MINIMAL SURFACES IN EUCLIDEAN N-SPACE

by

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A thesis presented to the faculty of the Rice Institute in partial fulfilment of the requirements for the degree of Doctor of Philosophy. June, 1931.

UMI Number: 3079619

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1. Introduction. Let

$$x_r = x_r(u, v), \quad r = 1, 2, \ldots, n, \tag{1.1}$$

define the general analytic 2-spread, or surface, in n dimensional Euclidean space, the x_r being analytic in the domain of definition. Put

$$x_{r,ab} = \frac{\partial^{a+b} x_r}{\partial^a u \, \partial^b v}; \tag{1.2}$$

$$E = \sum_{r=1}^{n} x_{r,10}^2, \quad F = \sum_{r=1}^{n} x_{r,10} x_{r,01}, \quad G = \sum_{r=1}^{n} x_{r,01}^2; \tag{1.3}$$

$$H = (EG - F^2)^{1/2}. \tag{1.4}$$

The element of area of the surface is defined to be

$$d\sigma = H \, du \, dv, \tag{1.5}$$

so that the area of a portion of the surface is

$$\sigma = \iint H \, du \, dv. \tag{1.6}$$

A minimal surface is, by definition, a surface for which the first variation of the above integral (1.6) vanishes for a given contour curve. This variation is

$$\delta\sigma = \iint \sum_{r=1}^{m} \left[\frac{\partial H}{\partial x_{r,10}} \delta x_{r,10} + \frac{\partial H}{\partial x_{r,01}} \delta x_{r,01} \right] du\, dv$$

$$= -\iint \sum_{r=1}^{m} \left[\frac{\partial}{\partial u} \frac{\partial H}{\partial x_{r,10}} + \frac{\partial}{\partial v} \frac{\partial H}{\partial x_{r,01}} \right] \delta x_r \, du\, dv; \tag{1.7}$$

in order that

$$\delta\sigma = 0, \tag{1.8}$$

it is necessary that the quantity in the latter brackets vanish for each r. Using the identities

$$\frac{\partial H}{\partial x_{r,10}} = \frac{1}{H}\left[x_{r,10} G - x_{r,01} F \right],$$

$$\frac{\partial H}{\partial x_{r,01}} = \frac{1}{H}\left[x_{r,01} E - x_{r,10} F \right], \tag{1.9}$$

we write this condition as

$$x_{r,20} \frac{G}{H} + x_{r,02} \frac{E}{H} - 2 x_{r,11} \frac{F}{H}$$

$$+ x_{r,10}\left[\frac{\partial}{\partial u}\left(\frac{G}{H}\right) - \frac{\partial}{\partial v}\left(\frac{F}{H}\right) \right] + x_{r,01}\left[\frac{\partial}{\partial v}\left(\frac{E}{H}\right) - \frac{\partial}{\partial u}\left(\frac{F}{H}\right) \right] = 0. \tag{1.10}$$

Thus far the parametric curves have been perfectly general. We shall make use of the following two choices of parameters.

If we choose isothermic parameters, so that

$$E = G, \quad F = 0, \tag{1.11}$$

then our equations of condition (1.10) become

$$x_{r,20} + x_{r,02} = 0, \quad r = 1, 2, \cdots, n, \tag{1.12}$$

that is, the x_r are harmonic functions. Since the converse of this result evidently is true, we have the

Theorem: A necessary and sufficient condition that a surface, given in terms of isothermic parameters, be minimal is that (1.12) be satisfied.

If we choose minimal curves as parametric, so that

$$E = G = 0, \tag{1.13}$$

then our equations reduce to

$$x_{r,11} = 0, \quad r = 1, 2, \cdots, n. \tag{1.14}$$

The converse of this also is true, so that we have the

Theorem: A necessary and sufficient condition that a surface, its minimal curves being parametric, be minimal is that (1.14) be satisfied.

If we take the coordinates of a minimal surface as satisfying (1.14) and integrate this equation, we obtain

$$x_r = u_r(u) + v_r(v); \tag{1.15}$$

and since, in this case, the parametric curves are minimal, we have

$$\sum_{r=1}^{n} x_{r,10}^2 = 0, \quad \sum_{r=1}^{n} x_{r,01}^2 = 0. \tag{1.16}$$

2. <u>Parametric representation of minimal curves.</u> Let a curve in n dimensions be represented by the analytic functions

$$x_r = x_r(u), \; r = 1, 2, \ldots, n. \tag{2.1}$$

A minimal curve, or curve of zero length, is a curve for which

$$\sum_{r=1}^{n} x_r'^2 = 0, \tag{2.2}$$

the primes denoting differentiation with respect to u.

Equation (2.2) can be written as

$$(x_1' + i x_2')(x_1' - i x_2') = - \sum_{r=3}^{n} x_r'^2, \tag{2.3}$$

so that

$$\frac{x_1' + i x_2'}{-\left(\sum_{r=3}^{n} x_r'^2\right)^{1/2}} = \frac{\left(\sum_{r=3}^{n} x_r'^2\right)^{1/2}}{x_1' - i x_2'} = \left[f_1(u)\right]^{1/2}. \tag{2.4}$$

This gives

$$x_1' : x_2' : \left(\sum_{r=3}^{n} x_r'^2\right)^{1/2} = \tfrac{1}{2}(1-f_1) : \tfrac{i}{2}(1+f_1) : f_1^{1/2}. \tag{2.5}$$

Consequently,

$$\begin{aligned} x_1' &= \tfrac{1}{2}(1-f_1)f_2, \\ x_2' &= \tfrac{i}{2}(1+f_1)f_2, \\ \left(\sum_{r=3}^{n} x_r'^2\right)^{1/2} &= f_1^{1/2} f_2, \end{aligned} \tag{2.6}$$

where f_2 (u) is the function of proportionality defined by

$$f_2(u) = \frac{2x_1'}{1-f_1} = \frac{-2i\,x_2'}{1+f_1} = \left(\frac{\sum_{r=3}^{n} x_r'^2}{f_1}\right)^{1/2}. \tag{2.7}$$

We start now with the equation

$$\sum_{r=3}^{n} x_r'^2 - f_1 f_2^2 = 0 \tag{2.8}$$

and proceed exactly as before to determine x_3' and x_4' , and so on. In general, we start with

$$\sum_{r=2s-1}^{n} x_r'^2 - \sum_{r=1}^{s-1} f_{2r-1} f_{2r}^2 = 0 \tag{2.9}$$

and define

$$f_{2s-1}(u) = \frac{(x'_{2s-1} + i\, x'_{2s})^2}{\sum_{r=2s+1}^{n} x_r'^2 - \sum_{r=1}^{s-1} f_{2r-1} f_{2r}^2}$$

$$= \frac{\sum_{r=2s+1}^{n} x_r'^2 - \sum_{r=1}^{s-1} f_{2r-1} f_{2r}^2}{(x'_{2s-1} - i\, x'_{2s})^2},$$

$$f_{2s}(u) = \frac{2x'_{2s-1}}{1 - f_{2s-1}} = \frac{-2i\, x'_{2s}}{1 + f_{2s-1}} \tag{2.10}$$

$$= \left(\frac{\sum_{r=2s+1}^{n} x_r'^2 - \sum_{r=1}^{s-1} f_{2r-1} f_{2r}^2}{f_{2s-1}} \right)^{1/2},$$

so that

$$x'_{2s-1} = \tfrac{1}{2}(1 - f_{2s-1}) f_{2s},$$

$$x'_{2s} = \tfrac{i}{2}(1 + f_{2s-1}) f_{2s}. \tag{2.11}$$

Finally, if n is even, n = 2 m, we have

$$f_{2m} = \left(\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}} \right)^{1/2}; \tag{2.12}$$

while if n is odd, $n = 2m + 1$, we have

$$x'_{2m+1} = \left(\sum_{r=1}^{m} f_{2r-1} f_{2r}^{2}\right)^{1/2}. \tag{2.13}$$

It is to be noted that we do not have two alternatives in selecting the roots appearing in (2.12) and (2.13), since we must choose those roots which yield the given x_r.

Whether n is even or odd, then, we are able to express the n analytic derivative functions

$$x_r'(u), \quad r = 1, 2, \cdots, n,$$

by means of the $n - 1$ analytic functions

$$f_r(u), \quad r = 1, 2, \cdots, n-1. \tag{2.14, 2.15}$$

Integrating, we have

$$x_{2s-1} = \frac{1}{2}\int (1 - f_{2s-1}) f_{2s}\, du,$$

$$x_{2s} = \frac{i}{2}\int (1 + f_{2s-1}) f_{2s}\, du; \tag{2.16}$$

if $n = 2m$,

$$x_{2m+1} = \frac{1}{2}\int (1 - f_{2m-1}) \left(\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^{2}}{f_{2m-1}}\right)^{1/2} du, \tag{2.17A}$$

$$x_{2m} = \frac{i}{2}\int (1+f_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right)^{1/2} du; \tag{2.17B}$$

while if n = 2 m + 1,

$$x_{2m+1} = \int \left(\sum_{r=1}^{m} f_{2r-1} f_{2r}^2\right)^{1/2} du. \tag{2.18}$$

Conversely, any n - 1 arbitrary analytic functions put into these equations determine a minimal curve in n dimensions.

Our parametric representation serves also for the expression of the coordinates of a space curve in n - 1 dimensions, for if we let s represent the arc length and set

$$x_n = i\, s, \tag{2.19}$$

then (2.2) becomes

$$ds^2 = \sum_{r=1}^{n-1} dx_r^2. \tag{2.20}$$

3. Normal parametric representation. The above representation is not at all unique, for we might

replace u by an arbitrary analytic function

$$u = u(v) \tag{3.1}$$

and express the x_r as functions of v.

We choose the particular parameter

$$U = f_1(u) = \frac{(dx_1 + i\,dx_2)^2}{\sum_{r=3}^{n} dx_r^2} = \frac{\sum_{r=3}^{n} dx_r^2}{(dx_1 - i\,dx_2)^2}\,, \tag{3.2}$$

neglecting for the present the possibility of the exceptional case

$$f_1(u) = c \tag{3.3}$$

where c is a constant, and the possibility of $f_1(u)$ being indeterminate. The equations (2.1) become

$$x_r = X_r(U), \quad r = 1, 2, \cdots, n, \tag{3.4}$$

and from these we can deduce the parametric equations

$$\begin{aligned} x_{2s-1} &= \frac{1}{2}\int (1 - F_{2s-1})\, F_{2s}\, dU, \\ x_{2s} &= \frac{i}{2}\int (1 + F_{2s-1})\, F_{2s}\, dU\,; \end{aligned} \tag{3.5}$$

if n = 2 m,

$$x_{2m-1} = \frac{i}{2}\int (1 - F_{2m-1}) \left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^{2}}{F_{2m-1}} \right)^{1/2} dU,$$

$$x_{2m} = \frac{i}{2}\int (1 + F_{2m-1}) \left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^{2}}{F_{2m-1}} \right)^{1/2} dU; \tag{3.6}$$

while if n = 2 m + 1,

$$x_{2m+1} = \int \left(\sum_{r=1}^{m} F_{2r-1} F_{2r}^{2} \right)^{1/2} dU. \tag{3.7}$$

Here we have

$$F_1(U) = U, \tag{3.8}$$

and

$$F_{2s-1}(U) = \frac{(x'_{2s-1} + i x'_{2s})^2}{\sum_{r=2s+1}^{n} x_r'^{2} - \sum_{r=1}^{s-1} F_{2r-1} F_{2r}^{2}}$$

$$= \frac{\sum_{r=2s+1}^{n} x_r'^{2} - \sum_{r=1}^{s-1} F_{2r-1} F_{2r}^{2}}{(x'_{2s-1} - i x'_{2s})^2},$$

$$F_{2s}(U) = \frac{2x'_{2s-1}}{1 - F_{2s-1}} = \frac{-2i\, x'_{2s}}{1 + F_{2s-1}} \tag{3.9}$$

$$= \left(\frac{\sum_{r=2s+1}^{n} x_r'^{2} - \sum_{r=1}^{s-1} F_{2r-1} F_{2r}^{2}}{F_{2s-1}} \right)^{1/2},$$

the primes denoting differentiation with respect to U.

We call U the normal parameter and the $F_r(U)$ the normal functions of the curve.

We have, by (2.4),

$$F_1(U) = \frac{\left(\frac{dx_1}{dU} + i\,\frac{dx_2}{dU}\right)^2}{\sum_{r=3}^{n}\left(\frac{dx_r}{dU}\right)^2} = \frac{(dx_1 + i\,dx_2)^2}{\sum_{r=3}^{n} dx_r^{\,2}}$$

$$= \frac{\left(\frac{dx_1}{du} + i\,\frac{dx_2}{du}\right)^2}{\sum_{r=3}^{n}\left(\frac{dx_r}{du}\right)^2} = f_1(u), \tag{3.10}$$

and, by (2.7),

$$F_2(U) = \frac{2\,\frac{dx_1}{dU}}{1-F_1} = \frac{2\,\frac{dx_1}{du}}{1-f_1}\,\frac{du}{dU} = f_2(u)\,\frac{du}{dU}. \tag{3.11}$$

Assuming

$$\left.\begin{aligned} F_{2r-1}(U) &= f_{2r-1}(u), \\ F_{2r}(U) &= f_{2r}(u)\,\frac{du}{dU}, \end{aligned}\right\} \quad r < \rho, \tag{3.12}$$

we use (3.9) and (2.10) to prove by induction that these formulae hold for r = s:

$$F_{2s-1}(U) = \frac{\left(\frac{dx_{2s-1}}{dU} + i\,\frac{dx_{2s}}{dU}\right)^2}{\sum_{r=2s+1}^{n}\left(\frac{dx_r}{dU}\right)^2 - \sum_{r=1}^{s-1} F_{2r-1} F_{2r}^2}$$

$$= \frac{\left[\left(\frac{dx_{2s-1}}{du} + i\,\frac{dx_{2s}}{du}\right)\frac{du}{dU}\right]^2}{\sum_{r=2s+1}^{n}\left(\frac{dx_r}{du}\,\frac{du}{dU}\right)^2 - \sum_{r=1}^{s-1} f_{2r-1}\left(f_{2r}\frac{du}{dU}\right)^2}$$

$$= \frac{\left(\frac{dx_{2s-1}}{du} + i\,\frac{dx_{2s}}{du}\right)^2}{\sum_{r=2s+1}^{n}\left(\frac{dx_r}{du}\right)^2 - \sum_{r=1}^{s-1} f_{2r-1} f_{2r}^2} = f_{2s-1}(u),$$

$$F_{2s}(U) = \frac{2\,\frac{dx_{2s-1}}{dU}}{1 - F_{2s-1}} = \frac{2\,\frac{dx_{2s-1}}{du}}{1 - f_{2s-1}}\,\frac{du}{dU} = f_{2s}(u)\frac{du}{dU}. \tag{3.13}$$

If n = 2 m + 1, (3.7) and (2.18) are equivalent:

$$x_{2m+1} = \int \left(\sum_{r=1}^{m} F_{2r-1} F_{2r}^{2} \right)^{1/2} dU$$

$$= \int \left[\sum_{r=1}^{m} f_{2r-1} \left(f_{2r} \frac{du}{dU} \right)^{2} \right]^{1/2} dU \qquad (3.14)$$

$$= \int \left(\sum_{r=1}^{m} f_{2r-1} f_{2r}^{2} \right)^{1/2} du;$$

while if n = 2 m, (3.6) and (2.17) are equivalent, by the same substitution.

We turn now to the exceptional case (3.3). Equations (2.6) give

$$x_1 = \tfrac{1}{2}(1-c)\, g(u),$$

$$x_2 = \tfrac{i}{2}(1+c)\, g(u), \qquad (3.15)$$

where

$$g(u) = \int f_2(u)\, du. \qquad (3.16)$$

Manifestly, then, the projection of the minimal curve on the (x_1 , x_2) plane is a straight line. If $f_1(u)$ is indeterminate, then x_1 and x_2 are both constant or $x_2 = \pm i x_1 + k$ and the projection on the (x_1 , x_2) plane is a point or a straight line.

Conversely, if the projection of the minimal curve on the (x_1, x_2) plane is a straight line or a point, either we have the exceptional case (3.3), or $f_1(u)$ is indeterminate. If either x_1 or x_2 is constant, but not both, then (2.4) gives respectively either

$$f_1 = 1 = c, \quad \text{or} \quad f_1 = -1 = c; \tag{3.17}$$

if both are constant, then $f_1(u)$ is indeterminate; if

$$x_2 = a x_1 + b, \tag{3.18}$$

then, by (2.16),

$$\frac{i}{2}\int (1+f_1) f_2 \, du = \frac{a}{2}\int (1-f_1) f_2 \, du + b,$$

$$\frac{i}{2}(1+f_1) f_2 = \frac{a}{2}(1-f_1) f_2, \tag{3.19}$$

so that either

$$f_1 = \frac{a-i}{a+i} = c, \tag{3.20}$$

or

$$f_2 = 0. \tag{3.21}$$

But if (3.21) holds, then, by (2.7),

$$x_1' = x_2' = 0, \tag{3.22}$$

and this is the indeterminate case we have just mentioned.

If the projection on the (x_1, x_2) plane is a straight line or a point, we define

$$U = \frac{(dx_1 + i\,dx_3)^2}{\sum_{r \neq 1,3} dx_r^2}, \tag{3.23}$$

provided this quantity is neither a constant nor indeterminate, and proceed to determine the normal parameter and normal functions for the sequence

$$x_1, x_3, x_2, x_4, x_5, \ldots, x_n, \tag{3.24}$$

just as we would do otherwise for the sequence

$$x_1, x_2, \ldots, x_n. \tag{3.25}$$

In general, if x_{n_1} is the coordinate of lowest rank for which there exists at least one other coordinate x_s such that

$$\frac{(dx_{n_1} + i\,dx_s)^2}{\sum_{r \neq n_1, s} dx_r^2} \tag{3.26}$$

is neither constant nor indeterminate, and if x_{n_2} is the coordinate of lowest rank among all such x_s, we determine the normal parameter and normal functions for the sequence

$$x_{n_1}, x_{n_2}, x_1, x_2, \cdots, x_n, \qquad (3.27)$$

and define these to be the normal parameter and normal functions for the sequence (3.25), that is, for the minimal curve.

If for all x_{n_1}, x_s, (3.26) is either constant or indeterminate, then each coordinate is a linear function of each other coordinate, excepting that some might be identically constant, and the minimal curve is a straight line. Conversely, if the minimal curve is a straight line, then for all x_{n_1}, x_s, (3.26) is either constant or indeterminate.

To sum up: the coordinates of the minimal curve defined by (2.1), (2.2) may be given in terms of the unique normal parameter U and the n - 2 unique normal functions $F_r(U)$, r = 2, 3,..., n - 1, unless the minimal curve is a straight line. If the curve is a straight line, the normal parametric representation is impossible, but the simple parametric representation of section 2 still is valid.

4. <u>Reflections in the coordinate hyperplanes.</u>

If we reflect our minimal curve in the hyperplane

$$x_{2s-1} = 0, \tag{4.1}$$

we obtain a minimal curve the equations of which are the same as those of the original curve except the one for the (2 s - 1)th coordinate, which differs from the original in sign only. Let us designate the coordinates of the original curve by

$$F_r(U),\ r = 1, 2, \dots, n-1,\ F_1(U) = U, \tag{4.2}$$

and those of the reflected curve by

$$\mathcal{F}_r(\mathcal{U}),\ r = 1, 2, \cdots, n-1,\ \mathcal{F}_1(\mathcal{U}) = \mathcal{U}. \tag{4.3}$$

We inquire the relations between the $\mathcal{F}_r(\mathcal{U})$ and the $F_r(U)$.

If 2 s - 1 = 1, then, by (3.2), (3.9),

$$\mathcal{U} = \frac{(-dx_1 + i\,dx_2)^2}{\sum_{r=3}^{n} dx_r^2} = \frac{(dx_1 - i\,dx_2)^2}{\sum_{r=3}^{n} dx_r^2} = \frac{1}{U},$$

$$\mathcal{F}_2 = \frac{-2\frac{dx_1}{d\mathcal{U}}}{1-\mathcal{U}} = \frac{2U\frac{dx_1}{dU}\frac{dU}{d\mathcal{U}}}{1-U} = UF_2\frac{dU}{d\mathcal{U}} = -U^3F_2. \tag{4.4}$$

The expressions (3.9) for the F_{2r-1}, F_{2r}, r > 1, are unaltered except for the appearances in them of the F_{2t-1}, F_{2t}, t < r. Since

$$\mathcal{U}\mathcal{F}_2^2 = UF_2^2\left(\frac{dU}{d\mathcal{U}}\right)^2, \tag{4.5}$$

we see by (3.9) that

$$\mathfrak{J}_3 = F_3 ,$$
$$\mathfrak{J}_4 = F_4 \frac{dU}{d\mathcal{U}} , \tag{4.6}$$

and so on; in general,

$$\mathfrak{J}_{2r-1} = F_{2r-1} ,$$
$$\mathfrak{J}_{2r} = F_{2r} \frac{dU}{d\mathcal{U}} , \quad r > 1. \tag{4.7}$$

If 2 s - 1 $\neq$ 1, then $\mathcal{U}$ = U, and, by (3.5),

$$(1 - \mathfrak{J}_{2s-1})\mathfrak{J}_{2s} = -(1 - F_{2s-1}) F_{2s} ,$$
$$(1 + \mathfrak{J}_{2s-1})\mathfrak{J}_{2s} = (1 + F_{2s-1}) F_{2s} . \tag{4.8}$$

Solving these equations, we obtain

$$\mathfrak{J}_{2s-1} = \frac{1}{F_{2s-1}} ,$$
$$\mathfrak{J}_{2s} = F_{2s-1} F_{2s} . \tag{4.9}$$

It is to be noted that F_{2s-1} and F_{2s} are not involved in the expressions for the x_r, r < 2 s - 1, which are unaltered by the reflection. The expressions for the x_r, r > 2 s, also are unaltered except for the appearances in them of F_{2s-1} and F_{2s}, which are changed according to (4.9); but these appearances always are of the form $F_{2s-1}\ F_{2s}^2$, which satisfies the identity

$$F_{2s-1}\, F_{2s}^2 = \mathcal{F}_{2s-1}\, \mathcal{F}_{2s}^2 . \tag{4.10}$$

The only normal functions which are altered by the reflection are therefore F_{2s-1} and F_{2s}. We have

$$\begin{aligned} \mathcal{F}_r &= F_r\ , \quad r \neq 2s-1,\ r \neq 2s, \\ \mathcal{F}_{2s-1} &= \frac{1}{F_{2s-1}}\ , \\ \mathcal{F}_{2s} &= F_{2s-1}\, F_{2s}\ . \end{aligned} \tag{4.11}$$

Quite similarly, we show that a reflection in the hyperplane

$$x_2 = 0 \tag{4.12}$$

is effected by means of the equations

$$\begin{aligned} \mathcal{U} &= \frac{1}{U}\ , \\ \mathcal{F}_2 &= -U F_2 \frac{dU}{d\mathcal{U}}\ , \\ \mathcal{F}_{2r-1} &= F_{2r-1}\ , \\ \mathcal{F}_{2r} &= F_{2r}\frac{dU}{d\mathcal{U}}\ , \qquad r > 1; \end{aligned} \tag{4.13}$$

while a reflection in the hyperplane

$$x_{2s} = 0, \quad s > 1, \tag{4.14}$$

is effected by means of the equations

$$\begin{aligned} y_r &= F_r, \quad r \neq 2s-1, \ r \neq 2s, \\ y_{2s-1} &= \frac{1}{F_{2s-1}}, \\ y_{2s} &= -F_{2s-1} F_{2s}. \end{aligned} \tag{4.15}$$

If n = 2 m, and we reflect in the hyperplane

$$x_{2m-1} = 0, \tag{4.16}$$

we obtain

$$\begin{aligned} y_r &= F_r, \quad r = 1, 2, \cdots, 2m-2, \\ y_{2m-1} &= \frac{1}{F_{2m-1}}, \\ \left(\frac{-\sum_{r=1}^{m-1} y_{2r-1} y_{2r}^2}{y_{2m-1}} \right)^{1/2} &= F_{2m-1} \left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^2}{F_{2m-1}} \right)^{1/2}; \end{aligned} \tag{4.17}$$

while if we reflect in the hyperplane

$$x_{2m} = 0 \tag{4.18}$$

we obtain

$$y_r = F_r \ , \ r = 1, 2, \cdots, 2m-2,$$

$$y_{2m-1} = \frac{1}{F_{2m-1}}, \tag{4.19}$$

$$\left(\frac{-\sum_{r=1}^{m-1} y_{2r-1} y_{2r}^2}{y_{2m-1}} \right)^{1/2} = -F_{2m-1} \left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^2}{F_{2m-1}} \right)^{1/2}.$$

If n = 2 m + 1 and we wish to reflect in the hyperplane

$$x_{2m+1} = 0, \tag{4.20}$$

we have but to choose the other value of the square root appearing in (3.7).

In solving for the F_r in any of the above reflections, we note that the F_r are the same functions of the y_r as the respective y_r are of the F_r.

We can reflect in as many of the coordinate hyperplanes as we wish, by making the reflections one at a time; the equations of the transformation result from the succession of the separate sets of equations.

Another way of considering the above development is this: we have shown that any minimal curve, given by the normal set of parametric equations (3.5)-(3.7), can, by a suitable change of the normal functions, be expressed by equations which are of the form (3.5)-(3.7) except that such of the integrals as we please are multiplied by minus one.

5. <u>Normal parametric representation of minimal surfaces.</u> According to (1.16), the two sets of functions $u_r(u)$ and $v_r(v)$ of (1.15) can be given normal representation; let the normal functions be $F_r(U)$ and $\Psi_r(T)$. According to section 4, we can replace the functions $\Psi_r(T)$ by the functions $\Phi_r(V)$ so that the functions (1.1) representing a minimal surface in n dimensions can be written in the form:

$$\begin{aligned} x_{2s-1} &= \frac{1}{2}\int(1-F_{2s-1})F_{2s}\,dU + \frac{1}{2}\int(1-\Phi_{2s-1})\Phi_{2s}\,dV, \\ x_{2s} &= \frac{i}{2}\int(1+F_{2s-1})F_{2s}\,dU - \frac{i}{2}\int(1+\Phi_{2s-1})\Phi_{2s}\,dV; \end{aligned} \tag{5.1}$$

if n = 2 m,

$$\begin{aligned} x_{2m-1} = \frac{1}{2}\int(1-F_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^2}{F_{2m-1}}\right)^{1/2} dU \\ + \frac{1}{2}\int(1-\Phi_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^2}{\Phi_{2m-1}}\right)^{1/2} dV, \end{aligned} \tag{5.2A}$$

$$x_{2m} = \frac{i}{2}\int (1+F_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^2}{F_{2m-1}}\right)^{1/2} dU$$

$$-\frac{i}{2}\int (1+\Phi_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^2}{\Phi_{2m-1}}\right)^{1/2} dV; \qquad (5.2B)$$

if $n = 2m + 1$,

$$x_{2m+1} = \int \left(\sum_{r=1}^{m} F_{2r-1}F_{2r}^2\right)^{1/2} dU + \int \left(\sum_{r=1}^{m} \Phi_{2r-1}\Phi_{2r}^2\right)^{1/2} dV. \qquad (5.3)$$

Here the F_r are given by

$$F_1(U) = U = \frac{(u_1' + i u_2')^2}{\sum_{r=3}^{m} u_r'^2} = \frac{\sum_{r=3}^{m} u_r'^2}{(u_1' - i u_2')^2},$$

$$F_{2s-1}(U) = \frac{(u'_{2s-1} + i u'_{2s})^2}{\sum_{r=2s+1}^{m} u_r'^2 - \sum_{r=1}^{s-1} F_{2r-1}F_{2r}^2}$$

$$= \frac{\sum_{r=2s+1}^{m} u_r'^2 - \sum_{r=1}^{s-1} F_{2r-1}F_{2r}^2}{(u'_{2s-1} - i u'_{2s})^2}, \qquad (5.4)$$

$$F_{2s}(U) = \frac{2u'_{2s-1}}{1 - F_{2s-1}} = \frac{-2i u'_{2s}}{1 + F_{2s-1}}$$

$$= \left(\frac{\sum_{r=2s+1}^{m} u_r'^2 - \sum_{r=1}^{s-1} F_{2r-1}F_{2r}^2}{F_{2s-1}}\right)^{1/2},$$

the primes denoting differentiation with respect to U.

In accordance with (4.13), (4.15), the Φ_r are given by

$$\Phi_1(V) = V = \frac{\sum_{r=3}^{n} v_r'^2}{(v_1' + i v_2')^2} = \frac{(v_1' - i v_2')^2}{\sum_{r=3}^{n} v_r'^2},$$

$$\Phi_{2s-1}(V) = \frac{\sum_{r=2s+1}^{n} v_r'^2 - \sum_{r=1}^{s-1} \Phi_{2r-1}\Phi_{2r}^2}{(v_{2s-1}' + i v_{2s}')^2}$$

$$= \frac{(v_{2s-1}' - i v_{2s}')^2}{\sum_{r=2s+1}^{n} v_r'^2 - \sum_{r=1}^{s-1} \Phi_{2r-1}\Phi_{2r}^2}, \tag{5.5}$$

$$\Phi_{2s}(V) = \frac{2 v_{2s-1}'}{1 - \Phi_{2s-1}} = \frac{2i\, v_{2s}'}{1 + \Phi_{2s-1}}$$

$$= \left(\frac{\sum_{r=2s+1}^{n} v_r'^2 - \sum_{r=1}^{s-1} \Phi_{2r-1}\Phi_{2r}^2}{\Phi_{2s-1}} \right)^{1/2},$$

the primes denoting differentiation with respect to V.

If and only if the minimal surface is given by these equations (5.1)-(5.3), we say that U, V are the normal parameters, and the F_r and Φ_r the normal functions, of the surface.

According to the discussion of the latter part of section 3, if it is impossible thus to choose one of the normal parameters, say U, then the curves, v = constant, on the surface are parallel straight lines: the surface is a cylinder. If neither parameter can be determined, the cylinder is a plane.

We proceed to the determination of the fundamental quantities E, F, G:

$$x^2_{2s-1,10} = \left[\tfrac{1}{2}(1-F_{2s-1})F_{2s}\right]^2 = \frac{F^2_{2s}}{4}\left(1-2F_{2s-1}+F^2_{2s-1}\right),$$

$$x^2_{2s,10} = \left[\tfrac{i}{2}(1+F_{2s-1})F_{2s}\right]^2 = \frac{F^2_{2s}}{4}\left(-1-2F_{2s-1}-F^2_{2s-1}\right), \qquad (5.6)$$

$$x^2_{2s-1,10} + x^2_{2s,10} = -F_{2s-1}F^2_{2s}.$$

If n = 2 m,

$$x^2_{2m-1,10} = \left[\tfrac{1}{2}(1-F_{2m-1})\left(\frac{-\sum'^{m-1}_{r=1} F_{2r-1}F^2_{2r}}{F_{2m-1}}\right)^{1/2}\right]^2$$

$$= \frac{-\sum'^{m-1}_{r=1} F_{2r-1}F^2_{2r}}{4F_{2m-1}}\left(1-2F_{2m-1}+F^2_{2m-1}\right), \qquad (5.7A)$$

$$x^2_{2m,10} = \left[\tfrac{i}{2}(1+F_{2m-1})\left(\frac{-\sum'^{m-1}_{r=1} F_{2r-1}F^2_{2r}}{F_{2m-1}}\right)^{1/2}\right]^2$$

$$= \frac{-\sum'^{m-1}_{r=1} F_{2r-1}F^2_{2r}}{4F_{2m-1}}\left(-1-2F_{2m-1}-F^2_{2m-1}\right),$$

$$x^2_{2m-1,10} + x^2_{2m,10} = \sum_{r=1}^{m-1} F_{2r-1} F^2_{2r}, \tag{5.7B}$$

so that in this case

$$E = \sum_{r=1}^{2m} x^2_{r,10} = -\sum_{r=1}^{m-1} F_{2r-1} F^2_{2r} + \sum_{r=1}^{m-1} F_{2r-1} F^2_{2r} = 0; \tag{5.8}$$

if n = 2 m + 1,

$$x^2_{2m+1,10} = \sum_{r=1}^{m} F_{2r-1} F^2_{2r}, \tag{5.9}$$

so that again

$$E = \sum_{r=1}^{2m+1} x^2_{r,10} = -\sum_{r=1}^{m} F_{2r-1} F^2_{2r} + \sum_{r=1}^{m} F_{2r-1} F^2_{2r} = 0. \tag{5.10}$$

Quite similarly, G = 0. These results for E and G were known a priori, since the minimal curves were taken to be parametric.

As for F:

$$\begin{aligned} x_{2s-1,10}\, x_{2s-1,01} &= \left[\tfrac{1}{2}(1-F_{2s-1})F_{2s}\right]\left[\tfrac{1}{2}(1-\Phi_{2s-1})\Phi_{2s}\right] \\ &= \frac{F_{2s}\Phi_{2s}}{4}\left(1-F_{2s-1}-\Phi_{2s-1}+F_{2s-1}\Phi_{2s-1}\right), \end{aligned} \tag{5.11A}$$

$$\chi_{2s,10}\,\chi_{2s,01} = \left[\frac{i}{2}\left(1+F_{2s-1}\right)F_{2s}\right]\left[\frac{-i}{2}\left(1+\bar{\Phi}_{2s-1}\right)\bar{\Phi}_{2s}\right]$$

$$= \frac{F_{2s}\bar{\Phi}_{2s}}{4}\left(1+F_{2s-1}+\bar{\Phi}_{2s-1}+F_{2s-1}\,\bar{\Phi}_{2s-1}\right), \qquad (5.11B)$$

$$\chi_{2s-1,10}\,\chi_{2s-1,01}+\chi_{2s,10}\,\chi_{2s,01} = \frac{F_{2s}\bar{\Phi}_{2s}}{2}\left(1+F_{2s-1}\,\bar{\Phi}_{2s-1}\right).$$

If n = 2 m,

$$\chi_{2m-1,10}\,\chi_{2m-1,01} = \left[\frac{1}{2}\left(1-F_{2m-1}\right)\left(\frac{-\sum_{r=1}^{m-1}F_{2r-1}\,F_{2r}^{2}}{F_{2m-1}}\right)^{1/2}\right]\times$$

$$\times\left[\frac{1}{2}\left(1-\bar{\Phi}_{2m-1}\right)\left(\frac{-\sum_{r=1}^{m-1}\bar{\Phi}_{2r-1}\,\bar{\Phi}_{2r}^{2}}{\bar{\Phi}_{2m-1}}\right)^{1/2}\right]$$

$$= \frac{1}{4}\left[\frac{\left(-\sum_{r=1}^{m-1}F_{2r-1}\,F_{2r}^{2}\right)\left(-\sum_{r=1}^{m-1}\bar{\Phi}_{2r-1}\,\bar{\Phi}_{2r}^{2}\right)}{F_{2m-1}\,\bar{\Phi}_{2m-1}}\right]^{\frac{1}{2}}\left[1-F_{2m-1}-\bar{\Phi}_{2m-1}+F_{2m-1}\bar{\Phi}_{2m-1}\right],$$

$$\chi_{2m,10}\,\chi_{2m,01} = \left[\frac{i}{2}\left(1+F_{2m-1}\right)\left(\frac{-\sum_{r=1}^{m-1}F_{2r-1}\,F_{2r}^{2}}{F_{2m-1}}\right)^{1/2}\right]\times \qquad (5.12A)$$

$$\times\left[\frac{-i}{2}\left(1+\bar{\Phi}_{2m-1}\right)\left(\frac{-\sum_{r=1}^{m-1}\bar{\Phi}_{2r-1}\,\bar{\Phi}_{2r}^{2}}{\bar{\Phi}_{2m-1}}\right)^{1/2}\right]$$

$$= \frac{1}{4}\left[\frac{\left(-\sum_{r=1}^{m-1}F_{2r-1}\,F_{2r}^{2}\right)\left(-\sum_{r=1}^{m-1}\bar{\Phi}_{2r-1}\,\bar{\Phi}_{2r}^{2}\right)}{F_{2m-1}\,\bar{\Phi}_{2m-1}}\right]^{1/2}\left[1+F_{2m-1}+\bar{\Phi}_{2m-1}+F_{2m-1}\,\bar{\Phi}_{2m-1}\right],$$

$$x_{2m-1,10}\,x_{2m-1,01}+x_{2m,10}\,x_{2m,01}$$

$$=\frac{1}{2}\left[\frac{\left(-\sum_{r=1}^{m-1}F_{2r-1}F_{2r}^{2}\right)\left(-\sum_{r=1}^{m-1}\Phi_{2r-1}\Phi_{2r}^{2}\right)}{F_{2m-1}\,\Phi_{2m-1}}\right]^{1/2}\left[1+F_{2m-1}\,\Phi_{2m-1}\right], \tag{5.12B}$$

so that in this case

$$F=\sum_{r=1}^{2m}x_{r,10}\,x_{r,01}=\sum_{r=1}^{m-1}\frac{F_{2r}\Phi_{2r}}{2}\left(1+F_{2r-1}\,\Phi_{2r-1}\right)$$

$$+\frac{1}{2}\left[\frac{\left(-\sum_{r=1}^{m-1}F_{2r-1}F_{2r}^{2}\right)\left(-\sum_{r=1}^{m-1}\Phi_{2r-1}\Phi_{2r}^{2}\right)}{F_{2m-1}\,\Phi_{2m-1}}\right]^{1/2}\left[1+F_{2m-1}\,\Phi_{2m-1}\right]; \tag{5.13}$$

if n = 2 m + 1,

$$x_{2m+1,10}\,x_{2m+1,01}=\left[\left(\sum_{r=1}^{m}F_{2r-1}F_{2r}^{2}\right)\left(\sum_{r=1}^{m}\Phi_{2r-1}\Phi_{2r}^{2}\right)\right]^{1/2}, \tag{5.14}$$

so that in this case

$$F=\sum_{r=1}^{2m+1}x_{r,10}\,x_{r,01}=\sum_{r=1}^{m}\frac{F_{2r}\Phi_{2r}}{2}\left(1+F_{2r-1}\,\Phi_{2r-1}\right)$$

$$+\left[\left(\sum_{r=1}^{m}F_{2r-1}F_{2r}^{2}\right)\left(\sum_{r=1}^{m}\Phi_{2r-1}\Phi_{2r}^{2}\right)\right]^{1/2}. \tag{5.15}$$

We have

$$EG-F^2 = \sum_{r=1}^{n} x_{r,10}^2 \sum_{r=1}^{n} x_{r,01}^2 - \left(\sum_{r=1}^{n} x_{r,10}\, x_{r,01}\right)^2$$

$$= \sum_{r,s=1}^{n} \left(x_{r,10}^2\, x_{s,01}^2 - x_{r,10}\, x_{r,01}\, x_{s,10}\, x_{s,01}\right)$$

$$\cancel{= \sum_{r<s} \left(x_{r,10}^2\, x_{s,01}^2 - 2\, x_{r,10}\, x_{r,01}\, x_{s,10}\, x_{s,01}\right)} \tag{5.16}$$

$$= \sum_{r<s} \left(x_{r,10}\, x_{s,01} - x_{r,01}\, x_{s,10}\right)^2$$

$$= \sum_{r<s} \left[\frac{\partial(x_r, x_s)}{\partial(u, v)}\right]^2 = \sum_{r<s} J_{r,s}^2 = H^2.$$

Let

$$P_{r,s} = -P_{s,r} = \frac{J_{r,s}}{H}\,; \tag{5.17}$$

then

$$\sum_{r<s} P_{r,s}^2 = 1. \tag{5.18}$$

The quantities $P_{r,s}$ are called the direction-cosines of the tangent plane.

6. Real minimal surfaces. The above particular form (5.1)-(5.3) of the equations of a minimal surface

was chosen, as regards plus and minus signs, because if $F_r(U)$ and $\Phi_r(V)$, r = 1, 2,..., n, are conjugates, and if, for n = 2 m,

$$\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^2}{F_{2m-1}}\right)^{1/2} \quad \text{and} \quad \left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1} \Phi_{2r}^2}{\Phi_{2m-1}}\right)^{1/2} \tag{6.1}$$

or, for n = 2 m + 1,

$$\left(\sum_{r=1}^{m} F_{2r-1} F_{2r}^2\right)^{1/2} \quad \text{and} \quad \left(\sum_{r=1}^{m} \Phi_{2r-1} \Phi_{2r}^2\right)^{1/2} \tag{6.2}$$

are conjugates, then the surface is real. This follows at once from the fact that to each element of the integral relative to U there corresponds, in each of the n equations, an element in the integral relative to V which is its conjugate imaginary. In this case we may write

$$\begin{aligned} x_{2s-1} &= R\int (1-F_{2s-1}) F_{2s}\, dU, \\ x_{2s} &= R\int i(1+F_{2s-1}) F_{2s}\, dU; \end{aligned} \tag{6.3}$$

if n = 2 m,

$$\begin{aligned} x_{2m-1} &= R\int (1-F_{2m-1}) \left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^2}{F_{2m-1}}\right)^{1/2} dU, \\ x_{2m} &= R\int i(1+F_{2m-1}) \left(\frac{-\sum_{r=1}^{m-1} F_{2r-1} F_{2r}^2}{F_{2m-1}}\right)^{1/2} dU; \end{aligned} \tag{6.4}$$

if n = 2 m + 1,

$$x_{2m+1} = R\int 2\left(\sum_{r=1}^{m} F_{2r-1}\, F_{2r}^{\,2}\right)^{1/2} dU, \tag{6.5}$$

where the R designates that we take only the real part of the function indicated.

We turn now to the converse question, as to whether or not these conjugate relations necessarily exist if the surface is real.

For a real surface, (1.4) can vanish at most at isolated points. In a small region about any other point, then, by (5.16),

$$J_{a,b} \neq 0 \tag{6.6}$$

for some a, b; consequently, U and V are functions of x_a and x_b in this region. Let

$$dx_t = p_t\, dx_a + q_t\, dx_b\,, \tag{6.7}$$

where

$$p_t = \frac{\partial x_t}{\partial x_a}\,, \quad q_t = \frac{\partial x_t}{\partial x_b}\,, \quad p_a = q_b = 1, \quad p_b = q_a = 0. \tag{6.8}$$

Along the minimal curves, which we are taking to be parametric, we have

$$\sum_{r=1}^{n} dx_r^2 = 0. \tag{6.9}$$

Equations (6.7) and (6.9) determine the following two systems of values of the differentials dx_1, dx_2, ..., dx_n:

$$dx_a : dx_b : dx_t = \left\{ -\sum_{r=1}^{n} p_r q_r \pm i \left[\sum_{\substack{r,s=1 \\ r<s}}^{n} (p_r q_s - p_s q_r)^2 \right]^{1/2} \right\} : \tag{6.10}$$

$$\sum_{r=1}^{n} p_r^2 : \left\{ -p_t \sum_{r=1}^{n} p_r q_r + q_t \sum_{r=1}^{n} p_r^2 \pm i p_t \left[\sum_{\substack{r,s=1 \\ r<s}}^{n} (p_r q_s - p_s q_r)^2 \right]^{1/2} \right\}.$$

Since for real surfaces the p_r, q_r are real, the corresponding terms of the two systems of ratios in the right-hand member of (6.10) are conjugate imaginaries. By the equations of definition (5.4), (5.5), then, F_r and Φ_r are conjugates.

If n = 2 m, then, by (5.2), if the two terms in (6.1) were negative conjugates, x_{2m-1} and x_{2m} would be pure imaginaries, contrary to hypothesis; the terms in (6.1) therefore must be conjugates. Similarly, if n = 2 m + 1, then the two terms in (6.2) must be conjugate imaginaries, for otherwise, by (5.3), x_{2m+1} would be a pure imaginary.

We have shown that a necessary and sufficient condition that a minimal surface be real is that F_r and Φ_r, r = 1, 2,..., n, and, if n = 2 m, the terms in (6.1), or, if n = 2 m + 1, the terms in (6.2), be conjugates. The surface then is given by (6.3)-(6.5).

A second way of obtaining equations (6.3)-(6.5) is the following. The normal parameters of a real minimal surface are conjugates, as we just pointed out:

$$U = \zeta + i\eta, \qquad V = \zeta - i\eta. \tag{6.11}$$

Since U and V refer to the minimal curves,

$$ds^2 = 2F\,dU\,dV. \tag{6.12}$$

Substituting from (6,11) in (6.12), we obtain

$$ds^2 = 2F(d\zeta^2 + d\eta^2) \tag{6.13}$$

so that ζ, η are real isothermic parameters of the minimal surface; therefore, by the first theorem of section 1, the x_r are harmonic functions of ζ, η. As is well known, then, the x_r are the real parts of analytic complex functions:

$$x_r = R\{\Omega_r(U)\}, \quad U = \zeta + i\eta. \tag{6.14}$$

Now

$$\Omega_r' = \frac{d\Omega}{dU} = \frac{\partial x_r}{\partial \zeta} - i\,\frac{\partial x_r}{\partial \eta} \tag{6.15}$$

so that

$$\sum_{r=1}^{n} \Omega_r'^2 = E - 2iF - G; \tag{6.16}$$

consequently, by (1.11),

$$\sum_{r=1}^{n} \Omega_r'^2 = 0. \tag{6.17}$$

Treating the identity (6.17) as we treated (2.2), we obtain expressions for the Ω_r analogous to (2.16)-(2.18). Absorbing the inconsequential factors 1/2 into the expressions of the $F_{2\alpha}$, we now use (6.14) to convert these equations to equations (6.3)-(6.5).

7. <u>Associate minimal surfaces.</u> The linear element of a minimal surface S_0 given by (1.15) is given by

$$ds^2 = \sum_{r=1}^{n} dx_r^2 = 2\sum_{r=1}^{n} du_r\, dv_r, \tag{7.1}$$

since the minimal curves are parametric. The surface S_α, defined by

$$x_{r,\alpha} = e^{i\alpha} u_r(u) + e^{-i\alpha} v_r(v), \tag{7.2}$$

where α is any constant, also is minimal, according to the second theorem of section 1; and its linear element obviously is given by (7.1). Accordingly, S_0 and S_α are applicable to each other. Equation (7.2) defines a one parameter family of applicable minimal surfaces, called associate minimal surfaces.

The normal functions defining S_α are, by (5.4)-(5.5),

$$F_{2s-1},\ e^{i\alpha}F_{2s},\ \Phi_{2s-1},\ e^{-i\alpha}\Phi_{2s}, \tag{7.3}$$

where

$$F_{2s-1},\ F_{2s},\ \Phi_{2s-1},\ \Phi_{2s} \tag{7.4}$$

are the normal functions defining S_0 . A glance at (5.13), (5.15) shows that F is the same for S_α as for S_0 .

Also, according to the definition (5.16), $J_{r,s}$ is the same for the two surfaces. Finally, if, by (1.13), we take $+iF$ for H , then

$$P_{r,s} = \frac{J_{r,s}}{+iF} \tag{7.5}$$

for our present parameters, so that $P_{r,s}$ is the same for the two surfaces.

The equations of the tangent plane to S_0 are

$$\left\| \begin{matrix} dx_1, & dx_2, & \dots, & dx_n \\ x_{1,10}, & x_{2,10}, & \dots, & x_{n,10} \\ x_{1,01}, & x_{2,01}, & \dots, & x_{n,01} \end{matrix} \right\| = 0. \tag{7.6}$$

Since $J_{r,s}$ is the same for S_0 and S_α , then, the tangent planes at corresponding points of a family of associate minimal surfaces are parallel.

The surface $S_{\pi/2}$ is called the adjoint of S_o.
By (5.1)-(5.3), its coordinates are

$$
\begin{aligned}
y_{2s-1} &= \frac{i}{2}\int (1-F_{2s-1})F_{2s}\,dU - \frac{i}{2}\int (1-\Phi_{2s-1})\Phi_{2s}\,dV,\\
y_{2s} &= \frac{-1}{2}\int (1+F_{2s-1})F_{2s}\,dU - \frac{1}{2}\int (1+\Phi_{2s-1})\Phi_{2s}\,dV;
\end{aligned}
\tag{7.7}
$$

if n = 2 m,

$$
\begin{aligned}
y_{2m-1} &= \frac{i}{2}\int (1-F_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^{2}}{F_{2m-1}}\right)^{1/2} dU\\
&\quad - \frac{i}{2}\int (1-\Phi_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^{2}}{\Phi_{2m-1}}\right)^{1/2} dV,\\
y_{2m} &= -\frac{1}{2}\int (1+F_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^{2}}{F_{2m-1}}\right)^{1/2} dU\\
&\quad - \frac{1}{2}\int (1+\Phi_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^{2}}{\Phi_{2m-1}}\right)^{1/2} dV;
\end{aligned}
\tag{7.8}
$$

if n = 2 m + 1,

$$
y_{2m+1} = i\int \left(\sum_{r=1}^{m} F_{2r-1}F_{2r}^{2}\right)^{1/2} dU - i\int \left(\sum_{r=1}^{m} \Phi_{2r-1}\Phi_{2r}^{2}\right)^{1/2} dV. \tag{7.9}
$$

We have

$$
\begin{aligned}
x_{r,\alpha} &= (\cos\alpha + i\sin\alpha)\,u_r + (\cos\alpha - i\sin\alpha)\,v_r \\
&= (u_r + v_r)\cos\alpha + (i u_r - i v_r)\sin\alpha \qquad (7.10) \\
&= x_r\cos\alpha + y_r\sin\alpha .
\end{aligned}
$$

The plane P determined by the origin and two corresponding points $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ and $(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n)$ of S_0 and $S_{\pi/2}$ is given by the equations

$$
\left\| \begin{matrix} z_1, & z_2, & \cdots, & z_n \\ \bar{x}_1, & \bar{x}_2, & \cdots, & \bar{x}_n \\ \bar{y}_1, & \bar{y}_2, & \cdots, & \bar{y}_n \end{matrix} \right\| = 0, \qquad (7.11)
$$

the z_r being the current coordinates. The coordinates $\bar{x}_{r,\alpha}$ of the corresponding point of S_α satisfy (7.11), by (7.10), so that the point $(\bar{x}_{1,\alpha}, \bar{x}_{2,\alpha}, \ldots, \bar{x}_{n,\alpha})$ is on P.

We might remark that $S_{\pi/2}$ and S_α are not fixed in space, since replacing $u_r(u)$ by $u_r(u) + a_r$ and $v_r(v)$ by $v_r(v) - a_r$ leaves S_0 unaltered but adds $2i\,a_r$ to y_r and $a_r(e^{i\alpha} - e^{-i\alpha})$ to $x_{r,\alpha}$. The discussion of the above paragraph assumes the same values of the a_r have been used to determine S_α as have been used to determine $S_{\pi/2}$.

For the surface $S_{\pi/2}$, as for all other surfaces of the family,

$$\sum_{r=1}^{n} dy_r^2 = \sum_{r=1}^{n} dx_r^2, \qquad (7.12)$$

as we already have remarked. We have also, for this particular surface,

$$\begin{aligned}\sum_{r=1}^{n} dx_r\, dy_r &= \sum_{r=1}^{n} (du_r + dv_r)(i\,du_r - i\,dv_r) \\ &= i \sum_{r=1}^{n} \left(du_r^2 - dv_r^2\right) = 0.\end{aligned} \qquad (7.13)$$

Equation (7.13) shows that corresponding curves on a minimal surface and on its adjoint are perpendicular to one another at corresponding points.

Since the tangent planes to S_0 and $S_{\pi/2}$ are parallel at corresponding points, we have

$$\left\| \begin{matrix} dy_1, & dy_2, \cdots, & dy_n \\ x_{1,10}, & x_{2,10}, \cdots, & x_{n,10} \\ x_{1,01}, & x_{2,01}, \cdots, & x_{n,01} \end{matrix} \right\| = 0. \qquad (7.17)$$

Consider the determinant formed from the first two columns and any other column of (7.17):

$$J_{2r}\, dy_1 + J_{r1}\, dy_2 + J_{12}\, dy_r = 0. \tag{7.18}$$

Solving (7.13) for dy_r and substituting this value in (7.18), we have

$$J_{2r}\, dy_1 + J_{r1}\, dy_2 - J_{12} \sum_{s \neq r} \frac{dx_s\, dy_s}{dx_r} = 0,$$

$$\left(J_{2r}\, dx_r + J_{21}\, dx_1\right) dy_1 - \left(J_{1r}\, dx_r + J_{12}\, dx_2\right) dy_2 - J_{12} \sum_{s \neq 1,2,r} dx_s\, dy_s = 0 \tag{7.19}$$

Adding the n - 2 of these equations, r = 3, 4,..., n, we obtain

$$\left[\sum_{r=3}^{n} J_{2r}\, dx_r + (n-2)\, J_{21}\, dx_1\right] dy_1 - \left[\sum_{r=3}^{n} J_{1r}\, dx_r + (n-2)\, J_{12}\, dx_2\right] dy_2$$
$$- (n-3)\, J_{12} \sum_{r=3}^{n} dx_r\, dy_r = 0. \tag{7.20}$$

Now, by (7.13),

$$-(n-3)\, J_{12} \sum_{r=3}^{n} dx_r\, dy_r = -(n-3)\, J_{21}\, dx_1\, dy_1 + (n-3)\, J_{12}\, dx_2\, dy_2; \tag{7.21}$$

substituting (7.21) in (7.20), and noting that

$$J_{rr} = 0, \tag{7.22}$$

we have

$$\sum_{r=1}^{n} J_{2r}\, dx_r\, dy_1 - \sum_{r=1}^{n} J_{1r}\, dx_r\, dy_2 = 0. \tag{7.23}$$

Since the same reasoning holds for other subscripts as well as for 1, 2, we have the equations

$$\frac{dy_1}{\sum_{r=1}^{n} J_{1r}\, dx_r} = \frac{dy_2}{\sum_{r=1}^{n} J_{2r}\, dx_r} = \cdots = \frac{dy_n}{\sum_{r=1}^{n} J_{nr}\, dx_r} = \lambda. \qquad (7.24)$$

We now shall determine λ as given by the first member of (7.24):

$$J_{1,2s-1} = \tfrac{1}{4}\Big[F_2 \Phi_{2s}\left(1 - F_1 - \Phi_{2s-1} + F_1 \Phi_{2s-1}\right)$$
$$+ F_{2s} \Phi_2\left(-1 + F_{2s-1} + \Phi_1 - F_{2s-1} \Phi_1\right)\Big],$$

$$dx_{2s-1} = \tfrac{1}{2}\Big[(1 - F_{2s-1}) F_{2s}\, dU + (1 - \Phi_{2s-1}) \Phi_{2s}\, dV\Big],$$

$$(7.25A)$$

$$J_{1,2s-1}\, dx_{2s-1} =$$
$$\tfrac{1}{8}\Big\{F_{2s}\, dU\Big[F_2 \Phi_{2s}\left(1 - F_1 - \Phi_{2s-1} + F_1 \Phi_{2s-1} - F_{2s-1} + F_1 F_{2s-1} + F_{2s-1}\Phi_{2s-1} - F_1 F_{2s-1} \Phi_{2s-1}\right)$$
$$+ F_{2s} \Phi_2\left(-1 + F_{2s-1} + \Phi_1 - F_{2s-1} \Phi_1 + F_{2s-1} - F_{2s-1}^2 - F_{2s-1}\Phi_1 + F_{2s-1}^2 \Phi_1\right)\Big]$$
$$+ \Phi_{2s}\, dV\Big[F_2 \Phi_{2s}\left(1 - F_1 - \Phi_{2s-1} + F_1 \Phi_{2s-1} - \Phi_{2s-1} + F_1 \Phi_{2s-1} + \Phi_{2s-1}^2 - F_1 \Phi_{2s-1}^2\right)$$
$$+ F_{2s} \Phi_2\left(-1 + F_{2s-1} + \Phi_1 - F_{2s-1}\Phi_1 + \Phi_{2s-1} - F_{2s-1}\Phi_{2s-1} - \Phi_1 \Phi_{2s-1} + F_{2s-1}\Phi_1 \Phi_{2s-1}\right)\Big]\Big\};$$

$$J_{1,2s} = -\tfrac{1}{4}\Big[F_2 \Phi_{2s}\left(1 - F_1 + \Phi_{2s-1} - F_1 \Phi_{2s-1}\right)$$
$$+ F_{2s} \Phi_2\left(1 + F_{2s-1} - \Phi_1 - F_{2s-1} \Phi_1\right)\Big],$$

$$dx_{2s} = \frac{i}{2}\Big[(1+F_{2s-1})F_{2s}\,dU - (1+\Phi_{2s-1})\Phi_{2s}\,dV\Big],$$

$$J_{1,2s}\,dx_{2s} = \qquad (7.25B)$$

$$\tfrac{1}{8}\Big\{F_{2s}\,dU\Big[F_2\Phi_{2s}\big(1-F_1+\Phi_{2s-1}-F_1\Phi_{2s-1}+F_{2s-1}-F_1F_{2s-1}+F_{2s-1}\Phi_{2s-1}-F_1F_{2s-1}\Phi_{2s-1}\big)$$

$$+F_{2s}\Phi_2\big(1+F_{2s-1}-\Phi_1-F_{2s-1}\Phi_1+F_{2s-1}+F_{2s-1}^2-F_{2s-1}\Phi_1-F_{2s-1}^2\Phi_1\big)\Big]$$

$$+\Phi_{2s}\,dV\Big[F_2\Phi_{2s}\big(-1+F_1-\Phi_{2s-1}+F_1\Phi_{2s-1}-\Phi_{2s-1}+F_1\Phi_{2s-1}-\Phi_{2s-1}^2+F_1\Phi_{2s-1}^2\big)$$

$$+F_{2s}\Phi_2\big(-1-F_{2s-1}+\Phi_1+F_{2s-1}\Phi_1-\Phi_{2s-1}-F_{2s-1}\Phi_{2s-1}+\Phi_1\Phi_{2s-1}+F_{2s-1}\Phi_1\Phi_{2s-1}\big)\Big]\Big\};$$

adding these two products, we obtain

$$J_{1,2s-1}\,dx_{2s-1} + J_{1,2s}\,dx_{2s} =$$

$$\tfrac{1}{4}\Big\{\Big[(1-F_1)F_2F_{2s}\Phi_{2s}(1+F_{2s-1}\Phi_{2s-1})+2(1-\Phi_1)\Phi_2F_{2s-1}F_{2s}^2\Big]dU \qquad (7.26)$$

$$-\Big[(1-\Phi_1)\Phi_2F_{2s}\Phi_{2s}(1+F_{2s-1}\Phi_{2s-1})+2(1-F_1)F_2\Phi_{2s-1}\Phi_{2s}^2\Big]dV\Big\}.$$

If n = 2 m, we find that

$$J_{1,2m-1}\,dx_{2m-1} + J_{1,2m}\,dx_{2m} = \qquad (7.27A)$$

$$= \frac{1}{4}\Bigg\{\Bigg[(1-F_1)F_2\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^2}{F_{2m-1}}\right)^{1/2}\left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^2}{\Phi_{2m-1}}\right)^{1/2}\left(1+F_{2m-1}\Phi_{2m-1}\right)$$

$$+2(1-\Phi_1)\Phi_2\left(-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^2\right)\Bigg]dU \qquad (7.27B)$$

$$-\Bigg[(1-\Phi_1)\Phi_2\left(\frac{-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^2}{F_{2m-1}}\right)^{1/2}\left(\frac{-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^2}{\Phi_{2m-1}}\right)^{1/2}\left(1+F_{2m-1}\Phi_{2m-1}\right)$$

$$+2(1-F_1)F_2\left(-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^2\right)\Bigg]dV\Bigg\},$$

so that

$$\sum_{r=1}^{2m} J_{1,r}\,dx_r = \frac{1}{4}\left\{(1-F_1)F_2\,dU-(1-\Phi_1)\Phi_2\,dV\right\}\times \qquad (7.28)$$

$$\times\left\{\sum_{r=1}^{m-1} F_{2r}\Phi_{2r}\left(1+F_{2r-1}\Phi_{2r-1}\right)+\left[\frac{\left(-\sum_{r=1}^{m-1} F_{2r-1}F_{2r}^2\right)\left(-\sum_{r=1}^{m-1} \Phi_{2r-1}\Phi_{2r}^2\right)}{F_{2m-1}\Phi_{2m-1}}\right]^{1/2}\left[1+F_{2m-1}\Phi_{2m-1}\right]\right\}.$$

By (7.7), the first of the above brackets is $-2i\,dy_1$, while, by (5.13), the second is 2 F. Consequently, if n = 2 m,

$$\frac{dy_1}{\sum_{r=1}^{n} J_{1,r}\,dx_r} = \frac{1}{-iF} = \frac{1}{-H} = \lambda. \qquad (7.29).$$

If n = 2 m + 1,

$$J_{1,2m+1}\,dx_{2m+1} =$$

$$\tfrac{1}{2}\left\{\left[(1-F_1)F_2\left(\sum_{r=1}^{m} F_{2r-1}F_{2r}^2\right)^{1/2}\left(\sum_{r=1}^{m}\Phi_{2r-1}\Phi_{2r}^2\right)^{1/2} - (1-\Phi_1)\Phi_2\left(\sum_{r=1}^{m} F_{2r-1}F_{2r}^2\right)\right]dU\right.$$
$$\left. - \left[(1-\Phi_1)\Phi_2\left(\sum_{r=1}^{m} F_{2r-1}F_{2r}^2\right)^{1/2}\left(\sum_{r=1}^{m}\Phi_{2r-1}\Phi_{2r}^2\right)^{1/2} - (1-F_1)F_2\left(\sum_{r=1}^{m}\Phi_{2r-1}\Phi_{2r}^2\right)\right]dV\right\}, \qquad (7.30)$$

so that

$$\sum_{r=1}^{2m+1} J_{1,r}\,dx_r = \frac{1}{4}\left\{(1-F_1)F_2\,dU - (1-\Phi_1)\Phi_2\,dV\right\}\times$$
$$\times\left\{\sum_{r=1}^{m} F_{2r}\Phi_{2r}\left(1+F_{2r-1}\Phi_{2r-1}\right) + 2\left[\left(\sum_{r=1}^{m} F_{2r-1}F_{2r}^2\right)\left(\sum_{r=1}^{m}\Phi_{2r-1}\Phi_{2r}^2\right)\right]^{1/2}\right\}. \qquad (7.31)$$

Again, by (7.7), the first of the above brackets is $-2i\,dy_1$, while, by (5.15), the second is 2 F. Consequently, (7.29) holds for n = 2 m + 1 as well as for n = 2 m.

By (5.17), we may write (7.29) as

$$\frac{dy_1}{\sum_{r=1}^{n} P_{1,r}\,dx_r} = \frac{dy_2}{\sum_{r=1}^{n} P_{2,r}\,dx_r} = \cdots = \frac{dy_n}{\sum_{r=1}^{n} P_{n,r}\,dx_r} = -1. \qquad (7.32)$$

Equations (1.15), (7.7)-(7.9), and (7.32) yield

$$x_s - i y_s = x_s + i \sum_{r=1}^{n} \int P_{s,r}\, dx_r = 2u_s,$$
$$x_s + i y_s = x_s - i \sum_{r=1}^{n} \int P_{s,r}\, dx_r = 2v_s. \qquad (7.33)$$

These formulae are analogous to the formulae of Schwarz for minimal surfaces in ordinary space. By them we are able to solve the following problem:

To determine the minimal surface passing through a given analytic curve and admitting at each point of the curve a given tangent plane.

Let the coordinates of the given curve be given by the analytic functions $x_s(t)$, and let the direction-cosines of the given tangent plane be given by the analytic functions $P_{s,r}(t)$. Being direction-cosines of a plane, the $P_{s,r}$ must satisfy (5.18) and

$$P_{r,s}\, dx_t + P_{s,t}\, dx_r + P_{t,r}\, dx_s = 0. \qquad (7.34)$$

Substituting the given values of the x_s and the $P_{s,r}$ in (7.33), we determine the forms of the functions u_s and v_s ; then substituting parameters $t = u$ and $t = v$ in these equations and adding, we obtain the coordinates of a point on the surface:

$$X_s = \frac{x_s(u) + x_s(v)}{2} + \frac{i}{2} \sum_{r=1}^{n} \int_v^u P_{s,r}\, dx_r. \tag{7.35}$$

We shall show that this actually is the desired surface.

We have, by (7.33),

$$4 du_s^2 = dx_s^2 + 2i\, dx_s \sum_{r=1}^{n} P_{s,r}\, dx_r - \left(\sum_{r=1}^{n} P_{s,r} dx_r\right)^2. \tag{7.36}$$

Now, by (5.17),

$$+2i \sum_{s=1}^{n} \sum_{r=1}^{n} P_{s,r}\, dx_s\, dx_r = 0, \tag{7.37}$$

so that

$$4 \sum_{s=1}^{n} du_s^2 = \sum_{s=1}^{n} dx_s^2 - \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{t=1}^{n} P_{s,r} P_{s,t}\, dx_r\, dx_t. \tag{7.38}$$

Solving (7.34) for $P_{t,r}\, dx_s$, squaring, and summing, we see that

$$-\sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{t=1}^{n} P_{s,r} P_{s,t}\, dx_r\, dx_t = -\frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{t=1}^{n} P_{t,r}^2\, dx_s^2 = -\sum_{s=1}^{n} dx_s^2, \tag{7.39}$$

the last expression resulting from (5.18). Putting (7.39) in (7.38), we obtain

$$4 \sum_{s=1}^{n} du_s^2 = \sum_{s=1}^{n} dx_s^2 - \sum_{s=1}^{n} dx_s^2 = 0. \tag{7.40}$$

Similarly,

$$\sum_{s=1}^{n} dv_s^2 = 0. \tag{7.41}$$

Again, by (7.33) and (7.34), along the given curve,

$$\begin{aligned} P_{r,s}du_t + P_{s,t}du_r + P_{t,r}du_s &= \tfrac{1}{2}\left(P_{r,s}dx_t + P_{s,t}dx_r + P_{t,r}dx_s\right) \\ &\quad + \tfrac{i}{2}\sum_{p=1}^{n}\left(P_{r,s}P_{t,p}dx_p + P_{s,t}P_{r,p}dx_p + P_{t,r}P_{s,p}dx_p\right) \\ = -\tfrac{i}{2}\sum_{p=1}^{n}\Big[\left(P_{s,p}P_{t,p}dx_r + P_{p,r}P_{t,p}dx_s\right) &+ \left(P_{t,p}P_{r,p}dx_s + P_{p,s}P_{r,p}dx_t\right) \\ &+ \left(P_{r,p}P_{s,p}dx_t + P_{p,t}P_{s,p}dx_r\right)\Big] = 0. \end{aligned} \tag{7.42}$$

Similarly, along this curve,

$$P_{r,s}dv_t + P_{s,t}dv_r + P_{t,r}dv_s = 0. \tag{7.43}$$

Adding, we get, along the given curve,

$$P_{r,s}dX_t + P_{s,t}dX_r + P_{t,r}dX_s = 0. \tag{7.44}$$

By (7.35), (7.40), (7.41), and the second theorem of section 1, the surface is minimal; when u = v = t, the surface is on the given curve, since the equations then define the curve; and, by (7.44), the surface has the given tangent plane at each point of the given curve. These are all the conditions which the surface was to fulfill. Finally, by (7.33), the surface affords the unique solution.

If the given curve is real and if t_0 corresponds to a real point of this given curve, then, by (7.35), the real part of the minimal surface is given by the equations

$$X_s = R\left[x_s(u) + i\sum_{r=1}^{n}\int_{t_0}^{u} P_{s,r}\,dx_r\right]. \tag{7.45}$$

We shall make the following two applications of formulae (7.35).

Suppose that a minimal surface is such that a straight line can be drawn upon it. Let this line be the x_1-axis. Along this line

$$x_r = 0, \quad P_{r,s} = 0, \quad r, s = 2, 3, \ldots, n, \tag{7.46}$$

so that, for this surface, (7.35) becomes

$$\begin{aligned} X_1 &= \frac{x_1(u) + x_1(v)}{2}, \\ X_s &= \frac{i}{2}\int_{v}^{u} P_{s,1}\,dx_1, \quad s = 2, 3, \ldots, n. \end{aligned} \tag{7.47}$$

When u and v are interchanged, X_1 is unaltered but X_s, $s > 1$, is changed in sign. Hence, if a straight line can be drawn on a minimal surface, it is an axis of symmetry of the surface.

Suppose, secondly, that the given curve lies in a hyperplane, say the $(x_1, x_2, \ldots, x_k)$ hyperplane. Then on the given curve

$$x_r = 0,\ dx_r = 0,\ r = k+1, k+2, \ldots, n, \tag{7.48}$$

so that the minimal surface is given by the equations (7.35) in the form

$$\begin{aligned} X_s &= \frac{x_s(u) + x_s(v)}{2} + \frac{i}{2}\sum_{r=1}^{k}\int_v^u P_{s,r}\,dx_r,\ s \le k, \\ X_s &= \frac{i}{2}\sum_{r=1}^{k}\int_v^u P_{s,r}\,dx_r,\ s > k. \end{aligned} \tag{7.49}$$

Suppose now that it is given that the surface is to be normal to the $(x_1, x_2, \ldots, x_k)$ hyperplane along the given curve; that is, suppose that along this curve

$$P_{s,r} = 0,\ s, r \le k. \tag{7.50}$$

Then

$$\begin{aligned} X_s &= \frac{x_s(u) + x_s(v)}{2},\ s \le k, \\ X_s &= \frac{i}{2}\sum_{r=1}^{k}\int_v^u P_{s,r}\,dx_r,\ s > k. \end{aligned} \tag{7.51}$$

In this case, when u and v are interchanged, the X_s, $s \le k$, are unaltered, while the X_s, $s > k$, are changed in sign. Hence, if a minimal surface cuts a hyperplane normally, it is symmetric with respect to the hyperplane.

8. Area theorems. The coordinates x_r, $r = 1, 2, \ldots, n$, of any real minimal surface can be given in terms of conjugate parameters (α, β), the minimal curves being parametric, in an infinity of ways, and also in terms of real isothermic parameters (u, v) in equally many ways, the general substitution between the two representations being

$$\alpha = u + iv, \qquad\qquad (8.1)$$
$$\beta = u - iv.$$

Here we have

$$ds^2 = \lambda\, d\alpha\, d\beta = \lambda(du^2 + dv^2), \qquad (8.2)$$

so that if E, F, G and $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the fundamental quantities (1.3) with respect to (α, β) and (u, v) respectively, we have

$$E = G = \mathcal{F} = 0, \quad \mathcal{E} = \mathcal{G} = 2F. \qquad (8.3)$$

By (2.16)-(2.18) and section 6,

$$x_{2s-1} = R\int(1 - f_{2s-1}) f_{2s}\, d\alpha, \qquad (8.4)$$
$$x_{2s} = R\int i(1 + f_{2s-1}) f_{2s}\, d\alpha;$$

if n = 2 m,

$$x_{2m-1} = R\int (1-f_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right)^{1/2} d\alpha,$$

$$x_{2m} = R\int i(1+f_{2m-1})\left(\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right)^{1/2} d\alpha; \tag{8.5}$$

if n = 2 m + 1,

$$x_{2m+1} = R\int 2\left(\sum_{r=1}^{m} f_{2r-1} f_{2r}^2\right)^{1/2} d\alpha; \tag{8.6}$$

the $f_r(\alpha)$ are given by (2.10).

If n = 2 m, by (8.3) and (5.13),

$$\mathcal{E} = \sum_{r=1}^{m-1} |f_{2r}|^2 (1+|f_{2r-1}|^2) + \left|\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right| (1+|f_{2m-1}|^2)$$

$$= \sum_{r=1}^{m-1} |f_{2r}|^2 + \sum_{r=1}^{m-1} |f_{2r-1} f_{2r}|^2 + \left|\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right| + \left|\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right| \cdot |f_{2m-1}|^2; \tag{8.7}$$

if n = 2 m + 1, by (8.3) and (5.15),

$$\mathcal{E} = \sum_{r=1}^{m} |f_{2r}|^2 (1+|f_{2r-1}|^2) + 2\left|\sum_{r=1}^{m} f_{2r-1} f_{2r}^2\right|$$

$$= \sum_{r=1}^{m} |f_{2r}|^2 + \sum_{r=1}^{m} |f_{2r-1} f_{2r}|^2 + 2\left|\sum_{r=1}^{m} f_{2r-1} f_{2r}^2\right|. \tag{8.8}$$

By their definitions, the f_r are analytic in the region of definition if the same is true of the x_r.

Further, by (2.10), if n = 2 m,

$$\left(\frac{-\sum_{r=1}^{m-1} f_{2r-1} f_{2r}^2}{f_{2m-1}}\right)^{1/2} = \frac{2x'_{2m-1}}{1-f_{2m-1}} = \frac{-2i\,x'_{2m}}{1+f_{2m-1}}, \tag{8.9}$$

so that the last two terms in (8.7) are the squares of analytic functions; similarly, if n = 2 m + 1, the last term in (8.8) is the square of an analytic function, according to (2.13). Then

$$\mathcal{E} = \sum_{r=1}^{m} |\phi_r(\alpha)|^2, \tag{8.10}$$

the ϕ_r being analytic functions of α. Let

$$\psi_r(\alpha) = \int \phi_r(\alpha)\, d\alpha. \tag{8.11}$$

We now can prove the following theorem:

If the isothermic harmonic functions (1.1) map the interior of a circle of radius a and center (u_0, v_0) on a surface, then the area A of the surface satisfies the inequality

$$A \geq \pi \mathcal{E}_0 a^2, \tag{8.12}$$

where $\mathcal{E}_0$ is the area deformation at (u_0, v_0). The equality holds if and only if the map is a circle.

In L. R. Ford's <u>Automorphic Functions</u>, page 167, it is proved that each of the functions ψ_r maps the circle on a region the area of which satisfies

$$A_r = \iint |\phi_r|^2 \, du \, dv \geq \pi \, |\phi_{r,0}|^2 a^2, \tag{8.13}$$

where $\phi_{r,0}$ is the value of ϕ_r at (u_0, v_0). The equality holds if and only if

$$\psi_r = a_r \alpha + b_r . \tag{8.14}$$

Adding the equations (8.13), we have the desired inequality:

$$A = \iint \sum_{r=1}^{\infty} |\phi_r|^2 \, du \, dv \geq \pi \sum_{r=1}^{\infty} |\phi_{r,0}|^2 a^2 = \pi \, \mathcal{E}_0 a^2. \tag{8.15}$$

By (8.14), the equality of (8.15) holds if and only if each

$$\phi_r = a_r ; \tag{8.16}$$

in this case, by (8.4)-(8.6),

$$x_r = R\{c_r \alpha\} = R\{c_r u\} + R\{c_r i v\} = p_r u + q_r v. \tag{8.17}$$

That this map is a circle becomes apparent when we make the map tangent to the (u, v) plane at (u_0, v_0) and make the x_1 and x_2 axes coincide with the u and v axes respectively. Then

$$\begin{aligned} x_{1,10} &= p_1, & x_{1,01} &= q_1 = 0, \\ x_{2,10} &= p_2 = 0, & x_{2,01} &= q_2, \\ x_{r,10} &= p_r = 0, & x_{r,01} &= q_r = 0, \quad r > 2; \end{aligned} \tag{8.18}$$

and since the parameters are isothermic,

$$p_1 = q_2 = E^{1/2} = E_0^{1/2}, \tag{8.19}$$

so that

$$x_1 = E_0^{1/2} u,$$

$$x_2 = E_0^{1/2} v, \tag{8.20}$$

$$x_r = 0 \ , \ r > 2:$$

the circle

$$u^2 + v^2 \leq a^2 \tag{8.21}$$

is mapped on the circle

$$x_1^2 + x_2^2 \leq E_0 a^2. \tag{8.22}$$

We could have proved this theorem directly without our parametric representation, as follows:

We can take (u_0 , v_0) = (0, 0) without loss of generality. Replacing (u, v) by polar coordinates, (ρ, θ), we have

$$x_s = \sum_{r=1}^{\infty} \rho^r \left(a_{s,r} \cos r\theta + b_{s,r} \sin r\theta\right); \tag{8.23A}$$

$$\frac{\partial x_s}{\partial u} = \frac{\partial x_s}{\partial \rho}\cos\theta - \frac{\partial x_s}{\partial \theta}\,\frac{\sin\theta}{\rho}$$

$$= \sum_{r=1}^{\infty} r\rho^{r-1}\left(a_{s,r}\cos r\theta + b_{s,r}\sin r\theta\right)\cos\theta$$

$$- \sum_{r=1}^{\infty} \rho^{r}\left(-r a_{s,r}\sin r\theta + r b_{s,r}\cos r\theta\right)\frac{\sin\theta}{\rho}$$

$$= \sum_{r=1}^{\infty} r\rho^{r-1}\left[a_{s,r}\cos(r-1)\theta + b_{s,r}\sin(r-1)\theta\right];$$

(8.23B)

$$\left(\frac{\partial x_s}{\partial u}\right)^2 = \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} r t \rho^{r+t-2}\left[a_{s,r}\cos(r-1)\theta + b_{s,r}\sin(r-1)\theta\right]\times$$

$$\times\left[a_{s,t}\cos(t-1)\theta + b_{s,t}\sin(t-1)\theta\right]$$

$$= \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} r t \rho^{r+t-2}\Big[a_{s,r}\,a_{s,t}\cos(r-1)\theta\cos(t-1)\theta$$

$$+2a_{s,r}\,b_{s,t}\cos(r-1)\theta\sin(t-1)\theta$$

$$+b_{s,r}\,b_{s,t}\sin(r-1)\theta\sin(t-1)\theta\Big].$$

The area of the map of the circle with center at (0, 0) and radius a', where $a' < a$, is

$$A' = \int_0^{a'} \rho\, d\rho \int_0^{2\pi} \mathcal{E}\, d\theta = \int_0^{a'} \rho\, d\rho \int_0^{2\pi} \sum_{s=1}^{n} \left(\frac{\partial x_s}{\partial u}\right)^2 d\theta$$

$$= \int_0^{a'} \left[2\pi \sum_{s=1}^{n} a_{s,1}^2 + \sum_{s=1}^{n} \sum_{r=2}^{\infty} r^2 \rho^{2r-2} \left(a_{s,r}^2 + b_{s,r}^2\right)\right] \rho\, d\rho \qquad (8.24)$$

$$= \pi \sum_{s=1}^{n} a_{s,1}^2 a'^2 + \frac{\pi}{2} \sum_{s=1}^{n} \sum_{r=2}^{\infty} r a'^{2r} \left(a_{s,r}^2 + b_{s,r}^2\right).$$

As a' approaches a, this area either becomes infinite or approaches the same quantity with a in place of a'. In either case,

$$A \geq \pi \sum_{s=1}^{n} a_{s,1}^2 a^2 = \pi \mathcal{E}_0 a^2, \qquad (8.25)$$

since

$$\mathcal{E}_0 = \sum_{s=1}^{n} a_{s,1}^2 = \sum_{s=1}^{n} b_{s,1}^2 = \mathcal{G}_0 . \qquad (8.26)$$

The equality in (8.25) holds if and only if

$$a_{s,r} = b_{s,r} = 0, \quad r > 1, \quad s = 1, 2, \ldots, n, \qquad (8.27)$$

or

$$x_s = \rho\left(a_{s,1} \cos\theta + b_{s,1} \sin\theta\right); \qquad (8.28)$$

and (8.28) is equivalent to (8.17).

We note, as a corollary* that the area of the map of a ring, concentric with the given circle, of radii a_1 and a_2, where $0 \le a_1 < a_2 \le a$, satisfies the inequality

$$A \geq \pi \mathcal{E}_0 (a_2^2 - a_1^2), \tag{8.29}$$

the equality holding, as before, if and only if the circle is mapped on a circle.

The proof of the above corollary consists of changing the limits of integration with respect to ρ in (8.24) from $(0, a')$ to (a_1, a_2).

As a second corollary, we prove the following theorem for minimal surfaces in ordinary space:

If n = 3, then the length L of the boundary of the map of the above circle satisfies the inequality

$$L \geq 2\pi \mathcal{E}_0^{1/2} a. \tag{8.30}$$

The equality holds if and only if the map is a circle.

This follows immediately from our theorem and from the known inequality

$$A \leq \frac{1}{4\pi} L^2, \tag{8.31}$$

due to Torsten Carleman, Zur Theorie der Minimalflächen, Mathematische Zeitschrift, vol. 9 (1921), pages 154-160.

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* This corollary was suggested by Dr. L. R. Ford.

From (8.25) and (8.31), we have

$$\pi \mathcal{E}_0 a^2 \leq A \leq \frac{1}{4\pi} L^2, \tag{8.32}$$

whence we obtain (8.30).

9. Minimal surfaces of planar derivation. If n = 3, (8.8) becomes

$$\mathcal{E} = |f_2|^2 + |f_1 f_2|^2 + 2|f_1 f_2^2| = \left(|f_2| + |f_1 f_2|\right)^2. \tag{9.1}$$

Let

$$M(\alpha) = \int f_2(\alpha)\, d\alpha \quad , \quad N(\alpha) = \int f_1(\alpha)\, f_2(\alpha)\, d\alpha. \tag{9.2}$$

When and only when a real minimal surface in Euclidean 3-space is such that $M(\alpha)$ and $N(\alpha)$ give plane maps of the region of definition, we say that the surface is of planar derivation.

This is a large but not exhaustive class of minimal surfaces. The following theorems are generalizations of known theorems concerning plane analytic maps: see Ford, loc. cit., pages 169-176.

Theorem: If the isothermic harmonic functions

$$x_r = x_r(u, v) \ , \quad r = 1, 2, 3, \tag{9.3}$$

map the circle

$$\left[(u-u_0)^2 + (v-v_0)^2\right]^{1/2} < \rho \tag{9.4}$$

on a finite surface of planar derivation, then the minimum distance on the surface from the image of the origin to the boundary is greater than or at least equal to $\mathcal{E}_0^{1/2}\rho/4$. No closer inequality holds for all surfaces of this type.

The above distance is given by the minimum, for all paths of integration, of the integral

$$\int_{r=0}^{\rho} \mathcal{E}^{1/2}\,|d\alpha| = \int_{r=0}^{\rho} \left(|M'| + |N'|\right)\,|d\alpha|. \tag{9.5}$$

Since $M(\alpha)$ and $N(\alpha)$ give plane maps of (9.4), we can apply Bieberbach's Theorem to these functions, getting, for any path of integration,

$$\begin{aligned} \int_{r=0}^{\rho} |M'|\,|d\alpha| &\geq \frac{|M_0'|\,\rho}{4}, \\ \int_{r=0}^{\rho} |N'|\,|d\alpha| &\geq \frac{|N_0'|\,\rho}{4}, \end{aligned} \tag{9.6}$$

where M_0' and N_0' are the values of M' and N' at (u, v) = (u_0, v_0); adding the two inequalities of (9.6), we obtain the desired inequality:

$$\int_{r=0}^{\rho} \mathcal{E}^{1/2}\,|d\alpha| \geq \frac{\mathcal{E}_0^{1/2}\rho}{4}. \tag{9.7}$$

That the inequality of the theorem is the closest possible for all such surfaces follows from the fact that, by Bieberbach's Theorem, it is the closest possible for all plane analytic maps. We can choose both $M(\alpha)$ and $N(\alpha)$ as equal to a function $P(\alpha)$ satisfying the inequality of the theorem for a certain path of integration. For this path, the corresponding distance on the surface is $\mathcal{E}_0^{1/2}\rho/4$.

Precisely the same method of proof as the above generalizes the theorem in Ford, loc. cit., page 173, to the following deformation

<u>Theorem:</u> If the isothermic harmonic functions (9.3) map the circle (9.4) on a finite surface of planar derivation, then at any point within the circle the following inequalities hold:

$$\mathcal{E}_0^{1/2}\frac{1-r}{(1+r)^3} \leq \left[\mathcal{E}(u,v)\right]^{1/2} \leq \mathcal{E}_0^{1/2}\frac{1+r}{(1-r)^3}, \tag{9.8}$$

where

$$\left[(u-u_0)^2 + (v-v_0)^2\right]^{1/2} = r\rho. \tag{9.9}$$

Further, no closer inequalities hold for all surfaces of this type.

The next theorem has its counterpart in the theory of plane maps, yet needs some proof. See Ford, loc. cit., pages 174, 175.

<u>Theorem:</u> If the isothermic harmonic functions (9.3) map the circle (9.4) on a finite surface of planar derivation, then the minimum distance m(u, v) on the surface from the image of the origin to the point corresponding to (u, v) satisfies the inequalities

$$\frac{r}{(1+r)^2}\,\mathcal{E}_0^{1/2}\,\rho \le m(u,v) \le \frac{r}{(1-r)^2}\,\mathcal{E}_0^{1/2}\,\rho, \tag{9.10}$$

where r is given by (9.9). No closer inequalities hold for all surfaces of this type.

The second inequality of (9.8) gives

$$\int_{(u_0,v_0)}^{(u,v)} \mathcal{E}^{1/2}\,ds \le \int_{(u_0,v_0)}^{(u,v)} \mathcal{E}_0^{1/2}\,\frac{1+r}{(1-r)^3}\,ds. \tag{9.11}$$

The first of these integrals, taken along the radius, is at least as great as m(u, v); hence,

$$
\begin{aligned}
m(u,v) &\le \int_0^r \frac{1+r}{(1-r)^3}\,\rho\,\mathcal{E}_0^{1/2}\,dr \\
&= \frac{r}{(1-r)^2}\,\rho\,\mathcal{E}_0^{1/2}.
\end{aligned} \tag{9.12}
$$

If the curve C minimizes the distance, then

$$m(u,v) = \int_C \mathcal{E}^{1/2}\,ds. \tag{9.13}$$

This integral is diminished if we substitute for

$$ds = \left[\rho^2 dr^2 + \rho^2 r^2\, d\theta^2\right]^{1/2} \tag{9.14}$$

the smaller quantity $\rho\,|dr|$:

$$m(u,v) \ge \int_C \rho\,\mathcal{E}^{1/2}\,|dr|. \tag{9.15}$$

The first inequality of (9.8) gives, then,

$$
\begin{aligned}
m(u,v) &\ge \int_C \frac{1-r}{(1+r)^3}\,\rho\,\mathcal{E}_0^{1/2}\,|dr| \\
&\ge \int_0^r \frac{1-r}{(1+r)^3}\,\rho\,\mathcal{E}_0^{1/2}\,dr \\
&= \frac{r}{(1+r)^2}\,\rho\,\mathcal{E}_0^{1/2}.
\end{aligned} \tag{9.16}
$$

From (9.12) and (9.16) we get (9.10). That there are no closer inequalities follows from the fact that we can take both $M(\alpha)$ and $N(\alpha)$ as equal to a function $P(\alpha)$ which attains both limits; then the surface map attains both limits.

The preceding two theorems yield the following pair of theorems. The proofs, depending on a division of the (u, v) domain into squares, are exactly the same as those of the corresponding theorems in the plane, and therefore are not given here. See Ford, loc. cit., pages 175, 176.

Theorem: Let Σ' be a plane finite region and let Σ be a subregion whose boundary consists of interior points of Σ'. Let the isothermic harmonic functions (9.3) map Σ' on a finite surface of planar derivation. Then there exists a constant K, dependent on Σ and Σ' but independent of the mapping functions, such that if (u_1, v_1) and (u_2, v_2) are any two interior or boundary points of Σ,

$$\frac{1}{K} < \frac{\mathcal{E}(u_1, v_1)}{\mathcal{E}(u_2, v_2)} < K \tag{9.17}$$

Theorem: In the mapping of the preceding theorem there exists a constant L, independent of the mapping functions, such that if (u_1, v_1), (u_2, v_2), and (u_3, v_3) are any three interior or boundary points of Σ, then

$$m[(u_1,v_1),(u_2,v_2)] < L\left[\sigma(u_3,v_3)\right]^{1/2}, \qquad (9.18)$$

where

$$m\,[(u_1,v_1),\,(u_2,v_2)] \qquad (9.19)$$

denotes the minimum distance on the surface between the points (u_1, v_1) and (u_2, v_2).

10. Theorem: The distance from any point, line, plane, or hyperplane of less than n dimensions, to the surface determined by the isothermic harmonic functions (1.1) cannot have a maximum within the region of definition, unless the distance is identically constant.

Let the point, line, plane, or hyperplane in question be the origin, the x_n axis, the (x_{n-1}, x_n) plane, or the $(x_{n-k+1}, x_{n-k+2}, \ldots, x_n)$ hyperplane. Then the distance in question is

$$\delta = \left(\sum_{r=1}^{n-k} x_r^2\right)^{1/2}, \quad 0 \le k < n. \qquad (10.1)$$

The function δ^2 is subharmonic, for its Laplacian is

$$\frac{\partial^2(\delta^2)}{\partial u^2} + \frac{\partial^2(\delta^2)}{\partial v^2} = 2\sum_{r=1}^{n-k}\left[x_r\left(\frac{\partial^2 x_r}{\partial u^2} + \frac{\partial^2 x_r}{\partial v^2}\right) + \left(\frac{\partial x_r}{\partial u}\right)^2 + \left(\frac{\partial x_r}{\partial v}\right)^2\right]$$

$$= 2\sum_{r=1}^{n-k}\left[\left(\frac{\partial x_r}{\partial u}\right)^2 + \left(\frac{\partial x_r}{\partial v}\right)^2\right] \geq 0. \tag{10.2}$$

The theorem follows from the fact that, as is well known, a subharmonic function cannot have an interior maximum, unless it is identically constant.

The function (10.1) would have served our purpose as well, for it also is subharmonic. Its Laplacian is more tedious to evaluate, but is actually

$$\frac{1}{\delta^3}\sum_{r>s}\left\{\begin{vmatrix} x_r & x_s \\ \frac{\partial x_r}{\partial u} & \frac{\partial x_s}{\partial u}\end{vmatrix}^2 + \begin{vmatrix} x_r & x_s \\ \frac{\partial x_r}{\partial v} & \frac{\partial x_s}{\partial v}\end{vmatrix}^2\right\} \geq 0. \tag{10.3}$$

It is possible, on the other hand, that the surface should attain its minimum distance from a point, line, plane, or hyperplane. Consider, for example, the set of isothermic harmonic functions

$$\begin{aligned} x_{n-1} &= u \\ x_n &= v \\ x_r &= 1, \quad r < n-1. \end{aligned} \tag{10.4}$$

These functions map the unit circle with center at the origin on a congruent circle; the minimum distance from this surface to the origin, to the x_n axis, or to the (x_{n-1}, x_n) plane is $(n - 2)^{1/2}$, to the (x_{n-2}, x_{n-1}, x_n) hyperplane is $(n - 3)^{1/2}, \ldots,$ to the $(x_{n-k+1}, x_{n-k+2} \ldots, x_n)$ hyperplane is $(n - k)^{1/2}$, and these minima are attained at the interior point (0,0).