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Inverse Measures, the Inversion Formula, and Discontinuous Multifractals

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Abstract

The present paper is part I of a series of three closely related papers in which the *inverse measure* $\mu^*(dt)$ of a given measure $\mu(dt)$ on $[0,1]$ is introduced. In the first case discussed in detail, μ and μ^* are multifractal in the usual sense, that is, both are linearly self-similar and continuous but not differentiable and both are non–zero for every interval of $[0,1]$. Under these assumptions the Hölder spectra of $\mu(dt)$ and $\mu^*(dt)$ are shown to be linked by the 'inversion formula' $f^*(\alpha) = \alpha f(1/\alpha)$.

The inversion formula is then subjected to several diverse variations, which reveal telling details of interest to the full understanding of multifractals. The inverse of the uniform measure on a Cantor dust leads us to argue that this inversion formula applies to the Hölder spectra f_{H} even if the measures μ and μ^* are not continuous while it may fail for the spectrum f_{L} obtained by the Legendre path. This phenomenon goes along with a loss of concavity in the spectrum f_{H}. Moreover, with the examples discussed it becomes natural to include the degenerate Hölder exponents 0 and ∞ in the Hölder spectra.

This present paper is the first of three closely related papers on inverse measures, introducing the new notion in a language adopted for the physicist. Parts II and III [RM2, RM3] make rigorous what is argued with intuitive arguments here. Part II extends the common scope of the notion of self-similar measures. With this broader class of invariant measures part III shows that the multifractal formalism may fail.

1 Heuristic proof of the inversion formula

To begin, let us state once again that a multifractal is not a set but a measure. Many multifractals of interest in physics are supported by fractal sets. However, to gain a

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full intuitive understanding of the notion of multifractal, unencumbered by extraneous complication relative to its support, is best achieved in terms of a measure supported by the interval $[0,1]$.

One begins by defining the measure $\mu(dt)$ for the closed intervals of the form $[\,0,t\,]$, in other words, by giving a positive non decreasing function $\mu([0,t])=M(t)$. For other intervals, μ is defined via $\mu(]s,t])=M(t)-M(s)$, $\mu([s,t])=M(t)-M(s-)$, etc. When $M(t)$ has a derivative $M'(t)$, the measure of an infinitesimal interval $\mu(]t,t+dt])$ is the ordinary differential $dM(t)=M'(t)dt$ and μ has the density $M'(t)$. When $M(t)$ is discontinuous at t, $dM(t)$ is the value of that discontinuity $M(t)-M(t-)$. In addition, M is right-continuous. Conversely, any right-continuous, non–decreasing function M with $M(0)=0$, $M(1)=1$ defines a measure μ as above.

Definition of the inverse of a 'basic' multifractal. The usual multifractals are measures that are continuous but not differentiable. In a first stage we require in addition that $M(t)$ is *strictly increasing* so that every interval of t's, however small, has a non–vanishing measure. This is equivalent to saying that the measure is supported on the whole interval $[0,1]$. In a widely used notation, it means that $D_0=1$. In this case, the function $M(t)$ has a well defined inverse function $M^*(\theta)$ that is right-continuous and non–decreasing, hence, defines a second measure $\mu^*(d\theta)$. More precisely, denoting the length of an interval $I=]s,t]$ by $|I|=t-s$ we have

$$\begin{array}{rcccl} \mu(I) & = & M(t)-M(s) & = & |M(I)| \\ \mu^*(M(I)) & = & t-s & = & |I|. \end{array}$$

Picking a point t at random on $[0,1]$ with respect to the measure μ amounts to taking θ at random on $[0,1]$ with uniform probability, and then taking for t the value $M^*(\theta)$. Picking a point θ at random on $[0,1]$ with the measure μ^* amounts to taking t at random on $[0,1]$ with uniform probability, and then taking for θ the value $M(t)$.

Heuristic argument for the inversion formula. Given a multifractal μ described by $f(\alpha)$ let us show that the function $f^*(\alpha)$ of the measure μ^* is given by the *inversion formula*

$$f^*(\alpha)=\alpha f(1/\alpha). \tag{1}$$

First, note that a point t of μ-Hölder exponent α corresponds to a point $\theta=M(t)$ of μ^*-Hölder exponent $\alpha^*=1/\alpha$:

$$\alpha=\lim_{dt\to\{t\}}\frac{\log\mu(dt)}{\log|dt|}=\lim_{M(dt)\to\{\theta\}}\frac{\log|M(dt)|}{\log\mu^*(M(dt))}=\frac{1}{\alpha^*},$$

where the limit is taken over all intervals dt shrinking down to $\{t\}$. Now, divide the interval $]\,0,1]$ on the t-axis into small 'ε-intervals' of length ε. By the definition of $f(\alpha)$, the set K_α of μ-Hölder exponent α can be covered by $N(\varepsilon,\alpha) \simeq \varepsilon^{-f(\alpha)}$ ε-intervals. The μ measure of each of these ε-intervals is approximately ε^α. In other words, the function $M(t)$ maps these ε-intervals to $N(\varepsilon,\alpha)$ intervals, each of length ε^α, covering the set $K^*_{\alpha^*}$ of points θ with μ^*-Hölder exponent $\alpha^* = 1/\alpha$. The dimension of this set is, therefore,

$$f^*(\alpha^*) = \dim(K^*_{\alpha^*}) = -\frac{\log N(\varepsilon,\alpha)}{\log \varepsilon^\alpha} = \frac{f(\alpha)}{\alpha}.$$

It follows that $f^*(\alpha^*) = \alpha^* f(1/\alpha^*)$, as asserted.

2 Examples and Comments

It is crucial to distinguish between the multifractal spectrum $f_{\mathrm{H}} = \dim(K_\alpha)$ and the coarse grained spectrum f_{G}: the former is the Hausdorff dimension of the 'set of Hölder exponent α, while $\delta^{-f_G(\alpha)}$ is roughly equal to the number of cubes C from a δ-grid with $\log\mu(C)/\log\delta \simeq \alpha$. These terms are introduced in more detail in part II and III [RM2, RM3]. There, it is shown that the argument in section 1 holds indeed for both types of spectra provided the measure μ is continuous.

The inversion formula preserves straight lines, therefore exchanges the universal linear bounds of f. Under the transformation $y(\alpha) \to y^*(\alpha) = \alpha y(1/\alpha)$, the straight line $y = A\alpha + B$ becomes the straight line $y^* = A + B\alpha$. Therefore, the well–known inequality $f(\alpha) \le 1$ implies $f^* \le \alpha$. The well–known inequality $f(\alpha) \le \alpha$ implies $f(1/\alpha) \le 1/\alpha$ and $f^* \le 1$. That is, the two lines that provide upper bounds to all f are exchanged in the operation $f \to f^*$. In particular, due to our assumption that $D_0 = 1$, α_0 and α_1 are the values where f and f^* reach these universal upper bounds. Therefore, $\alpha_0^* = 1/\alpha_1$ and $\alpha_1^* = 1/\alpha_0$. The important quantity α_1 characterizes the measure-theoretical support of $\mu(dt)$ and is the the μ-Hölder exponent of almost every point t picked randomly with distribution M. It transforms into α_0^*, which is of far lesser importance but which is the μ^*-Hölder exponent of almost every θ picked randomly with uniform distribution. Finally, $\alpha_{\min}$ and $\alpha_{\max}$ are also interchanged, meaning that ${\alpha^*}_{\min} = 1/\alpha_{\max}$ and ${\alpha^*}_{\max} = 1/\alpha_{\min}$.

Left-sided multifractals (see Mandelbrot [M90]) In a first sub-case $\alpha_0 = \infty$; if so, $f^*(0) = 0$, and $f^*(\alpha^*)$ is tangent to $f^* = \alpha^*$ at ${\alpha^*}_0 = 0$. In a second sub-case $\alpha_0 < \infty$ and $f(\alpha) = f(\alpha_0)$ for $\alpha > \alpha_0$; if so, $f^*(\alpha^*) = \alpha^*$ for $\alpha^* < {\alpha^*}_1$, and $f^*(\alpha^*)$ is tangent to $f^* = \alpha^*$ at ${\alpha^*}_1$. These facts are discussed in Mandelbrot [M90, Section 7] as well as in [RM1].

The degenerate case when $M(t)$, hence $M^*(\theta)$, is continuous and differentiable. In multifractal terms, $\alpha = \lim_{dt\to\{t\}} \log dM/\log dt = 1$ for all t, hence $f(\alpha)$ is defined only for

$\alpha = 1$, where $f(1) = 1$. The same is true of $f^*(\alpha)$. That is, these functions satisfy the inversion formula, trivially.

The "inverse binomial" measure. The binomial is the simplest multiplicative measure. It divides $[0,1]$ into two parts of equal lengths and assigns them masses m_0 and m_1. The usual brute force approach sets up the generating function

$$\tau(q) = -\log_2 \left({m_0}^q + {m_1}^q \right)$$

and obtains f via Legendre transform. An other way is to calculate f in explicit form by solving a 2×2 equation system for α and f [R1].

The next simplest multiplicative measure is the inverse binomial. This measure divides $[0,1]$ into two parts of lengths m_0 and m_1 and assigns them equal masses. The inversion formula yields $f^*(\alpha)$ explicitly, starting with the binomial measure μ. This function $f^*(\alpha)$ yields $\tau^*(q)$, but only in implicit form, and $\tau(q)$ yields $f(\alpha)$.

The inverse of a random multifractal This is not the place to describe in full the general theory of random multifractals presented by Mandelbrot [M89, M90, M95]. This theory introduces functions $f = f_{\mathrm{G}}$ which may have negative values. While the positive $f(\alpha)$ are still Hausdorff Besicovitch dimensions this is *not* true for the negative ones: Their importance lies in their discribing the fluctuations between coarse grained samples of those multifractals. (More precisely, $\delta^{1-f_G(\alpha)}$ is roughly equal to the probability of finding a cube C from a δ-grid with $\log \mu(C) / \log \delta \simeq \alpha$.) In particular, $f_{\mathrm{G}} \neq f_{\mathrm{H}}$ here. Nevertheless, even when $f < 0$, the inversion formula holds for conservative self-similar random multifractals with $D_0 = 1$. This is an immediate consequence of f_{G} being the Legendre transform of the function $\tau(q)$ and of the relation $\tau^* = -q$, $q^* = -\tau$, which is derived from the conservation of mass ($\sum p_i = 1$ almost surely) [R2].

As an example, it is instructing to invert a measure introduces by Mandelbrot [M89, section 3.3.], for which $f(\alpha)$ is defined for all $\alpha > 0$ and equals

$$f(\alpha) = c + \log_2 \alpha - \alpha, \text{ with } c = 1 - \log_2(\log_e 2) + 1/\log_e 2.$$

For this measure, $q = f'(\alpha)$ ranges from an upper bound $q_{\mathrm{top}} = \infty$ down to a lower bound $q_{\mathrm{bottom}} = -1$. Now we see that

$$f^*(\alpha) = \alpha c - \alpha \log_2 \alpha - 1.$$

First consider the unbounded right tail of $f(\alpha)$, where $f'(\alpha) \simeq -1$, so that $q_{\mathrm{bottom}} = -1$. The operation $f \to f^*$ replaces this right tail of f by a bounded left tail of f^* satisfying $f^*(0) = -1$ and also $f^{*\prime}(0) = \infty$, so that ${q^*}_{\mathrm{top}} = \infty$, and $f^{*\prime\prime}(0) = -\infty$. Next consider the unbounded left tail of $f(\alpha)$, where $q_{\mathrm{top}} = \infty$. The operation $f \to f^*$ replaces it by a very steep, unbounded right tail of f^* where $f^*(\alpha^*) \to -\infty$ ($\alpha^* \to \infty$), so that $q_{\mathrm{bottom}} = -\infty$. In other words, μ^* is less 'anomalous' than the original μ.

3 Discontinuous multifractal measures and an ambiguity in the definition of $f(\alpha)$

Our next topic concerns what happens, not only to the inversion formula, but also to the definition of multifractality, when some intervals of t, called *gaps*, have zero measure. Self-similarity then requires the measure μ to concentrate on a fractal dust of measure 0 and dimension $D_0 < 1$. We begin by a very special case.

The devil staircase. The inverse of the uniform Cantor measure $\mu_C(dt)$ is a purely discontinuous measure. It is well-known that for the uniform measure on the Cantor dust, the graph of the function $M(t)$ is the Cantor devil staircase. The devil function is constant over every gap of the Cantor dust or of $\mu(dt)$. Each dyadic value θ of $M(t)$ corresponds to a step of the staircase. The mirror image of the graph of $M(t)$ with respect to the diagonal is the graph of a function that is many valued for each of the dyadic θ. In other words, being a many-to-one function, $M(t)$ does not have a proper unique inverse function $M^*(\theta)$.

It is natural, however, to generalize the notion of inversion to wider classes of multifractals, hoping it will preserve the validity of the inversion formula. To achieve this goal, it suffices to define the measure $\mu^*(d\theta)$ as equal to the sum of the lengths of the all gaps of $\mu(dt)$ such that $\theta \leq M(t) < \theta + d\theta$. This defines the inverse function $M^*(\theta)$ as being continuous to the right, and constructed as follows: Take the mirror image of the graph of $M(t)$ with respect to the diagonal, and when θ is dyadic so that $M^*(\theta)$ was ambiguous, take the highest value in the 'interval' suggested by the mirror image graph.

For the uniform Cantor measure $\mu_C(dt)$, the function $f(\alpha)$ is not as simple as it seems. The conventional wisdom is that this measure is characterized by $f(D) = D$ and $f(\alpha) = 0$ for $\alpha \neq D$. This explains why the homogeneous measure is called *unifractal*. While this conventional wisdom is usually harmless, it is unjustified, and in the present context it would be very misleading. Indeed, the above assertion only takes into account the points in the Cantor set. But we must be more careful and also take into account the points t that lie in the gaps of the Cantor set. For those points

$$\alpha = \lim_{dt \to \{t\}} \frac{\log 0}{\log |dt|} = \infty.$$

Since gaps are of positive (Lebesgue) measure, $f(\infty) = 1$. Thus, we conclude that $f(\alpha)$ includes not one but two points.

Formal application of the inversion formula to the uniform Cantor measure $\mu_C(dt)$. If we start with the usual $f(\alpha)$ limited to one point with $f(D) = D$, a formal application of the inversion formula yields $f^*(1/D) = 1$ and $f^*(\alpha) = 0$ for $\alpha \neq 1/D$. Clearly, this result is completely inadequate. A more careful analysis must add the point of coordinates $\alpha^* = 0$ and $f^* = 0$. This second point on the spectrum expresses that, when μ^* is discontinuous, each discontinuity corresponds to

$$\alpha^* = \lim_{d\theta \to \{\theta\}} \frac{\text{constant}}{\log |d\theta|} = 0,$$

and discontinuities are denumerable and hence form a set of dimension 0. Since the measure μ^* reduces to its discontinuities, one should pay foremost attention to the point of $f^*(\alpha^*)$ which accounts for them, namely $\alpha^* = 0$, $f^* = 0$. In other words, we have $\alpha_1^* = 0$. Recall that the graph of the $f(\alpha)$ of a 'normal' multifractal is tangent to the bisector defined by $f(\alpha) = \alpha$, and that the point of tangency describes the measure–theoretical support of the measure. For $\mu_C(dt)$, this role is played by the point $f(D) = D$ lying on the bisector. Now we see that the same is true of $\mu_C^*(dt)$.

This being granted, the fact that $f^*(1/D) = 1$ seems highly 'anomalous'. But it is easy to explain. It expresses an almost sure property, namely a property of all the non–dyadic points θ. Such a point is defined as the limit of a sequence of dyadic intervals in which the k^{th} interval is of length 2^{-k}. The argument is simplest when these intervals contain the mass 3^{-k}, hence

$$\alpha_0^* = \lim_{d\theta \to \{\theta\}} \frac{\log(1/\mu^*)}{\log(1/|d\theta|)} = \lim_{d\theta \to \{\theta\}} \frac{\log 3}{\log 2} = \frac{1}{D}.$$

Two facts are worth noting : First, non–dyadic points belong to the closure of the set where μ^* is concentrated, and hence to the measure theoretical support of μ^*. Second, since μ is continuous, picking θ randomly with uniform probability amounts to choosing t randomly with respect to μ and letting $\theta = M(t)$. This explains why it is not only natural but even necessary to consider non–dyadic points θ.

These considerations bring us back to the relation

$$\alpha_1 = 1/\alpha_0^* = D \qquad\qquad \alpha_0 = 1/\alpha_1^* = \infty.$$

In other words, the μ-almost sure Hölder exponent α_1 corresponds to the uniformly almost sure μ^*-Hölder exponent α_0^* and vice versa. Obviously, this must have implications to 'realworld' applications and the issue arises how a numerical analysis reflects this drastical change of 'how to choose random points'. The uniform Cantor measure is, however, not suitable for this investigation and the issue becomes more clear at the end of this section.

Multifractal measures supported by the Cantor dust. The conventional wisdom is that such a measure is represented by a function $f(\alpha)$ whose graph is shaped like the symbol $\cap$ (perhaps a bit skewed). The maximum of $f(\alpha)$ is D, and the graph of $f(\alpha)$ has a point of contact with the bisector $f = \alpha$. To be complete, however, it is necessary to add to the graph the point $\alpha = \infty, f = 1$. The resulting shape may seem odd, because it negates the notion that the graph of $f(\alpha)$ is cap convex. Thus, depending on one's purpose, $f(\alpha)$ may take either its conventional form $\cap$, or the form of the left side of its $\cap$ combined with the point $\alpha = \infty, f = 1$.

Inverse of multifractal measures supported by the Cantor dust. Now to $f^*(\alpha^*)$. The graph obtained by applying the inversion formula separately to the two parts of $f(\alpha)$ is made of the origin and of a curve that is again shaped like $\cap$. The $\cap$-curve does have a point of contact with $f^* = 1$, but it *fails* to contact with $f^* = \alpha^*$ since the horizontal tangent to f, namely the line $f = D_0 = D$, is transformed into the line $f^* = D\alpha^*$. The contact with $f^* = \alpha^*$ takes place at $\alpha^* = 0$.

This shape is the correct form of the Hausdorff spectrum f^*_{H} of μ^*. As for the coarse grained spectrum f^*_{G}, negative q's no longer raise any problem. But positive q's do pose a serious difficulty when $q \geq q_{\mathrm{top}} = D$. The reason is that $f^*_{\mathrm{G}}(\alpha)$ can be evaluated from $\tau^*(q)$ since μ^* is self-similar, even though not in the strict sense since some of the ratios vanish (see [RM3]). The partition function $\tau^*(q)$ is evaluated as

$$\tau^*(q) = \lim_{\varepsilon \to 0} \frac{\log \chi(q, \varepsilon)}{\log \varepsilon} \qquad \text{with} \qquad \chi(q, \varepsilon) := \sum \mu^*(I)^q.$$

Here, the sum runs over all ε-intervals from a grid. Since the gaps of the Cantor dust have total length 1 the corresponding atoms of μ^* have total mass 1 and, thus, determine μ^* completely. Let us consider the simple case when μ is constructed by assigning mass 1/2 both to $[0, r_0]$ and $[r_1, 1]$, leaving the middle interval of length $r_2 := 1 - r_0 - r_1 > 0$ without mass. In this gap, $M(t) \equiv 1/2$. In the next step, additional gaps of lengths $r_0 r_2$ and $r_1 r_2$ are created where M takes the values 1/4 and 3/4, respecively. In a further step, gaps of length $r_0 r_0 r_2, \ldots, r_1 r_1 r_2$ are added, the value of M going in steps of 1/8 on the whole family of gaps. Thus, the partition function $\chi(q, \varepsilon)$ of μ^* at stage n is

$$\chi(q, 2^{-n}) = {r_2}^q \sum_{k=0}^{n-1} \sum_{\varepsilon_1 \ldots \varepsilon_k \in \{0,1\}^k} (r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_k})^q = {r_2}^q \Big(1 - ({r_0}^q + {r_1}^q)^n\Big) / (1 - {r_0}^q - {r_1}^q).$$

From this,

$$\tau^*(q) = \begin{cases} -\log_2 ({r_0}^q + {r_1}^q) & \text{for } q \leq D \\ 0 & \text{otherwise,} \end{cases}$$

Conclusions on the multifractal formalism The first half of the so-called multifractal formalism states that f_{G} is the Legendre transform of $\tau(q)$. While this is not true for a general measure it can be shown to hold for self-similar measures [AP, O, R1], even discontinuous ones [RM3]. We conclude that our f^*_{G} is concave. However, it can no longer take the conventional form $\cap$. It must take the form of the top and right portions of its

$\cap$ down to $\alpha^*{}_1 = 1/\alpha_0$, combined with a straight line to the point $\alpha^* = 0, f^* = 0$. This is a consequence of the presence of a whole hierarchy of atoms which produces a non-trivial range of 'frequently occuring' coarse Hölder exponents.

The more important second half of the multifractal formalism states that $f_{\mathrm{H}} = f_{\mathrm{G}}$. Note, that the full multifractal formalism has been shown to hold for quite general constructions of random self-similar measures (see [AP, O, L] and also [KP, CM, F]) as well as in the context of dynamical systems (see [R, PW] and also [BMP, CLP]).

In the presence of gaps, as we have seen, f^*_{H} is not concave. Consequently, $f^*_{\mathrm{H}} \neq f^*_{\mathrm{G}}$, moreover, f^*_{G} is the concave hull of f^*_{H}. Thus, the multifractal formalism does not hold for μ^*. The difference between Hölder spectra and coarse grained spectrum expresses, therefore, the strong dependence of the convergence rate of $\log \mu^*(I)/\log|I| \to \alpha(t)$ on t. In addition, this fact confirms our point of view which is to include all points of $[0,1]$ in the Hölder spectra. Otherwise, a convincing connection between f^*_{G} and f^*_{H} would not exist.

In summary, the inversion formula holds for the Hölder spectra f_{H} in general and for the coarse grained spectrum f_{G} only for continuous measures.

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References

[AP] M. Arbeiter and N. Patzschke, Self-Similar Random Multifractals, *Math. Nachr.* **181** (1996) pp 5–42

[BMP] G. Brown, G. Michon and J. Peyrière, On the Multifractal Analysis of Measures, *J. Stat. Phys.* **66** (1992) pp 775–790

[CM] R. Cawley and R. D. Mauldin, Multifractal Decompositions of Moran Fractals, *Adv. in Math.* **92** (1992) pp 196–236

[CLP] P. Collet, J. Lebovitc and A. Porcio, The Dimension Spectrum of Some Dynamical Systems, *J. Stat. Phys.* **47** (1987) pp 609–644.

[F] K. J. Falconer, The Multifractal Spectrum of Statistically Self-Similar Measures, *Report PM-93-01, University of Bristol* (1993)

[KP] J.-P. Kahane and J. Peyrière, Sur Certaines Martingales de Benoit Mandelbrot, *Adv. Math.* **22** (1976) pp131–145

[L] K.-S. Lau and S.-M. Ngai, Multifractal Measures and a Weak Separation Condition, *Adv. Math.* accepted for publication (1995).

[M89] B. B Mandelbrot A Class of Multifractal Measures with Negative (Latent) Values for the 'Dimension' $f(\alpha)$. *Fractals' Physical Origin and Properties* edited by Luciano Pietronero, New York Plenum, 1989, pp 3–29

[M90] B. B Mandelbrot New 'Anomalous' Multiplicative Multifractals: Left-Sided $f(\alpha)$ and the Modelling of DLA. *Physica* **A 168**, 1990, pp 95–111

[M95] B. B Mandelbrot, Negative dimensions and Hölders, multifractals and their Hölder spectra, and the role of lateral preasymptotics in science, *Fourier Anal. Appl.* **2**, 1995

[O] L. Olsen, Random geometrically graph directed self-similar multifractals, submitted to *Mem. Amer. Math. Soc.*

[PW] Y. Pesin and H. Weiss, A Multifractal Analysis o fEquilibrium Measures for Conformal Expanding Maps and Markov Moran Geometric Constructions, *preprint Pen State Univ.* (1995)

[R] D. A. Rand, The Singularity Spectrum $f(\alpha)$ for Cookie-Cutters, *Erg. Th. Dyn. Sys.* bf 9 (1989) pp 527–541

[R1] R. H. Riedi, An Improved Multifractal Formalism and Self-Similar Measures, *J. Math. Anal. Appl.* **189** (1995) pp 462–490

[R2] R. H. Riedi, Seven Definitions of Multifractal Scaling Exponents *forthcoming*

[RM1] R. H. Riedi and B. B Mandelbrot, Multifractal Formalism for Infinite Multinomial Measures, *Adv. Appl. Math.* **16** (1995) pp 132–150

[RM2] R. H. Riedi and B. B Mandelbrot, Inversion formula for Continuous Multifractals, *Adv. Appl. Math.* **19** (1997) pp 332–354

[RM3] R. H. Riedi and B. B Mandelbrot, Exceptions to the Multifractal Formalism for Discontinuous Measures, *Math. Proc. Cambr. Phil. Soc.* **123** (1998) 133-157.