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Spectral Analysis of One-Dimensional Operators

by

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Abstract

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We study the spectral analysis of one-dimensional operators, motivated by a desire to understand three phenomena: dynamical characteristics of quantum walks, the interplay between inverse and direct spectral problems for limit-periodic operators, and the fractal structure of the spectrum of the Thue-Morse Hamiltonian. Our first group of results comprises several general lower bounds on the spreading rates of wave packets defined by the iteration of a unitary operator on a separable Hilbert space. By using tools within the class of CMV matrices, we apply these general lower bounds to deduce quantitative lower bounds for the spreading of the time-homogeneous Fibonacci quantum walk. Second, we construct several classes of limit-periodic operators with homogeneous Cantor spectrum, which connects problems from inverse and direct spectral analysis for such operators. Lastly, we precisely characterize the gap structure of the canonical periodic approximants to the Thue-Morse Hamiltonian, which constitutes a first step towards understanding the fractal structure of its spectrum. This thesis contains joint work with David Damanik, Milivoje Lukic, Paul Munger, and Robert Vance.

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Contents

Chapter 1

Introduction

1.1 Quantum Dynamics

The Schrödinger equation is the foundational equation in quantum mechanics. In its most general form, it reads

$$i\frac{d\psi}{dt} = H\psi, \quad \psi(0) = \psi_0 \tag{1.1}$$

where ψ_0 is a member of $\mathcal{H}$, a separable Hilbert space, and H is a self-adjoint operator on $\mathcal{H}$. By the spectral theorem for self-adjoint operators, one can solve (1.1) explicitly via

$$\psi(t) = e^{-itH}\psi_0, \quad t \in \mathbb{R}. \tag{1.2}$$

The operators e^{-itH} comprise a strongly continuous one-parameter unitary group on $\mathcal{H}$. Thus, the study of the dynamical characteristics of solutions to (1.1) leads us naturally to a study of the dynamics of unitary group actions. There are many papers devoted to the study of the behavior of solutions given by (1.2); see [23, 38, 39, 40, 46] and the references therein. We henceforth refer to this as the *self-adjoint setting*.

In Chapter 2, we consider a closely related scenario, namely, the iteration of a unitary operator on a separable Hilbert space (we will call this the *unitary setting*, in contrast to the aforementioned self-adjoint setting). This class of dynamical systems is relevant to the study of so-called *quantum walks* (see [8, 9, 42, 43, 44] and references therein). Specifically, a quantum walk is defined by the iteration of a specific type of unitary operator, so our results on general unitary dynamics are immediately relevant to problems in the quantum walk setting. Moreover, the update rule for a time-homogeneous quantum walk on $\mathbb{Z}$ can be related to a CMV matrix [8]. Since there are tools available which enable one to verify the hypotheses of our general theorems for CMV matrices, this connection is quite profitable for the analysis of spreading in quantum walks.

Despite the obvious analogies with the self-adjoint case, significantly less is known about the unitary setting than the self-adjoint setting – see [6, 37, 50], for example. One may be tempted to reduce questions about the unitary setting to known results for solutions to (1.2), but this has at least two drawbacks. First, the study of (1.2) leads naturally to Cesàro averages with respect to the continuous parameter t; such results have no meaning in the unitary setting, where the time-parameter is discrete. In particular, one cannot deduce our results in a straightforward fashion from known results in the self-adjoint setting. Secondly, a problem in the unitary case is often given by an explicit unitary operator and one wants to take advantage of the structure of the operator, which may not be present in any associated self-adjoint operator.

Returning to the study of (1.1), it turns out that directly studying (1.2) is quite difficult in general. However, there is a beautiful link between the Lebesgue decomposition of the spectral measure of H and ψ_0 and transport characteristics of $\psi(t)$.

Theorem 1.1 (RAGE Theorem)**.** *Suppose H is a self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and $\psi \in \mathcal{H}$. Let $\mu = \mu_\psi^H$ denote the corresponding spectral measure. That is,*

$$\langle \psi, e^{-itH}\psi \rangle = \widehat{\mu}(t) := \int_{\sigma(H)} e^{-it\lambda}\, d\mu(\lambda)$$

for each $t \in \mathbb{R}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. Let

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$$

denote the Lebesgue decomposition of μ into pieces which are, respectively, absolutely continuous with respect to Lebesgue measure, singular continuous, and pure point.

1. *We have $\mu = \mu_{\mathrm{pp}}$ if and only if for every $\varepsilon > 0$, there is $N \in \mathbb{Z}_+$ such that*

$$\sum_{|n| \geq N} |\langle \delta_n, \psi(t) \rangle|^2 < \varepsilon \quad \text{for every } t \in \mathbb{R}.$$

2. *We have $\mu_{\mathrm{pp}} = 0$ if and only if for every $N \in \mathbb{Z}_+$,*

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \sum_{|n| \leq N} |\langle \delta_n, \psi(t) \rangle|^2 \, dt = 0.$$

3. *If $\mu = \mu_{\mathrm{ac}}$, then for every $N \in \mathbb{Z}_+$,*

$$\lim_{|t| \to \infty} \sum_{|n| \leq N} |\langle \delta_n, \psi(t) \rangle|^2 = 0.$$

Colloquially, we say that μ is pure point if and only if ψ remains almost localized, μ is continuous if and only if ψ exhibits transport in a time-averaged sense, and, if μ

is purely absolutely continuous, then ψ exhibits transport without averaging. Notice that the first two statements are equivalences, while the third is one-directional. Thus, as a first step towards understanding the dynamics of wave functions, one can begin by trying to understand the spectral type of various classes of operators. In the remainder of the introduction, we describe some of the operators that we will analyze in this thesis.

1.2 Schrödinger Operators

An extremely well-studied class of self-adjoint operators are given by considering a single particle in a one-dimensional medium subject to an external electrostatic potential V; in this case, one takes

$$\mathcal{H} = L^2(\mathbb{R}), \quad H = -\frac{d^2}{dx^2} + V. \tag{1.3}$$

Of course, operators of the form (1.3) are unbounded, which means that they are not self-adjoint until one prescribes the domain upon which they act. In this thesis, the potentials with which we work will be bounded, so all operators of the form (1.3) may be realized as self-adjoint operators on the Sobolev space $H^2(\mathbb{R}) = W^{2,2}(\mathbb{R})$ by the Kato-Rellich theorem (see, e.g., [11, Theorem 1.4] or [60, Theorem X.12]).

Since unbounded operators are often difficult to study, one is naturally led to the study of so-called discrete Schrödinger operators. More precisely, given a bounded sequence $V : \mathbb{Z} \to \mathbb{R}$, we can define an associated bounded self-adjoint operator

$H = H_V$ on $\ell^2(\mathbb{Z})$ by

$$(Hu)_n = u_{n-1} + u_{n+1} + V_n u_n, \quad n \in \mathbb{Z},\, u \in \ell^2(\mathbb{Z}).$$

There are two well-studied regimes of potentials. On one hand, almost-periodic potentials [3, 4, 24, 65] serve as models of materials with long-range order, while random potentials [25, 31, 32, 41, 48, 69] function as models for disordered, amorphous media. In Chapter 3, we study a subclass of the almost-periodic potentials, namely, the class of *limit-periodic* potentials; a potential is said to be *limit-periodic* if it can be written as a *uniform* limit of periodic potentials; see [1, 15, 16, 17, 29, 30, 33, 34]. A typical example of such a potential is furnished by

$$V(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} \cos\left(\frac{2\pi x}{j!}\right),\ x \in \mathbb{R}.$$

The main results of Chapter 3 concern homogeneity of the spectrum as a subset of $\mathbb{R}$. Loosely speaking, a homogeneous closed subset of $\mathbb{R}$ is one which has a uniform positive density in arbitrarily small neighborhoods of each of its points. The motivation for studying homogeneity of the spectrum arises from inverse spectral problems; this connection is elucidated in more detail in the introduction to Chapter 3. We prove several results which show that homogeneity of the spectrum is dense in several classes of limit periodic operators. The techniques developed therein are robust – they can be applied to discrete Schrödinger operators, Jacobi matrices, CMV matrices, and continuum Schrödinger operators. Moreover, the density results still hold if one fixes the combinatorial structure of the hull of the potentials in advance. Finally, motivated by a desire to produce specific examples, we prove that any continuum potential which satisfies a Pastur-Tkachenko convergence condition has homogeneous spectrum.

The regime in between almost-periodic and random potentials is not as well-understood as these two extreme regions, and this is the area from which we shall draw some further examples to study. Consider a finite set $\mathcal{A}$ endowed with the discrete topology, and $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ a closed subset which is invariant under the action of the left-shift:

$$(T\omega)_n = \omega_{n+1}. \tag{1.4}$$

Such an Ω is called a *subshift*. Discrete potentials generated by finite alphabets have been heavily studied since the 1980's (see the survey [12]). Recently, there have been several works considering Jacobi matrices [76] and CMV matrices [20, 21, 49, 52] generated by subshifts. In the final chapter, we explore a very specific class of discrete subshift models, specifically, those which are generated by the Thue-Morse substitution. We can exactly characterize the open and closed gaps of the canonical family of periodic approximants of this family of Hamiltonians.

Chapter 2

Unitary Dynamics

2.1 Introduction

2.1.1 Setting

This chapter is concerned with the dynamics of unitary operators acting on Hilbert spaces. Specifically, we fix a (separable) Hilbert space $\mathcal{H}$, an orthonormal basis $(\varphi_n)_{n\in A}$ for $\mathcal{H}$, a unitary operator $U : \mathcal{H} \to \mathcal{H}$, and a unit vector $\psi \in \mathcal{H}$ and consider the discrete-time evolution $\psi(k) = U^k\psi$. The orthonormal basis $(\varphi_n)_{n\in A}$ will be indexed by a suitable countable set A; in general we may always take $A = \mathbb{Z}_+$, but other countable sets may be more natural in certain settings. For example, if $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, it is natural to use the orthonormal basis $(\delta_n)_{n\in\mathbb{Z}^d}$. In this chapter, we always assume that $A = \mathbb{Z}_+^d$ or $A = \mathbb{Z}^d$ for some $d \in \mathbb{Z}_+$.

Our goal is to give as complete a dynamical picture as possible for the spreading of $U^k\psi$ with respect to the basis $(\varphi_n)_{n\in A}$ in terms of spectral characteristics of U, so we begin by clarifying what we mean by "spectral characteristics." First, recall that the *spectrum* of U is the set

$$\sigma(U) = \{z \in \mathbb{C} : U - z \text{ does not have a bounded, two-sided inverse }\}.$$

By standard arguments, unitarity of U implies that

$$\sigma(U) \subseteq \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}.$$

Here (and throughout the thesis), we use $\mathbb{D}$ to denote the open unit disk in $\mathbb{C}$. The spectral theorem for unitary operators tells us that there is a Borel probability measure μ_ψ^U on the circle such that

$$\langle \psi, f(U)\psi \rangle = \int_{\sigma(U)} f(z)\, d\mu_\psi^U(z)$$

for any bounded, Borel measurable function f on $\partial\mathbb{D}$. Typically, we will suppress the dependence of μ_ψ^U on U and simply write μ_ψ, or even just μ, if the initial state ψ is clear from context.

We will prove quantitative lower bounds for the spreading of the wave packet $U^k\psi$ as $k \to \infty$ in terms of continuity properties of μ_ψ. The basic underlying theme of our results can (roughly) be stated thusly: as μ_ψ becomes less singular, $U^k\psi$ spreads more quickly (in a time-averaged sense). However, our estimates are lower bounds, and hence, they are one-sided. Consequently, the converse of our results is false, i.e., fast transport of the wave packet need not imply continuity of μ_ψ [13].

2.1.2 Structure of Chapter 2

In Section 2.2 we introduce some basic quantities that capture the aspects of the dynamics in which we are interested. In particular, we define the transport exponents, which measure the rate at which the wave packet spreads.

In Sections 2.3, 2.4, and 2.5 we establish lower bounds for the transport exponents in terms of fractal regularity properties of the associated spectral measure. In particular, the Hausdorff dimension and the packing dimension of the spectral measure play a crucial role.

In Section 2.6 we discuss the application of our method to quantum walks on $\mathbb{Z}$. In this special case, the unitary operator in question is unitarily equivalent to an extended CMV matrix. We discuss how subordinacy theory provides an elegant way of establishing the spectral regularity that was shown to imply lower transport bounds. We then describe the CGMV formalism, which establishes a connection between quantum walks on the integers and CMV matrices; this connection enables us to explain how the material from earlier sections gives a useful framework in which the spreading rates of a quantum walk on the integers may be studied systematically. Finally, we use the Fibonacci quantum walk as an example for which we implement this overall strategy. We derive explicit lower bounds for the spreading rates associated with this model using the connection with CMV matrices, an analysis of the solutions which provides the input to subordinacy theory and hence implies spectral regularity, and the derivation of the lower bounds for the transport exponents from these spectral regularity properties.

This chapter is joint work with David Damanik and Robert Vance [14].

2.2 Preliminaries and Basic Definitions

We shall mirror the notation and development found in [23]. Let $\mathcal{H}$ be a complex separable Hilbert space, U a unitary operator on $\mathcal{H}$, and $\psi \in \mathcal{H}$ such that $\|\psi\| = 1$. We are interested in the time evolution of the vector ψ, that is, $\psi(k) = U^k\psi$, where $k \in \mathbb{Z}$. Let $(\varphi_n)_{n\in A}$ be an orthonormal basis for $\mathcal{H}$, indexed by a suitable countable set A – here, we will consider $A = \mathbb{Z}_+^d$, $\mathbb{Z}^d$ as appropriate. To describe the spreading of ψ with respect to the basis $(\varphi_n)_{n\in A}$, we first define

$$a_\psi(n,k) = |\langle \varphi_n, \psi(k)\rangle|^2, \quad n \in A,\, k \in \mathbb{Z},$$

which can be interpreted as the probability that ψ is in the state φ_n at time k. We shall also be interested in the Cesàro time-averaged probabilities, given by

$$\tilde{a}_\psi(n,K) = \frac{1}{K}\sum_{k=0}^{K-1} a_\psi(n,k), \quad n \in A,\, K \in \mathbb{Z}_+.$$

To concisely denote Cesàro averages, we introduce the following notation for a function $f : \mathbb{Z}_+ \to \mathbb{R}$:

$$\langle f\rangle(K) = \frac{1}{K}\sum_{j=0}^{K-1} f(j), \quad K \in \mathbb{Z}_+.$$

For example, in this notation, one could write $\tilde{a}_\psi(n,K) = \langle a_\psi(n,\cdot)\rangle(K)$. For fixed $k \in \mathbb{Z}$ and $K \in \mathbb{Z}_+$, unitarity of U implies

$$\sum_{n\in A} a_\psi(n,k) = \sum_{n\in A} \tilde{a}_\psi(n,K) = 1 \tag{2.1}$$

since $(\varphi_n)_{n\in A}$ is an orthonormal basis for $\mathcal{H}$. Given $R \geq 0$, we are interested in the probability of finding the wave packet ψ within a ball of radius R at time k. In order to quantify this, we define the inside and outside probabilities

$$P_{\text{in}}^{\psi}(R,k) = \sum_{|n|\leq R} a_\psi(n,k),$$

$$P_{\text{out}}^{\psi}(R,k) = \sum_{|n|>R} a_\psi(n,k) = 1 - P_{\text{in}}^{\psi}(R,k),$$

and their time-averaged counterparts

$$\tilde{P}_{\text{in}}^{\psi}(R,K) = \sum_{|n|\leq R} \tilde{a}_\psi(n,K) = \langle P_{\text{in}}(R,\cdot)\rangle(K),$$

$$\tilde{P}_{\text{out}}^{\psi}(R,K) = \sum_{|n|>R} \tilde{a}_\psi(n,K) = 1 - \tilde{P}_{\text{in}}^{\psi}(R,K).$$

In the formulae above, $|n|$ denotes the ℓ^1 norm of $n \in \mathbb{Z}^d$, that is, $|n| = |n_1|+\cdots+|n_d|$.

We shall also describe transport behavior of U and ψ in terms of the moments of the position operator. More specifically, (2.1) implies that for each $k \in \mathbb{Z}$, $\psi(k)$ defines a probability distribution on A, namely $a_\psi(\cdot,k)$, so we may describe the spreading of ψ by measuring the moments of these probability distributions. More precisely, given $p > 0$ and $k \in \mathbb{Z}$, define

$$|X|_\psi^p(k) = \sum_n (|n|^p + 1)\, a_\psi(n,k),$$

with the time-averaged counterparts

$$\left\langle |X|_\psi^p \right\rangle(K) = \sum_n (|n|^p + 1)\, \tilde{a}_\psi(n,K) = \frac{1}{K}\sum_{k=0}^{K-1} |X|_\psi^p(k).$$

The "+1" occuring in the definitions of $|X|^p$ and $\langle |X|^p\rangle$ is to ensure that these quantities are nonzero, since we will be taking logarithms in a later definition.

The following relationship between the moments and the outside probabilities will be useful in the sequel.

Proposition 2.1. *Let U and ψ be given. For all $k \in \mathbb{Z}$, $R, K \geq 0$ and all $p > 0$, we have*

$$|X|^p_\psi(k) \geq R^p P^\psi_{\text{out}}(R,k) \tag{2.2}$$

$$\left\langle |X|^p_\psi \right\rangle (K) \geq R^p \tilde{P}^\psi_{\text{out}}(R,K). \tag{2.3}$$

Proof. These follow from straightforward calculations. For example, to prove (2.2) simply write the definition of $|X|^p$, cut off the sum, and estimate the terms from below:

$$\begin{aligned} |X|^p_\psi(k) &= \sum_{n \in A} \left(|n|^p + 1\right) a_\psi(n,k) \\ &\geq \sum_{|n|>R} \left(|n|^p + 1\right) a_\psi(n,k) \\ &\geq R^p \sum_{|n|>R} a_\psi(n,k) \\ &= R^p P^\psi_{\text{out}}(R,k). \end{aligned}$$

One can prove (2.3) similarly (or by taking (2.2) and averaging over $k \in \mathbb{Z}$ such that $0 \leq k < K$). $\square$

Example 2.2 (Shift Operator)**.** Let $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by $(U\psi)_n = \psi_{n-1}$, and (for simplicity), consider $\psi = \delta_0$. It is easy to see that $U^k\psi = \delta_k$ for every k, and hence,

$$|X|^p_\psi(k) = |k|^p + 1$$

for all $k \in \mathbb{Z}$. With this choice of U and ψ, μ_ψ^U is precisely normalized Lebesgue measure on $\partial\mathbb{D}$, which one can easily verify by examining the Fourier coefficients of both measures.

In light of the foregoing example, we are motivated to compare the growth of $|X|_\psi^p(k)$ to polynomial growth of the form $k^{\beta p}$ for a suitable exponent β. Consequently, the following transport exponents are natural objects to consider

$$\begin{aligned}
\beta_\psi^+(p) &= \limsup_{k\to\infty} \frac{\log\left(|X|_\psi^p(k)\right)}{p\log(k)}, \\
\beta_\psi^-(p) &= \liminf_{k\to\infty} \frac{\log\left(|X|_\psi^p(k)\right)}{p\log(k)}, \\
\tilde{\beta}_\psi^+(p) &= \limsup_{K\to\infty} \frac{\log\left(\left\langle |X|_\psi^p\right\rangle(K)\right)}{p\log(K)}, \\
\tilde{\beta}_\psi^-(p) &= \liminf_{K\to\infty} \frac{\log\left(\left\langle |X|_\psi^p\right\rangle(K)\right)}{p\log(K)}.
\end{aligned}$$

In particular, with U and ψ as in Example 2.2, one has

$$\beta_\psi^\pm(p) = \tilde{\beta}_\psi^\pm(p) = 1$$

for all $p > 0$. One typically refers to this situation (in which $\beta^-(p) = 1$ for all p) as *ballistic transport* of the corresponding wave packet.

By Jensen's inequality, $\beta_\psi^\pm$ and $\tilde{\beta}_\psi^\pm$ are non-decreasing functions of p. For a detailed proof of this, the interested reader may consult [23, Lemma 2.7].

Usually, the initial state ψ will be explicitly given or clear from context, so we will often suppress the dependence of the dynamical quantities on ψ, and simply write $a(n,k)$, $P_{\text{in}}(R,k)$, $|X|^p(k)$, etc.

2.3 Uniformly Hölder Continuous Spectral Measures

In this section, we prove our first lower bound for the transport exponents introduced in the previous section. Specifically, we consider the case in which the spectral measure is uniformly α-Hölder continuous for some $\alpha > 0$. The key observation here is that we can obtain good upper bounds on the Fourier coefficients of uniformly Hölder continuous measures, which enables us to deduce lower bounds on spreading when μ_ψ^U is uniformly Hölder continuous.

2.3.1 Fourier Coefficients of Uniformly Hölder Continuous Measures

We shall first be interested in a description of continuity and singularity of measures supported on $\partial\mathbb{D}$. To that end, let us first recall the notion of uniform Hölder continuity for such measures.

Definition 2.3. We will say that a measure μ on $\partial\mathbb{D}$ is *uniformly α-Hölder continuous* (UαH) if there exists a constant $C > 0$ such that for every arc $I \subseteq \partial\mathbb{D}$, we have $\mu(I) \leq C|I|^\alpha$, where $|\cdot|$ shall be taken to mean one-dimensional Lebesgue measure on $\partial\mathbb{D}$ and an arc is the image of an interval $J \subseteq \mathbb{R}$ under the canonical projection $\pi : x \mapsto e^{2\pi i x}$.

We remark that a measure μ on $\partial\mathbb{D}$ which is UαH must necessarily be finite, for $\mu(\partial\mathbb{D}) \leq C|\partial\mathbb{D}|^\alpha < \infty$.

The following lemma estimates integrals of the Dirichlet kernel against UαH measures; it is the critical estimate for the results that follow.

Lemma 2.4. *Suppose μ is a UαH measure on $\partial\mathbb{D}$ for $0 \le \alpha < 1$. There exists a constant $\gamma > 0$ such that*

$$\int_{\partial\mathbb{D}} \left| \frac{\overline{z}^K w^K - 1}{\overline{z} w - 1} \right| d\mu(w) \le \gamma K^{1-\alpha}$$

for all $z \in \partial\mathbb{D}$ and all $K \in \mathbb{Z}_+$. In particular, γ depends on neither z nor K.

Proof. By uniformity of μ, it is no loss of generality to assume that $z = 1$, so we may consider the integral

$$\int_{\partial\mathbb{D}} \left| \frac{w^K - 1}{w - 1} \right| d\mu(w).$$

Case 1: $\alpha = 0$. This case is trivial: take $\gamma = \mu(\partial\mathbb{D})$ and observe that

$$\left| \frac{w^K - 1}{w - 1} \right| = \left| 1 + w + w^2 + \cdots + w^{K-1} \right| \le K \tag{2.4}$$

for all $w \in \partial\mathbb{D}$ by the triangle inequality, so that the integral in question is bounded by γK.

Case 2: $0 < \alpha < 1$. For each K, there are three parts of the integral that we will control:

$$S_1 = \{z \in \partial\mathbb{D} : \mathrm{Re}(z) \le 0\},$$

$$S_2 = \left\{ e^{i\theta} : -\frac{\pi}{2K} \le \theta \le \frac{\pi}{2K} \right\},$$

$$S_3 = \left\{ e^{i\theta} : \frac{\pi}{2K} < \theta \le \frac{\pi}{2} \text{ or } -\frac{\pi}{2} \le \theta < -\frac{\pi}{2K} \right\}.$$

It is easy to see that $\left|\frac{w^K-1}{w-1}\right| \le \sqrt{2}$ on S_1, and hence

$$\int_{S_1} \left|\frac{w^K-1}{w-1}\right| \, d\mu(w) \le \sqrt{2}\,\mu(\partial\mathbb{D})$$

Since μ is UαH, choose $C > 0$ such that $\mu(I) \le C|I|^\alpha$ whenever $I \subseteq \partial\mathbb{D}$ is an arc. In particular,

$$\mu(S_2) \le C\left(\frac{\pi}{K}\right)^\alpha = C_1 K^{-\alpha}$$

with $C_1 = C\pi^\alpha$. By (2.4),

$$\left|\frac{w^K-1}{w-1}\right| \le K$$

for all $w \in S_2$ (indeed for all $w \in \partial\mathbb{D}$). Consequently, we have

$$\int_{S_2} \left|\frac{w^K-1}{w-1}\right| \, d\mu(w) \le K\mu(S_2) \le C_1 K^{1-\alpha}.$$

Lastly, we estimate the integral over S_3. Consider the "upper half" of S_3, namely, $S_3^+ = \{w \in S_3 : \mathrm{Im}(w) > 0\}$. We can decompose

$$S_3^+ \subseteq \bigcup_{\ell=1}^{N} A_\ell,$$

where $N = N(K) = \left\lfloor \sqrt{K} \right\rfloor$, and $A_\ell = \left\{ e^{i\theta} : \frac{\ell^2\pi}{2K} < \theta \le \frac{(\ell+1)^2\pi}{2K} \right\}$. For each $1 \le \ell \le N$, we have

$$\mu(A_\ell) \le C\left(\frac{(\ell+1)^2\pi}{2K} - \frac{\ell^2\pi}{2K}\right)^\alpha \le C_2 \ell^\alpha K^{-\alpha}$$

with $C_2 = C\left(\frac{3\pi}{2}\right)^\alpha$. Additionally, for every $1 \le \ell \le N$ and every $w \in A_\ell$, one has $\left|w^K - 1\right| \le 2$ and

$$|w-1| \ge \mathrm{Im}(w) \ge \sin\left(\frac{\ell^2\pi}{2K}\right) \ge \frac{\ell^2}{K},$$

where we have used the bound $\sin(x) \geq \frac{2}{\pi}x$, which holds for all $0 \leq x \leq \pi/2$. Since $\alpha < 1$, we have

$$\begin{aligned}\int_{S_3^+} \left| \frac{w^K - 1}{w - 1} \right| \, d\mu(w) &\leq \sum_{\ell=1}^{N} \int_{A_\ell} \left| \frac{w^K - 1}{w - 1} \right| \, d\mu(w) \\ &\leq \sum_{\ell=1}^{N} \frac{2K}{\ell^2} \cdot C_2 \ell^\alpha K^{-\alpha} \\ &\leq 2C_2 K^{1-\alpha} \sum_{\ell=1}^{\infty} \ell^{\alpha-2} \\ &\leq C_3 K^{1-\alpha},\end{aligned}$$

where $C_3 = 2C_2 \sum_{\ell=1}^{\infty} \ell^{-(2-\alpha)}$. Note that the assumption $\alpha < 1$ guarantees convergence of the series

$$\sum_{\ell=1}^{\infty} \ell^{-(2-\alpha)}.$$

By symmetry and uniformity, we obtain the same estimate for the integral along the "lower half" of S_3, i.e., $S_3^- = \{w \in S_3 : \operatorname{Im}(w) < 0\}$. Combining the estimates for S_1, S_2, and S_3 gives us the desired bound. □

Remark. It is relatively easy to see that the proof of Lemma 2.4 yields an upper bound of constant times $\log(K)$ in the case when $\alpha = 1$. Moreover, it is well-known and not hard to verify that this upper bound is attained when μ is one-dimensional Lebesgue measure on $\partial\mathbb{D}$. However, the factor of $\log(K)$ would not be optimal in the following lemma, which is why a separate argument is necessary therein.

Lemma 2.4 is helpful, because it gives quantitative decay estimates for weighted Fourier coefficients on the unit circle; compare [72].

Lemma 2.5. *Suppose μ is a finite Borel measure on $\partial\mathbb{D}$. For each $f \in L^2(\partial\mathbb{D}, d\mu)$, define the μ-weighted Fourier coefficients of f by*

$$\widehat{f\mu}(k) = \int_{\partial\mathbb{D}} z^{-k} f(z)\, d\mu(z), \quad k \in \mathbb{Z}. \tag{2.5}$$

If μ is UαH for some $\alpha \in [0,1]$, then there exists a constant $C = C_\mu > 0$ such that for all $f \in L^2(\partial\mathbb{D}, d\mu)$ and $K > 0$, we have

$$\left\langle \left|\widehat{f\mu}\right|^2 \right\rangle (K) < C\|f\|^2_{L^2(\mu)} K^{-\alpha}.$$

Proof. **Case 1:** $\alpha < 1$**.** Using the definition of $\widehat{f\mu}$ and Fubini's theorem, we have

$$\begin{aligned} \left\langle \left|\widehat{f\mu}\right|^2 \right\rangle (K) &= \frac{1}{K}\sum_{j=0}^{K-1} \left|\widehat{f\mu}(j)\right|^2 \\ &= \frac{1}{K}\sum_{j=0}^{K-1} \int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \overline{z}^j w^j f(z)\overline{f(w)}\, d\mu(z)\, d\mu(w) \\ &= \frac{1}{K}\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \frac{\overline{z}^K w^K - 1}{\overline{z}w - 1} f(z)\overline{f(w)}\, d\mu(z)\, d\mu(w) \end{aligned}$$

for every $K \in \mathbb{Z}_+$. Recall that $|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$ for all $a, b \in \mathbb{C}$. Using this observation and Fubini's Theorem, we can bound the third line in the previous calculation from above by

$$\frac{1}{2K}\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \left|\frac{\overline{z}^K w^K - 1}{\overline{z}w - 1}\right| \left(|f(z)|^2 + |f(w)|^2\right)\, d\mu(z)\, d\mu(w),$$

and it is easy to see that this is equal to

$$\frac{1}{K}\int_{\partial\mathbb{D}}\int_{\partial\mathbb{D}} \left|\frac{\overline{z}^K w^K - 1}{\overline{z}w - 1}\right| |f(z)|^2\, d\mu(z)\, d\mu(w).$$

Now, we integrate with respect to w and apply the previous lemma to see that the previous expression is less than or equal to

$$\frac{1}{K}\gamma K^{1-\alpha}\int_{\partial\mathbb{D}} |f(z)|^2\, d\mu(z) = \gamma K^{-\alpha}\|f\|^2_{L^2(d\mu)}.$$

Case 2: $\alpha = 1$**.** This case is elementary. Suppose μ is U1H, and let λ denote (normalized) Lebesgue measure on $\partial\mathbb{D}$. Evidently, μ is absolutely continuous with respect to λ. Moreover, by the Lebesgue Differentiation Theorem, $g = \frac{d\mu}{d\lambda}$ is in $L^\infty(\lambda)$. Notice that $f\sqrt{g} \in L^2(\lambda)$ and, by definition, $\left\|f\sqrt{g}\right\|_{L^2(\lambda)} = \|f\|_{L^2(\mu)}$. Thus, by Plancherel, we have

$$\begin{aligned}
\left\langle \left|\widehat{f\mu}\right|^2 \right\rangle (K) &= \frac{1}{K}\sum_{j=0}^{K-1}\left|\widehat{f\mu}(j)\right|^2 \\
&= \frac{1}{K}\sum_{j=0}^{K-1}\left|\widehat{fg}(j)\right|^2 \\
&\le \frac{1}{K}\|fg\|^2_{L^2(\lambda)} \\
&\le \frac{\|g\|_\infty}{K}\|f\sqrt{g}\|^2_{L^2(d\lambda)} \\
&= \frac{\|g\|_\infty}{K}\|f\|^2_{L^2(\mu)}.
\end{aligned}$$

Notice that $\widehat{\cdot}$ has two different meanings in the above argument. In the first line, it is as defined in (2.5), while, in the second line, it denotes the usual Fourier transform from $L^2(\partial\mathbb{D}, d\lambda)$ to $\ell^2(\mathbb{Z})$. $\square$

2.3.2 Proofs of Lower Bounds

Proposition 2.6. *Let $\mathcal{H}$, U, ψ, and $(\varphi_n)_{n\in A}$ be given, with $A = \mathbb{Z}^d_+$ or $A = \mathbb{Z}^d$ as appropriate. If the spectral measure μ_ψ is UαH for some $0 \le \alpha \le 1$, then there is a uniform constant $C_0 > 0$ such that*

$$\tilde{P}_{\mathrm{in}}(N, K) \le C_0 N^d K^{-\alpha} \text{ for all } N,\ K \in \mathbb{Z}_+. \tag{2.6}$$

Consequently, for each $p > 0$, there exists a constant $C_p > 0$ such that

$$\left\langle |X|^p_\psi \right\rangle (K) \ge C_p K^{\frac{p\alpha}{d}} \text{ for all } K \in \mathbb{Z}_+.$$

Proof. Let $\mathcal{H}_\psi$ denote the cyclic subspace spanned by U and ψ, with the corresponding orthogonal projection $P_\psi : \mathcal{H} \to \mathcal{H}_\psi$. Next, let $V : \mathcal{H}_\psi \to L^2(\partial\mathbb{D}, d\mu_\psi)$ denote the natural unitary equivalence sending $f(U)\psi$ to f. Put $u_\psi^n = VP_\psi\varphi_n$. We may observe that

$$a(n,k) = \left|\widehat{u_\psi^n \mu_\psi}(k)\right|^2$$

by Fubini's theorem and the spectral theorem. Hence, we obtain

$$\begin{aligned}
\tilde{P}_{\text{in}}(N,K) &= \frac{1}{K}\sum_{k=0}^{K-1}\sum_{|n|\le N} a(n,k) \\
&= \sum_{|n|\le N}\left\langle \left|\widehat{u_\psi^n \mu_\psi}\right|^2 \right\rangle (K) \\
&\le \sum_{|n|\le N} C_{\mu_\psi} K^{-\alpha} \|u_\psi^n\|^2_{L^2(\partial\mathbb{D}, d\mu_\psi)},
\end{aligned}$$

where C_{μ_ψ} is the constant from Lemma 2.5. Since V is unitary and P_ψ is a projection,

$$\|u_\psi^n\|_{L^2(\partial\mathbb{D}, d\mu_\psi)} = \|P_\psi\varphi_n\| \le \|\varphi_n\| = 1.$$

Using this and $\#\{n : |n| \le N\} \sim N^d$, we get

$$\sum_{|n|\le N} C_{\mu_\psi} K^{-\alpha} \|u_\psi^n\|^2_{L^2(\partial\mathbb{D}, d\mu_\psi)} \le C_0 K^{-\alpha} N^d,$$

for all K, $N \in \mathbb{Z}_+$, which proves (2.6).

Using (2.6), we get

$$\tilde{P}_{\text{in}}\left(\left(\frac{K^\alpha}{2C_0}\right)^{1/d}, K\right) \le \frac{1}{2}.$$

Equivalently,

$$\tilde{P}_{\text{out}}\left(\left(\frac{K^\alpha}{2C_0}\right)^{1/d}, K\right) \ge \frac{1}{2}.$$

As a consequence of Proposition 2.1, we then obtain the estimate

$$\left\langle |X|_{\psi}^{p} \right\rangle (K) \geq \left(\frac{K^{\alpha}}{2C_0} \right)^{p/d} \cdot \frac{1}{2}.$$

With $C_p = \frac{1}{2(2C_0)^{p/d}}$, we obtain the desired lower bound. □

We can use the previous proposition to prove an upper bound on (time-averaged) expectations of general compact observables; compare [46, Theorem 3.2].

Proposition 2.7. *Suppose that μ_ψ is UαH for some $0 \leq \alpha \leq 1$. There then exists a constant $C = C_\psi$ such that the following holds for any compact operator A, any $p \in \mathbb{Z}_+$, and any $K > 0$:*

$$\frac{1}{K} \sum_{j=0}^{K-1} |\langle \psi(j), A\psi(j) \rangle| \leq C_{\psi}^{1/p} \|A\|_p K^{-\alpha/p}.$$

The expression $\|A\|_p$ denotes the pth Schatten trace norm of A, that is,

$$\|A\|_p = \left(\operatorname{tr}\left(|A|^p \right) \right)^{1/p}.$$

We allow the possibility that $\|A\|_p = \infty$, in which case the conclusion of the proposition is trivial.

Proof. We follow the strategy of [46]. We provide the details for the case $p > 1$. The result when $p = 1$ is significantly easier to prove, so we omit the proof in that case.

As before, let $P_\psi : \mathcal{H} \to \mathcal{H}_\psi$ denote the orthogonal projection onto the cyclic subspace spanned by U and ψ, and $V : \mathcal{H}_\psi \to L^2(\partial\mathbb{D}, d\mu_\psi)$ the standard unitary equivalence. Given $\phi \in \mathcal{H}$ with $\|\phi\| = 1$, put $f_\phi = VP_\psi\phi$. Since P_ψ is a projection and V is a unitary operator, we have $\|f_\phi\|_{L^2(\partial\mathbb{D},d\mu_\psi)} \leq 1$. We may observe that

$$\begin{aligned}|\langle\phi,\psi(k)\rangle| &= |\langle\phi,U^k\psi\rangle| \\ &= |\langle f_\phi(z), z^k\rangle_{L^2(\partial\mathbb{D},d\mu_\psi)}| \\ &= \left|\widehat{f_\phi\mu_\psi}(k)\right|.\end{aligned}$$

In particular, Lemma 2.5 implies that there exists a constant C_ψ (which does not depend on ϕ) such that

$$\frac{1}{K}\sum_{k=0}^{K-1}|\langle\phi,\psi(k)\rangle|^2 \leq C_\psi\|f_\phi\|^2_{L^2(d\mu_\psi)}K^{-\alpha} \leq C_\psi K^{-\alpha} \text{ for all } K\in\mathbb{Z}_+. \tag{2.7}$$

By the singular value decomposition, there exist real numbers $s_n \geq 0$ and orthonormal bases (x_n) and (y_n) of $\mathcal{H}$ such that A can be written as

$$A\phi = \sum_n s_n\langle x_n,\phi\rangle y_n.$$

Moreover, it is well known that $\|A\|_p = (\sum_n s_n^p)^{1/p}$. Let $q\in(1,\infty)$ denote the exponent conjugate to p, so that $\frac{1}{p}+\frac{1}{q}=1$. We may observe that, for each k, one has

$$\begin{aligned}|\langle\psi(k),A\psi(k)\rangle| &= \left|\left\langle\psi(k),\sum_n s_n\langle x_n,\psi(k)\rangle y_n\right\rangle\right| \\ &\leq \sum_n s_n|\langle x_n,\psi(k)\rangle\langle y_n,\psi(k)\rangle|.\end{aligned}$$

Thus, we obtain

$$\frac{1}{K}\sum_{k=0}^{K-1}|\langle\psi(k),A\psi(k)\rangle| \leq \frac{1}{K}\sum_n s_n\sum_{k=0}^{K-1}|\langle x_n,\psi(k)\rangle\langle y_n,\psi(k)\rangle|.$$

Applying Cauchy-Schwarz to the summation over k, we see that the expression on the right hand side is bounded above by

$$\frac{1}{K}\sum_n s_n\left(\sum_{k=0}^{K-1}|\langle x_n,\psi(k)\rangle|^2\right)^{1/2}\left(\sum_{k=0}^{K-1}|\langle y_n,\psi(k)\rangle|^2\right)^{1/2}.$$

Applying Hölder's inequality to the summation over n, this is in turn bounded above by

$$\frac{1}{K}\left(\sum_n s_n^p\right)^{1/p}\left(\sum_n\left[\left(\sum_{k=0}^{K-1}|\langle x_n,\psi(k)\rangle|^2\right)^{q/2}\left(\sum_{k=0}^{K-1}|\langle y_n,\psi(k)\rangle|^2\right)^{q/2}\right]\right)^{1/q}.$$

Using Cauchy-Schwarz one last time on the right-hand summation in n and the definition of the pth trace norm, this expression is less than or equal to

$$\|A\|_p\left(\left(\sum_n\left(\frac{1}{K}\sum_{k=0}^{K-1}|\langle x_n,\psi(k)\rangle|^2\right)^q\right)\left(\sum_n\left(\frac{1}{K}\sum_{k=0}^{K-1}|\langle y_n,\psi(k)\rangle|^2\right)^q\right)\right)^{\frac{1}{2q}}.$$

Since (x_n) and (y_n) are orthonormal bases of $\mathcal{H}$, unitarity of U and (2.7) imply that the expression above is in turn bounded above by

$$\|A\|_p\left(\left(C_\psi K^{-\alpha}\right)^{q-1}\left(C_\psi K^{-\alpha}\right)^{q-1}\right)^{\frac{1}{2q}}=\|A\|_p C_\psi^{1/p}K^{-\alpha/p},$$

where we have used $p^{-1}+q^{-1}=1$.

□

2.4 Spectral Measures with a Non-Trivial α-Continuous Component

The results of the previous section can be strengthened further. Specifically, we now turn to the case of spectral measures that have a non-trivial α-continuous component, that is, measures that are not singular with respect to α-dimensional Hausdorff measure. The key observation in this case is that the Rogers-Taylor theory implies that α-continuous measures can be approximated by UαH measures, which allows us to bootstrap the dynamical esimates of the previous section into this setting. Consequently, we obtain quantitative estimates in terms of the most continuous component of the spectral measure. One should emphasize that these estimates are strictly one-sided. That is, in some cases, transport can be fast even if the spectral measure is highly singular [13].

2.4.1 Hausdorff Measure

We begin by briefly reviewing the definition of α-dimensional Hausdorff measure.

Definition 2.8. Fix $\alpha \geq 0$, let $E \subseteq \partial\mathbb{D}$, and suppose that $\mathbf{S} = \{S_1, S_2, \ldots\}$ is a countable collection of subsets of $\partial\mathbb{D}$. Given $\delta > 0$, we shall say that $\mathbf{S}$ is a δ-*cover* of E if $\mathrm{diam}(S_n) < \delta$ for all n and

$$E \subseteq \bigcup_{n=1}^{\infty} S_n.$$

The collection of all δ-covers of E will be denoted $\mathcal{I}_\delta(E)$. The α-*dimensional Hausdorff measure* of E is then defined by

$$h^\alpha(E) = \lim_{\delta \to 0^+} \inf_{\mathbf{S} \in \mathcal{I}_\delta(E)} \sum_{n=1}^{\infty} \left(\mathrm{diam}(S_n)\right)^\alpha .$$

Note that the infimum of such sums over δ-covers is monotone in δ so that the indicated limit indeed exists. The *Hausdorff dimension* of a non-empty subset S of $\partial\mathbb{D}$ is defined by

$$\begin{aligned} \dim_{\mathrm{H}}(S) &= \sup\{\alpha : h^\alpha(S) > 0\} = \sup\{\alpha : h^\alpha(S) = \infty\} \\ &= \inf\{\alpha : h^\alpha(S) < \infty\} = \inf\{\alpha : h^\alpha(S) = 0\}. \end{aligned}$$

We shall say that a measure μ on $\partial\mathbb{D}$ is *α-continuous* (αc) if $\mu(E) = 0$ for all sets $E \subseteq \partial\mathbb{D}$ having $h^\alpha(E) = 0$. One can easily check that a UαH measure on $\partial\mathbb{D}$ must necessarily be α-continuous. The converse need not hold in general, but by results of Rogers and Taylor, any α-continuous measure on $\partial\mathbb{D}$ is "almost" a UαH measure. The precise formulation follows.

Lemma 2.9. *Suppose μ is a finite α-continuous measure on $\partial\mathbb{D}$. Then, for each $\varepsilon > 0$, there exist mutually singular Borel measures μ_1^ε and μ_2^ε on $\partial\mathbb{D}$ such that $\mu = \mu_1^\varepsilon + \mu_2^\varepsilon$, μ_1^ε is UαH, and $\mu_2^\varepsilon(\partial\mathbb{D}) < \varepsilon$.*

Proof. See [62, 63]. □

We shall say that μ is *α-singular* if it is supported on a set having zero α-dimensional Hausdorff measure. This leads to a natural decomposition of our Hilbert space, $\mathcal{H} = \mathcal{H}_{\alpha c} \oplus \mathcal{H}_{\alpha s}$, where $\mathcal{H}_\bullet = \{\psi \in \mathcal{H} : \mu_\psi \text{ is } \bullet\}$. One can check that these are closed, mutually orthogonal subspaces of $\mathcal{H}$. As usual, let us denote by $P_\bullet$ the orthogonal projection onto $\mathcal{H}_\bullet$.

2.4.2 Proofs of Lower Bounds

Using the Rogers-Taylor theorem, we can strengthen Proposition 2.6; compare [46, Theorem 6.1].

Proposition 2.10. *Suppose that $P_{\alpha c}\psi \neq 0$ for some $\alpha \in [0,1]$. Choose d as in Proposition 2.6. Then, for each $p > 0$, there exists a constant $C = C_{\psi,p}$ such that*

$$\left\langle |X|_\psi^p \right\rangle(K) > C_{\psi,p} K^{\alpha p/d} \text{ for every } K \in \mathbb{Z}_+.$$

Proof. The proof of this result follows the general strategy of Last from [46]. Put $\psi_{\alpha c} = P_{\alpha c}\psi$ and $\psi_{\alpha s} = P_{\alpha s}\psi = \psi - \psi_{\alpha c}$. By Lemma 2.9, we may choose mutually singular Borel measures μ_1 and μ_2 on $\partial\mathbb{D}$ such that $\mu_{\psi_{\alpha c}} = \mu_1 + \mu_2$, μ_1 is UαH, and $\mu_2(\partial\mathbb{D}) < \frac{1}{2}\|\psi_{\alpha c}\|^2$. Choose $S \subseteq \partial\mathbb{D}$ such that $\mu_1(\partial\mathbb{D}\backslash S) = \mu_2(S) = 0$.

Let $\psi_1 = \chi_S(U)\psi_{\alpha c}$, $\psi_2 = \psi - \psi_1$. By the spectral theorem, we may observe that

$$\begin{aligned}\mu_{\psi_1}(E) &= \langle \psi_1, \chi_E(U)\psi_1\rangle \\ &= \langle \psi_{\alpha c}, \chi_{E\cap S}(U)\psi_{\alpha c}\rangle \\ &= \mu_{\psi_{\alpha c}}(E \cap S) \\ &= \mu_1(E).\end{aligned}$$

Thus, $\mu_{\psi_1} = \mu_1$. In particular, μ_{ψ_1} is UαH.

For each $R > 0$, define the projection onto a ball of radius R via

$$P_R x = \sum_{|n|\le R} \langle \varphi_n, x\rangle \varphi_n.$$

We may choose a constant C which depends solely on d such that $\#\{n : |n| \le R\} \le CR^d$ for all $R \ge 1$. In particular, $\|P_R\|_1 = \mathrm{tr}(P_R) \le CR^d$ for every $R \ge 1$. By Proposition 2.7, we may choose a constant C_{ψ_1} such that

$$\frac{1}{K}\sum_{j=0}^{K-1}\langle \psi_1(j), P_R\psi_1(j)\rangle \le C_{\psi_1}\|P_R\|_1 K^{-\alpha} \text{ for all } K \in \mathbb{Z}_+,\ R \ge 1.$$

Using the fact that P_R is a projection, we see that

$$\begin{aligned}\frac{1}{K}\sum_{k=0}^{K-1}\|P_R\psi_1(k)\|^2 &= \frac{1}{K}\sum_{k=0}^{K-1}\langle\psi_1(k),P_R\psi_1(k)\rangle\\ &\le C_{\psi_1}\|P_R\|_1 K^{-\alpha}\\ &\le C_1 R^d K^{-\alpha}\end{aligned}$$

for all $K\in\mathbb{Z}_+$ and $R\ge 1$, where $C_1 = C_{\psi_1}C$. Consequently,

$$\begin{aligned}\tilde{P}_{\text{in}}(R,K) &= \frac{1}{K}\sum_{k=0}^{K-1}\|P_R\psi(k)\|^2\\ &\le \frac{1}{K}\sum_{k=0}^{K-1}\big(\|P_R\psi_1(k)\|+\|P_R\psi_2(k)\|\big)^2\\ &\le \frac{1}{K}\sum_{k=0}^{K-1}\big(\|P_R\psi_1(k)\|+\|\psi_2\|\big)^2\\ &\le \left(\sqrt{\frac{1}{K}\sum_{k=0}^{K-1}\|P_R\psi_1(k)\|^2}+\|\psi_2\|\right)^2\\ &\le \left(\sqrt{C_1R^dK^{-\alpha}}+\|\psi_2\|\right)^2\end{aligned}$$

for all $K\in\mathbb{Z}_+$ and $R\ge 1$. We have used projectivity of P_R and unitarity of U in the third line and Cauchy-Schwarz in the fourth. Choose $\eta>0$ with $\eta<\frac{\sqrt{6}-2}{2}$. Applying the previous calculation, we get

$$\begin{aligned}\tilde{P}_{\text{in}}\left(\left(\frac{\eta^2\|\psi_1\|^4K^\alpha}{C_1}\right)^{1/d},K\right) &\le \big(\eta\|\psi_1\|^2+\|\psi_2\|\big)^2\\ &< 1-\frac{1}{2}\|\psi_1\|^2\end{aligned}$$

for all $K\in\mathbb{Z}_+$, where we have used $\|\psi_1\|,\|\psi_2\|\le 1$ and the upper bound on η to obtain the second line of the estimate. It follows that

$$\tilde{P}_{\text{out}}\left(\left(\frac{\eta^2\|\psi_1\|^4K^\alpha}{C_1}\right)^{1/d},K\right) > \frac{1}{2}\|\psi_1\|^2 \text{ for all } K\in\mathbb{Z}_+.$$

Using Proposition 2.1 to bound the moments from below, we have

$$\left\langle |X|_{\psi}^{p} \right\rangle (K) \geq \left(\frac{\eta^2 \|\psi_1\|^4 K^{\alpha}}{C_1} \right)^{p/d} \tilde{P}_{\text{out}} \left(\left(\frac{\eta^2 \|\psi_1\|^4 K^{\alpha}}{C_1} \right)^{1/d}, K \right)$$
$$\geq C_{\psi,p} K^{\frac{\alpha p}{d}}$$

for all $p > 0$ and $K \in \mathbb{Z}_+$, where we take $C_{\psi,p} = \frac{\|\psi_1\|^2}{2} \left(\frac{\eta^2 \|\psi_1\|^4}{C_1} \right)^{p/d}$. □

Remark. One should note that Proposition 2.10 is indeed stronger than the second part of Proposition 2.6, since μ_ψ UαH $\implies$ μ_ψ is α-continuous $\implies$ $P_{\alpha c}\psi = \psi \neq 0$.

Proposition 2.10 immediately yields a lower bound on the transport exponents $\tilde{\beta}_{\psi}^{\pm}(p)$.

Corollary 2.11. *Suppose that ψ is such that $P_{\alpha c}\psi \neq 0$ for some $0 \leq \alpha \leq 1$. One then has*

$$\tilde{\beta}_{\psi}^{-}(p) \geq \frac{\alpha}{d}.$$

Proof. This is immediate from the definitions. □

A succinct way of restating this last result involves the concept of the (upper) Hausdorff dimension of a measure. Recall the following definition; see [28] for background and more information.

Definition 2.12. Let μ be a finite Borel measure on $\partial\mathbb{D}$. The *upper Hausdorff dimension* of μ is given by

$$\dim_{\mathrm{H}}^{+}(\mu) = \inf\{\dim_{\mathrm{H}}(S) : S \subset \partial\mathbb{D} \text{ measurable}, \ \mu(S) = \mu(\partial\mathbb{D})\}.$$

Loosely speaking, the upper Hausdorff dimension of the measure μ is the smallest Hausdorff dimension of a set which supports μ.

Corollary 2.13. *We have*

$$\tilde{\beta}_\psi^-(p) \geq \frac{\dim_{\mathrm{H}}^+\left(\mu_\psi^U\right)}{d}. \tag{2.8}$$

Proof. If $\dim_{\mathrm{H}}^+\left(\mu_\psi^U\right) = 0$, there is nothing to prove. Thus, let us assume that $\dim_{\mathrm{H}}^+\left(\mu_\psi^U\right) > 0$ and choose $\alpha \in \left(0, \dim_{\mathrm{H}}^+\left(\mu_\psi^U\right)\right)$. Since $\alpha < \dim_{\mathrm{H}}^+(\mu_\psi^U)$, the definition of the upper Hausdorff dimension implies that in the Rogers-Taylor decomposition of μ_ψ^U into an α-continuous piece and an α-singular piece, the former must be nontrivial (for otherwise we could choose a suitable support of the latter to derive a contradiction). This implies that Corollary 2.11 is applicable with the α in question and hence yields $\tilde{\beta}_\psi^\pm(p) \geq \frac{\alpha}{d}$. Since this estimate holds for every $\alpha \in \left(0, \dim_{\mathrm{H}}^+\left(\mu_\psi^U\right)\right)$, (2.8) follows. □

2.5 Packing Dimensions of Spectral Measures

Lastly, we bound $\tilde{\beta}^+$ from below by the packing dimension of the relevant spectral measure (compare [40] for a self-adjoint version).

We can prove a version of [40, Proposition 1] in the present context.

Lemma 2.14. *Let $F \subseteq A$ be a subset of the indexing set of the orthonormal basis $(\varphi_n)_{n\in A}$. Given $K \in \mathbb{Z}_+$ and $0 < \varepsilon < 1$, choose $N = N(K, \varepsilon)$ so that*

$$2^{N-2} \leq \frac{K\pi}{\sqrt{\varepsilon}} < 2^{N-1}.$$

Partition $\partial\mathbb{D}$ into dyadic arcs as follows: for $0 \le j \le 2^N - 1$, put

$$\theta_{j,N} = \frac{j\pi}{2^{N-1}},$$

$$\gamma_{j,N} = e^{i\theta_{j,N}},$$

$$\Gamma_{j,N} = \left\{e^{i\theta} : \theta_{j,N} \le \theta < \theta_{j+1,N}\right\}.$$

For all $\varepsilon > 0$ and $K \in \mathbb{Z}_+$, one has

$$\frac{1}{K}\sum_{n\in F}\sum_{l=0}^{K-1} |\langle \varphi_n, \psi(l)\rangle|^2 \le 2\varepsilon + \frac{8\pi}{\sqrt{\varepsilon}}\sum_{n\in F}\sum_{j=0}^{2^N-1}\left|\left\langle \varphi_n, \chi_{\Gamma_{j,N}}(U)\psi\right\rangle\right|^2.$$

Proof. We can see that $\partial\mathbb{D}$ is the disjoint union of the $\Gamma_{j,N}$ as j runs from 0 to $2^N - 1$. We may then approximate $\psi(l) = U^l\psi$ by

$$\psi_K(l) = \sum_{j=0}^{2^N-1} \gamma_{j,N}^l \chi_{\Gamma_{j,N}}(U)\psi.$$

Indeed, one readily observes the following for $0 \le l \le K$:

$$\begin{aligned}
\|\psi(l) - \psi_K(l)\|^2 &= \left\| U^l\psi - \sum_{j=0}^{2^N-1} \gamma_{j,N}^l \chi_{\Gamma_{j,N}}(U)\psi \right\|^2 \\
&= \sum_{j=0}^{2^N-1} \int_{\Gamma_{j,N}} \left|z^l - \gamma_{j,N}^l\right|^2 \, d\mu_\psi(z) \\
&\le \sum_{j=0}^{2^N-1} \int_{\Gamma_{j,N}} \left(\frac{l\pi}{2^{N-1}}\right)^2 \, d\mu_\psi(z) \\
&= \left(\frac{l\pi}{2^{N-1}}\right)^2 \\
&< \varepsilon.
\end{aligned}$$

The first line is a definition, the second follows from the spectral theorem, the third by construction of the $\Gamma_{j,N}$, the fourth from $\mu_\psi(\partial\mathbb{D}) = \|\psi\| = 1$, and the fifth from our choice of N and $0 \le l \le K$. It follows that

$$\begin{aligned}
&\frac{1}{K}\sum_{n\in F}\sum_{l=0}^{K-1}|\langle\varphi_n,\psi(l)\rangle|^2 \\
&\le \frac{2}{K}\sum_{n\in F}\sum_{l=0}^{K-1}|\langle\varphi_n,\psi_K(l)-\psi(l)\rangle|^2+\frac{2}{K}\sum_{n\in F}\sum_{l=0}^{K-1}|\langle\varphi_n,\psi_K(l)\rangle|^2 \\
&< 2\varepsilon+\frac{2}{K}\sum_{n\in F}\sum_{l=0}^{2^N-1}|\langle\varphi_n,\psi_K(l)\rangle|^2 . \qquad (2.9)
\end{aligned}$$

We have used the elementary inequality $|a|^2 \le 2|a-b|^2 + 2|b|^2$ in the second line. The third line is a consequence of previous estimates, nonnegativity of summands, and $K < 2^N$. By expanding ψ_K and performing some algebraic manipulations, we see that the expression in (2.9) is equal to

$$2\varepsilon+\frac{2}{K}\sum_{n\in F}\sum_{l=0}^{2^N-1}\sum_{j=0}^{2^N-1}\sum_{k=0}^{2^N-1}\gamma_{j,N}^l\overline{\gamma_{k,N}^l}\left\langle\varphi_n,\chi_{\Gamma_{j,N}}(U)\psi\right\rangle\overline{\left\langle\varphi_n,\chi_{\Gamma_{k,N}}(U)\psi\right\rangle}. \qquad (2.10)$$

Notice that each γ is a 2^Nth root of unity, so

$$\sum_{l=0}^{2^N-1}\gamma_{j,N}^l\overline{\gamma_{k,N}^l}=\begin{cases}2^N & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}$$

Consequently, we see that the expression in (2.10) is equivalent to

$$\begin{aligned}
&2\varepsilon+\frac{2}{K}\sum_{n\in F}\sum_{j=0}^{2^N-1}\sum_{k=0}^{2^N-1}2^N\delta_{j,k}\left\langle\varphi_n,\chi_{\Gamma_{j,N}}(U)\psi\right\rangle\overline{\left\langle\varphi_n,\chi_{\Gamma_{k,N}}(U)\psi\right\rangle} \\
&=2\varepsilon+\frac{2}{K}\sum_{n\in F}\sum_{j=0}^{2^N-1}2^N\left|\left\langle\varphi_n,\chi_{\Gamma_{j,N}}(U)\psi\right\rangle\right|^2 .
\end{aligned}$$

By using the the relationship between N, K, and ε, the above is at most

$$2\varepsilon+\frac{8\pi}{\sqrt{\varepsilon}}\sum_{n\in F}\sum_{j=0}^{2^N-1}\left|\left\langle\varphi_n,\chi_{\Gamma_{j,N}}(U)\psi\right\rangle\right|^2,$$

which completes the proof of the lemma. □

Proposition 2.15. *Choose d as in Propositions 2.6 and 2.10. Given $N \in \mathbb{Z}_+$ and $0 < \alpha < 1$, let $I_{N,\alpha} = \{j : \mu(\Gamma_{j,N}) < 2^{-N\alpha}\}$, $A_{N,\alpha} = \bigcup_{j \in I_{N,\alpha}} \Gamma_{j,N}$, and $b_{N,\alpha} = \mu(A_{N,\alpha})$. If $b_{N,\alpha} > 0$, then there exists a constant $M_{\alpha,d}$ depending only on α and d such that for all K with $b_{N,\alpha}2^{N-2} \leq 9\pi K < b_{N,\alpha}2^{N-1}$, one has*

$$\tilde{P}_{\text{out}}\left(M_{\alpha,d}\left(b_{N,\alpha}^{3-\alpha}K^{\alpha}\right)^{1/d}, K\right) > \frac{b_{N,\alpha}}{2}.$$

Proof. Put $\psi_N = \chi_{A_{N,\alpha}}(U)\psi$. Evidently, we have

$$\|\psi_N\|^2 = \langle \psi, \chi_{A_{N,\alpha}}(U)\psi \rangle = \mu(A_{N,\alpha}) = b_{N,\alpha}.$$

Put $\eta = 1/9$ and $\varepsilon = (\eta b_{N,\alpha})^2$. Given $m \geq 1$, take $F_m = \{n : |n| \leq m\}$. We note then that $b_{N,\alpha}2^{N-2} \leq 9\pi K < b_{N,\alpha}2^{N-1}$ is equivalent to $2^{N-2} \leq \frac{K\pi}{\sqrt{\varepsilon}} < 2^{N-1}$. Thus, applying Lemma 2.14 to ε, F_m and ψ_N, we see that

$$\frac{1}{K}\sum_{|n|\leq m}\sum_{l=0}^{K-1}\left|\langle \varphi_n, U^l\psi_N\rangle\right|^2 \leq 2\eta^2 b_{N,\alpha}^2 + \frac{8\pi}{\eta b_{N,\alpha}}\sum_{|n|\leq m}\sum_{j=0}^{2^N-1}\left|\left\langle \varphi_n, \chi_{\Gamma_{j,N}}(U)\psi_N\right\rangle\right|^2.$$

We can control the sum on the right hand side as follows:

$$\begin{aligned}
\sum_{|n|\leq m}\sum_{j=0}^{2^N-1}\left|\left\langle \varphi_n, \chi_{\Gamma_{j,N}}(U)\psi_N\right\rangle\right|^2 &= \sum_{|n|\leq m}\sum_{j\in I_{N,\alpha}}\left|\left\langle \varphi_n, \chi_{\Gamma_{j,N}}(U)\psi\right\rangle\right|^2 \\
&\leq \sum_{|n|\leq m}\sum_{j\in I_{N,\alpha}}\left\|\chi_{\Gamma_{j,N}}(U)\varphi_n\right\|^2\left\|\chi_{\Gamma_{j,N}}(U)\psi\right\|^2 \\
&< \sum_{|n|\leq m}\sum_{j\in I_{N,\alpha}} 2^{-N\alpha}\left\|\chi_{\Gamma_{j,N}}(U)\varphi_n\right\|^2 \\
&\leq \sum_{|n|\leq m} 2^{-N\alpha} \\
&\leq C_d m^d 2^{-N\alpha}.
\end{aligned}$$

The first line holds because $\psi_N = \chi_{A_{N,\alpha}}(U)\psi$. The second line follows from Cauchy-Schwarz, the third by definition of $I_{N,\alpha}$, and the fourth from $\|\varphi_n\| = 1$. In the fifth line, C_d is a constant which only depends on d.

Now, take

$$m = \left(\frac{1}{4\pi C_d} \left(\eta b_{N,\alpha}\right)^3 2^{N\alpha} \right)^{1/d}.$$

Substituting this value of m into the above inequality yields

$$\begin{aligned} \frac{1}{K} \sum_{|n|\le m} \sum_{l=0}^{K-1} \left|\left\langle \varphi_n, U^l \psi_N \right\rangle\right|^2 &\le 2\eta^2 b_{N,\alpha}^2 + \frac{8\pi}{\eta b_{N,\alpha}} C_d m^d 2^{-N\alpha} \\ &= \left(2\eta b_{N,\alpha}\right)^2. \end{aligned}$$

Now, let P_m be the projection onto a ball of radius m, that is, $P_m = \sum_{|n|\le m} \langle \cdot, \varphi_n \rangle \varphi_n$. With $\psi'_N = \psi - \psi_N$, we have

$$\begin{aligned} \tilde{P}_{\text{in}}(m, K) &= \frac{1}{K} \sum_{l=0}^{K-1} \|P_m \psi(l)\|^2 \\ &= \frac{1}{K} \sum_{l=0}^{K-1} \|P_m U^l (\psi_N + \psi'_N)\|^2 \\ &\le \frac{1}{K} \sum_{l=0}^{K-1} \left(\|P_m U^l \psi_N\|^2 + 2\|P_m U^l \psi_N\| \|P_m U^l \psi'_N\| + \|P_m U^l \psi'_N\|^2 \right) \\ &\le \left(2\eta b_{N,\alpha}\right)^2 + 4\eta b_{N,\alpha} + \|\psi'_N\|^2. \end{aligned}$$

In the final line, we have used the previous estimate, projectivity of P_m, unitarity of U, $\|\psi'_N\| \le 1$, and Cauchy-Schwarz. Thus, we see that

$$\begin{aligned} \tilde{P}_{\text{out}}(m, K) &= 1 - \tilde{P}_{\text{in}}(m, K) \\ &\ge 1 - \left(2\eta b_{N,\alpha}\right)^2 - 4\eta b_{N,\alpha} - \|\psi'_N\|^2 \\ &= b_{N,\alpha} - \left(2\eta b_{N,\alpha}\right)^2 - 4\eta b_{N,\alpha} \\ &> \frac{b_{N,\alpha}}{2}. \end{aligned}$$

The first line is trivial, the second follows from the estimate above, the third from orthogonality of ψ_N and ψ'_N, and the final line follows from $b_{N,\alpha} \leq 1$ and the choice of η. In particular, we may deduce that $\tilde{P}_{\mathrm{out}}(R,K) > b_{N,\alpha}/2$ whenever $R \leq m$. Recalling the relationships between the variables, we have $2^N > \frac{18\pi K}{b_{N,\alpha}}$, which yields

$$m = \left(\frac{1}{4\pi C_d} \left(\eta b_{N,\alpha}\right)^3 2^{N\alpha} \right)^{1/d} > M_{\alpha,d} \left(K^\alpha b_{N,\alpha}^{3-\alpha} \right)^{1/d},$$

with

$$M_{\alpha,d} = \left(\frac{\eta^3 (18\pi)^\alpha}{4\pi C_d} \right)^{1/d}.$$

The proposition follows. □

Recall the definition of the α-dimensional packing measure p^α; compare [28].

Definition 2.16. Suppose $S \subseteq \partial\mathbb{D}$ and $\delta > 0$. A δ-*packing* with centers in S is a countable collection of mutually disjoint closed arcs, $\{I_j\}_{j\in\mathbb{Z}_+}$, each of which has length bounded by δ and center belonging to S. We set

$$p_\delta^\alpha(S) = \sup \left\{ \sum_{j=1}^\infty |I_j|^\alpha : \{I_j\}_{j\in\mathbb{Z}_+} \text{ is a } \delta\text{-packing with centers in } S \right\}$$

and

$$\tilde{p}^\alpha(S) = \lim_{\delta\to 0} p_\delta^\alpha(S) = \inf_{\delta>0} p_\delta^\alpha(S).$$

We also set

$$p^\alpha(S) = \inf \left\{ \sum_{k=1}^\infty \tilde{p}^\alpha(S_k) : S = \bigcup_{k=1}^\infty S_k \right\}.$$

Note that $\tilde{p}_\delta^\alpha(S)$ decreases as δ decreases. This shows that the limit and the infimum above are indeed equal. Restricted to Borel sets S, p^α is a Borel measure. It is not hard to show that $p^\alpha(S) = 0$ whenever $p^{\alpha'}(S) < \infty$ and $0 \leq \alpha' < \alpha$.

Definition 2.17. The *packing dimension* of a non-empty subset S of $\partial\mathbb{D}$ is given by

$$\begin{aligned}\dim_{\mathrm{P}}(S) &= \sup\{\alpha : p^{\alpha}(S) > 0\} = \sup\{\alpha : p^{\alpha}(S) = \infty\} \\ &= \inf\{\alpha : p^{\alpha}(S) < \infty\} = \inf\{\alpha : p^{\alpha}(S) = 0\}.\end{aligned}$$

Definition 2.18. Let μ be a finite Borel measure on $\partial\mathbb{D}$. The *upper packing dimension* of μ is given by

$$\dim_{\mathrm{P}}^{+}(\mu) = \inf\{\dim_{\mathrm{P}}(S) : S \subset \partial\mathbb{D} \text{ measurable}, \ \mu(S) = \mu(\partial\mathbb{D})\}.$$

For the proof of the corollary below, the following characterization of the upper packing dimension is useful; compare Chapter 10 of [28] and the appendix of [40].

Proposition 2.19. *The upper packing dimension of μ is also given by*

$$\dim_{\mathrm{P}}^{+}(\mu) = \mu - \operatorname{esssup}_{E\in\mathbb{R}} \left(\limsup_{\varepsilon\to 0} \frac{\log(\mu([E-\varepsilon, E+\varepsilon]))}{\log(\varepsilon)} \right).$$

Corollary 2.20. *We have*

$$\tilde{\beta}_{\psi}^{+}(p) \geq \frac{\dim_{\mathrm{P}}^{+}(\mu_{\psi}^{U})}{d}. \tag{2.11}$$

Proof. We follow the general strategy implemented in [40]. If $\dim_{\mathrm{P}}^{+}(\mu_{\psi}^{U}) = 0$, then there is nothing to prove, so assume given $0 \leq \alpha < \dim_{\mathrm{P}}^{+}(\mu_{\psi}^{U})$. One easily verifies that

$$E \in \limsup_{N\to\infty} A_{N,\alpha} \iff \limsup_{\varepsilon\to 0} \frac{\log(\mu([E-\varepsilon, E+\varepsilon]))}{\log(\varepsilon)} > \alpha.$$

In particular, Proposition 2.19 implies that $\mu\left(\limsup_{N\to\infty} A_{N,\alpha}\right) > 0$, which, by Borel-Cantelli, implies that

$$\sum_{N\in\mathbb{Z}_+} \mu(A_{N,\alpha}) = \infty \tag{2.12}$$

Hence, there is a sequence $(N_j)_{j=1}^{\infty}$ of integers such that $b_{N_j} = \mu(A_{N_j,\alpha}) > N_j^{-2}$ (for, if not, a simple comparison argument would imply convergence of the sum in (2.12)). Lemma 2.15 then implies that

$$\tilde{P}_{\text{out}}\left(M_{\alpha,d}\left(b_{N_j}^{3-\alpha}K_j^{\alpha}\right)^{1/d}, K_j\right) \geq \frac{N_j^{-2}}{2}.$$

Of course, we have chosen an increasing subsequence of sampling times $K_1 < K_2 < \cdots$ so that Proposition 2.15 is relevant, that is, we have $b_{N_j}2^{N_j-2} \leq 9\pi K_j < b_{N_j}2^{N_j-1}$ for each $j \in \mathbb{Z}_+$. We can then make use of Proposition 2.1 to see that

$$\left\langle |X|_{\psi}^{p}\right\rangle (K) \geq M_{\alpha,d}^{p}\left(b_{N_j}^{3-\alpha}K_j^{\alpha}\right)^{p/d}\frac{N_j^{-2}}{2} \geq \frac{1}{2}M_{\alpha,d}^{p}K_j^{\alpha p/d}N_j^{(2\alpha-6)\frac{p}{d}-2}. \tag{2.13}$$

Note that we used $b_{N_j} \geq N_j^{-2}$ to obtain the second inequality. We claim that $\tilde{\beta}_{\psi}^{+}(p) \geq \alpha/d$. As a consequence of (2.13), it suffices to prove that

$$\lim_{j\to\infty}\frac{\log(N_j)}{\log(K_j)} = 0. \tag{2.14}$$

To that end, we begin by noticing that $\frac{\log(N_j)}{\log(K_j)}$ is uniformly bounded above. To see this, simply choose j large enough that $2^{N_j} \geq 36\pi N_j^3$, and observe that one has $K_j \geq \frac{b_{N_j,\alpha}}{36\pi}2^{N_j} \geq N_j$ for such j $\left(\text{by using } b_{N_j,\alpha} \geq N_j^{-2}\right)$. Thus, $\frac{\log(N_j)}{\log(K_j)} \leq 1$ for sufficiently large j, from which the boundedness observation follows.

Now, let $\varepsilon > 0$ be given, and choose j sufficiently large so that $\log(N_j) < \varepsilon N_j$. We can take the logarithm of the relationship $b_{N_j}2^{N_j-2} \leq 9\pi K_j$, use $b_{N_j} > N_j^{-2}$ and rearrange to obtain

$$\frac{N_j\log(2)}{\log(K_j)} \leq 1 + \frac{\log(36\pi) + 2\log(N_j)}{\log(K_j)}.$$

Using boundedness of $\frac{\log(N_j)}{\log(K_j)}$, we can see that the expression on the right is uniformly bounded for all j by a positive constant, say, $C > 0$. Thus, for j chosen sufficiently large as above, we have

$$\begin{aligned} 0 &\leq \frac{\log(N_j)}{\log(K_j)} \\ &\leq \frac{\varepsilon N_j}{\log(K_j)} \\ &\leq \frac{C\varepsilon}{\log(2)}, \end{aligned}$$

which proves (2.14). Thus, we obtain $\tilde{\beta}_\psi^+(p) \geq \frac{\alpha}{d}$. Since this holds for all $\alpha < \dim_{\mathrm{P}}^+(\mu_\psi)$, (2.11) follows. □

2.6 Applications

2.6.1 CMV Matrices

In this section we consider the special case where the unitary operator is given by a CMV matrix. Specifically, given a sequence α of complex numbers such that $\alpha_n \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for every $n \in \mathbb{Z}_{\geq 0}$, the associated CMV matrix $\mathcal{C} = \mathcal{C}_\alpha$ is defined by the matrix representation

$$\mathcal{C}_\alpha = \begin{pmatrix} a_0 & b_1 & d_1 & & & & & \\ c_0 & a_1 & c_1 & & & & & \\ & b_2 & a_2 & b_3 & d_3 & & & \\ & d_2 & c_2 & a_3 & c_3 & & & \\ & & & b_4 & a_4 & b_5 & d_5 & \\ & & & d_4 & c_4 & a_5 & c_5 & \\ & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

with respect to the standard basis of $\ell^2(\mathbb{Z})$, where

$$\rho_n = \sqrt{1 - |\alpha_n|^2} \tag{2.15}$$

$$a_n = -\overline{\alpha_n}\alpha_{n-1} \tag{2.16}$$

$$b_n = \overline{\alpha_n}\rho_{n-1} \tag{2.17}$$

$$c_n = -\rho_n\alpha_{n-1} \tag{2.18}$$

$$d_n = \rho_n\rho_{n-1}. \tag{2.19}$$

Note that we assume that any unspecified matrix element is zero and we take $\alpha_{-1} = -1$ by convention; consequently, $a_0 = \overline{\alpha_0}$ and $c_0 = \rho_0$. One can readily verify that $\mathcal{C}$ defines a unitary operator on $\ell^2(\mathbb{Z}_{\geq 0})$.

These particular unitary operators play an important role in the theory of orthogonal polynomials on the unit circle as well as in the study of quantum walks in one dimension. Moreover they provide a canonical representation of general unitary operators. Specifically, given any unitary operator U in $\mathcal{H}$ and an initial state $\psi \in \mathcal{H}$, the evolution $U^n\psi$ takes place inside the cyclic subspace $\mathcal{H}_\psi$ generated by U and ψ. The action of U on $\mathcal{H}_\psi$ is unitarily equivalent to multiplication by z in $L^2(\partial\mathbb{D}, d\mu_\psi^U)$. Choosing the so-called CMV basis of $L^2(\partial\mathbb{D}, d\mu_\psi^U)$, the matrix representation of the latter operator with respect to this basis is then given by a CMV matrix.

Sometimes it makes sense to consider so-called *extended CMV matrices*, which act on $\ell^2(\mathbb{Z})$. They have the same form, but are two-sided infinite and may be put in one-to-one correspondence with two-sided infinite sequences $\alpha : \mathbb{Z} \to \mathbb{D}$. More precisely, given such a sequence α, the associated extended CMV operator $\mathcal{E} = \mathcal{E}_\alpha$ is defined by the matrix representation

$$\mathcal{E} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots &&&&&& \\ & b_0 & a_0 & b_1 & d_1 &&&&& \\ & d_0 & c_0 & a_1 & c_1 &&&&& \\ &&& b_2 & a_2 & b_3 & d_3 &&& \\ &&& d_2 & c_2 & a_3 & c_3 &&& \\ &&&&& b_4 & a_4 & b_5 & d_5 & \\ &&&&& d_4 & c_4 & a_5 & c_5 & \\ &&&&&& \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.20}$$

with respect to the standard basis of $\ell^2(\mathbb{Z})$, where a_n, b_n, c_n, and d_n are defined by (2.15)–(2.19). The setting of two-sided CMV matrices occurs naturally when the sequence α is obtained by sampling along the orbit of an invertible ergodic transformation. Special cases of this scenario that are of great interest include the periodic case, the almost periodic case, and the random case. In the next chapter, we will study limit-periodic CMV matrices, a subclass of almost-periodic operators.

2.6.2 Extracting the α-Continuous Component of a Measure on the Circle

The results in this chapter show that the presence of a non-trivial α-continuous component in a spectral measure leads to a corresponding dynamical lower bound on the (time-averaged) transport exponents. This motivates us to find a useful way to extract and study the α-continuous component of a finite measure on $\partial\mathbb{D}$. In this subsection we summarize some known results and tools that allow us to do just that. This provides a bridge to the discussion of CMV matrices.

Let μ be a finite measure on $\partial\mathbb{D}$. Given $\alpha \in (0,1)$ and $z_0 \in \partial\mathbb{D}$, define the *upper α derivative* of μ at z_0 by

$$D_\mu^\alpha(z_0) = \limsup_{\varepsilon \downarrow 0} \frac{\mu\{z_0 e^{i\varphi} : \varphi \in (-\varepsilon, \varepsilon)\}}{(2\varepsilon)^\alpha} \in [0, \infty].$$

Let

$$S_\alpha = \{z \in \partial\mathbb{D} : D_\mu^\alpha(z) = \infty\}.$$

The following result is due to Rogers and Taylor [62, 63]; see also [67, Theorem 10.8.7] and its discussion therein.

Theorem 2.21. *Consider the restrictions*

$$\mu_{\alpha c} = \mu\Big|_{\partial\mathbb{D} \setminus S_\alpha}, \quad \mu_{\alpha s} = \mu\Big|_{S_\alpha}.$$

Then, $\mu_{\alpha c}$ gives zero weight to measurable $S \subseteq \partial\mathbb{D}$ with $h^\alpha(S) = 0$ and $\mu_{\alpha s}$ is supported by a measurable set $S \subseteq \partial\mathbb{D}$ with $h^\alpha(S) = 0$. In particular,

$$\mu = \mu_{\alpha c} + \mu_{\alpha s}$$

is the decomposition of μ into an α-continuous piece and an α-singular piece.

This shows that the upper α-derivative D_μ^α of μ may be used to extract the α-continuous component of μ. The following connection is also very useful. Recall that the Carathéodory function F associated with μ is given by

$$F(z) = \int_{\partial\mathbb{D}} \frac{w+z}{w-z} \, d\mu(w), \quad z \in \mathbb{D}.$$

The following equivalence is established in [67, Lemma 10.8.6].

Proposition 2.22. *For $z_0 \in \partial\mathbb{D}$, we have*

$$D_\mu^\alpha(z_0) = \infty \;\Leftrightarrow\; \limsup_{r\uparrow 1}(1-r)^{1-\alpha}|F(rz_0)| = \infty.$$

In the next subsection, we will see that the rate of divergence of $|F(rz_0)|$ as $r \uparrow 1$ can be studied by quite effective means in the case of CMV matrices. In particular, this provides a direct path toward dynamical lower bounds for such operators.

2.6.3 Spectral Regularity via Subordinacy Theory

Here we describe conditions on a CMV matrix that imply that some of the dynamical results presented in the previous section are applicable. That is, we state criteria for local and global regularity of spectral measures that are effective in the sense that for any given CMV matrix, there is clear path toward establishing these sufficient conditions since they are phrased in terms of solution estimates, which can be obtained in a variety of ways from the coefficients of the given matrix.

We begin with the half-line case. Suppose that one is given a CMV matrix $\mathcal{C}$ with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ (where $\alpha_n \in \mathbb{D}$ for each $n \geq 0$). The associated probability measure μ on the unit circle is given by the spectral measure associated with the unitary operator $\mathcal{C}$ on $\ell^2(\mathbb{Z}_{\geq 0})$ and the unit vector $\delta_0 \in \ell^2(\mathbb{Z}_{\geq 0})$. Recall that the Carathéodory function F associated with μ is given by

$$F(z) = \int_{\partial\mathbb{D}} \frac{w+z}{w-z}\, d\mu(w), \quad z \in \mathbb{D}.$$

The (Szegő) transfer matrices associated with these Verblunsky coefficients are defined as follows. For $z \in \partial\mathbb{D}$, $\alpha \in \mathbb{D}$, let $\rho = (1 - |\alpha|^2)^{1/2}$, and write

$$T(z,\alpha) = \rho^{-1} \begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix}.$$

Then, for $n \geq 1$, put

$$T_n(z) = T(z, \alpha_{n-1}) \cdots T(z, \alpha_0).$$

We also set $T_0(z) = I$.

The orthonormal polynomials of the first and second kind are defined by

$$\begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} = T_n(z) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \psi_n(z) \\ \psi_n^*(z) \end{pmatrix} = T_n(z) \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

respectively; compare [66, Proposition 3.2.1].

For a sequence $a_0, a_1, a_2, \ldots$ of complex numbers and $L \in (0, \infty)$, let

$$\|a\|_L = \sum_{n=0}^{\lfloor L \rfloor} |a_n|^2 + (L - \lfloor L \rfloor)|a_{\lfloor L \rfloor + 1}|^2.$$

That is, $\|\cdot\|_L$ is a local ℓ^2 norm for integer values of L, and $\|\cdot\|_L^2$ is linearly interpolated in between.

The following result is [67, Theorem 10.8.2]:

Theorem 2.23 (OPUC Version of the Jitomirskaya-Last Inequality)**.** *Suppose $z \in \partial\mathbb{D}$ and $r \in [0,1)$. Define $L(r)$ to be the unique solution of*

$$(1 - r)\|\varphi(z)\|_{L(r)} \|\psi(z)\|_{L(r)} = \sqrt{2}.$$

Then, for some universal constant $A \in (1, \infty)$, we have

$$A^{-1} \frac{\|\psi(z)\|_{L(r)}}{\|\varphi(z)\|_{L(r)}} \leq |F(rz)| \leq A \frac{\|\psi(z)\|_{L(r)}}{\|\varphi(z)\|_{L(r)}}.$$

Recall that by Proposition 2.22, the divergence rate of $|F(rz)|$ is connected to the α-derivative of μ at z. Combining this with Theorem 2.23 one arrives at the following equivalence, which is [67, Theorem 10.8.5].

Corollary 2.24. *Given $\alpha \in (0,1)$, let $\beta = \frac{\alpha}{2-\alpha}$. Then, for $z_0 \in \partial\mathbb{D}$, we have*

$$D_\mu^\alpha(z_0) = \infty \;\Leftrightarrow\; \liminf_{L \to \infty} \frac{\|\varphi(z_0)\|_L}{\|\psi(z_0)\|_L^\beta} = 0.$$

This result has the following immediate consequence:

Corollary 2.25. *Suppose that for $z_0 \in \partial\mathbb{D}$, we have*

$$\|\varphi(z_0)\|_L \gtrsim L^{\gamma_1}, \quad \|\psi(z_0)\|_L \lesssim L^{\gamma_2}$$

for $L \geq 1$, where $0 < \gamma_1 < \gamma_2 < \infty$. Then, with

$$\alpha = \frac{2\gamma_1}{\gamma_1 + \gamma_2},$$

we have

$$D_\mu^\alpha(z_0) < \infty.$$

In particular, the restriction of μ to the set

$$P(\gamma_1, \gamma_2) = \{z \in \partial\mathbb{D} : \|\varphi(z)\|_L \gtrsim L^{\gamma_1},\ \|\psi(z)\|_L \lesssim L^{\gamma_2}\}$$

(with implicit constants that may depend on z) is α-continuous for this choice of α.

Proof. We have

$$\frac{\|\varphi(z_0)\|_L^{2-\alpha}}{\|\psi(z_0)\|_L^{\alpha}} \gtrsim \frac{L^{\gamma_1(2-\alpha)}}{L^{\gamma_2\alpha}} = L^{-\alpha(\gamma_1+\gamma_2)+2\gamma_1} = L^0 = 1.$$

This shows that

$$\liminf_{L\to\infty} \frac{\|\varphi(z_0)\|_L^{2-\alpha}}{\|\psi(z_0)\|_L^{\alpha}} > 0,$$

which in turn implies

$$\liminf_{L\to\infty} \frac{\|\varphi(z_0)\|_L}{\|\psi(z_0)\|_L^{\beta}} > 0.$$

The result therefore follows from Corollary 2.24. □

Let us now turn to the whole-line case and consider extended CMV matrices $\mathcal{E}$, determined by a two-sided infinite sequence of coefficients $\alpha : \mathbb{Z} \to \mathbb{D}$. There is a close analog of Corollary 2.25. Namely, suitable power-law estimates imply continuity properties of spectral measures. In fact, it suffices to have such power-law estimates on one half-line, say the right half-line for definiteness. However, these estimates need to hold "uniformly in the boundary condition." That is, one has to consider all vector-valued sequences of the form

$$\begin{pmatrix} \xi_n \\ \zeta_n \end{pmatrix} = T_n(z) \begin{pmatrix} \xi_0 \\ \zeta_0 \end{pmatrix}, \tag{2.21}$$

where

$$|\xi_0| = |\zeta_0| = 1. \tag{2.22}$$

The following result was shown in [49]. It is an adaptation of a result shown by Damanik, Killip, and Lenz in the Schrödinger context [19].

Proposition 2.26. *Suppose that for $z \in \partial\mathbb{D}$, there are constants $0 < \gamma_1(z) < \gamma_2(z) < \infty$ and $0 < C_1(z), C_2(z) < \infty$ so that*

$$C_1(z) L^{\gamma_1(z)} \le \|\xi\|_L \le C_2(z) L^{\gamma_2(z)}, \quad L \ge 1$$

for every solution of (2.21) that is normalized in the sense of (2.22). Then, for every spectral measure μ of $\mathcal{E}$, we have $D_\mu^\alpha(z) < \infty$, where $\alpha = \frac{2\gamma_1(z)}{\gamma_1(z)+\gamma_2(z)}$.

In particular, if $S \subset \partial\mathbb{D}$ is a Borel set such that there are constants $0 < \gamma_1 < \gamma_2 < \infty$ and, for each $z \in S$, there are constants $0 < C_1(z), C_2(z) < \infty$ so that

$$C_1(z) L^{\gamma_1} \le \|\xi\|_L \le C_2(z) L^{\gamma_2}, \quad L \ge 1$$

for every $z \in S$ and every solution of (2.21) that is normalized in the sense of (2.22), then the restriction of every spectral measure of $\mathcal{E}$ to S is purely $\frac{2\gamma_1}{\gamma_1+\gamma_2}$-continuous, that is, it gives zero weight to sets of zero $h^{\frac{2\gamma_1}{\gamma_1+\gamma_2}}$ measure.

2.6.4 Quantum Walks on the Line

Let us recall the standard formalism for quantum walks on the integral lattice. The state space is given by the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. If we denote the standard basis of $\mathbb{C}^2$ by $\{e_\uparrow, e_\downarrow\}$, then $\mathcal{H}$ is equipped with a natural orthonormal basis given by the set of all elementary tensors of the form $\delta_n \otimes e_\uparrow$, $\delta_n \otimes e_\downarrow$ with $n \in \mathbb{Z}$. A time-homogeneous quantum walk scenario is given as soon as unitary coins

$$C_n = \begin{pmatrix} c_n^{11} & c_n^{12} \\ c_n^{21} & c_n^{22} \end{pmatrix} \in \mathrm{U}(2), \quad n \in \mathbb{Z}, \tag{2.23}$$

are specified. As one passes from time t to time $t+1$, the update rule of the quantum walk, denoted U, is defined as follows:

$$U : \delta_n \otimes e_\uparrow \mapsto c_n^{11} \delta_{n+1} \otimes e_\uparrow + c_n^{21} \delta_{n-1} \otimes e_\downarrow, \tag{2.24}$$

$$U : \delta_n \otimes e_\downarrow \mapsto c_n^{12} \delta_{n+1} \otimes e_\uparrow + c_n^{22} \delta_{n-1} \otimes e_\downarrow. \tag{2.25}$$

Extend this by linearity and continuity to general elements of $\mathcal{H}$. One can check that U defines a unitary operator on $\mathcal{H}$. Next, order the basis of $\mathcal{H}$ as follows:

$$\ldots \delta_{-1} \otimes e_\uparrow,\ \delta_{-1} \otimes e_\downarrow,\ \delta_0 \otimes e_\uparrow,\ \delta_0 \otimes e_\downarrow,\ \delta_1 \otimes e_\uparrow,\ \delta_1 \otimes e_\downarrow, \ldots. \tag{2.26}$$

With respect to this ordered basis of $\mathcal{H}$, the matrix representation of $U : \mathcal{H} \to \mathcal{H}$ is given by

$$U = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & & & & \\ & c_{-1}^{21} & 0 & 0 & c_{-1}^{11} & & & & & & \\ & c_{-1}^{22} & 0 & 0 & c_{-1}^{12} & & & & & & \\ & & & c_0^{21} & 0 & 0 & c_0^{11} & & & & \\ & & & c_0^{22} & 0 & 0 & c_0^{12} & & & & \\ & & & & & c_1^{21} & 0 & 0 & c_1^{11} & & \\ & & & & & c_1^{22} & 0 & 0 & c_1^{12} & & \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.27}$$

as can be checked readily using the update rule (2.24)–(2.25); compare [8, Section 4].[1] Notice that, if all Verblunsky coefficients with odd index vanish, the extended CMV

[1] Note that we follow the conventions of [8] here. One could argue that the correct matrix to consider is the transpose of U in (2.27). To conform with [8] and subsequent papers, we will consider the matrix U as given above in what follows. In the Fibonacci example we discuss below, this does not make a difference since the matrix entries will be real and hence the transpose of U is

matrix in (2.20) becomes (recall that all unspecified entries are zero)

$$\mathcal{E} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & & & & \\ & \overline{\alpha_0}\rho_{-1} & 0 & 0 & \rho_1\rho_0 & & & & & & \\ & \rho_0\rho_{-1} & 0 & 0 & -\rho_1\alpha_0 & & & & & & \\ & & & \overline{\alpha_2}\rho_1 & 0 & 0 & \rho_3\rho_2 & & & & \\ & & & \rho_2\rho_1 & 0 & 0 & -\rho_3\alpha_2 & & & & \\ & & & & & \overline{\alpha_4}\rho_3 & 0 & 0 & \rho_5\rho_4 & & \\ & & & & & \rho_4\rho_3 & 0 & 0 & -\rho_5\alpha_4 & & \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.28}$$

The matrix in (2.28) strongly resembles the matrix representation of U in (2.27). Note, however, that all ρ_n's need to be real and non-negative for genuine CMV matrices, and this property is not guaranteed when matching (2.27) and (2.28). However, this can be easily resolved, as shown in [8]. Concretely, given U as in (2.27), write

$$c_n^{kk} = \left|c_n^{kk}\right| e^{i\sigma_n^k}, \quad n \in \mathbb{Z},\ k \in \{1,2\},\ \sigma_n^k \in [0, 2\pi)$$

and define $\{\lambda_n\}_{n\in\mathbb{Z}}$ by

$$\lambda_0 = 1,\ \lambda_{-1} = 1,\ \lambda_{2n+2} = e^{-i\sigma_n^1}\lambda_{2n},\ \lambda_{2n+1} = e^{i\sigma_n^2}\lambda_{2n-1}.$$

With the unitary matrix $\Lambda = \mathrm{diag}(\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots)$, we then have

$$\mathcal{E} = \Lambda^* U \Lambda,$$

the inverse of U. Since our argument is based on spectral continuity of U, and the spectral continuity properties of U and U^{-1} are the same, the final result does not depend on the choice one makes at this juncture.

where $\mathcal{E}$ is the extended CMV matrix corresponding to the Verblunsky coefficients

$$\alpha_{2n+1} = 0,\ \alpha_{2n} = \frac{\lambda_{2n}}{\lambda_{2n-1}} \bar{c}_n^{21}, \quad n \in \mathbb{Z}. \tag{2.29}$$

In order for one to prove lower bounds for the spreading rates of a quantum walk on the line, the strategy is now clear. One needs to establish solution estimates for a given model that feed into Proposition 2.26. Once Proposition 2.26 is shown to be applicable, its output provides the input for an application of Proposition 2.10 and its corollaries. In the next subsection we present a non-trivial example where this strategy may be implemented and yields lower bounds for the spreading rates of the quantum walk discussed there, which are explicit in terms of the parameters of the model.

2.6.5 The Fibonacci Quantum Walk

We discuss the special case of the Fibonacci quantum walk, which is an example that requires the full extent of the machinery developed thus far in conjunction with the subordinacy result from [49] described in Subsection 2.6.3. In this example the sequence of coins takes only two different values, and the order in which these two unitary 2×2 matrices occur is determined by an element of the Fibonacci subshift.

Let us recall how the latter is generated. Consider two symbols, a and b. The Fibonacci substitution S sends a to ab and b to a. This substitution rule can be extended by concatenation to finite and one-sided infinite words over the alphabet $\{a, b\}$. There is a unique one-sided infinite word that is invariant under S: denote it by u. It is, in an obvious sense, the limit as $n \to \infty$ of the words $s_n = S^n(a)$. That is, $s_0 = a$, $s_1 = ab$, $s_2 = aba$, etc., so that $u = abaababaabaab\ldots$. The Fibonacci subshift Ω is given by

$$\Omega = \{\omega \in \{a,b\}^{\mathbb{Z}} : \text{every finite subword of } \omega \text{ occurs in } u\}.$$

Take $\theta_a, \theta_b \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and consider the rotations

$$C_a = \begin{pmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{pmatrix}, \quad C_b = \begin{pmatrix} \cos\theta_b & -\sin\theta_b \\ \sin\theta_b & \cos\theta_b \end{pmatrix}.$$

Given $\omega \in \Omega$, the associated sequence of coins $\{C_{\omega,n}\}_{n\in\mathbb{Z}}$ is given by $C_{\omega,n} = C_{\omega_n}$. The associated unitary operator will be denoted by U_ω. Inspecting (2.29) one sees that U_ω already has the form of an extended CMV matrix and we will therefore denote it by $\mathcal{E}_\omega$ to emphasize this fact.

Let us relabel the basis elements, ordered as in (2.26), and write them as $(\varphi_n)_{n\in\mathbb{Z}}$. We consider a non-zero finitely supported initial state $\psi \in \ell^2(\mathbb{Z})$ and study the spreading in space of $\mathcal{E}_\omega^n \psi$ as $|n| \to \infty$ with respect to this basis.

Implementing the strategy outlined at the end of the previous subsection, [22] establishes solution estimates in the form needed in Proposition 2.26. Thus, Proposition 2.10 and Corollaries 2.11 and 2.13 may be applied, and one obtains the following result:

Theorem 2.27. *Define:*

1.

$$I(z) = \mathrm{Re}\,(z)^2(\sec^2\theta_a + \sec^2\theta_b) + (\mathrm{Re}\,(z^2)\sec\theta_a\sec\theta_b - \tan\theta_a\tan\theta_b)^2$$
$$- 2(\mathrm{Re}\,(z)^2\sec^2\theta_a\sec^2\theta_b(\mathrm{Re}\,(z^2) - \sin\theta_a\sin\theta_b)) - 1;$$

2. $C(z) = \max\{2 + \sqrt{8 + I(z)}, (\sec\theta_a)^{-1}, (\sec\theta_b)^{-1}\}$*;*

3. $\gamma_1(z) = \frac{\log\left(1+\frac{1}{4C(z)^2}\right)}{16\log\phi}$*, where* ϕ *is the golden mean;*

4. $\gamma_2(z) = 4\log_2 K(z)$*, where* K *is a* z*-dependent constant;*[2]

5. $\beta(z) = \frac{2\gamma_1(z)}{\gamma_1(z)+2\gamma_2(z)+1}$*.*

Then, for all ψ, ω, p *as above, we have*

$$\tilde{\beta}^{\pm}_{\omega,\psi}(p) \geq \max\left\{\beta(z) : z \in \text{supp } \mu_{\mathcal{E}_\omega,\psi}\right\},$$

where $\mu_{\mathcal{E}_\omega,\psi}$ *denotes the spectral measure associated with the unitary operator* $\mathcal{E}_\omega$ *and the state* ψ*.*

This theorem was stated and proved in [22]. The proof given there used Proposition 2.26, proved in [49], and Proposition 2.10 and Corollaries 2.11 and 2.13. Thus, these two papers work together with [14] in establishing Theorem 2.27.

[2]The paper [22] contains an explicit expression for $K(z)$. Since this description is somewhat involved, we do not reproduce it here and refer the interested reader to [22].

Chapter 3

Spectral Homogeneity for Limit-Periodic Operators

3.1 Introduction

3.1.1 Background

In this chapter, we explore the spectral theory of several classes of limit-periodic operators. Let us first recall the definitions of limit-periodic potentials in the discrete and continuum settings.

Definition 3.1. A complex-valued sequence $s : \mathbb{Z} \to \mathbb{C}$ is called *periodic* if there exists some $p \in \mathbb{Z}_+$ (the *period*) such that

$$s(n + p) = s(n) \text{ for all } n \in \mathbb{Z}.$$

A sequence is said to be *limit-periodic* if it belongs to the closure of the set of periodic sequences in $\ell^\infty(\mathbb{Z})$. Equivalently, s is limit-periodic if and only if there exist periodic sequences $s^{(n)}$ such that

$$\lim_{n\to\infty} \left\| s - s^{(n)} \right\|_\infty = 0.$$

Given $\mathbb{S} \subseteq \mathbb{C}$, denote by $\mathrm{LP}(\mathbb{Z}, \mathbb{S})$ the set of limit-periodic sequences taking values in $\mathbb{S}$. In practice, we will only deal with $\mathbb{S} = \mathbb{R}$ (Jacobi operators) and $\mathbb{S} = \mathbb{D}$ (CMV matrices).

The continuum setting is particularly rich here, since there are several notions of what one can mean by a limit-periodic function on $\mathbb{R}$. We choose the definition that ensures that our potentials are Bohr almost-periodic functions on $\mathbb{R}$, but one can also work in other classes, e.g., the Besicovitch and Stepanov classes [54].

Definition 3.2. We say that $V : \mathbb{R} \to \mathbb{R}$ is *limit-periodic* if it belongs to the closure of periodic functions in $C(\mathbb{R})$. More precisely, there exist continuous periodic functions $V^{(n)} : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \left\| V - V^{(n)} \right\|_\infty = 0.$$

We denote by $\mathrm{LP}(\mathbb{R})$ the limit-periodic functions from $\mathbb{R}$ to $\mathbb{R}$.

Our main goal is to describe several classes of limit-periodic operators which have homogeneous spectrum; the precise definition of homogeneity follows (compare [10]).

Definition 3.3. We say that a closed set $K \subseteq \mathbb{R}$ is *homogeneous* (in the sense of Carleson) if there exist τ, $\delta_0 > 0$ such that

$$|B_\delta(x) \cap K| \geq \tau\delta \quad \text{for every } 0 < \delta \leq \delta_0 \text{ and } x \in K, \tag{3.1}$$

where $B_\delta(x) = (x - \delta, x + \delta)$ denotes the δ-neighborhood of x. If we want to emphasize the relative density of K, we will say that a closed set which satisfies (3.1) for some $\delta_0 > 0$ is *τ-homogeneous.*

Homogeneity of closed subsets of $\mathbb{R}$ is important from the point of view of inverse spectral theory. In particular, if K is a homogeneous compact set, then the space of Jacobi matrices which have spectrum K and are reflectionless thereupon is known to consist of almost-periodic operators by a theorem of Sodin and Yuditskii [71]; moreover, Poltoratski and Remling have proved that the spectral measures of such Jacobi matrices will be purely absolutely continuous [57]. There are analogous results for the inverse spectral theory of continuum Schrödinger operators and CMV matrices in [35, 70] and [36], respectively. On the other hand, it is known that results of this form need not hold in general if the underlying set is not homogeneous; see [26, 74].

Generically, the spectra of limit-periodic Schrödinger operators are of zero Lebesgue measure and so cannot be homogeneous in this sense [1, Corollary 1.2]. On the other hand, the spectrum corresponding to any periodic potential will be a union of non-degenerate closed intervals; such a set is clearly 1-homogeneous. In order to examine the interplay between inverse and direct spectral perspectives, it is of interest to apply direct spectral methods to construct almost-periodic examples with more exotic spectra which are nonetheless homogeneous in the sense of Carleson and which have purely absolutely continuous spectrum. This goal has been pursued in the setting of continuum quasi-periodic potentials in the regime of small coupling [18]. We can accomplish this in the class of limit-periodic operators because they are approximated by periodic operators in the operator norm topology. It turns out that a careful perturbative argument proves that the set of potentials with homogeneous Cantor spectrum is dense in the space of limit-periodic potentials. Moreover, by using work of Egorova, we are able to control the spectral type and produce purely absolutely continuous spectrum [27]. In the following subsections, we state precise results for various classes of operators

3.1.2 Jacobi Operators

Since spectral homogeneity is of interest from the point of view of inverse spectral theory, the natural family of self-adjoint tri-diagonal operators with which one should work is that of *Jacobi operators*, i.e., operators of the form $J = J_{a,b} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined by

$$(J\psi)(n) = a(n-1)\psi(n-1) + a(n)\psi(n+1) + b(n)\psi(n),\ n \in \mathbb{Z}, \tag{3.2}$$

where a and b are bounded sequences of real numbers; see [73]. Of course, discrete Schrödinger operators are a special case of (3.2) obtained by insisting that $a(n) = 1$ for every n. We will always assume that a is positive and bounded away from zero. In this context, our first theorem takes the following form.

Theorem 3.4. *Fix a periodic sequence $a > 0$, and denote by $\mathcal{H}_\tau^a$ the set of $b \in \mathrm{LP}(\mathbb{Z}, \mathbb{R})$ so that $\sigma(J_{a,b})$ is a τ-homogeneous Cantor set and such that the spectrum of $J_{a,b}$ is purely absolutely continuous. For every $\tau < 1$, $\mathcal{H}_\tau^a$ is dense in $\mathrm{LP}(\mathbb{Z}, \mathbb{R})$ with respect to the ℓ^∞ topology.*

Remark. By a Cantor set, we mean a totally disconnected closed set with no isolated points. In particular, each element of $\mathcal{H}_\tau^a$ is obviously aperiodic. Moreover, a Cantor set clearly cannot be 1-homogeneous, so Theorem 3.4 is optimal.

As an immediate corollary, we see that the set of limit-periodic Jacobi parameters which produce purely absolutely continuous spectrum supported on a homogeneous Cantor set is dense in the natural space of Jacobi parameters. More precisely, define

$$\mathcal{J}_C = \left\{(a,b) \in \ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z}) : C^{-1} \leq a(n) \leq C,\, -C \leq b(n) \leq C \text{ for all } n \in \mathbb{Z}\right\}$$

for each $C > 0$, and endow $\mathcal{J}_C$ with the relative topology that it inherits as a subspace of $\ell^\infty(\mathbb{Z}) \times \ell^\infty(\mathbb{Z})$. Let $\mathcal{L}_C \subseteq \mathcal{J}_C$ denote the set of Jacobi parameters which are limit-periodic (i.e. a and b are both limit-periodic sequences). One then has the following corollary of Theorem 3.4.

Corollary 3.5. *For each $\tau < 1$, $\mathcal{H}_{\tau,C}$ is dense in $\mathcal{L}_C$ with respect to the ℓ^∞ topology, where $\mathcal{H}_{\tau,C}$ denotes the set of $(a,b) \in \mathcal{L}_C$ for which $\sigma(J_{a,b})$ is a τ-homogeneous Cantor set and $J_{a,b}$ has purely absolutely continuous spectrum.*

By taking $a \equiv 1$ in Theorem 3.4, we see that the set of limit-periodic Schrödinger operators with purely absolutely continuous spectrum supported on a Carleson-homogeneous Cantor set is ℓ^∞-dense in the space of all limit-periodic potentials. Recall that H_V denotes the Schrödinger operator on $\ell^2(\mathbb{Z})$ defined by

$$(H_V u)_n = u_{n-1} + u_{n+1} + V_n u_n, \quad u \in \ell^2(\mathbb{Z}),\, n \in \mathbb{Z}.$$

Corollary 3.6. *For each $\tau < 1$, let $\mathcal{H}_\tau^{\mathrm{S}} \subseteq \mathcal{L}$ be the set of $V \in \mathrm{LP}(\mathbb{Z}, \mathbb{R})$ such that $\sigma(H_V)$ is a τ-homogeneous Cantor set and H_V has purely absolutely continuous spectrum. Then $\mathcal{H}_\tau^{\mathrm{S}}$ is ℓ^∞-dense in $\mathrm{LP}(\mathbb{Z}, \mathbb{R})$ for every $\tau < 1$.*

3.1.3 CMV Matrices

If one considers inverse spectral theory of unitary operators rather than the inverse spectral theory of self-adjoint operators, one is naturally led to the class of CMV operators, which were defined in the previous chapter. Specifically, given a sequence α of complex numbers such that $\alpha(n) \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for every $n \in \mathbb{Z}$, the associated extended CMV operator $\mathcal{E} = \mathcal{E}_\alpha$ is defined by the matrix representation

$$\mathcal{E}_\alpha = \begin{pmatrix} \ddots & \ddots & \ddots &&&&&& \\ & a(0) & b(1) & d(1) &&&&& \\ & c(0) & a(1) & c(1) &&&&& \\ && b(2) & a(2) & b(3) & d(3) &&& \\ && d(2) & c(2) & a(3) & c(3) &&& \\ &&&& b(4) & a(4) & b(5) && \\ &&&& d(4) & c(4) & a(5) && \\ &&&&& \ddots & \ddots & \ddots & \end{pmatrix}$$

with respect to the standard basis of $\ell^2(\mathbb{Z})$, where a, b, c, and d are defined by (2.15)–(2.19). By straightforward modifications to the proof of Theorem 3.4, one obtains the following analog in the realm of CMV operators.

Theorem 3.7. *For each $\tau < 1$, denote by $\mathcal{H}_\tau^{\mathrm{CMV}}$ the set of $\alpha \in \mathrm{LP}(\mathbb{Z}, \mathbb{D})$ such the* *that* $\sigma(\mathcal{E}_\alpha)$ *is a* τ*-homogeneous Cantor set.*[1] *Then* $\mathcal{H}_\tau^{\mathrm{CMV}}$ *is dense in* $\mathrm{LP}(\mathbb{Z}, \mathbb{D})$ *for every* $\tau < 1$.

Remark. Notice that we do not claim to produce purely absolutely continuous spectrum in this setting. It is likely true that our construction gives purely absolutely continuous spectrum in the CMV setting, but the paper of Egorova on which we rely to control the spectral type focuses on the Jacobi case.

3.1.4 Dynamical Hulls

It is frequently profitable to imbed limit-periodic sequences into a dynamical context. Specifically, any limit-periodic sequence is Bohr almost-periodic, and so its dynamical hull naturally enjoys the structure of a compact abelian topological group (the hull of a sequence is the closure of the set of its translates in ℓ^∞). Moreover, it is well-known that an almost-periodic sequence is limit-periodic if and only if its hull is totally disconnected; a detailed discussion of this may be found in [1, Section 2]. In light of this, the following definition is natural.

Definition 3.8. A *Cantor group* is a compact, abelian, totally disconnected topological group. A *monothetic group* is a topological group which contains a dense cyclic subgroup. A generator of this dense subgroup is referred to as a *topological generator* of the monothetic group.

Standard examples of monothetic Cantor groups include the additive group of

[1] We call a closed subset $K \subseteq \partial\mathbb{D}$ τ*-homogeneous* if and only if it satisfies a bound of the form (3.1) with $|\cdot|$ interpreted as arc-length measure on $\partial\mathbb{D}$.

p-adic integers and the profinite completion of $\mathbb{Z}$. More generally, the class of Cantor groups precisely coincides with the class of infinite profinite abelian groups; see [61, 75], for example.

As a consequence of this characterization of limit-periodic sequences via their hulls, it follows that limit-periodic sequences are precisely those which can be generated by continuously sampling along orbits of a minimal translation of a monothetic Cantor group; compare [1, Lemma 2.2]. More precisely, a complex-valued sequence s is limit-periodic if and only if one can produce a monothetic Cantor group Ω, a topological generator θ of Ω, an element $\omega \in \Omega$, and $f \in C(\Omega, \mathbb{C})$ such that

$$s(n) = s_\omega^f(n) := f(n\theta + \omega),\ n \in \mathbb{Z}. \tag{3.3}$$

Given $f \in C(\Omega, \mathbb{R})$ and a p-periodic positive sequence a, one obtains Jacobi operators $J_{a,\omega}^f$ with Jacobi parameters (a, b_ω^f), where $b_\omega^f = s_\omega^f$, as in (3.3). By a standard argument using minimality and strong operator convergence, there exists a deterministic compact set $\Sigma_a^f \subseteq \mathbb{R}$ with $\Sigma_a^f = \sigma(J_{a,\omega}^f)$ for every $\omega \in \Omega$.

Similarly, if we take $g \in C(\Omega, \mathbb{D})$, then, for each $\omega \in \Omega$, we obtain a limit-periodic CMV operator $\mathcal{E}_\omega^g$ defined by $\alpha_\omega^g = s_\omega^g$ with s defined by (3.3). As in the Jacobi case, there is a fixed compact set $\Sigma^g \subseteq \partial\mathbb{D}$ with $\sigma(\mathcal{E}_\omega^g) = \Sigma^g$ for every $\omega \in \Omega$.

This point of view is particularly pleasant, since one may fix the underlying dynamics (i.e. Ω and θ) and consider the dependence of spectral properties on $f, g \in C(\Omega)$. Once one fixes the Cantor group Ω, one restricts the limit-periodic potentials which one can generate; in essence, the choice of Ω restricts the "admissible periods" of periodic approximans to sequences of the form (3.3). However, our proofs of homogeneity are robust enough to pass to the dynamical setting and produce a dense set of elements of $C(\Omega)$ which produce τ-homogeneous Cantor spectrum. Heuristically, one may say that we are allowed to fix the "combinatorial structure" of the potential in advance.

Theorem 3.9. *Fix a monothetic Cantor group Ω, a topological generator $\theta \in \Omega$, a positive periodic sequence a, and $\tau < 1$. Then there is a dense subset $\mathcal{H}_\tau^a \subseteq C(\Omega, \mathbb{R})$ such that Σ_a^f is a τ-homogeneous Cantor set and $J_{a,\omega}^f$ has purely absolutely continuous spectrum for every $f \in \mathcal{H}_\tau^a$ and every $\omega \in \Omega$.*

Theorem 3.10. *Fix a monothetic Cantor group Ω, a topological generator θ, and denote by $\mathcal{H}_\tau^{\mathrm{CMV}} \subseteq C(\Omega, \mathbb{D})$ the set of g such that Σ^g is a τ-homogeneous Cantor set. For each $\tau < 1$, $\mathcal{H}_\tau^{\mathrm{CMV}}$ is dense in $C(\Omega, \mathbb{D})$.*

3.1.5 Continuum Schrödinger Operators

Finally, we can also describe homogeneity for limit-periodic continuum potentials. Recall that a continuum Schrödinger operator with potential V is defined by $H = -\nabla^2 + V$, that is,

$$H\phi = -\phi'' + V\phi.$$

Once it is restricted to a suitable dense subspace of $L^2(\mathbb{R})$, H becomes an unbounded self-adjoint operator, provided V is sufficiently nice; as noted in Chapter 1, our potentials are bounded, and so H is self-adjoint on the Sobolev space $H^2(\mathbb{R})$.

Definition 3.11. We say $V \in \mathrm{LP}(\mathbb{R})$ is of *Pastur-Tkachenko type* if there exist periodic V_n of period T_n such that

$$\lim_{n\to\infty} e^{bT_{n+1}} \|V - V_n\|_\infty = 0 \text{ for every } b > 0, \tag{3.4}$$

and T_n divides T_{n+1} for all n; compare [55, 56]. We denote by $\mathrm{PT}(\mathbb{R})$ the set of all Pastur-Tkachenko potentials.

Every member of the Pastur-Tkachenko class has homogeneous spectrum.

Theorem 3.12. *If $V \in \mathrm{PT}(\mathbb{R})$, then $\sigma(-\nabla^2 + V)$ is τ-homogeneous for every $\tau \in (0,1)$.*

As a consequence of Theorem 3.12, we easily recover a continuum Schrödinger analog of Theorem 3.4.

Corollary 3.13. *For each $0 < \tau < 1$, denote by $\mathcal{H}_\tau$ the set of $V \in \mathrm{LP}(\mathbb{R})$ such that $H = -\nabla^2 + V$ has purely absolutely continuous spectrum and $\sigma(-\nabla^2 + V)$ is a τ-homogeneous Cantor set. Then $\mathcal{H}_{1^-} = \bigcap_{0<\tau<1} \mathcal{H}_\tau$ is dense in $\mathrm{LP}(\mathbb{R})$ (with respect to the topology of uniform convergence).*

The structure of the chapter is as follows. In Section 3.2, we recall a few standard facts from functional analysis and some necessary pieces of Floquet theory and use these ingredients to prove a gap-opening lemma. This lemma is then used in Section 3.3 to prove Theorems 3.4 and 3.9. Clearly, these theorems imply Corollaries 3.5 and 3.6. Section 3.3 then discusses the necessary modifications to the proofs in the CMV case to obtain Theorems 3.7 and 3.10. Finally, Section 3.4 proves Theorem 3.12 and Corollary 3.13.

3.2 Preliminaries

3.2.1 The Hausdorff Metric

For our proof of Theorem 3.4, we will make use of two facts about the Hausdorff metric, whose definition we briefly recall. Given two *closed* subsets $F, K \subseteq \mathbb{R}$, put

$$d_{\mathrm{H}}(F,K) := \inf\{\varepsilon > 0 : F \subseteq B_\varepsilon(K) \text{ and } K \subseteq B_\varepsilon(F)\}, \tag{3.5}$$

where $B_\varepsilon(X)$ denotes the open ε-neighborhood of the set $X \subseteq \mathbb{R}$. Notice that we allow F and K to be unbounded, and hence d_{H} may take the value $+\infty$. The function d_{H} defines a metric on the space of *compact* subsets of $\mathbb{R}$, known as the *Hausdorff metric*. The following propositions are standard fare.

Proposition 3.14. *Suppose that $(F_n)_{n=1}^\infty$ and $(K_n)_{n=1}^\infty$ are sequences of compact subsets of $\mathbb{R}$. If there exist compact sets F and K such that $F_n \to F$ and $K_n \to K$ with respect to d_{H} as $n \to \infty$, then*

$$|F \cap K| \geq \limsup_{n\to\infty} |F_n \cap K_n|.$$

Proof. Given $\varepsilon > 0$, we may use compactness of $F \cap K$ to choose finitely many open intervals $I_1, \ldots, I_m$ with $F \cap K \subseteq O := \bigcup_{j=1}^m I_j$ and $|O| < |F \cap K| + \varepsilon/2$. Now, take

$$\delta = \frac{\varepsilon}{4m}.$$

Since $F_n \to F$ and $K_n \to K$ with respect to the Hausdorff metric, we have $F_n \cap K_n \subseteq B_\delta(F \cap K)$ for all sufficiently large n. For such large n, one then has $F_n \cap K_n \subseteq B_\delta(O)$, which yields

$$|F_n \cap K_n| \leq |B_\delta(O)| \leq |O| + 2m\delta < |F \cap K| + \varepsilon.$$

This argument clearly implies the desired semicontinuity statement. □

Proposition 3.15. *If S and T are bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, then*

$$d_{\mathrm{H}}(\sigma(S), \sigma(T)) \le \|S - T\|, \tag{3.6}$$

where $\|\cdot\|$ denotes the operator norm.

Proof. Let $\delta = \|T - S\|$, and suppose $x \in \mathbb{R}$ satisfies $d(x, \sigma(T)) > \delta$. In particular, $T - x$ is invertible and, by the spectral theorem, one has

$$\left\|(T - x)^{-1}\right\|^{-1} = d(x, \sigma(T)) > \delta.$$

By an easy geometric series argument, it follows that

$$S - x = (T - x) + (S - T) = (T - x)\left(I + (T - x)^{-1}(S - T)\right)$$

is invertible, i.e., $x \notin \sigma(S)$. Thus, the δ-neighborhood of $\sigma(T)$ contains $\sigma(S)$. By symmetry, one may run the previous argument with the roles of S and T reversed, which suffices to establish (3.6). □

3.2.2 Floquet Theory

In order to describe our main gap-opening lemma, we give a very brief overview of the necessary highlights of Floquet theory for periodic Jacobi operators and prove a minor variant of a gap-opening lemma due to Avila. Suppose $a, b \in \ell^\infty(\mathbb{Z})$ are p-periodic for some $p \in \mathbb{Z}_+$, that is,

$$a(n + p) = a(n), \;\; b(n + p) = b(n) \quad \text{for all } n \in \mathbb{Z}.$$

Given $E \in \mathbb{R}$, the study of the eigenvalue equation

$$a(n-1)u(n-1) + a(n)u(n+1) + b(n)u(n) = Eu(n) \quad \text{for all } n \in \mathbb{Z} \tag{3.7}$$

leads one to define the transfer matrices $T_E = T_E^{(a,b)}$ and $A_E = A_E^{(a,b)}$ via

$$T_E(n) = \frac{1}{a(n)} \begin{pmatrix} E - b(n) & -1 \\ a(n)^2 & 0 \end{pmatrix}, \quad A_E(n) = \begin{cases} T_E(n) \cdots T_E(1) & n \geq 1 \\ I & n = 0 \\ T_E(n+1)^{-1} \cdots T_E(0)^{-1} & n \leq -1 \end{cases}$$

Specifically, a complex-valued sequence u satisfies (3.7) if and only if

$$\begin{pmatrix} u(n+1) \\ a(n)u(n) \end{pmatrix} = A_E(n) \begin{pmatrix} u(1) \\ a(0)u(0) \end{pmatrix} \text{ for every } n \in \mathbb{Z}.$$

The *monodromy matrix* of $J_{a,b}$ is the transfer matrix over a full period; more precisely,

$$\Phi_E = A_E^{(a,b)}(p) = T_E(p) \cdots T_E(1).$$

The *discriminant* of $J_{a,b}$ is defined by $D(E) = \operatorname{tr}(\Phi_E)$.

One can also consider restrictions of $J_{a,b}$ with suitable periodic or antiperiodic boundary conditions. Specifically, let

$$J_{a,b}^{p,\pm} = \begin{pmatrix} b(1) & a(1) & & & \pm a(p) \\ a(1) & b(2) & a(2) & & \\ & \ddots & \ddots & \ddots & \\ & & a(p-2) & b(p-1) & a(p-1) \\ \pm a(p) & & & a(p-1) & b(p) \end{pmatrix}. \tag{3.8}$$

It is easy to see that E is an eigenvalue of $J_{a,b}^{p,+}$ if and only if there is a nontrivial p-periodic solution u of (3.7) and E is an eigenvalue of $J_{a,b}^{p,-}$ if and only if there is a nontrivial p-antiperiodic solution of (3.7). Specifically, if u is a nontrivial eigenvector of $J_{a,b}^{p,\pm}$ with eigenvalue E, then u can be extended to a two-sided sequence on $\mathbb{Z}$ such that (3.7) holds and $u(n+p) = \pm u(n)$ for all $n \in \mathbb{Z}$.

It is well-known that the spectrum of $J_{a,b}$ can be determined either from the polynomial D or from the matrices $J_{a,b}^{p,\pm}$. We summarize the relevant facts in the following theorem. Proofs and further details can be found in [68, Chapter 5].

Theorem 3.16. *If $a \in (-2, 2)$, then all solutions of the equation $D(z) = a$ are real and simple. If $a = \pm 2$, then all solutions of $D(z) = a$ are real and of multiplicity at most two. A solution E of $D(E) = \pm 2$ is of multiplicity two if and only if $\Phi_E = \pm I$. If α_j and β_j denote the solutions of $D = \pm 2$ (with multiplicity), ordered so that*

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_{p-1} \leq \beta_{p-1} \leq \alpha_p \leq \beta_p,$$

then $\alpha_j < \beta_j$ for each $1 \leq j \leq p$, and

$$\sigma(J_{a,b}) = \bigcup_{j=1}^{p} [\alpha_j, \beta_j] = \{E \in \mathbb{R} : |D(E)| \leq 2\}.$$

Moreover, the α's and β's comprise the set of all eigenvalues of $J_{a,b}^{p,\pm}$. Specifically, the eigenvalues of $J_{a,b}^{p,+}$ are $\beta_p, \alpha_{p-1}, \beta_{p-2}, \alpha_{p-3}, \ldots$, while the eigenvalues of $J_{a,b}^{p,-}$ are $\alpha_p, \beta_{p-1}, \alpha_{p-2}, \beta_{p-3}, \ldots$.

Definition 3.17. We call intervals of the form $[\alpha_j, \beta_j]$ with $1 \leq j \leq p$ *bands* of $\sigma(J_{a,b})$, while intervals of the form (β_j, α_{j+1}) with $1 \leq j \leq p-1$ are called *gaps* of the spectrum. Notice that the unbounded components of the resolvent set are not considered gaps. If $\beta_j = \alpha_{j+1}$, we say that the jth gap of $\sigma(J_{a,b})$ is *closed*. To avoid repeating the phrase "with all gaps open," we will say that $J_{a,b}$ is *p-generic* if both a and b are p-periodic and $\sigma(J_{a,b})$ has precisely p connected components. The *band-interior* of the spectrum will be defined by

$$\sigma_{\text{int}}(J_{a,b}) = \bigcup_{j=1}^{p} (\alpha_j, \beta_j).$$

Of course, this is different from the topological interior of $\sigma(J_{a,b})$ whenever the spectrum has closed gaps.

Since we fix a periodic off-diagonal sequence a, we will think of genericity of $J_{a,b}$ as a property of b; specifically, we will say that b is a (p, a)-generic potential if $J_{a,b}$ is p-generic.

The following gap-opening lemma is a straightforward modification of [1, Claim 3.4]. We include the proof with cosmetic alterations to the Jacobi case for the convenience of the reader.

Lemma 3.18. *Suppose $J = J_{a,b}$ is a p-periodic Jacobi matrix, and $k \geq 2$. For each $t \in \mathbb{R}$, define a kp-periodic sequence b_t by*

$$b_t(n) = \begin{cases} b(n) & 1 \leq n \leq kp - 1 \\ b(kp) + t & n = kp \end{cases}$$

Then b_t is (kp, a)-generic for all but finitely many choices of $t \in \mathbb{R}$. In particular, for any $\delta > 0$, there exists a (kp, a)-generic potential b' with $\|b - b'\|_\infty < \delta$, so the generic potentials are dense in the space of periodic potentials.

Proof. If $\sigma(J_{a,b_t})$ has a closed gap at energy $E_t \in \mathbb{R}$, then it follows from Theorem 3.16 that the matrix $A_{E_t}^{(a,b_t)}(kp)$ must be equal to $\pm I$. In particular, examining the unperturbed transfer matrices, we have

$$\begin{aligned} A_{E_t}^{(a,b)}(kp) &= \frac{1}{a(kp)} \begin{pmatrix} E_t - b(kp) & -1 \\ a(kp)^2 & 0 \end{pmatrix} \cdot A_{E_t}^{(a,b)}(kp-1) \\ &= \begin{pmatrix} E_t - b(kp) & -1 \\ a(kp)^2 & 0 \end{pmatrix} \begin{pmatrix} E_t - b(kp) - t & -1 \\ a(kp)^2 & 0 \end{pmatrix}^{-1} \cdot A_{E_t}^{(a,b_t)}(kp) \\ &= \pm \begin{pmatrix} 1 & \frac{t}{a(kp)^2} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

In particular, if $t \neq t'$, this forces $A_{E_t}^{(a,b)}(kp) \neq A_{E_{t'}}^{(a,b)}(kp)$ and hence $E_t \neq E_{t'}$. Moreover, if $t \neq 0$, then the result of the calculation above implies that

$$A_{E_t}^{(a,b)}(p) = \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix},$$

which means that the discriminant of $J_{a,b}$ is ± 2 at E_t. Since the discriminant of $J_{a,b}$ is a polynomial of degree p in E, there can be at most $2p$ distinct values of E for which it attains the values ± 2, and hence the lemma follows. □

3.2.3 Break Points

Next, we describe a device that will enable us to keep track of the locations at which gaps open in the spectrum of a limit-periodic operator. The idea here is the following: we start with a (p, a)-generic potential b and then perform a small perturbation of b to produce a (kp, a)-generic potential b'. Of course, if the perturbation is small enough, then the spectrum of $J_{a,b'}$ will inherit $p - 1$ gaps from the spectrum of $J_{a,b}$ and will produce $(k-1)p$ new gaps – this is a consequence of Proposition 3.15. We want to control the locations at which these new gaps form. From the point of view of logarithmic potential theory, it is natural to partition $\sigma(J_{a,b})$ into kp subintervals, each of which has harmonic measure $\frac{1}{kp}$; compare [68, Section 5.5]. The following definition precisely describes the endpoints of these subintervals.

Definition 3.19. Let $J = J_{a,b}$ be a periodic Jacobi matrix with corresponding discriminant D. We say that $E \in \mathbb{R}$ is a *k-break point* of J if it satsifies

$$D(E) = 2\cos\left(\frac{\pi j}{k}\right) \text{ for some integer } 0 \le j \le k. \tag{3.9}$$

We say that E is a *proper* break point if $1 \le j \le k-1$; equivalently, the improper break points of J are simply the edges of bands of the spectrum of J.

By Theorem 3.16, a p-periodic Jacobi operator will have precisely $(k-1)p$ *proper* k-break points. It is not hard to see that every solution of (3.7) is kp-periodic whenever E is a proper k-break point of J with j even; similarly, every solution of (3.7) is kp-antiperiodic whenever E is a proper k-break point of J with odd j. In particular, the k-break points of J are precisely the eigenvalues of J restricted to $[1, kp]$ with periodic or antiperiodic boundary conditions. More precisely, the set of k-break points of J is precisely the set of eigenvalues of $J_{a,b}^{kp,\pm}$ (where $J_{a,b}^{kp,\pm}$ is defined as in (3.8)). Moreover, from the discussion above, it is easy to see that any proper k-break point of J is a doubly degenerate eigenvalue of one of $J_{a,b}^{kp,\pm}$.

As the name suggests, when we perturb b slightly to produce b', then the $(k-1)p$ new small gaps form near the proper break points, provided b' is sufficiently close to b. This is a relatively straightforward consequence of standard eigenvalue perturbation theory, since the band edges of $\sigma(J_{a,b'})$ are precisely the eigenvalues of

$J^{kp,\pm}_{a,b'}$ by Theorem 3.16. Indeed, the breakup of the spectrum at the break points corresponds to the doubly degenerate eigenvalues of $J^{kp,\pm}_{a,b}$ splitting into two simple eigenvalues as b is perturbed to become b'. The precise statement follows.

Lemma 3.20. *Suppose $J = J_{a,b}$ is a p-generic Jacobi operator and $k \geq 2$. If $\varepsilon > 0$ is sufficiently small, b' is (kp, a)-generic, and*

$$\|b - b'\|_\infty < \varepsilon,$$

then, for each proper k-break point E of J, there exists a gap of $\sigma(J_{a,b'})$ entirely contained within $B_\varepsilon(E)$. Each of the remaining $p-1$ gaps of $\sigma(J_{a,b'})$ is contained in a ε-neighborhood of a gap of $\sigma(J)$. Indeed, this conclusion holds as soon as ε is less than one-half the minimum distance between distinct k-break points of J.

Remark. If one does not assume that b' is (kp, a)-generic, the proof still provides useful information about the structure of $\sigma(J_{a,b'})$. Specifically, if b' is simply kp-periodic and $\|b - b'\| < \delta$, then the proof shows that b' has at least $p - 1$ gaps, each of which is contained in an ε-neighborhood of a gap of $\sigma(J)$, while any other gaps of $\sigma(J_{a,b'})$ must form in ε-neighborhoods of proper break points, with at most one new gap in each ε-neighborhood of a proper break point of J.

Proof of Lemma 3.20. Let $\varepsilon > 0$ be given as in the statement of the lemma. Notice that the condition on ε means that the ε-neighborhoods of the k-break points (of J) are pairwise disjoint. Now, suppose that b' is (kp, a)-generic and satisfies $\|b-b'\|_\infty < \varepsilon$. Applying Proposition 3.15 to the symmetric matrices $J^{kp,+}_{a,b}$ and $J^{kp,+}_{a,b'}$ (resp., to $J^{kp,-}_{a,b}$

and $J^{kp,-}_{a,b'}$), we see that each eigenvalue of $J^{kp,+}_{a,b'}$ must be within ε of an eigenvalue of $J^{kp,+}_{a,b}$ (resp., each eigenvalue of $J^{kp,-}_{a,b'}$ must be within ε of an eigenvalue of $J^{kp,-}_{a,b}$). Using this, disjointness of the ε-neighborhoods of eigenvalues of $J^{kp,\pm}_{a,b}$, and Theorem 3.16, the lemma follows easily.

□

3.3 Spectral Homogeneity for Jacobi and CMV Matrices

3.3.1 Proof of Theorem 3.4

With the technical tools of Section 3.2 in hand, we are now able to prove Theorem 3.4. The main idea of the argument is to use Proposition 3.15 and Lemmas 3.18 and 3.20 inductively to construct sequences of generic potentials such that the spectral bands of each potential are very long compared to its "new" gaps (i.e. the gaps which form near the break points of the previous approximant). This gives us effective step-by-step estimates for the Lebesgue measure of the approximating spectra (at all sufficiently small length scales). We then use Proposition 3.14 to push these estimates through to the limit. The Cantor structure of the spectrum follows from genericity of the approximating potentials and a band length estimate from Appendix A. The precise details follow.

Proof of Theorem 3.4. Fix a (p_0, a)-generic potential b_0 and constants $\tau < 1$, $\varepsilon > 0$. By Lemma 3.18, the generic potentials are dense in $\mathrm{LP}(\mathbb{Z}, \mathbb{R})$, so, to prove the theorem, it suffices to construct an element of $\mathcal{H}_\tau^a$ in $B_\varepsilon(b_0)$. To that end, let λ_0 be the minimal length of a band of $\sigma_0 := \sigma(J_{a,b_0})$, and let γ_0 be the minimal length of a gap of σ_0. Put $\varepsilon_0 = \min(\gamma_0, 4\varepsilon)$ and fix a sequence[2] of integers $k_1, k_2, \ldots \geq 2$. The k's will control the periods of approximants viz. $p_n = k_n p_{n-1}$ for each $n \in \mathbb{Z}_+$. Now, let t_0 be the minimal distance between consecutive k_1-break points of J_{a,b_0} and choose a sequence $r_1 > r_2 > \cdots > 0$ so that

$$\sum_{\ell=1}^{\infty} r_\ell < 1 - \tau. \tag{3.10}$$

Now, we will inductively choose a sequence of positive numbers $(\varepsilon_j)_{j=1}^\infty$ and a sequence of potentials $(b_j)_{j=1}^\infty$ in such a way that the bands of $\sigma(J_{a,b_j})$ are very long relative to the spectral gaps which are introduced in passing from b_{j-1} to b_j. First, for the sake of notation, denote by λ_j the length of the shortest band of $\sigma_j := \sigma(J_{a,b_j})$, let γ_j be the minimal gap length of σ_j, and denote by t_j the minimal distance between consecutive k_{j+1}-break points of J_{a,b_j}. In this notation, we may choose the sequences $(\varepsilon_j)_{j=1}^\infty$ and $(b_j)_{j=1}^\infty$ so that the following properties hold:

- One has

$$\varepsilon_j < \min\left(\frac{\gamma_{j-1}}{5}, \frac{\varepsilon_{j-1}}{5}, \frac{t_{j-1}}{2}, \frac{r_j t_{j-1}}{2r_j + 5}, e^{-j \cdot p_{j+1}}\right) \text{ for all } j \geq 1. \tag{3.11}$$

- For every $j \in \mathbb{Z}_+$, b_j is (p_j, a)-generic.

- For all j, $\|b_j - b_{j-1}\|_\infty < \varepsilon_j$.

[2]For this theorem, it is not necessary to use an arbitrary sequence of k's. However, we will need this freedom to prove Theorem 3.9, since not all periodic potentials fiber over an arbitrary Cantor group, as discussed in the introduction.

We begin by noticing several consequences of these conditions. First, the condition $\varepsilon_j < t_{j-1}/2$ means that the conclusion of Lemma 3.20 holds with $b = b_{j-1}$, $b' = b_j$, and $\varepsilon = \varepsilon_j$. More precisely, $p_{j-1} - 1$ gaps of σ_j are contained in ε_j-neighborhoods of gaps of σ_{j-1}, each ε_j-neighborhood of a proper k_j-break point of $J_{a,b_{j-1}}$ contains exactly one gap of σ_j, and this is an exhaustive list of all gaps of σ_j.

As a consequence of the preceding paragraph, the assumptions on b_j, and Lemma 3.20, we get $\lambda_j \geq t_{j-1} - 2\varepsilon_j$ for each $j \geq 1$. In particular, using this and the fourth condition in (3.11), we obtain

$$\varepsilon_j \leq \frac{\varepsilon_j \lambda_j}{t_{j-1} - 2\varepsilon_j} < \frac{\lambda_j r_j}{5} \tag{3.12}$$

for all $j \in \mathbb{Z}_+$.

To establish the desired spectral homogeneity, we will prove the estimate

$$|B_\delta(E) \cap \sigma_n| \geq \delta \left(1 - \sum_{\ell=1}^{n} r_\ell\right) \text{ for all } 0 < \delta \leq \lambda_0,\ E \in \sigma_n \tag{3.13}$$

for all $n \in \mathbb{Z}_+$. Fix $n \in \mathbb{Z}_+$, $E \in \sigma_n$, and $0 < \delta \leq \lambda_0$. If $\delta \leq \lambda_n$, then the estimate in (3.13) is trivial. Specifically, this implies that $B_\delta(E)$ contains a subinterval of length δ which is completely contained in σ_n, which implies

$$|B_\delta(E) \cap \sigma_n| \geq \delta.$$

Next, assume that $\lambda_j < \delta \leq \lambda_{j-1}$ for some $1 \leq j \leq n$. By Proposition 3.15, there exists $E_0 \in \sigma_{j-1}$ with $|E - E_0| \leq \varepsilon_{j,n} := \sum_{\ell=j}^{n} \varepsilon_\ell$. The key inequality in this step is

$$\varepsilon_{j,n} + \sum_{\ell=j}^{n} 2\varepsilon_\ell \left(\frac{\delta}{\lambda_\ell} + 1\right) < \delta \sum_{\ell=j}^{n} r_\ell, \tag{3.14}$$

which follows from (3.12), and $\delta > \lambda_j$ (hence $\delta > \lambda_\ell$ for every $\ell \geq j$). It is easy to see that there exists an interval I_0 of length $\delta - \varepsilon_{j,n}$ which contains E_0 such that $I_0 \subseteq \sigma_{j-1} \cap B_\delta(E)$. Consequently, we have the following estimates:

$$\begin{aligned} |B_\delta(E) \cap \sigma_n| &\geq |I_0 \cap \sigma_n| \\ &\geq |I_0 \cap \sigma_{j-1}| - \sum_{\ell=j}^{n} |I_0 \cap (\sigma_{\ell-1} \setminus \sigma_\ell)| \\ &= (\delta - \varepsilon_{j,n}) - \sum_{\ell=j}^{n} |I_0 \cap (\sigma_{\ell-1} \setminus \sigma_\ell)| \,. \end{aligned}$$

The third line uses $I_0 \subseteq \sigma_{j-1}$. Obviously, I_0 completely contains fewer than δ/λ_ℓ bands of σ_ℓ for each $j \leq \ell \leq n$, so, by Proposition 3.15 and Lemma 3.20, we have

$$\begin{aligned} (\delta - \varepsilon_{j,n}) - \sum_{\ell=j}^{n} |I_0 \cap (\sigma_{\ell-1} \setminus \sigma_\ell)| &\geq (\delta - \varepsilon_{j,n}) - \sum_{\ell=j}^{n} 2\varepsilon_\ell \left(\frac{\delta}{\lambda_\ell} + 1\right) \\ &> \delta \left(1 - \sum_{\ell=j}^{n} r_\ell\right) \\ &\geq \delta \left(1 - \sum_{\ell=1}^{n} r_\ell\right), \end{aligned}$$

where the penultimate line follows from (3.14). Thus, (3.13) holds for every $n \in \mathbb{Z}_+$. Consequently, by (3.11) and our choice of ε_0, we have a limiting potential $b_\infty := \lim b_n$ with

$$\|b_0 - b_\infty\|_\infty < \sum_{\ell=1}^{\infty} \varepsilon_\ell < \varepsilon_0 \sum_{\ell=1}^{\infty} 5^{-\ell} = \frac{\varepsilon_0}{4} \leq \varepsilon.$$

Moreover, with $\sigma_\infty := \sigma(J_{a,b_\infty})$, we have

$$|B_\delta(E) \cap \sigma_\infty| \geq \delta \left(1 - \sum_{\ell=1}^{\infty} r_\ell\right) > \delta\tau$$

for all $E \in \sigma_\infty$ and $0 < \delta \leq \lambda_0$ by (3.13) and Propositions 3.14 and 3.15. Thus, σ_∞ is τ-homogeneous.

To see that σ_∞ is a Cantor set, it suffices to check that it is nowhere dense, since it cannot have isolated points by general principles [53, Theorem 1]. To that end, let $U \subseteq \mathbb{R}$ be an open interval, and choose n so that $4\pi A^2/p_n < |U|$, where $A = \max(1, \|a\|_\infty)$. By Theorem A.1, U must contain an open subinterval G of length γ_n with $G \cap \sigma_n = \emptyset$, since b_n is (p_n, a)-generic. Notice that (3.11) implies that

$$\|J_{a,b_n} - J_{a,b_\infty}\| = \|b_n - b_\infty\|_\infty < \sum_{\ell=n+1}^{\infty} \varepsilon_\ell < \gamma_n \sum_{k=1}^{\infty} 5^{-k} = \frac{\gamma_n}{4}.$$

Consequently, if c denotes the center of G, then $c \notin \sigma_\infty$ by Proposition 3.15. Thus, σ_∞ is nowhere dense, as desired.

Finally, purely absolutely continuous spectrum is an immediate consequence of (3.11) and a discrete analog of the theorem of Pastur-Tkachenko due to Egorova [27]. Specifically, (3.11) implies that

$$\|J_{a,b_\infty} - J_{a,b_n}\| \le e^{-n\cdot p_{n+1}} \cdot \sum_{j=1}^{\infty} 5^{-j} < e^{-np_{n+1}}.$$

Consequently, one obtains

$$\lim_{n\to\infty} e^{\tilde{C}p_{n+1}} \|J_{a,b_\infty} - J_{a,b_n}\| = 0$$

for every $\tilde{C} > 0$, which implies that J_{a,b_∞} has purely absolutely continuous spectrum by [27]. □

3.3.2 Proof of Theorem 3.9

To prove Theorem 3.9, we need to introduce some more machinery. Throughout this subsection, let Ω be a fixed monothetic Cantor group with topological generator θ.

We say that $f \in C(\Omega, \mathbb{R})$ is a *p-periodic sampling function* if $f \circ T^p = f$, where $T : x \mapsto x + \theta$. This is obviously equivalent to the statement that s_ω^f is a p-periodic sequence for every $\omega \in \Omega$, where s_ω^f is defined as in (3.3). Since Ω is profinite and monothetic, there exists a sequence $\Omega_1 \supseteq \Omega_2 \supseteq \cdots$ of compact finite-index subgroups of Ω with the property that

$$\bigcap_{j=1}^{\infty} \Omega_j = \{0\}.$$

Let n_j denote the index of Ω_j in Ω. The following proposition is not hard to prove; compare [1, Section 3].

Proposition 3.21. *Let $f \in C(\Omega, \mathbb{R})$. Then f is an n_j-periodic sampling function if and only if it descends to a well-defined function on the quotient Ω/Ω_j. Moreover, any periodic sampling function is defined over some quotient of the form Ω/Ω_j with $j \geq 1$. Consequently, if b is a periodic sequence with period p which divides n_j for some j, then $b = b_0^f$ for suitable $f \in C(\Omega, \mathbb{R})$, where $b_0^f(n) = f(n\theta)$, as usual.*

Proof of Theorem 3.9. Suppose $\varepsilon > 0$ and f is an (n_q, a)-generic sampling function for some $q \geq 1$. As before, the generic sampling functions are dense, so it suffices to find an element of $\mathcal{H}_\tau^a$ in $B_\varepsilon(f)$. Let $b_0 = s_0^f$ as in (3.3). Define $k_j = n_{q+j}/n_{q+j-1}$ so that $p_j = n_{q+j}$, and choose p_j-generic potentials b_j exactly as in the proof of Theorem 3.4. In particular, $b_\infty = \lim b_j$ is such that J_{a,b_∞} has all of the desired properties. By Proposition 3.21, there exist $f_j \in C(\Omega, \mathbb{R})$ such that $b_j = b_0^{f_j}$ for each j. It is not hard to see that $\|f_n - f_m\|_\infty = \|b_n - b_m\|_\infty$ for all n, m. Consequently, $f_\infty = \lim f_j$ exists and $b_0^{f_\infty} = b_\infty$, so the theorem is proved. □

3.3.3 The CMV Case

In this subsection, we discuss the modifications to the proofs of Theorems 3.4 and 3.9 necessary to obtain Theorems 3.7 and 3.10. In essence, no extra work is needed – one simply needs to find suitable replacements for the various pieces which comprise the proofs and then re-run the entire machine.

First, we replace Lebesgue measure on $\mathbb{R}$ with arc-length measure on $\partial\mathbb{D}$, that is, the pushforward of Lebesgue measure on $[0, 2\pi)$ under the map $t \mapsto \exp(it)$. Equivalently, arc-length measure on $\partial\mathbb{D}$ can simply be thought of as one-dimensional Hausdorff measure. Clearly, there is a version of the Hausdorff metric for compact subsets of $\partial\mathbb{D}$, also defined by the formula (3.5). Here, the ε-neighborhoods of sets should of course be thought of as ε-neighborhoods with respect to the usual metric on $\mathbb{C}$. It is then trivial to modify Propositions 3.14 and 3.15 to fit this setting. The precise statements follow.

Proposition 3.22. *If $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ are sequences of compact subsets of $\partial\mathbb{D}$ such that $A_n \to A$ and $B_n \to B$ with respect to the Hausdorff metric, then*

$$|A \cap B| \geq \limsup_{n\to\infty} |A_n \cap B_n|,$$

where $|\cdot|$ denotes arc-length measure on $\partial\mathbb{D}$.

Proposition 3.23. *If U and V are unitary operators on a Hilbert space $\mathcal{H}$, then*

$$d_{\mathrm{H}}(\sigma(U), \sigma(V)) \leq \|U - V\|.$$

One also has a version of Floquet theory for periodic CMV matrices; that is, if $\alpha \in \mathbb{D}^{\mathbb{Z}}$ is p-periodic, then the spectrum of $\mathcal{E} = \mathcal{E}_\alpha$ consists of p nonoverlapping closed subarcs of $\partial\mathbb{D}$, which can be found by examining a degree p-polynomial D, just as in the Jacobi case. As before, we say that $\mathcal{E}$ is p-generic if $\sigma(\mathcal{E})$ consists of precisely p connected components. In this setting, it is known that the p-generic CMV operators are dense in the space of all p-periodic CMV operators [67, Theorem 11.13.1]. There is a slight combinatorial difference here, namely, that p-generic CMV matrices have p spectral gaps, not $p-1$.

The analog of the band-length estimate in Theorem A.1 is proved in [51, Lemma 5]. Specifically, if α is p-periodic, then

$$|A| \leq \frac{2\pi}{p} \tag{3.15}$$

for each band $A \subseteq \sigma(\mathcal{E}_\alpha)$. Using Floquet theory for periodic CMV matrices, we can define k-break points of $\mathcal{E}$ in exactly the same way, namely, by partitioning each band of the spectrum into k closed subarcs, each of which has harmonic measure $\frac{1}{kp}$. One can then prove a straightforward modification of Lemma 3.20.

Lemma 3.24. *Suppose $\mathcal{E} = \mathcal{E}_\alpha$ is a p-generic CMV matrix and $k \geq 2$. For all $\varepsilon > 0$ sufficiently small, there exists $\delta > 0$ such that if α' is kp-generic and*

$$\|\alpha - \alpha'\| < \delta,$$

then, for each proper k-break point z of $\mathcal{E}$, there exists a gap of $\sigma(\mathcal{E}_{\alpha'})$ entirely contained within $B_\varepsilon(z)$. Each of the remaining p gaps of $\sigma(\mathcal{E})$ is contained in an ε-neighborhood of some gap of $\sigma(\mathcal{E})$.

The proof is essentially the same as before, with mostly cosmetic variations on the main theme. There is one minor annoyance in this case. Specifically, in the Jacobi case, we (implicitly) used the obvious identity

$$\|J_{a,b} - J_{a,b'}\| = \|b - b'\|_\infty,$$

and this does not translate directly to the CMV context. Instead, one has

$$\|\mathcal{E}_\alpha - \mathcal{E}_{\alpha'}\|^2 \leq 72\|\alpha - \alpha'\|_\infty, \tag{3.16}$$

by [66, (4.3.11)]. This simply introduces some constants which have no qualitative impact on the structure of the proof. With this variant of Lemma 3.20 in hand, the proofs from Section 3.3 can be rerun with minor changes.

3.4 Spectral Homogeneity for Pastur-Tkachenko Potentials

In this section, we prove Theorem 3.12. We begin by recalling the necessary preliminaries from Floquet theory for periodic continuum potentials and proving an estimate for the corresponding discriminants.

3.4.1 Preliminaries

First, we note that Proposition 3.15 extends to the setting of general unbounded self-adjoint operators.

Lemma 3.25. *If A and B are unbounded self-adjoint operators with $\mathcal{D}(A) = \mathcal{D}(B) = \mathcal{D}$ such that $A - B$ is a bounded operator on $\mathcal{D}$, then*

$$d_{\mathrm{H}}(\sigma(A), \sigma(B)) \leq \|A - B\|_\infty.$$

Proof. The proof is an easy modification of the proof of Proposition 3.15. □

Next, let us briefly recapitulate some aspects of Floquet theory for periodic continuum potentials. Suppose V is continuous and T-periodic. Given $z \in \mathbb{C}$, let y_D and y_N denote the Dirichlet and Neumann solutions of the Schrödinger equation

$$-y'' + Vy = zy. \tag{3.17}$$

Specifically, y_D and y_N solve (3.17) subject to the initial conditions

$$y_D(0) = y_N'(0) = 0, \quad y_D'(0) = y_N(0) = 1.$$

The associated *transfer matrices* A_z are defined by

$$A_z(x) = \begin{pmatrix} y_N(x) & y_D(x) \\ y_N'(x) & y_D'(x) \end{pmatrix}.$$

If u is any other solution of (3.17), then it is not hard to check that

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = A_z(x) \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

for every $x \in \mathbb{R}$. The *monodromy matrix* is $M_z := A_z(T)$ and the *discriminant* is given by $\Delta(z) = \mathrm{tr}(M_z)$. One has

$$\sigma(-\nabla^2 + V) = \{E \in \mathbb{R} : |\Delta(E)| \le 2\} = \bigcup_{j=1}^{\infty} [\alpha_j, \beta_j], .$$

where $\alpha_1 < \beta_1 \le \alpha_2 < \cdots$ denote the solutions of $\Delta = \pm 2$. As before, we define a k-break point of V to be a point $E \in \mathbb{R}$ for which

$$\Delta(E) = 2\cos\left(\frac{2\pi j}{k}\right), \quad j \in \mathbb{Z},$$

and say that E is *proper* if j is not congruent to 0 modulo k.

3.4.2 Stability of the Discriminant

To prove Theorem 3.12, we need to control the band lengths of periodic Schrödinger operators. The following lemma provides the key estimate to this effect.

Lemma 3.26. *Let V be a T-periodic potential. Then*

$$|\Delta'(E)| \leq C \exp\left(CT\|V\|_\infty^{1/2}\right) \tag{3.18}$$

for all $E \in \sigma(-\nabla^2 + V)$, where C is a universal constant which does not depend on V or T.

Proof. In this proof, C will stand for different universal constants; its value will increase at most finitely many times over the course of the argument.

We begin by proving (3.18) for $T = \pi$. Denote $Q = \|V\|_\infty$. Using [54, Corollary 17.8], we have

$$|\Delta(\mu^2)| \leq C \exp\left(CQ^{1/2} + \pi|\operatorname{Im}\mu|\right). \tag{3.19}$$

Put $f(\mu) = \Delta(\mu^2)$. Using the Cauchy differentiation formula on a circle of radius 1, we have

$$|f'(\mu)| \leq \int_0^{2\pi} |f(\mu + e^{i\theta})| \, \frac{d\theta}{2\pi}.$$

Consequently, we observe:

$$|2\mu\Delta'(\mu^2)| \leq C \exp\left(CQ^{1/2} + \pi|\operatorname{Im}\mu| + \pi\right). \tag{3.20}$$

We can also conclude from (3.19) that

$$|\Delta(z)| \leq C \exp\left(CQ^{1/2} + 2\pi\right), \qquad |z| \leq 2,$$

and then by the Cauchy differentiation formula for $\Delta(z)$, similarly as above, that

$$|\Delta'(z)| \leq C \exp\left(CQ^{1/2} + 2\pi\right), \qquad |z| \leq 1. \tag{3.21}$$

Using (3.20) for $|z| > 1$ and (3.21) for $|z| \leq 1$, we conclude that for all $z \in \mathbb{C}$,

$$|\Delta'(z)| \leq C \exp\left(CQ^{1/2} + \pi|\operatorname{Im}\sqrt{z}|\right).$$

If $E \in \sigma(-\nabla^2 + V)$, then $E \geq -Q$. Consequently, $\left|\operatorname{Im}\sqrt{E}\right| \leq Q^{1/2}$, so we conclude that the statement of the lemma holds for $T = \pi$.

The result for arbitrary potentials follows by a simple rescaling argument. For a potential V of arbitrary period T, introduce the rescaled π-periodic potential $V_\pi(x) = (T/\pi)^2 V(Tx/\pi)$. Its discriminant $\Delta_\pi(E)$ obeys $\Delta_\pi(E) = \Delta((T/\pi)^2 E)$, so applying the lemma to V_π and noting $\|V_\pi\|_\infty = (T/\pi)^2\|V\|_\infty$, the lemma follows. □

Corollary 3.27. *If C denotes the universal constant from Lemma 3.26 and V is a T-periodic potential with $\|V\|_\infty \leq Q$, then the length of any band of $\sigma(-\nabla^2+V)$ is at least $4C^{-1}e^{-CQ^{1/2}T}$. Consequently, any interval $I \subseteq \mathbb{R}$ completely contains no more than $\frac{1}{4}C|I|e^{CQ^{1/2}T}$ bands of $\sigma(-\nabla^2 + V)$.*

Proof. This follows from Lemma 3.26 and the Mean Value Theorem. □

3.4.3 Proof of Theorem 3.12

The proof of Theorem 3.12 is similar to the proof of Theorem 3.4, in that we use perturbative arguments to obtain effective step-by-step estimates and then Proposition 3.14 to pass to the limit. However, this time, we are not free to choose the periodic potentials which approximate our limiting potential. Instead, we have to use the Pastur-Tkachenko condition and Lemma 3.26 to show that (eventually) one has fast enough convergence to get bands which are very large compared to gaps. The precise details follow.

Proof of Theorem 3.12. Suppose $0 < \tau < 1$, $V \in \mathrm{PT}(\mathbb{R})$, and let $V = \lim V_n$, where $V_n \in C(\mathbb{R})$ is T_n-periodic and $\|V - V_n\|_\infty$ is $o\left(e^{-bT_{n+1}}\right)$ for every $b > 0$. Notice that this property (i.e. equation (3.4)) is preserved if one removes finitely many terms of the sequence $(V_n)_{n=1}^\infty$ and consecutively renumbers the resulting sequence. Put $k_n = T_{n+1}/T_n$. Using the Pastur-Tkachenko condition (3.4), it is easy to see that

$$\sum_{n=1}^\infty e^{CQ^{1/2}T_{n+1}} \|V_n - V_{n+1}\|_\infty < +\infty,$$

where C is the constant from Lemma 3.26 and $Q = \sup_n \|V_n\|_\infty$. Thus, by removing finitely many terms of the sequence $(V_n)_{n=1}^\infty$ and consecutively renumbering, we may assume that

$$\sum_{n=1}^\infty e^{CQ^{1/2}T_{n+1}} \|V_n - V_{n+1}\|_\infty < \frac{2-2\tau}{3C}. \tag{3.22}$$

Using (3.4) once more, we may truncate and renumber $(V_n)_{n=1}^\infty$ again to ensure that

$$\|V - V_n\|_\infty < 16C^{-2}e^{-2CQ^{1/2}T_{n+1}} \text{ for all } n \geq 1. \tag{3.23}$$

Denote $\Sigma = \sigma(-\nabla^2 + V)$ and $\Sigma_n = \sigma(-\nabla^2 + V_n)$, and put

$$\delta_0 = \min\left(4C^{-1}e^{-CQ^{1/2}T_1}, \frac{16}{C^2}, \frac{1-\tau}{13-\tau}\right).$$

Given $x \in \Sigma$ and $0 < \delta \leq \delta_0$, choose $n \geq 1$ such that

$$4C^{-1}e^{-CQ^{1/2}T_{n+1}} < \delta \leq 4C^{-1}e^{-CQ^{1/2}T_n}.$$

Note that such an n exists by our first restriction on δ_0. By Lemma 3.25, there exists $x_0 \in \Sigma_n$ with $|x - x_0| \leq \|V - V_n\|_\infty$. Using (3.23), we have

$$\|V - V_n\|_\infty < 16C^{-2}e^{-2CQ^{1/2}T_{n+1}} < \delta^2 < \frac{1-\tau}{3}\delta,$$

where we have used

$$\delta \leq \frac{1-\tau}{13-\tau} < \frac{1-\tau}{3}$$

in the last step. Thus, since $\delta \leq 4C^{-1}e^{-CQ^{1/2}T_n}$, Corollary 3.27 implies that there exists an interval I_0 with $x_0 \in I_0 \subseteq B_\delta(x) \cap \Sigma_n$ such that

$$|I_0| = \delta - \frac{1-\tau}{3}\delta = \frac{2+\tau}{3}\delta.$$

For any $\ell > n$, we obviously have

$$|B_\delta(x) \cap \Sigma_\ell| \geq |I_0 \cap \Sigma_n| - \sum_{j=n}^{\ell-1} |I_0 \cap (\Sigma_j \setminus \Sigma_{j+1})|$$

By Corollary 3.27, the interval I_0 completely contains at most $\frac{1}{4}C\delta e^{CQ^{1/2}T_{\ell+1}}$ bands of $\Sigma_{\ell+1}$ for each $\ell \geq n$. Thus,

$$|I_0 \cap (\Sigma_\ell \setminus \Sigma_{\ell+1})| \leq \left(\frac{1}{4}C\delta e^{CQ^{1/2}T_{\ell+1}} + 1\right) \cdot 2\|V_\ell - V_{\ell+1}\|_\infty$$

Combining this with (3.22), we obtain

$$\begin{aligned}
|B_\delta(x)\cap\Sigma_\ell| &\geq |I_0\cap\Sigma_n| - \sum_{j=n}^{\ell-1}|I_0\cap(\Sigma_j\setminus\Sigma_{j+1})| \\
&> \frac{2+\tau}{3}\delta - 2\sum_{j=n}^{\ell-1}\left(\frac{1}{4}C\delta e^{CQ^{1/2}T_{j+1}}+1\right)\|V_j-V_{j+1}\|_\infty \\
&> \frac{2+\tau}{3}\delta - \frac{1-\tau}{3}\delta - 2\sum_{j=n}^{\ell-1}\|V_j-V_{j+1}\|_\infty \\
&> \frac{1+2\tau}{3}\delta - 4\sum_{j=n}^{\infty}16C^{-2}e^{-2CQ^{1/2}T_{j+1}} \\
&\geq \frac{1+2\tau}{3}\delta - 4\sum_{m=2}^{\infty}\delta^m \\
&= \frac{1+2\tau}{3}\delta - \frac{4\delta^2}{1-\delta} \\
&\geq \tau\delta,
\end{aligned}$$

for all $\ell \geq n$. The fifth line uses $\delta \leq 16C^{-2}$ and $4C^{-1}e^{-CQ^{1/2}T_{n+1}} < \delta$, and the final line uses $\delta \leq \frac{1-\tau}{13-\tau}$ to get $\frac{4\delta}{1-\delta} \leq \frac{1}{3}(1-\tau)$. Using Proposition 3.14 and Lemma 3.25, we obtain

$$|B_\delta(x)\cap\Sigma| \geq \tau\delta,$$

to wit, Σ is τ-homogeneous. □

With Theorem 3.12 in hand, the proof of Corollary 3.13 is a straightforward combination of standard material.

Proof of Corollary 3.13. It is easy to see that $\mathrm{PT}(\mathbb{R})$ is dense in $\mathrm{LP}(\mathbb{R})$ with respect to the topology of uniform convergence. Using the main theorem of [64], we see that the set of $V \in \mathrm{PT}(\mathbb{R})$ such that $\sigma(-\nabla^2+V)$ is a Cantor set is dense in $\mathrm{PT}(\mathbb{R})$.

By Theorem 3.12 and [55, 56], $\mathrm{PT}(\mathbb{R}) \subseteq \mathcal{H}_{1^-}$. Thus, $\mathcal{H}_{1^-}$ is dense in $\mathrm{LP}(\mathbb{R})$, as desired. □

Chapter 4

The Thue-Morse Hamiltonian

4.1 Background

4.1.1 The Thue-Morse Substitution

Given a finite set $\mathcal{A}$, called the *alphabet*, a *substitution* on $\mathcal{A}$ is a map $S : \mathcal{A} \to \mathcal{A}^*$, where $\mathcal{A}^*$ denotes the free monoid on $\mathcal{A}$, i.e., the set of all finite words formed by concatenating elements of $\mathcal{A}$. Such a function extends naturally to maps (denoted by the same letter) $S : \mathcal{A}^* \to \mathcal{A}^*$ and $S : \mathcal{A}^{\mathbb{Z}_+} \to \mathcal{A}^{\mathbb{Z}_+}$. The Thue-Morse substitution acts on an alphabet with two symbols, say $\mathcal{A} = \{a, b\}$, and is defined by

$$S(a) = ab,\ S(b) = ba.$$

By repeatedly applying S to a, one obtains a sequence of words, each of which is a prefix of the next. In particular, it makes sense to define

$$x_a = \lim_{n\to\infty} S^n(a) = abbabaabbaababba\ldots.$$

More precisely, the kth letter of x_a is the kth letter of $S^n(a)$ for any $n \geq \log_2(k)$. Since $S(x_a) = x_a$, one calls x_a a *substitution word* for the Thue-Morse substitution. We can also produce another substitution word, x_b, by iterating S on b instead.

The word x_a generates a subshift, namely the set of all bi-infinite sequences which have the same local factor structure as x_a. More precisely, we define

$$\Omega_{\mathrm{TM}} = \left\{\omega \in \mathcal{A}^{\mathbb{Z}} : \omega(n) \cdots \omega(n+k) \in W_{x_a} \text{ for all } n \in \mathbb{Z}, k \geq 0\right\},$$

where W_{x_a} denotes the collection of all subwords of x_a. This a closed (hence compact) subset of $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ is given the discrete topology and $\mathcal{A}^{\mathbb{Z}}$ is given the product topology. Moreover, Ω_{TM} is invariant under the action of the left shift $T : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, defined by $(T\omega)(n) = \omega(n+1)$. It is well known that the dynamical system $(\Omega_{\mathrm{TM}}, T)$ is strictly ergodic, i.e. both minimal and uniquely ergodic (e.g., see [12, Theorem 2.21]).

4.1.2 Schrödinger Operators Defined by the Thue-Morse Substitution

We are interested in the spectra of Schrödinger operators associated to the Thue-Morse subshift. In particular, given $\lambda \in \mathbb{R}$ (often called the *coupling constant*) and $\omega \in \Omega_{\mathrm{TM}}$, we produce a potential V_ω^λ defined by

$$V_\omega^\lambda(n) = \lambda g(\omega(n)),$$

where $g(a) = -1, g(b) = 1$. This produces a self-adjoint operator H_ω^λ acting on $\ell^2(\mathbb{Z})$ by

$$(H_\omega^\lambda u)_n = u_{n-1} + u_{n+1} + V_\omega^\lambda(n) u_n.$$

It is well-known that the spectrum of H_ω^λ is independent of $\omega \in \Omega_{\mathrm{TM}}$ (compare [11, Theorem 9.2]; see also [53, Corollary 1]), so we will make a judicious choice, namely $\omega_0 = x_a^R x_a$, where x_a^R denotes the word x_a written backwards. If one requires an explicit formula,

$$\omega_0(k) = \begin{cases} x_a(k) & \text{if } k \geq 1 \\ \\ x_a(-k+1) & \text{if } k \leq 0 \end{cases}.$$

It is a straightforward exercise to show that one indeed has $\omega_0 \in \Omega_{\mathrm{TM}}$. We are interested in characteristics of the spectrum $\Sigma_\lambda := \sigma\left(H_{\omega_0}^\lambda\right)$.

Remark. It is no loss of generality to consider $\lambda > 0$. To see this, consider the operator $H_{\tilde{\omega}}$, where $\tilde{\omega}$ is obtained from ω_0 by reversing the roles of a and b. It is an easy exercise to see that $\tilde{\omega}$ is also a member of the subshift Ω_{TM}. Moreover, it is easy to see that $H_{\omega_0}^{-\lambda} = H_{\tilde{\omega}}^\lambda$, so, by uniformity of the spectrum, we have

$$\sigma(H_{\omega_0}^{-\lambda}) = \Sigma_\lambda.$$

Theorem 4.1 (Axel, Peyrière 1988; Bovier, Ghez 1993)**.** *For all $\lambda \neq 0$, Σ_λ is a Cantor set of zero Lebesgue measure.*

Theorem 4.1 was conjectured based on numerical evidence in [5], and was rigorously proved in [7]. This motivates the study of the fractal characteristics of Σ_λ. The preprint [47] suggests that the fractal dimensions of Σ_λ behave qualitatively differently than those for the Fibonacci Hamiltonian. Additionally, the calculations from [58] suggest that the structure of Σ_λ is rather exotic.

4.1.3 The Trace Map

In order to understand the spectrum of the operator $H^{\lambda}_{\omega_0}$, we consider the associated difference equation, as before:

$$u_{n-1} + u_{n+1} + V^{\lambda}_{\omega_0}(n)u_n = Eu_n, \quad n \in \mathbb{Z},\ E \in \mathbb{R}. \tag{4.1}$$

In this setting, the transfer matrices exhibit a useful hierarchical structure, which we will now describe. Specifically, for all $E, \lambda \in \mathbb{R}$, let

$$S_0(E,\lambda) = \begin{pmatrix} E+\lambda & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.2}$$

$$T_0(E,\lambda) = \begin{pmatrix} E-\lambda & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.3}$$

and define S_n and T_n inductively for $n \geq 1$ by

$$S_n(E,\lambda) = T_{n-1}(E,\lambda)S_{n-1}(E,\lambda) \tag{4.4}$$

$$T_n(E,\lambda) = S_{n-1}(E,\lambda)T_{n-1}(E,\lambda). \tag{4.5}$$

These unimodular matrices have the property that

$$S_n(E,\lambda)\begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} u_{2^n+1} \\ u_{2^n} \end{pmatrix}$$

for all $n \geq 0$ whenever $u : \mathbb{Z} \to \mathbb{C}$ satisfies (4.1). We define x_n for $n \geq 1$ by

$$x_n(E,\lambda) = \operatorname{tr}(S_n(E,\lambda)) = \operatorname{tr}(T_n(E,\lambda)), \tag{4.6}$$

where the second equality follows from (4.4), (4.5), and cyclicity of the trace. The following well-known recursive relationship amonst the traces is quite useful.

Proposition 4.2. *For all* $n \geq 1$*, we have*

$$x_{n+2} = x_n^2(x_{n+1} - 2) + 2. \tag{4.7}$$

Consequently,

$$x'_{n+2} = 2x_n x'_n(x_{n+1} - 2) + x_n^2 x'_{n+1}. \tag{4.8}$$

In these identities, we suppress the dependence of x *on* E *and* λ *and use primes to denote derivatives with respect to* E*.*

Proof. By the Cayley-Hamilton theorem, cyclicity of the trace, (4.4), and (4.5), we have

$$\begin{aligned}
x_{n+2} &= \mathrm{tr}(S_{n+2}) \\
&= \mathrm{tr}(S_n T_n T_n S_n) \\
&= \mathrm{tr}(S_n^2 T_n^2) \\
&= \mathrm{tr}((x_n S_n - I)(x_n T_n - I)) \\
&= \mathrm{tr}(x_n^2 S_n T_n - x_n(S_n + T_n) + I) \\
&= x_n^2 x_{n+1} - 2x_n^2 + 2,
\end{aligned}$$

as desired. □

Since we have a two-term recursion relation amongst the x_n, it behooves us to compute explicit formulae for x_1 and x_2, which is easily done. Using the definitions of S_n and T_n ((4.2), (4.3), (4.4), and (4.5)), we get

$$S_1 = \begin{pmatrix} E^2 - \lambda^2 - 1 & -E + \lambda \\ E + \lambda & -1 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} E^2 - \lambda^2 - 1 & -E - \lambda \\ E - \lambda & -1 \end{pmatrix},$$

and

$$S_2 = \begin{pmatrix} (E^2 - \lambda^2 - 1)^2 - (E + \lambda)^2 & -(E^2 - \lambda^2 - 1)(E - \lambda) + E + \lambda \\ (E^2 - \lambda^2 - 1)(E - \lambda) - E - \lambda & -(E - \lambda)^2 + 1 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} (E^2 - \lambda^2 - 1)^2 - (E - \lambda)^2 & -(E^2 - \lambda^2 - 1)(E + \lambda) + E - \lambda \\ (E^2 - \lambda^2 - 1)(E + \lambda) - E + \lambda & -(E + \lambda)^2 + 1 \end{pmatrix}$$

Thus,

$$x_1(E, \lambda) = E^2 - \lambda^2 - 2 \tag{4.9}$$

$$x_2(E, \lambda) = E^4 - (2\lambda^2 + 4)E^2 + \lambda^4 + 2. \tag{4.10}$$

Inductively, x_n is an even function of E for every $n \geq 1$ (by (4.7)).

4.2 Periodic Approximants to the Thue-Morse Hamiltonian

4.2.1 Preliminaries

For each $n \in \mathbb{Z}_+$, consider H_n^λ, the Schrödinger operator whose potential agrees with that of $H_{\omega_0}^\lambda$ from $k = 1$ to $k = 2^n$ and is 2^n-periodic. By our judicious choice of ω_0, we know that H_{2n}^λ converges to $H_{\omega_0}^\lambda$ in the strong operator topology as $n \to \infty$. On the other hand, H_{2n+1}^λ strongly converges to the operator whose matrix has λ and $-\lambda$ switched on the nonpositive half of the diagonal. This potential corresponds to the sequence $x_b^R x_a$, which is also readily seen to be an element of Ω_{TM}). As noted in Theorem 3.16, the spectrum of H_n^λ is precisely given by

$$\sigma(H_n^\lambda) = \Sigma_{n,\lambda} := \{E \in \mathbb{R} : |x_n(E, \lambda)| \le 2\}.$$

By strong operator convergence (cf. [59, Theorem VIII.24]), we have

$$\Sigma_\lambda \subseteq \bigcap_{n=1}^{\infty} \overline{\bigcup_{k \ge n} \Sigma_{k,\lambda}}. \tag{4.11}$$

In fact, we can simplify the right hand side of (4.11) considerably by making the following observation.

Proposition 4.3. *For any $n \ge 1$ and $\lambda > 0$,*

$$\Sigma_{n+2,\lambda} \subseteq \Sigma_{n,\lambda} \cup \Sigma_{n+1,\lambda}.$$

Proof. The proof is a simple argument by cases which makes use of (4.7), the trace recursion formula. By way of contraposition, assume that $|x_n|, |x_{n+1}| > 2$. If $x_{n+1} > 2$, then

$$x_{n+2} = x_n^2(x_{n+1} - 2) + 2 > 2.$$

Otherwise, $x_{n+1} < -2$, which yields $x_{n+2} < 4(-4) + 2 < -2$. □

Corollary 4.4. *The containment* (4.11) *reduces to*

$$\Sigma_\lambda \subseteq \bigcap_{n=1}^{\infty} \left(\Sigma_{n,\lambda} \cup \Sigma_{n+1,\lambda}\right). \tag{4.12}$$

Proof. It suffices to show that the right hand sides of (4.11) and (4.12) are equal. One inclusion is trivial. The nontrivial inclusion is a consequence of Proposition 4.3, closedness of $\Sigma_{n,\lambda}$ for all n and λ, and the fact that finite unions of closed sets are closed. □

4.2.2 Gap Opening

As $\Sigma_{n,\lambda}$ is the spectrum of a 2^n-periodic operator, it consists of 2^n bands (possibly touching at closed gaps). Let $B_{n,\lambda}$ denote the number of connected components of $\Sigma_{n,\lambda}$ and $G_{n,\lambda}$ the number of closed gaps. Evidently, $B_{n,\lambda}+G_{n,\lambda} = 2^n$. We are heading towards the proof of our main result: one can explicitly compute $B_{n,\lambda}$ for all n and the result is λ-independent.

Theorem 4.5. *Fix $\lambda > 0$. We have $G_{1,\lambda} = 0$ and $B_{1,\lambda} = 2$. For any $n \geq 2$,*

$$G_{n,\lambda} = 2^{n-1} - 2$$

$$B_{n,\lambda} = 2^{n-1} + 2.$$

First, notice that the statement in the $n = 1$ case is a trivial consequence of (4.9), so, for the remainder of this section, we will concern ourselves with $n \geq 2$. We will prove the remaining content of the theorem by understanding multiplicities of solutions of $x_n = \pm 2$. In particular, Theorem 4.5 (for $n \geq 2$) will follow relatively easily once we prove that the following facts hold true:

- Any solution of $x_2 = 2$ has multiplicity one. Moreover, any such solution is also a simple solution of $x_n = 2$ for all $n \geq 3$.
- For each $n \geq 3$, any solution of $x_n = 2$ which is not a solution of $x_2 = 2$ has multiplicity two.
- For each $n \geq 2$, every solution of $x_n = -2$ is simple.

Throughout this subsection, $\lambda > 0$ is regarded as fixed, so the dependence of various quantities thereon is suppressed to make the notation more readable.

Lemma 4.6. *Fix $\lambda > 0$. All roots of $x_2 - 2$ are simple. Moreover, any such root is also a simple root of $x_n - 2$ for all $n \geq 2$.*

Proof. The identity (4.10) makes it plain to see that $x_2 = 2$ has four distinct solutions whenever $\lambda > 0$, namely

$$E = \pm\sqrt{\lambda^2 + 2 \pm 2\sqrt{\lambda^2 + 1}}.$$

Since x_2 has degree four, it immediately follows that all solutions are simple.

For the second claim of the lemma, assume that $x_2(E) = 2$. It follows that $x_n = 2$ for all $n \geq 2$ by (4.7). Looking at (4.9), we see also that $x_1(E) \neq 0$, so (4.8) yields

$$x_3'(E) = x_1(E)^2 x_2'(E) \neq 0.$$

Using (4.8) inductively, $x_n'(E) \neq 0$ for every $n \geq 2$, which concludes the proof of the lemma. □

Lemma 4.7. *Fix $\lambda > 0$. For all $n \geq 3$, any root of $x_n - 2$ which is not a root of $x_2 - 2$ necessarily has multiplicity two.*

Proof. Suppose $n \geq 3$ and $x_n(E) = 2 \neq x_2(E)$. Using (4.7), we see that either $x_{n-1} = 2$ or $x_{n-2} = 0$. Iterating this (and using $x_2(E) \neq 2$ and $n \geq 3$), we may choose $1 \leq j \leq n-2$ with $x_j(E) = 0$. In particular,

$$x_{j+2}'(E) = 2x_j(E)x_j'(E)(x_{j+1}(E) - 2) + x_j^2(E)x_{j+1}'(E) = 0$$

by (4.8). Using (4.8) inductively, we get $x_n'(E) = 0$, so $x_n - 2$ has a root of multiplicity 2 at E. □

The purpose of the next two lemmas is to show that $x_n = -2$ never has a solution of multiplicity two, which is the same thing as saying that all gaps at band edges where $x_n = -2$ are open. The overall idea underlying the proof is as follows: If $x_n = -2$ has a double root at E, then Floquet theory implies that $T_n(E) = S_n(E) = -I$. The meat of the proof is then showing that this equality cannot happen.

Lemma 4.8. *Fix $\lambda > 0$. If $S_n(E) = T_n(E)$ for some $E \in \mathbb{R}$ and $n \geq 1$, then we must have $x_n(E) = 2$.*

Proof. One can verify by hand that the statement of the lemma holds for $n = 1, 2$. In fact, for all $\lambda > 0, E \in \mathbb{R}$, we have $S_1(E) \neq T_1(E)$ and $S_2(E) \neq T_2(E)$.

Arguing by way of contradiction, assume that we have $n \geq 3$ minimal such that there exists $E \in \mathbb{R}$ with $S_n(E) = T_n(E)$ but $x_n(E) \neq 2$. By (4.4) and (4.5), we know that $S_{n-1}(E)$ and $T_{n-1}(E)$ commute. Of course, we cannot have $x_{n-1}(E) = 2$, for this would force $x_n(E) = 2$, contrary to our hypothesis. We also cannot have $x_{n-1}(E) = -2$. For, if this is true, S_{n-1} and T_{n-1} enjoy a common eigenvector $v \neq 0$ corresponding to the eigenvalue -1. But then

$$T_n v = S_{n-1} T_{n-1} v = -S_{n-1} v = v,$$

which implies that 1 is an eigenvalue of T_n. But this would yield $x_n = 2$, contrary to our assumption thereon.

Thus, it must be the case that $x_{n-1} \neq \pm 2$. In particular, S_{n-1} and T_{n-1} are $\mathrm{SL}_2(\mathbb{R})$ matrices with the same non-parabolic trace, so they must have the same eigenvalues $z \neq z^{-1}$. Let $v_\pm$ be unit vectors such that

$$S_{n-1} v_\pm = z^{\pm 1} v_\pm.$$

By commutativity, we must have either

$$T_{n-1} v_\pm = z^{\pm 1} v_\pm \text{ or } T_{n-1} v_\pm = z^{\mp 1} v_\pm.$$

Unpacking, we have $S_{n-1} = T_{n-1}$ in the first case and $S_{n-1} = T_{n-1}^{-1}$ in the second. The first possibility contradicts minimality of n. The second possibility implies that $T_n = S_{n-1} T_{n-1} = I$, contradicting $x_n \neq 2$. Thus, it is impossible to have $T_n = S_n$ and $x_n \neq 2$.

□

Corollary 4.9. *Fix $\lambda > 0$. For all $n \geq 1$, $x_n = -2$ has only simple solutions.*

Proof. Notice that having a double root of $x_n = -2$ implies that $S_n = T_n = -I$, contrary to Lemma 4.8.

□

With this preparatory work, the proof of Theorem 4.5 is easy.

Proof of Theorem 4.5. By Floquet theory, there are exactly 2^{n+1} solutions of $x_n = \pm 2$, all of which are real and have multiplicity one or two. Lemma 4.6, Lemma 4.7, and Corollary 4.9 imply that $x_n = \pm 2$ has precisely $2^n + 4$ simple solutions and $2^{n-1} - 2$ solutions of multiplicity two. Since closed gaps form precisely at double roots of $x_n = \pm 2$, the claimed identities for $B_{n,\lambda}$ and $G_{n,\lambda}$ follow immediately.

□

Appendix A

A Band Length Estimate for Periodic Jacobi Operators

In this appendix, we provide a proof of a band length estimate for periodic Jacobi operators which is analogous to (3.15) and [1, Lemma 2.4(1)]. Specifically, we have the following upper bound.

Theorem A.1. *Suppose J is a p-periodic Jacobi matrix, and let*

$$A = \max(1, a_1, \ldots, a_p).$$

The Lebesgue measure of any band of $\sigma(J)$ is bounded above by $\frac{2\pi A^2}{p}$.

In order to prove the desired band length estimate, we need to discuss the integrated density of states for periodic Jacobi operators. In particular, we will elucidate a point of view on the IDS of periodic operators discussed in [2]. This is a special case of general, powerful formulas for absolutely continuous spectrum; see [24]. The material in this appendix is standard and well-known within the community of spectral theorists, but we opted to present it here, since [1] and [2] do not work out the details explicitly, and these references work exclusively in the discrete Schrödinger setting, where $a \equiv 1$ (except [24], which also works out a similar framework for continuum Schrödinger operators).

In general, for a Jacobi matrix J, the corresponding integrated density of states, k, is defined by the limit

$$k(E) = \lim_{N \to \infty} \frac{1}{2N+1} \#\{\lambda \in \sigma(J_N) : \lambda \leq E\}, \tag{A.1}$$

whenever the limit exists, where J_N denotes the restriction of J to the interval $[-N, N]$ with Dirichlet boundary conditions, i.e.

$$J_N = \begin{pmatrix} b(-N) & a(-N) & & & \\ a(-N) & b(-N+1) & a(-N+1) & & \\ & \ddots & \ddots & \ddots & \\ & & a(N-2) & b(N-1) & a(N-1) \\ & & & a(N-1) & b(N) \end{pmatrix}.$$

It is a well-known fact that the the limit on the right-hand side of (A.1) exists whenever J is a p-periodic Jacobi matrix. Using Floquet theory, one can explicitly describe k in terms of the discriminant, D. The following theorem is standard; see [68, Theorem 5.4.5].

Theorem A.2. *Suppose J is a p-periodic Jacobi matrix, with corresponding discriminant D and integrated density of states k. Then k is differentiable on on the interior of the spectrum. We have*

$$\frac{dk}{dE} = \frac{1}{\pi p}\left|\frac{d\theta}{dE}\right|, \tag{A.2}$$

where $\theta = \theta(E)$ is chosen continuously so that

$$2\cos(\theta) = D(E) \tag{A.3}$$

for $E \in \sigma_{\text{int}}(H)$. In particular, if B is any band of the spectrum,

$$\int_B dk(E) = \frac{1}{p}. \tag{A.4}$$

Recall that any element of $\mathrm{SL}(2,\mathbb{R})$ induces a linear fractional transformation on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

For $\theta \in \mathbb{R}$, define

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Obviously,

$$\mathrm{SO}(2) = \{R_\theta : 0 \le \theta < 2\pi\} = \{R \in \mathrm{SL}(2,\mathbb{R}) : R \cdot i = i\}. \tag{A.5}$$

Lemma A.3. *A matrix $A \in \mathrm{SL}(2,\mathbb{R})$ satisfies $|\mathrm{tr}(A)| < 2$ if and only if its action on $\mathbb{C}_+$ has a unique fixed point. Whenever $|\mathrm{tr}(A)| < 2$, there exists $M \in \mathrm{SL}(2,\mathbb{R})$ such that*

$$MAM^{-1} = R_\theta \in \mathrm{SL}(2,\mathbb{R}),$$

where $2\cos(\theta) = \mathrm{tr}(A)$. Moreover, such a conjugacy is unique modulo left-multiplication by an element of $\mathrm{SO}(2)$.

Proof. This is a consequence of straightforward calculations. □

Now, for E in the interior of a band, $|D(E)| < 2$, so the monodromy matrix is conjugate to R_θ, where θ satisfies $2\cos(\theta) = D(E)$. From Theorem A.2, we know that the derivative of the integrated density of states can be related to $|d\theta/dE|$, so we would like to find some other way to recover this derivative. By way of motivation, suppose θ is a smooth function of t. It is then easy to check that

$$R_\theta^{-1}\frac{dR_\theta}{dt} = \begin{pmatrix} 0 & -d\theta/dt \\ d\theta/dt & 0 \end{pmatrix}.$$

This motivates us to define the *anti-trace* of a 2×2 matrix by

$$\operatorname{atr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = c - b.$$

Like the usual trace, the anti-trace is a linear functional in the sense that

$$\operatorname{atr}(A + \lambda B) = \operatorname{atr}(A) + \lambda \operatorname{atr}(B)$$

for all $\lambda \in \mathbb{R}$ and all $A, B \in \mathbb{R}^{2\times 2}$. However, unlike the trace, the anti-trace is not cyclic, i.e., one can have $\operatorname{atr}(AB) \neq \operatorname{atr}(BA)$. Despite this, we still have the following weakened variant of cyclicity.

Lemma A.4. *If $R \in \mathrm{SO}(2)$ and $A \in \mathbb{R}^{2\times 2}$,*

$$\operatorname{atr}\left(R^{-1}AR\right) = \operatorname{atr}(A). \tag{A.6}$$

Proof. This is an easy calculation. □

Lemma A.5. *Suppose I is an open interval and $\Phi : I \to \mathrm{SL}(2,\mathbb{R})$ is a smooth map such that $\left|\operatorname{tr}(\Phi(t))\right| < 2$ for all $t \in I$. Under these conditions, there exists a smooth* *choice of* $M \in \mathrm{SL}(2,\mathbb{R})$ *such that*

$$M\Phi M^{-1} = R_\theta, \tag{A.7}$$

where $2\cos(\theta) = \mathrm{tr}(\Phi)$*. Moreover, the angle* θ *can be chosen to be a smooth function of* t*; in this case, it satisfies*

$$\frac{d\theta}{dt} = \frac{1}{2}\mathrm{atr}\left(M\Phi^{-1}\frac{d\Phi}{dt}M^{-1}\right),$$

Proof. To construct the conjugacy M, first notice that the unique fixed point $z = z(t) \in \mathbb{C}_+$ of Φ varies smoothly with t. We then define

$$M(t) = \big(\mathrm{Im}(z(t))\big)^{-1/2}\begin{pmatrix} 1 & -\mathrm{Re}(z(t)) \\ 0 & \mathrm{Im}(z(t)) \end{pmatrix},$$

Evidently, the linear fractional transformation corresponding to $M\Phi M^{-1}$ fixes i, which implies $M\Phi M^{-1} \in \mathrm{SO}(2)$. By cyclicity of the trace, $M\Phi M^{-1}$ must be of the claimed form. Differentiating the relation (A.7) using the product rule, one obtains

$$\frac{dM}{dt}\Phi M^{-1} + M\frac{d\Phi}{dt}M^{-1} + M\Phi\frac{dM^{-1}}{dt} = \frac{dR}{dt} = R\begin{pmatrix} 0 & -d\theta/dt \\ d\theta/dt & 0 \end{pmatrix}.$$

Multiply on the left by R^{-1} and simplify using (A.7) to obtain

$$R^{-1}\frac{dM}{dt}M^{-1}R + M\Phi^{-1}\frac{d\Phi}{dt}M^{-1} + M\frac{dM^{-1}}{dt} = \begin{pmatrix} 0 & -d\theta/dt \\ d\theta/dt & 0 \end{pmatrix}. \tag{A.8}$$

By (A.6), linearity of the anti-trace, and the product rule,

$$\begin{aligned} \mathrm{atr}\left(R^{-1}\frac{dM}{dt}M^{-1}R + M\frac{dM^{-1}}{dt}\right) &= \mathrm{atr}\left(\frac{dM}{dt}M^{-1} + M\frac{dM^{-1}}{dt}\right) \\ &= \mathrm{atr}\left(\frac{d}{dt}(MM^{-1})\right) \\ &= 0. \end{aligned}$$

Thus, (A.7) follows by taking the anti-trace of (A.8). □

We can use the preceding lemma to find another way to view the integrated density of states of a periodic Jacobi operator via Hilbert-Schmidt norms of conjugacies between monodromy matrices and rotations. Specifically, suppose J is p-periodic and denote

$$T_j = \frac{1}{a(j)} \begin{pmatrix} E - b(j) & -1 \\ a(j)^2 & 0 \end{pmatrix}, \quad A_j = T_j \cdots T_1, \quad \Phi_j = A_{j-1} A_p A_{j-1}^{-1}, \quad \text{for } j \geq 1,$$

where we adopt the convention $A_0 = I$ in the $j = 1$ case of the final definition and suppress the dependence of all quantities on E for notational simplicity. For $E \in \sigma_{\mathrm{int}}(H)$, choose $M_j \in \mathrm{SL}(2,\mathbb{R})$ such that $M_j \Phi_j M_j^{-1} \in \mathrm{SO}(2)$.

Theorem A.6. *Let J be a p-periodic Jacobi operator with corresponding integrated density of states k, and put*

$$A = \max(a_1, \ldots, a_p, 1).$$

We have

$$\frac{dk}{dE} \geq \frac{1}{4\pi A^2 p} \sum_{j=1}^{p} \|M_j\|_2^2$$

on $\sigma_{\mathrm{int}}(H)$, where $\|M\|_2 = \sqrt{\mathrm{tr}(M^* M)}$ denotes the Hilbert-Schmidt norm of M.

Proof. First, notice that $\|M_j\|_2$ does not depend on the choice of conjugacy, for any other conjugacy from Φ_j to a rotation must take the form OM_j for some $O \in \mathrm{SO}(2)$ by Lemma A.3. Since we may take M_j to be given by the explicit formula

$$M_j = \big(\mathrm{Im}(z_j)\big)^{-1/2} \begin{pmatrix} 1 & -\mathrm{Re}(z_j) \\ 0 & \mathrm{Im}(z_j) \end{pmatrix}$$

we see that

$$\|M_j\|_2^2 = \frac{1 + |z_j|^2}{\mathrm{Im}(z_j)},$$

where z_j is the unique fixed point of the action of Φ_j on $\mathbb{C}_+$. Notice that $T_j z_j = z_{j+1}$ and hence $M_{j+1} T_j M_j^{-1}$ fixes i, so $M_{j+1} T_j M_j^{-1} =: Q_j \in \mathrm{SO}(2)$. One can easily compute

$$T_j^{-1} \frac{dT_j}{dE} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

Thus, by the product rule, we have

$$\Phi_1^{-1} \frac{d\Phi_1}{dE} = \sum_{j=1}^{p} A_{j-1}^{-1} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} A_{j-1}$$

With $R_j = Q_j \cdots Q_1$ and $R_0 = I$, we have

$$\Phi_1^{-1} \frac{d\Phi_1}{dE} = \sum_{j=1}^{p} M_1^{-1} R_{j-1}^{-1} M_j \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} M_j^{-1} R_{j-1} M_1 \tag{A.9}$$

To find the rate of change of θ with respect to E, we apply Lemma A.5 and compute

$$\begin{aligned} \frac{d\theta}{dE} &= \frac{1}{2} \mathrm{atr} \left(M_1 \Phi_1^{-1} \frac{d\Phi_1}{dE} M_1^{-1} \right) \\ &= \frac{1}{2} \mathrm{atr} \left(\sum_{j=1}^{p} M_j \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} M_j^{-1} \right) \\ &= \frac{1}{2} \sum_{j=1}^{p} \frac{|z_j|^2}{\mathrm{Im}(z_j)}. \end{aligned}$$

The second line follows from (A.9) and Lemma A.4, and the final line is a straightforward computation from the explicit form of M_j. An easy calculation using $T_j z_j = z_{j+1}$ reveals

$$\mathrm{Im}(z_{j+1}) = \mathrm{Im} \left(\frac{E - b_j}{a_j^2} - \frac{1}{a_j^2 z_j} \right) = \frac{\mathrm{Im}(z_j)}{a_j^2 |z_j|^2},$$

which implies

$$\begin{aligned}\left|\frac{d\theta}{dE}\right| &= \frac{1}{2}\sum_{j=1}^{p}\frac{|z_j|^2}{\mathrm{Im}(z_j)} \\ &= \frac{1}{4}\sum_{j=1}^{p}\left(\frac{1}{a_{j-1}^2\mathrm{Im}(z_j)} + \frac{|z_j|^2}{\mathrm{Im}(z_j)}\right) \\ &\geq \frac{1}{4A^2}\sum_{j=1}^{p}\frac{1+|z_j|^2}{\mathrm{Im}(z_j)} \\ &= \frac{1}{4A^2}\sum_{j=1}^{p}\|M_j\|_2^2.\end{aligned}$$

Thus, the conclusion of the Theorem follows from Theorem A.2. □

With this fact in hand, the desired estimate on the bands is easy.

Proof of Theorem A.1. Let B denote a band of $\sigma(J)$. Using (A.4) and Theorem A.6, one has

$$\frac{1}{p} = \int_B dk(E) \geq \int_B \left(\frac{1}{4\pi A^2 p}\sum_{j=1}^{p}\|M_j\|_2^2\right) dE \geq \frac{|B|}{2\pi A^2},$$

where we have used the bound $\|M\|_2^2 \geq 2$ which holds for any $M \in \mathrm{SL}(2,\mathbb{R})$ (by Cauchy-Schwarz) in the final inequality. The theorem follows. □

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