves; for the $+$-components it equals the S_α given above and differs only in sign for the $-$-components. To them must be added

$$S(0) = \begin{Vmatrix} i & 0 \\ 0 & i \end{Vmatrix}$$

The law of transformation of the components of ψ is again to be described as one under which the four quantities $\bar{\psi}S(\alpha)\psi$ transform like the components of a four-vector under transition from one Cartesian coordinate system to another.

The Dirac wave equations arise from an *action principle* which is additively composed of the two integral invariants:

$$(4) \qquad \frac{h}{i}\int \bar{\psi}S(\alpha)\frac{\partial\psi}{\partial x_\alpha}dx \qquad (dx = dx_0 dx_1 dx_2 dx_3)$$

(α summed over 0, 1, 2, 3), and

$$(5) \quad cm\int(\psi_1^+\bar{\psi}_1^- + \psi_2^+\bar{\psi}_2^- + \psi_1^-\bar{\psi}_1^+ + \psi_2^-\bar{\psi}_2^+)dx = cm\int(\bar{\psi}^-\psi^+ + \bar{\psi}^+\psi^-)dx$$

m being the mass of the electron. The components of ψ are the quantities which are to be varied. This holds as long as there is no electromagnetic field present; if there be such, the operator $\frac{\partial}{\partial x_\alpha}$ acting on ψ must be replaced by $\frac{\partial}{\partial x_\alpha} + \frac{ie}{h}\varphi_\alpha$ ($-e$ the charge of the electron, $\frac{h}{2\pi}$ the quantum of action and φ the components of the electromagnetic potential). In addition to Dirac's equation we have Maxwell's equations, in which the four-vector of charge-current density is given by

$$(6) \qquad \rho_\alpha = -e\bar{\psi}S(\alpha)\psi$$

in particular the charge density is

$$(7) \qquad -e(\psi_1^+\bar{\psi}_1^+ + \psi_2^+\bar{\psi}_2^+ + \psi_1^-\bar{\psi}_1^- + \psi_2^-\bar{\psi}_2^-)$$

This corresponds to the circumstances that the Maxwellian action of the electromagnetic field is to be added to (4) and (5) as a third constituent and the φ_α also varied.

These field equations differ strikingly from those of classical physics in that *they contain the fundamental atomic constants* e, h, m. Denoting $\frac{e}{h}\varphi_\alpha$ by φ_α they only occur in the two combinations: e^2/hc, a pure number (the so-called fine structure constant) and h/mc, an atomic length (the "wave length of the electron"). The meaning of the field equations differs radically from those of classical physics. It is not meant that the charge of the electron is really smeared over all space in accordance with (7), as Schrödinger held for a time, but rather that (7) is the probability of localization multiplied by the total charge $-e$. Before the equations can yield correct statistical predictions they must be subjected to the *process of quantization*, which is not entirely cleared up as yet. This process introduces the action constant h again. But that is obviously necessary if the theory is to be in a position to give an account of the atomistic constitution of matter, for in order to determine the three atomistic units of length, time, and mass we need in addition to the velocity of light c two further independent dimensional constants; and the field equations as they appear above contain but one, the length h/mc.

By this new situation, which introduces an atomic radius into the field equations themselves—but not until this step—my principle of *gauge-invariance*, with which I had hoped to relate gravitation and electricity, is robbed of its support. But it is now very agreeable to see that this principle has an equivalent in the quantum-theoretical field equations which is exactly like it in formal respects; the laws are invariant under the simultaneous replacement of ψ by $e^{i\lambda}\psi$, φ_α by $\varphi_\alpha - \frac{\partial\lambda}{\partial x_\alpha}$ where λ is an arbitrary real function of position and time. Also the relation of this property of invariance to the law of conservation of electricity remains exactly as before. The fact that the action integral remains unchanged by the infinitesimal variation

$$\delta\psi = i\lambda\psi, \qquad \delta\varphi_\alpha = -\frac{\partial\lambda}{\partial x_\alpha}$$

(λ an arbitrary function) signifies a dependence between the laws of matter and electromagnetism—which arise from the action integral by variation of the ψ and φ respectively. This identical relation consists in the fact that the law of conservation of electricity

$$\frac{\partial \rho_\alpha}{\partial x_\alpha} = 0$$

follows from the material as well as from the electromagnetic equations. The principle of gauge-invariance has the character of general relativity since it contains an arbitrary function λ, and can certainly only be understood in terms of it.

The tensor calculus is not the proper mathematical instrument to use in translating the quantum-theoretic equations of the electron over into the *general theory of relativity.* Vectors and terms are so constituted that the law which defines the transformation of their components from one Cartesian set of axes to another can be extended to the most general linear transformation, to an affine set of axes. That is not the case for quantity ψ, however; this kind of quantity belongs to a representation of the rotation group which cannot be extended to the affine group. Consequently we cannot introduce components of ψ relative to an arbitrary coordinate system in general relativity as we can for the electromagnetic potential and field strengths. We must rather describe the metric at a point P by local Cartesian axes $e(\alpha)$ instead of by the g_{pq}. The wave field has definite components ψ_1^+, ψ_2^+; ψ_1^-, ψ_2^- relative to such axes, and we know how they transform on transition to any other Cartesian axes in P. The laws shall naturally be invariant under arbitrary rotation of the axes in P, and the axes at different points can be rotated independently of each other; they are in no way bound together. The formal aspects of our theory are similar to Einstein's recent attempts to unify electricity and gravitation; he too employs local Cartesian axes—n-legs, as he calls them—in place of the g_{pq}. But he assumes "*distant parallelism*," i.e., the axes in different points shall be so bound to one another that when rotated at one of them the axes in all other points automatically undergo the same rotation. I do not believe in this distant parallelism at all; there is no indication that Nature has availed herself of such an artificial geometry. I am convinced that if there is a physical content in Einstein's latest formal development it must come to light in the present connection. It now seems to me hopeless to seek a unification of gravitation and electricity without taking the material waves—the ψ field—into account.

For the purpose of analytic expression we need four *coordinates* x_p in addition to these local Cartesian axes. Let $e^p(\alpha)$ be the components of the vector $e(\alpha)$ in this coordinate system; the 4·4 quantities $e^p(\alpha)$ characterize *the gravitational field*. If $t(\alpha)$ be the components of a vector relative to the Cartesian axes, its contravariant components t^p relative to the coordinate system are given by

$$t^p = \sum_\alpha e^p(\alpha) t(\alpha).$$

Conversely the $t(\alpha)$ are obtained from the covariant components t_p, relative to the coordinate system by

$$t(\alpha) = \sum_p e^p(\alpha) t_p.$$

We consequently have upper and lower Latin indices, which refer to the coordinate system, and Greek indices, which belong to the axes. I have described how we are to juggle them.

Let $\int \mathfrak{H} dx$ be the action quantity of matter and the electromagnetic field ("matter in the extended sense"). We subject the $e^p(\alpha)$ to an arbitrary infinitesimal variation which vanishes outside a finite portion of the world and obtain an equation

$$\delta \int \mathfrak{H} dx = \int \mathfrak{t}_p(\alpha) \delta e^p(\alpha) dx.$$

It is here immaterial if or how the ψ and φ are varied, as the material and electromagnetic equations are assumed to hold; these latter state that the change in the action integral brought about by unrestricted variations of the ψ and φ vanishes. As we know, the general theory of relativity first enables us to give a general definition of the tensor density $\mathfrak{t}_p(\alpha)$ of energy by varying the metric field as above. The action integral is invariant under infinitesimal transformations of the coordinates; its variation consequently vanishes if the changes $\delta e^p(\alpha)$ are brought about by deformation of the coordinate system whereas the axes $e(\alpha)$ are held fast. This leads, as is well known, to the four components of the law of conservation of energy and linear momentum

$$\frac{\partial \mathfrak{t}_p{}^q}{\partial x^q} + \mathfrak{t}_q(\alpha) \frac{\partial e^q(\alpha)}{\partial x^p} = 0 \tag{8}$$

(This is admittedly not a true law of conservation in the general theory of relativity on account of the second term occurring in addition to the divergence.) Further, the action is invariant under an infinitesimal rotation, depending arbitrarily on position, of the local axes. This yields, as can be seen immediately, the law of symmetry of the energy tensor; it is here not identically fulfilled, but only in consequence of the material and electromagnetic equations. As we know, this symmetry is essentially identical with the law of conservation of angular momentum. Conservation theorems always result—and this is a general rule—from properties of invariance. We have also found an invariance property—gauge invariance—corresponding to the law of conservation of energy.

I must be more concrete. We wish to take over the principal term (4) of Dirac's action into general relativity. For this purpose we must know how a quantity of kind ψ is to be differentiated covariantly. Consider the fixed point $P=(x_p)$ and a neighboring point $P'=(x_p+dx_p)$; in both P and P' there is a set of axes $e(\alpha)$ and we denote the components of ψ referred to these by $\psi_p=\psi_p(P)$ and $\psi_p(P')$ respectively. The metric determines an infinitesimal parallel displacement which enables us to carry the axes $e(\alpha)$ in P over to P'; we thus obtain a set of Cartesian axes $e'(\alpha)$ in P', and we call the components of ψ in P' relative to these axes ψ'_p. The four differences $\delta\psi_p=\psi'_p-\psi_p$ depend only on the axes in P and transform with them in the same way as the ψ_p themselves. But $e'(\alpha)$ can be obtained from the axes $e(\alpha)$ in P' by an infinitesimal rotation do:

$$\delta e(\alpha)=e'(\alpha)-e(\alpha)=do(\alpha\beta)e(\beta)$$

and the ψ'_p arise from the $\psi_p(P')$ by the corresponding infinitesimal linear transformation dE. $\psi_p(P')-\psi_p(P)$ is the differential in the ordinary sense. Hence we finally have

$$\delta\psi=d\psi+dE\cdot\psi$$

dE depends linearly on the displacement $\overrightarrow{PP'}$ with the components $dx_p=(dx)^p$

$$dE=E_p(dx)^p.$$

Hence our formula yields

$$\psi_{(p)}=\frac{\partial\psi}{\partial x_p}+E_p\psi$$

as the components of the covariant derivative.

$$\tilde{\psi}e^p(\alpha)S(\alpha)\psi_{(p)}$$

is an invariant which becomes a scalar density on division by the absolute value ϵ of the determinant $|e^p(\alpha)|$. Its integral must replace Dirac's quantity (4). After some difficulty we obtain as the action density of matter

$$(9) \qquad \frac{1}{i}\left\{\tilde{\psi}\, \mathfrak{e}^p(\alpha) S(\alpha)\, \frac{\partial \psi}{\partial x_p^\alpha} + \tfrac{1}{2} \frac{\partial \mathfrak{e}^p(\alpha)}{\partial x^p} \tilde{\psi} S(\alpha) \psi\right\} + \frac{1}{\epsilon} f(\alpha) \psi S(\alpha) \psi.$$

$\mathfrak{e}^p(\alpha)$ is the quotient $e^p(\alpha)/\epsilon$; $f(\alpha)$ is expressed in terms of the $e^p(\alpha)$ and their first derivatives in such a way that it vanishes with the derivatives.

On calculating the energy tensor belonging to this action quantity by the rules given above and then from it the total energy, the momentum and the moment of momentum by integration over the spacial cross-section $x_0 = \text{const.}$, we arrive back at the familiar assumptions of quantum theory. In particular the components of momentum are given by the spacial integral of

$$\frac{1}{i} \tilde{\psi} \frac{\partial \psi}{\partial x_p} (p = 1, 2, 3)$$

and this is in accord with the basic association of the operators $\frac{1}{i} \tilde{\psi} \frac{\partial}{\partial x_1}$, with momentum p_1, as employed by Schrödinger. Also, the portion of the angular momentum due to the electron spin is not lacking, although one might at first be inclined to fear that by defining the components of angular momentum as the spacial integral of

$$x_2 \mathfrak{t}_3(0) - x_3 \mathfrak{t}_2(0), \cdots$$

only the orbital momentum would appear. This is naturally a powerful support for the assumption that (9) is actually the contribution to the action in so far as its dependence on the ψ is concerned.

Gravitation may be represented by the same action quantity as in Einstein's classical theory of gravitation. There then exists, as is well known a gravitational energy such that the total energy satisfies a true conservation law. If κ be the Einsteinian constant of gravitation the additive composition of the two terms in the action is accompanied by multiplying the gravitational contribution by $\frac{\kappa h}{c}$. This constant is the square of a length d. But d is much smaller than atomic dimensions—it is of the order 10^{-32} cm.

Now for the critical part, the electromagnetic field! Is it an appendage of gravitation or of the material field? A promising suggestion for the development of the first of these two viewpoints here offers itself. In the Dirac theory the influence of the electromagnetic potentials φ_p on matter is represented by the term

$$\varphi(\alpha)\tilde{\psi}S(\alpha)\psi. \tag{10}$$

Now we find that there is an additional term of exactly the same structure already in (9), without it being necessary to introduce a new entity into the theory in addition to gravitation and matter: we need only consider the $f(\alpha)$ which there appears, but which I did not write down explicitly, as the electromagnetic potential. This term arose from the gravitational potentials in such a way that, although it is invariant with respect to coordinate transformations, it is only invariant under rotation of the local axes in the restricted case in which the axis-systems in all points undergo the same rotation. Therefore, if we disregard the material field, there appears to be "distant parallelism" and we have a theory of exactly the same kind as Einstein's latest. But the calculations which I have made on this assumption seem to me to prove that it cannot be correct: we find no connection with Maxwell's equations, which are so well founded on observation; contrary to all experience we find that the potentials themselves, and not merely the field intensities, are of physical significance and our gauge invariance remains totally ununderstandable.

It is my firm conviction that we must seek the origin of the electromagnetic field in another direction. We have already mentioned that it is impossible to connect the transformations of the ψ in a unique manner with the rotations of the axis system; however we may attempt to accomplish this by means of invariants which can be used as constituents of an action quantity we always find that there remains an arbitrary "gauge factor" $e^{i\lambda}$. Hence the local axis-system does not determine the components of ψ uniquely, but only within such a factor of absolute magnitude 1. In the special theory of relativity, in which the axis system is not tied up to any particular point, this factor is a constant. But it is otherwise in the general theory of relativity when we remove the restriction binding the local axis-systems to each other; we cannot avoid allowing the gauge factor to depend arbitrarily on position. There then remains in the infinitesimal linear transformation dE of ψ, which corresponds to the given infinitesimal rotation do of the axis-systems, an arbitrary additive term $+id\varphi\cdot 1$. The complete determination of the covariant differential $\delta\psi$ of ψ requires that such a $d\varphi$ be given. But it must depend linearly on the displacement PP': $d\varphi = \varphi_p(dx)^p$, if $\delta\psi$ shall depend linearly on the displacement. On altering ψ by multiplying it by the gauge factor $e^{i\lambda}$ we must at the same time replace $d\varphi$ by $d\varphi - d\lambda$ as is immediately seen from this formula of the covariant differential. The principle of gauge invariance becomes self-evident. Our φ are the components of the electromagnetic potential, for they influence matter in the same way as these latter are known to do. Also conversely, they are influenced by matter in accordance with the law which experience has shown to hold for electricity.

$$f_{pq} = \frac{\partial \varphi_q}{\partial x_k} - \frac{\partial \varphi_p}{\partial x_q}$$

is indeed a gauge-invariant tensor. We build from it the familiar Maxwellian action, the analogue of the Dirichlet integral, the integrand of which we shall call $\mathfrak{D}(\varphi)$, and add it to the material term (9). The Maxwellian equations then result from varying the φ. Both terms in the action quantity have the same dimensions; the combining constant, a pure number, is the fine structure constant α. Our theory does not at present indicate any mathematical reason why this number has the numerical value which it actually has.

Until now we have stood, I believe, on solid ground; matter, electricity and gravitation are represented by the Dirac, the Maxwell and the classical Einstein action quantities, all of which have well withstood the test of observation (there could at most be a doubt concerning gravitation). The term (5) of the Dirac theory is, however, more doubtful. It must be admitted that if we retain it we can obtain all details of the line spectrum of the hydrogen atom—of one electron moving in the electrostatic field of a nucleus—in accord with what is known from experiment. But we obtain twice too much; if we replace the electron by a particle of the same mass and positive charge $+e$ (which admittedly does not exist in nature) the Dirac theory gives, contrary to all reason and experience, the same energy terms as for a negative electron, except for a change in sign. Obviously an essential change is here necessary. Furthermore, the theory in its present form contains only the *electron;* there can be no doubt that the *proton* must be introduced into the field equations before they are quantized. In place of one law of conservation of electricity we should have two, expressing the conservation of the number of electrons and protons separately. Our earthly physics sets before us the puzzle why an electron and a proton do not neutralize each other and release their total energy in the form of radiation —although astronomers, in the search for a more copious source of energy, play with the thought that this may actually occur in the stars.

The transformation of ψ under the influence of a rotation of the axis-system was described by the fact that the four quantities

$$\bar{\psi} S(\alpha) \psi$$

then transform among themselves as the four components of a vector fixed relative to the axes. But this description leaves open an even greater indeterminateness than was expressed by the gauge-factor $e^{i\lambda}$ above; the two pairs ψ^+ and ψ^- can indeed each be multiplied by an arbitrary number $e^{i\lambda^+}$, $e^{i\lambda^-}$; only then have we completely exhausted the arbitrariness inherent in the ψ. If we make use of it we find a double gauge invariance, correspondingly two laws of conservation of electricity and two electromagnetic potentials φ^+ and φ^- instead of one. The two components $\psi_1{}^+$, $\psi_2{}^+$ are to be ascribed to the proton, the remaining two $\psi_1{}^-$, $\psi_2{}^-$ to the electron. It is not at all unpleasant that the wave quantity ψ of a single particle is again, as in the Pauli theory, reduced to the smaller ration of 2 components. But the term (5) involving the mass is now untenable, as it does not have the required gauge-invariance. The mass must be brought into the theory in some other way.

A possibility, which at first sight seems plausible, is to consider the gravitational term as the substitute for the mass. For mass is a gravitational effect; it is the flux of the gravitational field through a surface enclosing the particle in the same sense that charge is the flux of electric field. In a satisfactory theory it should be as impossible to introduce a non-vanishing mass without gravitational field as it already is to introduce charge without electromagnetic field. And the gravitational term actually introduces a constant d of the dimensions of length; but unfortunately it lies far under the order of magnitude of the wave length of the electron $\frac{h}{mc} \backsim 10^{-10}$ cm. This is the old dilemma—that the constant of gravitation falls so far out of the range of the other constants of nature. For this reason it seems impossible that gravitation could help us out. I suggest the following way: Be bold enough to leave the term involving mass entirely out of the field equations. But the integral of the total energy density over space yields an invariant, and at the same time constant, mass; *require of it that its value be an absolute constant of nature* m which cannot vary in value from case to case. This introduction of mass is born of the idea that the inertia of matter is due to its energy content.

The two electromagnetic fields may excite some doubt, but as far as I can see one can only thus obtain two conservation theorems, and from the group-theoretic standpoint their appearance seems almost unavoidable. In place of the Maxwellian $\mathfrak{D}(\varphi)$ we should introduce the electromagnetic action density

$$a\mathfrak{D}(\varphi^+) + 2b\mathfrak{D}(\varphi^+, \varphi^-) + a\mathfrak{D}(\varphi^-).$$

The middle term is the symmetric bilinear form corresponding to the quadratic $\mathfrak{D}(\varphi)$. This quadratic form with coefficients a, b, a must be definite, $a^2 - b^2 > o$, if the energy of radiation is to be always positive; one wishes indeed to retain this property, for according to the testimony of experience a field of light has a lowest energy level—darkness. The middle term is responsible for the interaction between protons and electrons. To the same approximation in which the electromagnetic energy arising from this middle term and the gravitational energy can be neglected in comparison with the remaining energy, the mass m introduced above is broken up into two parts m^+ and m^-, the masses of the proton and the electron, which contain only ψ^+, φ^+; ψ^-, φ^- respectively. The masses of the proton and the electron will consequently not be exactly constant but will vary somewhat from quantum state to quantum state. Mass already behaves in this way in the Dirac theory; Dirac's constant m is not the inertial mass of the electron, but rather a "mass factor," a number which occurs as a common factor in the inertial masses of the electron in various quantum states.

I believe that these ideas point in the direction of the future development of quantum theory; on the other hand, I am not certain that the above sketch is correct in every point. I have not as yet found a satisfactory approach to that approximation which reduces the problem of a hydrogen atom to a linear one by neglecting the field of radiation and its reaction on the atom in comparison with that portion of the electromagnetic field, known *a priori*, which binds the proton and the electron to one another in accordance with the Coulomb law.

Another difficulty which stands in the way of a comparison with experience is that the field equations must first be quantized before they can be applied as a basis for the statistics of quantum transitions. But our theory is also hopeful in this respect inasmuch as the anti-symmetric Fermi statistics of the electrons, corresponding to the Pauli exclusion principle, here necessarily leads to the symmetric Bose-Einstein statistics of photons.

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