RICE UNIVERSITY

Obstructions to the Concordance of Satellite Knots

by

Bridget Dawn Franklin

A Thesis Submitted
in Partial Fulfillment of the
Requirements for the Degree

Doctor of Philosophy

Approved, Thesis Committee:

Tim D. Cochran

Tim D. Cochran,
Professor of Mathematics, Chair

Shelly L. Harvey

Shelly L. Harvey,
Associate Professor of Mathematics

David W. Scott

David W. Scott,
Noah G. Harding Professor of Statistics

Houston, Texas
April, 2012

Abstract

Obstructions to the Concordance of Satellite Knots

by

Bridget Dawn Franklin

Well-known concordance invariants for a satellite knot $\mathcal{R}(\eta, J)$ tend to be functions of R and J but depend only weakly on the axis η. The Alexander polynomial, the Blanchfield linking form, and Casson-Gordon invariants all fail to distinguish concordance classes of satellites obtained by slightly varying the axes. By applying higher-order invariants and using filtrations of the knot concordance group, satellite concordance may be distinguished by determining the term of the derived series of $\pi_1(S^3 \setminus \mathcal{R})$ in which the axes lie. There is less hope when the axes lie in the same term. We introduce new conditions to distinguish these latter classes by considering the axes in higher-order Alexander modules in three situations. In the first case, we find that $\mathcal{R}(\eta_1, J)$ and $\mathcal{R}(\eta_2, J)$ are non-concordant when η_1 and η_2 have distinct orders in the classical Alexander module of $\mathcal{R}$. In the second, we show that even when η_1 and η_2 have the same order, $\mathcal{R}(\eta_1, J)$ and $\mathcal{R}(\eta_2, J)$ may be distinguished when the classical Blanchfield form of η_1 with itself differs from that of η_2 with itself. Ultimately, this allows us to find infinitely many concordance classes of $\mathcal{R}(-, J)$ whenever $\mathcal{R}$ has nontrivial Alexander polynomial. Finally, we find sufficient conditions to distinguish these satellites when the axes represent equivalent elements of the classical Alexander module by analyzing higher order Alexander modules and localizations thereof.

Contents

List of Figures

Chapter 1

Motivation

Knot theory has become increasingly important in 3- and 4-manifold topology. We begin our study of knot concordance and the effect of satellite operations on concordance by discussing the important implications of this field.

A knot is the embedding of an oriented circle into the 3-dimensional sphere, $S^1 \hookrightarrow S^3$. If there exists a continuous map $h : S^1 \times [0,1] \to S^3$ such that $h_t = h(\cdot, t)$ is an embedding for all t, then $K_0 = h_0(S^1)$ and $K_1 = h_1(S^1)$ are said to be isotopic. The exterior of K is the bounded 3-manifold $S^3 \setminus K$ obtained by removing an open tubular neighborhood of K from S^3, and the knot group is the fundamental group of the knot exterior. As a consequence of Gordon-Lueke, isotopy classes of (unoriented) knots are determined by the oriented homeomorphism type of their exteriors [GL89]. Thus, knot theory is to a great extent the study of knot exteriors and knot groups.

For any K_0 and K_1, the connected sum, $K_0 \# K_1$, is the knot obtained by removing an unknotted arc from each K_i and joining the endpoints via two unknotted arcs such that orientation respects that of K_0 and K_1, as in Figure 1.1. The set of knots under

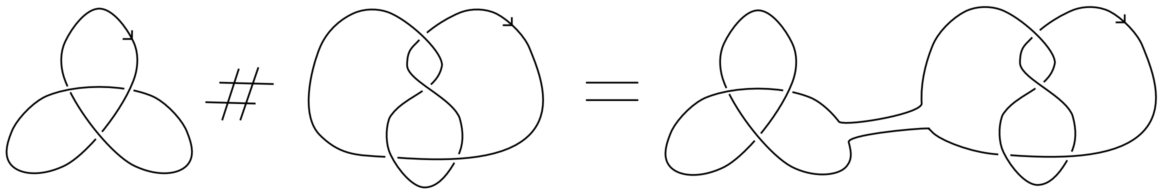
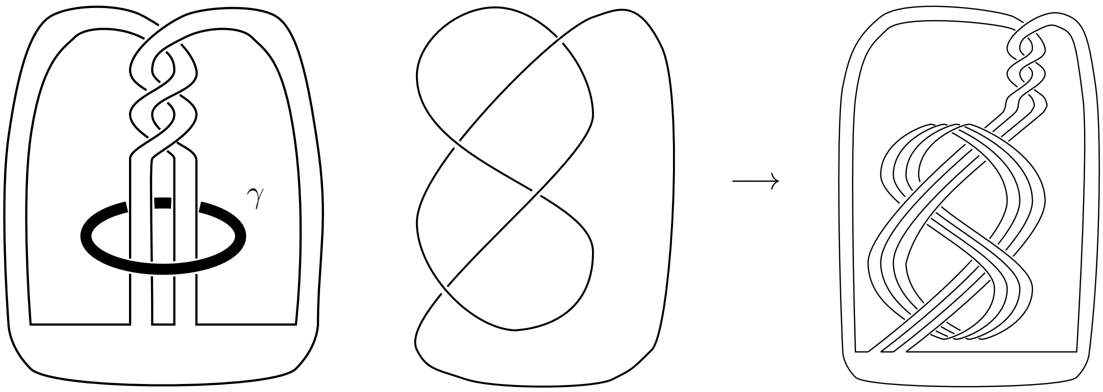


Figure 1.1: The connected sum of the Trefoil and Figure 8 knots

connected sum forms a commutative monoid, but there is no connected sum inverse.

A generalization of connected sum is a satellite operation, a process which is often described via infections. In this procedure, one begins with any knot $\mathcal{R}$ along with an unknotted circle η in the complement of $\mathcal{R}$, as on the left-hand side of Figure 1.2. There are two ways to envision infection; the first is more intuitive though less rigourous. Since η is unknotted in S^3, it bounds a disk D^2 that is intersected transversely by strands of $\mathcal{R}$. Given any J, we "infect $\mathcal{R}$ by J along η" to yield the satellite $\mathcal{R}(\eta, J)$. This is done by cutting strands of $\mathcal{R}$ intersecting D and tying the parallel strands into the knot J, as shown on the right-hand side of Figure 1.2 for $J = 4_1$. Equivalently, we may think of forming $\mathcal{R}(\eta, J)$ by removing a tubular neighborhood of η in $S^3 \setminus \mathcal{R}$ creating a second toroidal boundary component $\partial_\eta(S^3 \setminus \mathcal{R} \cup \eta)$. Identify $\partial S^3 \setminus J$ and $\partial_\eta(S^3 \setminus \mathcal{R} \cup \eta)$ such that the longitude $\lambda(J)$ is identified with the meridian $\mu(\eta)$ and $\mu(J)$ is identified with $\lambda(\eta)^{-1}$. The curve η is the axis, $\mathcal{R}$ is the pattern, and J is the companion of the satellite operation.

The unique knot which bounds a disk in S^3 is the trivial knot. When $K \subset S^3 = \partial B^4$ bounds an embedded disk in B^4, K is said to be a slice knot. The slice disk may be a topologically locally-flat or smoothly embedding of D^2 in B^4, and K is called topologically or smoothly slice respectively. We say that K_0 and K_1 are



$\mathcal{R}$ $\quad$ $J = 4_1$ $\quad$ $\mathcal{R}(\gamma, J)$

Figure 1.2: The infection of $\mathcal{R}$ by the Figure 8 knot.

concordant if there is a (topologically locally-flat or smoothly) embedded annulus $A : S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ such that $A(S^1 \times \{0\}) = K_0$ and $A(S^1 \times \{1\}) = K_1$. Using this concordance annulus, it follows that if K_0 and K_1 are concordant, then $K_0 \# r\overline{K_1}$ is slice, where $r\overline{K_1}$ is the reverse mirror image of K_1. This implies that the concordance inverse of K is $r\overline{K}$ justifying the notation $-K$. The set of knots modulo concordance forms a commutative group $\mathcal{C}$ under connected sum called the knot concordance group. It may be referred to as the smooth $\mathcal{C}^\infty$ or topological $\mathcal{C}^{\text{top}}$ knot concordance group depending on whether whether the concordance annulus is required to be smoothly or topologically locally-flat embedded respectively. In either case the structure of $\mathcal{C}$ is not well understood.

Most, if not all, known examples of smoothly slice knots are in fact ribbon knots. A ribbon knot is one which bounds an immersed disk $D^2 \looparrowright S^3$ whose singularities are comprised of pairs of arcs $\{\gamma_i, \gamma_i'\} \subset D$ such that $\gamma_i \subset \text{Int}D$ and $\gamma_i' \cap \partial D = \partial \gamma_i'$. A ribbon knot is shown to be slice by pushing an open neighborhood in D of each γ_i into B^4. The ribbon-slice conjecture states that every smoothly slice knot is also ribbon [Fox61]. This conjecture remains open.

Recently, knot concordance has become increasingly important to the study of low-dimensional topology. Efforts to understand the structure of the knot concordance group have uncovered a great amount of complexity highlighting the difficulty of classifying 3- and 4-manifolds.

1.1 Knots and Dehn Surgery

Like the fundamental group of a 3-manifold M^3, the set of isotopy classes of embeddings of S^1 in M^3 gives clues to the structure of M^3. Primarily, knots are thought of as embeddings in S^3, and given any such embedding, one may perform p/q Dehn surgery to yield a closed 3-manifold.

Definition 1.1. [Rol90, p. 258] The Dehn surgery of a knot K in S^3 with surgery coefficient $p/q \in \mathbb{Q}$ is obtained by removing a tubular neighborhood of K and replacing it with a solid torus, $\mathbf{T} = D^2 \times S^1$, such that the meridian of $\partial\mathbf{T}$ is identified with the curve in $\partial S^3 \setminus K$ given by $p\mu(K) + q\lambda(K)$ where $\mu(K)$ and $\lambda(K)$ are the meridian and longitude of K respectively.

This technique may be used to produce a wide variety of interesting 3-manifolds, including the Poincaré homology sphere, which is given by $+1$ surgery on the right-handed trefoil. In later arguments, we consider the 0-surgery, where the meridian of $\partial\mathbf{T}$ is identified with $\lambda(K)$. This closed manifold, denoted by M_K, shares many important properties with the knot exterior. In particular,

$$\frac{\pi_1(M_K)}{\pi_1(M_K)^{(2)}} \cong \frac{\pi_1(S^3 \setminus K)}{\pi_1(S^3 \setminus K)^{(2)}},$$

where $G^{(2)}$ refers to the second term of the derived series of a group G.

Dehn surgery on knots produces closed manifolds with first Betti number $\beta_1 \leq 1$. By allowing Dehn surgery on each component of a link, $L = S^1 \sqcup S^1 \sqcup \cdots \sqcup S^1 \hookrightarrow S^3$, more complicated manifolds may be constructed.

Theorem 1.2 (Lickorish-Wallace [Rol90])**.** *Every closed, orientable, connected* 3-*manifold may be obtained by surgery on an oriented link $L \subset S^3$. Furthermore, a surgery presentation may be found such that each L_i is unknotted and all surgery coefficients are* $+1$*.*

Theorem 1.2 exemplifies the importance of knot (and link) theory in 3-dimensional topology.

1.2 The Whitney Embedding Theorem

The information encoded by knots not only supplies a wealth of information for the study of 3-manifolds but is also integral to the study of higher-dimensional spaces. A major advancement in this field was obtained by Hassler Whitney in 1944.

Theorem 1.3 (Whitney Embedding Theorem [Whi44])**.** *Any n-manifold X may be smoothly embedded in the Euclidean space* $\mathbb{R}^{2n}$*.*

The proof of this theorem relies upon first being able to find an immersion

$$f : X \looparrowright \mathbb{R}^{2n}$$

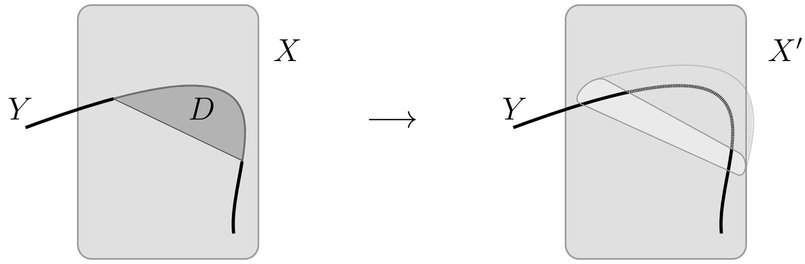


Figure 1.3: The Whitney Trick.

with only transverse self-intersections. Pairs of singularities are then removed via the Whitney Trick.

Proposition 1.4 (Whitney Trick)**.** *Suppose P^p and Q^q are locally flat connected submanifolds of M^{p+q} embedded such that P and Q intersect transversely. Suppose that $r, s \in P \cap Q$ have opposite signs and that either*

1. *$p \geq 3, q \geq 3$ and $\pi_1(M) = 0$, or*
2. *$p = 2$, $q \geq 3$ and $\pi_1(M \setminus Q) = 0$.*

Then there exist arcs α and β such that Int(α) $\subset P \setminus Q$, Int(β) $\subset Q \setminus P$, and $\alpha \cap \beta = \{r, s\}$. These arcs bound a locally flat disk D^2 with $D \cap (P \cup Q) = \partial D$.

Pairs of singularities of the immersion $X \looparrowright \mathbb{R}^{2n}$ are then removed by pushing X off itself along the embedded disk guaranteed by the Whitney Trick, as in Figure 1.3. The Whitney Trick and the Whitney Embedding Theorem have vast implications for geometric topology of manifolds in dimensions $n \geq 5$. For instance, Smale used the Whitney Trick to prove the h-Cobordism Theorem from which follows the Generalized Poincaré Conjecture for $n \geq 5$ [Sma62]. Unfortunately, the strategy fails in lower dimensions since an immersed disk in $\mathbb{R}^n$ cannot be isotoped to an embedded disk for $n \leq 4$. One major application of the knot theory is to measure the failure of the Whitney Trick in dimensions 3 and 4. This raises the question of which embedded curves in the boundary of a simply connected 4-manifold bound embedded disks. Given technical issues raised by the smooth 4-dimensional Poincaré Conjecture, we simplify this question to ask which embedded curves in S^3 bound embedded disks in the 4-ball. This is precisely the study of knot concordance.

1.3 Knot Concordance and Homology Cobordism

In addition to its implications on the Whitney embedding theorem, knot concordance has an interesting relationship with the study of homology cobordism of 3-manifolds. For $n \geq 5$, the h-cobordism theorem [Sma62] shows that h-cobordism, or "homotopy cobordism", plays an important role in the classification of smooth n-manifolds. For $n = 4$, the result holds only in the topological category, and in dimension 3, the question is more ambiguous and is largely dependent on the smooth 4-dimensional Poincaré conjecture. Instead, low-dimensional topologists study homology cobordism classes of 3-manifolds.

Definition 1.5. Two n-manifolds M and N are G-homology cobordant if there exists an $(n+1)$-manifold W with boundary $\partial W = M \sqcup -N$ such that inclusion of each boundary component induces isomorphisms on homology with G coefficients.

$$\begin{tikzcd} H_*(M;G) \arrow[dr, "\cong"] & & H_*(N;G) \arrow[dl, "\cong"'] \\ & H_*(W;G) & \end{tikzcd}$$

It is well-known that if K and K' are concordant, there exists a $\mathbb{Z}$-homology cobordism between M_K and $M_{K'}$. This cobordism is constructed by first removing a tubular neighborhood of the concordance annulus $A \subset S^3 \times [0,1]$. The exterior of A may be shown using excision to be a homology cobordism between the knot exteriors, and the meridians of K and K' are isotopic in $(S^3 \times [0,1]) \setminus A$. Replacing the annulus with $D^2 \times S^1 \times [0,1]$ such that $\partial D^2 \times \cdot \times [0,1]$ is identified with $\lambda(K)$ and $\lambda(K')$ when $t = 0, 1$ respectively yields the M_K and $\overline{M_{K'}}$ along the boundary.

The converse of this fact is a long-standing open question. If the 0-surgeries of K and K' are homology cobordant, is K concordant to K' (modulo changing the orientation of K')? Recent work of Cochran, Hedden, Horn, and the author resolves the question in certain categories.

Theorem 1.6 (Cochran-Franklin-Hedden-Horn [CFHH12])**.** *There exist knots whose zero surgeries are $\mathbb{Q}$-homology cobordant but which are not $\mathbb{Q}$-concordant. There exist topologically slice knots whose zero surgeries are smoothly $\mathbb{Z}$-homology cobordant but which are not concordant.*

The results of Theorem 1.6 were found by methods similar to those of Chapters 3, 4, and 5 by studying the concordance of satellite knots. The first result, that there exist knots K and K' whose zero-surgeries are $\mathbb{Q}$-homology cobordant but which are not rationally concordant, was found by taking account of the linking number of the axis of the satellite operation with the pattern. It follows that if the pattern is slice, and the linking number of the axis with the pattern is p, then there exists a $\mathbb{Z}[1/p]$-homology cobordism between the zero-surgeries of the companion and satellite. By choosing a similar pattern with winding number 1, this construction yields an integral homology cobordism, and careful choices of K and K' contradict concordance.

Theorem 1.6 implies that the theory of knot concordance is stronger than homology cobordism in the classification of 3-manifolds, and the methods used exemplify the importance of satellite constructions.

Chapter 2

Knot Theory and Its History

2.1 Classical Invariants

Many results in knot theory have been obtained by building complicated knots by the use of satellite operations. This operation is a generalization of the well-known connected-sum operation and is a simple way of constructing a new knot from two old knots. Satellite operations are also closely related to the JSJ-decomposition of 3-manifolds by torii, and many classical and even higher-order knot invariants behave nicely under the satellite operation.

Invariants of the concordance class of the satellite $\mathcal{R}(\eta, J)$ depend upon those of $\mathcal{R}$ and J but often depend only on the linking number of η with $\mathcal{R}$. By applying higher-order invariants when $\ell k(\eta, \mathcal{R}) = 0$, concordance information may be recovered by considering at which level the class represented by η lies in the derived series of the knot group $\pi_1(S^3 \setminus \mathcal{R})$.

In this thesis, we seek to differentiate the concordance classes of satellites while varying the axis but fixing the pattern and companion knots. In particular, we take $\mathcal{R}$ to be some fixed ribbon knot and give sufficient conditions such that infection upon two distinct axes, η_1 and η_2, by a chosen knot J yields distinct concordance classes. The methods are based upon classical concordance invariants such as the Alexander module and Blanchfield linking-form, their higher-order analogues, as well as higher-order ρ-invariants.

2.1.1 The Alexander Module

One classical invariant used in distinguishing concordance is the Alexander module. Since each knot exterior has the homology type of a circle, classical homology theory fails to yield any useful information. Instead, we consider the homology of $\widetilde{S^3 \setminus K}$, the infinite cyclic cover of the knot exterior. The group of deck translations is $\pi_1(S^3 \setminus K)/\pi_1(S^3 \setminus K)^{(1)} = \langle t \rangle \cong \mathbb{Z}$ and is generated by $\mu(K)$. We define

$$H_*(S^3 \setminus K; \mathbb{Z}[t, t^{-1}]) \equiv H_*(\widetilde{S^3 \setminus K}; \mathbb{Z}).$$

This is the homology of $S^3 \setminus K$ with coefficients twisted by $\mathbb{Z}[t, t^{-1}]$ and is viewed as a right $\mathbb{Z}[t, t^{-1}]$-module via the action $[\alpha]t = [\mu(K)^{-1}\alpha\mu(K)]$ where $\widetilde{\mu(K)}$ is the lift of $\mu(K)$ in $\widetilde{S^3 \setminus K}$.

Definition 2.1. The Alexander module of K is $\mathcal{A}^{\mathbb{Z}}(K) \cong H_1(S^3 \setminus K; \mathbb{Z}[t, t^{-1}])$.

Let Σ_g be a Seifert surface for K, that is, a surface of genus g embedded in S^3 with boundary K. The first homology of Σ_g is generated by $2g$ curves $\{e_1, \ldots, e_{2g}\}$ in its interior. For any i, we denote by e_i^+ the curve in $S^3 \setminus \Sigma_g$ given by the pushing e_i

slightly off Σ_g in the positlve direction as determined by the orientation of Σ_g. The Seifert matrix of V is the matrix with entries $V_{i,j} = \ell k(e_i, e_j^+)$. The Seifert matrix yields a presentation matrix for $\mathcal{A}^{\mathbb{Z}}(K)$.

Theorem 2.2 ([Lic97, Theorem 6.5]). *If V is a Seifert matrix for K, then $tV - V^{\intercal}$ is a presentation matrix for $\mathcal{A}^{\mathbb{Z}}(K)$ as a right $\mathbb{Z}[t, t^{-1}]$-module.*

Definition 2.3. The Alexander polynomial of K, $\Delta_K(t)$, is $\det(tV - V^{\intercal})$.

The Alexander polynomial is invariant of the choice of Seifert surface Σ_g for K as well as the choice of basis for $H_1(\Sigma_g)$ up to multiplication by a unit of $\mathbb{Z}[t, t^{-1}]$. We have the following properties of $\Delta_K(t)$, where $\doteq$ denotes equivalence up to multiplication by $\pm t^k$ for some $k \in \mathbb{Z}$.

Theorem 2.4 ([Lic97, Theorem 6.10, Proposition 6.11]). *If K is a knot with Alexander polynomial $\Delta_K(t)$, then*

1. $\Delta_K(1) = \pm 1$,

2. $\Delta_K(t) \doteq \Delta_K(t^{-1})$, *and*

3. $\Delta_{K \# K'}(t) \doteq \Delta_K(t)\Delta_{K'}(t)$.

Further results concerning the Alexander module will be useful in our calculations. Consider the satellite $\mathcal{R}(\eta, J)$. If η has linking number w with $\mathcal{R}$, then the Alexander polynomial of $\mathcal{R}(\eta, J)$ is given by

$$\Delta_{\mathcal{R}(\eta,J)}(t) = \Delta_{\mathcal{R}}(t)\Delta_J(t^w) \tag{2.1}$$

[Sei34]. Many classical knot invariants have similar behavior under satellite operations. In the case that $\ell k(\eta, \mathcal{R}) = 0$, η represents an element of $\pi_1(S^3 \setminus \mathcal{R})^{(1)}$, and hence η lifts to the infinite cyclic cover $\widetilde{S^3 \setminus \mathcal{R}}$. The meridian of J normally generates $\pi_1(S^3 \setminus J)$ and is identified with a push-off of η in $S^3 \setminus \mathcal{R}(\eta, J)$. Thus the image of $\pi_1(S^3 \setminus J)$ is contained in $\pi_1(S^3 \setminus \mathcal{R}(\eta, J))^{(1)}$ and lifts to $\widetilde{S^3 \setminus \mathcal{R}(\eta, J)}$. Hence as a special case of (2.1) when $w = 0$, $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}(\eta, J)) \cong \mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ and $\Delta_{\mathcal{R}(\eta,qJ)}(t) = \Delta_{\mathcal{R}}(t)$.

2.1.2 The Algebraic Concordance Group

Using Seifert matrices of knots, Levine defined the algebraic knot concordance group, $\mathcal{AC}$. To do so, Levine constructed a homomorphism $\mathcal{C} \xrightarrow{\phi} G_- \equiv \mathcal{AC}$ where G_- is the cobordism group of Seifert matrices given by

$$G_- = \{A \in M_{2n}(\mathbb{Z}) | \det(A - A^{\intercal}) = \pm 1\}/\sim .$$

The operation on G_- is the block sum of matrices denoted by $A \oplus A'$. A matrix $A \in M_{2n}(\mathbb{Z})$ is trivial if it is null-cobordant, that is, if there exists an invertible square matrix $Q \in M_{2n}(\mathbb{Z})$ such that $QAQ^{\intercal}$ has the form

$$\begin{pmatrix} 0_{n\times n} & X_{n\times n} \\ Y_{n\times n} & Z_{n\times n} \end{pmatrix}. \tag{2.2}$$

Finally A and A' are cobordant if $A \oplus -A'$ is null-cobordant. Levine's homomorphism ϕ maps a knot to the cobordism class of its Seifert matrix.

Theorem 2.5 (Levine [Lev69]). *$\phi : \mathcal{C} \to G_- \equiv \mathcal{AC}$ is an epimorphism. Furthermore, $G_- \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_4^\infty \oplus \mathbb{Z}_2^\infty$.*

Knots in the kernel of ϕ are said to be algebraically slice and knots with cobordant Seifert matrices are algebraically concordant. Since algebraically slice knots have null-cobordant Seifert matrices, their Alexander polynomials have the form

$$\Delta_K(t) \doteq f(t)f(t^{-1}).$$

This is easily shown using Theorem 2.4 and (2.2). However, by considering only the algebraic concordance group, a great amount of structure of $\mathcal{C}$ is inevitably lost.

Casson and Gordon were the first to show that Levine's homomorphism ϕ was not an injection. They did this by finding a non-slice knot whose Seifert matrix was null-cobordant. Their method involved finding characters on the first homology of $M_{K,m}$, the m-fold branched cyclic covers of S^3 branched over K. The Casson-Gordon invariants can be used as a secondary obstruction to concordance beyond the algebraic concordance group. For any charachter $\chi_K : H_1(M_{K,m}) \to \mathbb{Z}/p\mathbb{Z}$ where p and m are prime powers, Casson and Gordon define two invariants [CG78, CG86].

$$\tau(K, \chi_K) \in W(\mathbb{C}(t), j) \otimes \mathbb{Z},$$
$$\sigma(M_{K,n}, \chi_K) \in \mathbb{Q}.$$

Here, $W(\mathbb{C}(t), j)$ is the Witt group of $\mathbb{C}(t)$ under the involution $j : f(t) \mapsto f(t^{-1})$ [Lan02, p. 594]. Unfortunately, like the Alexander module, the Casson-Gordon invariants do little to distinguish concordance of satellites with a fixed pattern and companion. When $K = \mathcal{R}(\eta, J)$, Litherland gives a formula for $\tau(K, \chi_K)$ and $\sigma(M_{K,n})$ based only upon the invariants of $\mathcal{R}$, J and the linking number of η with $\mathcal{R}$ [Lit84]. Since our examples involve satellites where η has linking number 0 with $\mathcal{R}$, these invariants will not be useful. Furthermore, all Casson-Gordon invariants of our examples from Chapters 4 and 5 are zero because the satellites will be 2-solvable as described in Subsection 2.2.3.

2.1.3 The Classical Blanchfield Form

The Blanchfield form is a sesquilinear form on the Alexander module. Recall that $\mathcal{A}^{\mathbb{Z}}(K)$ is given by the first homology of the universal abelian cover of the knot exterior (equivalently that of the zero surgery M_K) with coefficients in $\mathbb{Z}[t,t^{-1}]$. The Blanchfield linking form is given by the composition of the following maps.

$$
\begin{aligned}
\mathcal{B}\ell^{\mathbb{Z}}_K : H_1(S^3\setminus K;\mathbb{Z}[t,t^{-1}]) &\xrightarrow{\pi} H_1(S^3\setminus K,\partial(S^3\setminus K);\mathbb{Z}[t,t^{-1}])\\
&\xrightarrow{P.D.} \overline{H^2(S^3\setminus K;\mathbb{Z}[t,t^{-1}])}\\
&\xrightarrow{\mathcal{B}^{-1}} \overline{H^1(S^3\setminus K;\mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}])}\\
&\xrightarrow{\mathcal{K}} \overline{\mathrm{Hom}\left(H_1(S^3\setminus K;\mathbb{Z}[t,t^{-1}]);\mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]\right)}\\
&= H_1(S^3\setminus K;\mathbb{Z}[t,t^{-1}])^{\#}
\end{aligned}
$$

Here, $P.D.$ denotes the Poincaré Duality isomorphism, $\mathcal{B}$ is the Bochstein isomorphism, and $\mathcal{K}$ is the Kronecker evaluation map. The Blanchfield form, $\mathcal{B}\ell^{\mathbb{Z}}_K(x)$: $\mathcal{A}^{\mathbb{Z}}(K)\to\mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$, is nonsingular, and we denote $[\mathcal{B}\ell^{\mathbb{Z}}_K(x)](y)$ by $\mathcal{B}\ell^{\mathbb{Z}}_K(x,y)$.

$$\mathcal{B}\ell^{\mathbb{Z}}_K : \mathcal{A}^{\mathbb{Z}}(K)\times\mathcal{A}^{\mathbb{Z}}(K)\to\mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$$

One may calculate the Blanchfield form using a formula of Kearton [Kea78]. Let V be a Seifert matrix for K and r,s be elements of $\mathcal{A}^{\mathbb{Z}}(K)$. Then,

$$\mathcal{B}\ell^{\mathbb{Z}}_K(r,s) = (1-t)s^{\intercal}(V-tV^{\intercal})^{-1}r,$$

and the image of $\mathcal{B}\ell$ lies in the subring

$$\frac{\mathbb{Z}[t,t^{-1}]}{\Delta_K(t)\mathbb{Z}[t,t^{-1}]} \subset \mathbb{Q}(t) \mod \mathbb{Z}[t,t^{-1}].$$

That $\mathcal{B}\ell_K^{\mathbb{Z}}$ is sesquilinear means that it satisfies the following properties. For $\alpha_i, \beta_j \in \mathcal{A}^{\mathbb{Z}}(K)$,

$$\mathcal{B}\ell_K^{\mathbb{Z}}(\alpha_1+\alpha_2,\beta) = \mathcal{B}\ell_K^{\mathbb{Z}}(\alpha_1,\beta) + \mathcal{B}\ell_K^{\mathbb{Z}}(\alpha_2,\beta), \text{ and}$$

$$\mathcal{B}\ell_K^{\mathbb{Z}}(\alpha,\beta_1+\beta_2) = \mathcal{B}\ell_K^{\mathbb{Z}}(\alpha,\beta_1) + \mathcal{B}\ell_K^{\mathbb{Z}}(\alpha,\beta_2).$$

For $\alpha, \beta \in \mathcal{A}^{\mathbb{Z}}(K)$ and $x(t), y(t) \in \mathbb{Z}[t,t^{-1}]$,

$$\mathcal{B}\ell_K^{\mathbb{Z}}(x(t)\alpha,\beta) = x(t)\mathcal{B}\ell_K^{\mathbb{Z}}(\alpha,\beta), \text{ and}$$

$$\mathcal{B}\ell_K^{\mathbb{Z}}(\alpha,y(t)\beta) = \overline{y(t)}\mathcal{B}\ell_K^{\mathbb{Z}}(\alpha,\beta).$$

Here $\overline{y(t)}$ denotes the image of $y(t)$ under the group ring involution

$$\overline{\sum_i n_i t^i} = \sum_i n_i t^{-i}.$$

For any knot K with nontrivial Alexander polynomial, let a be the leading coefficient of $\Delta_K(t)$ and set $\mathcal{Q} = \mathbb{Z}[1/a]$. The Alexander module of K with $\mathcal{Q}$-coefficients is defined by

$$\mathcal{A}^{\mathcal{Q}}(K) \equiv H_1(M_K;\mathcal{Q}[t,t^{-1}]) \cong \mathcal{A}^{\mathbb{Z}}(K) \otimes_{\mathbb{Z}} \mathcal{Q}.$$

As a $\mathcal{Q}$-module, $\mathcal{A}^{\mathcal{Q}}(K)$ is finitely generated and free, that is $\mathcal{A}^{\mathcal{Q}}(K) \cong \mathcal{Q}^d$ where d is the degree of $\Delta_K(t)$. Thus, any element $\gamma \in \mathcal{A}^{\mathcal{Q}}(K)$ may be described as a vector $(\gamma_1,\ldots,\gamma_d) \in \mathcal{Q}^d$.

Although classically the Blanchfield form is a sesquilinear form on the integral Alexander module of K, this form extends to the Alexander module with coefficients in $\mathcal{Q}$ in a natural way.

Theorem 2.6 ([COT03, Theorem 2.13])**.** *If $\mathcal{Q}$ is any ring such that $\mathbb{Z} \subseteq \mathcal{Q} \subseteq \mathbb{Q}$, then there is a nonsingular symmetric linking form*

$$\mathcal{B}\ell_K^{\mathcal{Q}} : \mathcal{A}^{\mathcal{Q}}(K) \times \mathcal{A}^{\mathcal{Q}}(K) \to \mathbb{Q}(t) \mod \mathcal{Q}[t, t^{-1}].$$

We later employ the Blanchfield form with $\mathcal{Q}$-coefficients above for arbitrary subrings of $\mathbb{Q}$ and sometimes alternate between coefficient systems. As we will be primarily concerned with examples of $x, y \in \mathcal{A}^{\mathcal{Q}}(K)$ such that $\mathcal{B}\ell_K^{\mathcal{Q}}(x, x) \neq \mathcal{B}\ell_K^{\mathcal{Q}}(y, y)$, this distinction is actually unnecessary for our purposes. Suppose K has a nontrivial Alexander polynomial $\Delta_K(t)$ and x and y are unknotted curves in $S^3 \setminus K$ with $\ell k(x, \mathcal{R}) = \ell k(y, \mathcal{R}) = 0$. Then x and y lift to $\widetilde{S^3 \setminus K}$ and let x and y also denote the corresponding elements of $\mathcal{A}^{\mathbb{Z}}(K)$. Then $x \otimes 1, y \otimes 1$ are the respective images of x and y under the map

$$\mathcal{A}^{\mathbb{Z}}(K) \to \mathcal{A}^{\mathcal{Q}}(K) \cong \mathcal{A}^{\mathbb{Z}}(K) \otimes_{\mathbb{Z}} \mathcal{Q}. \tag{2.3}$$

given by $z \mapsto z \otimes 1$. Since $\mathcal{A}^{\mathbb{Z}}(K)$ has no $\mathbb{Z}$-torsion, this map is injective. The following proposition, though easy to show, was not found in the literature. We prove it here for clarity.

Proposition 2.7. *For any ring $\mathcal{Q}$ such that $\mathbb{Z} \subseteq \mathcal{Q} \subseteq \mathbb{Q}$,*

$$\mathcal{B}\ell_K^{\mathbb{Z}}(x, x) = \mathcal{B}\ell_K^{\mathbb{Z}}(y, y) \iff \mathcal{B}\ell_K^{\mathcal{Q}}(x \otimes 1, x \otimes 1) = \mathcal{B}\ell_K^{\mathcal{Q}}(y \otimes 1, y \otimes 1).$$

Proof. Notice that the field of fractions of both $\mathbb{Z}[t, t^{-1}]$ and $\mathcal{Q}[t, t^{-1}]$ is $\mathbb{Q}(t)$ and the ring monomorphism $\mathbb{Z} \hookrightarrow \mathcal{Q}$ induces the following $\mathbb{Z}[t,t^{-1}]$-module homomorphisms.

$$\begin{aligned} h: &\quad \mathbb{Q}(t) \hookrightarrow \mathbb{Q}(t) \\ \overline{h}: &\quad \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}] \to \mathbb{Q}(t)/\mathcal{Q}[t,t^{-1}] \\ h_*: &\quad \mathcal{A}^{\mathbb{Z}}(K) \to \mathcal{A}^{\mathcal{Q}}(K) \cong \mathcal{A}^{\mathbb{Z}}(K) \otimes_{\mathbb{Z}} \mathcal{Q} \end{aligned}$$

The first map is the identity and the third is equivalent to the map of (2.3). Given any element $z \in \mathcal{A}^{\mathbb{Z}}(K)$, we have

$$\mathcal{B}\ell_K^{\mathcal{Q}}(z \otimes 1, z \otimes 1) = \overline{h}(\mathcal{B}\ell_K^{\mathbb{Z}}(z,z))$$

[Lei06, Theorem 4.7].

From here, the $\implies$ direction is obvious. We prove the $\impliedby$ direction by contradiction. Suppose

$$\mathcal{B}\ell^{\mathbb{Z}}(x,x) - \mathcal{B}\ell^{\mathbb{Z}}(y,y) = \frac{p(t)}{\delta_K(t)} \in \mathbb{Q}(t) \mod \mathbb{Z}[t,t^{-1}]$$

where $(p(t), \delta_K(t)) = 1$ and $\delta_K(t)$ divides $\Delta_K(t)$. If

$$\mathcal{B}\ell^{\mathcal{Q}}(x \otimes 1, x \otimes 1) - \mathcal{B}\ell^{\mathcal{Q}}(y \otimes 1, y \otimes 1) = 0,$$

this implies

$$\overline{h}\left(\frac{p(t)}{\delta_K(t)}\right) = 0.$$

The map $\overline{h}$ is given by modding out by the subring $\mathcal{Q}[t,t^{-1}]/\mathbb{Z}[t,t^{-1}] \subset \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$. This means $\frac{p(t)}{\delta_K(t)}$ reduces to a polynomial $F(t) \in \mathcal{Q}[t,t^{-1}]$. After multiplying through by some constant $q \in \mathbb{Z}$ that is a unit in $\mathcal{Q}$, we obtain the following equation in $\mathbb{Z}[t,t^{-1}]$:

$$q \cdot p(t) = f(t)\delta_K(t),$$

where $q \cdot F(t) = f(t) \in \mathbb{Z}[t, t^{-1}]$. Since $\delta_K(1) = \pm 1$ and by regarding q as a constant polynomial in $\mathbb{Z}[t, t^{-1}]$, q and $\delta_K(t)$ are coprime. Hence q divides $f(t)$, and

$$\frac{p(t)}{\delta_K(t)} = \frac{f(t)}{q} \in \mathbb{Z}[t, t^{-1}].$$

This implies $\mathcal{B}\ell^{\mathbb{Z}}(x, x) - \mathcal{B}\ell^{\mathbb{Z}}(y, y) \in \mathbb{Z}[t, t^{-1}]$. $\square$

As an important implication of Proposition 2.7, we are free to suppress the distinction between the integral and rational Blanchfield forms in comparing the Blanchfield linking form of two elements with themselves. We will frequently pass between the two and, by an abuse of notation, allow $\mathcal{B}\ell_K(x, x)$ to identify both $\mathcal{B}\ell_K^{\mathbb{Z}}(x, x)$ and $\mathcal{B}\ell_K^{\mathbb{Q}}(x \otimes 1, x \otimes 1)$ where understood.

2.2 Higher-Order Invariants

Although the classical invariants provide the motivation for our study, they will not be sufficient to prove that two satellites $\mathcal{R}(\eta_1, J)$ and $\mathcal{R}(\eta_2, J)$ represent distinct concordance classes. If η_1 and η_2 have distinct orders as elements of the Alexander module $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$, the situation is easier and treated in Chapter 3. Our main results, however, apply even while η_1 and η_2 generate the same submodule of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ (Chapter 4) and sometimes while η_1 and η_2 are equivalent as elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ (Chapter 5). Obstructions are ultimately found using a version of the Blanchfield form generalized to understand the structure of higher-order Alexander modules.

2.2.1 Commutator Series and Localizations of Rings

Before delving into the higher-order invariants and methods necessary to our proofs, it will first be beneficial to establish some basic results and definitions of group and ring theory which have important applications to our work.

Definition 2.8 ([CHL10, Definition 2.1]). A commutator series is a function $*$ which assigns to each group G a sequence of normal subgroups

$$\cdots \lhd G_*^{(n+1)} \lhd G_*^{(n)} \lhd \cdots \lhd G_*^{(0)} = G,$$

such that $G_*^{(n)}/G_*^{(n+1)}$ is a torsion-free abelian group for each n. The commutator series is weakly functorial if for any homomorphism $f : A \to B$ inducing an isomorphism on $H_1(-;\mathbb{Q})$, $f(A_*^{(n)}) \subset B_*^{(n)}$ for each $n \geq 0$. More generally, if $G_*^{(n)}$ is only defined for $0 \leq n \leq N$, then $*$ is called a partial commutator series.

One example of a commutator series is the rational derived series, defined recursively by

$$G_r^{(n+1)} = \left\{x \in G_r^{(n)} | x^k \in \left[G_r^{(n)}, G_r^{(n)}\right] \text{ for some } k \in \mathbb{Z}\right\}. \tag{2.4}$$

Note that for any commutator series $*$, $G_r^{(n)} \subset G_*^{(n)}$ for all n. Similarly, the canonical epimorphism $G_*^{(n)} \to G_*^{(n)}/G_*^{(n+1)}$ must factor through

$$G_*^{(n)} \xrightarrow{\phi_n} \frac{G_*^{(n)}/[G_*^{(n)}, G_*^{(n)}]}{\mathbb{Z}-\text{torsion}} \xrightarrow{\sigma_n} \frac{G_*^{(n)}}{G_*^{(n+1)}}$$

where the first map quotients out by

$$\left\{x \in G_*^{(n)} | x^k \in \left[G_*^{(n)}, G_*^{(n)}\right] \text{ for some } k \in \mathbb{Z}\right\}.$$

Hence, one may recursively define a commutator series by the kernels of σ_n. An important property of these commutator series is the following definition.

Definition 2.9. A group Γ is said to be poly-(torsion-free abelian) if it admits a subnormal series

$$\{e\} = \Gamma^n \lhd \Gamma^{n-1} \lhd \cdots \lhd \Gamma^0 = \Gamma$$

such that each quotient group G^i/G^{i+1} is torsion-free abelian.

Note that by Definition 2.8, for any G and and any commutator series $*$, $\Gamma_k = G/G_*^k$ is poly-(torsion-free abelian).

We are able to distinguish higher-order (localized) Alexander modules and Blanchfield forms (Subsection 2.2.2) by carefully choosing the kernels of these σ_n. This will establish a weakly functorial commutator series which will refine the n-solvable filtration of the knot concordance group (Subsection 2.2.3).

Let S be a multiplicative subset of a domain R. Then S is called a right divisor set if it satisfies the right Ore condition. That is, given $r \in R$ and $s \in S$, there exists $r' \in R$, $s' \in S$ such that $sr' = rs'$, i.e. $sR \cap rS \neq 0$. When S is a right divisor set, one may define the localization of R at S as $R_S = RS^{-1}$. R_S is uniquely determined by the homomorphism $\phi : R \to R_S$ which satisfies

1. $\phi(s)$ is invertible in R_S for every $s \in S$, and

2. every element of R_S has the form rs^{-1} for some $r \in R, s \in S$

[Coh00, Proposition 5.3]. More generally, if $S = R^\times$ is a right divisor set, R is called a right Ore domain and has a classical right ring of fractions.

One may use the localization of rings in order to recursively define a commutator series. Suppose a partial commutator series $G_*^{(k)}$ has been defined for all $k \leq n$. Then

$G/G_*^{(n)}$ is poly-(torsion-free abelian) and $\mathbb{Q}\left[G/G_*^{(n)}\right]$ is an Ore domain [COT03]. Suppose S_n is a right divisor set of $\mathbb{Q}\left[G/G_*^{(n)}\right]$. Note that

$$\frac{G_*^{(n)}}{\left[G_*^{(n)}, G_*^{(n)}\right]}$$

is a right $\mathbb{Z}[G/G_*^{(n)}]$-module with the action induced by conjugation. Define

$$G_*^{(n+1)} = \ker\left\{G_*^{(n)} \to \frac{G_*^{(n)}}{\left[G_*^{(n)}, G_*^{(n)}\right]} \underset{\mathbb{Z}[G/G_*^{(n)}]}{\otimes} \mathbb{Q}\left[G/G_*^{(n)}\right] S_n^{-1}\right\}. \tag{2.5}$$

It follows that $G_*^{(n)}/G_*^{(n+1)}$ is torsion-free abelian, and $G_*^{(n+1)}$ extends the partial commutator series $*$. Finding an appropriate right divisor set in $\mathbb{Q}[G/G_*^{(n)}]$ becomes an important objective. We will make frequent use of the following proposition.

Proposition 2.10 ([CHL10, Proposition 4.1])**.** *Suppose $A \triangleleft G$ where $\mathbb{Q}A$ is a domain and S is a right divisor set of $\mathbb{Q}A$ which is G-invariant. Then S is a right divisor set of $\mathbb{Q}G$.*

This weak functoriality of the commutator series is determined by the choice of right divisor sets S_i. That is, suppose $f : A \to B$ is a group homomorphism inducing an isomorphism on rational homology and $f(S_i(A)) \subseteq S_i(B)$ for $0 \leq i \leq n$. Then $f(A_*^{(n+1)}) \subseteq B_*^{(n+1)}$ [CHL10, Proposition 3.2].

2.2.2 Higher-Order Analogues of the Alexander Module and Blanchfield Form

The classical knot invariants, the Alexander module and Blanchfield form, provide many details on the structure of the knot concordance group. We remark that the classical Alexander module as defined in 2.1.1 may be described via the fundamental group of the knot exterior as

$$\frac{\pi_1(S^3 \setminus K)^{(1)}}{\pi_1(S^3 \setminus K)^{(2)}} \text{ as a } \mathbb{Z}\left[\frac{\pi_1(S^3 \setminus K)}{\pi_1(S^3 \setminus K)^{(1)}}\right] \text{ module.}$$

Much information is lost, however, while restricting to these abelian invariants, and one may recover a great amount of information by accounting for deeper noncommutative structure in $\pi_1(S^3 \setminus K)$ beyond the second term of its derived series.

Suppose $\pi_1(X) \xrightarrow{\phi} \Gamma$ where Γ is poly-(torsion-free abelian). Associated to the kernel of ϕ is a regular cover of X, denoted X_Γ, and Γ is the group of deck translations. The homology of X_Γ has the structure of a right $\mathbb{Z}\Gamma$-module. When ϕ is surjective, X_Γ is connected and the module structure is given by

$$h \cdot g \mapsto g^{-1}hg$$

for $g \in \Gamma$ and $h \in \pi_1(X_\Gamma) = \ker \phi$. Thus, $H_1(X_\Gamma; \mathbb{Z})$ is given by $\ker(\phi)/\ker(\phi)^{(1)}$ and has a $\mathbb{Z}[\Gamma]$-module structure. We define

$$H_1(X; \mathbb{Z}\Gamma) \equiv H_1(X_\Gamma; \mathbb{Z}).$$

Frequently, Γ is taken to be $\pi_1(X)/\pi_1(X)^{(n)}_*$ for $n \geq 0$ and some commutator series $*$ as in Definition 2.8. Note that when $n = 1$ and $*$ is the derived series, this is simply the classical Alexander module.

Definition 2.11 ([Coc04, Definition 2.6]). The n^{th} integral higher-order Alexander module, $\mathcal{A}^{\mathbb{Z}}_n(K)$, $n \geq 0$, of a knot K is the first integral homology of the cover of $S^3 \setminus K$ corresponding to the kernel of

$$\phi_n : \pi_1(S^3 \setminus K) \to \Gamma_n = \frac{\pi_1(S^3 \setminus K)}{\pi_1(S^3 \setminus K)^{(n+1)}}$$

considered as a right $\mathbb{Z}\Gamma_n$-module.

$$\mathcal{A}_n^{\mathbb{Z}}(K) \cong H_1\left(\left(S^3 \setminus K\right)_{\Gamma_n}; \mathbb{Z}\right)$$

Higher order Alexander modules provide great clarity in understanding knot concordance. In fact, $\mathcal{A}_n^{\mathbb{Z}}(K)$ is a torsion $\mathbb{Z}\Gamma_n$-module for every $n \geq 0$ and is nontrivial whenever $\Delta_K(t) \neq 1$ [Coc04, Theorem 3.1, Corollary 4.8]. These modules also behave nicely under satellite operations. This behavior allows us to find knots K and K' such that $\mathcal{A}_k^{\mathbb{Z}}(K) \cong \mathcal{A}_k^{\mathbb{Z}}(K')$ for each $0 \leq k < n$ but which are distinguished by $\mathcal{A}_n^{\mathbb{Z}}$.

Proposition 2.12. *Suppose $\mathcal{R}$ is a knot and η is an unknotted circle in $\pi_1(S^3 \setminus \mathcal{R})^{(n)}$. Then the k^{th} higher-order Alexander module of $\mathcal{R}(\eta, J)$ is given by*

$$\mathcal{A}_k^{\mathbb{Z}}(\mathcal{R}(\eta, J)) \cong \begin{cases} \mathcal{A}_k^{\mathbb{Z}}(\mathcal{R}) & 0 \leq k < n \\ \mathcal{A}_k^{\mathbb{Z}}(\mathcal{R}) \oplus \left(\mathcal{A}_0^{\mathbb{Z}}(J) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}\Gamma_n\right) & k = n \end{cases}$$

where $\Gamma_n = \pi_1(M_{\mathcal{R}})/\pi_1(M_{\mathcal{R}})^{(n+1)}$ and $\mathbb{Z}\Gamma_n$ is considered as a left $\mathbb{Z}[t, t^{-1}]$-module via the action $t \mapsto \eta$.

These modules are not typically finitely generated and the noncommutativity of $\mathbb{Z}\Gamma_n$ provides an added layer of difficulty. In order to distinguish these invariants, it becomes necessary to define a localized version of the higher order Alexander modules. Let $\mathcal{K}\Gamma_n$ be the field of fractions of $\mathbb{Q}\Gamma_n$, and suppose P_n is a localization of $\mathbb{Q}\Gamma_n$ given by $\mathbb{Q}\Gamma_n S^{-1}$ for some right divisor set $S \subset \mathbb{Q}\Gamma_n$. So $\mathbb{Q}\Gamma_n \subset P_n \subset \mathcal{K}\Gamma_n$.

Definition 2.13 ([Coc04, Definitions 3.5, 4.1])**.** An n^{th}-order "localized" Alexander module of a knot K is the homology of $S^3 \setminus K$ with coefficients induced by the map $\pi_1(S^3 \setminus K) \to \mathbb{Z}\Gamma_n \to P_n$. That is,

$$\mathcal{A}^{P_n}(K) = H_1(E(K); P_n) = H_1\left(C_*\left((S^3 \setminus K)_{\Gamma_n}; \mathbb{Z}\right) \otimes_{\mathbb{Z}\Gamma_n} P_n\right),$$

and $\mathcal{A}^{P_n}(K)$ may be regarded as a finitely-generated torsion module over P_n.

Note that these are not the localized Alexander modules of [Coc04, COT03] since here P_n is taken to be an arbitrary localization of $\mathbb{Q}\Gamma_n$ and is not necessarily a principal ideal domain. However P_n is a flat left $\mathbb{Z}\Gamma_n$-module [Ste75, Proposition 3.4], and hence, $H_1(S^3 \setminus K; P_n) \cong \mathcal{A}^{\mathbb{Z}} \otimes_{\mathbb{Z}\Gamma_n} P_n$ [Coc04, Proposition 4.4].

Although a knot K is determined by its exterior [GL89], it is often convenient to work with closed manifolds . Hence, we will often study the 3-manifold obtained by zero-framed surgery on K. This distinction could have an adverse affect on the homology groups presented. Note that the kernel of the map

$$\pi_1(S^3 \setminus K) \to \pi_1(M_K)$$

is normally generated by the longitude of K, which lies in the second term of the derived series of $\pi_1(S^3 \setminus K)$. Hence, as long as $\pi_1(M_K) \to \Gamma$ factors through

$$\pi_1(M_K) \to \pi_1(M_K)/\pi_1(M_K)^{(1)} \to \Gamma,$$

we have the following proposition, which is a slight generalization of [Lei06, Proposition 6.1] and [Coc04, Lemma 8.3].

Proposition 2.14. *Suppose $\pi_1(S^3 \setminus K) \to \Gamma$ is a map where Γ is torsion-free and which factors nontrivially through $\pi_1(S^3 \setminus K)/\pi_1(S^3 \setminus K)^{(1)}$. Then*

$$H_1(M_K; \mathbb{Z}\Gamma) \cong H_1(S^3 \setminus K; \mathbb{Z}\Gamma).$$

Proof. Since the kernel of $\pi_1(S^3 \setminus K) \to \Gamma$ contains $\pi_1(S^3 \setminus K)^{(1)}$, its image is cyclic generated by $\mu(K)$, and the image of $\pi_1(\partial S^3 \setminus K)$ in Γ is the same as the image of

$\pi_1(S^3 \setminus K)$. Thus, $i : \partial S^3 \setminus K \to S^3 \setminus K$ induces an isomorphism on $H_0(-;\mathbb{Z}\Gamma)$ by [Coc04, Proposition 3.7]. Consider the Mayer Vietoris sequence

$$H_1(M_K;\mathbb{Z}\Gamma) \xrightarrow{\partial_*} H_0(\partial S^3 \setminus K;\mathbb{Z}\Gamma) \to H_0(S^3 \setminus K;\mathbb{Z}\Gamma) \oplus H_0(D^2 \times S^2;\mathbb{Z}\Gamma).$$

Here, ∂_* must be the trivial map. Since $\pi_1(S^3 \setminus K) \to \Gamma$ is nontrivial, the image of $\mu(K)$ is nonzero and $\mu(K)$ is "unwound" in the induced Γ cover of $\partial S^3 \setminus K$ and $H_1(\partial S^3 \setminus K;\mathbb{Z}\Gamma)$ is generated by $\lambda(K)$. Moreover, $\lambda(K)$ bounds a surface in $S^3 \setminus K$ and this surface lifts to the Γ cover since each curve on the surface lies in $\pi_1(S^3 \setminus K)^{(1)}$. Hence, $i : \partial S^3 \setminus K \to S^3 \setminus K$ induces the zero map on $H_1(-;\mathbb{Z}\Gamma)$. Consider the following portion of the Mayer Vietoris sequence

$$H_1(\partial S^3 \setminus K;\mathbb{Z}\Gamma) \xrightarrow{(i_*,j_*)} H_1(S^3 \setminus K;\mathbb{Z}\Gamma) \oplus H_1(D^2 \times S^1;\mathbb{Z}\Gamma) \xrightarrow{i'_*+j'_*} H_1(M_K;\mathbb{Z}\Gamma).$$

Note that $H_1(D^2 \times S^1;\mathbb{Z})$ is generated by $\mu(K)$ and we've seen that $\mu(K)$ is unwound in the Γ cover, so $H_1(D^2 \times S^1;\mathbb{Z}\Gamma) = 0$ and $H_1(S^3 \setminus K;\mathbb{Z}\Gamma) \xrightarrow{j'_*} H_1(M_K;\mathbb{Z}\Gamma)$ is an isomorphism. $\square$

The classical Blanchfield linking form generalizes to symmetric linking forms on the localized higher-order Alexander modules. Recall that a linking form λ on $\mathcal{A}$ is <u>symmetric</u> if $\mathcal{A}$ is a torsion P-module and

$$\lambda : \mathcal{A} \to \overline{\mathrm{Hom}_P(\mathcal{A};\mathcal{K}P/P)},$$

is a P-module map with $\lambda(x,y) = \overline{\lambda(y,x)}$ and where $\overline{M}$ denotes the right P-module obtained from the left P-module M given by involution on P. The linking form is said to be nonsingular when λ is an isomorphism.

Theorem 2.15 ([COT03]). *Suppose M is a compact, oriented, connected 3-manifold with $\beta_1(M) = 1$, $\phi : \pi_1(M) \to \Gamma$ is a nontrivial coefficient system, and Γ is poly-(torsion-free abelian). Suppose P is a ring with involution extending that of $\mathbb{Z}\Gamma$ such that $\mathbb{Z}\Gamma \subseteq P \subseteq \mathcal{K}\Gamma$. Then there exists a symmetric linking form*

$$\mathcal{B}\ell_M^P : H_1(M; P) \to \overline{Hom(H_1(M; P); \mathcal{K}\Gamma/P)} \equiv H_1(M; P)^{\#}.$$

This form is nonsingular if P is a principal ideal domain.

When M is the knot exterior or zero surgery, this linking form is called a higher-order Blanchfield linking form. As in the classical case of Subsection 2.1.1, it is defined via the composition of the following maps.

$$\begin{aligned} H_1(M; P_n) &\xrightarrow{\pi} H_1(M, \partial M; P_n) \\ &\xrightarrow{P.D.} \overline{H^2(M; P_n)} \\ &\xrightarrow{\mathcal{B}^{-1}} \overline{H^1(M; \mathcal{K}_n/P_n)} \\ &\xrightarrow{\kappa} \overline{\mathrm{Hom}_{P_n}(H_1(M; P_n); \mathcal{K}_n/P_n)} \end{aligned}$$

If $P_n = \mathbb{Z}[\pi_1(M)/\pi_1(M)^{(n)}]$, we denote $\mathcal{B}\ell_M^{P_n}$ simply by $\mathcal{B}\ell_M^n$.

Like higher-order Alexander modules, the higher-order Blanchfield forms also behave predictably under satellite operations. Suppose $\eta \in \pi_1(S^3 \setminus \mathcal{R})^{(n)}$ for some knot $\mathcal{R}$. Then $\mathcal{B}\ell_{\mathcal{R}}^k \cong \mathcal{B}\ell_{\mathcal{R}(\eta,J)}^k$ for all $0 \leq k < n$. When $k = n$, we have the following theorem.

Theorem 2.16 ([Lei06, Theorems 4.6, 4.7]). *Suppose $x_1, x_2 \in \mathcal{A}_n^{\mathbb{Z}}(\mathcal{R})$ and $y_1, y_2 \in \mathcal{A}_0^{\mathbb{Z}}(J)$, then*

$$\mathcal{B}\ell_{\mathcal{R}(\eta,J)}^n(i(x_1) + j(y_1), i(x_2) + j(y_2)) = \mathcal{B}\ell_{\mathcal{R}}^n(x_1, x_2) + \overline{j}\left(\mathcal{B}\ell_J^0(y_1, y_2)\right).$$

Here $i : \mathcal{A}_n^{\mathbb{Z}}(\mathcal{R}) \hookrightarrow \mathcal{A}_n^{\mathbb{Z}}(\mathcal{R}(\eta, J))$, and $j : \mathcal{A}_0^{\mathbb{Z}}(E(J)) \to \mathcal{A}_n^{\mathbb{Z}}(\mathcal{R}(\eta, J))$. Inclusion induces the map

$$\bar{j} : \frac{\mathbb{Q}(t)}{\mathbb{Z}[t, t^{-1}]} \to \frac{\mathcal{K}\Gamma_n}{\mathbb{Z}\Gamma_n}$$

given by $t \mapsto \eta$.

2.2.3 The n-Solvable Filtration of Cochran-Orr-Teichner

Our results build upon and help clarify the structure of the n-solvable filtration of the knot concordance group.

Definition 2.17 ([COT03, Definition 1.2]). A knot K is n-solvable if there exists a 4-manifold V with boundary $\partial V = M_K$ such that the following hold.

1. Inclusion induces an isomorphism $H_1(M_K; \mathbb{Z}) \xrightarrow{\cong} H_1(V; \mathbb{Z})$.

2. There exists a basis for $H_2(V; \mathbb{Z})$, $\{L_i, D_j | i, j = 1, \ldots, r\}$, consisting of compact, connected, embedded surfaces with trivial normal bundles which are pairwise disjoint, except that for each i, L_i intersects D_i transversely once with positive sign.

3. Inclusion induces $\pi_1(L_i) \to \pi_1(V)^{(n)}$ and $\pi_1(D_i) \to \pi_1(V)^{(n)}$.

The knot is $n.5$-solvable if, in addition,

4. $\pi_1(L_i) \to \pi_1(V)^{(n+1)}$.

V is called the n-solution (respectively the $n.5$-solution) for K. The subset of $\mathcal{C}$ consisting of all n-solvable knots is denoted $\mathcal{F}_n$.

If instead of the usual derived series, we employ a weakly functorial commutator series $*$ on $\pi_1(V)$ and property 3 (and 4) holds for $\pi_1(V)_*^{(n)}$, we say that K is $(n,*)$-solvable (respectively $(n.5,*)$-solvable) [CHL10, Definition 2.1]. The set of $(n,*)$-solvable knots is denoted by $\mathcal{F}_n^*$. These definitions induce a filtration on the concordance group indexed by half integers, where $\mathcal{F}_n \subset \mathcal{F}_n^*$ for each nonnegative $n \in \frac{1}{2}\mathbb{Z}$ [CHL10, Proposition 2.5]. That is,

$$0 \subset \bigcap \mathcal{F}_n^* \subset \cdots \subset \mathcal{F}_{n.5}^* \subset \mathcal{F}_n^* \subset \cdots \subset \mathcal{F}_1^* \subset \mathcal{F}_{0.5}^* \subset \mathcal{F}_0^* \subset \mathcal{C}.$$

Many recent results in concordance have relied on this filtration[COT03, CHL10, CHL11, CHL09], and the n-solvable filtration contains many previous concordance results as well. Knots which are 0-solvable are precisely those which have Arf-invariant zero, 0.5-solvable knots are algebraically slice, and any topologically slice knot is in $\mathcal{F}_n$ for every $n \geq 0$. Furthermore, knots in $\mathcal{F}_{1.5}$ have vanishing Casson-Gordon invariants [COT03].

The following theorem shows how satellite operations affect the n-solvable filtration and will be integral to later results.

Theorem 2.18 ([CHL10, Proposition 2.7])**.** *Suppose $J \in \mathcal{F}_n$, $\mathcal{R}$ is a ribbon knot, and $\eta \subset S^3 \backslash \mathcal{R}$ is an unknotted curve. If $\eta \in \pi_1(M_{\mathcal{R}})_*^{(k)}$, then $\mathcal{R}(\eta, J)$ is $(n+k,*)$-solvable.*

2.2.4 Cheeger-Gromov Constants and the von Neumann ρ-Invariant

The definition of an n-solvable knot relies heavily on properties of the putative n-solution V. Therefore, we must look to invariants associated to the 4-manifold V in order to obstruct $n.5$-solvability and hence sliceness.

Given a compact orientable 4-manifold X with boundary $\partial X = M_K$, let $\Phi : \pi_1(X) \to \Lambda$ be a coefficient system such that Λ is a poly-(torsion-free abelian) group. If $\partial(X, \Phi) = (M_K, \phi)$, Cheeger and Gromov studied the ρ-invariant, denoted $\rho(M_K, \phi)$, associated to Φ and showed that it is equal to the "von Neumann signature defect" [CG85],

$$\rho(M_K, \phi) = \sigma^{(2)}_\Lambda(X, \Phi) - \sigma(X).$$

In this equation, $\sigma^{(2)}_\Lambda(X, \Phi)$ is the $L^{(2)}$-signature of the equivariant intersection form defined on $H_2(X; \mathbb{Z}\Lambda)$ twisted by Φ, and $\sigma(X)$ is the ordinary signature (See [COT03, Section 5]).

Proposition 2.19 ([CHL11, Proposition 4.1, Theorem 4.2])**.**

1. *If ϕ factors through $\phi' : \pi_1(M_K) \to \Lambda'$ where Λ' is a subgroup of Λ, then $\rho(M_K, \phi') = \rho(M_K, \phi)$.*

2. *If Φ is trivial on the restriction to $M_K \subset \partial X$, then $\rho(M_K, \phi) = 0$.*

3. *If $\phi : \pi_1(M_K) \to \mathbb{Z}$ is the abelianization homomorphism, then $\rho(M_K, \phi)$ is denoted by $\rho_0(K)$ and is equal to the integral of the Levine-Tristram signature function of K.*

4. *The von Neumann signature defect satisfies Novikov additivity, i.e. if X_1 and X_2 intersect along a common boundary component and Φ_i is the restriction of $\Phi : X_1 \cup X_2 \to \Lambda$ to X_i, then $\sigma^{(2)}_\Lambda(X_1 \cup X_2, \Phi) = \sigma^{(2)}_\Lambda(X_1, \Phi_1) + \sigma^{(2)}_\Lambda(X_2, \Phi_2)$.*

5. *There is a positive real number C_K called the* Cheeger-Gromov constant *of M_K such that, for any $\phi : \pi_1(M_K) \to \Lambda$, $|\rho(M_K, \phi)| < C_K$.*

6. *Let $*$ be an arbitrary commutator series and suppose $K \in \mathcal{F}^*_{n.5}$ via X with $G = \pi_1(X)$. If $\Phi : \pi_1(X) \to G/G_*^{(n+1)} = \Lambda$, then*

$$\sigma_\Lambda^{(2)}(X, \Phi) - \sigma(X) = 0 = \rho(M_K, \phi).$$

2.3 Satellite Concordance

It seems clear that satellites obtained by distinct axes should produce distinct concordance classes, but this is not always true. First, the companion knot J must satisfy some restrictions. In particular, if J is slice, $\mathcal{R}(\eta, J)$ will always be concordant to $\mathcal{R}$. The following examples illustrate the complexity of satellite concordance.

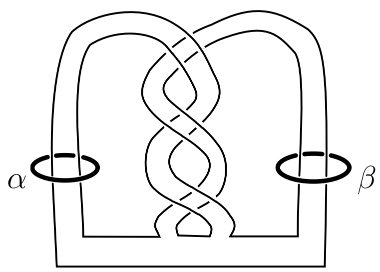


Figure 2.1: The ribbon knot of Example 2.20, $\mathcal{R}_1 = 9_{46}$ with curves α and β

Example 2.20. Take $\mathcal{R}_1 = 9_{46}$ and let α and β be the curves shown in Figure 2.1. Let J be any knot, and set $K_1 \equiv \mathcal{R}_1(\alpha, J)$ and $K_2 \equiv \mathcal{R}_1(\beta, J)$. Notice that α and β have different orders and generate different submodules as elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}_1)$. However, both α and β encircle ribbon bands of $\mathcal{R}$. If we cut along the band encircled by α in K_1 we obtain the two-component trivial link shown on the right-hand side of Figure 2.2. This proves that K_1 is ribbon and K_2 is shown to be ribbon similarly.

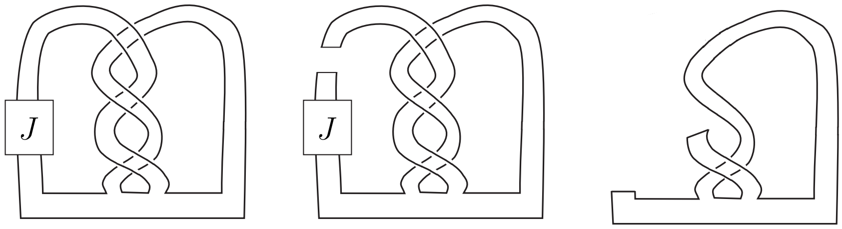


Figure 2.2: Cutting a ribbon band yields a trivial link in S^3.

In Example 2.20, α and β generate different submodules and have different orders as elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}_1)$, but both submodules are isotropic with respect to the classical Blanchfield linking form, that is $\mathcal{B}\ell_{\mathcal{R}_1}(\alpha, \alpha) = \mathcal{B}\ell_{\mathcal{R}_1}(\beta, \beta) = 0$. This motivates our inquiry into how restrictions on the Blanchfield form of the axes with themselves could obstruct concordance of satellites. These restrictions prove lucrative even if the axes generate the same submodule and have the same order in the Alexander module. The following example illustrates that the question of which η_i lead to distinct concordance classes is complicated even when the η_i do not lie in isotropic submodules.

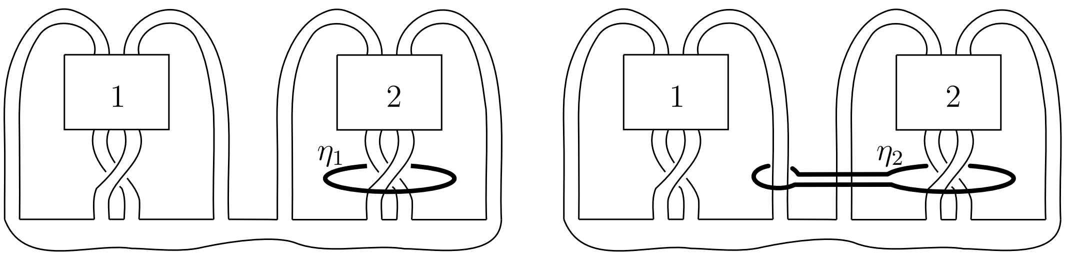


Figure 2.3: The ribbon knot of Example 2.21, $\mathcal{R}_2 = R_1 \# R_2$ and curves η_1, η_2.

Example 2.21. Let $\mathcal{R}_2 = R_1 \# R_2$ be the ribbon knot of Figure 2.3 formed by taking the connected sum of ribbon knots R_1 and R_2. The numbers 1 and 2 inside the boxes indicate 1 and 2 negative full twists respectively. Let η_1 and η_2 be the curves in

$S^3 \setminus \mathcal{R}_2$ shown on the left- and right-hand sides of Figure 2.3. Again, take J to be any knot, and set $K_1 \equiv \mathcal{R}_2(\eta_1, J)$ and $K_2 \equiv \mathcal{R}_2(\eta_2, J)$. The Alexander module of $\mathcal{R}_2$ is given by

$$\mathcal{A}^{\mathbb{Z}}(\mathcal{R}_2) = \mathcal{A}^{\mathbb{Z}}(R_1) \oplus \mathcal{A}^{\mathbb{Z}}(R_2) = \frac{\mathbb{Q}[t, t^{-1}]}{(1-2t)(2-t)} \oplus \frac{\mathbb{Q}[t, t^{-1}]}{(2-3t)(3-2t)}.$$

One easily shows that η_1 and η_2 generate different submodules of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}_2)$, and neither is isotropic with respect to the Blanchfield form. The orders of η_1 and η_2 are $(2-3t)(3-2t)$ and $(2-3t)(3-2t)(2-t)$ respectively, but a quick calcluation gives that

$$\mathcal{B}\ell_{\mathcal{R}_2}(\eta_1, \eta_1) = \mathcal{B}\ell_{\mathcal{R}_2}(\eta_2, \eta_2) = \frac{5(-1+t)^2}{6-13t+6t^2}.$$

In fact, K_1 and K_2 are concordant because the "extra band" encircled by η_2 is a ribbon band of R_1. By cutting this band in a similar process to that depicted in Figure 2.2, we see that K_1 and K_2 are concordant.

Chapter 3

Satellites Distinguished by Orders in the Alexander Module

Many previous findings on the structure of the knot concordance group, and in particular the n-solvable filtration, have been found using satellite operations, $\mathcal{R}(\eta, J)$, though hese results typically rely only on the linking number of η with $\mathcal{R}$ [CHL10, CHL11, CHL09]. In this section, by building upon previous results, we show that when η_1 and η_2 have different orders as elements of $\mathcal{A}_0^{\mathbb{Z}}(\mathcal{R})$, distinct concordance classes are easily obtained (as compared to our results in Chapters 4 and 5). Although closely related to those of [CHL10], these do not appear in the literature and are detailed here.

We suppose $\mathcal{R}$ is a ribbon knot with $\Delta_{\mathcal{R}}(t) \neq 1$. Let η_1 and η_2 be unknotted curves representing elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ with orders $o_1(t)$ and $o_2(t)$, respectively. Suppose $p(t)$ is a prime polynomial such that $p(t)$ divides $o_1(t)$ but $(o_2(t), p(t)) = (o_2(t), p(t^{-1})) = 1$.

Then define P to be the multiplicative subset of $\mathbb{Q}[t, t^{-1}]$ given by

$$P = \{q(t)|(q_i, p) = (q_i, \overline{p}) = 1\},$$

and define Ψ to be the inclusion induced homomorphism $\Psi : \mathbb{Q}[t, t^{-1}] \to \mathbb{Q}[t, t^{-1}]P^{-1}$. Then Ψ induces the map

$$\overline{\Psi} : \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]} \to \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]P^{-1}}.$$

We require $\overline{\Psi}(\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1)) \neq 0$, which yields the following theorem.

Theorem 3.1. *Let $\mathcal{R}$ be a ribbon knot with $\Delta_{\mathcal{R}}(t) \neq 1$ and J be a 0-solvable knot such that $|\rho_0(J)| > 2C_{\mathcal{R}}$. Let η_1 and η_2 be unknotted curves with $\ell k(\eta_i, \mathcal{R}) = 0$ and such that the orders of η_1,and η_2 in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ are $o_1(t)$ and $o_2(t)$ respectively. If there exists a prime $p(t)$ dividing $o_1(t)$ such that $(p(t), o_2(t)) = (\overline{p(t)}, o_2(t)) = 1$ and $\overline{\Psi}(\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1)) \neq 0$, then given any knot L which is 0-solvable, $K_1 = \mathcal{R}(\eta_1, J)$ and $K_2 = \mathcal{R}(\eta_2, L)$ represent distinct classes in $\mathcal{C}$.*

Proof. Construct K_1 and K_2 as indicated in the statement of the theorem. Certainly, both K_i are 1-solvable. Note that from the satellite operation arises a natural cobordism between zero surgeries on the knots involved. Given that $K_1 \equiv \mathcal{R}(\eta_1, J)$, denote by E_1 the cobordism obtained by first taking the disjoint union of $M_{\mathcal{R}} \times [0, 1]$ and $M_J \times [0, 1]$. Then identify a neighborhood of $\eta_1 \times \{1\}$, denoted by $\nu(\eta_1)$, in $M_{\mathcal{R}} \times \{1\}$ with $\nu(J)$, a neighborhood of $J \times \{1\}$ in $M_J \times \{1\}$ given by $(M_J \setminus (S^3 \setminus J)) \times \{1\}$ as shown in Figure 3.1. This identification is done such that the longitude of J is identified with the meridian of $\nu(\eta_1)$ and the meridian of J is identified with the reverse of the longitude of $\nu(\eta_1)$. That is,

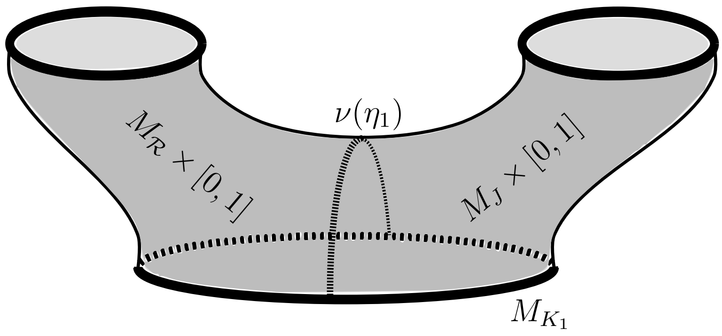


Figure 3.1: The cobordism E_1 given by the satellite operation $K_1 = \mathcal{R}(\eta_1, J)$

$$F_1 \equiv \frac{(M_{\mathcal{R}} \times [0,1]) \cup (M_J \times [0,1])}{\nu(\eta_1) \sim \nu(J)}.$$

The boundary of E_1 is then given by $\partial E_1 = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{K_1}}$, where by $\overline{X}$, we mean the manifold X with opposite orientation. Similarly, we let E_2 denote the satellite cobordism given by $K_2 = \mathcal{R}(\eta_2, L)$. Since connected sum $K_1\# - K_2$ may also be viewed as a satellite of K_1 by $-K_2$ with axis given a meridian, form a cobordism F between zero surgeries on K_1, $-K_2$, and $K_1\# - K_2$ in a similar manner. We show by contradiction that $K_1\# - K_2$ is not slice. If $K_1\# - K_2$ is slice, there exists a slice disk complement V with boundary $\partial V = M_{K_1\#-K_2}$. Let W be the manifold obtained by adjoining V to F along $M_{K_1\#-K_2}$ and similarly Z is obtained by adjoining W to E_1 and $\overline{E_2}$ along M_{K_1} and $\overline{M_{K_2}}$ respectively shown in Figure 3.2. Then $\partial Z = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{\mathcal{R}}} \sqcup \overline{M_L}$.

Take $\mathcal{P}$ to be a partial commutator series on the class of groups G with $\beta_1 = 1$, given by

$$\begin{aligned}
G_{\mathcal{P}}^{(0)} &= G \\
G_{\mathcal{P}}^{(1)} &= G_r^{(1)} \\
G_{\mathcal{P}}^{(2)} &= \ker\left\{ G_{\mathcal{P}}^{(1)} \to \tfrac{G_{\mathcal{P}}^{(1)}}{[G_{\mathcal{P}}^{(1)}, G_{\mathcal{P}}^{(1)}]} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}[t,t^{-1}]P^{-1} \right\}.
\end{aligned}$$

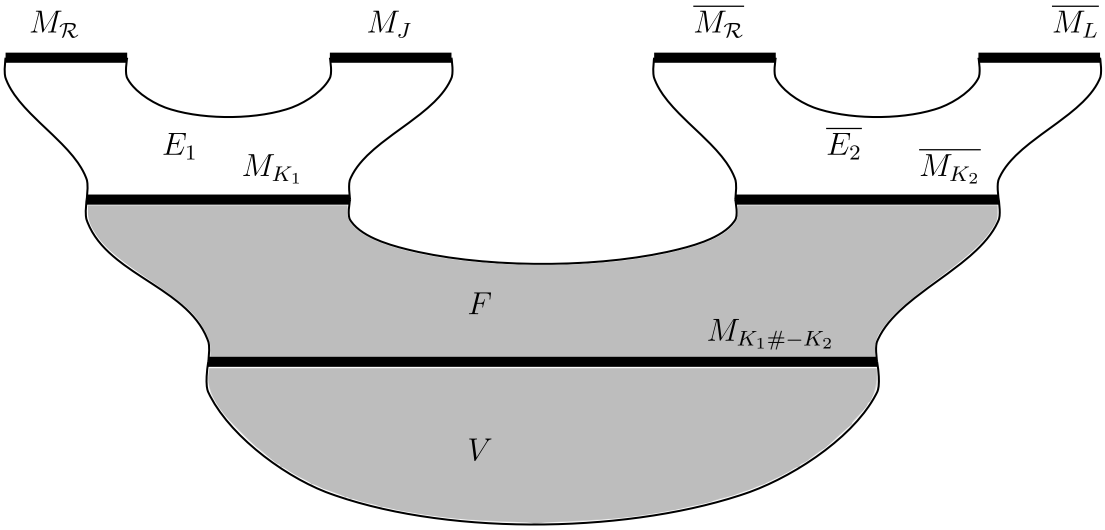


Figure 3.2: The 4-manifold Z constructed from a tower of cobordisms. The shaded region is W.

Let ϕ be the projection

$$\phi : \pi_1(Z) \to \frac{\pi_1(Z)}{\pi_1(Z)^{(2)}} \to \frac{\pi_1(Z)}{\pi_1(Z)^{(2)}_{\mathcal{P}}}.$$

We consider the von Neumann signature defect of Z given by this coefficient system. By Proposition 2.19, we have

$$\begin{aligned} 0 = \ & \sigma^{(2)}(Z,\phi) - \sigma(Z) \\ = \ & \rho(\partial Z, \phi|_{\pi_1(\partial Z)}) \\ = \ & \rho(M_{\mathcal{R}}, \phi|_{\pi_1(M_{\mathcal{R}})}) + \rho(M_J, \phi|_{\pi_1(M_J)}) + \rho(\overline{M_{\mathcal{R}}}, \phi|_{\pi_1(\overline{M_{\mathcal{R}}})}) + \rho(\overline{M_L}, \phi|_{\pi_1(\overline{M_L})}). \end{aligned} \tag{3.1}$$

We claim the restriction of ϕ to $\pi_1(M_J)$ factors nontrivially through $\mathbb{Z}$, and the restriction to $\pi_1(\overline{M_L})$ is trivial, yielding

$$\rho(M_J, \phi|_{\pi_1(M_J)}) = \rho_0(J), \text{ and } \rho(\overline{M_L}, \phi|_{\pi_1(\overline{M_L})}) = 0.$$

We first show that the restriction of ϕ to $\pi_1(\overline{M_L})$ is trivial. Since $\pi_1(\overline{M_L})$ is normally generated by its meridian which is isotopic in Z to η_2, it suffices to show that η_2 is trivial in $\pi_1(Z)^{(1)}/\pi_1(Z)^{(2)}_{\mathcal{P}}$. For any space X, we denote by $\mathcal{A}^{\mathcal{P}}(X)$ the localized Alexander module of X, given by

$$\mathcal{A}^{\mathcal{P}}(X) \equiv \mathcal{A}^{\mathbb{Z}}(X) \otimes \mathbb{Q}[t,t^{-1}]P^{-1} \cong \frac{\pi_1(X)^{(1)}}{\pi_1(X)^{(2)}} \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}[t,t^{-1}]P^{-1}.$$

Consider the following diagram where $\phi_*, f_*, g_*, \phi'_*, f'_*$ and g'_* are all induced by inclusion and the vertical maps by projection.

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{Z}}(K_1\# -K_2) \arrow[r, "\phi_*"] \arrow[d, "\psi"'] & \mathcal{A}^{\mathbb{Z}}(V) \arrow[r, "f_*"] \arrow[d] & \mathcal{A}^{\mathbb{Z}}(W) \arrow[r, "g_*"] \arrow[d] & \mathcal{A}^{\mathbb{Z}}(Z) \arrow[r] \arrow[d] & \dfrac{\pi_1(Z)^{(1)}}{\pi_1(Z)^{(2)}_{\mathcal{P}}} \arrow[dl, "i"'] \\
\mathcal{A}^{\mathcal{P}}(K_1\# -K_2) \arrow[r, "\phi'_*"] & \mathcal{A}^{\mathcal{P}}(V) \arrow[r, "f'_*"] & \mathcal{A}^{\mathcal{P}}(W) \arrow[r, "g'_*"] & \mathcal{A}^{\mathcal{P}}(Z) &
\end{tikzcd}$$

By definition, $\pi_1(Z)^{(2)}_{\mathcal{P}}$ is the kernel of $\pi_1(Z)^{(1)} \to \mathcal{A}^{\mathcal{P}}(Z)$ and i is injective. Under the map ψ, $\eta_2 \mapsto \eta_2 \otimes 1$. Since $o_2(t)$ is relatively prime to both $p(t)$ and $p(t^{-1})$, $o_2(t) \in P$. Hence

$$\eta_2 \otimes 1 = \eta_2 \cdot o_2(t) \otimes \frac{1}{o_2(t)} = 0,$$

and η_2 is trivial in $\pi_1(Z)^{(1)}/\pi_1(Z)^{(2)}_{\mathcal{P}}$ as desired.

Next consider $\pi_1(M_J)$ which is normally generated by its meridian, $\mu(J)$, which is isotopic in Z to η_1. The kernel of ψ is the P-torsion submodule of $\mathcal{A}^{\mathbb{Z}}(K_1\# -K_2) \cong \mathcal{A}^{\mathbb{Z}}(K_1) \oplus \mathcal{A}^{\mathbb{Z}}(K_2)$. However, η_1 is $o_1(t)$-torsion, and $o_1(t) \notin P$ by definition. Therefore, $\psi(\eta_1)$ is nontrivial. Since we assumed V to be a slice disk complement for $K_1\# -K_2$, the kernel of ϕ'_* is an isotropic submodule of $\mathcal{A}^{\mathcal{P}}(K_1\# -K_2)$ with respect to the localized Blanchfield form $\mathcal{B}\ell^{\mathcal{P}}_{\mathcal{R}}$ which is given by

$$\mathcal{B}\ell^{\mathcal{P}}_{\mathcal{R}}(\psi(\eta_1), \psi(\eta_1)) = \Psi(\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1))$$

[Lei06, Theorem 4.7]. Since $\Psi(\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1)) \neq 0$ by hypothesis, η_1 must survive in $\mathcal{A}^{\mathcal{P}}(V)$.

The kernels of both $\pi_1(V) \to \pi_1(W)$ and $\pi_1(W) \to \pi_1(Z)$ are normally generated by longitudes of the companion knots. These lie in the second term of the derived series of $\pi_1(V)$ and $\pi_1(W)$ and therefore in $\pi_1(V)^{(2)}_{\mathcal{P}}$ and $\pi_1(W)^{(2)}_{\mathcal{P}}$. Hence, inclusion induces the isomorphisms

$$\mathcal{A}^{\mathcal{P}}(V) \cong \mathcal{A}^{\mathcal{P}}(W) \cong \mathcal{A}^{\mathcal{P}}(Z),$$

and $g'_* \circ f'_*$ is injective. This implies $\mu(J)$ represents a nontrivial element of

$$\pi_1(Z)^{(1)}/\pi_1(Z)^{(2)}_{\mathcal{P}},$$

and the map

$$\phi : \pi_1(M_J) \to \frac{\pi_1(Z)}{\pi_1(Z)^{(2)}_{\mathcal{P}}}$$

must factor through $\pi_1(M_J)/\pi_1(M_J)^{(1)} \cong \mathbb{Z}$. Therefore, $\rho(M_J, \phi) = \rho_0(J)$, and equation 3.1 reduces to

$$\rho_0(J) = -\rho(M_{\mathcal{R}}, \phi|_{\pi_1(M_{\mathcal{R}})}) - \rho(\overline{M_{\mathcal{R}}}, \phi|_{\pi_1(\overline{M_{\mathcal{R}}})}) \leq 2C_{\mathcal{R}}.$$

By hypothesis $|\rho_0(J)| > 2C_{\mathcal{R}}$ yielding the desired contradiction. $\square$

Chapter 4

Satellites Distinguished by Classical Blanchfield Form

In this chapter, we go further by distinguishing concordance of satellites $\mathcal{R}(\eta_1, J)$ and $\mathcal{R}(\eta_2, J)$ when η_1 and η_2 have the same order in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. We see that a sufficient condition to distinguish concordance classes is provided by the value of the classical Blanchfield form of the η_i with themselves, $\mathcal{B}\ell_{\mathcal{R}}(\eta_i, \eta_i)$.

We suppose $\mathcal{R}$ is a ribbon knot with nontrivial Alexander polynomial and η_1 and η_2 are unknotted curves in $S^3 \setminus \mathcal{R}$ with $\ell k(\eta_i, \mathcal{R}) = 0$ and such that $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq \mathcal{B}\ell_{\mathcal{R}}(\eta_2, \eta_2)$ when viewed as elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. The companions J and L will be chosen to be 1-solvable knots which may or may not be distinct. Then we define $K_1 = \mathcal{R}(\eta_1, J)$, and $K_2 = \mathcal{R}(\eta_2, L)$. Under certain conditions for J, and L, we show that K_1 and K_2 are not concordant. Since both J and L are 1-solvable, by Theorem 2.18, both K_i will lie in $\mathcal{F}_2$. We show that $K_1\# -K_2$ is not slice by showing that it is not $(2.5, \mathcal{S})$-solvable where $\mathcal{S}$ is a commutator series defined in Definition 4.6.

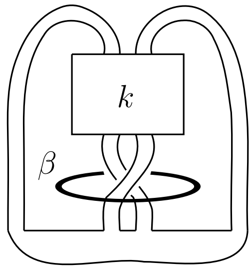


Figure 4.1: The ribbon knot R_k and axis β

Property 6 of Proposition 2.19 is integral to providing obstructions to solvability. If we assume V is a (2.5)-solution for $K_1\# - K_2$, and $\Phi : \pi_1(V) \to \Lambda$ is trivial on $\pi_1(V)^{(3)}$, then $\rho(M_{K_1\#-K_2}, \phi)$ is trivial.

Using properties of ρ-invariants, we make the choice of J explicit. First, J_0 will be an Arf-invariant zero knot. Take R to be a ribbon knot with nontrivial Alexander polynomial, and let β be an unknotted curve in $S^3 \setminus R$, which generates the rational Alexander module of R. An example of such a ribbon knot is shown in Figure 4.1, where the k in the box denotes k negative full twists, and $\Delta_R(t) = (kt - (k+1))\,((k+1)t - k))$. We will require that J_0 have $|\rho_0(J_0)| > C_R + 2C_{\mathcal{R}}$ where C_R and $C_{\mathcal{R}}$ are the Cheeger-Gromov constants of R and $\mathcal{R}$ respectively (properties 3 and 5 of Proposition 2.19). We define $J = R(\beta, J_0)$. By Theorem 2.18, $J \in \mathcal{F}_1$.

Choose L to be any 1-solvable knot with Alexander polynomial $\Delta_L(t)$ satisfying one of the two following conditions:

1. Δ_R and Δ_L are strongly coprime, i.e. $\Delta_R(t^n), \Delta_L(t^m)$ are relatively prime for every $n, m \in \mathbb{Z}$, or

2. $\Delta_R(t^m)$ and $\Delta_L(t^n)$ have no common roots unless $n = \pm m$.

Certainly (1) implies (2). If (1) holds, we prove that $K_1 = \mathcal{R}(\eta_1, J)$ and $K_2 = \mathcal{R}(\eta_2, L)$ are distinct (and even linearly independent) in $\mathcal{C}$ by a generalization of Cochran-Harvey-Leidy [CHL10]. If (2), a secondary restriction will be given by the Blanchfield form of the curves η_1 and η_2 with themselves. In particular, we require that $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq \mathcal{B}\ell_{\mathcal{R}}(\eta_2, \eta_2)$.

Theorem 4.1. *Let R and $\mathcal{R}$ be ribbon knots with nontrivial Alexander polynomials, and let J_0 be an Arf-invariant zero knot such that $|\rho_0(J_0)| > C_R + 2C_{\mathcal{R}}$. Suppose $J = R(\beta, J_0)$ where β generates the rational Alexander module of R. Then form $K_1 = \mathcal{R}(\eta_1, J)$ where $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq 0$ and $K_2 = \mathcal{R}(\eta_2, L)$ where L is some 1-solvable knot. Suppose that one of the following conditions hold.*

1. *$\Delta_L(t)$ and $\Delta_R(t)$ are strongly coprime, or*

2. *$\Delta_L(t^m)$ and $\Delta_R(t^n)$ share a common root only when $n = \pm m$ and $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq \mathcal{B}\ell_{\mathcal{R}}(\eta_2, \eta_2)$.*

Then K_1 and K_2 are distinct in $\mathcal{C}$. In particular, $\mathcal{R}(\eta_1, -)$ and $\mathcal{R}(\eta_2, -)$ are distinct maps on $\mathcal{C}$.

Before discussing its proof, we introduce the following corollaries which illustrate the impact of Theorem 4.1.

Corollary 4.2. *Let $\mathcal{R}$ be a ribbon knot. Suppose $J = R(\beta, J_0)$ where J_0 is an Arf-invariant zero knot and R is the ribbon knot from Figure 4.1 with β as shown. Let $K_1 = \mathcal{R}(\eta_1, J)$ and $K_2 = \mathcal{R}(\eta_2, J)$ where η_1 and η_2 are unknotted curves in the* *complement of* $\mathcal{R}$ *with* $\ell k(\eta_1, \mathcal{R}) = \ell k(\eta_2, \mathcal{R}) = 0$*. If* $|\rho_0(J_0)| > C_R + 2C_{\mathcal{R}}$ *and* $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq \mathcal{B}\ell_{\mathcal{R}}(\eta_2, \eta_2)$*, then* K_1 *and* K_2 *are not concordant.*

Proof that Theorem 4.1 implies Corollary 4.2. We assume without loss of generality $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq 0$. Since $\Delta_R(t) = \Delta_J(t) = (kt - (k+1))((k+1)t - k)$ has roots $\{\frac{k}{k+1}, \frac{k+1}{k}\}$, $\Delta_R(t^m)$ and $\Delta_R(t^n)$ share no common roots unless $n = \pm m$. The result follows from Theorem 4.1. □

The above stresses the distinction between the axes of a satellite operation and shows that given suitable choices of axes, the influence of the companion knot is less necessary to obstruct concordance. We next generalize these results to produce infinitely many distinct concordance classes.

Corollary 4.3. *Suppose* $\mathcal{R}$ *is any ribbon knot with* $\Delta_{\mathcal{R}} \neq 1$*. Then there exists a countably infinite set of curves* $\{\eta_i\}$ *in* $S^3 \setminus \mathcal{R}$ *which are unknotted in* S^3 *and have linking number* 0 *with* $\mathcal{R}$*, and also a knot* J *such that each* $K_i = \mathcal{R}(\eta_i, J)$ *represents a distinct concordance class.*

Proof. In order to employ Corollary 4.2, we must ensure the existence of an infinite family of unknotted curves $\{\eta_i\}$ which have give distinct values of $\mathcal{B}\ell_{\mathcal{R}}(\eta_i, \eta_i)$, that is $\mathcal{B}\ell_{\mathcal{R}}(\eta_i, \eta_i) = \mathcal{B}\ell_{\mathcal{R}}(\eta_j, \eta_j)$ only when $i = j$. Since $\mathcal{R}$ has nontrivial Alexander polynomial and the Blanchfield form on $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ is nonsingular, there must exist some curve $\eta \subset S^3 \setminus \mathcal{R}$ such that $\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) \neq 0$. We use the following lemma.

Lemma 4.4. *Suppose* $\eta \subset S^3 \setminus \mathcal{R}$ *is an unknotted curve in* S^3 *with* $lk(\eta, \mathcal{R}) = 0$ *and* $\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) \neq 0$*. For each* $i \in \mathbb{Z}_{\geq 0}$*, set* $\eta_i = i\eta \in \mathcal{A}^{\mathbb{Z}}(\mathcal{R})$*. Then* $\mathcal{B}\ell_{\mathcal{R}}(\eta_i, \eta_i) =$

$\mathcal{B}\ell_{\mathcal{R}}(\eta_j, \eta_j)$ only when $i = j$, and each η_i is represented by an unknotted curve in $S^3 \setminus \mathcal{R}$.

Proof. Suppose $\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) = \frac{p(t)}{\delta_{\mathcal{R}}(t)} \notin \mathbb{Z}[t, t^{-1}]$ such that $(p(t), \delta_{\mathcal{R}}(t)) = 1$ and $\delta_{\mathcal{R}}(t)$ divides $\Delta_{\mathcal{R}}(t)$. Then we have

$$\mathcal{B}\ell_{\mathcal{R}}(\eta_i, \eta_i) = \mathcal{B}\ell_{\mathcal{R}}(i\eta, i\eta) = i^2 \mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) = i^2 \frac{p(t)}{\delta_{\mathcal{R}}(t)}.$$

If $\mathcal{B}\ell_{\mathcal{R}}(\eta_i, \eta_i) = \mathcal{B}\ell_{\mathcal{R}}(\eta_j, \eta_j)$, this implies $(i^2 - j^2)\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) = f(t) \in \mathbb{Z}[t, t^{-1}]$. We have the following equation

$$(i^2 - j^2)p(t) = f(t)\delta_{\mathcal{R}}(t),$$

where since $\frac{p(t)}{\delta_{\mathcal{R}}(t)} \notin \mathbb{Z}[t, t^{-1}]$, we can assume that $i^2 - j^2$ does not divide $f(t)$ over $\mathbb{Z}[t, t^{-1}]$. Since $\delta_{\mathcal{R}}(1) = \pm 1$, $i^2 - j^2 \in \{0, \pm 1\}$. If $i^2 - j^2 = \pm 1$, this contradicts $\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) \neq 0$. As $i, j \geq 0$, $i^2 - j^2$ is zero only when $i = j$. We must next show that each η_i is unknotted in S^3. But notice that the element $i\eta \in \mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ is realized by the $(i, 1)$-cable of η. This completes the proof. □

By taking J to be the knot given in the statement of Corollary 4.2, we obtain a family of pairwise distinct concordance classes $\{K_i = \mathcal{R}(\eta_i, J)\}$ for $i \geq 0$. □

The following corollary illustrates how uncommon it is for two unknotted curves, η and γ in $S^3 \setminus \mathcal{R}$, to yield concordant knots. By viewing them as elements of $\mathcal{A}^{\mathbb{Q}}(\mathcal{R}) \cong \mathbb{Q}^d$ where $d = \deg \Delta_{\mathcal{R}}(t)$, we get an approximate answer to this question by seeing that a subset of axes $\{\gamma | \gamma \text{ is unknotted in } S^3, \ell k(\gamma, \mathcal{R}) = 0\}$ which yield knots concordant to $K = \mathcal{R}(\eta, J)$ must lie on a quadratic hypersurface in $\mathbb{Q}^d$.

Proposition 4.5. *Let $\mathcal{R}$ be a ribbon knot such that* $\deg \Delta_{\mathcal{R}} = d \neq 0$ *and* $J = R(\beta, J_0)$ *as above. Fix some unknotted curve* $\eta \subset S^3 \setminus \mathcal{R}$ *such that* $\ell k(\eta, \mathcal{R}) = 0$ *and let* $K = \mathcal{R}(\eta, J)$*. Then,*

$$\{\gamma | \mathcal{B}\ell_{\mathcal{R}}(\gamma, \gamma) = \mathcal{B}\ell_{\mathcal{R}}(\eta, \eta)\}$$

is the subset of a quadric hypersurface in $\mathbb{Q}^d$*, and thus*

$$\{\gamma | K' = \mathcal{R}(\gamma, J) \text{ is not concordant to } K\}$$

is dense as a subset of $\mathbb{Q}^d$*.*

Proof. Following work of Trotter [Tro78, Tro73], let $z = (1-t)^{-1}$ and note that $\mathbb{Q}(t) = \mathbb{Q}(z)$. Furthermore, since z gives an automorphism of $\mathcal{A}^{\mathbb{Z}}(K)$, enlarging coefficients from $\mathbb{Z}[t, t^{-1}]$ to $\mathbb{Z}[t, t^{-1}, z]$ has no effect on the module structure. Consider the map

$$\frac{\mathbb{Q}(t)}{\mathbb{Z}[t, t^{-1}]} \xrightarrow{j} \frac{\mathbb{Q}(t)}{\mathbb{Z}[t, t^{-1}, z]}$$

given by inclusion. The form given by $\widehat{\mathcal{B}\ell}(x, y) = j(\mathcal{B}\ell_{\mathcal{R}}(x, y))$ is a nonsingular sesquilinear form and j maps the image of $\mathcal{B}\ell_{\mathcal{R}}(-,-)$ one-to-one onto the image of $\widehat{\mathcal{B}\ell}(-,-)$ [Tro78].

Using a partial fraction decomposition, any element in $\mathbb{Q}(t)$ may be written uniquely as the sum of a polynomial and a proper fraction. Thus, $\mathbb{Q}(t)$ splits over $\mathbb{Q}$ as the direct sum of $\mathbb{Q}[t, t^{-1}, z]$ and a subspace P consisting of 0 and proper fractions with denominator coprime to t and $1-t$. Then we have a $\mathbb{Q}$-linear map $\chi : \mathbb{Q}(t) \to \mathbb{Q}$ defined by

$$\chi(f) = \begin{cases} f'(1) & f \in P \\ 0 & f \in \mathbb{Q}[t, t^{-1}, z] \end{cases}.$$

Since χ is trivial on $\mathbb{Q}[t, t^{-1}, z]$, it is well defined on $\mathbb{Q}(z)/\mathbb{Q}[t, t^{-1}, z]$ and thus on the image of $\widehat{\mathcal{B}\ell}$. Note that the value of $\widehat{\mathcal{B}\ell}(x, y)$ is uniquely determined by the value of $\chi(\lambda\widehat{\mathcal{B}\ell}(x, y))$ for all $\lambda \in \mathbb{Z}[t, t^{-1}, z]$, and furthermore, χ satisfies

$$\chi(\overline{f}) = \chi(f) \qquad \chi((t-1)f) = f(1)$$

for any $f \in P$ [Tro73, Section 2]. Since $\mathcal{B}\ell_{\mathcal{R}}(x, y) = \overline{\mathcal{B}\ell_{\mathcal{R}}(y, x)}$ for any $x, y \in \mathcal{A}^{\mathbb{Z}}(K)$, $\chi(\widehat{\mathcal{B}\ell}(\gamma, \gamma)) = 0$ for all γ. This is also seen by noting that a formula for $\mathcal{B}\ell_{\mathcal{R}}$ is given by [Kea78]

$$\mathcal{B}\ell_{\mathcal{R}}(x, y) = \overline{y}(1-t)\left(tV - V^{\intercal}\right)^{-1} x.$$

Since $\mathcal{B}\ell_{\mathcal{R}}$ is nonsingular, there must exist some $\lambda_0 \in \mathbb{Z}[t, t^{-1}, z]$ and some $\gamma_0 \in \mathcal{A}^{\mathbb{Z}}(K)$ such that $\chi(\lambda_0 \mathcal{B}\ell_{\mathcal{R}}(\gamma_0, \gamma_0))$ is nonzero. Define $\widehat{\chi} : \mathcal{A}^{\mathbb{Q}}(K) \cong \mathbb{Q}^d \to \mathbb{Q}$ by

$$\widehat{\chi}(\gamma) \equiv \chi(\lambda_0 \widehat{\mathcal{B}\ell}(\gamma, \gamma)).$$

Suppose $\widehat{\chi}(\eta) = c \in \mathbb{Q}$. Fix a $\mathbb{Q}$-basis $\{e_i\}$ for $\mathcal{A}^{\mathbb{Q}}(K)$ such that $\sum x_i e_i = (x_1, \ldots, x_d)$. Then $\widehat{\chi}(x_1, \ldots, x_d) = c$ is a rational equation in d variables and the left-hand side is a homogeneous polynomial of degree 2. That is,

$$\begin{aligned}
\widehat{\chi}(x_1, \ldots, x_d) &= \sum_{i,j} \chi\left(\lambda_0 \mathcal{B}\ell_{\mathcal{R}}(x_i, x_j)\right) \\
&= \sum_{i,j} x_i, x_j \chi\left(\lambda_0 \mathcal{B}\ell_{\mathcal{R}}(e_i, e_j)\right) \\
&= \sum_{i,j} a_{i,j} x_i x_j = c
\end{aligned}$$

where $a_{i,j} = \chi\left(\lambda_0 \mathcal{B}\ell_{\mathcal{R}}(e_i, e_j)\right)$. Since $\mathcal{B}\ell_{\mathcal{R}}$ is nonsingular, not all $a_{i,j} = 0$. By Theorem 4.1, the set of axes $\gamma \subset S^3 \backslash \mathcal{R}$ with $\ell k(\gamma, \mathcal{R}) = 0$ such that $K' = \mathcal{R}(\gamma, J)$ is concordant to $K = \mathcal{R}(\eta, J)$ must satisfy $\mathcal{B}\ell_{\mathcal{R}}(\gamma,\gamma) = \mathcal{B}\ell_{\mathcal{R}}(\eta,\eta)$. Therefore $\gamma = (\gamma_1, \ldots, \gamma_d)$ must be a solution to $\widehat{\chi}(x_1, \ldots, x_d) = c$.

Consider the polynomial $F(x_1, \ldots, x_d) = \widehat{\chi}(x_1, \ldots, x_d) - c = 0$. If $c \neq 0$, this polynomial is clearly nonconstant since $F(\vec{0}) = -c$. Otherwise, given the choice of λ_0 and that $\mathcal{B}\ell_{\mathcal{R}}$ is nonsingular, there exists an element $\gamma_0 \in \mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ such that $\widehat{\chi}(\gamma_0) \neq 0$. Hence $F(\gamma_0) \neq 0$ and F is a nonconstant polynomial. The zero locus of $\widehat{\chi}(x_1, \ldots, x_d) - c$ is a quadric hypersurface in $\mathbb{Q}^d$ whose compliment is dense. $\square$

In the proof Proposition 4.5, we distinguish axes γ with $\ell k(\gamma, \mathcal{R}) = 0$ by evaluating Trotter's trace function χ on $\lambda_0 \widehat{\mathcal{B}\ell_{\mathcal{R}}}(\gamma,\gamma)$ for one particular value $\lambda_0 \in \mathbb{Q}[t, t^{-1}, z]$. Since $\widehat{\mathcal{B}\ell_{\mathcal{R}}}(\gamma,\gamma)$ is uniquely determined by the value of $\chi(\lambda\widehat{\mathcal{B}\ell}(\gamma,\gamma))$ for all $\lambda \in \mathbb{Q}[t, t^{-1}, z]$, one could attempt to distinguish the curves γ and η by using multiple values of λ when $\chi(\lambda_0\widehat{\mathcal{B}\ell}(\gamma,\gamma)) = \chi(\lambda_0\widehat{\mathcal{B}\ell}(\eta,\eta))$.

We now proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. We show the stronger fact that $K_1\# - K_2$ is not 2.5-solvable by contradiction. Note that $K_1\# - K_2$ is 2-solvable by [CHL10, Proposition 2.7], and we suppose it is 2.5-solvable via V. We construct a tower of cobordisms for $M_{K_1\#-K_2}$. Let F_1 and F_2 be the satellite cobordisms corresponding to $K_1 = \mathcal{R}(\eta_1, J)$ and $K_2 = \mathcal{R}(\eta_2, J)$ respectively, and let E denote the cobordism given by the connected sum $K_1\# - K_2$. The satellite $J = R(\beta, J_0)$ will yield a cobordism denoted G. Define W' to be the union of V and E along their common boundary. Similarly, W is the union $W' \cup F_1 \cup \overline{F_2}$. Then, let Z be the manifold obtained by joining the cobordism G to W along M_J. The boundary of Z is given by $\partial Z = M_{\mathcal{R}} \sqcup M_R \sqcup M_{J_0} \sqcup \overline{M_{\mathcal{R}}} \sqcup \overline{M_L}$. A

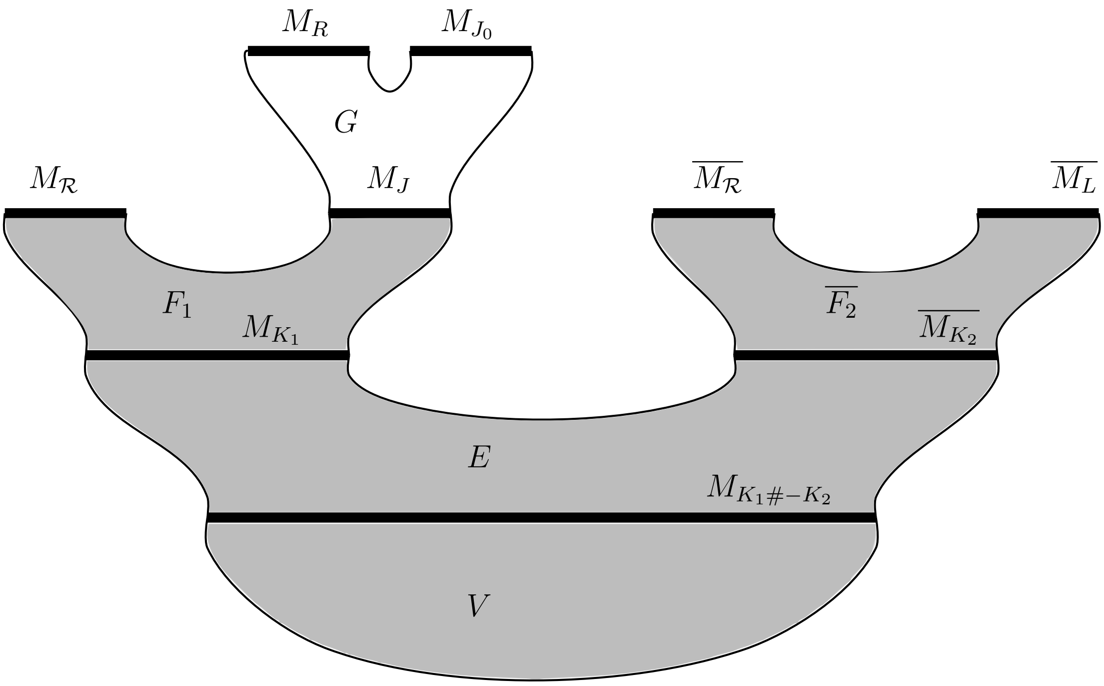


Figure 4.2: The 4-manifold Z, constructed by a tower of cobordisms

complete picture of Z is shown in Figure 4.2. In overview, we have the following.

$$\partial V = M_{K_1\#-K_2}$$

$$\partial E = M_{K_1} \sqcup M_{K_2} \sqcup \overline{M_{K_1\#-K_2}} \qquad W' = V \bigcup_{M_{K_1\#-K_2}} E$$

$$\partial F_1 = M_J \sqcup M_{\mathcal{R}} \sqcup \overline{M_{K_1}} \qquad W = W' \bigcup_{M_{K_1}} F_1 \bigcup_{\overline{M_{K_2}}} \overline{F_2}$$

$$\partial F_2 = M_L \sqcup M_{\mathcal{R}} \sqcup \overline{M_{K_2}} \qquad Z = W \bigcup_{M_J} G$$

$$\partial G = M_{J_0} \sqcup M_R \sqcup \overline{M_J}$$

Unfortunately, the derived series itself will not be useful in finding an obstruction to the 2.5-solvablity of $K_1\# - K_2$. Instead, we define a partial commutator series, $\mathcal{S}$, which will be slightly larger than the rational derived series so that

$$\pi_1(Z)^{(3)} \subset \pi_1(Z)^{(3)}_{\mathcal{S}}.$$

Notice in Definition 4.6, $\mathcal{S}$ will be equivalent to the rational derived series on the first two terms.

Definition 4.6. Let G be a group with $G/G^{(1)} = \langle \mu \rangle \cong \mathbb{Z}$ and let $\eta \in G^{(1)}/G_r^{(2)}$ and $q(t) \in \mathbb{Q}[t, t^{-1}]$. Then the derived series localized at $\mathcal{S}$ is defined recursively by

$$\begin{aligned} G_{\mathcal{S}}^{(0)} &\equiv G \\ G_{\mathcal{S}}^{(1)} = G_r^{(1)} &\equiv \ker\left(G \to \frac{G}{[G,G]} \otimes_{\mathbb{Z}} \mathbb{Q}\right) \\ G_{\mathcal{S}}^{(2)} = G_r^{(2)} &\equiv \ker\left(G_{\mathcal{S}}^{(1)} \to \frac{G_{\mathcal{S}}^{(1)}}{[G_{\mathcal{S}}^{(1)}, G_{\mathcal{S}}^{(1)}]} \otimes_{\mathbb{Z}[G/G_{\mathcal{S}}^{(1)}]} \mathbb{Q}[G/G_{\mathcal{S}}^{(1)}]\right) \\ G_{\mathcal{S}}^{(3)} &\equiv \ker\left(G_{\mathcal{S}}^{(2)} \to \frac{G_{\mathcal{S}}^{(2)}}{[G_{\mathcal{S}}^{(2)}, G_{\mathcal{S}}^{(2)}]} \otimes_{\mathbb{Z}[G/G_{\mathcal{S}}^{(2)}]} \mathbb{Q}[G/G_{\mathcal{S}}^{(2)}]S^{-1}\right). \end{aligned}$$

The set $S \subset \mathbb{Q}[G_{\mathcal{S}}^{(1)}/G_{\mathcal{S}}^{(2)}] \subset \mathbb{Q}[G/G_{\mathcal{S}}^{(2)}]$ is the multiplicative set generated by $\{q(\mu^i \eta \mu^{-i}) | i \in \mathbb{Z}, \eta \in G^{(1)}/G_r^{(2)}\}$. S is a multiplicatively closed set with unity by definition, and 0 is not an element of $\mathcal{S}$. Since $\mathbb{Q}[G^{(1)}/G_{\mathcal{S}}^{(2)}]$ is commutative, this verifies S is a right divisor set of $\mathbb{Q}[G^{(1)}/G_{\mathcal{S}}^{(2)}]$. Furthermore, let $\gamma \in G/G_{\mathcal{S}}^{(2)}$. If $q(a) \in S$, then $\gamma q(a) \gamma^{-1} = q(\gamma a \gamma^{-1}) \in S$. Therefore, S is invariant under conjugation by $\mathbb{Q}[G/G_{\mathcal{S}}^{(2)}]$. By Theorem 2.10, $\mathcal{S}$ is a right divisor set of $\mathbb{Q}[G/G_{\mathcal{S}}^{(2)}]$. In the case that $G = \pi_1(Z)$ or $G = \pi_1(W)$, we choose $q(t)$ to be $\Delta_L(t)$ and η to be η_2', the image of η_2 in $M_{-\mathcal{R}} \subset M_{\mathcal{R}\#-\mathcal{R}}$ considered as an element of $\pi_1(Z)$ or $\pi_1(W)$ by inclusion.

Consider the coefficient system on W given by the projection

$$\Phi : \pi_1(Z) \to \pi_1(Z)/\pi_1(Z)^{(3)} \to \pi_1(Z)/\pi_1(Z)_{\mathcal{S}}^{(3)} \equiv \Lambda.$$

Because of property (4) of Proposition 2.19 (and after suppressing notation by $\sigma_\Lambda^{(2)} =$

$\sigma^{(2)}$ and $\Phi|_X = \Phi$ where understood), we have

$$\begin{aligned}\sigma^{(2)}(Z,\Phi) - \sigma(Z) = \big(\sigma^{(2)}(V,\Phi) - \sigma(V)\big) + \big(\sigma^{(2)}(E,\Phi) - \sigma(E)\big)\\ + \big(\sigma^{(2)}(F_1,\Phi) - \sigma(F_1)\big) + \big(\sigma^{(2)}(\overline{F_2},\Phi) - \sigma(\overline{F_2})\big)\\ + \big(\sigma^{(2)}(G,\Phi) - \sigma(G)\big)\,.\end{aligned} \tag{4.1}$$

By assumption, V is a 2.5-solution. Property (6) of Proposition 2.19 yields $\sigma^{(2)}(V,\Phi) - \sigma(V) = 0$. For E, F_1, F_2, and G, all of the (integral and twisted) second homology comes from the boundary [CHL09, Lemma 2.4], and

$$\sigma^{(2)}(E,\Phi) - \sigma(E) = \sigma^{(2)}(F_1,\Phi) - \sigma(F_1) = \sigma^{(2)}(F_2,\Phi) - \sigma(F_2) = \sigma^{(2)}(G,\Phi) - \sigma(G) = 0.$$

However, $\sigma^{(2)}(Z,\Phi) - \sigma(Z) = \rho(\partial Z, \Phi|_\partial)$, and

$$0 = \rho(\partial Z,\Phi) = \rho(M_{J_0},\Phi) + \rho(\overline{M_L},\Phi) + \rho(M_{\mathcal{R}},\Phi) + \rho(\overline{M_{\mathcal{R}}},\Phi) + \rho(M_R,\Phi).$$

We employ the following lemmas but delay their proof.

Lemma 4.8 *The restriction of Φ to $\pi_1(M_{J_0})$ factors non-trivially through $\mathbb{Z}$.*

Lemma 4.10 *The restriction of Φ to $\pi_1(\overline{M_L})$ also factors through $\mathbb{Z}$ and yields $\rho(\overline{M_L},\Phi) = 0$.*

After proving Lemma 4.8 and using properties (1) and (3) of Proposition 2.19, we will have $\rho(M_{J_0},\Phi) = \rho_0(J_0)$. Secondly, by Lemma 4.10 and property (2) of Proposition 2.19, $\rho(\overline{M_L},\Phi) = -\rho(M_L,\Phi) = 0$. Together with equation 4.1, this yields the following equation:

$$\rho_0(J_0) = -\rho(M_{\mathcal{R}},\Phi) - \rho(M_{-\mathcal{R}},\Phi) - \rho(M_R,\Phi).$$

This is a contradiction since, by hypothesis,

$$|\rho_0(J_0)| > C_R + 2C_{\mathcal{R}} \geq |\rho(M_{\mathcal{R}},\Phi)| + |\rho(M_{-\mathcal{R}},\Phi)| + |\rho(M_R,\Phi)|.$$

This completes the proof of Theorem 4.1 modulo the proofs of Lemmas 4.8 and 4.10 □

We are now prepared to prove the lemmas needed for the completion of the proof of Theorem 4.1. Before proving Lemma 4.8, we must first show that the curve η_1 represents a nontrivial element of $\pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}}$ by inclusion. Note that $\pi_1(M_{J_0})$ is normally generated by the meridian $\mu(J)$ which is isotopic in Z to $\beta \in \pi_1(M_R)^{(1)}$. Similarly, $\pi_1(M_R)$ is normally generated by its meridian $\mu(R)$ which is identified with η_1. Inclusion induces

$$\eta_1 \in \pi_1(M_{\mathcal{R}})^{(1)} \to \pi_1(W)^{(1)} \to \pi_1(Z)^{(1)}$$

which implies that $\mu(J) \sim \beta$ is in $\pi_1(Z)^{(2)}$. If $\eta_1 \in \pi_1(Z)^{(2)}$, then $\pi_1(M_{J_0})$ is mapped to a subset of $\pi_1(Z)^{(3)}$ and the restriction of Φ to $\pi_1(M_{J_0})$ is trivial.

Continue to let $\eta_2' \subset M_{\mathcal{R}\#-\mathcal{R}}$ denote the image of η_2 after reversing the orientation of $M_{\mathcal{R}}$ and taking the connected sum to form $M_{\mathcal{R}\#-\mathcal{R}}$. By an added abuse of notation, η_1 and η_2' also represent the corresponding elements in the Alexander module and fundamental group. Let $\mathcal{A}(X)$ denote the Alexander module of the space X with rational coefficients. The following proofs closely follow the methodology of [CHL11, Lemmas 7.5, 7.6].

Lemma 4.7. *The curve η_1 represents a nontrivial element of $A \equiv \pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}}$.*

Proof. Consider the following commutative diagram of Alexander modules.

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\#-\mathcal{R}) \arrow[r, "\phi_*"] \arrow[d, "i_1"] & \mathcal{A}^{\mathbb{Z}}(V) \arrow[r, "f_*"] \arrow[d, "i_2"] & \mathcal{A}^{\mathbb{Z}}(W') \arrow[r, "g_*"] \arrow[d, "i_3"] & \mathcal{A}^{\mathbb{Z}}(W) \arrow[r] \arrow[d, "i_5"] & A \arrow[dl, "i_6"] \\
\mathcal{A}(\mathcal{R}\#-\mathcal{R}) \arrow[r, "\phi_*'"] & \mathcal{A}(V) \arrow[r, "f_*'"] & \mathcal{A}(W') \arrow[r, "g_*'"] & \mathcal{A}(W) &
\end{tikzcd} \tag{4.2}$$

The validity of this diagram is supported by the fact that $\mathcal{A}^{\mathbb{Z}}(K_1\# - K_2) \cong \mathcal{A}^{\mathbb{Z}}(\mathcal{R}\# - \mathcal{R})$. The horizontal maps are induced by inclusion. Since $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\# - \mathcal{R})$ is $\mathbb{Z}$-torsion free, i_1 is injective. By Definition 4.6, $\pi_1(W)^{(2)}_{\mathcal{S}} = \pi_1(W)^{(2)}_r$, and therefore $i_6 : \pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}} \to \mathcal{A}(W)$ is well-defined.

The kernel of ϕ'_* is an isotropic submodule of $\mathcal{A}(\mathcal{R}\# - \mathcal{R})$ with respect to the Blanchfield form. Since the rational Alexander module of $\mathcal{R}\# - \mathcal{R}$ and its Blanchfield form decompose under connected sum, $\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq 0$ implies η_1 must be mapped to a nontrivial element of $\mathcal{A}(V)$.

It remains to show that the lower maps f'_* and g'_* are injective; that is, the rational Alexander module of V injects into that of W. Since the connected sum operation may be described as the satellite operation $K_1\# - K_2 = K_1(\mu(K_1), -K_2)$, the kernel of $f'_* : \pi_1(M_{K_1\# - K_2}) = \pi_1(\partial V) \to \pi_1(E)$ is normally generated by the longitude of $-K_2$ considered as an element of $\pi_1(M_{K_1})$ [CHL09, Lemma 2.5(1)]. The longitude lies in the second derived subgroup of $\pi_1(K_2)$ and also in the second derived subroup of $\pi_1(M_{K_1\# - K_2})$. Since the rational Alexander module of a space, X, with $H_1(X) \cong \mathbb{Z}$ is given by $\mathcal{A}(X) \cong \pi_1(X)^{(1)}/\pi_1(X)^{(2)} \otimes_{\mathbb{Z}} \mathbb{Q}$, f'_* is an isomorphism between the rational Alexander modules of V and W'.

Similarly, to show g'_* is injective, we note that its kernel is normally generated by the longitudes of J and L considered as curves in M_{K_1} and $\overline{M_{K_2}}$ respectively. These lie in $\pi_1(M_J)^{(2)}$ and $\pi_1(\overline{M_L})^{(2)}$, contained via inclusion in $\pi_1(M_{K_1})^{(3)}$ and $\pi_1(\overline{M_{K_2}})^{(3)}$ respectively, and thus g'_* is an isomorphism. $\square$

For the contradiction used in the proof of Theorem 4.1, we show that $\mu(J_0) \sim \beta$

is nontrivial as an element of $\pi_1(Z)^{(2)}/\pi_1(Z)^{(3)}_{\mathcal{S}}$.

Lemma 4.8. *The meridian of J_0, $\mu(J_0)$, which is isotopic in Z to β, is nontrivial as an element of*
$$\frac{\pi_1(Z)^{(2)}}{\pi_1(Z)^{(3)}_{\mathcal{S}}}.$$
Therefore, the restriction $\Phi : \pi_1(M_{J_0}) \to \pi_1(Z)/\pi_1(Z)^{(3)}_{\mathcal{S}} = \Lambda$ factors nontrivially through $\mathbb{Z}$.

Proof. Recall that the kernel of
$$\pi_1(W) \to \pi_1(W \cup G) = \pi_1(Z)$$
is the normal closure in $\pi_1(W)$ of the kernel of $\pi_1(M_J) \to \pi_1(G)$. This is normally generated by the longitude of the companion knot J_0 considered as a curve in $S^3 \setminus J_0 \subset M_J \subset \partial W$ [CHL09, Lemma 2.5 (1)] which lies in $\pi_1(M_{J_0})^{(2)}$. Inclusion induces
$$\pi_1(M_{J_0})^{(2)} \to \pi_1(M_J)^{(3)} \to \pi_1(W)^{(3)} \subseteq \pi_1(W)^{(3)}_{\mathcal{S}}$$
as well as the following isomorphism:
$$\frac{\pi_1(W)}{\pi_1(W)^{(3)}_{\mathcal{S}}} \cong \frac{\pi_1(Z)}{\pi_1(Z)^{(3)}_{\mathcal{S}}} = \Lambda.$$
Therefore, it suffices to show β is nontrivial $\pi_1(W)/\pi_1(W)^{(3)}_{\mathcal{S}}$. Consider the following commutative diagram, where we set $\Gamma \equiv \pi_1(W)/\pi_1(W)^{(2)}_{\mathcal{S}}$ and $\mathcal{Q} \equiv \mathbb{Q}\Gamma S^{-1}$. Here, S is the multiplicative set from Definition 4.6.

$$\begin{array}{ccccccc}
\pi_1(M_J)^{(1)} & \xrightarrow{\qquad\qquad j_* \qquad\qquad} & & & \pi_1(W)^{(2)} & \xrightarrow{\quad\Phi\quad} & \dfrac{\pi_1(W)^{(2)}_{\mathcal{S}}}{\pi_1(W)^{(3)}_{\mathcal{S}}} \\
\big\downarrow & & & & \big\downarrow & & j\big\downarrow \\
\mathcal{A}(J) \otimes \mathcal{Q} & \xrightarrow{\quad\cong\quad} & H_1(M_J;\mathcal{Q}) & \xrightarrow{\quad j_* \quad} & H_1(W;\mathcal{Q}) & \xrightarrow{\cong} & \dfrac{\pi_1(W)^{(2)}_{\mathcal{S}}}{[\pi_1(W)^{(2)}_{\mathcal{S}}, \pi_1(W)^{(2)}_{\mathcal{S}}]} \otimes \mathcal{Q}
\end{array} \tag{4.3}$$

We will now justify certain maps of the diagram. Here, the horizontal map j_* is given by functoriality of the derived series and inclusion which induces $\pi_1(M_J) \to \pi_1(W)^{(1)}$. Since $\pi_1(M_J)$ is normally generated by the meridian $\mu(J)$ which is identified with η_1 in W which by Lemma 4.7 is nontrivial in $A = \pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}}$, the map

$$\pi_1(M_J) \to \frac{\pi_1(W)^{(1)}}{\pi_1(W)^{(2)}_{\mathcal{S}}} \hookrightarrow \frac{\pi_1(W)}{\pi_1(W)^{(2)}_{\mathcal{S}}} \equiv \Gamma$$

must factor nontrivially through $\pi_1(M_J)/\pi_1(M_J)^{(1)} = \langle \mu(J) \rangle \cong \mathbb{Z}$. It follows that

$$H_1(M_J;\mathbb{Q}\Gamma) \cong H_1(M_J;\mathbb{Q}[t,t^{-1}]) \otimes \mathbb{Q}\Gamma \cong \mathcal{A}(J) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Gamma$$

where $\mathbb{Q}[t,t^{-1}]$ acts on $\mathbb{Q}\Gamma$ by $t \mapsto \eta_1$. Thus, $H_1(M_J;\mathcal{Q}) \cong \mathcal{A}(J) \otimes \mathcal{Q}$. To justify the map

$$H_1(W;\mathcal{Q}) \xrightarrow{\cong} \frac{\pi_1(W)^{(2)}_{\mathcal{S}}}{[\pi_1(W)^{(2)}_{\mathcal{S}}, \pi_1(W)^{(2)}_{\mathcal{S}}]} \otimes \mathcal{Q}, \tag{4.4}$$

note that we may interpret $H_1(W;\mathbb{Z}\Gamma)$ as the first homology of the Γ covering space of W, so

$$H_1(W;\mathbb{Z}\Gamma) \xrightarrow{\cong} \frac{\pi_1(W)^{(2)}_{\mathcal{S}}}{[\pi_1(W)^{(2)}_{\mathcal{S}}, \pi_1(W)^{(2)}_{\mathcal{S}}]}.$$

Since $\mathcal{Q}$ is a flat $\mathbb{Z}\Gamma$-module, equation (4.4) is justified. Moreover, by the definition of $\pi_1(W)^{(3)}_{\mathcal{S}}$ in Definition 4.6, the vertical map j is well-defined. Recall that by hypothesis, β generates the rational Alexander module of R, and hence J, which implies $\beta \otimes 1$ is the generator of $H_1(M_J;\mathcal{Q})$. Therefore, in order to finish the proof, it suffices to show that $\beta \otimes 1$ is not in the kernel of the bottom row of (4.3).

Note that W is given by $V \cup E \cup F_1 \cup \overline{F_2}$ with $\partial W = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{\mathcal{R}}} \sqcup \overline{M_L}$. Since E, F_1, F_2 have no second homology relative boundary,

$$\frac{H_2(W)}{i_*\left(H_2(\partial W)\right)} \cong H_2(V).$$

Furthermore, V is a 2-solution and therefore $H_2(W)/i_*\left(H_2(\partial W)\right)$ has a basis which satisfies conditions 2 and 3 of Definition 2.17 though it fails condition 1. Therefore, W is called a 2-bordism for ∂W [CHL10, Definition 7.11].

Suppose $P \equiv \ker\{j_* : H_1(M_J;\mathcal{Q}) \to H_1(W;\mathcal{Q})\}$. Then, since W is a 2-bordism, by [CHL10, Theorem 7.15], P is an isotropic submodule of $H_1(M_J;\mathcal{Q})$ with respect to the Blanchfield form on $H_1(\partial W;\mathcal{Q})$. However, we have already shown that $\beta \otimes 1$ is a generator of $H_1(M_J;\mathcal{Q})$, and if $\beta \otimes 1 \in P$, then $\mathcal{B}\ell^{\mathcal{Q}}_J(\beta \otimes 1, \beta \otimes 1) = 0$. Since $\mathcal{B}\ell^{\mathcal{Q}}_J$ is nonsingular [CHL10, Lemma 7.16], this means $H_1(M_J;\mathcal{Q}) \cong 0$. In order to give a contradiction, we show

$$\mathcal{A}(J) \otimes \mathcal{Q} \cong \left(\frac{\mathbb{Q}\Gamma}{\Delta_R(\eta_1)\mathbb{Q}\Gamma}\right) S^{-1} \neq 0.$$

By the hypotheses of Theorem 4.1, the rational Alexander module of R is nontrivial, and $\Delta_R(t)$ is not a unit in $\mathbb{Q}[t,t^{-1}]$. The map $\mathbb{Z} \to \Gamma$ given by $t \mapsto \eta_1$ is nontrivial, since we showed in Lemma 4.7 that $\eta_1 \neq 0$ in $\pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}}$. Since Γ is torsion-free, $\mathbb{Q}\Gamma$ is a free left $\mathbb{Q}[\eta_1,\eta_1^{-1}]$-module on the set of right cosets of $\langle \eta_1 \rangle \subset \Gamma$, where $\langle \eta_1 \rangle$ denotes the submodule of $\mathbb{Q}\Gamma$ generated by η_1. We may then fix a set of coset representatives so that any $x \in \mathbb{Q}\Gamma$ has a unique decomposition

$$x = \sum_{\xi} x_{\xi}\xi,$$

where each $x_{\xi} \in \mathbb{Q}[\eta_1, \eta_1^{-1}]$ and each ξ is a coset representative in Γ. Notice that if $\Delta_R(\eta_1)x = 1$ then

$$\Delta_R(\eta_1)x = \Delta_R(\eta_1)\sum_{\xi} x_{\xi}\xi = \sum_{\xi} \Delta_R(\eta_1)x_{\xi}\xi = 1.$$

This implies that on the coset $\xi = e$, we have $\Delta_R(\eta_1)x_e = 1$ in $\mathbb{Q}[\eta_1, \eta_1^{-1}]$, contradicting the fact that $\Delta_R(t)$ is not a unit in $\mathbb{Q}[t, t^{-1}]$. Therefore, $\Delta_R(\eta_1)$ has no right inverse in $\mathbb{Q}\Gamma$. Since Γ is poly-torsion-free abelian, $\mathbb{Q}\Gamma$ is a domain [Str74] and

$$\frac{\mathbb{Q}\Gamma}{\Delta_R(\eta_1)\mathbb{Q}\Gamma} \ncong 0.$$

Next, we consider the localization of this module at S. The kernel of

$$\frac{\mathbb{Q}\Gamma}{\Delta_R(\eta_1)\mathbb{Q}\Gamma} \to \frac{\mathbb{Q}\Gamma}{\Delta_R(\eta_1)\mathbb{Q}\Gamma}S^{-1}$$

is the S-torsion submodule [Ste75, Cor 3.3, p 57]. So to establish the desired result, it suffices to show that the generator of $\mathbb{Q}\Gamma/\Delta_R(\eta_1)\mathbb{Q}\Gamma$ is not S-torsion. If this generator, which we denote by 1, is S-torsion, then $1s = \Delta_R(\eta_1)y$ for some $s \in S$ and $y \in \mathbb{Q}\Gamma$.

Remember that $\Gamma \equiv \pi_1(W)/\pi_1(W)^{(2)}_{\mathcal{S}}$ and $A \equiv \pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}} \lhd \Gamma$. Since $A \subset \Gamma$, we may view $\mathbb{Q}\Gamma$ as a free left $\mathbb{Q}A$-module on the set of right cosets of A in Γ. So any $y \in \mathbb{Q}\Gamma$ has a unique decomposition

$$y = \sum_{\xi} y_{\xi}\xi,$$

where the sum is over a set of coset representatives $\{\xi \in \Gamma\}$ and y_ξ is an element of $\mathbb{Q}A$. Then

$$\begin{aligned} s =& \Delta_R(\eta_1)y \\ =& \Delta_R(\eta_1)\sum_\xi y_\xi \xi. \end{aligned}$$

Since $s \in S \subset \mathbb{Q}A$ and $\Delta_R(\eta_1) \in \mathbb{Q}A$, it must be that each coset representative $\xi \neq e$ yields $0 = \Delta_R(\eta_1)y_\xi$. Note that $\mathbb{Q}[\eta_1, \eta_1^{-1}] \subset \mathbb{Q}\Gamma$ and hence $\Delta_R(\eta_1) \neq 0$. Since $\mathbb{Q}A \subset \mathbb{Q}\Gamma$ is a domain, it must be that $y_\xi = 0$ for all $\xi \neq e$. Therefore $y \in \mathbb{Q}A$ and $s = \Delta_R(\eta_1)y$ is an equation in $\mathbb{Q}A$. Because of Definition 4.6, each element of S can be written as the product of terms of the form $\Delta_L(\mu^i\eta_2'\mu^{-i})$.

Moreover, since A is a torsion-free abelian group, we may view $s = \Delta_R(\eta_1)y$ as an equation in the group ring $\mathbb{Q}F$ for some free abelian group $F \subset A$ of finite rank r. Since $\mathbb{Q}F$ is a UFD, we apply the following proposition.

Proposition 4.9 ([CHL10, Proposition 4.5]). *Suppose $\Delta_R(t), \Delta_L(t) \in \mathbb{Q}[t, t^{-1}]$ are non zero. Then Δ_R and Δ_L are strongly coprime if and only if, for any finitely generated free abelian group F and any nontrivial $a, b \in F$, $\Delta_R(a)$ is relatively prime to $\Delta_L(b)$ in $\mathbb{Q}F$.*

Recall if $s = \Delta_R(\eta_1)y$ is an equation in S, $\Delta_R(\eta_1)$ must divide a product of terms of the form $\Delta_L(\mu^i\eta_2'\mu^{-i})$. If Δ_R and Δ_L are strongly coprime, we already arrive at a contradiction, since Proposition 4.9 implies $\Delta_R(\eta_1)$ is relatively prime to $\Delta_L(\mu^i\eta_2'\mu^{-i})$ for any i. Otherwise, choose some basis $\{x_1, x_2, \ldots, x_r\}$ for F such that $\eta_1 = x_1^m$ for some positive $m \in \mathbb{Z}$. Then $\mu^i\eta_2'\mu^{-i} = x_1^{n_{i,1}}x_2^{n_{i,2}}\cdots x_r^{n_{i,r}}$, and we may view $\mathbb{Q}F$ as a Laurent Polynomial ring in the variables $\{x_1, x_2, \ldots, x_r\}$. Since $\Delta_R \neq 0$ and is not a unit, there exists some nonzero complex root, ζ, of $\Delta_R(x_1^m)$. Suppose that $\tilde{p}(x_1)$ is a nonzero irreducible factor of $\Delta_R(x_1^m)$ of which ζ is a root. Then for some i, $\tilde{p}(x_1)$ divides $\Delta_L(x_1^{n_{i,1}}x_2^{n_{i,2}}\cdots x_r^{n_{i,r}})$ and so ζ must be a zero of $\Delta_L(x_1^{n_{i,1}}x_2^{n_{i,2}}\cdots x_r^{n_{i,r}})$ for every complex value of $x_2,\ldots,x_r$ which is impossible unless $n_{i,j}=0$ for each $j>1$. Therefore, $\mu^i\eta_2'\mu^{-i}=x_1^{n_i}$ for some $n_i\neq 0$. Recall that $\Delta_R(t^m)$ and $\Delta_L(t^n)$ share no common roots unless $n=\pm m$. Thus $n_i=\pm m$ and $\mu^i\eta_2'\mu^{-i}=(\eta_1)^{\pm 1}$ for some i.

This equation holds in A but each of η_1,η_2', and μ are given by circles in $M_{\mathcal{R}\#-\mathcal{R}}$ where $\mu^i\eta_2'\mu^{-i}$ and η_1 represent elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\#-\mathcal{R})$. Therefore, the validity of the equation $\mu^i\eta_2'\mu^{-i}=(\eta_1)^{\pm 1}$ may be considered in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\#-\mathcal{R})$ as long as $(\mu^i\eta_2'\mu^{-i})\eta_1^{\mp 1}$ does not lie in the kernel of

$$\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\#-\mathcal{R})\to\mathcal{A}^{\mathbb{Z}}(W)\to\frac{\pi_1(W)^{(1)}}{\pi_1(W)^{(2)}_{\mathcal{S}}}\equiv A.$$

Notice however, that in the module notation for $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\#-\mathcal{R})$,

$$(\mu^i\eta_2'\mu^{-i})\eta_1^{\mp 1}=\tau_*^i(\eta_2')\mp\eta_1,$$

and we consult the Blanchfield form:

$$\begin{aligned}\mathcal{B}\ell_{\mathcal{R}\#-\mathcal{R}}(\tau_*^i(\eta_2')\mp\eta_1,\tau_*^i(\eta_2')\mp\eta_1) &= \mathcal{B}\ell_{-\mathcal{R}}(\tau_*^i(\eta_2'),\tau_*^i(\eta_2'))+\mathcal{B}\ell_{\mathcal{R}}(\eta_1,\eta_1)\\ &= \mathcal{B}\ell_{-\mathcal{R}}(\eta_2',\eta_2')+\mathcal{B}\ell_{\mathcal{R}}(\eta_1,\eta_1)\\ &= -\mathcal{B}\ell_{\mathcal{R}}(\eta_2,\eta_2)+\mathcal{B}\ell_{\mathcal{R}}(\eta_1,\eta_1)\\ &\neq 0.\end{aligned}$$

The last inequality holds since the requirement imposed upon η_1,η_2 was that $\mathcal{B}\ell_{\mathcal{R}}(\eta_1,\eta_1)\neq\mathcal{B}\ell_{\mathcal{R}}(\eta_2,\eta_2)$. Therefore, if the equality $\mu^i\eta_2'\mu^{-i}=(\eta_1)^{\pm 1}$ holds in A, it must hold in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\#-\mathcal{R})$ where it is written as $\tau_*^i(\eta_2')=\eta_1^{\pm 1}$. Let U and U' be

Seifert matrices for $\mathcal{R}$ and $-\mathcal{R}$ respecitvely. We remark that although $U' = -U$, this distinction is made to emphasize the different contributions from the respective basis elements coming from the Seifert surfaces of $\mathcal{R}$ and $-\mathcal{R}$. A presentation matrix for the Alexander module $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\# - \mathcal{R})$ is given by

$$\begin{pmatrix} U - \tau_* U^\intercal & 0 \\ 0 & U' - \tau_* U'^\intercal \end{pmatrix}.$$

The automorphism τ_* decomposes under connected sum $\mathcal{R}\# - \mathcal{R}$. Thus $\tau_*(\mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \oplus 0) \subset \mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \oplus 0$ and $\tau_*(0 \oplus \mathcal{A}^{\mathbb{Z}}(-\mathcal{R})) \subset 0 \oplus \mathcal{A}^{\mathbb{Z}}(-\mathcal{R})$, invalidating the equation $\tau_*^i(\eta_2') = \eta_1^{\pm 1}$. This contradicts the equality of the statement $\mu^i \eta_2' \mu^{-i} = \eta_1^{\pm 1}$ in A and therefore contradicts the assumption that the generator of $\mathbb{Q}\Gamma/\Delta_R(\eta_1)\mathbb{Q}\Gamma$ is S-torsion. Thus $\mathcal{A}(J) \otimes \mathcal{Q}$ is nontrivial, and $\beta \otimes 1$ cannot lie in the kernel of the bottom row of 4.3. This completes the proof that $\mu(J_0) \sim \beta$ is nontrivial in $\pi_1(Z)^{(2)}/\pi_1(Z)^{(3)}_{\mathcal{S}}$ so the restriction of Φ to $\pi_1(M_{J_0})$ factors nontrivially through $\mathbb{Z}$. $\square$

Our last task is to show that $\rho(M_L, \Phi) = 0$, which we complete in the following short lemma.

Lemma 4.10. *The restriction of Φ to $\pi_1(\overline{M_L})$ also factors through $\mathbb{Z}$ and $\rho(M_L, \Phi) = 0$.*

Proof. Similar to the beginning of Lemma 4.8, we begin with the following commutative diagram.

$$\begin{tikzcd}
\pi_1(\overline{M_L})^{(1)} \arrow[rr, "j_*"] \arrow[d] & & \pi_1(W)^{(2)} \arrow[r, "\Phi"] \arrow[d] & \dfrac{\pi_1(W)^{(2)}_{\mathcal{S}}}{\pi_1(W)^{(3)}_{\mathcal{S}}} \arrow[d, "j"'] \\
\mathcal{A}(-L) \otimes \mathcal{Q} \arrow[r, "\cong"] & H_1(\overline{M_L};\mathcal{Q}) \arrow[r, "j_*"] & H_1(W;\mathcal{Q}) \arrow[r, "\cong"] & \dfrac{\pi_1(W)^{(2)}_{\mathcal{S}}}{[\pi_1(W)^{(2)}_{\mathcal{S}}, \pi_1(W)^{(2)}_{\mathcal{S}}]} \otimes \mathcal{Q}
\end{tikzcd} \tag{4.5}$$

Again, j_* is given by functoriality of the comutator series and inclusion given that $\pi_1(\overline{M_L}) \to \pi_1(W)^{(1)}$. Again, $\pi_1(\overline{M_L})$ is normally generated by its meridian, $\overline{\mu(L)}$, which is identified with η_2'. Suppose that η_2' is nontrivial in $\pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}}$. Then the map

$$\pi_1(\overline{M_L}) \to \frac{\pi_1(W)^{(1)}}{\pi_1(W)^{(2)}_{\mathcal{S}}} \hookrightarrow \frac{\pi_1(W)}{\pi_1(W)^{(2)}_{\mathcal{S}}}$$

must factor through $\pi_1(\overline{M_L})/\pi_1(\overline{M_L})^{(1)} = \langle\overline{\mu(L)}\rangle \cong \mathbb{Z}$, and

$$H_1(\overline{M_L};\mathbb{Q}\Gamma) \cong H_1(\overline{M_L};\mathbb{Q}[t,t^{-1}]) \otimes \mathbb{Q}\Gamma \cong \mathcal{A}(-L) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}\Gamma.$$

Here $\mathbb{Q}[t,t^{-1}]$ acts on $\mathbb{Q}\Gamma$ by $t \mapsto \overline{\mu(L)} \simeq \eta_2'$. This implies $H_1(\overline{M_L};\mathcal{Q}) \cong \mathcal{A}(-L) \otimes \mathcal{Q}$. Since the rational Alexander module of L is $\Delta_L(t)$-torsion and $\Delta_L(\eta_2') \in S$ by definition, this module is trivial. Since j is injective, this implies that the map along the top row of Diagram 4.5 is zero.

On the other hand, suppose η_2' is trivial in $\pi_1(W)^{(1)}/\pi_1(W)^{(2)}_{\mathcal{S}}$. Since $\pi_1(\overline{M_L})$ is normally generated by $\overline{\mu(L)} \simeq \eta_2'$ in Z, this implies $j_*(\pi_1(\overline{M_L})) \subset \pi_1(W)^{(2)}_{\mathcal{S}}$ by inclusion and the map along the top row of the diagram is again zero.

Finally, consider the restriction of Φ to $\pi_1(\overline{M_L})$:

$$\Phi : \pi_1(\overline{M_L}) \to \frac{\pi_1(W)}{\pi_1(W)^{(3)}_{\mathcal{S}}}.$$

By the above arguments, this map is trivial on the subgroup $\pi_1(\overline{M_L})^{(1)} \subset \pi_1(\overline{M_L})$ and must factor through $\pi_1(\overline{M_L})/\pi_1(\overline{M_L})^{(1)} \cong \mathbb{Z}$. There are two easy cases to consider. If the map is trivial, we have $\rho(\overline{M_L}; \Phi) = 0$. Otherwise, the map factors nontrivially through $\mathbb{Z}$ and $\rho(\overline{M_L}; \Phi) = \rho_0(-L) = 0$ since L is a 1-solvable knot. This finishes the proof of the Lemma 4.10 and completes the proof of Theorem 4.1. □

4.1 Example: Satellites of the 9_{46} Knot

In this section, we give an explicit example of Corollary 4.3. We take $\mathcal{R} = 9_{46}$ so that $\Delta_{\mathcal{R}}(t) = -2t^2 + 5t - 2$. Our axes, however, will not be the same as those constructed in the proof of Corollary 4.3. Note that the curves a and b, as shown in Figure 4.3, generate the integral Alexander module of $\mathcal{R}$, and $\eta = a + b$ generates the rational Alexander module. In $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$, we have the relations:

$$2ta = a \Rightarrow (2t - 1)a = 0, \tag{4.6}$$

$$tb = 2b \Rightarrow (t - 2)b = 0. \tag{4.7}$$

Any element, γ, of the integral Alexander module may be written as a polynomial combination of a and b, that is $\gamma = x(t)a + y(t)b \in \mathcal{A}^{\mathbb{Z}}$, where $x(t), y(t) \in \mathbb{Z}[t, t^{-1}]$. Let $\mathcal{Q}$ denote the subring $\mathbb{Z}[2^{-1}] \subset \mathbb{Q}$, and consider the map

$$\mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \to \mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \otimes_{\mathbb{Z}} \mathcal{Q}.$$

Because of identities 4.6 and 4.7,

$$t^r a \mapsto 2^{-r} a, \qquad t^r b \mapsto 2^r b.$$

Therefore,

$$x(t)a \mapsto x(2^{-1})a, \qquad y(t)b \mapsto y(2)b,$$

and $\gamma \mapsto x(2^{-1})a + y(2)b$, where $x(2^{-1}), y(2) \in \mathcal{Q} \subset \mathbb{Q}$. These equations hold as we map to the rational Alexander module:

$$\mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \to \mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \otimes_{\mathbb{Z}} \mathcal{Q} \to \mathcal{A}^{\mathbb{Z}}(\mathcal{R}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathcal{A}(\mathcal{R}).$$

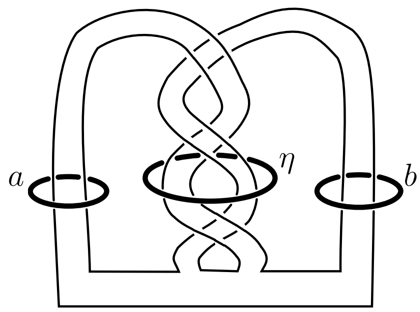


Figure 4.3: The ribbon knot $\mathcal{R} = 9_{46}$. Note $\eta = a + b$ in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$.

Since η is the generator of $\mathcal{A}(\mathcal{R})$ which is nontrivial, $\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta) \neq 0$. Let $K_1 = \mathcal{R}(\eta; J)$, where J is constructed as in the statement of Corollary 4.2. Suppose $\gamma = x(t)a + y(t)b \in \mathcal{A}^{\mathbb{Z}}(\mathcal{R})$, and let $K_2 = \mathcal{R}(\gamma; J)$. The rational Blanchfield self-linking of γ is given by

$$\begin{aligned}
\mathcal{B}\ell_{\mathcal{R}}(\gamma, \gamma) = \; & \mathcal{B}\ell_{\mathcal{R}}\big(x(t)a + y(t)b, x(t)a + y(t)b\big) \\
= \; & \mathcal{B}\ell_{\mathcal{R}}\left(x\left(2^{-1}\right)a + y\left(2\right)b, x\left(2^{-1}\right)a + y\left(2\right)b\right) \\
= \; & \left[x\left(2^{-1}\right)^2 \mathcal{B}\ell_{\mathcal{R}}(a, a)\right] + \left[x\left(2^{-1}\right)y\left(2\right)\mathcal{B}\ell_{\mathcal{R}}(a, b)\right] \\
& + \left[x\left(2^{-1}\right)y(2)\mathcal{B}\ell_{\mathcal{R}}(b, a)\right] + \left[y\left(2\right)^2 \mathcal{B}\ell_{\mathcal{R}}(b, b)\right] \\
= \; & x\left(2^{-1}\right)y\left(2\right)\big(\mathcal{B}\ell_{\mathcal{R}}(a, b) + \mathcal{B}\ell_{\mathcal{R}}(b, a)\big) \\
= \; & x\left(2^{-1}\right)y(2)\mathcal{B}\ell_{\mathcal{R}}(\eta, \eta).
\end{aligned} \tag{4.8}$$

Here $\mathcal{B}\ell_{\mathcal{R}}(a,a) = \mathcal{B}\ell_{\mathcal{R}}(b,b) = 0$ since a and b both generate isotropic submodules of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. Corollary 4.2 states that K_1 and K_2 are distinct up to concordance as long as $\mathcal{B}\ell_{\mathcal{R}}(\eta,\eta) \neq \mathcal{B}\ell_{\mathcal{R}}(\gamma,\gamma)$ which from (4.8) is equivalent to $(1 - x(2^{-1})y(2))\,\mathcal{B}\ell_{\mathcal{R}}(\eta,\eta) \neq 0$. Recall that a formula for the Blanchfield form can be given by a Seifert matrix U for $\mathcal{R}$:

$$\mathcal{B}\ell(r,s) = \overline{s}(1-t)\left(tU - U^{\mathsf{T}}\right)^{-1} r$$

where $\overline{s}$ is the image of s under the involution $t \mapsto t^{-1}$ [Kea78]. The Seifert matrix for $\mathcal{R}$ yielding a presentation matrix for $\mathcal{A}(\mathcal{R})$ with respect to the basis $\{a,b\}$ is

$$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix},$$

and by a simple calculation,

$$\mathcal{B}\ell_{\mathcal{R}}(\eta,\eta) = \frac{3(t-1)^2}{\Delta_{\mathcal{R}}(t)}, \text{ where } (3(t-1)^2, \Delta_{\mathcal{R}}(t)) = 1.$$

This implies $(1-x(2^{-1})y(2))\mathcal{B}\ell_{\mathcal{R}}(\eta,\eta)$ is zero if and only if $1-x(2^{-1})y(2)$ is a multiple of $\Delta_{\mathcal{R}}(t)$. This is only possible if $x(2^{-1})$ and $y(2)$ are inverses in $\mathcal{Q} \subset \mathbb{Q}$, and it must be that $x(2^{-1}) = \pm 2^{-r}$ and$y(2) = \pm 2^{r}$ with the same sign. Therefore, $x(t)a$ and $y(t)b$ are equivalent in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$ to $\pm t^r a$ and $\pm t^r b$ respectively and with the same sign. Therefore, $x(t)a + y(t)b = \pm(t^r a + t^r b) = \pm t^r \eta$. Since the element $\pm t^r \eta$ is represented by the curve $\pm\eta$ in $S^3 \setminus \mathcal{R}$, regardless of r, we see that infection upon η and γ may yield concordant satellites only if $\gamma = \pm\eta$.

More generally, let $\gamma_i = x_i(t)a + y_i(t)b$ where $x_i(t), y_i(t) \in \mathbb{Z}[t,t^{-1}]$ for $i = 1,2$. Then by (4.8), $\mathcal{B}\ell_{\mathcal{R}}(\gamma_1,\gamma_1) = \mathcal{B}\ell_{\mathcal{R}}(\gamma_2,\gamma_2)$ if and only if $(x_1y_1 - x_2y_2)\mathcal{B}\ell_{\mathcal{R}}(\eta,\eta) = 0$, where for simplicity we set $x_i \equiv x_i(2^{-1})$ and $y_i \equiv y_i(2) \in \mathcal{Q} \subset \mathbb{Q}$. This is zero in

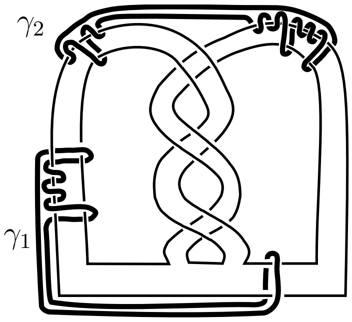


Figure 4.4: These curves, $\gamma_1 = (t + t^{-1})a + b$, and $\gamma_2 = ta + (t^2 + 1)b$ as elements of $\mathcal{A}^{\mathbb{Z}}(9_{46})$, have the property that $\mathcal{B}\ell_{\mathcal{R}}(\gamma_1, \gamma_1) = \mathcal{B}\ell_{\mathcal{R}}(\gamma_2, \gamma_2)$.

$\mathbb{Q}(t)/\mathcal{Q}[t, t^{-1}]$ when $x_1y_1 = x_2y_2$ in $\mathcal{Q}$. For every distinct value $c_i \in \mathbb{Z}[1/2]$, we can find an unknotted curve $\gamma_i \subset S^3 \setminus \mathcal{R}$ with $\ell k(\mathcal{R}, \gamma_i) = 0$ and such that $\mathcal{B}\ell^{\mathbb{Q}}_{\mathcal{R}}(\gamma_i, \gamma_i) = c_i\mathcal{B}\ell^{\mathbb{Q}}_{\mathcal{R}}(\eta, \eta)$. If $c_i = \hat{c}_i 2^{-k_i}$ for $\hat{c}_i, k_i \in \mathbb{Z}$, γ_i may be given by $\gamma_i = t^{k_i}a + \hat{c}_i b$. Thus, each c_i yields a distinct concordance class $K_i = \mathcal{R}(\gamma_i, J)$. We summarize these results in the following lemma and also in the graph of Figure 4.5.

Lemma 4.11. *Let $\mathcal{R}$ be the 9_{46} knot and J the knot given in Corollary 4.2. For every $c_i \in \mathbb{Z}[1/2]$, we obtain an unknotted curve $\eta_i \subset S^3 \setminus \mathcal{R}$ with $\ell k(\eta_i, \mathcal{R}) = 0$. The $\{\eta_i\}$ yield infinitely many distinct concordance classes of satellite knots $K_i = \mathcal{R}(\eta_i, J)$.*

Nonetheless, there are many combinations of $x_1, y_1, x_2, y_2 \in \mathbb{Z}[1/2]$ for which $x_1y_1 = x_2y_2$. For instance, take $\gamma_1 = (t+t^{-1})a+b, \gamma_2 = ta+(t^2+1)b$ as in Figure 4.4. Although these curves are not isotopic in $S^3 \setminus \mathcal{R}$, $x_1y_1 = x_2y_2 = 5/2$ implying that $\gamma_1 + \gamma_2'$ lies in an isotropic submodule of the rational Alexander module, $\mathcal{A}(\mathcal{R}\# - \mathcal{R})$, and thus potentially in the kernel of the map

$$\mathcal{A}^{\mathbb{Z}}(\mathcal{R}\# - \mathcal{R}) \xrightarrow{\phi_*} \mathcal{A}^{\mathbb{Z}}(V)$$

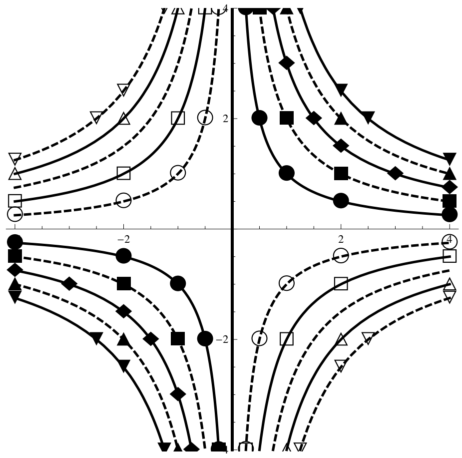


Figure 4.5: Each level curve in this graph is given by $xy = c \in \mathbb{Z}[1/2]$. An isotopy class of $\gamma = x(t)a + y(t)b$ in $S^3 \setminus \mathcal{R}$ with $\mathcal{B}\ell_{\mathcal{R}}(\gamma, \gamma) = c$ are represented by shaded points $(x(2^{-1}), y(2))$ on the level curves. Choices of γ_i lying on different level curves lead to nonconcordant knots $\mathcal{R}(\gamma_i, J)$.

for some potential 2.5-solution of $K_1\# - K_2$. Infection upon γ_1 and γ_2 by J may thus produce concordant knots as we saw in Example 2.21.

Chapter 5

Satellites Distinguished by Higher-Order Blanchfield Forms

The overarching goal is to distinguish knots obtained by satellite operations

$$K_1 = \mathcal{R}(\eta_1, J), \text{ and } K_2 = \mathcal{R}(\eta_2, J),$$

where η_1 and η_2 are "different" axes in $S^3 \setminus \mathcal{R}$. In Theorem 3.1, we distinguish the concordance classes of K_1 and K_2 when η_1 and η_2 have distinct orders in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. More generally, in Theorem 4.1, we prove K_1 and K_2 often represent distinct concordance classes when η_1 and η_2, viewed as elements of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$, are distinguished by the classical Blanchfield linking form, that is,

$$\mathcal{B}\ell_{\mathcal{R}}(\eta_1, \eta_1) \neq \mathcal{B}\ell_{\mathcal{R}}(\eta_2, \eta_2).$$

We are now prepared to strengthen these results. Suppose η_1 and η_2 are equivalent when viewed as elements of the classical Alexander module. Our goal is to find sufficient conditions to distinguish K_1 and K_2.

First, define a partial commutator series on the class of groups G with $G/G^{(1)} = \langle \mu \rangle \cong \mathbb{Z}$. This commutator series will be weakly functorial and each quotient $G/G_x^{(n)}$ will be poly-(torsion-free abelian). For the first term, take

$$G_x^{(1)} = G_r^{(1)} = \ker\left\{ G \to \frac{G}{[G,G]} \otimes_{\mathbb{Z}} \mathbb{Q} \right\}. \tag{5.1}$$

In the usual derived series, whenever $\beta_1(G) = 1$, $G_r^{(1)}/G_r^{(2)}$ is a torsion right $\mathbb{Q}[t,t^{-1}]$-module where the t action is given by conjugation. Fix some element $\mathfrak{g} \in G_x^{(1)}$ which is nonzero in $\mathcal{A}^{\mathbb{Z}}(G)$. Further terms of the commutator series x will be a function of $\mathfrak{g}$. There exists some unique minimal polynomial $p_1(t) \in \mathbb{Z}[t,t^{-1}]$ such that $\mathfrak{g} \cdot p_1(t)$ is trivial in $\mathcal{A}^{\mathbb{Z}}(G)$. Let $S_1 \subset \mathbb{Q}[t,t^{-1}]$ be the multiplicative set generated by $p_1(t)$ and $p_1(t^{-1})$. Since $\mathbb{Q}[t,t^{-1}]$ is a commutative ring, it follows immediately that S_1 is a divisor set of $\mathbb{Q}[t,t^{-1}]$, and furthermore $\mathbb{Q}[t,t^{-1}]S_1^{-1}$ inherits the natural involution. Define the second term of the commutator series as

$$G_x^{(2)} = \ker\left\{ G_x^{(1)} \to \frac{G_x^{(1)}}{\left[G_x^{(1)}, G_x^{(1)}\right]} \xrightarrow{\sigma_2} \frac{G_x^{(1)}}{\left[G_x^{(1)}, G_x^{(1)}\right]} \underset{\mathbb{Z}[t,t^{-1}]}{\otimes} \mathbb{Q}[t,t^{-1}]S_1^{-1} \right\}. \tag{5.2}$$

This map σ_2 kills all S_1-torsion in $G_x^{(1)}/[G_x^{(1)}, G_x^{(1)}]$ and hence $\mathfrak{g} \in G_x^{(2)}$. For the third term of the commutator series, we need only annihilate $\mathbb{Z}$-torsion:

$$G_x^{(3)} = \ker\left\{ G_x^{(2)} \to \frac{G_x^{(2)}}{\left[G_x^{(2)}, G_x^{(2)}\right]} \underset{\mathbb{Z}[G/G_x^{(2)}]}{\otimes} \mathbb{Q}\left[G/G_x^{(2)}\right] \right\}. \tag{5.3}$$

At the fourth level, choose some symmetric Laurent polynomial $p_2(t) \in \mathbb{Z}[t,t^{-1}]$ such that $p_2(1) = \pm 1$. Let $S_3 \subset \mathbb{Q}[G_x^{(2)}/G_x^{(3)}]$ be the multiplicative set with unity generated by polynomials of the form

$$\left\{ p_2(\mathfrak{g}^{\pm g}) | g \in G/G_x^{(3)} \right\}, \tag{5.4}$$

where $\mathfrak{g}^{\pm g} = g^{-1}\mathfrak{g}^{\pm 1}g$. Since $G_x^{(2)}/G_x^{(3)}$ is a normal subgroup of $G/G_x^{(3)}$, each $\mathfrak{g}^{\pm g} = g^{-1}\mathfrak{g}^{\pm 1}g$ is an element of $G_x^{(2)}/G_x^{(3)}$ and each $p_2(\mathfrak{g}^{\pm g}) \in \mathbb{Q}[G_x^{(2)}/G_x^{(3)}]$. The image of $p_2(\mathfrak{g}^{\pm g})$ under the augmentation map is ± 1 since $p_2(1) = \pm 1$, which implies that $0 \notin S_3$. Since $\mathbb{Q}[G_x^{(2)}/G_x^{(3)}]$ is a commutative domain, S_3 is a right divisor set of $\mathbb{Q}[G_x^{(2)}/G_x^{(3)}]$. If $h \in G/G_x^{(3)}$, we have

$$h^{-1}p_2(\mathfrak{g}^{\pm g})h = p_2(h^{-1}\mathfrak{g}^{\pm g}h) = p_2(\mathfrak{g}^{\pm gh})$$

where $\mathfrak{g}^{\pm gh} = h^{-1}(g^{-1}\mathfrak{g}^{\pm 1}g)h = (gh)^{-1}\mathfrak{g}^{pm1}(gh)$. So S_3 is $G/G_x^{(3)}$-invariant and is a right divisor set of $\mathbb{Q}[G/G_x^{(3)}]$ by Proposition 2.10. We localize at S_3 to obtain the fourth term of the commutator series:

$$G_x^{(4)} = \ker\left\{ G_x^{(3)} \to \frac{G_x^{(3)}}{\left[G_x^{(3)}, G_x^{(3)}\right]} \underset{\mathbb{Z}[G/G_x^{(3)}]}{\otimes} \mathbb{Q}\left[G/G_x^{(3)}\right] S_3^{-1} \right\}. \tag{5.5}$$

Note that since S_3 is closed under involution, $\mathbb{Q}[G/G_x^{(3)}]S_3^{-1}$ inherits the natural involution from $\mathbb{Q}[G/G_x^{(3)}]$.

Proposition 5.1. *Suppose $H_1(G,\mathbb{Z}) \cong H_1(H,\mathbb{Z}) \cong \mathbb{Z}$. The commutator series x is functorial with respect to homomorphisms $G \to H$ which are isomorphisms on $\mathbb{Z}$-homology and send $\mathfrak{g}_G$ to $\mathfrak{g}_H$.*

Proof. Suppose $\iota : G \to H$ induces an isomorphism $H_1(G;\mathbb{Z}) \xrightarrow{\cong} H_1(H;\mathbb{Z})$ and $\iota(\mathfrak{g}_G) = \mathfrak{g}_H$. The commutator series is defined using the right divisor sets $S_i(G)$ and $S_i(H)$. By [CHL10, Proposition 3.2], we need only check that ι sends the right divisor sets $S_i(G)$ to right divisor sets $S_i(H)$. Note that ι induces $H_1(G,\mathbb{Z}) \xrightarrow{\cong} H_1(H,\mathbb{Z}) \equiv \langle t\rangle \cong \mathbb{Z}$ sending $t_G \mapsto t_H$, and $S_1(G)$ is generated by $p_1(t_G)$ and $p_1(t_G^{-1})$. Since $\mathfrak{g}_G$ is

$p_1(t_G)$-torsion, $\mathfrak{g}_G \cdot p_1(t_G) = 0$ in $\mathcal{A}^{\mathbb{Z}}(G)$, and $\mathfrak{g}_G \cdot p_1(t_G)$ is represented by an element in $G^{(2)}$,

$$\mathfrak{g}_G \cdot p_1(t_G) = \prod \mu_G^{-i} \mathfrak{g}_G^{c_i} \mu_G^i$$

where $p_1(t) = \sum_i c_i t^i$. Since the derived series is functorial, $\iota_*(\mathfrak{g}_G \cdot p_1(t_G)) = \mathfrak{g}_H \cdot p_1(t_H)$ is represented by an element of $H^{(2)}$, and so $\mathfrak{g}_H \in H_x^{(2)}/H_x^{(3)}$. Hence, the order of $\mathfrak{g}_H$ divides $p_1(t_G)$ and $\iota_*(S_1(G)) \subset S_1(H)$.

By [CHL10, Proposition 3.2], $\iota(G_x^{(3)}) \subset H_x^{(3)}$. We must next check that $\iota_*(S_3(G))$ is contained in $S_3(H)$, but this is clear since S_3 is the multiplicative set in $\mathbb{Q}[G/G_x^{(3)}]$ generated by

$$\left\{p_2(\mathfrak{g}_G^{\pm g}) | g \in G/G_x^{(3)}, \mathfrak{g}_G^{\pm g} = g^{-1}\mathfrak{g}_G^{\pm 1} g\right\},$$

and $\mathfrak{g}_G^{\pm g} \mapsto \mathfrak{g}_H^{\pm \iota(g)}$ for $\iota(g) \in H/H_x^{(3)}$. Hence, $\iota_*(S_3(G)) \subset S_3(H)$. □

In the construction of the commutator series x above, we were motivated by the choice of the "special element" $\mathfrak{g}$. Before proceeding, note that the polynomial $p_1(t)$ may be chosen and thus the second term of the commutator series defined for all groups with first homology isomorphic to $\mathbb{Z}$ before specifying $\mathfrak{g}$. Furthermore, the partial commutator series $G_*^{(n)}$ is weakly functorial for $n \leq 3$ if $p_1(t)$ chosen to be some fixed polynomial in $\mathbb{Q}[t, t^{-1}]$ by [CHL10, Corollary 4.3]. In the statement of Theorem 5.2 below, we need only the first three terms of the commutator series, and the exact choice of $\mathfrak{g}$ needed for the construction of the right divisor set S_3, and hence the fourth term of the commutator series, will be revealed in the proof.

Let $\mathcal{R}$ be a ribbon knot. In order to apply this series to $\pi_1(M_{\mathcal{R}})$, choose some unknotted curve η which has $\ell k(\eta, \mathcal{R}) = 0$ and is nontrivial in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. Let $p_1(t)$ be the order of η as an element of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. Hence $p_1(t)$ divides $\Delta_{\mathcal{R}}(t)$, and $\eta \in \pi_1(M_{\mathcal{R}})^{(2)}_x$. Since the choice of the polynomial $p_2(t)$ (or even the fourth term of the commutator series) will not be necessary for the statement of our theorem, it will be chosen later.

Let Γ_2 denote $\pi_1(M_{\mathcal{R}})/\pi_1(M_{\mathcal{R}})^{(2)}_x$ and $\mathcal{K}\Gamma_2$ denote the fraction field of $\mathbb{Q}\Gamma_2$. Consider the first homology of $M_{\mathcal{R}}$ under the coefficient system induced by $\pi_1(M_{\mathcal{R}}) \to \Gamma_2$, $H_1(M_{\mathcal{R}}; \mathbb{Q}\Gamma_2)$. By [COT03, Theorem 2.13], there exists a symmetric linking form

$$\mathcal{B}\ell^{\Gamma_2}_{\mathcal{R}} : H_1(M_{\mathcal{R}}; \mathbb{Q}\Gamma_2) \times H_1(M_{\mathcal{R}}; \mathbb{Q}\Gamma_2) \to \mathcal{K}\Gamma_2/\mathbb{Q}\Gamma_2. \tag{5.6}$$

Suppose $\tilde{P}$ is an isotropic submodule of $H_1(M_{\mathcal{R}}; \mathbb{Q}[t, t^{-1}]S_1^{-1})$ with respect to the localized Blanchfield form, and let P be the subgroup of $\pi_1(M_{\mathcal{R}})^{(1)}/\pi_1(M_{\mathcal{R}})^{(2)}_x$ which maps to $\tilde{P}$ in the localized Alexander module. Since

$$\Gamma_2 = \frac{\pi_1(M_{\mathcal{R}})^{(1)}}{\pi_1(M_{\mathcal{R}})^{(2)}_x} \rtimes \frac{\pi_1(M_{\mathcal{R}})}{\pi_1(M_{\mathcal{R}})^{(1)}},$$

every element of Γ_2 may be written uniquely as the product $g = h\mu^k$ where $h \in \pi_1(M_{\mathcal{R}})^{(1)}/\pi_1(M_{\mathcal{R}})^{(2)}_x$ and $\mu^k \in \pi_1(M_{\mathcal{R}})/\pi_1(M_{\mathcal{R}})^{(1)} \cong \langle \mu(\mathcal{R}) \rangle$. Let p be an arbitrary element of P. Then

$$g^{-1}pg = (h\mu^k)^{-1}p(h\mu^k) = \mu^{-k}p\mu^k.$$

As an element of the Alexander module, $\mu^{-k}p\mu^k$ is written as pt^k, and since $\tilde{P}$ is a submodule of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$, $pt^k \in \tilde{P}$. Hence P is a normal subgroup of Γ_2. Denote Γ_2/P by Γ_2^P. The map $\pi_1(M_{\mathcal{R}}) \to \Gamma_2 \to \Gamma_2^P$ yields the higher-order module $H_1(M_{\mathcal{R}}; \mathbb{Q}\Gamma_2^P)$, and by [COT03, Theorem 2.13], there exists a symmetric linking form on $H_1(M_{\mathcal{R}}; \mathbb{Q}\Gamma_2^P)$, which we denote by $\mathcal{B}\ell^{\mathbb{Q}\Gamma_2^P}_{\mathcal{R}}$. This linking form will provide the necessary obstruction to the concordance of the satellites $\mathcal{R}(\eta_1, J)$ and $\mathcal{R}(\eta_2, J)$ when the axes are equivalent to η the classical Alexander module.

Theorem 5.2. *Let $\mathcal{R}$ be a ribbon knot and η be an unknotted curve in $S^3 \setminus \mathcal{R} \subset M_{\mathcal{R}}$ with $\ell k(\mathcal{R}, \eta) = 0$ and which represents a nontrivial element of $\mathcal{A}^{\mathbb{Z}}(\mathcal{R})$. Suppose that γ is an unknotted curve in $S^3 \setminus \mathcal{R} \subset M_{\mathcal{R}}$ which represents a nontrivial element of $\pi_1(M_{\mathcal{R}})^{(2)}/\pi_1(M_{\mathcal{R}})^{(3)}$. Then let $\eta\gamma$ denote the unknotted curve in $S^3 \setminus \mathcal{R}$ which is equivalent to $\eta\gamma$ in $\pi_1(M_{\mathcal{R}})$. Suppose*

$$\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P}(\eta\gamma, \eta\gamma) \neq \mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P}(\eta, \eta) \tag{5.7}$$

holds for any subgroup

$$P \subset \frac{\pi_1(M_{\mathcal{R}})^{(1)}}{\pi_1(M_{\mathcal{R}})_x^{(2)}},$$

mapping to an isotropic submodule of $H_1(M_{\mathcal{R}}; \mathbb{Q}[t, t^{-1}]S_1^{-1})$ and where S_1 is defined by setting $p_1(t)$ to be the order of η. Then the knots $K_1 = \mathcal{R}(\eta_1, J)$ and $K_2 = \mathcal{R}(\eta_2, J)$, are distinct in $\mathcal{C}$ for some knot J where $\eta_1 = \eta\gamma$ and $\eta_2 = \eta$. In particular, $\mathcal{R}(\eta, -)$ and $\mathcal{R}(\eta\gamma, -)$ are distinct maps on $\mathcal{C}$.

Remark 5.3.
1. *Before beginning the proof, we want to emphasize that the knot J is independent of both the choice of η and γ and is dependent only on $\mathcal{R}$. This will be shown in the proof of Theorem 5.2.*
2. *We assume without loss of generality that $\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P}(\eta\gamma, \eta\gamma) \neq 0$, since otherwise we reverse the roles of η and $\eta\gamma$ by setting $\eta_1 = \eta\gamma$ and $\eta_2 = \eta = \eta_1\gamma^{-1}$ since γ^{-1} satisfies the hypotheses of the theorem.*

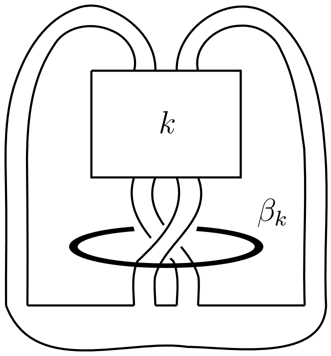


Figure 5.1: The ribbon knot R_k and axis β_k.

Proof. This proof is similar to that of Theorem 4.1 and is also by contradiction. Suppose K_1 and K_2 are concordant and that W_0 is a slice disk complement for $K_1\#-K_2$. We construct a tower of cobordisms for the zero-framed surgery $M_{K_1\#-K_2}$. The necessary J will be found via a satellite operation $J = R_k(\beta_k, J_0)$ as in Theorem 4.1 where R_k is the ribbon knot of Figure 5.1. We let G be the satellite cobordism for $J = R_k(\beta_k, J_0)$. Define F_1 and F_2 to be the satellite cobordism for $K_1 = \mathcal{R}(\eta_1, J)$ and $K_2 = \mathcal{R}(\eta_2, J)$ respectively. Since connected sum may be described as the satellite $K_1\# - K_2 = K_1(\mu(K_1), -K_1)$, let E to be the satellite cobordism for $K_1\# - K_2$.

$$\partial G = M_{R_k} \sqcup M_{J_0} \sqcup \overline{M_J} \tag{5.8}$$

$$\partial F_1 = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{K_1}} \tag{5.9}$$

$$\partial F_2 = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{K_2}} \tag{5.10}$$

$$\partial E = M_{K_1} \sqcup \overline{M_{K_2}} \sqcup \overline{M_{K_1\#-K_2}} \tag{5.11}$$

$$\partial V = M_{K_1\#-K_2} \tag{5.12}$$

Finally, define W to be be the union of $W_0, E, F_1, \overline{F_2}, G$ and $\overline{G}$ along their common boundary components as shown in Figure 5.2. Define W_1 to be $W_0 \cup E$ and W_2 to be

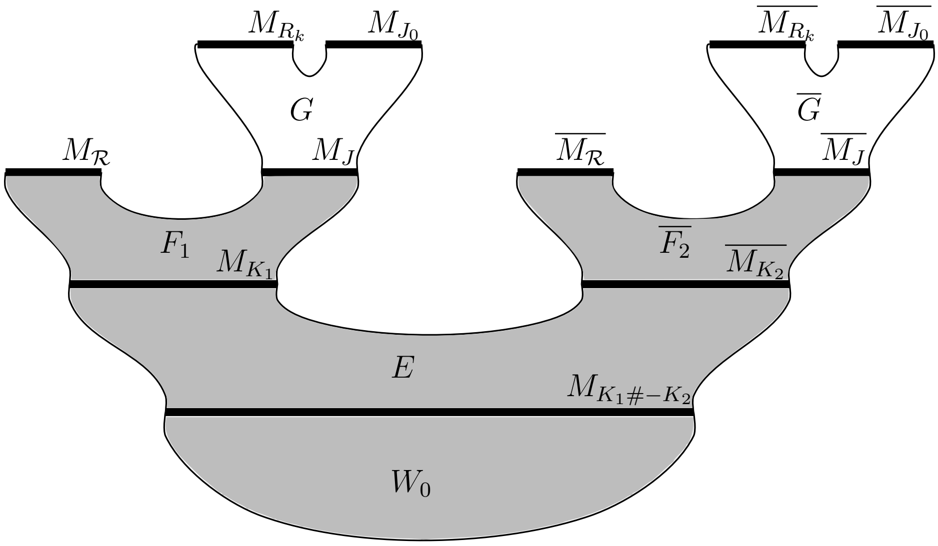


Figure 5.2: This figure represents the cobordism W. The shaded region is the submanifold W_2.

$W_1 \cup F_1 \cup F_2$. Then W is simply $W_2 \cup G \cup \overline{G}$. In order to define the commutator series $\pi_1(W)_x^{(n)}$, let $\mathfrak{g}$ be the image of η under the inclusion map $\pi_1(\overline{M_{\mathcal{R}}}) \to \pi_1(W)$, denoted by $\tilde{\eta}$. To define the fourth term as in (5.5), let $p_2(t)$ be the Alexander polynomial of J, $\Delta_J(t) = \Delta_{R_k}(t) = (k^2+k)t^2 - (2k^2+2k+1)t + (k^2+k)$. That is,

$$\pi_1(W)_x^{(4)} = \ker \left\{ \pi_1(W)_x^{(3)} \to \frac{\pi_1(W)_x^{(3)}}{\left[\pi_1(W)_x^{(3)}, \pi_1(W)_x^{(3)}\right]} \otimes \mathbb{Q}[\pi_1(W)/\pi_1(W)_x^{(3)}]S_3^{-1} \right\}$$

where

$$S_3 = \{\Delta_{R_k}(\tilde{\eta}^g) | g \in \pi_1(W)/\pi_1(W)_x^{(3)}\}.$$

Note that $\Delta_{R_k}(t)$ is not a unit in $\mathbb{Q}[t,t^{-1}]$ and $\Delta_{R_k}(t^n)$ and $\Delta_{R_k}(t^m)$ share no common roots unless $m = \pm n$ (as in the proof of Theorem 4.1). Note that the commutator series is functorial with respect to the inclusion $\pi_1(\overline{M_{\mathcal{R}}}) \to \pi_1(W)$ by Proposition 5.1.

Consider the coefficient system on W given by the projection

$$\Phi : \pi_1(W) \to \pi_1(W)/\pi_1(W)_x^{(4)} \equiv \Lambda_4. \tag{5.13}$$

Since the von Neumann signature defect satisfies Novikov additivity, we have

$$\begin{aligned}\sigma^{(2)}(W,\Phi)-\sigma(W) = &\left(\sigma^{(2)}(W_0,\Phi)-\sigma(W_0)\right)+\left(\sigma^{(2)}(E,\Phi)-\sigma(E)\right)\\ &+\left(\sigma^{(2)}(F_1,\Phi)-\sigma(F_1)\right)+\left(\sigma^{(2)}(\overline{F_2},\Phi)-\sigma(\overline{F_2})\right)\\ &+\left(\sigma^{(2)}(G,\Phi)-\sigma(G)\right)+\left(\sigma^{(2)}(\overline{G},\Phi)-\sigma(\overline{G})\right). \qquad (5.14)\end{aligned}$$

Since W_0 is a slice disk complement for $K_1\#-K_2$, by Proposition 2.19 (6),

$$\sigma^{(2)}(W_0,\Phi)-\sigma(W_0)=0.$$

All of the ordinary and twisted homology of the cobordisms E, F_1, F_2, G and $\overline{G}$ comes from the boundary [CHL09, Lemma 2.4], and

$$\sigma^{(2)}(X,\Phi)-\sigma(X)=0$$

for $X=E,F_1,F_2,G$ and $\overline{G}$. Since the von Neumann signature defect of W may be computed by using the corresponding ρ-invariant of its boundary [CG85], we have

$$\sigma^{(2)}(W,\Phi)-\sigma(W)=\rho(\partial W,\Phi|_{\partial W})=0.$$

Since the ρ-invariant is additive, (5.14) reduces to

$$\begin{aligned}0=&\rho(M_{J_0},\Phi|_{M_{J_0}})+\rho(M_{R_k},\Phi|_{M_{R_k}})+\rho(M_{\mathcal{R}},\Phi|_{M_{\mathcal{R}}})\\ &+\rho(\overline{M_{J_0}},\Phi|_{\overline{M_{J_0}}})+\rho(\overline{M_{R_k}},\Phi|_{\overline{M_{R_k}}})+\rho(\overline{M_{\mathcal{R}}},\Phi|_{\overline{M_{\mathcal{R}}}}). \qquad (5.15)\end{aligned}$$

In order to prove Theorem 5.2, it suffices to find a contradiction to (5.15). We do so by proving the following claims.

Claim 5.4.

$$\rho(M_{J_0},\Phi|_{M_{J_0}})=\rho_0(J_0).$$

Claim 5.5.

$$\rho(\overline{M_{J_0}}, \Phi|_{M_{J_0}}) = 0.$$

Modulo the proofs of Claims 5.4 and 5.5, (5.15) reduces to

$$|\rho_0(J_0)| = \rho(M_{\mathcal{R}}, \Phi|_{M_{\mathcal{R}}}) + \rho(M_{R_k}, \Phi|_{M_{R_k}}) + \rho(\overline{M_{\mathcal{R}}}, \Phi|_{\overline{M_{\mathcal{R}}}}) + \rho(\overline{M_{R_k}}, \Phi|_{\overline{M_{R_k}}}).$$

In order to obtain a contradiction, we need only choose J_0 such that

$$|\rho_0(J_0)| > 2C_{\mathcal{R}} + 2C_{R_k}.$$

Proof of Claim 5.4. In order to show $\rho(M_{J_0}, \Phi|_{M_{J_0}}) = \rho_0(J_0)$, we must show that the restriction of Φ to $\pi_1(M_{J_0})$ factors nontrivially through abelianization. Since $\pi_1(M_{J_0})$ is normally generated by $\mu(J_0)$, it suffices to show that the inclusion of $\mu(J_0)$ is nontrivial in $\pi_1(W)^{(3)}_x/\pi_1(W)^{(4)}_x$. However, $\mu(J_0)$ is identified with $\beta_k \in \pi_1(M_{R_k})^{(1)}$. Similarly, the meridian of R_k is isotopic in W to $\mu(J)$ which normally generates $\pi_1(M_J)$. Since $\mu(J)$ is identified with $\eta\gamma \in \pi_1(M_{\mathcal{R}})^{(2)}_x$, we see that

$$\pi_1(M_{J_0}) \subset \pi_1(M_J)^{(1)} \subset \pi_1(M_{K_1})^{(3)}_x \subset \pi_1(W_2)^{(3)}_x.$$

The kernel of $\pi_1(W_2) \to \pi_1(W)$ is generated by the longitudes of J_0 and $\overline{J_0}$. Since $\lambda(J_0)$ and $\lambda(\overline{J_0})$ are elements of $\pi_1(M_{J_0})^{(2)}$ and $\pi_1(\overline{M_{J_0}})^{(2)}$ respectively, $\lambda(J_0)$ and $\lambda(\overline{J_0})$ are represented by elements of $\pi_1(W)^{(4)}_x$. By [CHL10, Proposition 4.17],

$$\frac{\pi_1(W)}{\pi_1(W)^{(4)}_x} \cong \frac{\pi_1(W_2)}{\pi_1(W_2)^{(4)}_x}. \tag{5.16}$$

We show $\pi_1(M_{J_0})$ is not mapped to $\pi_1(W)^{(4)}_x$ under inclusion in two steps. In the first step which follows, we ensure that the element represented by the axis $\eta\gamma$ is not contained in $\pi_1(W_2)^{(3)}$. In the second step, we show β_k is nontrivial as an element of $\pi_1(W_2)^{(3)}_x/\pi_1(W_2)^{(4)}_x$.

Consider temporarily, the coefficient system induced on W by

$$\Phi' : \pi_1(W) \to \frac{\pi_1(W)}{\pi_1(W)^{(2)}_x} \equiv \Lambda_2. \tag{5.17}$$

Note that $\Phi'|_{W_2}$ also represents a coefficient system on W_2 and we have the following commutative diagram, which we justify below.

$$\begin{tikzcd}
\pi_1(M_{\mathcal{R}})^{(2)}_x \arrow[r] \arrow[d] & \pi_1(W_2)^{(2)}_x \arrow[r] \arrow[dd] & \dfrac{\pi_1(W_2)^{(2)}_x}{\pi_1(W_2)^{(3)}_x} \arrow[dd] \\
\dfrac{\pi_1(M_{\mathcal{R}})^{(2)}_x}{\pi_1(M_{\mathcal{R}})^{(3)}_x} \arrow[d] & & \\
H_1(M_{\mathcal{R}};\mathbb{Q}\Lambda_2) \arrow[r] & H_1(W_2;\mathbb{Q}\Lambda_2) \arrow[r, "\cong"] & \dfrac{\pi_1(W_2)^{(2)}_x}{\left[\pi_1(W_2)^{(2)}_x, \pi_1(W_2)^{(2)}_x\right]} \otimes_{\mathbb{Z}} \mathbb{Q}
\end{tikzcd} \tag{5.18}$$

By definition, $\pi_1(W_2)^{(3)}_x$ is the kernel of

$$\pi_1(W_2)^{(2)}_x \to \frac{\pi_1(W_2)^{(2)}_x}{\left[\pi_1(W_2)^{(2)}_x, \pi_1(W_2)^{(2)}_x\right]} \otimes \mathbb{Q},$$

and the vertical map on the right-handed side is a monomorphism. Since $\Lambda_2 \cong \pi_1(W_2)/\pi_1(W_2)^{(2)}_x$, it suffices to show the homology class represented by $\eta\gamma$ does not lie in the kernel of the bottom row of the diagram. The first homology of W_2 with $\mathbb{Q}\Lambda_2$-coefficients may be interpreted as the rational first homology of its Λ_2-cover, $\widetilde{W_2}$, which has $\pi_1(\widetilde{W_2}) = \pi_1(W_2)^{(2)}_x$. Thus,

$$H_1(W_2;\mathbb{Q}\Lambda_2) \cong H_1(\widetilde{W_2};\mathbb{Q}) \cong \frac{\pi_1(W_2)^{(2)}_x}{\left[\pi_1(W_2)^{(2)}_x, \pi_1(W_2)^{(2)}_x\right]} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This validates that the second map in the bottom row of (5.18) is an isomorphism. Note that $\partial W_2 = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{\mathcal{R}}} \sqcup \overline{M_J}$. Since W_0 is a slice disk complement, it is also a $(3,x)$-solution for $K_1\# - K_2$. Furthermore, we've already noted that

$$\frac{H_2(E;\mathbb{Z})}{H_2(\partial E;\mathbb{Z})} \cong \frac{H_2(F_1;\mathbb{Z})}{H_2(\partial F_1;\mathbb{Z})} \cong \frac{H_2(F_2;\mathbb{Z})}{H_2(\partial F_2;\mathbb{Z})} \cong 0,$$

and hence W_2 satisfies the definitions to be $(3,x)$-bordism for ∂W_2 [CHL10, Definition 7.11] (as discussed in the proof of Lemma 4.8). Let $\mathcal{P}$ be the kernel of the map

$$H_1(\partial W_2;\mathbb{Q}\Lambda_2) \to H_1(W_2;\mathbb{Q}\Lambda_2).$$

By [COT03, Theorem 2.13] there exists a symmetric linking form on $H_1(\partial W_2;\mathbb{Q}\Lambda_2)$ which we denote by $\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\partial W_2}$, and this form decomposes under the disjoint union $\partial W_2 = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{\mathcal{R}}} \sqcup \overline{M_J}$. Then $\mathcal{P}$ must be an isotropic submodule of $H_1(\partial W_2;\mathbb{Q}\Lambda_2)$ with respect to $\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\partial W_2}$ by [CHL10, Theorem 7.15]. It suffices to show that

$$\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta\gamma,\eta\gamma) \neq 0.$$

Recall that $\Gamma_2 \equiv \pi_1(M_{\mathcal{R}})/\pi_1(M_{\mathcal{R}})^{(2)}_x$, and consider the kernel of the map

$$\Gamma_2 \xrightarrow{\phi} \frac{\pi_1(W_2)}{\pi_1(W_2)^{(2)}_x}.$$

Note that $\pi_1(M_{\mathcal{R}})/\pi_1(M_{\mathcal{R}})^{(1)}$ is generated by $\mu(\mathcal{R})$ which is isotopic in W_2 to the meridian of $K_1\# - K_2$). Since

$$\langle \mu(K_1\# - K_2)\rangle \cong H_1(M_{K_1\#-K_2}) \xrightarrow{\cong} H_1(W_0) \cong H_1(W_2),$$

$\ker\phi \subset \pi_1(M_{\mathcal{R}})^{(1)}/\pi_1(M_{\mathcal{R}})^{(2)}_x$ and is equal to the kernel of

$$\frac{\pi_1(M_{\mathcal{R}})^{(1)}}{\pi_1(M_{\mathcal{R}})^{(2)}_x} \xrightarrow{\bar{\phi}} \frac{\pi_1(W_2)^{(1)}}{\pi_1(W_2)^{(2)}_x}.$$

Note that W_2 is a $(3,x)$-bordism for ∂W_2, and when viewed as $\mathbb{Q}[t,t^{-1}]S_1^{-1}$-modules, this kernel $\tilde{P}$ is isotropic with respect to the localized Blanchfield form. Let $P \subset \pi_1(M_{\mathcal{R}})^{(1)}/\pi_1(M_{\mathcal{R}})^{(2)}_x$ denote the normal subgroup of Γ_2 which is mapped to $\tilde{P}$. This yields a monomorphism,

$$\Gamma_2/P = \Gamma_2^P \overset{\psi}{\hookrightarrow} \frac{\pi_1(W_2)}{,}\pi_1(W_2)^{(2)}_x.$$

which gives rise to the following ring and module homomorphisms respectively.

$$\begin{array}{ll} \psi : \mathbb{Q}\Gamma_2^P \hookrightarrow \mathbb{Q}\left[\frac{\pi_1(W_2)}{\pi_1(W_2)^{(2)}_x}\right] \cong \mathbb{Q}\Lambda_2 & \Psi : \mathcal{K}\Gamma_2^P \to \mathcal{K}\Lambda_2 \\ \psi_* : H_1(M_{\mathcal{R}};\mathbb{Q}\Gamma_2^P) \to H_1(M_{\mathcal{R}};\mathbb{Q}\Lambda_2) & \overline{\Psi} : \mathcal{K}\Gamma_2^P/\mathbb{Q}\Gamma_2^P \to \mathcal{K}\Lambda_2/\mathbb{Q}\Lambda_2 \end{array}$$

By hypothesis, $\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P}(\eta\gamma,\eta\gamma) \neq 0$ for any $P \lhd \pi_1(M_{\mathcal{R}})^{(1)}/\pi_1(M_{\mathcal{R}})^{(2)}_x$ mapping to an isotropic submodule of $H_1(M_{\mathcal{R}};\mathbb{Q}[t,t^{-1}]S_1^{-1})$. We wish to show that $\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Lambda_2}(\eta\gamma,\eta\gamma) \neq 0$ where $\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Lambda_2}$ is the Blanchfield linking form on the $\mathbb{Q}\Lambda_2$-module $H_1(M_{\mathcal{R}};\mathbb{Q}\Lambda_2)$. We employ the following proposition.

Proposition 5.6. *Suppose $f : A \to B$ is a monomorphism between poly-(torsion-free abelian) groups and that f induces the ring and module homomorphisms $f : \mathbb{Q}A \hookrightarrow \mathbb{Q}B$ and $f_* : H_1(M_K;\mathbb{Q}A) \to H_1(M_K;\mathbb{Q}B)$ respectively. Then if $x,y \in H_1(M_K;\mathbb{Q}A)$, we have*

$$\overline{f}\left(\mathcal{B}\ell_K^{\mathbb{Q}A}(x,y)\right) = \mathcal{B}\ell_K^{\mathbb{Q}B}(f_*(x),f_*(y)).$$

where $\overline{f}$ is the induced ring homomorphism $\overline{f} : \mathcal{K}A/\mathbb{Q}A \to \mathcal{K}B/\mathbb{Q}B$.

Proof. Recall that the Blanchfield form $\mathcal{B}\ell_K^{\mathbb{Q}A}$ is given by the composition of the following maps,

$$\begin{aligned} H_1(M_K;\mathbb{Q}A) &\xrightarrow{P.D.} \overline{H^2(M_K;\mathbb{Q}A)} \\ &\xrightarrow{\mathfrak{B}^{-1}} \overline{H^1(M_K;\mathcal{K}A/\mathbb{Q}A)} \\ &\xrightarrow{\kappa} \overline{\mathrm{Hom}_{\mathbb{Q}A}(H_1(M_K;\mathbb{Q}A);\mathcal{K}A/\mathbb{Q}A)}, \end{aligned}$$

where $x \mapsto \mathcal{B}\ell_K^{\mathbb{Q}A}(x,-)$. Here, $P.D.$ refers to the Poincaré Dualitity isomorphism, $\mathfrak{B}^{-1}$ is the inverse of the Bochstein homomorphism, and κ is the Kronecker evaluation map. Note that since $\mathbb{Q}A$ is not necessarily a principal ideal domain, κ is not an isomoprhism and $\mathcal{B}\ell_K^{\mathbb{Q}A}$ may be singular. Consider the following commutative diagram induced by f.

$$\begin{tikzcd} H_1(M_K;\mathbb{Q}A) \arrow[r, "f_*"] \arrow[d, "P.D."'] & H_1(M_K;\mathbb{Q}B) \arrow[d, "P.D"] \\ \overline{H^2(M_K;\mathbb{Q}A)} \arrow[r, "f_*"] \arrow[d, "B^{-1}"'] & \overline{H^2(M_K;\mathbb{Q}B)} \arrow[d, "B^{-1}"] \\ \overline{H^1(M_K;\mathcal{K}A/\mathbb{Q}A)} \arrow[r, "f_*"] \arrow[d, "\kappa"'] & \overline{H^1(M_K;\mathcal{K}B/\mathbb{Q}B)} \arrow[d, "\kappa"] \\ \overline{\mathrm{Hom}_{\mathbb{Q}A}(H_1(M_K;\mathbb{Q}A);\mathcal{K}A/\mathbb{Q}A)} \arrow[d, "\overline{f}"'] & \overline{\mathrm{Hom}_{\mathbb{Q}B}(H_1(M_K;\mathbb{Q}B);\mathcal{K}B/\mathbb{Q}B)} \arrow[dl, "f^*"'] \\ \overline{\mathrm{Hom}_{\mathbb{Q}A}(H_1(M_K;\mathbb{Q}A);\mathcal{K}B/\mathbb{Q}B)} & \end{tikzcd} \tag{5.19}$$

Since the diagram commutes, we have $f^* \circ \mathcal{B}\ell_K^{\mathbb{Q}B} \circ f_* = \overline{f} \circ \mathcal{B}\ell_K^{\mathbb{Q}A}$ where

$$\left(f^* \circ \mathcal{B}\ell_K^{\mathbb{Q}B} \circ f_*(x)\right)(y) = \left(\mathcal{B}\ell_K^{\mathbb{Q}B} \circ f_*(x)\right)(f_*(y)) = \mathcal{B}\ell_K^{\mathbb{Q}B}(f_*(x), f_*(y)).$$

$\square$

Proposition 5.6 implies that $\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Lambda_2} = \overline{\Psi}(\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P})$. We must show that the value of $\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P}(\eta\gamma, \eta\gamma)$ is not in the kernel of $\overline{\Psi}$.

Proposition 5.7. *Suppose A is a subgroup of the poly-(torsion-free abelian) group B. The ring monomorphism $f : \mathcal{K}A \hookrightarrow \mathcal{K}B$ induced by the embedding $A \hookrightarrow B$ yields a ring monomorphism on the quotient*

$$\overline{f} : \mathcal{K}A/\mathbb{Q}A \hookrightarrow \mathcal{K}B/\mathbb{Q}B.$$

Proof. Since $\mathcal{K}A$ embeds as a subring of $\mathcal{K}B$, it suffices to show that

$$\mathcal{K}A \cap \mathbb{Q}B = \mathbb{Q}A.$$

Since $A < B$, fix a set of left coset representatives $\{b_i \in B\}$ such that b_0 is the identity of B. Then $\mathbb{Q}B$ is free as a right $\mathbb{Q}A$-module on the set of left cosets of A. Suppose there exist $r, s \in \mathbb{Q}A$, where $s \neq 0$, $t \in \mathbb{Q}B$, and such that $rs^{-1} = t$. Since t may be written uniquely as the sum

$$t = \sum_i b_i a_i$$

where $a_i \in \mathbb{Q}A$, the equation rs^{-1} may be rewritten as

$$r = \left(\sum_i b_i a_i\right) s = \sum_i b_i(a_i s).$$

Since $r \in \mathbb{Q}A$, it must be that $\sum_i b_i(a_i s) \in \mathbb{Q}A$ as well. Hence $a_i s = 0$ implying $a_i = 0$ for each $i \neq 0$ since $\mathbb{Q}A$ is a domain. Then $t = b_0 a \in \mathbb{Q}A$. $\square$

Proposition 5.7 implies $\overline{\Psi}$ is a ring monomorphism, and $\overline{\Psi}(\mathcal{B}\ell_{\mathcal{R}}^{\mathbb{Q}\Gamma_2^P}(\eta\gamma, \eta\gamma)) \neq 0$ as desired. Hence $\eta\gamma$ represents a nontrivial element of $\pi_1(W)_x^{(2)}/\pi_1(W)_x^{(3)}$ and $\mu(J)$ does as well, concluding the first step of the proof of Claim 5.4.

For the second step, we ensure that β_k is represented by a nontrivial element of $\pi_1(W_2)_x^{(3)}/\pi_1(W_2)_x^{(4)}$. For this, we make use of the following commutative diagram.

$$\begin{tikzcd}
\pi_1(M_J)^{(1)} \arrow[r] \arrow[d] & \pi_1(W_2)_x^{(3)} \arrow[r] \arrow[dd] & \dfrac{\pi_1(W_2)_x^{(3)}}{\pi_1(W_2)_x^{(4)}} \arrow[dd] \\
\mathcal{A}(J) \otimes \mathcal{Q}_3 \arrow[d, "\cong"'] & & \\
H_1(M_J;\mathcal{Q}_3) \arrow[r] & H_1(W_2;\mathcal{Q}_3) \arrow[r, "\cong"] & \dfrac{\pi_1(W_2)_x^{(3)}}{\left[\pi_1(W_2)_x^{(3)}, \pi_1(W_2)_x^{(3)}\right]} \otimes \mathcal{Q}_3
\end{tikzcd} \tag{5.20}$$

Here we denote $\Lambda_3 \equiv \pi_1(W)/\pi_1(W)_x^{(3)} \cong \pi_1(W_2)/\pi_1(W_2)_x^{(3)}$ (5.16) and $\mathcal{Q}_3 \equiv \mathbb{Q}\Lambda_3 S_3^{-1}$. The justification of this diagram follows that of (4.3). Since β_k generates the rational Alexander module of R_k, and $\mathcal{A}(R_k) \cong \mathcal{A}(J)$, $\beta_k \otimes 1$ is the generator of $H_1(M_J;\mathcal{Q}_3)$. We must show that $\beta_k \otimes 1$ is not in the kernel of the bottom row of (5.20).

By the same arguments as before, W_2 may also be viewed as a $(4,x)$-bordism for $\partial W_2 = M_{\mathcal{R}} \sqcup M_J \sqcup \overline{M_{\mathcal{R}}} \sqcup \overline{M_J}$. If $\beta_k \otimes 1 \in \ker\{H_1(M_J;\mathcal{Q}_3) \to H_1(W_2;\mathcal{Q}_3)\}$, this implies $\mathcal{B}\ell_J^{\mathcal{Q}_3}(\beta_k \otimes 1, \beta_k \otimes 1) = 0$. Since $\pi_1(M_J) \to \Lambda_3$ factors through $\mathbb{Z}$, $\mathcal{B}\ell_J^{\mathcal{Q}_3}$ is nonsingular by [CHL10, Lemma 7.16], and it must be that $H_1(M_J;\mathcal{Q}_3) = 0$. We show

$$\mathcal{A}(J) \otimes \mathcal{Q}_3 \cong \left(\frac{\mathbb{Q}\Gamma_3}{\Delta_{R_k}(\eta\gamma)\mathbb{Q}\Gamma_3}\right) S_3^{-1} \neq 0.$$

By hypothesis, $\mathcal{A}^{\mathbb{Z}}(R_k)$ is nontrivial and $\Delta_{R_k}(t)$ is not a unit in $\mathbb{Q}[t,t^{-1}]$. The map $t \mapsto \eta\gamma$ is nontrivial since we showed in the first step of the proof that $\eta\gamma \neq 0$ in $\pi_1(W_2)_x^{(2)}/\pi_1(W_2)_x^{(3)}$. Since $\Lambda_3 \cong \pi_1(W_2)/\pi_1(W_2)_x^{(3)}$ is torsion-free, $\mathbb{Q}\Lambda_3$ is a free left $\mathbb{Q}[\eta\gamma,(\eta\gamma)^{-1}]$-module on the set of right cosets of $\langle\eta\gamma\rangle \subset \Lambda_3$. Fix a set of coset representatives such that any $x \in \mathbb{Q}\Lambda_3$ has a uniqe decomposition as

$$x = \sum_\xi x_\xi \xi$$

where $x_\xi \in \mathbb{Q}[\eta\gamma, (\eta\gamma)^{-1}]$ and each ξ is a coset representiative in Γ_3. If $\Delta_{R_k}(\eta\gamma)$ has a right inverse in $\mathbb{Q}\Gamma_3$, then there exists some $x \in \mathbb{Q}\Gamma_3$ such that $\Delta_{R_k}(\eta\gamma)x = 1$ implying

$$\Delta_{R_k}(\eta\gamma)x = \Delta_{R_k}(\eta\gamma)\sum_\xi x_\xi \xi = \sum \Delta_{R_k}(\eta\gamma)x_\xi \xi = 1.$$

Then on the coset $\xi = e$, we have $\Delta_{R_k}(\eta\gamma)x_e = 1$ in $\mathbb{Q}[\eta\gamma, (\eta\gamma)^{-1}]$ contradicting the fact that $\Delta_{R_k}(t)$ is not a unit in $\mathbb{Q}[t, t^{-1}]$. Since Λ_3 is poly-(torsion-free abelian), $\mathbb{Q}\Lambda_3$ is a domain and

$$\frac{\mathbb{Q}\Lambda_3}{\Delta_{R_k}(\eta\gamma)\mathbb{Q}\Lambda_3} \neq 0.$$

The kernel of

$$\frac{\mathbb{Q}\Lambda_3}{\Delta_{R_k}(\eta\gamma)\mathbb{Q}\Lambda_3} \to \left(\frac{\mathbb{Q}\Lambda_3}{\Delta_{R_k}(\eta\gamma)\mathbb{Q}\Lambda_3}\right) S_3^{-1}$$

is simply the S_3-torsion submodule, and hence we must show that the generator of $\mathbb{Q}\Lambda_3/\Delta_{R_k}(\eta\gamma)\mathbb{Q}\Lambda_3$ is not S_3-torsion. Denote this generator by 1. If it is S_3-torsion, there must be some $s \in S_3$ and $y \in \mathbb{Q}\Lambda_3$ such that $1s = \Delta_{R_k}(\eta\gamma)y$.

Let $A_3 = \pi_1(W_2)_x^{(2)}/\pi_1(W_2)_x^{(3)}$ which is normal in Λ_3. We now view $\mathbb{Q}\Lambda_3$ as a free left $\mathbb{Q}A_3$-module on the set of right cosets of A_3 in Λ_3 where each $y \in \mathbb{Q}\Lambda_3$ now has a unique decomposition as

$$y = \sum_\xi y_\xi \xi$$

where $y_\xi \in \mathbb{Q}A_3$ and each ξ is a coset representative in Λ_3. Then

$$s = \Delta_{R_k}(\eta\gamma)\sum_\xi y_\xi \xi = \sum_\xi \Delta_{R_k}(\eta\gamma)y_\xi \xi.$$

Since $s \in S_3 \subset \mathbb{Q}A_3$ and $\Delta_{\mathcal{R}}(\eta\gamma) \in \mathbb{Q}A_3$, each coset representative $\xi \neq e$ gives $0 = \Delta_{R_k}(\eta\gamma)y_\xi$, but $\mathbb{Q}[\eta\gamma, (\eta\gamma)^{-1}] \subset \mathbb{Q}\Lambda_3$ and $\Delta_{R_k}(\eta\gamma)$ is nonzero in $\mathbb{Q}[\eta\gamma, (\eta\gamma)^{-1}]$ since $\Delta_{R_k}(t)$ is nonzero in $\mathbb{Q}[t, t^{-1}]$. Thus, $y_\xi = 0$ for each $\xi \neq e$. Hence $y \in \mathbb{Q}A_3$ and the equation $s = \Delta_{R_k}(\eta\gamma)y$ is one in $\mathbb{Q}A_3$. By definition, each element of S_3 may be written as a product of terms of the form $\Delta_{R_k}(\tilde{\eta}^g)$ where $\tilde{\eta}^g = g^{-1}\tilde{\eta}g$ for some $g \in \Lambda_3$ and so $\Delta_{R_k}(\eta\gamma)$ must divide a product of terms of the form $\Delta_{R_k}(\tilde{\eta}^{g_i})$.

Since A_3 is torsion-free abelian, we may view $s = \Delta_{R_k}(\eta\gamma)y$ as an equation in $\mathbb{Q}F$ where $F \subset A_3$ is a free abelian group of finite rank. Then $\mathbb{Q}F$ is a unique factorization domain. Choose some basis $\{x_1, x_2, \ldots, x_r\}$ of F such that $\eta\gamma = x_1^m$ for some $m \in \mathbb{Z}_+$. Then $\tilde{\eta}^{g_i} = g_i^{-1}\tilde{\eta}g_i = x_1^{n_{i,1}}x_2^{n_{i,2}}\cdots x_r^{n_{i,r}}$, and $\mathbb{Q}F$ may be viewed as a Laurent polynomial ring in the variables $\{x_1, \ldots, x_r\}$. There must exist some nonzero complex root ζ of $\Delta_{R_k}(x_1^m)$. Let $\tilde{f}(x_1)$ be an irreducible factor of $\Delta_{R_k}(x_1^m)$ of which ζ is a root. Then for some i, $\tilde{f}(x_1)$ must divide $\Delta_{R_k}(x_1^{n_{i,1}}, x_2^{n_{i,2}}, \ldots, x_r^{n_{i,r}})$. For every value of $n_{i,k}$ with $1 < k \leq r$, ζ must be a root of $\Delta_{R_k}(x_1^{n_{i,1}}, x_2^{n_{i,2}}, \ldots, x_r^{n_{i,r}})$, and so each $n_{i,k} = 0$ for $k = 2, \ldots, r$. Then we have $\tilde{\eta}^{g_i} = x_1^n$ for some $n \neq 0$. Recall that

$$\Delta_{R_k}(t) = (k^2+k)t^2 - (2k^2+2k+1)t + (k^2+k),$$

and $(\Delta_{R_k}(t^n), \Delta_{R_k}(t^m)) = 1$ whenever $n \neq \pm m$. Hence

$$\tilde{\eta}^{g_i} = (\eta\gamma)^{\pm 1} \tag{5.21}$$

for some $g_i \in \Lambda_3$. Since $\tilde{\eta}$ and $\eta\gamma$ originate as circles in ∂W_2, $\tilde{\eta}^{g_i}$ and $\eta\gamma$ represent elements of $H_1(\partial W_2; \mathbb{Q}\Lambda_2)$ where their difference can be written as $\tilde{\eta}^{g_i} \pm \eta\gamma$. Then (5.21) implies that this difference lies in the kernel of

$$H_1(\partial W_2; \mathbb{Q}\Lambda_2) \to H_1(W_2; \mathbb{Q}\Lambda_2), \tag{5.22}$$

and it must be that

$$\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\partial W_2}(\tilde{\eta}^{g_i} \mp \eta\gamma, \tilde{\eta}^{g_i} \mp \eta\gamma) = 0.$$

In order to provide a contradiction, we must show that $\tilde{\eta}^g_i \mp \eta\gamma$ does not lie in an isotropic submodule of $H_1(\partial W_2, \mathbb{Q}\Lambda_2)$. Note that since $\partial W_2 = M_{\mathcal{R}} \sqcup \overline{M_{\mathcal{R}}} \sqcup M_J \sqcup \overline{M_J}$, its twisted first homology decomposes as

$$H_1(\partial W_2; \mathbb{Q}\Lambda_2) \cong H_1(M_{\mathcal{R}}; \mathbb{Q}\Lambda_2) \oplus H_1(\overline{M_{\mathcal{R}}}; \mathbb{Q}\Lambda_2) \oplus H_1(M_J; \mathbb{Q}\Lambda_2) \oplus H_1(\overline{M_J}; \mathbb{Q}\Lambda_2),$$

and the action of Λ_2 is invariant on these summands.

Recall by hypothesis $\mathcal{B}\ell^{\mathbb{Q}\Gamma^P_2}_{\mathcal{R}}(\eta\gamma, \eta\gamma) \neq \mathcal{B}\ell^{\mathbb{Q}\Gamma^P_2}_{\mathcal{R}}(\eta, \eta)$. This implies $\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta\gamma, \eta\gamma) \neq \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta, \eta)$ since $\mathcal{K}\Gamma^P_2/\mathbb{Q}\Gamma^P_2 \hookrightarrow \mathcal{K}\Lambda_2/\mathbb{Q}\Lambda_2$ by Proposition 5.7. Recall that $\tilde{\eta}$ originates as a circle in $\overline{M_{\mathcal{R}}}$ and $\eta\gamma$ by an circle in $M_{\mathcal{R}}$. Since $\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}$ is a symmetric linking form,

$$\mathcal{B}\ell^{\mathbb{Q}\Lambda_2} : H_1(\overline{M_{\mathcal{R}}}; \mathbb{Q}\Lambda_2) \to \overline{\mathrm{Hom}(H_1(\overline{M_{\mathcal{R}}}; \mathbb{Q}\Lambda_2); \mathcal{K}\Lambda_2/\mathbb{Q}\Lambda_2)}$$

where $\overline{\mathrm{Hom}(H_1(\overline{M_{\mathcal{R}}}; \mathbb{Q}\Lambda_2); \mathcal{K}\Lambda_2/\mathbb{Q}\Lambda_2)}$ denotes the right $\mathbb{Q}\Lambda_2$-module resulting from involution of $\mathbb{Q}\Lambda_2$ and the left $\mathbb{Q}\Lambda_2$-module $\mathrm{Hom}(H_1(\overline{M_{\mathcal{R}}}; \mathbb{Q}\Lambda_2); \mathcal{K}\Lambda_2/\mathbb{Q}\Lambda_2)$. Hence, if $g \in \Lambda_2 \subset \mathbb{Q}\Lambda_2$, we have

$$\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}(\alpha \cdot g, \beta \cdot g) = \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}(\alpha, \beta)g\overline{g} = \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\alpha, \beta).$$

An easy calculation then yields the following, where $\overline{g_i}$ denotes the image of $g_i \in \Lambda_3$

in $\Lambda_2 \subset \mathbb{Q}\Lambda_2$.

$$\begin{aligned}
\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\partial W_2}(\tilde{\eta}^{g_i} \mp \eta\gamma, \tilde{\eta}^{g_i} \mp \eta\gamma) &= \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}(\tilde{\eta}^{g_i}, \tilde{\eta}^{g_i}) + \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}(\eta\gamma, \eta\gamma) \\
&= \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}(\tilde{\eta} \cdot \overline{g_i}, \tilde{\eta} \cdot \overline{g_i}) + \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta\gamma, \eta\gamma) \\
&= \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{-\mathcal{R}}(\tilde{\eta}, \tilde{\eta}) + \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta\gamma, \eta\gamma) \\
&= -\mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta, \eta) + \mathcal{B}\ell^{\mathbb{Q}\Lambda_2}_{\mathcal{R}}(\eta\gamma, \eta\gamma) \\
&\neq 0
\end{aligned}$$

This contradicts the assumption that the generator of $\mathbb{Q}\Lambda_3/\Delta_{R_k}(\eta\gamma)\mathbb{Q}\Lambda_3$ is S_3-torsion. Hence $H_1(M_J; \mathcal{Q}_3)$ is nontrivial and $\beta_k \otimes 1$ does not lie in the kernel of the bottom row of (5.20). This further implies that $\mu(J_0)$ must be nontrivial as an element of $\pi_1(W)^{(3)}_x/\pi_1(W)^{(4)}_x$ and the restriction of Φ to $\pi_1(M_{J_0})$ factors nontrivially through abelianization. This completes the proof of Claim 5.4. □

The final component of the proof of Theorem 5.2 is the proof of Claim 5.5.

Proof of Claim 5.5. In order to show $\rho(\overline{M_{J_0}}, \Phi|_{\overline{M_{J_0}}}) = 0$, we show that the restriction of Φ to $\pi_1(\overline{M_{J_0}})$ is trivial. This argument is similar to the beginning of the proof of Claim 5.4. Here, $\pi_1(\overline{M_{J_0}})$ is normally generated by the meridian, $\mu(\overline{J_0})$ which is identified with $\overline{\beta_k} \in \pi_1(\overline{M_{R_k}})^{(1)}$. The meridian of $-R_k$ is isotopic in W to $\mu(\overline{J})$ which normally generates $\pi_1(\overline{M_J})$ and is identified with $\tilde{\eta} \in \pi_1(\overline{M_{\mathcal{R}}})^{(2)}_x$. Hence,

$$\pi_1(\overline{M_{J_0}}) \subset \pi_1(\overline{M_J})^{(1)} \subset \pi_1(\overline{M_{K_2}})^{(3)}_x \subset \pi_1(W)^{(3)}_x.$$

If $\tilde{\eta} \in \pi_1(W)^{(3)}_x$, then we are done. Otherwise, suppose $\tilde{\eta}$ is nontrivial in $\pi_1(W)^{(2)}_x/\pi_1(W)^{(3)}_x$. We will show that $\overline{\beta_k}$ is trivial in $\pi_1(W_2)^{(3)}_x/\pi_1(W_2)^{(4)}_x$. We have the following commutative diagram, similar to 5.20.

$$\begin{tikzcd}
\pi_1(\overline{M_J})^{(1)} \arrow[r] \arrow[d] & \pi_1(W_2)_x^{(3)} \arrow[r] \arrow[dd] & \dfrac{\pi_1(W_2)_x^{(3)}}{\pi_1(W_2)_x^{(4)}} \arrow[dd] \\
\mathcal{A}(-J) \otimes \mathcal{Q}_3 \arrow[d, "\cong"'] & & \\
H_1(\overline{M_J}; \mathcal{Q}_3) \arrow[r] & H_1(W_2; \mathcal{Q}_3) \arrow[r, "\cong"] & \dfrac{\pi_1(W_2)_x^{(3)}}{\left[\pi_1(W_2)_x^{(3)}, \pi_1(W_2)_x^{(3)}\right]} \otimes \mathcal{Q}_3
\end{tikzcd} \tag{5.23}$$

Since $\overline{\beta_k}$ generates the rational Alexander module of $-R_k$ and $\mathcal{A}(R_k) \cong \mathcal{A}(J)$, $\overline{\beta_k} \otimes 1$ is the generator of $H_1(\overline{M_J}; \mathcal{Q}_3)$. However,

$$\mathcal{A}(J) \otimes \mathcal{Q}_3 \cong \left(\frac{\mathbb{Q}\Lambda_3}{\Delta_{R_k}(\tilde{\eta})\mathbb{Q}\Lambda_3} \right) S_3^{-1}, \tag{5.24}$$

and the generator is $\Delta_{R_k}(\tilde{\eta})$-torsion and $\Delta_{R_k}(\tilde{\eta}) \in S_3$ by definition. This implies $H_1(\overline{M_J}; \mathcal{Q}_3) = 0$ and $\overline{\beta_k}$ is in the kernel of the top row of the diagram. Since $\overline{\mu(J_0)}$ is identified with $\overline{\beta_k}$, we have $\overline{\mu(J_0)}$ is represented by an element of $\pi_1(W)_x^{(4)}$ and the restriction of Φ to $\pi_1(\overline{M_{J_0}})$ is trivial. $\square$

This concludes the proof of Theorem 5.2. $\square$

5.1 A Higher-Order Example

In this section, we give an example which illustrates the power of Theorem 5.2. Recall R_k is the ribbon knot shown in Figure 5.3 where the k in the box denotes k negative full twists and β_k generates the rational Alexander module of R_k. The Alexander polynomial of R_k is $\Delta_k = (k^2+k)t^2 - (2k^2+2k+1)t + (k^2+k)$. We take $\mathcal{R}$ to the

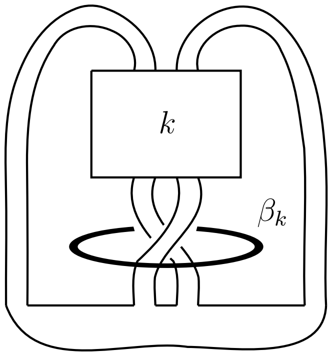


Figure 5.3: The ribbon knot R_k and infecting curve β_k.

ribbon knot $\mathcal{R}_\#$ of Figure 5.4, which may be described as the result of the following infections.

$$\mathcal{R}_\# = R_2(\beta_2, R_1)\# R_3 \tag{5.25}$$

The classical rational Alexander module of $\mathcal{R}_\#$ is given by

$$\mathcal{A}^{\mathbb{Q}}(\mathcal{R}_\#) \cong \mathcal{A}^{\mathbb{Q}}(R_2(\beta_2, R_1)) \oplus \mathcal{A}^{\mathbb{Q}}(R_3) \cong \mathcal{A}^{\mathbb{Q}}(R_2) \oplus \mathcal{A}^{\mathbb{Q}}(R_3)$$

and hence the Alexander polynomial is $(6t^2 - 13t + 6)(12t^2 - 25t + 12)$. Let η be the image of $\beta_3 \subset S^3 \setminus R_3$ in $M_{\mathcal{R}_\#}$. Note that the order of η in $\mathcal{A}^{\mathbb{Q}}(\mathcal{R}_\#)$ is $\Delta_3(t)$. We define the first few terms of the commutator series from the proof of Theorem 5.2 as follows for groups with $\beta_1(G) = 1$.

$$G_x^{(1)} = G_r^{(1)} \tag{5.26}$$

$$G_x^{(2)} = \ker\left\{ G_x^{(1)} \to \frac{G_x^{(1)}}{\left[G_x^{(1)}, G_x^{(1)}\right]} \otimes \mathbb{Q}[t, t^{-1}]\langle \Delta_3(t)\rangle^{-1} \right\} \tag{5.27}$$

$$G_x^{(3)} = \ker\left\{ G_x^{(2)} \to \frac{G_x^{(2)}}{\left[G_x^{(2)}, G_x^{(2)}\right]} \otimes \mathbb{Q}[G/G_x^{(2)}] \right\} \tag{5.28}$$

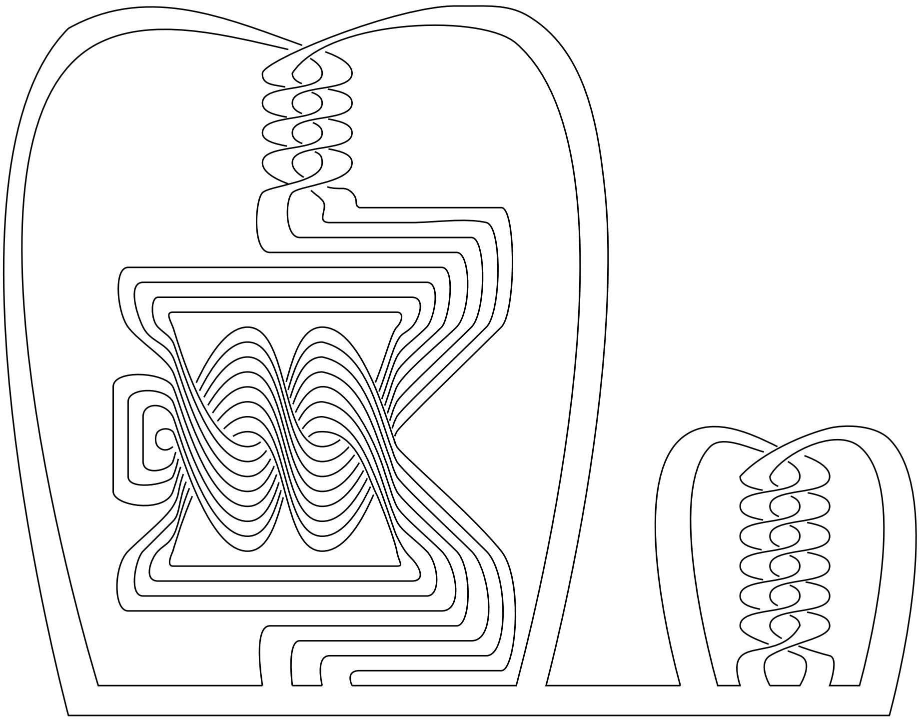


Figure 5.4: The ribbon knot $\mathcal{R}_\# = R_2(\beta_2, R_1)\#R_3$

Denote $\mathbb{Q}[t,t^{-1}]\langle\Delta_3(t)\rangle^{-1}$ by $\mathcal{Q}_1$. Note that the kernel of

$$\frac{\pi_1(M_{\mathcal{R}_\#})^{(1)}}{\pi_1(M_{\mathcal{R}_\#})^{(2)}} \otimes \mathbb{Q}[t,t^{-1}] \to \frac{\pi_1(M_{\mathcal{R}_\#})^{(1)}}{\pi_1(M_{\mathcal{R}_\#})^{(2)}} \otimes \mathcal{Q}_1$$

is the $\Delta_3(t)$-torsion submodule of $\mathcal{A}^{\mathbb{Q}}(\mathcal{R}_\#)$. Hence,

$$H_1(M_{\mathcal{R}_\#};\mathcal{Q}_1) \cong H_1(M_{R_2(\beta_2,R_1)};\mathcal{Q}_1) \oplus H_1(M_{R_3};\mathcal{Q}_1) \cong H_1(M_{R_2};\mathcal{Q}_1) \cong \mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1,$$

and isotropic submodules of $H_1(M_{\mathcal{R}_\#};\mathcal{Q}_1)$ with respect to the localized Blanchfield form correspond to isotropic submodules of $\mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1$.

Let $\Gamma_2 = \pi_1(M_{\mathcal{R}_\#})/\pi_1(M_{\mathcal{R}_\#})^{(2)}_x$ and $\mathcal{K}\Gamma_2$ be the field of fractions of $\mathbb{Q}\Gamma_2$.

In order to provide an example of Theorem 5.2, we must provide a $\gamma \in \pi_1(M_{\mathcal{R}_\#})^{(2)}$ such that

$$\mathcal{B}\ell^{\mathbb{Q}\Gamma_2^P}(\eta,\eta) \neq \mathcal{B}\ell^{\mathbb{Q}\Gamma_2^P}(\eta\gamma,\eta\gamma)$$

for any subgroup P of $\pi_1(M_{\mathcal{R}_\#})^{(1)}/\pi_1(M_{\mathcal{R}_\#})^{(2)}_x \subset \Gamma_2$ which maps to an isotropic submodule of $H_1(M_{R_\#};\mathcal{Q}_1)$ with respect to the localized classical Blanchfield form.

Let $j : \pi_1(M_{R_1}) \to \pi_1(M_{R_\#})$ be the map induced by inclusion. Then since $\mu(R_1)$ is identified with β_2 which represents a nontrivial element of $\pi_1(M_{R_2\# R_3})^{(1)}$, we have

$$j\left(\pi_1(M_{R_1})\right) \subset \pi_1(M_{R_\#})^{(1)}.$$

By an abuse of notation, let j also represent be the map given by taking the quotient

$$j : \frac{\pi_1(M_{R_1})}{\pi_1(M_{R_1})^{(1)}} \to \frac{\pi_1(M_{R_\#})^{(1)}}{\pi_1(M_{R_\#})^{(2)}_x} \to \frac{\pi_1(M_{R_\#})}{\pi_1(M_{R_\#})^{(2)}_x}.$$

Fixing any subgroup P of $\pi_1(M_{\mathcal{R}_\#})^{(1)}/\pi_1(M_{\mathcal{R}})^{(2)}_x \subset \Gamma_2$ which maps to an isotropic submodule with respect to the localized classical Blanchfield form on $H_1(M_{\mathcal{R}_\#};\mathcal{Q}_1)$, define the ring homomorphism

$$\psi : \mathbb{Q}[t,t^{-1}] \xrightarrow{j} \mathbb{Q}\Gamma_2 \twoheadrightarrow \mathbb{Q}\Gamma_2^P$$

given by $t \mapsto \beta_2$. Note that β_2 generates the rational Alexander module of R_2 and hence $\beta_2 \otimes 1$ is a generator of $\mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1$. Since $\mathcal{Q}_1$ is a principal ideal domain, the following composition is an isomorphism

$$\begin{aligned}
\mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1 \cong H_1(M_{R_2};\mathcal{Q}_1) &\underset{\cong}{\xrightarrow{P.D.}} \overline{H^2(M_{R_2};\mathcal{Q}_1)} \\
&\underset{\cong}{\xrightarrow{B^{-1}}} \overline{H^1(M_{R_2};\mathbb{Q}(t)/\mathcal{Q}_1)} \\
&\underset{\cong}{\xrightarrow{\kappa}} \overline{\mathrm{Hom}(H_1(M_{R_1};\mathcal{Q}_1);\mathbb{Q}(t)/\mathcal{Q}_1)},
\end{aligned}$$

and the localized Blanchfield form is nonsingular on $\mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1$. Since $\mathcal{A}^{\mathbb{Q}}(R_2)$ is not $\Delta_3(t)$-torsion, $\mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1 \ncong 0$ and $\mathcal{B}\ell^{\mathcal{Q}_1}_{R_2}(\beta_2\otimes 1, \beta_2\otimes 1) \neq 0$. This implies $\beta_2\otimes 1$ is not a member of any isotropic submodule of $\mathcal{A}^{\mathbb{Q}}(R_2)\otimes\mathcal{Q}_1 \cong H_1(M_{\mathcal{R}_\#};\mathcal{Q}_1)$. Hence,

β_2 is represented by a nontrivial element of $\mathbb{Q}\Gamma_2^P$, and ψ is a monomorphism. By Proposition 5.7, ψ induces the ring monomorphisms

$$\psi' : \mathbb{Q}(t) \hookrightarrow \mathcal{K}_2, \text{ and}$$
$$\overline{\psi} : \mathbb{Q}(t)/\mathbb{Q}[t,t^{-1}] \hookrightarrow \mathcal{K}\Gamma_2^P/\mathbb{Q}\Gamma_2^P.$$

By Proposition 5.6, we have

$$\mathcal{B}\ell_{R_1}^{\mathbb{Q}\Gamma_2^P}(\psi_*(\beta_1), \psi_*(\beta_1)) = \overline{\psi}\left(\mathcal{B}\ell_{R_1}^{\mathbb{Q}}(\beta_1,\beta_1)\right) \neq 0.$$

where $\psi_* : H_1(M_{R_1}; \mathbb{Q}[t,t^{-1}]) \to H_1(M_{R_1}; \mathbb{Q}\Gamma_2^P)$. Suppose

$$i : H_1(M_{R_2\# R_3}, \mathbb{Q}\Gamma_2^P) \to H_1(M_{\mathcal{R}_\#}, \mathbb{Q}\Gamma_2^P), \text{ and}$$
$$j : H_1(M_{R_1}, \mathbb{Q}\Gamma_2^P) \to H_1(M_{\mathcal{R}_\#}, \mathbb{Q}\Gamma_2^P).$$

By Theorem 2.16, the linking form

$$\mathcal{B}\ell_{R_\#}^{\mathbb{Q}\Gamma_2^P} : H_1(M_{\mathcal{R}_\#}; \mathbb{Q}\Gamma_2^P) \times H_1(M_{\mathcal{R}_\#}; \mathbb{Q}\Gamma_2^P) \to \mathcal{K}\Gamma_2^P/\mathbb{Q}\Gamma_2^P$$

is given by the formula

$$\mathcal{B}\ell_{\mathcal{R}_\#}^{\mathbb{Q}\Gamma_2^P}(i(x_1)+j(y_1), i(x_2)+j(y_2)) = \mathcal{B}\ell_{R_2\# R_3}^{\mathbb{Q}\Gamma_2^P}(x_2,y_2) + \overline{\psi}\left(\mathcal{B}\ell_{R_1}^{\mathbb{Q}}(y_1,y_2)\right). \tag{5.29}$$

Suppose γ is the image of $\beta_1 \subset S^3 \setminus R_1$. Since β_1 generates the rational Alexander module of R_1, we have $\mathcal{B}\ell_{R_1}(\beta_1,\beta_1) \neq 0$. Thus

$$\begin{aligned}\mathcal{B}\ell_{\mathcal{R}_\#}^{\mathbb{Q}\Gamma_2^P}(\eta\gamma,\eta\gamma) =& \mathcal{B}\ell_{\mathcal{R}_\#}^{\mathbb{Q}\Gamma_2^P}(i(\eta)+j(\gamma), i(\eta)+j(\gamma)) \\ =& \mathcal{B}\ell_{R_2\# R_3}^{\mathbb{Q}\Gamma_2^P}(\eta,\eta) + \overline{\psi}\left(\mathcal{B}\ell_{R_1}^{\mathbb{Q}}(\gamma,\gamma)\right) \\ \neq& \mathcal{B}\ell_{\mathcal{R}_\#}^{\mathbb{Q}\Gamma_2^P}(\eta,\eta).\end{aligned}$$

This example allows us to generalize the choice of η and γ to provide an infinite family of infecting curves η_i which are equivalent in the classical Alexander module $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}_{\#})$ but which provide distinct concordance classes of knots $\mathcal{R}_{\#}(\eta_i, J)$ when J is chosen as in the proof of Theorem 5.2.

Corollary 5.8. *Let $\mathcal{R}_{\#} = R_2(\beta_2, R_1)\#R_3$, There exists an infinite family of infecting curves η_i which are equivalent in $\mathcal{A}^{\mathbb{Z}}(\mathcal{R}_{\#})$ such that each $\mathcal{R}_{\#}(\eta_i, -)$ is a distinct map on $\mathcal{C}$.*

Proof. For $i \in \mathbb{Z}_{\geq 0}$, let $\eta_i = \eta\gamma^i$ where η be the image of β_3 under the inclusion $S^3 \setminus R_3 \to M_{\mathcal{R}_{\#}}$ and γ be the image of β_1 under the inclusion $S^3 \setminus R_1 \to M_{\mathcal{R}_{\#}}$. As an element of $H_1(M_{\mathcal{R}_{\#}}; \mathbb{Q}\Gamma_2^P)$, η_i may be written as $\eta + i\gamma$, and

$$\mathcal{B}\ell_{\mathcal{R}_{\#}}^{\mathbb{Q}\Gamma_2^P}(\eta_i, \eta_i) = i^2\overline{\psi}\left(\mathcal{B}\ell_{R_1}(\beta_1, \beta_1)\right) + \mathcal{B}\ell_{R_2\#R_3}^{\mathbb{Q}\Gamma_2^P}(\eta, \eta).$$

Hence $\mathcal{B}\ell^{\mathbb{Q}\Gamma_2^P}(\eta_i, \eta_i) = \mathcal{B}\ell^{\mathbb{Q}\Gamma_2^P}(\eta_j, \eta_j)$ if and only if $i^2 = j^2$. It suffices to find a knot J such that $\mathcal{R}_{\#}(\eta_i, J)$ is not concordant to $\mathcal{R}_{\#}(\eta_j, J)$ for $i \neq j$. Choose $J = R_k(\beta_k, J_0)$ such that

$$|\rho_0(J_0)| \geq 2C_{\mathcal{R}_{\#}} + 2C_{R_k}.$$

$\square$

Bibliography

[CFHH12] T. Cochran, B. Franklin, M. Hedden, and P. Horn, Knot concordance and homology cobordism, Proceedings of the American Mathematical Society (2012), To appear.

[CG78] A.J. Casson and C.McA. Gordon., On slice knots in dimension three, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII (Providence, R. I.), American Mathematical Society, 1978, pp. 39–53.

[CG85] J. Cheeger and M. Gromov, Bounds on the von Neumann dimension of l^2-cohomology and the Gauss-Bonnet theorem for open manifolds, Journal of Differential Geometry **21** (1985), no. 1, 1–34.

[CG86] A. J. Casson and C. McA. Gordon, Cobordism of classical knots, Progress in Mathematics, vol. 62, pp. 181–199, Birkhäuser Boston, Boston, MA, 1986, With an appendix by P. M. Gilmer.

[CHL09] T. Cochran, S. Harvey, and C. Leidy, Knot concordance and higher-order Blanchfield duality, Geometric Topology **13** (2009), no. 3, 1419–1482.

[CHL10] ______, Primary decomposition and the fractal nature of the knot concordance group, Mathematische Annalen **351** (2010), no. 2, 443–508.

[CHL11] ______, 2-torsion in the n-solvable filtration of the knot concordance group, Proceedings of the London Mathematical Society **102** (2011), no. 3, 257–290.

[Coc04] T. Cochran, Noncommutative knot theory, Algebraic & Geometric Topology **4** (2004), 347–398.

[Coh00] P. M. Cohn, An introduction to ring theory, Springer undergraduate mathematics series, Springer-Verlag, London, 2000.

[COT03] T. Cochran, K. Orr, and P. Teichner, Knot concordance, whitney towers, and l^2-signatures, Annals of Mathematics **157** (2003), no. 2, 433–519.

[Fox61] R. H. Fox, Some problems in knot theory, Topology of 3-manifolds and related topics, The University of Georgia Institute, 1961, pp. 168–176.

[GL89] C. McA. Gordon and J. Luecke, Knots are determined by their complements, Journal of the American Mathematical Society **2** (1989), no. 2, 371–415.

[Kea78] C. Kearton, Attempting to classify knot modules and their hermitian pairings, Lecture Notes in Mathematics, vol. 685, pp. 227–242, Springer, Berlin, 1978.

[Lan02] S. Lang, Algebra, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002, Revised third edition.

[Lei06] C. Leidy, Higher-order linking forms for knots, Commentarii Mathematici Helvetici **81** (2006), no. 4, 755–781.

[Lev69] J. P. Levine, Knot cobordism groups in codimension two, Commentarii Mathematici Helvetici **44** (1969), 229–244.

[Lic97] W. B. R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, 1997.

[Lit84] R. A. Litherland, Cobordism of satellite knots, Contemporary Mathematics **35** (1984), 327–362.

[Rol90] D. Rolfsen, Knots and links, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990, Corrected reprint of the 1976 original.

[Sei34] H. Seifert, Über das geschlecht von knoten, Mathematische Annalen **110** (1934), no. 1, 571–592.

[Sma62] S. Smale, On the structure of manifolds, American Journal of Mathematics **84** (1962), no. 3, 387–399.

[Ste75] B. Stenström, Rings of quotients, Springer-Verlag, New York, 1975.

[Str74] R. Strebel, Homological methods applied to the derived series of groups, Commentarii Mathematici Helvetici **49** (1974), 302–332.

[Tro73] H. F. Trotter, On s-equivalence of Seifert matrices, Inventiones Mathematicae **20** (1973), 173–207.

[Tro78] ______, Knot modules and Seifert matrices., Lecture Notes in Mathematics, vol. 685, pp. 291–299, Springer, Berlin, 1978.

[Whi44] H. Whitney, The singularities of a smooth n-manifold in $(2n-1)$ dimensions, Annals of Mathematics **45** (1944), no. 2, 247–293.