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The Spectrum of the Off-diagonal Fibonacci Operator

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Abstract

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The family of off-diagonal Fibonacci operators can be considered as Jacobi matrices acting in $\ell^2(\mathbb{Z})$ with diagonal entries zero and off-diagonal entries given by sequences in the hull of the Fibonacci substitution sequence. The spectrum is independent of the sequence chosen and thus the same for all operators in the family. The spectrum is purely singular continuous and has Lebesgue measure zero. We will consider the trace map and its relation to the spectrum. Upper and lower bounds for the Hausdorff and lower box counting dimensions of the spectrum can be found under certain restrictions of the elements of the Fibonacci substitution sequence, and results from hyperbolic dynamics can be used to show that equality can be achieved between the two dimensions.

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For Eliot.

Contents

List of Figures

Chapter 1

Introduction

The off-diagonal Fibonacci operator is of interest particularly due to its connection with quasicystals, which will be discussed more in Section 2.1. Let us construct the off-diagonal model: Following the notation in [7], let $a, b \in \mathbb{R}_+$ with $a \neq b$ and consider the Fibonacci substitution $S(a) = ab$, $S(b) = a$. Under the substitution, $S^2(a) = aba$, $S^3(a) = abaab$, $S^4(a) = abaababa$, etc. The substitution has the following property:

$$S^k(a) = S^{k-1}(S(a)) = S^{k-1}(ab) = S^{k-1}(a)S^{k-1}(b) = S^{k-1}(a)S^{k-2}(a). \tag{1.1}$$

Note that for $k \geq 1$, the sequence $S^{k-1}(a)$ has finite length F_k, the kth Fibonacci number, with $F_0 = F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$. Beginning with a and iterating this substitution gives a one-sided infinite sequence u that is invariant under the substitution; $u = abaababaabaababaababaabaab....$ Beginning with $b|a$, where $|$ denotes the origin, and iterating S^2 gives a bi-infinite sequence, which we will denote ω_s. If we let $\omega_s(n)$ denote the nth place in the sequence, we see $\omega_s(-2) = a, \omega_s(-1) = a, \omega_s(0) = b, \omega_s(1) = a, \omega_s(2) = b$, etc. Note that the iteration of S^2 is necessary, because for

$k > 0$, the sequence $S^{2k}(b)$ ends in ab and the sequence $S^{2k+1}(b)$ ends in ba. The following property holds:

$$S^{2k}(b) = S^{2k-1}(b)S^{2k-2}(b)$$

and thus we see that to the left of the origin, the sequence ends with $S^{2k}(b)$ for every $k \geq 0$.

The hull Ω is all two-sided sequences that locally look like u:

$$\Omega = \Omega(\omega) = \{\omega \in \{a,b\}^{\mathbb{Z}} \mid \text{every subword of } \omega \text{ is a subword of } u\}.$$

In the standard basis the operator of interest can be expressed as the Jacobi matrix

$$H_\omega = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & 0 & \omega(-1) & 0 & & \\ \ddots & \omega(-1) & 0 & \omega(0) & 0 & \\ & 0 & \omega(0) & 0 & \omega(1) & \ddots \\ & & 0 & \omega(1) & 0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

which acts on $\ell^2(\mathbb{Z})$ by

$$(H_\omega u)(n) = \omega(n+1)u(n+1) + \omega(n)u(n-1).$$

The family of operators $\{H_\omega\}_{\omega\in\Omega}$ is called the off-diagonal Fibonacci model.

To consider the spectra $\sigma(H_\omega)$, the following result from [7] is quite useful.

Theorem 1.1. *The spectrum of H_ω is independent of ω; $\sigma(H_\omega) = \Sigma_{a,b}$ for $\omega \in \Omega$. Moreover, $\Sigma_{a,b}$ is a compact set of Lebesgue measure zero.*

And thus we now denote the spectrum of H_{ω_s} as $\Sigma_{a,b}$. We will see that the spectrum $\Sigma_{a,b}$ is related to the so-called trace map, which is the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x, y, z) = (2xy - z, x, y)$. We can use the trace map to define the spectrum $\Sigma_{a,b}$. Given $x_1 = \frac{E}{2a}$, $x_0 = \frac{E}{ab}$, and $x_{-1} = \frac{a^2+b^2}{2ab}$, which we will see in Section 3.1 arise organically from the operator H_{ω_s}, we can define a bi-infinite sequence $\{x_n\}$ that depends on E and a and b from the trace map:

$$T^{-1}(x_{n+2}, x_{n+1}, x_n) = (x_{n+1}, x_n, x_{n-1}) = T(x_n, x_{n-1}, x_{n-2}).$$

Then we can make the following statement:

Theorem 1.2. *The spectrum $\Sigma_{a,b}$ is precisely given by the energies E such that the sequence $\{x_n(E)\}$ is bounded in the forward direction.*

Also from [7] comes the next theorem, though in Section 3.2 a slightly different proof of this statement will be given:

Theorem 1.3. *The spectrum $\Sigma_{a,b}$ is purely singular continuous.*

Note that while the absence of eigenvalues is shown in [7] for all $\omega \in \Omega$, we will just prove it for the sequence ω_s.

The zero Lebesgue measure of the spectrum naturally raises the question of the fractal dimension of the spectrum. Recall the definitions of the box counting and Hausdorff dimensions:

The lower box counting dimension of a bounded set $S \subset \mathbb{R}$ is defined as

$$\dim_{\bar{B}}(S) = \liminf_{\varepsilon \to 0} \frac{\log N_S(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where

$$N_s(\varepsilon) = \#\{j \in \mathbb{Z} \mid [j\varepsilon, (j+1)\varepsilon) \cap S \neq \emptyset\}.$$

The upper box counting dimension is defined similarly, with lim sup in place of lim inf. If the upper and lower box counting dimensions are equal, we say the box counting dimension $\dim_B(S)$ exists, and $\dim_B(S) = \dim_B^-(S) = \dim_B^+(S)$.

Hausdorff dimension in $\mathbb{R}$ is defined as follows: Consider a δ-cover of a set $S \subset \mathbb{R}$; i.e., a countable union of intervals $\bigcup_{i=1}^{\infty} I_i$ such that the length of each interval I_i is bounded above by $\delta > 0$ and $S \subset \bigcup_{i=1}^{\infty} I_i$. Then let

$$h^\alpha(S) = \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \sum_{i=1}^{\infty} |I_i|^\alpha,$$

and the Hausdorff dimension is

$$\dim_H(S) = \inf\{\alpha \mid h^\alpha(S) < \infty\} = \sup\{\alpha \mid h^\alpha(S) = \infty\}.$$

Note that if for some α we have $h^\alpha(S) = 0$, then $h^{\alpha'}(S) = 0$ for all $\alpha' > \alpha$. Also if for some α we have $h^\alpha(S) = \infty$, then $h^{\alpha'}(S) = \infty$ for all $\alpha' < \alpha$.

To consider the fractal dimension of the spectrum, first we must define the constant c to be the positive square root of $\frac{a^4 - 2a^2b^2 + b^4}{4a^2b^2}$, which is determined by whether $a > b$ or $b > a$, and also define $f^* = \log(1 + \sqrt{2})$. Again, it will be seen how these quantities arise in the later sections. We will find the following bounds on the lower box counting dimension and the Hausdorff dimension.

Theorem 1.4. *If $c > 4$, then*

$$\dim_B^-(\Sigma_{a,b}) \geq \frac{f^*}{\log(4c + 14)}.$$

Theorem 1.5. *If $c > 4$, then*

$$\dim_H(\Sigma_{a,b}) \leq \frac{f^*}{\log(c - 2 + \sqrt{c^2 - 4c + 1})}.$$

Finally, with the help of hyperbolic dynamics, we will see the following result.

Theorem 1.6. *For $c \geq 6$, the box counting dimension of the spectrum $\Sigma_{a,b}$ exists and* $\dim_H(\Sigma_{a,b}) = \dim_B(\Sigma_{a,b})$.

In Chapter 2, we will go into more depth about the discovery of quasicrystals and their importance, and how the off-diagonal model relates. We will also compare similar results of the (diagonal) Fibonacci operator, a model closely related to the off-diagonal model. Finally, we will state some important definitions and results from hyperbolic dynamics.

In Chapter 3, we start to consider the off-diagonal model in the specific case where $a > b$. First, in Section 3.1 we will consider basic properties of the sequence ω_s and see how it relates to the operator H_{ω_s}. Then in Section 3.2 and Section 3.3, the proofs of Theorem 1.2 and Theorem 1.3 and the proofs of Theorem 1.4 and Theorem 1.5 will be given. In Section 3.4, hyperbolicity will be considered, and results from hyperbolic dynamics will be used to prove Theorem 1.6. Finally, in Section 3.5, the necessary changes will be made to certain lemmas and proofs to show that the main results all hold for the case when $b > a$.

Chapter 2

Background

2.1 Quasicrystals

Quasicrystals are solids whose atomic arrangements have symmetries that are forbidden for periodic crystals. The rotational symmetries allowed for crystals are 2, 3, 4, and 6-fold [2]. The discovery of such alloys with sharp diffraction spots not consistent with crystallographic symmetries was reported in 1984 by Shechtman et al [23]. According to a 1985 paper by Zia and Dallas [27], a quasi-crystalline structure may be mathematically represented as a sum of delta functions located at a discrete set of points in D-dimensional space, distributed neither randomly nor periodically. With D=2, for example, these points may be located at vertices of a Penrose tiling pattern. Therefore, aperiodic tilings naturally arise in the study of quasicrystals. The so-called cut and project method from [11] can be used to construct an almost periodic tiling of the line, and thus a one-dimensional quasicrystal. The method is as follows: Start with a square lattice in the plane and choose a line with irrational slope. Then consider the set of lattice points that lie within a certain distance from the line, and project them onto the line. Particularly, with the lattice $\mathbb{Z}^2$ and the slope $\frac{1}{\tau}$, where $\tau = \frac{\sqrt{5}+1}{2}$ is the golden ratio, and with a strip that coincides with the unit square shifted along the line, one obtains the Fibonacci substitution sequence ω_s. That is, one obtains short and long intervals, and the arrangement of the intervals follows a Fibonacci substitution sequence. The cut and project method can be used to "create" a quasicrystal from a periodic pattern in higher dimensions as well [27].

More thoroughly studied has been the (diagonal) Fibonacci operator, going back to 1983 [16], [24], at first in a context unrelated to quasicrystals. Such Schrödinger operators with quasiperiodic potentials are used in characterizing the properties of quasicrystals. Though the diagonal model, which we will look at in the next section, has been more widely studied, the off-diagonal model is just as relevant in the study of quasicrystals, and perhaps even more fitting considering the ability to construct the Fibonacci substitution sequence via the cut and project method.

2.2 The Diagonal Model

In this section we will consider the (diagonal) Fibonacci operator. This operator is closely related to the off-diagonal operator, not only in construction but with their spectral properties as well.

The (diagonal) Fibonacci operator is the discrete one-dimensional Schrödinger operator acting in $\ell^2(\mathbb{Z})$ given by

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n). \tag{2.1}$$

The potential $V : \mathbb{Z} \to \mathbb{R}$ is

$$V(n) = \lambda \chi_{[1-\phi,1)}(n\phi + \theta \mod 1),$$

where $\lambda > 0$ is the coupling constant, $\phi = \frac{\sqrt{5}-1}{2}$ is the reciprocal of the golden ratio, and $\theta \in [0,1)$ is the phase. This is closely related to the off-diagonal model, mostly perhaps by its similar relation to the trace map.

The spectrum of H is independent of the phase, but does depend on the coupling [4]. Therefore the spectrum $\sigma(H)$ will be denoted by Σ_λ. It was shown by Sütő [21] that for $\theta = 0$ there are no eigenvalues in Σ_λ, and that the spectrum is a Cantor set for $\lambda \geq 4$. He later went on the show [22] that for any $\lambda \neq 0$ and for $\theta = 0$, the spectrum is a Cantor set of zero Lebesgue measure, and the spectrum is purely singular continuous. There are further partial results from Hof-Knill-Simon [13] and Kaminaga [14], and Damanik-Lenz [8] proved the absence of eigenvalues for all phases, thus implying a purely singular continuous spectrum. Then, of natural interest due to the zero measure of the spectrum, is the fractal dimension of the spectrum.

Let $f^* = \log(1+\sqrt{2})$ be defined as in the introduction. From [6], we see that for $\lambda > 4$ and $\lambda \geq 8$, respectively, the lower box counting dimension and the Hausdorff dimension of the spectrum have the following bounds:

$$\dim_{\overline{B}}(\Sigma_\lambda) \geq \frac{f^*}{\log(2\lambda + 22)};$$

$$\dim_H(\Sigma_\lambda) \leq \frac{f^*}{\log\left(\frac{1}{2}(\lambda - 4 + \sqrt{(\lambda-4)^2 - 12})\right)}.$$

Finally, in [6], it was shown, using hyperbolic dynamics and results from [5], that for $\lambda \geq 16$, that the box counting dimension of Σ_λ exists and $\dim_B(\Sigma_\lambda) = \dim_H(\Sigma_\lambda)$. Thus, as a corollary, $\lim\limits_{\lambda\to\infty} \dim(\Sigma_\lambda) \cdot \log \lambda = f^*$.

2.3 Hyperbolic Dynamics

We will see that the theory of hyperbolic dynamical systems is important for proving results about the fractal dimension of the spectrum of the off-diagonal Fibonacci operator. Therefore, provided in this section are pertinent definitions and theorems from the theory of hyperbolic dynamics. First, we define what it means to be a hyperbolic set.

Definition 2.1. Suppose M is a manifold and f is a map defined on M. Let $\Lambda \subset M$ be a compact invariant set; that is, let $f(\Lambda) = \Lambda$, on which f is invertible. Then Λ is said to be a hyperbolic set if the tangent bundle over Λ admits a decomposition

$$T_\Lambda M = E^u \oplus E^s,$$

invariant under Df and such that $\|Df^{-n}(x) \restriction_{E^u_x} \| \leq c\kappa^n$ and $\|Df^n(x) \restriction_{E^s_x} \| \leq c\kappa^n$ for every $x \in \Lambda, n \in \mathbb{N}$ and for some $c > 0$, $\kappa \in (0,1)$. Moreover, if there is an open neighborhood V of Λ such that $\Lambda = \Lambda^f_V := \bigcap_{n \in \mathbb{Z}} f^n(\overline{V})$, then Λ is said to be locally maximal, or basic.

Proving that a set is hyperbolic just from the definition might be tricky, but the following is a theorem giving a cone criterion that will be useful in proving hyperbolicity. From [12]:

Theorem 2.2. *A compact f-invariant set Λ is hyperbolic if and only if there exist $\lambda < 1 < \mu$ such that at every $x \in \Lambda$ there are complementary subspaces S_x and T_x (in general, not Df-invariant), a field of horizontal cones $H_x \supset S_x$, and a family of vertical cones $V_x \supset T_x$ associated with that decomposition such that*

$Df_x H_x \subset \mathrm{Int} H_{f(x)}, Df_x^{-1} V_{f(x)} \subset \mathrm{Int} V_x, \|Df_x \xi\| \geq \mu \|\xi\|$ for $\xi \in H_x$, *and* $\|Df_x^{-1}\xi\| \geq \lambda^{-1}\|\xi\|$ for $\xi \in V_{f(x)}$.

Next, we want to define the stable and unstable manifolds of a hyperbolic set. First, we define the stable and unstable sets of a point in the hyperbolic set.

Definition 2.3. For $x \in \Lambda$ and a small $\varepsilon > 0$, define

$$W_\varepsilon^s(x) = \{w \in U \mid d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \geq 0\}$$

and

$$W_\varepsilon^u(x) = \{w \in U \mid d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \leq 0\}$$

to be the local stable and unstable sets. Then the (global) stable and unstable sets for a point $x \in \Lambda$ are given by

$$W^s(x) = \bigcup_{n \in \mathbb{Z}_+} f^{-n}(W_\varepsilon^s(f^n(x))) \quad \text{and} \quad W^u(x) = \bigcup_{n \in \mathbb{Z}_+} f^n(W_\varepsilon^u(f^{-n}(x))).$$

Finally, the stable and unstable manifolds of a hyperbolic set Λ are given by

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x) \quad \text{and} \quad W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x).$$

Of interest will be certain results relating hyperbolic sets and dimension. But first, we have the following theorem relating foliations and locally maximal hyperbolic sets. For Λ, a stable foliation $\mathcal{F}^s$ is a foliation of a neighborhood of Λ such that for each $x \in \Lambda$, the foliation $\mathcal{F}^s(x)$ is tangent to E_x^s, and $f(\mathcal{F}^s(x)) \subset \mathcal{F}^s(f(x))$.

The following theorem comes from [19], Appendix 1, theorem 8.

Theorem 2.4. *Let $\mu \in \Sigma$ and M be an ambient manifold. Let $\Phi : \Sigma \times M \to \Sigma \times M$ be defined by $\Phi(\mu, x) = (\mu, \phi_\mu(x))$ where $\phi_\mu(x)$ is a diffeomorphic C^k function of (μ, x).*

Let $\Lambda_\mu \subset M$ be a basic set of the diffeomorphism and let U be a small neighborhood of Λ_μ. Then if Φ is C^2, there are transverse invariant foliations $\mathcal{F}^s_\mu(x), \mathcal{F}^u_\mu(x)$ defined on U such that the maps $(\mu, x) \mapsto T_x\mathcal{F}^s_\mu(x)$ and $(\mu, x) \mapsto T_x\mathcal{F}^u_\mu(x)$ are C^1.

Now let us define the limit capacity of a compact $X \subset \mathbb{R}$ to be

$$c(X) = \limsup_{\varepsilon \to 0} \frac{\log(C(X, \varepsilon))}{-\log \varepsilon},$$

where $C(X, \varepsilon)$ denotes the minimal number of ε-neighborhoods needed to cover X. Note that this corresponds to the upper box counting dimension. The next two theorems relating the Hausdorff dimension and the limit capacity are from [20].

Theorem 2.5. *Let $f : M \to M$ be a diffeomorphism and Λ a basic set for f. Suppose $\dim E^u = 1$, where E^u is the unstable subspace of the hyperbolic splitting of Λ for f. Then $\dim_H(W^u_\varepsilon(x) \cap \Lambda)$, $c(W^u_\varepsilon(x) \cap \Lambda)$ are continuous functions of f and are independent of $x \in \Lambda$. Moreover, $\dim_H(W^u_\varepsilon(x) \cap \Lambda) = c(W^u_\varepsilon(x) \cap \Lambda)$.*

As the lower box counting dimension is bounded between the upper box counting dimension and the Hausdorff dimension, this theorem implies that the box counting dimension exists, and that $\dim_H(W^u_\varepsilon(x) \cap \Lambda) = \dim_B(W^u_\varepsilon(x) \cap \Lambda)$. Now consider the same notation and setup as the previous theorem.

Theorem 2.6. *Let $\dim M = 2$, and $\dim E^u = \dim E^s = 1$. Then $\dim_H(\Lambda) = \dim_H(W^u_\varepsilon(x) \cap \Lambda) + \dim_H(W^s_\varepsilon(x) \cap \Lambda)$ and $c(\Lambda) = c(W^u_\varepsilon(x) \cap \Lambda) + c(W^s_\varepsilon(x) \cap \Lambda)$ and $\dim_H(\Lambda) = c(\Lambda)$ is a continuous function of f.*

Combining these results, it is clear that for a locally maximal hyperbolic set on a surface, the relations $\dim_H(W^s_\varepsilon(x) \cap \Lambda)$ and $\dim_B(W^s_\varepsilon(x) \cap \Lambda)$ are also continuous

x-independent functions of f, and $\dim_H(W^s_\varepsilon(x) \cap \Lambda) = \dim_B(W^s_\varepsilon(x) \cap \Lambda)$.

Chapter 3

The Off-diagonal Fibonacci Hamiltonian

3.1 Basic Properties of the Model

In this section we will discuss and derive some basic properties of the off-diagonal model. Particularly, we will see how the trace map arises in its association to the spectrum of the model. Ultimately we will derive results for the spectrum $\Sigma_{a,b}$ from these properties. From Theorem 1.1 we know $\Sigma_{a,b}$ is the spectrum of H_ω for all $\omega \in \Omega$, so we just wish to consider the sequence ω_s and the corresponding operator H_{ω_s}. Recall that the sequence ω_s is constructed by iterating the substitution S^2, where $S(a) = ab$, $S(b) = a$, on $b|a$. For ease of notation, let $\omega_s = \omega$ for the remainder of this paper.

Recall we have $a, b \in \mathbb{R}_+$ with $a \neq b$. Let us consider the off-diagonal model with $a > b$. All the results for the case $b > a$ will be addressed in Section 3.5.

Also recall from the introduction that in the standard basis the operator can be expressed as the Jacobi matrix

$$H_\omega = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & 0 & \omega(-1) & 0 & & \\ \ddots & \omega(-1) & 0 & \omega(0) & 0 & \\ & 0 & \omega(0) & 0 & \omega(1) & \ddots \\ & & 0 & \omega(1) & 0 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

which acts on $\ell^2(\mathbb{Z})$ by

$$(H_\omega u)(n) = \omega(n+1)u(n+1) + \omega(n)u(n-1).$$

Consider the difference equation

$$(H_\omega u)(n) = \omega(n+1)u(n+1) + \omega(n)u(n-1) = Eu(n). \tag{3.1}$$

If we define $U_n = \begin{pmatrix} u(n) \\ \omega(n)u(n-1) \end{pmatrix}$ and $T_n(\omega, E) = \frac{1}{\omega(n)} \begin{pmatrix} E & -1 \\ \omega(n)^2 & 0 \end{pmatrix}$, then u solves (3.1) for every $n \in \mathbb{Z}$ if and only if U_n solves $U_n = T_n(\omega, E)U_{n-1}$ for every $n \in \mathbb{Z}$. Then $U_n = M_n(\omega, E)U_0$, where $M_n(\omega, E) = T_n(\omega, E) \times \cdots \times T_1(\omega, E)$. Note that $\det T_n(\omega, E) = 1$ and thus $\det M_n(\omega, E) = 1$. Recall from (1.1) that the Fibonacci substitution has the property $S^k(a) = S^{k-1}(a)S^{k-2}(a)$. As the sequence given by $S^k(a)$ determines the non-E entries of $T_1(\omega, E), T_2(\omega, E), \ldots, T_{F_{k+1}}(\omega, E)$ and thus of $M_{F_{k+1}}(\omega, E)$, we have

$$M_{F_{k+1}}(\omega, E) = M_{F_{k-1}}(\omega, E)M_{F_k}(\omega, E). \tag{3.2}$$

Define $x_k := x_k(E) = \frac{1}{2}\text{tr } M_{F_k}(\omega, E)$ for $k \geq 1$. So as

$$
\begin{aligned}
M_{F_1}(\omega, E) &= \frac{1}{a}\begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix}, \\
M_{F_2}(\omega, E) &= \frac{1}{b}\begin{pmatrix} E & -1 \\ b^2 & 0 \end{pmatrix}\frac{1}{a}\begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix} = \frac{1}{ab}\begin{pmatrix} E^2 - a^2 & -E \\ Eb^2 & -b^2 \end{pmatrix}, \\
M_{F_3}(\omega, E) &= \frac{1}{a}\begin{pmatrix} E & -1 \\ a^2 & 0 \end{pmatrix}\frac{1}{ab}\begin{pmatrix} E^2 - a^2 & -E \\ Eb^2 & -b^2 \end{pmatrix} \\
&= \frac{1}{a^2 b}\begin{pmatrix} E^3 - Ea^2 - Eb^2 & -E^2 + b^2 \\ E^2 a^2 - a^4 & -Ea^2 \end{pmatrix},
\end{aligned}
$$

then

$$
x_1 = \frac{E}{2a}, \quad x_2 = \frac{E^2 - a^2 - b^2}{2ab}, \quad x_3 = \frac{E^3 - 2Ea^2 - Eb^2}{2a^2 b}. \tag{3.3}
$$

By taking the trace of the equation

$$
M_{F_{k+1}} + M_{F_{k-2}}^{-1} = M_{F_{k-1}} M_{F_k} + M_{F_{k-1}} M_{F_k}^{-1},
$$

which follows from (3.2), we see

$$
\begin{aligned}
\text{tr } M_{F_{k+1}} + \text{tr } M_{F_{k-2}}^{-1} &= \text{tr } (M_{F_{k-1}}(M_{F_k} + M_{F_k}^{-1})) \\
&= \frac{1}{2}\text{tr } M_{F_{k-1}} \text{tr } (M_{F_k} + M_{F_k}^{-1}) = \text{tr } M_{F_{k-1}} \text{tr } M_{F_k},
\end{aligned}
$$

with the last two equalities holding because the matrices are 2×2 with determinant 1. This gives, for $k \geq 2$, the map

$$
x_{k+1} = 2x_k x_{k-1} - x_{k-2}. \tag{3.4}
$$

This is basically the trace map introduced in Chapter 1. Using the trace map and induction, we can see that the following quantity is independent of E and k, for $k \geq 2$:

$$I_k := x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 = \left(\frac{a^2+b^2}{2ab}\right)^2 - 1. \tag{3.5}$$

Let $c := \frac{a^2-b^2}{2ab}$ so that $c^2 = \left(\frac{a^2+b^2}{2ab}\right)^2 - 1$ and we can write this as $I_k = c^2$. First, note that $x_3^2 + x_2^2 + x_1^2 - 2x_3x_2x_1 - 1 = \frac{(a^2+b^2)^2}{(2ab)^2} - 1 = c^2$. Assume that $x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 = c^2$. Then

$$\begin{aligned}
c^2 &= x_{k+1}^2 + x_k^2 + (2x_{k+1}x_k - x_{k+2})^2 - 2x_{k+1}x_k(2x_{k+1}x_k - x_{k+2}) - 1 \\
&= x_{k+1}^2 + x_k^2 + 4x_{k+1}^2x_k^2 + x_{k+2}^2 - 4x_{k+1}x_kx_{k+2} - 4x_{k+1}^2x_k^2 + 2x_{k+1}x_kx_{k+2} - 1 \\
&= x_{k+2}^2 + x_{k+1}^2 + x_k^2 - 2x_{k+2}x_{k+1}x_k - 1.
\end{aligned}$$

Note that starting with (3.3) and iterating the inverse trace map $x_{n-1} = 2x_nx_{n+1} - x_{n+2}$, we get

$$x_0 = \frac{E}{2b}, \qquad x_{-1} = \frac{a^2+b^2}{2ab}, \qquad x_{-2} = \frac{Ea}{2b^2}, \tag{3.6}$$

and so on, obtaining a bi-infinite sequence $\{x_n\}$. The invariant (3.5) holds for $k < 2$ as well. This follows similarly and simply by induction.

Proposition 3.1. *Let* $V = T_1^{-1} = \begin{pmatrix} 0 & 1/a \\ -a & E/a \end{pmatrix}$ *and* $N = T_0 = \begin{pmatrix} E/b & -1/b \\ b & 0 \end{pmatrix}$. *Then* $x_0 = \frac{1}{2}tr\ N$, $x_{-1} = \frac{1}{2}tr\ (VN)$, *and* $x_{-2} = \frac{1}{2}tr\ (VNN)$. *If we let* $S_0 = N$, $S_1 = VN$, $S_2 = VNN$ *and define* S_k *by the recurrence* $S_k := S_{k-1}S_{k-2}$, *then* $x_{-k} = \frac{1}{2}tr\ (S_k)$ *for* $k \geq 2$.

Proof. It can easily be seen from (3.6) that $x_0 = \frac{1}{2}\mathrm{tr}\ N$, $x_{-1} = \frac{1}{2}\mathrm{tr}\ (VN)$, and $x_{-2} = \frac{1}{2}\mathrm{tr}\ (VNN)$. Now we proceed by induction. For the base case, we note that

$S_2 = S_1 S_0$. Now assume that $x_{-k} = \frac{1}{2}\text{tr }(S_{k-1}S_{k-2})$, with $x_{-k+1} = \frac{1}{2}\text{tr } S_{k-1}$ and $x_{-k+2} = \frac{1}{2}\text{tr } S_{k-2}$. Note that as $S_k = S_{k-1}S_{k-2}$, then $S_{k-2}^{-1} = S_k^{-1}S_{k-1}$. Consider $x_{-k-1} = 2x_{-k}x_{-k+1} - x_{-k+2}$. We have

$$\begin{aligned} x_{-k-1} &= \frac{1}{2}\text{tr } S_k \text{tr } S_{k-1} - \frac{1}{2}\text{tr } S_{k-2} \\ &= \frac{1}{2}\left(\frac{1}{2}\text{tr }(S_k + S_k^{-1})\text{ tr } S_{k-1}\right) - \frac{1}{2}\text{tr } S_{k-2}^{-1} \\ &= \frac{1}{2}\text{tr }((S_k + S_k^{-1})S_{k-1}) - \frac{1}{2}\text{tr } S_{k-2}^{-1} \\ &= \frac{1}{2}\text{tr }(S_k S_{k-1} + S_k^{-1}S_{k-1}) - \frac{1}{2}\text{tr } S_{k-2}^{-1} = \frac{1}{2}\text{tr }(S_k S_{k-1}). \end{aligned}$$

Thus we see that $x_{-k-1} = \frac{1}{2}\text{tr }(S_{k+1}) = \frac{1}{2}\text{tr }(S_k S_{k-1})$, as desired. $\square$

Note that we also have $x_{-2} = \frac{1}{2}\text{tr }(NVN)$, and so we could also have defined $S_k = S_{k-2}S_{k-1}$ and still have $x_{-k} = \frac{1}{2}\text{tr } S_k$ for $k \geq 2$.

Proposition 3.2. *(a) We have*

$$\begin{aligned} \omega(-n) &= \omega(n-1), \quad n \geq 2, \\ \omega(F_2+1) &= \omega(1), \\ \omega(F_k+l) &= \omega(l), \quad n \geq 3, \quad 1 \leq l \leq F_k, \end{aligned}$$

and

$$\omega(-F_{2n}+l) = \omega(l), \quad n \geq 1, \quad 1 \leq l \leq F_{2n+1}.$$

(b) We have

$$U_{-N-2} = T_N^{-1} \cdots T_1^{-1} U_{-2} \quad \textit{for } N \geq 1.$$

Moreover, if we define $L_n := L_n(\omega, E) = T_n^{-1} \cdots T_1^{-1}$ *and* $y_k := \frac{1}{2}tr\ L_{F_k}$*, then we can write*

$$U_{-F_k-2} = L_{F_k} U_{-2},$$

and $y_k = x_k$ *for* $k \geq 1$.

Proof. (a). To prove $\omega(-n) = \omega(n-1)$, we will proceed by induction.

First recall that at the origin (denoted by $|$), the sequence ω looks like $S^{2k}(b)|S^{2k}(a)$, or $S^{2k-1}(a)|S^{2k}(a)$. By (1.1) this becomes $S^{2k-1}(a)|S^{2k-1}(a)S^{2k-2}(a)$. Just considering $S^{2k-1}(a)|S^{2k-1}(a)$, this amounts to the sequence

$$\omega(-F_{2k}+1)\omega(-F_{2k}+2)\ldots\omega(-2)\omega(-1)\omega(0)|\omega(1)\omega(2)\ldots\omega(F_{2k}-2)\omega(F_{2k}-1)\omega(F_{2k}),$$

with $\omega(-n) = \omega(F_{2k}-n)$ for $n \leq F_{2k}-1$. This is true for any whole number n; just choose a large enough k. Thus we want to show that $\omega(n-1) = \omega(F_{2k}-n)$ holds for $n \geq 2$. Just considering the sequence $S^{2k-1}(a)$ to the right of the origin,

$$\omega(1)\omega(2)\omega(3)\ldots\omega(F_{2k}-4)\omega(F_{2k}-3)\omega(F_{2k}-2)\omega(F_{2k}-1)\omega(F_{2k}),$$

it is clear that $\omega(n-1) = \omega(F_{2k}-n)$ holds if, once the last two numbers from the sequence are removed, the new sequence

$$\omega(1)\omega(2)\omega(3)\ldots\omega(F_{2k}-4)\omega(F_{2k}-3)\omega(F_{2k}-2)$$

is a palindrome. Therefore we want to show that for $n \geq 2$, the sequence $S^k(a)$ is a palindrome once the last two elements of the sequence have been removed. For the base case, consider $S^2(a) = aba$, $S^3(a) = abaab$ and $S^4(a) = abaababa$. Clearly, all are palindromes once the last two elements have been removed. Now assume $S^{2k}(a)$, $S^{2k-1}(a)$ and $S^{2k-2}(a)$ are such that $S^{2k}(a) = pba$, $S^{2k-1}(a) = p'ab$ and $S^{2k-2}(a) = p''ba$, where p, p' and p'' are all palindromes. Then

$$\begin{aligned} S^{2k+1}(a) = S^{2k}(a)S^{2k-1}(a) &= S^{2k-1}(a)S^{2k-2}(a)S^{2k-1}(a) \\ &= p'abp''bap'ab, \end{aligned}$$

which is a palindrome once the last two elements have been removed. Similarly,

$$S^{2k+2}(a) = S^{2k}(a)S^{2k-1}(a)S^{2k}(a) = pbap'abpab.$$

The relation $\omega(F_2+1) = \omega(1)$ is clear. The relation $\omega(F_k+l) = \omega(l)$ for $n \geq 3$ and $1 \leq l \leq F_k$ follows from (1.1). Recall that $S^k(a)$ is a sequence of length F_{k+1}. We have

$$\begin{aligned} S^{k+1}(a) &= S^k(a)S^{k-1}(a) = S^{k-1}(a)S^{k-2}(a)S^{k-1}(a) \\ &= S^{k-2}(a)S^{k-3}(a)S^{k-2}(a)S^{k-2}(a)S^{k-3}(a) \\ &= S^{k-2}(a)S^{k-3}(a)S^{k-2}(a)S^{k-3}(a)S^{k-4}(a)S^{k-3}(a) \end{aligned}$$

and it is clear that after F_k terms, the sequence repeats those F_k terms. This holds for all $k \geq 4$, and to see the case when $k=3$, just consider $S^4(a) = abaababa$.

The relation $\omega(-F_{2n}+l) = \omega(l)$ for $n \geq 1, 1 \leq l \leq F_{2n+1}$ is shown in a similar way. To the left of the origin, the sequence ends with $S^{2k}(b)$, or rather with $S^{2k-1}(a)$, for every $k \geq 0$ (setting $S^{-1}(a) = b$). To the right of the origin the sequence starts with $S^k(a)$ for every $k \geq 0$, so it is clear that $\omega(-F_{2n}+l) = \omega(l)$ for $1 \leq l \leq F_{2n}$. To see that it is true for $F_{2n} < l \leq F_{2n+1}$, note that $S^{2k}(a) = S^{2k-2}(a)S^{2k-3}(a)S^{2k-2}(a)$, and so between $F_{2n} < l \leq F_{2n+1}$ we have that $\omega(-F_{2n}+l)$ is given by the leftmost $S^{2k-2}(a)$ of $S^{2k}(a)$, while for $F_{2n} < l \leq F_{2n+1}$, we have $\omega(l)$ given by the $S^{2k-2}(a)$ on the right.

(b). Using the reflective property from (a), the first relation follows from the equation $U_{-2} = T_{-2}T_{-3}\cdots T_{-N}T_{-N-1}U_{-N-2}$. To see that $y_k = x_k$, note that

$$\begin{aligned} y_k &= \frac{1}{2}\text{tr } L_{F_k} = \frac{1}{2}\text{tr } (T_{F_k}^{-1}T_{F_k-1}^{-1}\cdots T_1^{-1}) = \frac{1}{2}(T_1T_2\cdots T_{F_k})^{-1} \\ &= \frac{1}{2}\text{tr } M_{F_k}^{-1} = \frac{1}{2}\text{tr } M_{F_k} = x_k, \end{aligned}$$

using the fact that M_{F_k} is a 2×2 matrix with determinant 1. ☐

3.2 Spectrum and Spectral Properties

Now we will start considering the spectrum $\Sigma_{a,b}$ of H_ω; specifically we will prove Theorem 1.2 and Theorem 1.3. Thus we want to consider the sequence $\{x_n\}$, and energies where the sequence is bounded.

Define $B_\infty := \{E \in \mathbb{R} \mid \{x_n\} \text{ is bounded}\}$, and B_∞ is contained in each of the following sets: $B_{-\infty} := \{E \in \mathbb{R} \mid \{x_n\} \text{ is bounded in the backward direction}\}$ and $B_{+\infty} := \{E \in \mathbb{R} \mid \{x_n\} \text{ is bounded in the forward direction}\}$. Define $\sigma_k := \{E \in \mathbb{R} \mid |x_k| \le 1\}$ and $\rho_k := \{E \in \mathbb{R} \mid |x_k| > 1\}$. The goal is to see that $\Sigma_{a,b} = B_{+\infty}$, which will prove Theorem 1.2.

Lemma 3.3. *The set of energies where the trace map is bounded in the forward direction is contained in the spectrum of H_ω; i.e., $B_{+\infty} \subseteq \Sigma_{a,b}$, and there is no eigenvalue in $B_{+\infty}$.*

Proof. Let $u \neq 0$ be a solution of $(H_\omega u)(n) = Eu(n)$. By Proposition 3.2, for $n \ge 3$

we have

$$\begin{aligned} U_{2F_n} &= T_{2F_n} \cdots T_{F_n+1} T_{F_n} \cdots T_1 U_0 \\ &= T_{F_n+F_n} \cdots T_{F_n+1} M_{F_n} U_0 = T_{F_n} \cdots T_1 M_{F_n} U_0 \\ &= M_{F_n}^2 U_0. \end{aligned}$$

Also,

$$U_{-F_n-2} = T_{-F_n-2} T_{-F_n-1} \cdots T_{-2F_n-1} U_{-2F_n-2},$$

so

$$\begin{aligned} U_{-2F_n-2} &= T^{-1}_{-2F_n-1} T^{-1}_{-2F_n} \cdots T^{-1}_{-F_n-2} U_{-F_n-2} \\ &= T^{-1}_{-(2F_n+1)} T^{-1}_{-2F_n} \cdots T^{-1}_{-(F_n+2)} L_{F_n} U_{-2} \\ &= T^{-1}_{2F_n} T^{-1}_{2F_n-1} \cdots T^{-1}_{F_n+1} L_{F_n} U_{-2} \\ &= T^{-1}_{F_n} T^{-1}_{F_n-1} \cdots T^{-1}_{1} L_{F_n} U_{-2} \\ &= L^2_{F_n} U_{-2} \end{aligned}$$

for $n \geq 3$. By the Cayley-Hamilton theorem, $M_n^2 - \text{tr } M_n \cdot M_n + \det M_n = 0$, so $M_n^2 x - \text{tr } M_n \cdot M_n x + x = 0$ for $x \in \mathbb{C}^2$. Thus $\|M_n^2 x - \text{tr } M_n \cdot M_n x\| = \|x\|$ and $\|M_n^2 x\| + |\text{tr } M_n| \cdot \|M_n x\| \geq \|x\|$. Therefore

$$\max\{\|M_n^2 x\|, |\text{tr } M_n| \cdot \|M_n x\|\} \geq \frac{1}{2}\|x\|,$$

and similarly

$$\max\{\|L_n^2 x\|, |\text{tr } L_n| \cdot \|L_n x\|\} \geq \frac{1}{2}\|x\|.$$

Let $E \in B_{+\infty}$, so there exists some $C < \infty$ such that $|x_n| \leq C$ if $n > 0$. Note here that this actually works for any fixed $N \in \mathbb{Z}$; i.e., there exists some $C < \infty$ such that

$|x_n| \leq C$ if $n > N$. Let $x = U_0$ and $n = F_k$, and we get $\max\{\|M_{F_k}^2 U_0\|, |\text{tr } M_{F_k}| \cdot \|M_{F_k} U_0\|\} \geq \frac{1}{2}\|U_0\|$, or $\max\{\|U_{2F_k}\|, |2x_k| \cdot \|U_{F_k}\|\} \geq \frac{1}{2}\|U_0\|$ for $k \geq 3$. Thus

$$\max\{\|U_{2F_k}\|, 2C\|U_{F_k}\|\} \geq \frac{1}{2}\|U_0\|, \tag{3.7}$$

and similarly

$$\max\{\|U_{-2F_k-2}\|, 2C\|U_{-F_k-2}\|\} \geq \frac{1}{2}\|U_{-2}\| \tag{3.8}$$

for $k \geq 3$. Therefore E is not an eigenvalue, as $u \notin \ell^2(\mathbb{Z})$. Now, to show that $E \in \Sigma_{a,b}$, suppose not. Then there is a unique $u \in \ell^2(\mathbb{Z})$ such that $((H_\omega - E)u)(n) = \delta_{n,-1}$. Note that for all $n \neq -1$, this is just $((H_\omega - E)u)(n) = 0$, and thus $U_0 = T_0 U_{-1}$ is the only transfer matrix relation that no longer holds. Thus (3.7) and (3.8) are true, and as $\omega(0)u(0) + \omega(-1)u(-2) - Eu(-1) = 1$, then one of $u(0), u(-1), u(-2)$ is nonzero and thus either U_0 or U_{-2} is nonzero. This is a contradiction, for then by (3.7) or (3.8), it is true that $u \notin \ell^2(\mathbb{Z})$. Therefore $E \in \Sigma_{a,b}$, as desired. $\square$

Now, in order to show the other containment, that $\Sigma_{a,b} \subseteq B_{+\infty}$, we need to consider sequences $\{x_n\}$ that are unbounded in either the forward or backward direction.

Lemma 3.4. *A sufficient condition that the sequence $\{x_n\}$ be unbounded in the backward direction is that there exists some $N \in \mathbb{Z}$ such that*

$$|x_{N-1}| > 1, \quad |x_N| > 1, \quad \textit{and} \ |x_{N+1}| \leq 1. \tag{3.9}$$

This N is unique, and moreover $|x_{n-2}| > |x_{n-1}x_n| > 1$ for $n \leq N$, and there is a $C > 1$ such that $|x_n| > C^{F_{N-n}}$ for $n \leq N$.

Similarly, a sufficient condition that the sequence $\{x_n\}$ be unbounded in the forward direction is that there exists some $N \in \mathbb{Z}$ such that

$$|x_{N-1}| \leq 1, \quad |x_N| > 1, \text{ and } |x_{N+1}| > 1. \tag{3.10}$$

This N is unique, and moreover $|x_{n+2}| > |x_{n+1}x_n| > 1$ for $n \geq N$, and there is a $C > 1$ such that $|x_n| > C^{F_{n-N}}$ for $n \geq N$.

Proof. Suppose that (3.9) is true for some $N \in \mathbb{Z}$. Then $|x_{N-2}| = |2x_{N-1}x_N - x_{N+1}| \geq |x_{N-1}x_N| + (|x_{N-1}x_N| - |x_{N+1}|) > |x_{N-1}x_N| > 1$. By induction, we can show that $|x_n|, |x_{n-1}| > 1$ and $|x_{n-1}| > |x_n x_{n+1}|$ for all $n \leq N$. For the base case of $n = N$, the inequalities are true by the assumptions and the argument above. Now consider x_{n-2} with $n \leq N$. From $|x_n| > 1$ it follows that $|x_n^2 x_{n+1}| > |x_{n+1}|$, and we have $|x_{n-2}| \geq |x_{n-1}x_n| + (|x_{n-1}x_n| - |x_{n+1}|) > |x_{n-1}x_n| + (|x_n^2 x_{n+1}| - |x_{n+1}|) > |x_{n-1}x_n|$. The induction hypothesis was used in the second inequality involving x_{n-2}. Also from the induction hypothesis we have $|x_{n-1}x_n| > 1$, as $|x_{n-1}|, |x_n| > 1$. Therefore $|x_{n-2}| > 1$, as desired. Now, considering the relation $|x_{n-2}| > |x_{n-1}x_n|$ and taking the log of both sides, we see that $\log |x_{n-2}| > \log |x_{n-1}| + \log |x_n|$ for $n \leq N$. Therefore, $\log |x_n|$ grows faster in the backwards direction than the Fibonacci sequence ($F_{n+2} = F_{n+1} + F_n$), and $\log |x_n| > F_{N-n} \log C$ for some $C > 1$, or $|x_n| > C^{F_{N-n}}$ for $n \leq N$. Therefore, (3.9) is a sufficient condition for $\{x_n\}$ to be unbounded in the backward direction. We have

$$|x_{N+1}| \leq 1 < |x_N|,\ 1 < |x_{N-1}| < |x_{N-2}| < \dots ,$$

and clearly $|x_{n+1}| \leq 1 < |x_n|$, $1 < |x_{n-1}| < |x_{n-2}| < \dots$ cannot hold for any $n \neq N$; therefore this N is unique.

The same technique is used for the proof in the forward direction. □

Corollary 3.5. *A sufficient condition for $\{x_n\}$ to be unbounded in both the backward and forward direction is that $|x_1| > 1$.*

Proof. Let $|x_1| > 1$. Recall that $x_0 = \frac{E}{2b}$ and $x_1 = \frac{E}{2a}$, so as $a > b > 0$, we have $|x_1| < |x_0|$. Also note that $x_{-1} = \frac{a^2+b^2}{2ab} > 1$. Then $|x_{-1}x_0| - |x_1| > 0$, and $|x_{-2}| \geq |x_{-1}x_0| + (|x_{-1}x_0| - |x_1|) > |x_{-1}x_0|$, and by the proof of Lemma 3.4, $\{x_n\}$ is unbounded in the backward direction.

To see that $\{x_n\}$ is unbounded in the forward direction, note that $|x_1| > 1$ implies that $|x_2| > 1$. This is true as $|x_1| > 1$ gives $|E| > 2a$, or $E^2 > 4a^2$. Then $x_2 = \frac{E^2-a^2-b^2}{2ab} > \frac{3a^2-b^2}{2ab} > \frac{2a^2}{2ab} = \frac{a}{b} > 1$. We want to show that $|x_3| > |x_2x_1|$. It suffices to show that $|x_2x_1| > |x_0|$. From $\frac{E^2-a^2-b^2}{2a^2} > \frac{2a^2}{2a^2} = 1$ it follows that $\frac{E^2-a^2-b^2}{4a^2b} > \frac{1}{2b}$. Multiplying both sides by $|E|$ gives $\left|\frac{E^3-Ea^2-Eb^2}{4a^2b}\right| > \left|\frac{E}{2b}\right|$, or $|x_2x_1| > |x_0|$, as desired. Then from the proof of Lemma 3.4, we see that $\{x_n\}$ is unbounded in the forward direction. □

Therefore we see that if $|x_n| > 1$ for all n, the sequence $\{x_n\}$ is unbounded in both the forward and backward directions.

Lemma 3.6. *A necessary condition for $\{x_n\}$ to be unbounded (in either the forward or backward direction) is that one of the following holds:*

$$|x_{n-1}| > 1, \quad |x_n| > 1, \quad \textit{and} \quad |x_{n+1}| \leq 1 \quad \textit{for some} \;\; n \in \mathbb{Z},$$

$$|x_{n-1}| \leq 1, \quad |x_n| > 1, \quad \textit{and} \quad |x_{n+1}| > 1 \quad \textit{for some} \;\; n \in \mathbb{Z},$$

or

$$|x_n| > 1 \quad \textit{for all } \ n \in \mathbb{Z}.$$

Proof. Suppose that these do not hold. Note that if $|x_n| \le 1$ for all $n \in \mathbb{Z}$ then we are done, so consider the case where there exists some $n \in \mathbb{Z}$ such that $|x_n| > 1$. Without loss of generality, choose $n \in \mathbb{Z}$ so that $|x_{n-1}| \le 1, |x_{n+1}| \le 1$. This can be done, otherwise one of the conditions in the statement of the lemma is satisfied. From (3.5) we see that $x_n = x_{n+1}x_{n-1} \pm \sqrt{c^2 + (1 - x_{k+1}^2)(1 - x_{k-1}^2)}$, and so $|x_n| \le 1 + c$. Thus, the sequence $\{x_n\}$ is bounded (by $1 + c$), as desired. □

Corollary 3.7. *The sequence $\{x_n\}$ is bounded in the forward direction if and only if (3.10) does not hold for all $N \in \mathbb{N}$ and $|x_1| \le 1$.*

Proof. Suppose $\{x_n\}$ is bounded in the forward direction. Then Lemma 3.4 and Corollary 3.5 imply the "only if" direction of the statement.

Now suppose that $|x_1| \le 1$ and (3.10) does not hold for all $N \in \mathbb{N}$; i.e., for each $N \in \mathbb{N}$ one of the following inequalities does not hold: $|x_{N-1}| \le 1, |x_N| > 1$, and $|x_{N+1}| > 1$. If $|x_n| \le 1$ for all $n \in \mathbb{N}$, the sequence is bounded in the forward direction. Thus let $|x_n| > 1$ for some $n \in \mathbb{N}$. Then $|x_{n+1}|, |x_{n-1}| \le 1$, otherwise (3.10) holds for some N with $2 \le N \le n$. Now the statement follows from the proof of Lemma 3.6. □

We now have necessary and sufficient conditions for a sequence $\{x_n\}$ to be unbounded or bounded in the forward direction. Similar conditions for boundedness in the backward direction and unboundedness in just one direction can be identified from the work above, but they are not of particular interest for our purposes. Note that $E \in \rho_n \cap \rho_{n+1}$ gives rise to a sequence $\{x_n\}$ that is either unbounded in forward or backward direction (or both), so $\rho_n \cap \rho_{n+1} \subset B^c_{+\infty}$ or $\rho_n \cap \rho_{n+1} \subset B^c_{-\infty}$ (or both).

Now we will see more results concerning the boundedness and unboundedness of the sequence $\{x_n\}$, particularly their relation to the sets ρ_k and σ_k.

Proposition 3.8. *For any $N \in \mathbb{Z}$, we have*

$$B^c_{+\infty} \subset \bigcup_{n=N}^{\infty} (\rho_n \cap \rho_{n+1}). \tag{3.11}$$

and

$$B^c_{-\infty} \subset \bigcup_{-\infty}^{n=N} (\rho_n \cap \rho_{n+1}). \tag{3.12}$$

Also, if $\rho_n \cap \rho_{n+1} \subset B^c_{+\infty}$, then $\rho_n \cap \rho_{n+1} = \bigcap_{k=n}^{\infty} \rho_k$ for all $n \in \mathbb{Z}$. Similarly, if $\rho_n \cap \rho_{n+1} \subset B^c_{-\infty}$, then $\rho_n \cap \rho_{n+1} = \bigcap_{-\infty}^{k=n+1} \rho_k$ for all $n \in \mathbb{Z}$.

Proof. From Lemma 3.4 and Lemma 3.6, we see that

$$B^c_{+\infty} = \bigcap_{-\infty}^{\infty} \rho_n \cup \bigcup_{-\infty}^{\infty} (\sigma_{n-1} \cap \rho_n \cap \rho_{n+1})$$

and

$$B^c_{-\infty} = \bigcap_{-\infty}^{\infty} \rho_n \cup \bigcup_{-\infty}^{\infty} (\rho_{n-1} \cap \rho_n \cap \sigma_{n+1}).$$

First, notice that $\bigcap_{-\infty}^{\infty} \rho_n \subset \rho_k \cap \rho_{k+1}$ for any $k \in \mathbb{Z}$. By Lemma 3.4, if $E \in \sigma_{n-1} \cap \rho_n \cap \rho_{n+1}$, then $E \in \sigma_{n-1} \cap \bigcap_{k=n}^{\infty} \rho_k$. Thus $E \in \rho_k \cap \rho_{k+1}$ for every $k \geq n$, or rather $E \in \bigcup_{n=N}^{\infty} (\rho_n \cap \rho_{n+1})$ for any $N \in \mathbb{Z}$. Therefore (3.11) holds. Similarly, if $E \in \rho_{n-1} \cap \rho_n \cap \sigma_{n+1}$, then $E \in \bigcap_{-\infty}^{n} \rho_k \cap \sigma_{n+1} \subset \bigcup_{-\infty}^{n=N} \rho_{n-1} \cap \rho_n$ for any $N \in \mathbb{Z}$, so (3.12) holds.

Now if $E \in \rho_n \cap \rho_{n+1} \subset B^c_{+\infty}$, then $E \in \bigcap_{-\infty}^{\infty} \rho_k$ or $E \in \sigma_m \cap \bigcap_{k=m+1}^{\infty} \rho_k$ for some $m < n$. Either way, $E \in \bigcap_{k=n}^{\infty} \rho_k$, so $\rho_n \cap \rho_{n+1} \subset \bigcap_{k=n}^{\infty} \rho_k$. And $\bigcap_{k=n}^{\infty} \rho_k \subset \rho_n \cap \rho_{n+1}$ is trivial.

If $E \in \rho_n \cap \rho_{n+1} \subset B^c_{-\infty}$, then $E \in \bigcap_{-\infty}^{\infty} \rho_k$ or $E \in \bigcap_{-\infty}^{k=m-1} \rho_k \cap \sigma_m$ for some $m > n+1$, so $E \in \bigcap_{-\infty}^{k=n+1} \rho_k$ and thus $\rho_n \cap \rho_{n+1} \subset \bigcap_{-\infty}^{k=n+1} \rho_k$. And the containment $\bigcap_{-\infty}^{k=n+1} \rho_k \subset \rho_n \cap \rho_{n+1}$ is trivial. $\square$

Corollary 3.9. *For $n \geq 1$, we have*

$$\sigma_n \cup \sigma_{n+1} \supset \sigma_{n+1} \cup \sigma_{n+2}.$$

Proof. Either $E \in \sigma_1$ or not. By Corollary 3.5, if $E \notin \sigma_1$ then $E \in B^c_{+\infty}$. In $B^c_{+\infty}$, as $\rho_n \cap \rho_{n+1} = \bigcap_{k=n}^{\infty} \rho_k$, clearly $\rho_n \cap \rho_{n+1} \subset \rho_{n+1} \cap \rho_{n+2}$ and the statement is true. Now let $E \in \sigma_1$. Consider $\rho_n \cap \rho_{n+1}$ for $n > 1$. Clearly $\rho_n \cap \rho_{n+1} \subset \sigma_1 \cap \rho_2 \cap \rho_3 \cup \bigcup_{k>1} \sigma_k \cap \rho_{k+1} \cap \rho_{k+2}$, and by the proof of Lemma 3.4, then $\rho_n \cap \rho_{n+1} \subset \rho_{n+1} \cap \rho_{n+2}$. Therefore $\sigma_n \cup \sigma_{n+1} \supset \sigma_{n+1} \cup \sigma_{n+1}$ for all $n > 1$. To see this when $n = 1$, note that $\sigma_3 = [-\frac{1}{2}(b + \sqrt{8a^2+b^2}), \frac{1}{2}(b - \sqrt{8a^2+b^2})] \cup [-b, b] \cup [\frac{1}{2}(-b + \sqrt{8a^2+b^2}), \frac{1}{2}(b + \sqrt{8a^2+b^2})] \subset [-2a, 2a] = \sigma_1$. $\square$

Now we construct periodic operators, which are approximations of the operator H_ω. We will then relate the resolvents of the approximations to the resolvent of our operator, ultimately obtaining more information about the spectrum $\Sigma_{a,b}$. First, let us consider some results about (periodic) Jacobi operators. Let J be a Jacobi matrix

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ a(n-2) & b(n-1) & a(n-1) & & \\ & a(n-1) & b(n) & a(n) & \\ & & a(n) & b(n+1) & a(n+1) \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

with $a(n) \in \mathbb{R} \setminus \{0\}$ and $b(n) \in \mathbb{R}$. Then we have the following lemma from [26]:

Lemma 3.10. *Let $c_\pm = b(n) \pm (|a(n)| + |a(n-1)|)$. Then $\sigma(J) \subseteq [\inf_{n\in\mathbb{Z}} c_-(n), \sup_{n\in\mathbb{Z}} c_+(n)]$.*

Now consider a periodic Jacobi matrix J_N with period N, so $a(n+N) = a(n)$ and $b(n+N) = b(n)$ for all $n \in \mathbb{Z}$. Define the modified monodromy matrix $M_N(E)$ by $M_N = T_N \times \cdots \times T_1$, where $T_n = \frac{1}{a(n)} \begin{pmatrix} E - b(n-1) & -1 \\ a(n)^2 & 0 \end{pmatrix}$. Then, with $U_n = \begin{pmatrix} u(n) \\ a(n)u(n-1) \end{pmatrix}$, it is true that $u \in \ell^2(\mathbb{Z})$ solves $J_N u = Eu$ if and only if $U_n = T_n U_{n-1}$ for every $n \in \mathbb{Z}$. Let $\Delta(E) = \frac{1}{2}\mathrm{tr}\, M_N(E)$.

Lemma 3.11. *For the periodic Jacobi matrix J_N, the spectrum is given by $\sigma(J_N) = \{E \mid |\Delta(E)| \leq 1\}$. Moreover, the spectrum consists of N non-overlapping bands, on each of which $\Delta(E)$ is monotone increasing or decreasing.*

Proof. As M_N has determinant one, its eigenvalues can be written as w and w^{-1} with $|w| \geq 1$. Then $2\Delta = w + w^{-1}$. Clearly, if $|w| = 1$, then $w = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, and $w + w^{-1} = 2\cos\theta \in [-2, 2]$. So if $|w| = 1$, then $\Delta \in [-1, 1]$. If $|w| \neq 1$, then $w + w^{-1} \notin [-2, 2]$. This is clear, as for $w \in \mathbb{R}$, the minimum value $w + w^{-1}$ takes is two, at $w = \pm 1$. If $w \notin \mathbb{R}$, then $w = re^{i\theta}$ for some $r \neq 1$ and $\theta \in (0, 2\pi)$ with $\theta \neq \pi$. Then $\Im(w + w^{-1}) = (r - \frac{1}{r}) \sin \theta \neq 0$, so $w + w^{-1}$ is not real, much less in the interval $[-2, 2]$. Thus $|w| = 1$ if and only if $\Delta \in [-1, 1]$.

So take $\Delta \notin [-1, 1]$, and then $|w| > 1$. Therefore M_N can be diagonalized, and $A^{-1} M_N A = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$ for some invertible matrix A. Therefore $A^{-1} M_{mN} A = \begin{pmatrix} w^m & 0 \\ 0 & w^{-m} \end{pmatrix}$. Note that the $a(n)$ in the definition of U_n does not affect the boundedness or unboundness of solutions u to $J_N u = Eu$, as $a(n)$ is bounded away from zero and infinity, and so we can find solutions u_+, u_- that decay exponentially at $\pm\infty$, respectively. This allows one to write down explicitly the Green function $G(E, n, m) =< \delta_n, (J_N - E)^{-1} \delta_m >$ (see [26]), and thus $E \notin \sigma(J_N)$.

Now consider $\Delta \in (-1, 1)$. We have that $|w| = 1$, but specifically $w \neq w^{-1}$, and again M_N is diagonalizable. Then $A^{-1} M_{mN} A = \begin{pmatrix} w^m & 0 \\ 0 & w^{-m} \end{pmatrix}$, and now we see that solutions of $J_N u = Eu$ are actually bounded. It can then be shown that E is in the spectrum of J_N via a Weyl sequence argument.

Finally, because the periodic operator must be bounded, the spectrum is closed, so $\sigma(J_N)$ also contains the E's such that $\Delta(E) = \pm 1$.

It is clear that Δ is a polynomial of degree N, so it just remains to show that for $z \in (-1, 1)$, the roots of $\Delta - z$ are simple. Let $\Delta(E) = z$. As J_N is a self-adjoint operator, the spectrum is a subset of $\mathbb{R}$, and thus $E \in \mathbb{R}$. If E is not a simple root, then there is a complex number $\tilde{E}$ near E such that $\Delta(\tilde{E}) \in (-1, 1)$, but this is a contradiction, as then $\tilde{E} \in \sigma(J_N) \subset \mathbb{R}$. Therefore roots of $\Delta - z$ are simple, and

$\Delta(E)$ is monotone increasing or decreasing as it goes between -1 and 1. $\square$

Proposition 3.12. *Define a sequence of operators $\{H_m\}_{m\geq 1}$ on $\ell^2(\mathbb{Z})$ by*

$$(H_m u)(n) = \omega_m(n+1)u(n+1) + \omega_m(n)u(n-1),$$

where

$$\omega_m(n) = \omega(n) \quad \textit{for } 1 \leq n \leq F_m$$

and

$$\omega_m(n + F_m) = \omega_m(n) \qquad \textit{for all } n \in \mathbb{Z}.$$

Then $H_\omega = s\text{-}\lim_{m\to\infty} H_{2m}$ and $\rho(H_m) = \rho_m$.

Proof. Let $u \in \ell^2(\mathbb{Z})$. Then

$$\begin{aligned}
\|(H - H_{2m})u\|^2 &\leq \\
&\leq \sum_{|n|>F_{2m}-1} |(\omega(n+1) - \omega_m(n+1))u(n+1) + (\omega(n) - \omega_m(n))u(n-1)|^2 \\
&\leq \sum_{|n|>F_{2m}-1} (\omega(n+1) - \omega_m(n+1))^2|u(n+1)|^2 + (\omega(n) - \omega_m(n))^2|u(n-1)|^2 \\
&\leq (a-b)^2 \left(\sum_{|n|>F_{2m}-1} |u(n+1)|^2 + \sum_{|n|>F_{2m}-1} |u(n-1)|^2 \right) \to 0
\end{aligned}$$

as $m \to \infty$. By definition H_m is a periodic operator with period F_m and transfer matrix M_{F_m}. Therefore, by Lemma 3.11, we know that $E \in \sigma(H_m)$ if and only if $|\frac{1}{2}\mathrm{tr}\ M_{F_m}| \leq 1$, and this is precisely the definition of E in σ_m. Therefore $\sigma(H_m) = \sigma_m$, and $\rho(H_m) = \rho_m$. $\square$

Note that we can similarly define a sequence of operators $\{H_m\}_{m\leq -1}$ on $\ell^2(\mathbb{Z})$ by

$$(H_m u)(n) = \omega_m(n+1)u(n+1) + \omega_m(n)u(n-1),$$

where

$$\omega_m(n) = \omega(n) \quad \text{for} \quad -F_m + 1 \le n \le 0$$

and

$$\omega_m(n + F_m) = \omega_m(n) \quad \text{for all} \;\; n \in \mathbb{Z}.$$

Then $H_\omega = \text{s-}\lim_{m\to\infty} H_{2m}$ and $\rho(H_m) = \rho_m$ as well.

The following lemma is from [21]:

Lemma 3.13. *Let A, $\{A_m\}$ be bounded self-adjoint operators on a Hilbert space such that $A = s\text{-}\lim A_m$. Then $\left(\bigcap \rho(A_m)\right)^\circ \subset \rho(A)$.*

We can now prove Theorem 1.2.

Proof of Theorem 1.2. In Lemma 3.3 we found that $B_{+\infty} \subseteq \Sigma_{a,b}$, so we want to show that $\Sigma_{a,b} \subseteq B_{+\infty}$, or rather $B_{+\infty}^c \subseteq \rho(H_\omega)$. Let $E \in B_{+\infty}^c$. Then, by Proposition 3.8, there exists an n such that $E \in \rho_n \cap \rho_{n+1} = \bigcap_{k\ge n} \rho_k$. From Proposition 3.12 we have that $\rho_k = \rho(H_k)$, so $\bigcap_{k\ge n} \rho_k = \bigcap_{k\ge n} \rho(H_k)$. Thus $\rho_n \cap \rho_{n+1} = \bigcap_{k\ge n} \rho(H_k)$. As the left hand side is an open set then the right hand side must be also, and $\bigcap_{k\ge n} \rho(H_k) = \left(\bigcap_{k\ge n} \rho(H_k)\right)^\circ$. It is clear that $\bigcap_{k\ge n} \rho(H_k) \subset \bigcap_{m\ge \frac{n}{2}} \rho(H_{2m})$, and thus $E \in \left(\bigcap_{k\ge n} \rho(H_k)\right)^\circ \subset \left(\bigcap_{m\ge \frac{n}{2}} \rho(H_{2m})\right)^\circ \subset \rho(H_\omega)$, with the last containment following from Lemma 3.13. $\square$

And with the proof of Theorem 1.2 complete, we can prove Theorem 1.3:

Proof of Theorem 1.3. The singularity of the spectrum is clear from the zero Lebesgue measure statement in Lemma 1.1, and the absence of eigenvalues follows from Theorem 1.2 and Lemma 3.3. $\square$

We can also show that for a certain set of c's, the spectrum is a Cantor set:

Proposition 3.14. *For $c > 2$, the spectrum $\Sigma_{a,b}$ is a Cantor set.*

Proof. To show that the spectrum is a Cantor set, as we already have that $\Sigma_{a,b}$ is a closed set, we want to show that the set contains no open intervals. For contradiction, assume that there exists an open interval $I \subset \Sigma_{a,b}$. Let $E \in I \subset \Sigma_{a,b}$. As E is in the spectrum it belongs to an infinite intersection of σ_n's and ρ_n's that locally looks like either $\sigma_k \cap \rho_{k+1} \cap \sigma_{k+2} \cap \sigma_{k+3} \cap \rho_{k+4}$ or $\sigma_k \cap \rho_{k+1} \cap \sigma_{k+2} \cap \rho_{k+3} \cap \sigma_{k+4}$. We can write this as $E \in \bigcap_{k=1}^{\infty}(\sigma_{n_k-1} \cap \rho_{n_k} \cap \sigma_{n_k+1})$ for some sequence $\{n_k\}_{k\in\mathbb{N}}$, where $n_1 \geq 2$ and $n_{k+1} - n_k$ is equal to 2 or 3. Note that $|x_{n_k}(E)|$ is actually bounded away from 1: From the invariant (3.5) we see that $x_{n_k} = x_{n_k-1}x_{n_k+1} \pm \sqrt{c^2 + (1 - x_{n_k-1}^2)(1 - x_{n_k+1}^2)}$, and as $|x_{n_k-1}(E)|, |x_{n_k+1}(E)| \leq 1$, we get that $|x_{n_k}(E)| \geq c - 1 > 1$. Thus, by the continuity of x_{n_k}, every point in I must be contained in the same infinite intersection (for the same sequence $\{n_k\}$), and $I \subset \left(\bigcap_{k=1}^{\infty}(\sigma_{n_k-1} \cap \rho_{n_k} \cap \sigma_{n_k+1})\right)^{\circ}$. It is clear that $\left(\bigcap_{k=1}^{\infty}(\sigma_{n_k-1} \cap \rho_{n_k} \cap \sigma_{n_k+1})\right)^{\circ} \subset \left(\bigcap_{k=1}^{\infty}\rho_{n_k}\right)^{\circ}$.

The sequence $\{n_k\}$ contains either an infinite number of odds or evens. Suppose it contains an infinite number of evens. Then we can choose a subsequence $\{m_k\}$ of the even integers such that $\bigcap_{k=1}^{\infty}\rho_{n_k} \subset \bigcap_{k=1}^{\infty}\rho_{m_k}$, and $H_\omega = \operatorname{s-lim}_{m\to\infty} H_{m_k}$, with H_{m_k} defined as in Proposition 3.12. Thus from Lemma 3.13 it follows that $\left(\bigcap_{k=1}^{\infty}\rho_{m_k}\right)^{\circ} \subset \rho(H_\omega) = \Sigma_{a,b}^c$, and $I \subset \Sigma_{a,b}^c$, which is a contradiction.

Now suppose $\{n_k\}$ contains an infinite number of odds. Then we want to consider the sequence $\omega_o \in \Omega$ formed by iterating S^{2k} on $a|a$, where $|$ denotes the eventual origin. Then, considering the same set of operators $\{H_m\}_{m\geq 1}$, we see that $H_{2m-1} \rightarrow$

H_{ω_o}, and thus $\left(\bigcap_{m=1}^{\infty} \rho_{2m-1}\right)^{\circ} \subset \rho(H_{\omega_o}) = \Sigma_{a,b}^c$. So we can find a subsequence s_k of odd integers such that $\left(\bigcap_{k=1}^{\infty} \rho_{n_k}\right)^{\circ} \subset \left(\bigcap_{k=1}^{\infty} \rho_{s_k}\right)^{\circ} \subset \rho(H_{\omega_o})$, as $H_{s_k} \to H_{\omega_o}$. Therefore $I \subset \Sigma_{a,b}^c$, a contradiction. □

In this last proof, we considered the invariant (3.5). From this we saw that for $c > 2$, it is clear that there cannot exist as pair E, k such that $|x_{k+1}(E)| \leq 1$, $|x_k(E)| \leq 1$ and $|x_{k-1}(E)| \leq 1$, and thus

$$\sigma_{k+1} \cap \sigma_k \cap \sigma_{k-1} = \emptyset. \tag{3.13}$$

Now let us consider a way to find the spectrum $\Sigma_{a,b}$ in terms of the σ_k's, which we saw in Proposition 3.12 are the spectra of periodic approximations of the operator H_ω. The structure of these σ_k's will provide information about the fractal dimension of $\Sigma_{a,b}$, which we will see in the next section.

Lemma 3.15. *The spectrum of H_ω is given by*

$$\Sigma_{a,b} = \bigcap_{n\geq 1} (\sigma_n \cup \sigma_{n+1}). \tag{3.14}$$

Proof. From Corollary 3.5 we know that if $|x_1| > 1$, then $\{x_n\}$ is unbounded; indeed, from Lemma 3.10 we have that $\|H_\omega\| \leq 2a$, so for $E \in \Sigma_{a,b}$, this implies $E \in [-2a, 2a]$ and $|x_1| = |\frac{E}{2a}| \leq 1$. From Proposition 3.8 we have $B_{+\infty}^c \subset \bigcup_{n\geq 1}(\rho_n \cap \rho_{n+1})$, so $\bigcap_{n\geq 1}(\sigma_n \cup \sigma_{n+1}) \subset B_{+\infty} = \Sigma_{a,b}$.

Now, if we restrict to the energies such that $|x_1| \leq 1$, we still have $B_{+\infty}$, the set of E's such that $\{x_n\}$ has bounded forward orbit. The complement of $B_{+\infty}$ under this restriction, which we'll call $\tilde{B}_{+\infty}$, is the set of all E's such that $|x_1| \leq 1$ and

$\{x_n\}$ has unbounded forward orbit. The claim is that $\tilde{B}_{+\infty} = \bigcup_{n>1}(\rho_n \cap \rho_{n+1})$, so by taking complements we obtain $B_{+\infty} = \bigcap_{n>1}(\sigma_n \cup \sigma_{n+1})$. It is clear from Corollary 3.7 that $\tilde{B}_{+\infty} = (\rho_2 \cap \rho_3) \cup \left(\bigcup_{n\geq 2}(\sigma_n \cap \rho_{n+1} \cap \rho_{n+2}) \right)$: Obviously if $E \in \rho_n \cap \rho_{n+1}$ for some $n > 1$, then as $|x_1| \leq 1$, it is true that (3.10) holds for some $N \in \mathbb{N}$ such that $2 \leq N \leq n$, and $E \in \tilde{B}_{+\infty}$. And if $E \in \tilde{B}_{+\infty}$, then again (3.10) holds for some $N \in \mathbb{N}$ such that $2 \leq N$, and $E \in \rho_n \cap \rho_{n+1}$ for all $n \geq N$, so the other containment is also obvious.

Now we show that $(\rho_2 \cap \rho_3) \cup \left(\bigcup_{n\geq 2}(\sigma_n \cap \rho_{n+1} \cap \rho_{n+2}) \right) = \bigcup_{n>1}(\rho_n \cap \rho_{n+1})$. The containment $\subset$ is obvious, as $\rho_2 \cap \rho_3 \subset \bigcup_{n>1}(\rho_n \cap \rho_{n+1})$, and for $n \geq 2$ it is true that $\sigma_n \cap \rho_{n+1} \cap \rho_{n+2} \subset \bigcup_{n>1}(\rho_n \cap \rho_{n+1})$.

For the other containment, consider $E \in \rho_m \cap \rho_{m+1}$ with $m > 1$. If $m = 2$, then $E \in \rho_2 \cap \rho_3$, and the containment is clear. Now, if $m > 2$, then either $E \in \sigma_{m-1}$ or $E \in \rho_{m-1}$. If the former is true, then $E \in \sigma_{m-1} \cap \rho_m \cap \rho_{m+1} \subset \left(\bigcup_{n\geq 2}(\sigma_n \cap \rho_{n+1} \cap \rho_{n+2}) \right)$ and the containment holds. If the latter is true, that $E \in \rho_{m-1}$, then we consider that either $E \in \sigma_{m-2}$ or $E \in \rho_{m-2}$. In general, either $E \in \sigma_k$ for some $1 < k < m$, or $E \in \rho_k$ for all $1 < k < m$. If $E \in \sigma_k$ for some $1 < k < m$, then $E \in \sigma_k \cap \rho_{k+1} \cap \rho_{k+2}$ and the containment is obvious. If $E \in \rho_k$ for all $1 < k < m$, then $E \in \rho_2 \cap \rho_3$, and the containment is obvious.

Thus $\tilde{B}_{+\infty} = \bigcup_{n>1}(\rho_n \cap \rho_{n+1})$. So $B_{+\infty} = \bigcap_{n>1}(\sigma_n \cup \sigma_{n+1})$, but this is all under the restriction that $|x_1| \leq 1$, so really we have $\Sigma_{a,b} = B_{+\infty} = \sigma_1 \cap \bigcap_{n>1}(\sigma_n \cup \sigma_{n+1})$. However, it can easily be seen from (3.3) that $\sigma_2 = [-a-b, -a+b] \cup [a-b, a+b] \subset [-2a, 2a] = \sigma_1$, so $\sigma_1 \cap \bigcap_{n>1}(\sigma_n \cup \sigma_{n+1}) = \bigcap_{n\geq 1}(\sigma_n \cup \sigma_{n+1})$, as desired. $\square$

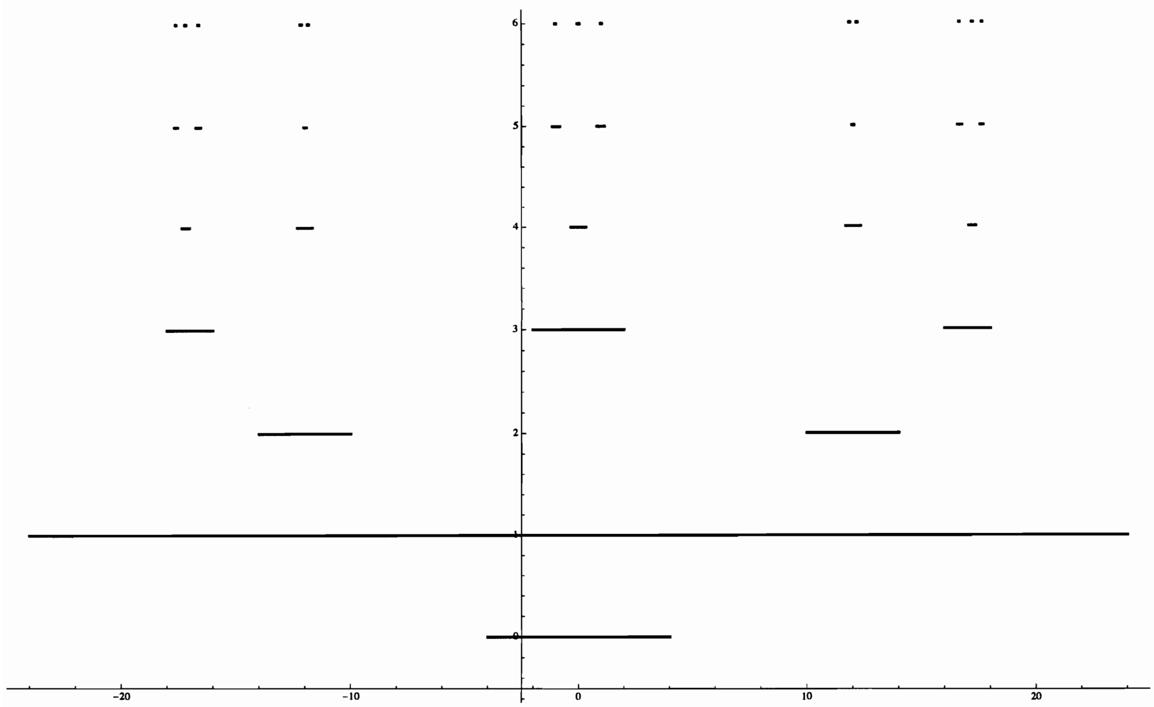

Figure 3.1: Band Structure with $a = 12, b = 2$.

Now let us look at the structure of these spectra. First note that from Lemma 3.11, we know σ_k consists of F_k non-overlapping bands.

Definition 3.16. Define a band $B_k \subset \sigma_k$ to be of type A if $B_k \subset \sigma_{k-1}$, and to be a type B band if $B_k \subset \sigma_{k-2}$.

From Corollary 3.9 and (3.13), it is clear that for $c > 2$, $k \geq 2$, each band B_k is in exactly one of $\sigma_{k-1}, \sigma_{k-2}$. Therefore type A and B bands are well-defined. Note also that for $k \geq 3, c > 2$, it follows from (3.13) that if B_k is a type A band then $B_k \cap (\sigma_{k+1} \cup \sigma_{k-2}) = \emptyset$, and similarly for a type B band that $B_k \cap \sigma_{k-1} = \emptyset$.

Lemma 3.17. *Let $c > 2$, and $k \geq 2$. Then*

(a) Each type A band $B_k \subset \sigma_k$ contains exactly one type B band $B_{k+2} \subset \sigma_{k+2}$ and no

other bands from σ_{k+1} and σ_{k+2}.

(b) Each type B band $B_k \subset \sigma_k$ contains exactly one type A band $B_{k+1} \subset \sigma_{k+1}$ and two type B bands from σ_{k+2} positioned around B_{k+1}, and no other bands from σ_{k+1} and σ_{k+2}.

Proof. This follows the proof of a similar statement in [15]. (a) Let $B_k \subset \sigma_k$ be a type A band, so $B_k \subset \sigma_{k-1}$. Then $|x_{k+1}| \geq 1$ and $B_k \cap \sigma_{k+1} = \emptyset$. On B_k, the function x_k changes monotonically between 1 and -1, so there is a unique value $E_k \in B_k$ such that $x_k(E_k) = 0$. By the trace map, $x_{k+2} = 2x_{k+1}x_k - x_{k-1}$, so

$$|x_{k+2}(E_k)| = |x_{k-1}(E_k)| \leq 1,$$

and thus $B_k \cap \sigma_{k+2} \neq \emptyset$. Note that when $x_k = \pm 1$ then $x_{k+2} = \pm 2x_{k+1} - x_{k-1}$ and

$$|x_{k+2}| \geq 2|x_{k+1}| - |x_{k-1}| > 1.$$

Therefore any bands of σ_{k+2} that intersect B_k lie strictly inside B_k. Also, in each band B_{k+2}, the function x_{k+2} changes continuously between -1 and 1. Let E_0, E_1 be the endpoints of B_{k+2} such that $x_{k+2}(E_0) = -1$ and $x_{k+2}(E_1) = 1$. Let $x_{k-1}(E_0) = \alpha$ and $x_{k-1}(E_1) = \beta$, so for $B_{k+2} \subset \sigma_k \subset \sigma_{k-1}$, we have $-1 < \alpha, \beta < 1$. Without loss of generality, assume $E_0 < E_1$. Then $x_{k+2} + x_{k-1}$ is a continuous function on $[E_0, E_1]$, and we have $x_{k+2}(E_0) + x_{k-1}(E_0) = \alpha - 1$ and $x_{k+2}(E_1) + x_{k-1}(E_1) = \beta + 1$. As $0 \in [\alpha - 1, \beta + 1]$, by the intermediate value theorem, there exists a value $E_{k+2} \in B_{k+2} \subset B_k$ such that $x_{k+2}(E_{k+2}) + x_{k-1}(E_{k+2}) = 0$. Then the trace map gives $2x_{k+1}(E_{k+2})x_k(E_{k+2}) = 0$, and as $|x_{k+1}| > 1$ on B_k, it must be true that $x_k(E_{k+2}) = 0$.

So every band $B_{k+2} \subset B_k$ contains an energy such that $x_k = 0$, and by monotonicity such an energy is unique, so there exists exactly one band of σ_{k+2} in B_k.

(b) Now let $B_k \subset \sigma_k$ be a type B band, so $B_k \subset \sigma_{k-2}$ and $|x_{k-1}| > 1$ on B_k. Again let $E_k \in B_k$ be such that $x_k(E_k) = 0$, so from $x_{k+1} = 2x_k x_{k-1} - x_{k-2}$ we get $|x_{k+1}(E_k)| = |x_{k-2}(E_k)| \leq 1$ and $\sigma_{k+1} \cap B_k \neq \emptyset$. Similar to above, at $x_k = \pm 1$ we have $|x_{k+1}| \geq 2|x_{k-1}| - |x_{k-2}| > 1$, so any band in σ_{k+1} that intersects B_k is contained strictly inside B_k. Also similar to above, any such band B_{k+1} must contain the unique energy in B_k at which $x_k = 0$, and so there is exactly one band of σ_{k+1} in B_k. Now consider σ_{k+2}. Iterating the trace map and substituting gives

$$x_{k+2} = (4x_k^2 - 1)x_{k-1} - 2x_k x_{k-2}. \tag{3.15}$$

When $x_k = \pm\frac{1}{2}$, then $|x_{k+2}| = |x_{k-2}| < 1$, so there are at least two bands in σ_{k+2} that intersect B_k and lie to the right and left of B_{k+1}, because B_{k+1} contains the energy where $x_k = 0$, and $\sigma_{k+2} \cap \sigma_{k+1} \cap \sigma_k = \emptyset$. Also, when $x_k = \pm 1$, then $|x_{k+2}| \geq 3|x_{k-1}| - 2|x_{k-2}| > 1$, so these bands are strictly contained in B_k. Now to show that there are exactly two such bands in σ_{k+2}, first define

$$T_\pm := (2x_k \pm 1)(x_{k+2} \pm x_{k-2}) = (4x_k^2 - 1)(x_{k+1} \pm x_{k-1}). \tag{3.16}$$

The equality holds as the left hand side expands to

$$\begin{aligned} & 2x_k x_{k+2} \pm x_{k+2} \pm 2x_k x_{k-2} + x_{k-2} = \\ = \quad & 4x_k^2 x_{k+1} - 2x_k x_{k-1} \pm 4x_k^2 x_{k-1} \mp x_{k-1} \mp 2x_k x_{k-2} \pm 2x_k x_{k-2} + x_{k-2} \\ = \quad & 4x_k^2 x_{k+1} \pm 4x_k^2 x_{k-1} \mp x_{k-1} - x_{k+1}, \end{aligned}$$

which is precisely the expansion of the right hand side. The first equality comes from substituting $x_{k+2} = 2x_{k+1}x_k - x_{k-1}$ and (3.15), and the second equality follows from $-x_{k+1} = -2x_k x_{k-1} + x_{k-2}$.

Consider a band B_{k+2} in B_k. As $|x_{k+1}|, |x_{k-1}| > 1$ on B_{k+2}, then $x_{k+1}(E), x_{k-1}(E)$ have fixed signs for $E \in B_{k+2}$. Choose either T_+ or T_- depending on the signs so that $x_{k+1} \pm x_{k-1} \neq 0$ for all energies in B_{k+2}. As before, the intermediate value theorem and monotonicity give that there are unique energies $E_\pm \in B_{k+2}$ at which $x_{k+2} \pm x_{k-2} = 0$. By (3.16), we have $4x_k^2 - 1 = 0$ at such an energy, and there are exactly two energies in B_k where $x_k = \pm\frac{1}{2}$, so there are at most two bands of σ_{k+2} in B_k. □

Lemma 3.18. *For every band I_k of σ_k, we have $I_k \cap \Sigma_{a,b} \neq \emptyset$.*

Proof. Let I_k be a band of σ_k. By the band structure presented in Lemma 3.17, we can choose a band I_{k+1} in $\sigma_{k+1} \cup \sigma_{k+2}$ such that $I_{k+1} \subset I_k$, and a band I_{k+2} in $\sigma_{k+2} \cup \sigma_{k+3}$ such that $I_{k+2} \subset I_{k+1}$, etc. Similarly, choose a band I_{k-1} in $\sigma_{k-2} \cup \sigma_{k-1}$ such that $I_k \subset I_{k-1}$ and iterate this procedure, producing a nested sequence of closed intervals

$$I_3 \supset \ldots \supset I_{k-1} \supset I_k \supset I_{k+1} \supset \ldots.$$

It follows that $\bigcap_{k \geq 3} I_k$ is nonempty, and

$$\bigcap_{k \geq 3} I_k \subset \bigcap_{k>1} \sigma_k \cup \sigma_{k+1} = \Sigma_{a,b}$$

by construction, so there exists a point $E \in I_k$ such that $E \in \Sigma_{a,b}$. □

Now armed with information about the structure of the bands of the σ_k's and how they relate to the spectrum $\Sigma_{a,b}$, we move on to the next section to consider dimension of the spectrum.

3.3 Fractal Dimension of the Spectrum.

In this section we will prove Theorem 1.4 and Theorem 1.5. To this end we will first consider various lemmas concerning the size and number of the type A and type B bands defined in the previous section. Recall that $c = \frac{a^2-b^2}{2ab}$.

Lemma 3.19. *Define the functions $f_\pm(x, y, c)$ by*

$$f_\pm(x,y,c) = xy \pm \sqrt{c^2 + (1-x^2)(1-y^2)}. \tag{3.17}$$

For $c > 2$ and $|x|, |y| \le 1$, we have

$$\left|\frac{\partial f_\pm}{\partial x}(x,y,c)\right|, \left|\frac{\partial f_\pm}{\partial y}(x,y,c)\right| \le 1. \tag{3.18}$$

Proof. It suffices to prove the bound for $\partial f_+/\partial x$, as $f_+(x,y,c) = f_+(y,x,c)$ and $-f_+(x,-y,c) = f_-(x,y,c)$. We have

$$\frac{\partial f_+}{\partial x}(x,y,c) = y - \frac{x(1-y^2)}{\sqrt{c^2 + (1-x^2)(1-y^2)}}.$$

Then as

$$\begin{aligned} \frac{\partial^2 f_+}{\partial x \partial y}(x,y,c) &= 1 + \frac{xy[2c^2 + (1-x^2)(1-y^2)]}{(c^2+(1-x^2)(1-y^2))^{3/2}} \\ &\ge 1 + \frac{-2c^2}{(c^2)^{3/2}} > 0, \end{aligned}$$

we see that

$$\max_{|x|,|y|\le 1} \left|\frac{\partial f_+}{\partial x}(x,y,c)\right| = \max_{|x|\le 1, |y|=1} \left|\frac{\partial f_+}{\partial x}(x,y,c)\right| = 1.$$

$\square$

Lemma 3.20. *Let $c > 4$ and $k \geq 1$. Then, with $\xi_{a,b} = c - 2 + \sqrt{c^2 - 4c + 1}$, we have the following inequalities.*

(a) For any type A band $B_{k+1} \subset \sigma_{k+1}$, $E \in B_{k+1}$ implies

$$\left| \frac{x'_{k+1}(E)}{x'_k(E)} \right| \geq \xi_{a,b}.$$

(b) For any type B band $B_{k+2} \subset \sigma_{k+2}$, $E \in B_{k+2}$ implies

$$\left| \frac{x'_{k+2}(E)}{x'_k(E)} \right| \geq \xi_{a,b}.$$

Proof. The proof is by induction. Let $c > 4$, which is the same as $a > (4 + \sqrt{17}b)$. This is important, as then $\xi_{a,b} > 1$ and real. Consider the base case for a type A band $\left|\frac{x'_2}{x'_1}\right| = \left|\frac{2E}{b}\right|$. As $E \in \sigma_2$ then $\left|\frac{E^2-a^2-b^2}{2ab}\right| \leq 1$, and $|E| \geq a - b$. Thus $\left|\frac{x'_2}{x'_1}\right| \geq \frac{2(a-b)}{b}$. Now the claim is that $\frac{2(a-b)}{b} > \xi_{a,b}$. Note that if we write $\xi_{a,b}$ in terms of a and b, we get

$$\xi_{a,b} = \frac{a^2 - b^2 - 4ab}{2ab} + \frac{\sqrt{a^4 + 2a^2b^2 + b^4 - 8a^3b + 8ab^3}}{2ab},$$

and since $-8a^3b + 8ab^3 < 0$, we see $\xi_{a,b} < \frac{a^2-b^2-4ab}{2ab} + \frac{\sqrt{(a^2+b^2)^2}}{2ab} = \frac{a-2b}{b}$. So $\xi_{a,b} < \frac{a-2b}{b} < \frac{2(a-b)}{b}$, as desired.

Similarly, for a type B band we see that $\left|\frac{x'_3}{x'_1}\right| = \left|\frac{3E^2-2a^2-b^2}{ab}\right|$. As $E \in \sigma_3$, we have $\left|\frac{E^3-2Ea^2-Eb^2}{2a^2b}\right| \leq 1$. From the roots of the equations $\frac{E^3-E(2a^2+b^2)\pm 2a^2b}{2a^2b} = 0$, one can see that $E \in [\frac{1}{2}(-b - \sqrt{8a^2+b^2}), \frac{1}{2}(b - \sqrt{8a^2+b^2})] \cup [-b, b] \cup [\frac{1}{2}(-b + \sqrt{8a^2+b^2}), \frac{1}{2}(b + \sqrt{8a^2+b^2})]$. On the interval $[-b, b]$, the numerator $3E^2 - 2a^2 - b^2 \in [-2a^2 - b^2, -2a^2 + 2b^2]$, so $\left|\frac{x'_3}{x'_1}\right| > \frac{2a^2-2b^2}{ab} = 4c > c - 2 + \sqrt{c^2 - 4c + 1}$. On the interval $[\frac{1}{2}(-b + \sqrt{8a^2+b^2}), \frac{1}{2}(b + \sqrt{8a^2+b^2})]$, the numerator $3E^2 - 2a^2 - b^2$ is increasing so the minimum occurs at the left endpoint. At $E = \frac{1}{2}(-b + \sqrt{8a^2 + b^2})$, the quantity $\frac{x_3'}{x_1'} = \frac{8a^2+b^2-3b\sqrt{8a^2+b^2}}{2ab}$, and the claim is that this is greater than $\xi_{a,b}$. We have $\xi_{a,b} < \frac{a-2b}{b}$, so it suffices to show that $8a^2 + b^2 - 3b\sqrt{8a^2 + b^2} \geq 2a^2 - 4ab$. The left hand side is bounded below by $8a^2+b^2-9ab$, and clearly $8a^2+b^2 > 2a^2+5ab$, as $a > b$. Thus $\frac{x_3'}{x_1'} \geq \xi_{a,b}$. The bound for E's in the interval $[\frac{1}{2}(-b-\sqrt{8a^2 + b^2}), \frac{1}{2}(b-\sqrt{8a^2 + b^2})]$ follows from symmetry.

Assume that $\left|\frac{x_k'(E)}{x_{k-1}'(E)}\right| \geq \xi_{a,b}$ and $\left|\frac{x_k'(E)}{x_{k-2}'(E)}\right| \geq \xi_{a,b}$ hold for the appropriate bands.

(a) Let $B_{k+1} \subset \sigma_{k+1}$ be a type A band, and let $E \in B_{k+1} \subset \sigma_k$. Then we have $|x_{k+1}|, |x_k|, |x_{k-2}| \leq 1$, and B_{k+1} is contained in a type B band at level k, so $\left|\frac{x_k'(E)}{x_{k-2}'(E)}\right| \geq \xi_{a,b}$. From the invariant (3.5) it follows that

$$x_{k-1} = x_{k-2}x_k \pm \sqrt{c^2 + (1 - x_{k-2}^2)(1 - x_k^2)} = f_\pm(x_k, x_{k-2}, c), \tag{3.19}$$

and thus

$$|x_{k-1}| \geq c - 1.$$

Also $x_{k-1}' = \frac{\partial f_\pm}{\partial x_k}(x_k, x_{k-2}, c) \cdot x_{k+1}' + \frac{\partial f_\pm}{\partial x_{k-2}}(x_k, x_{k-2}, c) \cdot x_k'$, where the $\pm$ means that either plus or minus can occur, and by Lemma 3.19, as $|x_k|, |x_{k-2}| \leq 1$, then

$$|x_{k-1}'| \leq |x_k'| + |x_{k-2}'|. \tag{3.20}$$

Differentiating (3.4) and dividing by x_k' gives

$$\frac{x_{k+1}'}{x_k'} = 2x_{k-1} + \frac{2x_k x_{k-1}'}{x_k'} - \frac{x_{k-2}'}{x_k'}.$$

Thus

$$\begin{aligned}
\left|\frac{x'_{k+1}}{x'_k}\right| &\geq 2|x_{k-1}| - 2\left|\frac{x_k x'_{k-1}}{x'_k}\right| - \left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 2(c-1) - 2|x_k|\left(1 + \left|\frac{x'_{k-1}}{x'_k}\right|\right) - \left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 2(c-1) - 2 - 3\left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 2c - 4 - \frac{3}{\xi_{a,b}},
\end{aligned}$$

where the second inequality follows from (3.20). So if $2c - 4 - \frac{3}{\xi_{a,b}} \geq \xi_{a,b}$, then $\left|\frac{x'_{k+1}}{x'_k}\right| \geq \xi_{a,b}$. The inequality holds if $c-2-\sqrt{c^2-4c+1} \leq \xi_{a,b} \leq c-2+\sqrt{c^2-4c+1}$, and this is true by definition.

(b) Let $B_{k+2} \subset \sigma_{k+2}$ be a type B band, and let $E \in B_{k+2} \subset \sigma_k$. There are two cases to consider. First, let $B_{k+2} \cap \sigma_{k-1} = \emptyset$, so $B_{k+2} \subset \sigma_{k-2}$. Then $|x_{k+2}|, |x_k|, |x_{k-2}| \leq 1$, and B_{k+2} is contained in a type B band at level k, so $\left|\frac{x'_{k+2}(E)}{x'_k(E)}\right| \geq \xi_{a,b}$. Note that, similar to part (a), we have

$$|x_{k+1}| = |f_{\pm}(x_{k+2}, x_k, c)| \geq c - 1 \tag{3.21}$$

and

$$x'_{k-1} = \frac{\partial f_{\pm}}{\partial x_k}(x_k, x_{k-2}, c) \cdot x'_k + \frac{\partial f_{\pm}}{\partial x_{k-2}}(x_k, x_{k-2}, c) \cdot x'_{k-2}. \tag{3.22}$$

From the invariant (3.5) we get the following equations:

$$\begin{aligned}
x'_{k+2} &= 2x'_{k+1}x_k + 2x_{k+1}x'_k - x'_{k-1}, \\
x'_{k+1} &= 2x'_k x_{k-1} + 2x_k x'_{k-1} - x'_{k-2}, \\
4x_k x_{k-1} &= 2x_{k+1} + 2x_{k-2},
\end{aligned}$$

and using these we see that

$$\begin{aligned}
\frac{x'_{k+2}}{x'_k} &= \frac{2x'_{k+1}x_k}{x'_k} + 2x_{k+1} - \frac{x'_{k-1}}{x'_k} \\
&= \frac{2x_k(2x'_k x_{k-1} + 2x_k x'_{k-1} - x'_{k-2})}{x'_k} + 2x_{k+1} - \frac{x'_{k-1}}{x'_k} \\
&= 4x_k x_{k-1} + \frac{4x_k^2 x'_{k-1}}{x'_k} - \frac{2x_k x'_{k-2}}{x'_k} + 2x_{k+1} - \frac{x'_{k-1}}{x'_k} \\
&= 4x_{k+1} + 2x_{k-2} + \frac{x'_{k-1}}{x'_k}(4x_k^2 - 1) - 2\frac{x_k x'_{k-2}}{x'_k}.
\end{aligned}$$

Thus by (3.21), (3.22), Lemma 3.19, the induction assumption and the bounds $|x_k|, |x_{k-2}| \leq 1$, we have

$$\begin{aligned}
\left|\frac{x'_{k+2}}{x'_k}\right| &\geq 4|x_{k+1}| - 2|x_{k-2}| - |4x_k^2 - 1|\left|\frac{x'_{k-1}}{x'_k}\right| - 2|x_k|\left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 4(c-1) - 2 - 3\left|\frac{x'_{k-1}}{x'_k}\right| - 2\left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 4c - 6 - 3\left|\frac{\partial f_\pm}{\partial x_k}(x_k, x_{k-2}, c)\right| - 3\left|\frac{\partial f_\pm}{\partial x_{k-2}}(x_k, x_{k-2}, c)\right|\left|\frac{x'_{k-2}}{x'_k}\right| - 2\left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 4c - 9 - 5\left|\frac{x'_{k-2}}{x'_k}\right| \\
&\geq 4c - 9 - \frac{5}{\xi_{a,b}}.
\end{aligned}$$

Solving $0 \geq \xi^2 + (9-4c)\xi + 5$, the inequality holds if ξ is between $\frac{1}{2}(4c - 9 - \sqrt{16c^2 - 72c + 61})$ and $\frac{1}{2}(4c - 9 + \sqrt{16c^2 - 72c + 61})$. It can easily be seen that for $c > 4$, it is true that $c - 2 + \sqrt{c^2 - 4c + 1}$ is between these two values.

Now consider the case where $B_{k+2} \subset \sigma_{k-1}$, so $B_{k+2} \cap \sigma_{k-2} = \emptyset$. Note that (3.21) still holds, $|x_k|, |x_{k-1}| \leq 1$, and B_{k+2} is contained in a type A band at level k, so $\left|\frac{x'_k(E)}{x'_{k-1}(E)}\right| \geq \xi_{a,b}$. Also

$$x'_{k+1} = \frac{\partial f_\pm}{\partial x_k}(x_k, x_{k-1}, c) \cdot x'_k + \frac{\partial f_\pm}{\partial x_{k-1}}(x_k, x_{k-1}, c) \cdot x'_{k-1},$$

so

$$\left|\frac{x'_{k+1}}{x'_k}\right| \leq 1 + \left|\frac{x'_{k-1}}{x'_k}\right|.$$

Thus

$$\begin{aligned}
\left|\frac{x'_{k+2}}{x'_k}\right| &= \left|2x_{k+1} + 2\frac{x'_{k+1}x_k}{x'_k} - \frac{x'_{k-1}}{x'_k}\right| \\
&\geq 2(c-1) - 2|x_k|\left(1 + \left|\frac{x'_{k-1}}{x'_k}\right|\right) - \left|\frac{x'_{k-1}}{x'_k}\right| \\
&\geq 2c - 4 - 3\left|\frac{x'_{k-1}}{x'_k}\right| \\
&\geq 2c - 4 - \frac{3}{\xi_{a,b}}.
\end{aligned}$$

The inequality is the same as in part (a). □

Lemma 3.21. *Let $c > 4$ and $k \geq 2$. Then*

(a) For any type A band $B_{k+1} \subset \sigma_{k+1}$, $E \in B_{k+1}$ implies

$$\left|\frac{x'_{k+1}(E)}{x'_k(E)}\right| \leq 2c + 7.$$

(b) For any type B band $B_{k+2} \subset \sigma_{k+2}$, $E \in B_{k+2}$ implies

$$\left|\frac{x'_{k+2}(E)}{x'_k(E)}\right| \leq 2(2c + 7).$$

Proof. This follows the proof of a similar statement in [9]. (a) Suppose $B_{k+1} \subset \sigma_{k+1}$ is a type A band. Then $|x_{k+1}|, |x_k| \leq 1$. From the invariant (3.5) it follows that $x_{k-1} = x_{k+1}x_k \pm \sqrt{c^2 + (1 - x_{k+1}^2)(1 - x_k^2)} = f_\pm(x_{k+1}, x_k, c)$, and

$$|x_{k-1}| \leq c + 1.$$

From Lemma 3.19 we have

$$|x'_{k-1}| = \left|\frac{\partial f_\pm}{\partial x_k}(x_k, x_{k-2}, c) \cdot x'_k + \frac{\partial f_\pm}{\partial x_{k-2}}(x_k, x_{k-2}, c) \cdot x'_{k-2}\right| \leq |x'_k| + |x'_{k-2}|,$$

and from Lemma 3.20 we have

$$\left|\frac{x'_{k-2}}{x'_k}\right| < 1.$$

Differentiating (3.4) and dividing by x'_k gives

$$\frac{x'_{k+1}}{x'_k} = 2x_{k-1} + 2\frac{x_k x'_{k-1}}{x'_k} - \frac{x'_{k-2}}{x'_k},$$

and so

$$\begin{aligned}
\left|\frac{x'_{k+1}}{x'_k}\right| &\leq 2|x_{k-1}| + 2\left|\frac{x_k x'_{k-1}}{x'_k}\right| + \left|\frac{x'_{k-2}}{x'_k}\right| \\
&\leq 2(c+1) + 2\left(\frac{|x'_k| + |x'_{k-2}|}{|x'_k|}\right) + \left|\frac{x'_{k-2}}{x'_k}\right| \\
&= 2c + 4 + 3\left|\frac{x'_{k-2}}{x'_k}\right| \\
&\leq 2c + 7.
\end{aligned}$$

(b) There are two cases to consider. First, let B_{k+2} be a type B band such that $B_{k+2} \cap \sigma_{k-1} = \emptyset$ and so $B_{k+2} \subset \sigma_{k-2}$. Similar to part (a) we get that

$$|x_{k+1}| \leq c + 1 \tag{3.23}$$

and

$$|x'_{k-1}| \leq |x'_k| + |x'_{k-2}|.$$

As in Lemma 3.20 we have that

$$\frac{x'_{k+2}}{x'_k} = 4x_{k+1} + 2x_{k-2} + \frac{x'_{k-1}}{x'_k}(4x_k^2 - 1) - 2\frac{x_k x'_{k-2}}{x'_k},$$

and so

$$\begin{aligned}\left|\frac{x'_{k+2}}{x'_k}\right| &\leq 4|x_{k+1}| + 2|x_{k-2}| + \left|\frac{x'_{k-1}}{x'_k}(4x_k^2 - 1)\right| + 2\left|\frac{x_k x'_{k-2}}{x'_k}\right| \\ &\leq 4(c+1) + 2 + 3\left(1 + \left|\frac{x'_{k-2}}{x'_k}\right|\right) + 2\left|\frac{x'_{k-2}}{x'_k}\right| \\ &\leq 4c + 9 + 5\left|\frac{x'_{k-2}}{x'_k}\right| \\ &\leq 4c + 14.\end{aligned}$$

Now let B_{k+2} be a type B band such that $B_{k+2} \subset \sigma_{k-1}$ and so $B_{k+2} \cap \sigma_{k-2} = \emptyset$. Still (3.23) holds, and now we have

$$|x'_{k+1}| \leq |x'_k| + |x'_{k-1}|.$$

Again, as in Lemma 3.20 we have that

$$\frac{x'_{k+2}}{x'_k} = 2x_{k+1} + 2\frac{x'_{k+1}x_k}{x'_k} - \frac{x'_{k-1}}{x'_k},$$

and so

$$\begin{aligned}\left|\frac{x'_{k+2}}{x'_k}\right| &\leq 2|x_{k+1}| + 2\left|\frac{x'_{k+1}x_k}{x'_k}\right| + \left|\frac{x'_{k-1}}{x'_k}\right| \\ &\leq 2(c+1) + 2|x_k|\left(1 + \left|\frac{x'_{k-1}}{x'_k}\right|\right) + \left|\frac{x'_{k-1}}{x'_k}\right| \\ &\leq 2c + 4 + 3\left|\frac{x'_{k-1}}{x'_k}\right|.\end{aligned}$$

Finally, in the proof of Lemma 3.20 we saw that $\left|\frac{x'_{k-1}}{x'_k}\right| < 1$, so

$$\left|\frac{x'_{k+2}}{x'_k}\right| \leq 2c + 7.$$

$\square$

From the previous two lemmas, we see that the size of a band can be estimated from the size of the band in which it lies at the previous two levels, so the lengths of a band at a certain level roughly depends on how many bands it intersects from lower levels. We start this process at level two, because that is where our desired structure begins.

Lemma 3.22. *Let $c > 2$. The following inequalities hold for $E \in \sigma_2, \sigma_3$, respectively:*

$$\frac{a-b}{ab} \leq |x_2'(E)|, |x_3'(E)| \leq \frac{9}{2b}.$$

Note that these bounds are not optimal; the optimal bounds can be found in the proof. However, it will really only be useful that they are bounded away from zero.

Proof. First consider $|x_2'| = |\frac{E}{ab}|$. The bands in σ_2 are the intervals $[-a-b, b-a]$ and $[a-b, a+b]$, so it's clear that $\frac{a-b}{ab} \leq |x_2'| \leq \frac{a+b}{ab}$. The bands of σ_3 are the intervals $[\frac{1}{2}(-b-\sqrt{8a^2+b^2}), \frac{1}{2}(b-\sqrt{8a^2+b^2})]$, $[-b, b]$ and $[\frac{1}{2}(-b+\sqrt{8a^2+b^2}), \frac{1}{2}(b+\sqrt{8a^2+b^2})]$. It is clear that the maximum value of $|x_3'| = |\frac{3E^2-2a^2-b^2}{2a^2b}|$ occurs at either $E = 0$ or $E = \frac{1}{2}(b+\sqrt{8a^2+b^2})$, and indeed it is the latter with $|x_3'| = \frac{4a^2+\frac{b^2}{2}+\frac{3b}{2}\sqrt{8a^2+b^2}}{2a^2b} \leq \frac{9}{2b}$.

Similarly, the minimum value of $|x_3'|$ occurs at either $E = b$ or $E = \frac{1}{2}(-b+\sqrt{8a^2+b^2})$. This is true because x_3' is symmetric in E, negative on the interval $[-b, b]$ and positive on the interval $[\frac{1}{2}(-b+\sqrt{8a^2+b^2}), \frac{1}{2}(b+\sqrt{8a^2+b^2})]$. It can be seen that the minimum occurs at $E = b$, where $|x_3'| = \frac{a^2-b^2}{a^2b} > \frac{a-b}{ab}$. □

Definition 3.23. Define

$$a_k := \text{number of type A bands in } \sigma_{k+2},$$

$$b_k := \text{number of type B bands in } \sigma_{k+2},$$

$$a_{k,m} := \text{number of type A bands } b \text{ in } \sigma_{k+2} \text{ with } \#\{2 \leq j < k+2 : b \cap \sigma_j \neq \emptyset\} = m,$$

$$b_{k,m} := \text{number of type B bands } b \text{ in } \sigma_{k+2} \text{ with } \#\{2 \leq j < k+2 : b \cap \sigma_j \neq \emptyset\} = m.$$

First, let us consider what these definitions actually mean: Starting at σ_2, we see that $a_{k,m}$ counts the number of type A bands k levels up that lie in exactly m bands at previous levels. The number $b_{k,m}$ similarly counts the type B bands.

Note that, based on the definitions of type A and type B bands, we have that $a_k = b_{k-1}$ and $b_k = 2b_{k-2} + a_{k-2}$ for $k \geq 2$, with initial values $a_0 = 2$ and $b_1 = 3$. Similarly we have that $a_{k,m} = b_{k-1,m-1}$ and $b_{k,m} = 2b_{k-2,m-1} + a_{k-2,m-1}$ with initial values $a_{0,0} = 2$ and $a_{0,m} = 0$ for $m \neq 0$, $a_{1,m} = 0$, $b_{0,m} = 0$, $b_{1,0} = 3$ and $b_{1,m} = 0$ for $m \neq 0$. It will be useful later to consider the recurrence with only the a's:

$$a_{k,m} = 2b_{k-3,m-2} + a_{k-3,m-2} = 2a_{k-2,m-1} + a_{k-3,m-2} \tag{3.24}$$

for $k \geq 3$.

This next result gives the number $a_{k,m}$ explicitly.

Lemma 3.24. *If* $\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{2k-1}{3} \rfloor$, *then*

$$\begin{aligned} a_{k,m} &= b_{k-1,m-1} \\ &= 2^{2k-3m-1} \left(\begin{array}{c} k-m-1 \\ \\ 2m-k \end{array} \right) \left(\frac{2k-m}{2k-3m} \right) \\ &= \frac{2^{2k-3m-1}(k-m-1)!}{(2m-k)!(2k-3m-1)!} \left(\frac{2k-m}{2k-3m} \right). \end{aligned} \tag{3.25}$$

Otherwise, $a_{k,m} = 0$.

Proof. The proof is in two parts, first considering the bands in the middle and then the bands along either side, and developing a relation with the Chebyshev polynomials.

The bands in the middle are generated by a type B band in σ_3. Let the number of these bands be denoted by $a'_k, b'_k, a'_{k,m}$ and $b'_{k,m}$, corresponding to $a_k, b_k, a_{k,m}$ and

$b_{k,m}$. The recurrence relation in (3.24) still holds, as we are just looking at a subset of bands:

$$a'_{k,m} = 2a'_{k-2,m-1} + a'_{k-3,m-2}. \tag{3.26}$$

The claim is that

$$a'_{k,m} = \begin{cases} 2^{2k-3m-1}\begin{pmatrix} k-m-1 \\ 2m-k \end{pmatrix}, & \text{when} \lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{2k-1}{3} \rfloor; \\ 0, & \text{otherwise.} \end{cases} \tag{3.27}$$

Note that the initial conditions are

$$a'_{2,1} = 1 \qquad \text{and} \qquad a'_{4,2} = 2. \tag{3.28}$$

The Chebyshev polynomials of the second kind can be defined by the recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \tag{3.29}$$

with initial values $U_0(x) = 1$ and $U_1(x) = 2x$. This can be written explicitly as

$$U_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m c_{m,n} x^{n-2m},$$

where

$$c_{m,n} = 2^{n-2m} \frac{(n-m)!}{m!(n-2m)!} = 2^{n-2m} \begin{pmatrix} n-m \\ m \end{pmatrix} \tag{3.30}$$

for $0 \leq m \leq \frac{n}{2}$; see [1]. The initial conditions are $c_{0,0} = 1$ and $c_{0,1} = 2$. Take $c_{m,n} = 0$

for $m < 0$ and $m > \frac{n}{2}$. From the recursion it follows that

$$\begin{aligned}
U_{n+1}(x) &= \sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^m c_{m,n+1}x^{n-2m+1} \\
&= 2x\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^m c_{m,n}x^{n-2m} - \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^m c_{m,n-1}x^{n-2m-1} \\
&= \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^m 2c_{m,n}x^{n-2m+1} + \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^{m+1} c_{m,n-1}x^{n-2m-1} \\
&= \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^m 2c_{m,n}x^{n-2m+1} + \sum_{m=1}^{\lfloor\frac{n+1}{2}\rfloor}(-1)^{m} c_{m-1,n-1}x^{n-2m+1}.
\end{aligned}$$

For n even this gives

$$\sum_{m=0}^{\frac{n}{2}}(-1)^m x^{n-2m+1}(c_{m,n+1} - 2c_{m,n} - c_{m-1,n-1}) = 0,$$

and so obviously

$$c_{m,n+1} = 2c_{m,n} + c_{m-1,n-1}.$$

For n odd we get

$$\sum_{m=0}^{\frac{n-1}{2}}(-1)^m x^{n-2m+1}(c_{m,n+1} - 2c_{m,n} - c_{m-1,n-1}) = (-1)^{\frac{n-1}{2}}(c_{\frac{n+1}{2},n+1} - c_{\frac{n-1}{2},n-1}),$$

but note that $c_{\frac{n}{2},n} = 2^0\begin{pmatrix} n/2 \\ n/2 \end{pmatrix} = 1$, so the right hand side is zero and

$$c_{m,n+1} = 2c_{m,n} + c_{m-1,n-1}$$

holds. Define $\tilde{a}_{k,m} := c_{2m-k,m-1}$. Then

$$\begin{aligned}
\tilde{a}_{k,m} &= 2c_{2m-k,m-2} + c_{2m-k-1,m-3} \\
&= 2c_{2(m-1)-(k-2),m-2} + c_{2(m-2)-(k-3),m-3} \\
&= 2\tilde{a}_{k-2,m-1} + \tilde{a}_{k-3,m-2};
\end{aligned}$$

compare with (3.26). Consider initial conditions $1 = c_{0,0} = \tilde{a}_{2,1}$ and $2 = c_{0,1} = \tilde{a}_{4,2}$, and compare with (3.28). Thus $a'_{k,m} = \tilde{a}_{k,m} = c_{2m-k,m-1}$, and with (3.30), the equation (3.27) follows.

Next consider the bands either to the right or the left; they are symmetric. Denote the number of bands here by $a''_{k,m}$, corresponding to $a_{k,m}$. The claim is that

$$a''_{k,m} = \begin{cases} 2^{2k-3m-1}\frac{m}{k-m}\begin{pmatrix} k-m \\ 2m-k \end{pmatrix} & \text{when} \lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{2k-1}{3} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Note the initial conditions are $a''_{0,0} = 1$ and $a''_{2,1} = 1$. This follows analogously using T_n, the Chebyshev polynomials of the first kind, which have the same recursion from (3.29) but initial conditions $T_0(x) = 1$ and $T_1 = x$. The explicit form (again see [1]) is $T_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m 2^{n-2m-1} \dfrac{n}{n-m} \begin{pmatrix} n-m \\ m \end{pmatrix} x^{n-2m}$. The details can be found in [6]. Adding together $2a''_{k,m}$ and $a'_{k,m}$ gives (3.25), as desired. $\square$

Let us now introduce the quantity f^* first mentioned in the introduction as part of Theorem 1.4 and Theorem 1.5. Then we will consider two more needed results concerning the number $a_{k,m}$, and finally the proofs of the theorems on fractal dimension of the spectrum will be given.

On the interval $(\frac{1}{2}, \frac{2}{3})$, define

$$\begin{aligned} f(x) := \frac{1}{x}[&(2-3x)\log 2 + (1-x)\log(1-x) \\ &- (2x-1)\log(2x-1) - (2-3x)\log(2-3x)]. \end{aligned}$$

The function f extends to a continuous function on $[\frac{1}{2}, \frac{2}{3}]$ with $f(\frac{1}{2}) = \log 2$ and

$f(\frac{2}{3}) = 0$ which attains its maximum at $x^* = \frac{12-2\sqrt{2}}{17}$ with $f^* = f(x^*) = \log(1+\sqrt{2})$.

Lemma 3.25. *If* $\frac{k}{2} \leq m \leq \frac{2k-1}{3}$*, then*

$$k^{-1} \exp\left(mf\left(\frac{m}{k}\right)\right) \lesssim a_{k,m} \lesssim k \exp\left(mf\left(\frac{m}{k}\right)\right). \tag{3.31}$$

Proof. First consider when $m = \frac{k}{2}$. From (3.25) it follows that $a_{k,\frac{k}{2}} = 3 \cdot 2^{\frac{k}{2}-1}$. Also $\exp\left(\frac{k}{2} f\left(\frac{1}{2}\right)\right) = 2^{\frac{k}{2}}$, so it is clear that (3.31) holds in this case.

Next consider $m = \frac{2k-1}{3}$. Equation (3.25) gives $a_{k,\frac{2k-1}{3}} = \frac{4k+1}{3}$. With a little work we can see that $\frac{2k-1}{3} \cdot f\left(\frac{2k-1}{3k}\right) = \log 2 + \frac{k+1}{3} \log \frac{k+1}{3k} - \frac{k-2}{3} \log \frac{k-2}{3k} - \log \frac{1}{k}$, or that $\exp\left(\frac{2k-1}{3} \cdot f\left(\frac{2k-1}{3k}\right)\right) = 2k\left(\frac{k+1}{3k}\right)^{\frac{k+1}{3}} \left(\frac{3k}{k-2}\right)^{\frac{k-2}{3}} = 2k\left(\frac{k+1}{k-2}\right)^{\frac{k-1}{3}} \left(\frac{k+1}{3k}\right)^{\frac{2}{3}}$. Clearly the expression holds.

Remark 3.26. *The notation $a \asymp b$ means that $a \lesssim b$ and $a \gtrsim b$.*

We will use Stirling's approximation, which says $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$, so

$$n! \asymp \sqrt{n}\left(\frac{n}{e}\right)^n.$$

Thus

$$a_{k,m} \asymp$$
$$2^{2k-3m} \frac{2k-m}{2k-3m} \left(\frac{k-m-1}{(2m-k)(2k-3m-1)}\right)^{\frac{1}{2}} \frac{\left(\frac{k-m-1}{e}\right)^{k-m-1}}{\left(\frac{2m-k}{e}\right)^{2m-k} \left(\frac{2k-3m-1}{e}\right)^{2k-3m-1}}.$$

Just consider the factors

$$\begin{aligned} \alpha_{k,m} &:= 2^{2k-3m} \frac{\left(\frac{k-m-1}{e}\right)^{k-m-1}}{\left(\frac{2m-k}{e}\right)^{2m-k} \left(\frac{2k-3m-1}{e}\right)^{2k-3m-1}} \\ &= 2^{2k-3m} \frac{(k-m-1)^{k-m-1}}{(2m-k)^{2m-k}(2k-3m-1)^{2k-3m-1}}. \end{aligned}$$

Letting $x = m/k$, we get

$$\begin{aligned}
\alpha_{k,m} &= 2^{2k-3m}\frac{(k-m-1)^{k-m-1}}{(2m-k)^{2m-k}(2k-3m-1)^{2k-3m-1}} \\
&= 2^{2k-3xk}\frac{(k-xk-1)^{k-xk-1}}{(2xk-k)^{2xk-k}(2k-3xk-1)^{2k-3xk-1}} \\
&= 2^{k(2-3x)}\frac{(1-x-\frac{1}{k})^{k(1-x-1/k)}}{(2x-1)^{k(2x-1)}(2-3x-\frac{1}{k})^{k(2-3x-1/k)}},
\end{aligned}$$

so

$$\begin{aligned}
\log \alpha_{k,m} = k[(2-3x)\log 2 \quad &+ \quad (1-x)\log(1-x-1/k) \\
&- \quad (2x-1)\log(2x-1) - (2-3x)\log(2-3x-1/k)] \\
&+ \quad \log(2-3x-1/k) - \log(1-x-1/k).
\end{aligned}$$

Thus, for large k, we have

$$\alpha_{k,m} \approx \exp\left(mf\left(\frac{m}{k}\right)\right)\left(\frac{2k-3m-1}{k-m-1}\right).$$

That is,

$$\begin{aligned}
\alpha_{k,m} = \exp\Big(mf\left(\frac{m}{k}\right) &+ k(1-x)[\log(1-x-1/k) - \log(1-x)] \\
&+k(2-3x)[\log(2-3x) - \log(2-3x-1/k)]\Big)\left(\frac{2k-3m-1}{k-m-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{k\to\infty} k(1-x)[\log(1-x-1/k) - \log(1-x)] + \\
&\lim_{k\to\infty} k(2-3x)[\log(2-3x) - \log(2-3x-1/k)] = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
a_{k,m} &\asymp \exp\left(mf\left(\frac{m}{k}\right)\right)\frac{(2k-3m-1)(2k-m)}{(k-m-1)(2k-3m)}\left(\frac{k-m-1}{(2m-k)(2k-3m-1)}\right)^{\frac{1}{2}} \\
&= \exp\left(mf\left(\frac{m}{k}\right)\right)\frac{(2k-3m-1)^{\frac{1}{2}}(2k-m)}{(k-m-1)^{\frac{1}{2}}(2k-3m)(2m-k)^{\frac{1}{2}}}.
\end{aligned}$$

Then consider $\frac{k}{2} < m < \frac{2k-1}{3}$, which gives the following inequalities:

$$\begin{aligned}
1 \leq 2m - k &< \frac{k-2}{3}, \\
1 \leq 2k - 3m - 1 &< \frac{k-2}{2}, \\
1 < 2k - 3m &< \frac{k}{2}, \\
\frac{k-2}{3} < k - m - 1 &< \frac{k-2}{2}, \\
\frac{4k+1}{3} < 2k - m &< \frac{3k}{2}.
\end{aligned}$$

Putting the first, third and forth inequalities in a more useful form, we have

$$\begin{aligned}
\frac{3}{k-2} < \frac{1}{2m-k} &\leq 1, \\
\frac{2}{k} < \frac{1}{2k-3m} &< 1, \\
\frac{2}{k-2} < \frac{1}{k-m-1} &< \frac{3}{k-2}.
\end{aligned}$$

These lead to

$$\begin{aligned}
a_{k,m} &\lesssim \left(\frac{k-2}{2} \cdot \frac{3}{k-2}\right)^{\frac{1}{2}} \frac{3k}{2} \cdot \exp\left(mf\left(\frac{m}{k}\right)\right) \\
&\lesssim k \exp\left(mf\left(\frac{m}{k}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
a_{k,m} &\gtrsim \left(\frac{3}{k-2} \cdot \frac{2}{k-2}\right)^{\frac{1}{2}} \frac{2}{k} \cdot \frac{4k+1}{s} \cdot \exp\left(mf\left(\frac{m}{k}\right)\right) \\
&\gtrsim \frac{1}{k} \exp\left(mf\left(\frac{m}{k}\right)\right),
\end{aligned}$$

so

$$k^{-1} \exp\left(mf\left(\frac{m}{k}\right)\right) \lesssim a_{k,m} \lesssim k \exp\left(mf\left(\frac{m}{k}\right)\right),$$

as desired. $\square$

Proposition 3.27. *Let* $\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{2k-1}{3} \rfloor$. *Then*

$$\lim_{k\to\infty} \max_m \frac{1}{m} \log a_{k,m} = f^*$$

Proof. From (3.31), for $\frac{k}{2} \leq m \leq \frac{2k-1}{3}$, we have

$$C_1 - \log k + mf\left(\frac{m}{k}\right) \leq \log a_{k,m} \leq C_2 + \log k + mf\left(\frac{m}{k}\right)$$

for some constants $C_1, C_2 \in \mathbb{R}$. The latter inequality, along with

$$\max_m \frac{1}{m}\left(C_2 + \log k + mf\left(\frac{m}{k}\right)\right) \leq \frac{2C_2}{k} + \frac{2}{k}\log k + f^*,$$

implies that

$$\limsup_{k\to\infty} \max_m \frac{1}{m} \log a_{k,m} \leq f^*.$$

As k grows we can take $m = m'$ such that $\frac{m'}{k}$ gets arbitrarily close to $x*$, as $\frac{1}{2} \leq \frac{m'}{k} \leq \frac{2}{3} - \frac{1}{3k}$. Thus

$$\frac{1}{k}\frac{C_1}{m'/k} - \frac{1}{2m'/k}\frac{1}{k}\log k + f\left(\frac{m'}{k}\right) \leq \max_m \left(\frac{C_1}{m} - \frac{1}{2m}\log k + f\left(\frac{m}{k}\right)\right),$$

and so

$$f^* \leq \liminf_{k\to\infty} \max_m \frac{1}{m} \log a_{k,m}.$$

$\square$

Now we can prove Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.4. Let $m_k = \lfloor 3kx^* \rfloor$. Note that $3kx^* - 1 < \lfloor 3kx^* \rfloor < 3kx^*$ and $\lim_{k\to\infty} \dfrac{3kx^*-1}{3k} = \lim_{k\to\infty} \dfrac{3kx^*}{3k} = x^*$, so

$$\lim_{k\to\infty} \frac{m_k}{3k} = x^*.$$

Also, considering m_{k+1}, we have $(3k+3)x^* - 1 < \lfloor (3k+3)x^* \rfloor < (3k+3)x^*$ and $\lim_{k\to\infty} \dfrac{(3k+3)x^*-1}{3kx^*} = \lim_{k\to\infty} \dfrac{(3k+3)x^*}{3kx^*-1} = 1$, so

$$\lim_{k\to\infty} \frac{m_{k+1}}{m_k} = 1. \tag{3.32}$$

Define

$$f_k := \frac{1}{m_k} \log a_{3k,m_k}.$$

From Lemma 3.25 it follows that

$$\lim_{k\to\infty} f_k = f^*. \tag{3.33}$$

For a given k, consider $N_k := a_{3k,m_k}$, the number of bands of type A in σ_{3k+2} that lie in m_k bands in previous levels. Combining the results of Lemma 3.21 with the fact that $|x_2'|, |x_3'| < \frac{9}{2b}$ from Lemma 3.22, and recalling that on each band of σ_k we have that x_k is monotone between ± 1 to ∓ 1, it is clear that each band has length at least $\varepsilon_k := 2\left(\frac{2b}{9}\right)(4c+14)^{-m_k}$. Let $\{A_{3k,j}\}_{j=1}^{N_k}$ be the type A bands of σ_{3k+2}, indexed so that $A_{3k,j}$ is to the immediate left of $A_{3k,j+1}$ for all j. From Lemma 3.18, each band has nonempty intersection with $\Sigma_{a,b}$. Therefore there exists an energy $E_{3k,j} \in A_{3k,j} \cap \Sigma_{a,b}$ for all j. Consider the $\{E_{3k,j}\}$ with j odd; i.e., $j = 2s+1$ for $0 \le s \le \lfloor N_k \rfloor$. These energies are separated by bands $A_{3k,2s}$ of length at least ε_k, so they lie in different ε-intervals as long as $\varepsilon < \varepsilon_k$. Thus $N_{\Sigma_{a,b}}(\varepsilon) \ge \frac{N_k}{2}$ for $\varepsilon < \varepsilon_k$. Take $\varepsilon > 0$, and choose k such that $\varepsilon_{k+1} \le \varepsilon < \varepsilon_k$. Then

$$\frac{\log N_{\Sigma_{a,b}}(\varepsilon)}{\log \frac{1}{\varepsilon}} \ge \frac{\log \frac{N_k}{2}}{\log \frac{1}{\varepsilon_{k+1}}} = \frac{\frac{1}{m_k}\log N_k - \frac{1}{m_k}\log 2}{\frac{1}{m_k}\log\frac{1}{\varepsilon_{k+1}}} = \frac{f_k - \frac{1}{m_k}\log 2}{\frac{1}{m_k}\log\frac{9}{4b} + \frac{m_{k+1}}{m_k}\log(4c+14)}.$$

As $\varepsilon \to 0$, we have $k, m_k \to \infty$. Using (3.33) and (3.32), we see

$$\lim_{k\to\infty} \frac{f_k - \frac{1}{m_k}\log 2}{\frac{1}{m_k}\log\frac{9}{4b} + \frac{m_{k+1}}{m_k}\log(4c+14)} = \frac{f^*}{\log(4c+14)},$$

which implies the bound on the dimension. □

Proof of Theorem 1.5. For ease of notation, let $c - 2 + \sqrt{c^2 - 4c + 1} = \xi_{a,b}$, as in Lemma 3.20.

To find a bound for the Hausdorff dimension of the spectrum $\Sigma_{a,b}$, first note that $\sigma_k \cup \sigma_{k+1}$ covers the spectrum as a finite union of compact intervals. Let us consider σ_k: There are exactly $a_{k-2,m} + b_{k-2,m}$ bands in σ_k that lie in exactly m bands in previous levels. We know $a_{k-2,m}$ is nonzero only for $\lceil \frac{k-2}{2} \rceil \leq m \leq \lfloor \frac{2k-5}{3} \rfloor$, and $b_{k-2,m} = a_{k-1,m+1}$ is nonzero only for $\lceil \frac{k-3}{2} \rceil \leq m \leq \lfloor \frac{2k-6}{3} \rfloor$. So to get every case where $a_{k-2,m}$ and $b_{k-2,m}$ are nonzero, we need $\lceil \frac{k-3}{2} \rceil \leq m \leq \lfloor \frac{2k-5}{3} \rfloor$. Combining the results of Lemma 3.20 and Lemma 3.22, we see that the length of each band in σ_k are bounded above by $\frac{2ab}{a-b}\xi_{a,b}^{-m}$. Thus,

$$
\begin{aligned}
h^{\alpha}(\Sigma_{a,b}) \leq\; & \left(\sum_{m=\lceil \frac{k-3}{2} \rceil}^{\lfloor \frac{2k-5}{3} \rfloor} (a_{k-2,m} + b_{k-2,m}) \left(\frac{2ab}{a-b} \xi_{a,b}^{-m} \right) \right. \\
& \left. + \sum_{m=\lceil \frac{k-2}{2} \rceil}^{\lfloor \frac{2k-3}{3} \rfloor} (a_{k-1,m} + b_{k-1,m}) \left(\frac{2ab}{a-b} \xi_{a,b}^{-m} \right) \right)^{\alpha} \\
\leq\; & \sum_{m=\lceil \frac{k-3}{2} \rceil}^{\lfloor \frac{2k-5}{3} \rfloor} (a_{k-2,m} + b_{k-2,m}) \left(\frac{2ab}{a-b} \xi_{a,b}^{-m} \right)^{\alpha} \\
& + \sum_{m=\lceil \frac{k-2}{2} \rceil}^{\lfloor \frac{2k-3}{3} \rfloor} (a_{k-1,m} + b_{k-1,m}) \left(\frac{2ab}{a-b} \xi_{a,b}^{-m} \right)^{\alpha}.
\end{aligned}
$$

Therefore, if the right hand side is goes to zero as $k \to \infty$ for some α, then $h^{\alpha'}(\Sigma_{a,b}) = 0$ for all $\alpha' > \alpha$, and $\dim_H(\Sigma_{a,b}) < \alpha$. Suppose $\alpha > \frac{f^*}{\log \xi_{a,b}}$. Note that this gives $f^* - \alpha \log \xi_{a,b} < 0$. We want to show

$$
\sum_{m=\lceil \frac{k-3}{2} \rceil}^{\lfloor \frac{2k-5}{3} \rfloor} (a_{k-2,m} + b_{k-2,m}) \left(\frac{2ab}{a-b} \xi_{a,b}^{-m} \right)^{\alpha} \to 0
$$

and

$$\sum_{m=\lceil\frac{k-2}{2}\rceil}^{\lfloor\frac{2k-3}{3}\rfloor}(a_{k-1,m}+b_{k-1,m})\left(\frac{2ab}{a-b}\xi_{a,b}^{-m}\right)^{\alpha}\to 0$$

as $k\to\infty$ to get that $\dim_H(\Sigma_{a,b})\leq\frac{f^*}{\log\xi_{a,b}}$.

First consider $A=\displaystyle\sum_{m=\lceil\frac{k-2}{2}\rceil}^{\lfloor\frac{2k-5}{3}\rfloor}a_{k-2,m}\left(\frac{2ab}{a-b}\xi_{a,b}^{-m}\right)^{\alpha}$. Using (3.31) we get

$$\begin{aligned}
A &\lesssim \sum_{m=\lceil\frac{k-2}{2}\rceil}^{\lfloor\frac{2k-5}{3}\rfloor}k\exp\left(mf\left(\frac{m}{k-2}\right)\right)\xi_{a,b}^{-m\alpha}\\
&= \sum_{m=\lceil\frac{k-2}{2}\rceil}^{\lfloor\frac{2k-5}{3}\rfloor}k\exp\left(m\left(f\left(\frac{m}{k-2}\right)-\alpha\log\xi_{a,b}\right)\right)\\
&\leq k\left(\frac{2k-5}{3}-\frac{k-2}{2}\right)\exp\left(\frac{k-2}{2}\left(f^*-\alpha\log\xi_{a,b}\right)\right)\\
&\lesssim k^2\exp\left(\frac{k}{2}(f^*-\alpha\log\xi_{a,b})\right),
\end{aligned}$$

which goes to 0 as $k\to\infty$, due to the bound on α.

Next consider $B=\displaystyle\sum_{m=\lceil\frac{k-3}{2}\rceil}^{\lfloor\frac{2k-6}{3}\rfloor}b_{k-2,m}\left(\frac{2ab}{a-b}\xi_{a,b}^{-m}\right)^{\alpha}=\sum_{m=\lceil\frac{k-3}{2}\rceil}^{\lfloor\frac{2k-6}{3}\rfloor}a_{k-1,m+1}\left(\frac{2ab}{a-b}\xi_{a,b}^{-m}\right)^{\alpha}$.

Similarly we see

$$\begin{aligned}
B &\lesssim \sum_{m=\lceil\frac{k-3}{2}\rceil}^{\lfloor\frac{2k-6}{3}\rfloor}k\exp\left((m+1)f\left(\frac{m+1}{k-1}\right)\right)\xi_{a,b}^{-m\alpha}\\
&= \sum_{m=\lceil\frac{k-2}{2}\rceil}^{\lfloor\frac{2k-5}{3}\rfloor}k\exp\left(m\left(f\left(\frac{m+1}{k-1}\right)-\alpha\log\xi_{a,b}\right)+f\left(\frac{m+1}{k-1}\right)\right)\\
&\leq k\left(\frac{2k-6}{3}-\frac{k-3}{2}\right)\exp\left(\frac{k-3}{2}(f^*-\alpha\log\xi_{a,b})+f^*\right)\\
&\lesssim k^2\exp\left(\frac{k}{2}(f^*-\alpha\log\xi_{a,b})\right)
\end{aligned}$$

which goes to 0 as $k\to\infty$. $\square$

3.4 Hyperbolicity of the Set $\Omega_{a,b}$

In this section we will prove Theorem 1.6. To do this we must first consider a surface $\mathbb{S}_{a,b}$, introduced below and related to the trace map, and prove that it contains a certain hyperbolic set. Recall from Section 2.3 the definition of a hyperbolic set:

Definition 2.1. Suppose M is a manifold and f is a map defined on M. Let $\Lambda \subset M$ be a compact invariant set; that is, let $f(\Lambda) = \Lambda$, on which f is invertible. Then Λ is said to be a hyperbolic set if the tangent bundle over Λ admits a decomposition

$$T_\Lambda M = E^u \oplus E^s,$$

invariant under Df and such that $\|Df^{-n}(x) \restriction_{E^u_x} \| \leq c\kappa^n$ and $\|Df^n(x) \restriction_{E^s_x} \| \leq c\kappa^n$ for every $x \in \Lambda, n \in \mathbb{N}$ and for some $c > 0$, $\kappa \in (0,1)$. Moreover, if there is an open neighborhood V of Λ such that $\Lambda = \Lambda^f_V := \bigcap_{n\in\mathbb{Z}} f^n(\overline{V})$, then Λ is said to be locally maximal, or basic.

Also recall the following theorem, which will be useful in proving hyperbolicity:

Theorem 2.2. *A compact f-invariant set Λ is hyperbolic if and only if there exist $\lambda < 1 < \mu$ such that at every $x \in \Lambda$ there are complementary subspaces S_x and T_x (in general, not Df-invariant), a field of horizontal cones $H_x \supset S_x$, and a family of vertical cones $V_x \supset T_x$ associated with that decomposition such that $Df_x H_x \subset \mathrm{Int} H_{f(x)}, Df_x^{-1} V_{f(x)} \subset \mathrm{Int} V_x, \|Df_x \xi\| \geq \mu \|\xi\| \,\mathrm{for}\, \xi \in H_x$, and $\|Df_x^{-1}\xi\| \geq \lambda^{-1}\|\xi\| \,\mathrm{for}\, \xi \in V_{f(x)}$.*

Now let us define $\mathbb{S}_{a,b}$, and see how it relates to the spectrum $\Sigma_{a,b}$. If $E \in \Sigma_{a,b}$ and $k \geq 2$, Corollary 3.7 and Theorem 1.2 imply if $|x_k| > 1$, then $|x_{k-1}|, |x_{k+1}| \leq 1$.

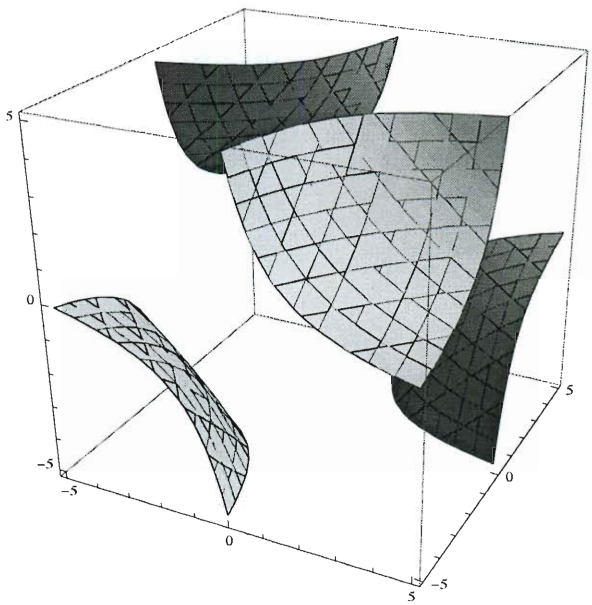

Figure 3.2: $\mathbb{S}_{a,b}$ for $c = 112/15$.

Thus, to obtain further results about $\Sigma_{a,b}$, we want to consider bi-infinite sequences $\{x_n\}$ generated by the trace map and its inverse such that no two consecutive terms have modulus greater than unity. First we restrict ourselves to the family of cubic surfaces

$$\mathbb{S}_{a,b} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - 2xyz - 1 = c^2\}. \tag{3.34}$$

Each of these surfaces is preserved under the trace map $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x, y, z) = (2xy - z, x, y)$.

Define property $\mathbb{P}$ to be that no two consecutive terms have modulus greater than unity. Consider the set

$$R_{a,b} = \{(x, y, z) \in \mathbb{S}_{a,b} \mid 2xy - z, x, y, z, 2yz - x \text{ has property } \mathbb{P}\}. \tag{3.35}$$

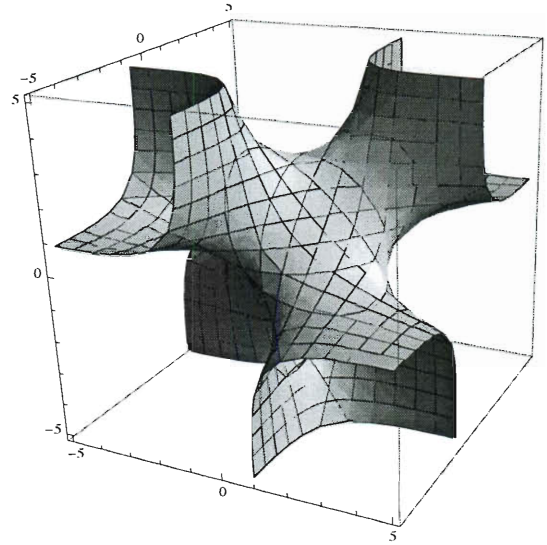

Figure 3.3: $\mathbb{S}_{a,b}$ for $c = 12/5$.

Lemma 3.28. *For $c > 2$, the set $R_{a,b}$ consists of ten disjoint regions defined by:*

$$R_1 = *sL^+s* \qquad R_2 = *sL^-s*$$

$$R_3 = L^-ssL^+s \quad R_4 = L^+ssL^-s \quad R_5 = sL^+ssL^- \quad R_6 = sL^-ssL^+$$

$$R_7 = sL^+sL^+s \quad R_8 = sL^+sL^-s \quad R_9 = sL^-sL^-s \quad R_{10} = sL^-sL^+s,$$

where $L^-, s, L^+, *$ respectively denote the intervals $(-\infty, -c+1], [-1, 1], [c-1, \infty)$, $(-\infty, \infty)$.

This notation closely follows that in [5]. For example, if $(x, y, z) \in R_3$, then $2xy - z \in (\infty, -c+1], x \in [-1, 1], y \in [-1, 1], z \in [c-1, \infty)$, and $2yz - x \in [-1, 1]$.

Proof. From property $\mathbb{P}$ we see that $L^{\pm}$ must have an s before and after it. Also, the combinations of $L^{\pm}ssL^{\pm}$ are not possible: Let $x, y \in s$ and $z \in L^{\pm}$. Considering $T(x, y, z) = (2xy - z, x, y)$ we notice that $2xy - z \in L^{\mp}$. And the combination of sss is not possible for $c > 2$ due to the invariant $x^2 + y^2 + x^2 - 2xyz - 1 = c^2$. Finally, we must show that if $(x, y, z) \in R_{a,b}$ and one of the terms $2xy - z, x, y, z, 2yz - x$

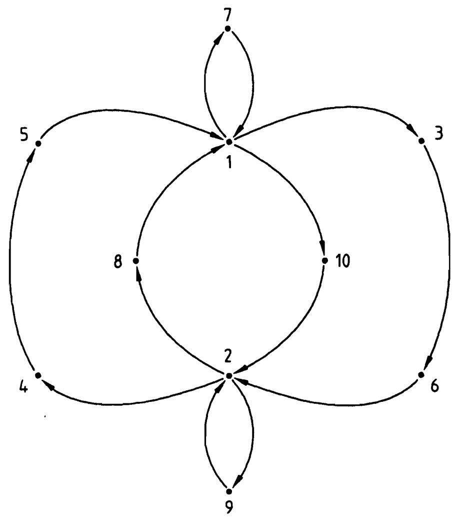

Figure 3.4: Movement between regions in $R_{a,b}$ under T.

has modulus greater than one, then it actually has modulus greater than or equal to $c-1$; i.e., if the term is not in s, then it is in $L^{\pm}$. Without loss of generality, let $|y|>1$. From the invariant we get that $y = xz \pm \sqrt{c^2+(1-x^2)(1-z^2)}$ and thus $|y| \geq c-1$. $\square$

Let $\Omega_{a,b}$ be the set of points in $\mathbb{S}_{a,b}$ with bounded full (forward and backward) orbits under T. We can see how a point in $\Omega_{a,b}$ can move between the regions in $R_{a,b}$ under iterations of T in the directed graph in Figure 3.4. The goal is to see that under the trace map, the set $\Omega_{a,b}$ is a locally maximal invariant hyperbolic set. From there, we will see that Theorem 1.6 follows easily.

Definition 3.29. Define the sets V_s for $s \in S = \{17, 110, 136, 29, 28, 245\}$ by $V_{s_0 s_1} = R_{s_0} \cap T^{-1} R_{s_1}$ and $V_{s_0 s_1 s_2} = R_{s_0} \cap T^{-1} R_{s_1} \cap T^{-2} R_{s_2}$.

We are concerned with these sets for the following reason: Consider a point $x \in$

$\Omega_{a,b} \subset \mathbb{R}^3$. Either $x \in V_s$ for some $s \in S$, or one of $T(x), T^2(x)$ is in V_s. Define ϕ on $\bigcup_{s \in S} V_s$ by $\phi(x) = \begin{cases} T^2(x) & \text{if } x \in V_{17} \cup V_{110} \cup V_{28} \cup V_{29} \\ T^3(x) & \text{if } x \in V_{136} \cup V_{245} \end{cases}$.

Lemma 3.30. *Let $\mu = \frac{1}{3}$. Consider the cone field $S^+ = \{(\xi, \eta) \mid |\eta| \leq \mu|\xi|\}$ defined over $\bigcup_{s \in S} V_s$, where (ξ, ζ, η) are the local coordinates for a point in the tangent space of $\mathbb{S}_{a,b}$ at (x, y, z). For $c \geq 6$ these cones are mapped into themselves by $D\phi$; i.e., for $(x, y, z) \in V_S$, and for ξ_0, η_0 with $|\eta_0| \leq \mu|\xi_0|$, then ξ_1 and η_1 defined by $(\xi_1, \zeta_1, \eta_1) = D\phi(\xi_0, \zeta_0, \eta_0)$ are such that $|\eta_1| \leq \mu|\xi_1|$. Also, the mapping is such that $|\xi_1| \geq \frac{1}{\mu}|\xi_0|$.*

Similarly, the cone field $S^- = \{(\xi, \eta) \mid |\eta| \geq \frac{1}{\mu}|\xi|\}$ is mapped into itself by $D\phi^{-1}$ with $|\eta_1| \geq \frac{1}{\mu}|\eta_0|$ where $(\xi_1, \zeta_1, \eta_1) = D\phi^{-1}(\xi_0, \zeta_0, \eta_0)$ for some $(\xi_0, \eta_0) \in S^-$.

Note that, by Theorem 2.2, this lemma proves that the set $\bigcup_{s \in S} V_s$ under the map ϕ is a hyperbolic set. This will be an important tool in showing that $\Omega_{a,b}$ is hyperbolic.

Proof. First note that $T^{-1} = \rho_{xz}^{-1} T \rho_{xz}$ where ρ_{xz} is the reflection given by $\rho_{xz}(x, y, z) = (z, y, x)$. Thus, we just need to show the statement of the lemma holds for S^+.

Consider $V_{136} = R_1 \cap T^{-1}R_3 \cap T^{-2}R_6$, though note that $V_{136} = R_1 \cap T^{-2}R_6$ is enough. As R_6 is represented by sL^-ssL^+, letting $y = 1$ and $z = t$ gives $x = t - c$, and the line $\{(t - c, 1, t) \mid t \in [-1, 1]\}$ is the right boundary for y in R_6. Then $T^{-2}((t-c, 1, t)) = (t, t + c, 2t^2 + 2ct - 1)$, and for this to be in R_1 we need $t \in [-1, 1]$, $t + c \in [c - 1, \infty)$, and $2t^2 + 2ct - 1 \in [-1, 1]$. Combining these we get

$$\begin{aligned} t &\in [-1, 1] \cap [-1, \infty) \cap \left([-\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2 + 4}, -c] \cup [0, -\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2 + 4}]\right) \\ &= [0, -\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2 + 4}]. \end{aligned}$$

Next, consider the left boundary for y in R_6 given by $\{(-t-c,-1,t) \mid t \in [-1,1]\}$. We have $T^{-2}((-t-c,-1,t)) = (t, c-t, -2t^2+2ct+1)$. For this to be in R_1 we need $t \in [-1,1]$, $c-t \in [c-1,\infty)$ and $-2t^2+2ct+1 \in [-1,1]$. Combining these we get

$$\begin{aligned} t &\in [-1,1] \cap (-\infty,1] \cap \left([\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+4}, 0] \cup [c, \tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2+4}]\right) \\ &= [\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+4}, 0]. \end{aligned}$$

So in $V_{136} = R_1 \cap T^{-2}R_6$ we have $x(=t) \in [\frac{c}{2} - \frac{1}{2}\sqrt{c^2+4}, -\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4}]$ and $z \in [-1,1]$. To get better bounds for y, note that $y = xz + \sqrt{c^2 + (1-x^2)(1-z^2)}$ from the invariant $x^2+y^2+z^2-2xyz-1 = c^2$, and we use interval analysis:

- $[a,b] + [c,d] = [a+c, b+d]$
- $[a,b] - [c,d] = [a-d, b-c]$
- $[a,b] \cdot [c,d] = [\min\{ac,ad,bc,bd\}, \max\{ac,ad,bc,bd\}]$
- $[a,b]/[c,d] = [a,b] \cdot [1/d, 1/c]$ provided that $0 \notin [c,d]$

With interval analysis we get that $y \in [\frac{3c}{2} - \frac{1}{2}\sqrt{c^2+4}, -\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4} + \sqrt{c^2+1}]$.

The tangent plane at a point (x,y,z), after canceling out a factor of 2, is given by the equation $(x-yz)\xi + (y-xz)\zeta + (z-xy)\eta = 0$ where $(\xi,\zeta,\eta) \in \mathbb{R}^3$ are points on the plane relative to (x,y,z). Note that in R_1 and R_2 it is true that $y - xz \neq 0$, and so any point on the tangent plane can be given just in terms of ξ and η by $(\xi, \zeta(\xi,\eta), \eta)$. Solving the tangent plane equation for ζ gives

$$\zeta = \frac{(yz-x)\xi + (xy-z)\eta}{y-xz}.$$

We want to show that for ξ_0, η_0 with $|\eta_0| \leq \frac{1}{3}|\xi_0|$, then ξ_1 and η_1 defined by $(\xi_1, \zeta_1, \eta_1) = D(T^3)(\xi_0, \zeta_0, \eta_0)$ are such that $|\eta_1| \leq \frac{1}{3}|\xi_1|$ and $|\xi_1| \geq 3|\xi_0|$. By linearity we can set $\xi_0 = 1$ and $\eta_0 \in [-\frac{1}{3}, \frac{1}{3}]$. Then, using interval analysis we see that $(yz - x)\xi_0 + (xy - z)\eta_0 \in [-\frac{c^2}{6} + \frac{c}{6}\sqrt{c^2+4} + \frac{c}{6}\sqrt{c^2+1} - \frac{1}{6}\sqrt{(c^2+4)(c^2+1)} - \frac{2}{3} + c - \sqrt{c^2+4} - \sqrt{c^2+1}, \frac{c^2}{6} - \frac{c}{6}\sqrt{c^2+4} - \frac{c}{6}\sqrt{c^2+1} + \frac{1}{6}\sqrt{(c^2+4)(c^2+1)} + \frac{2}{3} - c + \sqrt{c^2+4} + \sqrt{c^2+1}]$ and $y - xz \in [2c - \sqrt{c^2+4}, -c + \sqrt{c^2+4} + \sqrt{c^2+1}]$. This gives $\zeta_0 \in [-\frac{4}{3}, \frac{4}{3}]$ for $c \geq 6$.

Computing the differential, we get $D(T^3) =$

$$2\begin{pmatrix} 24x^2y^2 - 16xyz - 2y^2 + 2z^2 - \frac{1}{2} & 16x^3y - 8x^2z - 4xy + z & -8x^2y + 4xz + y \\ 4xy - z & 2x^2 & -x \\ y & x & -\frac{1}{2} \end{pmatrix}$$

Let us first consider $\xi_1 = 48x^2y^2 - 32xyz - 4y^2 + 4z^2 - 1 + \zeta_0(32x^3y - 16x^2z - 8xy + 2z) + \eta_0(-16x^2y + 8xz + 2y)$. We would like to find the minimum modulus for $c \geq 6$. Using Mathematica, for $c = 6$ and with the aforementioned constraints on x, y, z, ξ_0, η_0 and ζ_0, we see that the maximum value of ξ_1 is quite negative: $\xi_1 < -46$. We will see that actually $\xi_1 < -33$, and the claim now is that as c grows, this bound decreases.

First, substituting $y = xz + \sqrt{c^2 + (1-x^2)(1-z^2)}$ and setting $\delta := \delta(c, x, z) = \sqrt{c^2 + (1-x^2)(1-z^2)}$, we can break up ξ_1 as follows: $\xi_1 = A + B$, where $A = 96x^4z^2 + 52x^2 - 48x^4 - 88x^2z^2 - 5 + 8z^2 + 32\zeta_0x^4z - 24\zeta_0x^2z + 2\zeta_0z - 16\eta_0x^3z + 10\eta_0xz$ and $B = 96x^3z\delta + 48x^2c^2 - 40xz\delta - 4c^2 + 32\zeta_0x^3\delta - 8\zeta_0x\delta - 16\eta_0x^2\delta + 2\eta_0\delta$. Note that A is c independent, and B is c dependent. Considering A, as the intervals for z, ζ_0 and η_0 do not depend on c, and because the interval in which x is contained shrinks as c gets bigger, it is clear that as c grows, the maximum value of A is nonincreasing. Using Mathematica, one finds that the maximum value of A at $c = 6$ is less than 6, so $A < 6$ for $c \geq 6$.

Now consider B. Just noting that $x, z \in [-1, 1]$, one sees that $\delta \in [c, \sqrt{c^2+1}]$. For a fixed c, we can put a bound on the maximum of B by the sum of the maxima of each term in B. That is, we have $B \leq 96(-\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4})^3\sqrt{c^2+1} + 48(-\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4})^3 c^2 + 40(-\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4})\sqrt{c^2+1} - 4c^2 + \frac{128}{3}(-\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4})^3\sqrt{c^2+1} + \frac{32}{3}(-\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4})\sqrt{c^2+1} + \frac{16}{3}(-\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4})^2\sqrt{c^2+1} + \frac{2}{3}\sqrt{c^2+1} =: C(c) < -39$ for $c \geq 6$, and this bound is decreasing for $c > 0$. Thus, it is clear that for $c \geq 6$, we have $\xi_1 < -33$, and actually $\xi_1 < C(c) + 6$.

Now consider $\eta_1 = 2y\xi_0 + 2x\zeta_0 - \eta_0$. Using interval analysis one finds $\eta_1 \in [\frac{13c}{3} - \frac{7}{3}\sqrt{c^2+4} - \frac{1}{3}, -\frac{7c}{3} + \frac{7}{3}\sqrt{c^2+4} + 2\sqrt{c^2+1} + \frac{1}{3}]$. Then we can see that $|\eta_1/\xi_1| \leq \left|\frac{-\frac{7c}{3}+\frac{7}{3}\sqrt{c^2+4}+2\sqrt{c^2+1}+\frac{1}{3}}{C(c)+6}\right| < \frac{1}{3}$ for $c \geq 6$. Finally, it is clear that as $|\xi_1| > 30$ for $c \geq 6$, we have $|\xi_1| \geq 3|\xi_0| = 3$.

Consider $V_{245} = R_2 \cap T^{-1}R_4 \cap T^{-2}R_5 = R_2 \cap T^{-2}R_5$. Recall R_5 is represented by sL^+ssL^-, and the right boundary is $\{(t+c, 1, t) \mid t \in [-1,1]\}$. We see $T^{-2}((t+c, 1, t)) = (t, t-c, 2t^2 - 2tc - 1)$, and for this to be in R_2 we need $t \in [-1,1], t-c \in (-\infty, -c+1]$ and $2t^2 - 2tc - 1 \in [-1,1]$. Combining these we get

$$
\begin{aligned}
t &\in [-1,1] \cap (-\infty, 1] \cap \left([\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+4}, 0] \cup [c, \tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2+4}]\right) \\
&= [\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+4}, 0].
\end{aligned}
$$

The left boundary is $\{(-t+c, -1, t) \mid t \in [-1,1]\}$, and for $T^{-2}((-t+c, -1, t)) = (t, -t-c, -2t^2 - 2ct + 1)$ to be in R_2 we need $t \in [-1,1], -t-c \in (-\infty, -c+1]$, and

$-2t^2 - 2ct + 1 \in [-1, 1]$; i.e., we need

$$t \in [-1,1] \cap [-1,\infty) \cap \left([-\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+4}, -c] \cup [0, -\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2+4}]\right)$$
$$= [0, -\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2+4}].$$

So in V_{245} we have $x \in [\frac{c}{2} - \frac{1}{2}\sqrt{c^2+4}, -\frac{c}{2} + \frac{1}{2}\sqrt{c^2+4}]$ and $z \in [-1,1]$. As $y = xz - \sqrt{c^2 + (1-x^2)(1-z^2)}$, using interval analysis we get $y \in [\frac{c}{2} - \frac{1}{2}\sqrt{c^2+4} - \sqrt{c^2+1}, -\frac{3c}{2} + \frac{1}{2}\sqrt{c^2+4}]$. Again, with $(\xi_1, \zeta_1, \eta_1) = D(T^3)(\xi_0, \zeta_0, \eta_0)$, take $\xi_0 = 1$ and $\eta_0 \in [-\frac{1}{3}, \frac{1}{3}]$, and we find that the bounds for $(yz - x)\xi_0 + (xy - z)\eta_0$ are as before from V_{136}. Also $y - xz \in [c - \sqrt{c^2+4} - \sqrt{c^2+1}, -2c + \sqrt{c^2+4}]$, and again we find that $\zeta_0 \in [-\frac{4}{3}, \frac{4}{3}]$ for $c \geq 6$. Again, we want to consider ξ_1 by its c independent and dependent parts. Substituting in $y = xz - \sqrt{c^2 + (1-x^2)(1-z^2)}$, we see $\xi_1 = A' + B'$, where $A' = A$ from the V_{136} case and $B' = -96x^3z\delta + 48x^2c^2 + 40xz\delta - 4c^2 - 32\zeta_0 x^3\delta + 8\zeta_0 x\delta + 16\eta_0 x^2\delta - 2\eta_0\delta$. Note that up to some sign changes, these are the same terms in B. Recall that as c grows, the maximum of A is nonincreasing. We bound the maximum of B' by the sum of the maxima of each term, again getting $B' \leq C(c)$. So $\xi_1 < C(c) + 6$ for $c \geq 6$. Performing interval analysis on $\eta_1 = 2y\xi_0 + 2x\zeta_0 - \eta_0$ gives $\eta_1 \in [\frac{7c}{3} - \frac{7}{3}\sqrt{c^2+4} - 2\sqrt{c^2+1} - \frac{1}{3}, -\frac{13c}{7} + \frac{7}{3}\sqrt{c^2+4} + \frac{1}{3}]$. Then we can see that $|\eta_1/\xi_1| \leq \left|\frac{\frac{7c}{3} - \frac{7}{3}\sqrt{c^2+4} - 2\sqrt{c^2+1} - \frac{1}{3}}{C(c)+6}\right| < \frac{1}{3}$ for $c \geq 6$. Finally, it is clear that as $|\xi_1| > 30$ for $c \geq 6$, we have $|\xi_1| \geq 3|\xi_0| = 3$.

Consider $V_{17} = R_1 \cap T^{-1}R_7 = R_1 \cap T^{-2}R_1$. Recall that R_1 is represented by $*sL^+s*$. Letting $x = 1$ and $z = t$ gives $\{(1, c+t, t) \mid t \in [-1,1]\}$ as the right vertical boundary of R_1. For $T^{-2}(1, c+t, t) = (t, 2t^2 + 2ct - 1, 4t^3 + 4ct^2 - 3t - c)$ to be in R_1 we need that $t \in [-1,1]$, $2t^2 + 2ct - 1 \in [c-1, \infty)$, and $4t^3 + 4ct^2 - 3t - c \in [-1,1]$.

That is, we need

$$t \in [-1,1] \cap \left((-\infty, -\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+2c}] \cup [-\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2+2c}, \infty)\right)$$
$$\cap \left([-\tfrac{1}{4} - \tfrac{c}{2} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, \tfrac{1}{4} - \tfrac{c}{2} - \tfrac{1}{4}\sqrt{4c^2+4c+9}] \cup\right.$$
$$\left.[-\tfrac{1}{2}, -\tfrac{1}{4} - \tfrac{c}{2} + \tfrac{1}{4}\sqrt{4c^2-4c+9}] \cup [\tfrac{1}{2}, \tfrac{1}{4} - \tfrac{c}{2} + \tfrac{1}{4}\sqrt{4c^2+4c+9}]\right)$$
$$= [\tfrac{1}{2}, \tfrac{1}{4} - \tfrac{c}{2} + \tfrac{1}{4}\sqrt{4c^2+4c+9}].$$

Similarly, the left vertical boundary is given by $\{(-1, c-t, t) \mid t \in [-1,1]\}$ and $T^{-2}(-1, c-t, t) = (t, -2t^2 + 2ct + 1, -4t^3 + 4ct^2 + 3t - c)$, and for this to be in R_1 we need

$$t \in [-1,1] \cap [\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2-2c+4}, \tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2-2c+4}] \cap$$
$$\left([-\tfrac{1}{4} + \tfrac{c}{2} - \tfrac{1}{4}\sqrt{4c^2+4c+9}, -\tfrac{1}{2}] \cup [\tfrac{1}{4} + \tfrac{c}{2} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, \tfrac{1}{2}] \cup\right.$$
$$\left.[-\tfrac{1}{4} + \tfrac{c}{2} + \tfrac{1}{4}\sqrt{4c^2+4c+9}, \tfrac{1}{4} + \tfrac{c}{2} + \tfrac{1}{4}\sqrt{4c^2-4c+9}]\right)$$
$$= [\tfrac{1}{4} + \tfrac{c}{2} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, \tfrac{1}{2}].$$

Putting these together we get that $t(=x) \in [\frac{1}{4} + \frac{c}{2} - \frac{1}{4}\sqrt{4c^2-4c+9}, \frac{1}{4} - \frac{c}{2} + \frac{1}{4}\sqrt{4c^2+4c+9}] \subset [\frac{1}{3} - \frac{1}{2c}, \frac{1}{2} + \frac{1}{2c}]$.

Now $y = xz + \sqrt{c^2 + (1-x^2)(1-z^2)}$, and using $x \in [\frac{1}{3} - \frac{1}{2c}, \frac{1}{2} + \frac{1}{2c}]$ and $z \in [-1,1]$, we find $y \in [c - \frac{1}{2} - \frac{1}{2c}, \sqrt{c^2 + 8/9 + 1/3c - 1/4c^2} + \frac{1}{2} + \frac{1}{2c}] \subset [c - \frac{1}{2} - \frac{1}{2c}, c + \frac{3}{4} + \frac{1}{2c}]$. These bounds, along with $\xi_0 = 1$ and $\eta_0 \in [-1/3, 1/3]$, give, for $c > 5$, that $(yz - x)\xi_0 + (xy - z)\eta_0 \in [-\frac{7c}{6} - \frac{4}{3c} - \frac{1}{12c^2} - 2, \frac{7}{6c} + \frac{4}{3c} + \frac{1}{12c^2} + \frac{7}{6}]$ and $y - xz \in [\frac{c^2-c-1}{c}, \frac{c^2+5c/4+1}{c}]$. Then, for $c \geq 6$, we have $\zeta_0 = \frac{(yz-x)\xi_0 + (xy-z)\eta_0}{y-xz} \in [-2, 2]$.

Finally, $D(T^2)$ is given by $\begin{pmatrix} 8xy-2z & 4x^2-1 & -2x \\ 2y & 2x & -1 \\ 1 & 0 & 0 \end{pmatrix}$, and here we have

$$\begin{pmatrix} [\frac{8c}{3}+\frac{2}{3c}+\frac{2}{c^2}-\frac{16}{3}, 4c+\frac{8}{c}+\frac{2}{c^2}+10] & [-\frac{5}{9}-\frac{4}{3c}+\frac{1}{c^2}, \frac{2}{c}+\frac{1}{c^2}] & [-1-\frac{1}{c}, -\frac{2}{3}+\frac{1}{c}] \\ [2c-1-\frac{1}{c}, 2c+\frac{3}{2}+\frac{1}{c}] & [\frac{2}{3}-\frac{1}{c}, 1+\frac{1}{c}] & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $(\xi_1, \zeta_1, \eta_1) = D(T^2)(\xi_0, \zeta_0, \eta_0)$, and then, for $c > 5$, we have $\xi_1 \in [\frac{8c}{3} - \frac{7}{3c} + \frac{4}{c^2} - \frac{61}{9}, 4c + \frac{37}{3c} - \frac{1}{c^2} + 12]$. We also have $\eta_1 = 1$. Thus it can now easily be seen that for $c \geq 6$, we have the desired results of $\eta_1/\xi_1 \in [-\frac{1}{3}, \frac{1}{3}]$ and $|\xi_1| > 3$.

Consider $V_{29} = R_2 \cap T^{-1}R_9 = R_2 \cap T^{-2}R_2$. Recall that R_2 is represented by $*sL^-s*$. Letting $x = 1$ and $z = t$ gives $\{(1, t-c, t) \mid t \in [-1,1]\}$ as the right vertical boundary of R_2. For $T^{-2}((1, t-c, t)) = (t, 2t^2 - 2ct - 1, 4t^3 - 4ct^2 - 3t + c)$ to be in R_2 we need that $t \in [-1,1]$, $2t^2 - 2ct - 1 \in (-\infty, -c+1]$, and $4t^3 - 4ct^2 - 3t + c \in [-1,1]$. That is, we need

$$\begin{aligned} t \in [-1,1] \cap [\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2-2c+4}, \tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2-2c+4}] \cap \\ ([\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2+4c+9}, -\tfrac{1}{2}] \cup [\tfrac{c}{2} + \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, \tfrac{1}{2}] \cup \\ [\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2+4c+9}, \tfrac{c}{2} + \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2-4c+9^2}]) \\ = [\tfrac{c}{2} + \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, \tfrac{1}{2}]. \end{aligned}$$

The left vertical boundary of R_2 is $\{(-1, -t-c, t) \mid t \in [-1,1]\}$ and we need

$T^{-2}((-1,-t-c,t)) = (t, -2t^2-2ct+1, -4t^3-4ct^2+3t+c) \in R_2$. Thus

$$t \in [-1,1] \cap \left((-\infty, -\tfrac{c}{2}-\tfrac{1}{2}\sqrt{c^2+2c}] \cup [-\tfrac{c}{2}+\tfrac{1}{2}\sqrt{c^2+2c}, \infty)\right) \cap$$

$$\left([-\tfrac{c}{2}-\tfrac{1}{4}-\tfrac{1}{4}\sqrt{4c^2-4c+9}, -\tfrac{c}{2}+\tfrac{1}{4}-\tfrac{1}{4}\sqrt{4c^2+4c+9}] \cup\right.$$

$$\left.[-\tfrac{1}{2}, -\tfrac{c}{2}-\tfrac{1}{4}+\tfrac{1}{4}\sqrt{4c^2-4c+9}] \cup [\tfrac{1}{2}, -\tfrac{c}{2}+\tfrac{1}{4}+\tfrac{1}{4}\sqrt{4c^2+4c+9}]\right)$$

$$= [\tfrac{1}{2}, -\tfrac{c}{2}+\tfrac{1}{4}+\tfrac{1}{4}\sqrt{4c^2+4c+9}].$$

So we get $x = t \in [\frac{c}{2}+\frac{1}{4}-\frac{1}{4}\sqrt{4c^2-4c+9}, -\frac{c}{2}+\frac{1}{4}+\frac{1}{4}\sqrt{4c^2+4c+9}] \subset [\frac{1}{3}-\frac{1}{2c}, \frac{1}{2}+\frac{1}{2c}]$. As $y = xz - \sqrt{c^2+(1-x^2)(1-z^2)}$, using $x \in [\frac{1}{3}-\frac{1}{2c}, \frac{1}{2}+\frac{1}{2c}]$ and $z \in [-1,1]$, it follows that $y \in [-\sqrt{c^2+8/9+1/3c-1/4c^2}-\frac{1}{2}-\frac{1}{2c}, -c+\frac{1}{2}+\frac{1}{2c}] \subset [-c-\frac{3}{4}-\frac{1}{2c}, -c+\frac{1}{2}+\frac{1}{2c}]$.

Then, with $\xi_0 = 1$ and $\eta_0 \in [-\frac{1}{3}, \frac{1}{3}]$ and $c \geq 6$, we have $(yz-x)\xi_0 + (xy-z)\eta_0 \in [-\frac{7c}{6}-\frac{29}{24c}-\frac{1}{12c^2}-\frac{15}{8}, \frac{7c}{6}+\frac{29}{24c}+\frac{1}{12c^2}+\frac{25}{24}]$ and $y - xz \in [-c-\frac{5}{4}-\frac{1}{c}, -c+1+\frac{1}{c}]$, and thus $\zeta_0 \in [-2,2]$.

We have $D(T^2)$ in

$$\begin{pmatrix} [-4c-\frac{5}{c}-\frac{2}{c^2}-9, \frac{22}{3}-\frac{8c}{3}-\frac{2}{3c}-\frac{2}{c^2}] & [-\frac{5}{9}-\frac{4}{3c}+\frac{1}{c^2}, \frac{2}{c}+\frac{1}{c^2}] & [-1-\frac{1}{c}, -\frac{2}{3}+\frac{1}{c}] \\ [-2c-\frac{3}{2}-\frac{1}{c}, -2c+1+\frac{1}{c}] & [\frac{2}{3}-\frac{1}{c}, 1+\frac{1}{c}] & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

If $(\xi_1, \zeta_1, \eta_1) = D(T^2)(\xi_0, \zeta_0, \eta_0)$, we have $\xi_1 \in [-4c-\frac{28}{3c}+\frac{1}{c^2}-11, -\frac{8c}{3}+\frac{7}{3c}-\frac{4}{c^2}+\frac{79}{7}]$ and $\eta_1/\xi_1 \in [-\frac{1}{3}, \frac{1}{3}]$, $|\xi_1| > 3$ for $c \geq 6$.

Consider $V_{110} = R_1 \cap T^{-1}R_{10} = R_1 \cap T^{-2}R_2$. For $(t, 2t^2-2ct-1, 4t^3-4ct^2-3t+c) \in$

R_1 we need

$$t \in [-1,1] \cap \left((-\infty, \tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2+2c}] \cup [\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2+2c}, \infty)\right) \cap$$
$$\left([\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2+4c+9}, -\tfrac{1}{2}] \cup [\tfrac{c}{2} + \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, \tfrac{1}{2}] \cup\right.$$
$$\left.[\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2+4c+9}, \tfrac{c}{2} + \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2-4c+9^2}]\right)$$
$$= [\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2+4c+9}, -\tfrac{1}{2}],$$

and for $(t, -2t^2 - 2ct + 1, -4t^3 - 4ct^2 + 3t + c) \in R_1$ we need

$$t \in [-1,1] \cap [-\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2-2c+4}, -\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2-2c+4}] \cap$$
$$\left([-\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2-4c+9}, -\tfrac{c}{2} + \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2+4c+9}] \cup\right.$$
$$\left.[-\tfrac{1}{2}, -\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2-4c+9}] \cup [\tfrac{1}{2}, -\tfrac{c}{2} + \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2+4c+9}]\right)$$
$$= [-\tfrac{1}{2}, -\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2-4c+9}].$$

Together these give $x = t \in [\frac{c}{2} - \frac{1}{4} - \frac{1}{4}\sqrt{4c^2+4c+9}, -\frac{c}{2} - \frac{1}{4} + \frac{1}{4}\sqrt{4c^2-4c+9}] \subset [-\frac{1}{2} - \frac{1}{2c}, -\frac{1}{3} + \frac{1}{2c}]$.

Then with $z \in [-1,1]$ we find $y \in [c - \frac{1}{2} - \frac{1}{2c}, \sqrt{c^2 + 8/9 + 1/3c - 1/4c^2} + \frac{1}{2} + \frac{1}{2c}] \subset [c - \frac{1}{2} - \frac{1}{2c}, c + \frac{3}{4} + \frac{1}{2c}]$.

Putting these bounds together with $\xi_0 = 1$ and $\eta_0 \in [-\frac{1}{3}, \frac{1}{3}]$, we find, for $c > 5$, that $(yz - x)\xi_0 + (xy - z)\eta_0 \in [-\frac{7c}{6} - \frac{29}{24c} - \frac{1}{12c^2} - \frac{25}{24}, \frac{7c}{6} + \frac{29}{24c} + \frac{1}{4c^2} + \frac{15}{8}]$ and $y - xz \in [c - 1 - \frac{1}{c}, c + \frac{5}{4} + \frac{1}{c}]$, and thus for $c \geq 6$, we find $\zeta_0 \in [-2, 2]$.

Now $D(T^2)$ is in

$$\begin{pmatrix} [-4c - \frac{5}{c} - \frac{2}{c^2} - 7, -\frac{8c}{3} - \frac{2}{3c} - \frac{2}{c^2} + \frac{22}{3}] & [-\frac{4}{3c} + \frac{1}{c^2} - \frac{5}{9}, \frac{2}{c} + \frac{1}{c^2}] & [\frac{2}{3} - \frac{1}{c}, 1 + \frac{1}{c}] \\ [2c - 1 - \frac{1}{c}, 2c + \frac{3}{2} + \frac{1}{c}] & [-1 - \frac{1}{c}, -\frac{2}{3} + \frac{1}{c}] & -1 \\ 1 & 0 & 0 \end{pmatrix},$$

and for $(\xi_1, \zeta_1, \eta_1) = D(T^2)(\xi_0, \zeta_0, \eta_0)$ then $\xi_1 \in [-4c - \frac{28}{3c} + \frac{1}{c^2} - 11, -\frac{8c}{3} + \frac{7}{3c} - \frac{4}{c^2} + \frac{79}{9}]$ and $\eta_1 = 1$, so for $c \geq 6$, it follows that $\eta_1/\xi_1 \in [-\frac{1}{3}, \frac{1}{3}]$ and $|\xi_1| > 3$.

Consider $V_{28} = R_2 \cap T^{-1}R_8 = R_2 \cap T^{-2}R_1$. For $(t, 2t^2 + 2ct - 1, 4t^3 + 4ct^2 - 3t - c) \in R_2$ we need

$$\begin{aligned}
t \in [-1,1] \cap [-\tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2 - 2c + 4}, -\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2 - 2c + 4}] \cap \\
([-\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2 - 4c + 9}, -\tfrac{c}{2} + \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2 + 4c + 9}] \cup \\
[-\tfrac{1}{2}, -\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2 - 4c + 9}] \cup [\tfrac{1}{2}, -\tfrac{c}{2} + \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2 + 4c + 9}]) \\
= [-\tfrac{1}{2}, -\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2 - 4c + 9}],
\end{aligned}$$

and for $(t, -2t^2 + 2ct + 1, -4t^3 + 4ct^2 + 3t - c) \in R_2$ we need

$$\begin{aligned}
t \in [-1,1] \cap ((-\infty, \tfrac{c}{2} - \tfrac{1}{2}\sqrt{c^2 + 2c}] \cup [\tfrac{c}{2} + \tfrac{1}{2}\sqrt{c^2 + 2c}, \infty)) \cap \\
([\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2 + 4c + 9}, -\tfrac{1}{2}] \cup [\tfrac{c}{2} + \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2 - 4c + 9}, \tfrac{1}{2}] \cup \\
[\tfrac{c}{2} - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2 + 4c + 9}, \tfrac{c}{2} + \tfrac{1}{4} + \tfrac{1}{4}\sqrt{4c^2 - 4c + 9}]) \\
= [\tfrac{c}{2} - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{4c^2 + 4c + 9}, -\tfrac{1}{2}].
\end{aligned}$$

Together these give $x = t \in [\frac{c}{2} - \frac{1}{4} - \frac{1}{4}\sqrt{4c^2 + 4c + 9}, -\frac{c}{2} - \frac{1}{4} + \frac{1}{4}\sqrt{4c^2 + 4c + 9}] \subset [-\frac{1}{2} - \frac{1}{2c}, -\frac{1}{3} + \frac{1}{2c}]$. This, with $z \in [-1,1]$, gives $y \in [-\frac{1}{2} - \frac{1}{2c} - \sqrt{c^2 + 8/9 + 1/3c - 1/4c^2}, -c + \frac{1}{2} + \frac{1}{2c}] \subset [-c - \frac{3}{4} - \frac{1}{2c}, -c + \frac{1}{2} + \frac{1}{2c}]$ for $c > 2$. Together with $\xi_0 = 1$, $\eta_0 \in [-\frac{1}{3}, \frac{1}{3}]$ and $c > 5$, we find that $(yz - x)\xi_0 + (xy - z)\eta_0 \in [-\frac{7c}{6} - \frac{29}{24c} - \frac{1}{12c^2} - \frac{25}{24}, \frac{7c}{6} + \frac{29}{24c} + \frac{1}{12c^2} + \frac{15}{8}]$ and $y - xz \in [-c - \frac{5}{4} - \frac{1}{c}, -c + 1 + \frac{1}{c}]$. It can then be easily seen that $\zeta_0 \in [-2, 2]$ for $c \geq 6$. Then $D(T^2)$ is in

$$\begin{pmatrix} [\frac{8c}{3}+\frac{2}{3c}+\frac{2}{c^2}-\frac{22}{3}, 4c+\frac{5}{c}+\frac{2}{c^2}+9] & [-\frac{4}{3c}+\frac{1}{c^2}-\frac{5}{9}, \frac{2}{c}+\frac{1}{c^2}] & [\frac{2}{3}-\frac{1}{c}, 1+\frac{1}{c}] \\ [-2c-\frac{3}{2}-1, -2c+1+\frac{1}{c}] & [-1-\frac{1}{c}, -\frac{2}{3}+\frac{1}{c}] & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For $(\xi_1, \zeta_1, \eta_1) = D(T^2)(\xi_0, \zeta_0, \eta_0)$ then $\xi_1 \in [\frac{8c}{3}-\frac{7}{3c}+\frac{4}{c^2}-\frac{79}{9}, 4c+\frac{28}{3c}-\frac{1}{c^2}+11]$ and $\eta_1 = 1$, so for $c \geq 6$ we have $\eta_1/\xi_1 \in [-\frac{1}{3}, \frac{1}{3}]$ and $|\xi_1| > 3$. $\square$

Now we will use this result to show hyperbolicity of the set $\Omega_{a,b}$:

Theorem 3.31. *The set $\Omega_{a,b}$ is a locally maximal invariant hyperbolic set of $T : \mathbb{S}_{a,b} \to \mathbb{S}_{a,b}$ for $c \geq 6$.*

Proof. From Theorem 2.2 and Lemma 3.30 it is clear that the set $\bigcup_{s\in S} V_s$ is a hyperbolic set of $\phi(x)$. Consider a point $x \in \Omega_{a,b}$ such that $x \in V_s$ for some $s \in S$. If n is such that $T^n(x) \in \bigcup_{s\in S} V_s$, then by the definition of a hyperbolic set there exists a $\gamma > 0$, $\kappa \in (0,1)$ such that $\|DT^n(x) \restriction_{E^s_x}\| \leq \gamma\kappa^j$, where $T^n(x) = \phi^j(x)$. Define $E^{u,s}_{T(x)} := DT(E^{u,s}_x)$ and $E^{u,s}_{T^2(x)} := DT^2(E^{u,s}_x)$. It is clear that the subspaces $E^s_{T(x)}$ and $E^u_{T(x)}$ and the subspaces $E^s_{T^2(x)}$ and $E^u_{T^2(x)}$ are complementary, because if not then $E^{u,s}_{T^3(x)} = DT^3(E^{u,s}_x)$ would not be possible. As we are on a compact surface the differential is bounded; that is, $\|DT\| < \alpha < \infty$. Let $\tilde{\kappa} = \kappa^{j/n}$ and choose $\gamma' > 1$ such that $\alpha \leq \gamma'\tilde{\kappa}$. Then $\|DT^n(x) \restriction_{E^s_x}\| \leq \gamma\tilde{\kappa}^n$, $\|DT^{n+1}(x) \restriction_{E^s_x}\| \leq \alpha\gamma\tilde{\kappa}^n \leq \gamma\gamma'\tilde{\kappa}^{n+1}$, and $\|DT^{n+2}(x) \restriction_{E^s_x}\| \leq \alpha^2\gamma\tilde{\kappa}^n \leq \gamma\gamma'^2\tilde{\kappa}^{n+2}$. Choose $\beta = \gamma'^2\gamma$ and it is clear we have the desired relation $\|DT^k(x) \restriction_{E^s_x}\| \leq \beta\tilde{\kappa}^k$ for all $x \in \bigcup_{s\in S} V_s$. The same argument can be made in the unstable direction.

Now consider an $x \in \Omega_{a,b}$ such that $x \notin V_s$ for any s. Then either $T(x) \in V_s$ or $T^2(x) \in V_s$ for some $s \in S$. If $T(x) \in V_s$ and we have $T^{n+1}(x) \in \bigcup_{s\in S}$, then

$\|DT^{n+1}(x) \restriction_{E_x^s} \| \leq \alpha\gamma\tilde{\kappa}^n \leq \gamma\gamma'\tilde{\kappa}^{n+1}$. Similarly $\|DT^{n+2}(x) \restriction_{E_x^s} \| \leq \alpha^2\gamma\tilde{\kappa}^n \leq \gamma\gamma'^2\tilde{\kappa}^{n+2}$ if $T^2(x) \in V_s$, and the rest follows from above. The same argument can be made in the unstable direction. Therefore there exists a $\tilde{\kappa} \in (0,1)$ and a $\beta > 0$ such that $\|DT^m(x) \restriction_{E_x^s} \|, \|DT^{-m}(x) \restriction_{E_x^u} \| \leq \beta\tilde{\kappa}^m$ for all $m \in \mathbb{N}$, $x \in \Omega_{a,b}$.

The set $\Omega_{a,b}$ is clearly invariant under T by definition: For $x \in \mathbb{S}_{a,b}$, it follows that $x \in \Omega_{a,b}$ if its full orbit under T is bounded, so obviously then $T^n(x) \in \Omega_{a,b}$ as well. Now to show that $\Omega_{a,b}$ is locally maximal, we want to show that there exists an open neighborhood V of $\Omega_{a,b}$ such that $\Omega_{a,b} = \bigcap_{n\in\mathbb{Z}} T^n(\overline{V})$. Take V to be the ε-neighborhood of $\Omega_{a,b}$. Then, as $\Omega_{a,b}$ is a bounded set, V is also bounded. Let $x \in \bigcap_{n\in\mathbb{Z}} T^n(\overline{V})$. Suppose for contradiction that $x \notin \Omega_{a,b}$. Then $T^k(x)$ is unbounded either as $k \to \infty$ or as $k \to -\infty$. Without loss of generality, assume $T^k(x)$ is unbounded as $k \to \infty$. As $x \in T^n(\overline{V})$ for all $n \in \mathbb{Z}$, we have $T^k(x) \in \bigcap_{n\in\mathbb{Z}} T^n(\overline{V})$ for all k. This is true as T is invertible, so if $T^k(x) \notin T^n(\overline{V})$ for some $k, n \in \mathbb{Z}$, then we would have $x \notin T^{n-k}(\overline{V})$. So $T^k(x) \in \bigcap_{n\in\mathbb{Z}} T^n(\overline{V})$, but this is a contradiction, as $\bigcap_{n\in\mathbb{Z}} T^n(\overline{V})$ is a bounded set and $T^k(x)$ is unbounded as $k \to \infty$. So $x \in \Omega_{a,b}$, and $\bigcap_{n\in\mathbb{Z}} T^n(\overline{V}) \subset \Omega_{a,b}$. The other containment is obvious, as $\Omega_{a,b} = \bigcap_{n\in\mathbb{Z}} T^n(\Omega_{a,b}) \subset \bigcap_{n\in\mathbb{Z}} T^n(\overline{V})$. □

Now that we have shown $\Omega_{a,b}$ is a locally maximal invariant hyperbolic set of the trace map, we want to use this to prove Theorem 1.6. To this end recall Theorem 1.2, that the spectrum $\Sigma_{a,b}$ is related to the boundedness of $\{x_n\}$. Define $x_E := \left(\frac{E}{2a}, \frac{E}{2b}, \frac{a^2+b^2}{2ab}\right) = (x_1, x_0, x_{-1})$, so $E \in \Sigma_{a,b}$ if and only if the forward orbit of x_E under $T : \mathbb{R}^3 \to \mathbb{R}^3$ is bounded. The line $l_{a,b} = \{x_E \mid E \in \mathbb{R}\}$ interacts with the stable manifold of $\Omega_{a,b}$ in such a way that, using the results given in Section 2.3, the proof of Theorem 1.6 is relatively simple. So let us recall from Section 2.3 the definition of the stable manifold of a hyperbolic set:

Let Λ be a hyperbolic set under the map f. For $x \in \Lambda$ and a small $\varepsilon > 0$, define

$$W^s_\varepsilon(x) = \{w \in U \mid d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \geq 0\}$$

to be the local stable set. The the (global) stable set is given by

$$W^s(x) = \bigcup_{n \in \mathbb{Z}_+} f^{-n}(W^s_\varepsilon(f^n(x))),$$

and the stable manifold of Λ is given by

$$W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x).$$

Proposition 3.32. *If $E \in \Sigma_{a,b}$ and $c > 2$, then there exists some $k \in \mathbb{N}$ such that $T^k(x_E) \in R_1 \cup R_2$.*

Proof. First, recall from Theorem 1.2 and Corollary 3.7 that $E \in \Sigma_{a,b}$ implies that $|\frac{E}{2a}| \leq 1$. Also note that $x_{-1} = \frac{a^2+b^2}{2ab} > 1$. Now we will break this up into cases. The first case is when $|x_0| \leq 1$. By (3.13), this implies that $|x_2| > 1$. As $E \in \Sigma_{a,b}$, the sequence $\{x_n\}$ is bounded in the forward direction, so Corollary 3.7 implies $|x_3| \leq 1$. Thus we see that $T^2(x_E) \in R_1 \cup R_2$.

Now consider the case when $|x_0| > 1$. Either $|x_2| \leq 1$ or $|x_2| > 1$. Suppose the former is true, and then $|x_3| > 1$ and $|x_4| \leq 1$, so $T^3(x_E) \in R_1 \cup R_2$. If $|x_2| > 1$, then again we see $|x_3| \leq 1$, and $T^2(x_E) \in R_1 \cup R_2$. $\square$

Lemma 3.33. *For $c > 2$, the line $l_{a,b} = \{(\frac{E}{2a}, \frac{E}{2b}, \frac{a^2+b^2}{2ab}) \mid E \in \mathbb{R}\}$ intersects the stable manifold of the hyperbolic set $\Omega_{a,b}$ transversally.*

Proof. The set of points with bounded forward orbit is exactly $W^s(\Omega_{a,b})$, the stable manifold. Therefore we only need to worry about points on the line that are near points corresponding to energies in the spectrum. We saw in the previous proposition that if we iterate enough by T we can consider the line as a curve in $R_1 \cup R_2$, which is precisely where the cones determining the stable manifold are easily defined. Recall from Lemma 3.30 that these cones $|\eta| \geq 3|\xi|$ were given (in local coordinates) as projections in the xy plane, so we just need to show that the projections of these curves into the xy plane intersect the cones transversally.

Note that as $c > 2$, we have $a > (2+\sqrt{5})b$. Recall that $x_0 = \frac{E}{2b}, x_1 = \frac{E}{2a}, x_2 = \frac{E^2-a^2-b^2}{2ab}$ and $x_3 = \frac{E^3-2Ea^2-Eb^2}{2a^2b}$. Let us first look at the case when $|x_0| \leq 1$. Then $|x_2| > 1$ and $|x_3| \leq 1$, and $T^2(x_E) \in R_1 \cup R_2$. First consider $\left|\frac{E^2-a^2-b^2}{2ab}\right| > 1$. Solving $\frac{E^2-a^2-b^2}{2ab} = \pm 1$, we find that $E \in (-\infty, -a-b) \cup (b-a, a-b) \cup (a+b, \infty)$. Now, considering $\left|\frac{E}{2b}\right| \leq 1$, we get $E \in [-2b, 2b]$. Thus $E \in (b-a, a-b)$. Finally, considering $\left|\frac{E^3-2Ea^2-Eb^2}{2a^2b}\right| \leq 1$, we get that $E \in [-\frac{1}{2}(b+\sqrt{8a^2+b^2}), \frac{1}{2}(b-\sqrt{8a^2+b^2})] \cup [-b, b] \cup [\frac{1}{2}(-b+\sqrt{8a^2+b^2}), \frac{1}{2}(b+\sqrt{8a^2+b^2})]$. Thus

$$E \in [-b, b].$$

The stable manifold is in the cone $|\eta| \geq 3|\xi|$, so we want $|\xi|/|\eta| > \frac{1}{3}$. That is, if for a fixed $E_0 \in [-b, b]$ we have $\left|\frac{E^3-E_0^3-2(E-E_0)a^2-(E-E_0)b^2}{2a^2b}\right| / \left|\frac{E-E_0}{2a}\right| > \frac{1}{3}$ for $E \in [-b, b]$, then the curve (x_3, x_2, x_1) intersects the stable manifold transversally. This simplifies to $\left|\frac{E^2+EE_0+E_0^2-2a^2-b^2}{ab}\right|$, and we can instead consider $\left|\frac{x-2a^2-b^2}{ab}\right|$ for $x \in [-b^2, 3b^2]$. The minimum value of $\frac{x-2a^2-b^2}{ab}$ occurs at $x = -b^2$ and is given by $\frac{-2a^2-2b^2}{ab} < -4c < -\frac{1}{3}$. The maximum value on the interval thus occurs at $x = 3b^2$ and is $\frac{2b^2-2a^2}{ab} = -4c < -\frac{1}{3}$.

Therefore the curve intersects the stable manifold transversally.

Now consider the case where $|x_0| > 1$. Suppose $|x_2| > 1$. Then $|x_3| \leq 1$, and $T^2(x_E) \in R_1 \cup R_2$. Combining $|x_0| > 1, |x_1| \leq 1$ and $|x_2| > 1$ we get $E \in [-2a, -a-b) \cup (b-a, -2b) \cup (2b, a-b) \cup (a+b, 2a]$. Finally, considering $|x_3| \leq 1$, we get $E \in [-\frac{1}{2}(b+\sqrt{8a^2+b^2}), \frac{1}{2}(b-\sqrt{8a^2+b^2})] \cup [-b, b] \cup [\frac{1}{2}(-b+\sqrt{8a^2+b^2}), \frac{1}{2}(b+\sqrt{8a^2+b^2})]$, or

$$E \in [-\frac{1}{2}(b+\sqrt{8a^2+b^2}), \frac{1}{2}(b-\sqrt{8a^2+b^2})] \cup [\frac{1}{2}(-b+\sqrt{8a^2+b^2}), \frac{1}{2}(b+\sqrt{8a^2+b^2})].$$

This is clear, as $a+b < \frac{1}{2}(-b+\sqrt{8a^2+b^2})$ as long as $a > \frac{3+\sqrt{17}}{2}b$, and $\frac{1}{2}(b+\sqrt{8a^2+b^2}) < 2a$ for $a > b$. Again, we want to consider $\left|\frac{x-2a^2-b^2}{ab}\right|$. Now we have $x \in [3(\frac{1}{2}(-b+\sqrt{8a^2+b^2}))^2, 3(\frac{1}{2}(b+\sqrt{8a^2+b^2}))^2]$. The minimum of ξ/η occurs at $x = 3(\frac{1}{2}(-b+\sqrt{8a^2+b^2}))^2$ and is $\frac{4a^2+\frac{b^2}{2}-\frac{3}{2}b\sqrt{8a^2+b^2}}{ab}$, which is greater than $\frac{1}{3}$ when $c > 2$. Therefore the curve intersects the stable manifold transversally.

Instead, if for $|x_0| > 1$ we have $|x_2| \leq 1$, then $|x_3| > 1$ and $T^3(x_E) \in R_1 \cup R_2$. Then $|x_0| > 1, |x_1| \leq 1$ and $|x_2| \leq 1$ give

$$E \in [-a-b, -a+b] \cup [a-b, a+b].$$

Now we want to consider $|\xi/\eta|$, or

$$\left|\frac{E^5 - E_0^5 - (3a^2+2b^2)(E^3 - E_0^3) + (2a^4+2a^2b^2+b^4)(E - E_0)}{2a^3b^2} \cdot \frac{2ab}{E^2 - E_0^2}\right| =$$

$$\left|\frac{E^4 + E^3E_0 + E^2E_0^2 + EE_0^3 + E_0^4 - (3a^2+2b^2)(E^2 + EE_0 + E_0^2) + 2a^4 + 2a^2b^2 + b^4}{a^2b(E+E_0)}\right|$$

with $E, E_0 \in [-a-b, -a+b]$ or $E, E_0 \in [a-b, a+b]$.

For fixed values of a and b, such as $a = 30$ and $b = 2$, it is clear using Mathematica that the maximum value of ξ/η on the interval $[a-b, a+b]$ occurs at $E = E_0 = a+b$.

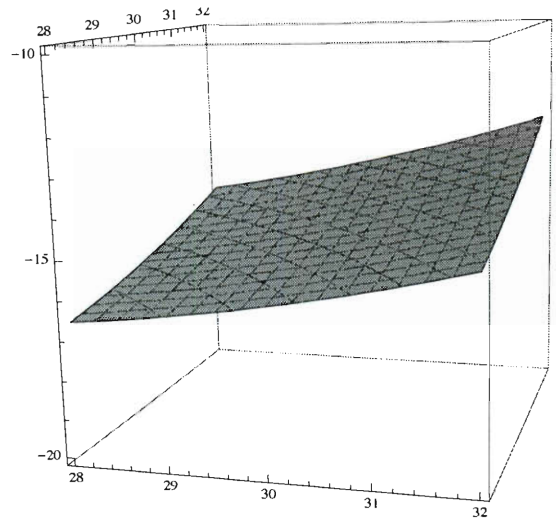

Figure 3.5: Graph of ξ/η when $|x_0| > 1$, $|x_2| \leq 1$ and $E, E_0 \in [a-b, a+b]$ with $a = 30$, $b = 2$.

With $E = E_0 = a + b$, we see

$$\begin{aligned}\frac{\xi}{\eta} &= \frac{5(a+b)^4 - 9a^2(a+b)^2 - 6b^2(a+b)^2 + 2a^4 + 2a^2b^2 + b^4}{a^2b(2a+2b)} \\ &= \frac{-2a^3 + 2a^2b + 17ab^2 + 8b^3}{ab(2a+2b)}.\end{aligned}$$

Now we want to show that for $c > 2$ we have $-6a^3 + 4a^2b + 49ab^2 + 24b^3 < 0$, as then $3(-2a^3 + 2a^2b + 17ab^2 + 8b^3) < -ab(2a+2b)$ and thus

$$\frac{-2a^3 + 2a^2b + 17ab^2 + 8b^3}{ab(2a+2b)} < -\frac{1}{3}.$$

We know that if $c > 2$ it is true that $a > 4b$, and so

$$-6a^3 + 4a^2b + 49ab^2 + 24b^3 < \left(-6 + 1 + \frac{49}{16} + \frac{24}{4^3}\right) a^3 < -1.5a^3 < 0,$$

as desired. So if this were true, that the maximum value of ξ/η on the interval $[a-b, a+b]$ occurs at $E = E_0 = a + b$ for all a, b with $c > 2$, then we would have

$\frac{\xi}{\eta} < -\frac{1}{3}$. Now if we can show just that the numerator

$$E^4 + E^3E_0 + E^2E_0^2 + EE_0^3 + E_0^4 - (3a^2 + 2b^2)(E^2 + EE_0 + E_0^2) + 2a^4 + 2a^2b^2 + b^4$$

is maximized by $E = E_0 = a + b$, and that the maximum is negative, then the same holds true. Maximizing the numerator subject to the constraints $a, b > 0$ and $a > 4b$, the latter being clear if $c > 2$, Mathematica indeed gives that the maximum value over the interval $[a - b, a + b]$ occurs at $E = E_0 = a + b$, and thus on the interval $[a - b, a + b]$, the curve intersects the stable manifold transversally.

Similarly, on the interval $[-a-b, -a+b]$, the minimum occurs at $E = E_0 = -a+b$ for given values of a and b like $a = 30$ and $b = 2$, and $\xi/\eta > \frac{1}{3}$. It is important to note here that now for $E, E_0 \in [-a - b, -a + b]$, the denominator is negative, so we are still interested in maximizing the numerator, which is a negative number. Indeed, with the above constraints, Mathematica gives the maximum value over the interval $[-a - b, -a + b]$ to be $-2a^4 + 2a^3b + 17a^2b^2 + 8ab^3$ at $E = E_0 = -a - b$. For $a > 4b$ this is clearly negative, and so we see that the the minimum value of ξ/η is given when the denominator is $-a^2b(2a + 2b)$, and the value is

$$\frac{\xi}{\eta} = \frac{2a^3 - 2a^2b - 17ab^2 - 8b^3}{ab(2a + 2b)},$$

which is greater than $1/3$ for $c > 2$. Therefore over the interval $[-a - b, -a + b]$, the curve intersects the stable manifold transversally.

Thus we see that for each of the cases, in an interval of energies around the spectrum, the pushforward of the line $(\frac{E}{2a}, \frac{E}{2b}, \frac{a^2+b^2}{2ab})$ intersects the stable manifold transversally. $\square$

And finally we have all the tools to prove Theorem 1.6.

Proof of Theorem 1.6. Take $c \geq 6$, so $\Omega_{a,b}$ is a locally maximal hyperbolic set. From Lemma 3.33 and the fact that the set of points with bounded forward orbits is $W^s(\Omega_{a,b})$, we see that the spectrum is affinely equivalent to $W^s(\Omega_{a,b}) \cap l_{a,b}$. By the existence of the C^1 foliations from Theorem 2.4, we see that $W^s(\Omega_{a,b}) \cap l_{a,b}$ is the C^1 image of $W^u_\varepsilon(\Omega_{a,b}) \cap \Omega_{a,b}$. Thus we can consider the dimension of $W^u_\varepsilon(\Omega_{a,b}) \cap \Omega_{a,b}$ to get results on the dimension of the spectrum $\Sigma_{a,b}$.

From the theorems in Section 2.3, we see that

$$\dim_H(W^u_\varepsilon(\Omega_{a,b}) \cap \Omega_{a,b}) = \dim_B(W^u_\varepsilon(\Omega_{a,b}) \cap \Omega_{a,b}),$$

and thus

$$\dim_H(\Sigma_{a,b}) = \dim_B(\Sigma_{a,b}),$$

as desired. □

Corollary 3.34. *It is true that*

$$\lim_{c \to \infty} \dim(\Sigma_{a,b}) \cdot \log c = f^*.$$

Proof. This follows from Theorem 1.6, Theorem 1.4 and Theorem 1.5. □

3.5 The Case $b > a$

Up until this section we have just considered the off-diagonal Fibonacci model when $a > b$. Now we consider what happens when $b > a$. We want to show that the main results all still hold true in this case. Indeed, most of the theorem statements and proofs are the same, as they just involve c and certain bounds on c. In this section we will state which results from the case $a > b$ need to be changed, and in situations where the results do not change but the proofs differ once we consider $b > a$, we will provide the altered proof.

First, though, we need to redefine c for this case, as we want it to be a positive parameter, which is no longer true of $c = \frac{a^2-b^2}{2ab}$ for $b > a$. So instead, we consider the invariant (3.5) and choose c to be the positive square root of $\frac{a^4-2ab+b^4}{4a^2b^2}$; that is, define $c = \frac{b^2-a^2}{2ab}$ so that

$$x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 = c^2.$$

Thus the same invariant holds with this new definition of c.

Now the first statement to be affected by this new definition of c is Corollary 3.5, a corollary to Lemma 3.4. Recall Lemma 3.4:

Lemma 3.4. *A sufficient condition that the sequence $\{x_n\}$ be unbounded in the backward direction is that there exists some $N \in \mathbb{Z}$ such that*

$$|x_{N-1}| > 1, \quad |x_N| > 1, \quad \text{and } |x_{N+1}| \leq 1.$$

This N is unique, and moreover $|x_{n-2}| > |x_{n-1}x_n| > 1$ for $n \leq N$, and there is a $C > 1$ such that $|x_n| > C^{F_{N-n}}$ for $n \leq N$.

Similarly, a sufficient condition that the sequence $\{x_n\}$ be unbounded in the forward direction is that there exists some $N \in \mathbb{Z}$ such that

$$|x_{N-1}| \leq 1, \quad |x_N| > 1, \quad \text{and } |x_{N+1}| > 1. \tag{3.36}$$

This N is unique, and moreover $|x_{n+2}| > |x_{n+1}x_n| > 1$ for $n \geq N$, and there is a $C > 1$ such that $|x_n| > C^{F_{n-N}}$ for $n \geq N$.

The new corollary is:

Corollary 3.35. *A sufficient condition for $\{x_n\}$ to be unbounded in the forward direction is that $|x_0| > 1$.*

Proof. Recall $x_0 = \frac{E}{2b}$ and $x_1 = \frac{E}{2a}$. Therefore for $b > a$, we have $|x_1| > |x_0|$ and $|E| > 2b$. Thus $x_2 = \frac{E^2-a^2-b^2}{2ab} > \frac{3b^2-a^2}{2ab} > \frac{2b^2}{2ab} = \frac{b}{a} > 1$. Then $|x_2x_1| - |x_0| > 0$, and $|x_3| \geq |x_2x_1| + (|x_2x_1| - |x_0|) > |x_2x_1|$, so by the proof of Lemma 3.4, $\{x_n\}$ is unbounded in the forward direction. $\square$

Similarly affected is Corollary 3.7, a corollary of Lemma 3.6. Recall Lemma 3.6 is the following:

Lemma 3.6. *A necessary condition for $\{x_n\}$ to be unbounded (in either the forward or backward direction) is that one of the following holds:*

$$|x_{n-1}| > 1, \quad |x_n| > 1, \quad \textit{and} \quad |x_{n+1}| \leq 1 \quad \textit{for some} \quad n \in \mathbb{Z},$$

$$|x_{n-1}| \leq 1, \quad |x_n| > 1, \quad \textit{and} \quad |x_{n+1}| > 1 \quad \textit{for some} \quad n \in \mathbb{Z},$$

or

$$|x_n| > 1 \quad \textit{for all} \quad n \in \mathbb{Z}.$$

Now Corollary 3.7 becomes:

Corollary 3.36. *The sequence $\{x_n\}$ is bounded in the forward direction if and only if (3.36) does not hold for all $N \in \mathbb{N}$ and $|x_0| \leq 1$.*

The proof is completely analogous to the proof of Corollary 3.7.

Proof. Suppose $\{x_n\}$ is bounded in the forward direction. Then Lemma 3.4 and Corollary 3.35 imply the "only if" direction of the statement.

Now suppose that $|x_0| \leq 1$ and (3.36) does not hold for all $N \in \mathbb{N}$; i.e., for each $N \in \mathbb{N}$ one of the following inequalities does not hold: $|x_{N-1}| \leq 1, |x_N| > 1$, and $|x_{N+1}| > 1$. If $|x_n| \leq 1$ for all $n \in \mathbb{N}$, the sequence is bounded in the forward direction. Thus let $|x_n| > 1$ for some $n \in \mathbb{N}$. Then $|x_{n+1}|, |x_{n-1}| \leq 1$, otherwise (3.10) holds for some N with $1 \leq N \leq n$. Now the statement follows from the proof of Lemma 3.6. □

Lemma 3.15 still holds, but now the proof is slightly different. Recall the lemma:

Lemma 3.15. *The spectrum of H_ω is given by*

$$\Sigma_{a,b} = \bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1}).$$

The new proof is as follows:

Proof. From Corollary 3.35 we know that if $|x_0| > 1$, then $\{x_n\}$ is unbounded; indeed, from Lemma 3.10 we have that $\|H_\omega\| \leq 2b$, so for $E \in \Sigma_{a,b}$, this implies $E \in [-2b, 2b]$ and $|x_0| = |\frac{E}{2b}| \leq 1$. From Proposition 3.8 we have $B^c_{+\infty} \subset \bigcup_{n \geq 1} (\rho_n \cap \rho_{n+1})$, so $\bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1}) \subset B_{+\infty} = \Sigma_{a,b}$.

Now, if we restrict to the energies such that $|x_0| \leq 1$, we still have $B_{+\infty}$, the set of E's such that $\{x_n\}$ has bounded forward orbit. The complement of $B_{+\infty}$ under this restriction, which we'll call $\tilde{B}_{+\infty}$, is the set of all E's such that $|x_0| \leq 1$ and $\{x_n\}$ has unbounded forward orbit. The claim is that $\tilde{B}_{+\infty} = \bigcup_{n \geq 1} (\rho_n \cap \rho_{n+1})$, so by taking complements we obtain $B_{+\infty} = \bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1})$. It is clear from Corollary 3.36

that $\tilde{B}_{+\infty} = (\rho_1 \cap \rho_2) \cup \left(\bigcup_{n \geq 1} (\sigma_n \cap \rho_{n+1} \cap \rho_{n+2}) \right)$: Obviously if $E \in \rho_n \cap \rho_{n+1}$ for some $n > 0$, then as $|x_0| \leq 1$, we have that (3.36) holds for some $N \in \mathbb{N}$ such that $N \leq n$, and $E \in \tilde{B}_{+\infty}$. And if $E \in \tilde{B}_{+\infty}$, then again (3.36) holds for some $N \in \mathbb{N}$, and $E \in \rho_n \cap \rho_{n+1}$ for all $n \geq N$, so the other containment is also obvious.

Now we show that $(\rho_1 \cap \rho_2) \cup \left(\bigcup_{n \geq 1} (\sigma_n \cap \rho_{n+1} \cap \rho_{n+2}) \right) = \bigcup_{n \geq 1} (\rho_n \cap \rho_{n+1})$. The containment $\subset$ is obvious, as $\rho_1 \cap \rho_2 \subset \bigcup_{n \geq 1} (\rho_n \cap \rho_{n+1})$, and for $n \geq 1$ it is true that $\sigma_n \cap \rho_{n+1} \cap \rho_{n+2} \subset \bigcup_{n \geq 1} (\rho_n \cap \rho_{n+1})$.

For the other containment, consider $E \in \rho_m \cap \rho_{m+1}$ with $m \geq 1$. If $m = 1$, then $E \in \rho_1 \cap \rho_2$, and the containment is clear. Now, if $m > 1$, then either $E \in \sigma_{m-1}$ or $E \in \rho_{m-1}$. If the former is true, then $E \in \sigma_{m-1} \cap \rho_m \cap \rho_{m+1} \subset \left(\bigcup_{n \geq 1} (\sigma_n \cap \rho_{n+1} \cap \rho_{n+2}) \right)$ and the containment holds. If the latter is true, that $E \in \rho_{m-1}$, then we consider that either $E \in \sigma_{m-2}$ or $E \in \rho_{m-2}$. In general, either $E \in \sigma_k$ for some $0 < k < m$, or $E \in \rho_k$ for all $0 < k < m$. If $E \in \sigma_k$ for some $0 < k < m$, then $E \in \sigma_k \cap \rho_{k+1} \cap \rho_{k+2}$ and the containment is obvious. If $E \in \rho_k$ for all $0 < k < m$, then $E \in \rho_1 \cap \rho_2$, and the containment is obvious.

Thus $\tilde{B}_{+\infty} = \bigcup_{n \geq 1} (\rho_n \cap \rho_{n+1})$, and $B_{+\infty} = \bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1})$. However, this is all under the restriction that $|x_0| \leq 1$, so really we have $\Sigma_{a,b} = B_{+\infty} = \sigma_0 \cap \bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1})$. But as $\sigma_1 = [-2a, 2a] \subset [-2b, 2b] = \sigma_0$ and $\sigma_2 = [-b-a, -b+a] \cup [b-a, b+a] \subset \sigma_0$, and the band structure is the same; i.e., for $n \geq 1$, $\sigma_n \cup \sigma_{n+1} \supset \sigma_{n+1} \cup \sigma_{n+2}$, then $\sigma_0 \cap \bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1}) = \bigcap_{n \geq 1} (\sigma_n \cup \sigma_{n+1})$ and the lemma holds. $\square$

So while structure of the bands σ_k is unchanged when $b > a$, the initial bands are

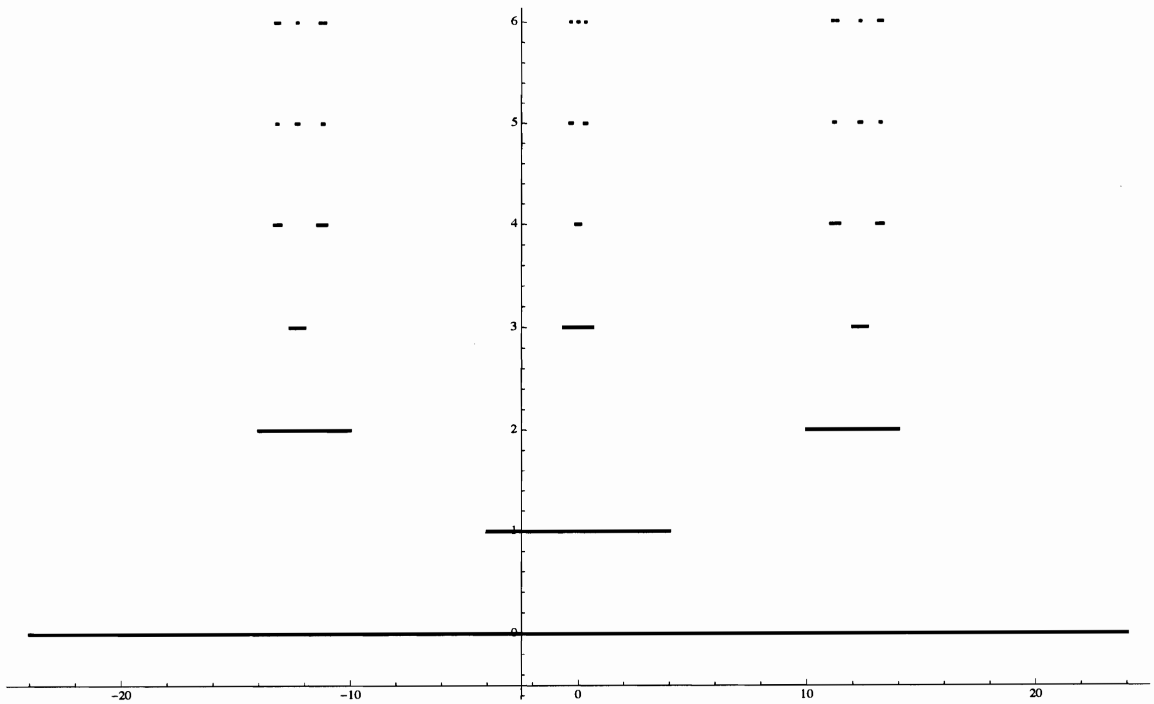

Figure 3.6: Band Structure with $a = 2, b = 12$.

different. This affects Lemma 3.20, as induction is used in the proof and thus the difference in the initial bands affects the base case of the induction argument. Recall Lemma 3.20:

Lemma 3.20. *Let $c > 4$ and $k \geq 1$. Then, with $\xi_{a,b} = c - 2 + \sqrt{c^2 - 4c + 1}$, we have the following inequalities.*

(a) For any type A band $B_{k+1} \subset \sigma_{k+1}$, $E \in B_{k+1}$ implies

$$\left| \frac{x'_{k+1}(E)}{x'_k(E)} \right| \geq \xi_{a,b}.$$

(b) For any type B band $B_{k+2} \subset \sigma_{k+2}$, $E \in B_{k+2}$ implies

$$\left| \frac{x'_{k+2}(E)}{x'_k(E)} \right| \geq \xi_{a,b}.$$

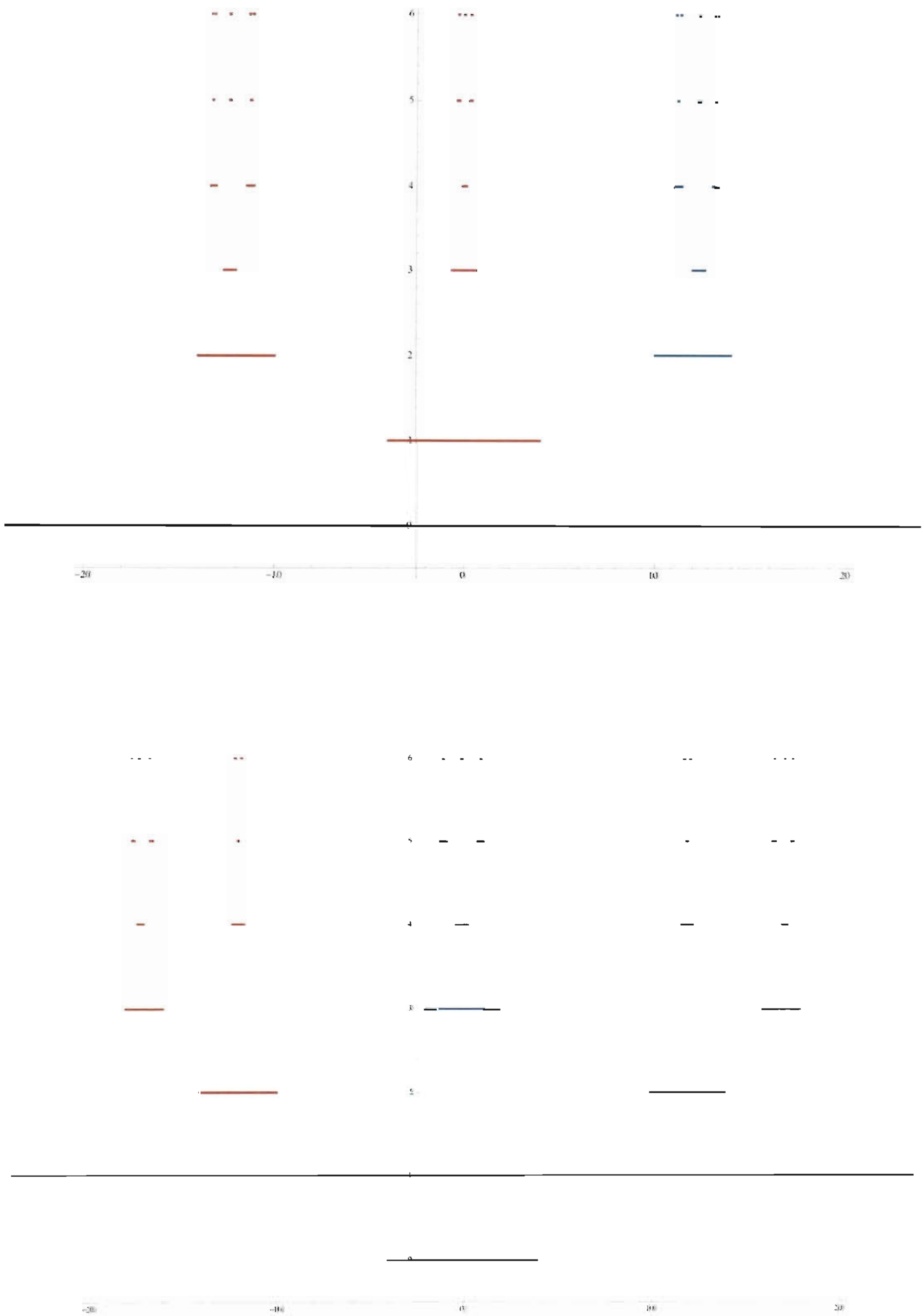

Figure 3.7: Band structure with $b > a$, $a > b$.

As only the base case of the induction argument is different, only the start to the proof will be given.

Proof. Consider the base case for a type A band, $\left|\frac{x_1'}{x_0'}\right| = \frac{b}{a}$. We want to see that this is greater than or equal to $\xi_{a,b} = c - 2 + \sqrt{c^2 - 4c + 1}$. In terms of a and b, we see, analogously to the case when $a < b$, that $\xi_{a,b} < \frac{b-2a}{a}$. Clearly $\frac{b}{a} \geq \frac{b-2a}{a}$.

Now consider the base case for a type B band, $\left|\frac{x_2'}{x_1'}\right| = \left|\frac{2E}{a}\right|$. As $E \in \sigma_2$, it follows that $|E| \geq b - a$, and so $\left|\frac{x_2'}{x_1'}\right| \geq \frac{2(b-a)}{a} > \frac{b-2a}{a}$, as desired. The remainder of the proof of the lemma is the same as given in Section 3.3. $\square$

The next lemma that changes is Lemma 3.22. We now have the following:

Lemma 3.37. *Let $c > 2$. The following inequalities hold for $E \in \sigma_1, \sigma_2$, respectively:*

$$\frac{1}{2a} \leq |x_1'(E)|, |x_2'(E)| \leq \frac{b+a}{ab}.$$

Note that these bounds come up in the proofs of Theorem 1.4 and Theorem 1.5, though the proofs of the theorems work the same with the new bounds.

Proof. First, we see that $|x_1'| = \frac{1}{2a}$. Now, considering $E \in \sigma_2$, we get that $E \in [-b-a, -b+a] \cup [b-a, b+a]$, so $b-a \leq |E| \leq b+a$. Then as $|x_2'(E)| = |\frac{E}{ab}|$, we have $\frac{b-a}{ab} \leq |x_2'(E)| \leq \frac{b+a}{ab}$. Finally, it must be shown that $\frac{1}{2a} \leq \frac{b-a}{ab}$. This can be see by observing, as $c > 2$, that $b > 4a$, so particularly, $2ab > 2a^2 + ab$, or $2a(b-a) > ab$. $\square$

Now, let us consider the differences in initial bands between the $a > b$ case and the $b > a$ case. Because of symmetry, this can be viewed two ways. The choice is arbitrary. For the case $a > b$, there are the bands on the right and on the left, which are symmetric, and the bands in the middle. The band structure in the middle is essentially the same found on the left for the case $b > a$, just moved down one level. The band structure on the right side for the case $a > b$ can be found of the right side for $b > a$, again moved down one level. See Figure 3.7.

Now, for simplicity, because the bands in the case $b > a$ can easily be related to the bands in $a > b$, though moved down a level, we want to redefine how we count the bands:

Definition 3.38. Define

$$a_k := \text{number of type A bands in } \sigma_{k+1},$$

$$b_k := \text{number of type B bands in } \sigma_{k+1},$$

$$a_{k,m} := \text{number of type A bands } b \text{ in } \sigma_{k+1} \text{ with } \#\{1 \leq j < k+1 : b \cap \sigma_j \neq \emptyset\} = m,$$

$$b_{k,m} := \text{number of type B bands } b \text{ in } \sigma_{k+1} \text{ with } \#\{1 \leq j < k+1 : b \cap \sigma_j \neq \emptyset\} = m.$$

Again, based on the definitions of type A and type B bands, we have that $a_k = b_{k-1}$ and $b_k = 2b_{k-2} + a_{k-2}$ with initial values $a_0 = 1$ and $b_1 = 2$. Similarly, it is true that $a_{k,m} = b_{k-1,m-1}$ and $b_{k,m} = 2b_{k-2,m-1} + a_{k-2,m-1}$ with initial values $a_{0,0} = 1$ and $a_{0,m} = 0$ for $m \neq 0$, $a_{1,m} = 0$, $b_{0,m} = 0$, $b_{1,0} = 2$ and $b_{1,m} = 0$ for $m \neq 0$.

Then, we have the following result.

Lemma 3.39. *If* $\lceil \frac{k}{2} \rceil \leq m \leq \lfloor \frac{2k-1}{3} \rfloor$, *then*

$$\begin{aligned} a_{k,m} &= b_{k-1,m-1} \\ &= 2^{2k-3m-1} \left(\begin{array}{c} k-m-1 \\ \\ 2m-k \end{array} \right) \left(\frac{2k-2m}{2k-3m} \right) \\ &= \frac{2^{2k-3m-1}(k-m-1)!}{(2m-k)!(2k-3m-1)!} \left(\frac{2k-2m}{2k-3m} \right). \end{aligned}$$

Otherwise, $a_{k,m} = 0$.

Proof. The proof follows exactly from the proof of Lemma 3.24. The only difference is in the bands $a''_{k,m}$. In the case where $a > b$, there is a factor of 2 for the $a''_{k,m}$, and for the $b > a$ case, there is only one. $\square$

The next thing to check is that Lemma 3.25 still holds for the case $b > a$. The statement of the lemma is that if $\frac{k}{2} \leq m \leq \frac{2k-1}{3}$, then

$$k^{-1} \exp\left(mf\left(\frac{m}{k}\right)\right) \lesssim a_{k,m} \lesssim k \exp\left(mf\left(\frac{m}{k}\right)\right),$$

which is (3.31). The differences in the proof are minor. First, checking the endpoints, we see that if $m = \frac{k}{2}$, then $a_{k,\frac{k}{2}} = 2^{\frac{k}{2}}$. As $\exp\left(\frac{k}{2} f\left(\frac{1}{2}\right)\right) = 2^{\frac{k}{2}}$, it is clear that (3.31) holds in this case.

Next, considering $m = \frac{2k-1}{3}$, we get $a_{k,\frac{2k-1}{3}} = \frac{2k+2}{3}$. We saw before in the proof of the lemma that $\exp\left(\frac{2k-1}{3} \cdot f\left(\frac{2k-1}{3k}\right)\right) = 2k \left(\frac{k+1}{3k}\right)^{\frac{k+1}{3}} \left(\frac{3k}{k-2}\right)^{\frac{k-1}{3}} = 2k \left(\frac{k+1}{k-2}\right)^{\frac{k-1}{3}} \left(\frac{k+1}{3k}\right)^{\frac{2}{3}}$, so clearly the expression holds.

The final difference is that we have

$$a_{k,m} \asymp \exp\left(mf\left(\frac{m}{k}\right)\right) \frac{(2k-3m-1)^{\frac{1}{2}}(2k-2m)}{(k-m-1)^{\frac{1}{2}}(2k-3m)(2m-k)^{\frac{1}{2}}}.$$

We get the following inequalities from $\frac{k}{2} < m < \frac{2k-1}{3}$:

$$\begin{aligned} \frac{3}{k-2} < \frac{1}{2m-k} &\leq 1, \\ 1 \leq 2k - 3m - 1 &< \frac{k-2}{2}, \\ \frac{2}{k} < \frac{1}{2k-3m} &< 1, \\ \frac{2}{k-2} < \frac{1}{k-m-1} &< \frac{3}{k-2}, \\ \frac{2k+2}{3} < 2k - 2m &< k, \end{aligned}$$

and as in the proof of the lemma, (3.31) clearly follows.

Finally, we must redo the proof of Lemma 3.33. We want to show that for $c > 2$, the line $l_{a,b} = \{(\frac{E}{2a}, \frac{E}{2b}, \frac{a^2+b^2}{2ab}) \mid E \in \mathbb{R}\}$ intersects the stable manifold transversally, and once again, we just need to show this in a neighborhood around the spectrum. The first thing to do is to break it up into cases. Either $|x_1| > 1$ or $|x_1| \leq 1$. We know that if $E \in \Sigma_{a,b}$, then $|x_0| = |\frac{E}{2b}| \leq 1$.

Consider first the case where $|x_1| > 1$. For $E \in \Sigma_{a,b}$, this means that $|x_2| \leq 1$. Therefore $T(x_1, x_0, x_{-1}) \in R_1 \cup R_2$. We want to show that $|\xi/\eta| > 1/3$. As $|x_0| \leq 1$ and $|x_1| > 1$, we get $E \in [-2b, -2a) \cup (2a, 2b]$. And

$$\frac{\xi}{\eta} = \frac{\frac{E^2 - E_0^2}{2ab}}{\frac{E - E_0}{2b}} = \frac{E + E_0}{a},$$

so $|\xi/\eta| > 4 > 1/3$, with the infimum occurring at $E = E_0 = \pm 2a$.

Now consider the case $|x_1| \leq 1$. This means, for $E \in \Sigma_{a,b}$, that $|x_2| > 1$ and $|x_3| \leq 1$, so $T^2(x_1, x_0, x_{-1}) \in R_1 \cup R_2$. From $|x_3| \leq 1$ we get that $E \in [\frac{1}{2}(-b - \sqrt{8a^2+b^2}), -b] \cup [\frac{1}{2}(b - \sqrt{8a^2+b^2}), \frac{1}{2}(-b + \sqrt{8a^2+b^2})] \cup [b, \frac{1}{2}(b + \sqrt{8a^2+b^2})]$, and from $|x_1| \leq 1$ we get $E \in [-2a, 2a]$. Thus, claiming that $\frac{1}{2}(-b + \sqrt{8a^2+b^2}) < 2a$, we get $E \in [\frac{1}{2}(b - \sqrt{8a^2 + b^2}), \frac{1}{2}(-b + \sqrt{8a^2 + b^2})]$. To prove the claim, we must show that $\sqrt{8a^2 + b^2} < 4a + b$. This is the same as $8a^2 + b^2 < 16a^2 + 8ab + b^2$, which is obviously true.

Now,

$$|\xi/\eta| = \left|\frac{E^2 + EE_0 + E_0^2 - 2a^2 - b^2}{ab}\right|,$$

but instead we can just consider $\frac{x-2a^2-b^2}{ab}$ for $x \in [-(\frac{1}{2}(-b + \sqrt{8a^2 + b^2}))^2, 3(\frac{1}{2}(-b + \sqrt{8a^2 + b^2}))^2]$. The claim is that on the interval, we have $\frac{x-2a^2-b^2}{ab} < -\frac{1}{3}$. The maximum occurs at the right endpoint of the interval, which gives $(4a^2 + \frac{b^2}{2} - \frac{3}{2}b\sqrt{8a^2 + b^2})/ab$. To show that this is less than $-\frac{1}{3}$, it suffices to show that $8a^2 + b^2 + \frac{2}{3}ab < 3b\sqrt{8a^2 + b^2}$. Squaring both sides, we want to show that $64a^4 + b^4 + \frac{148}{9}a^2b^a + \frac{4}{3}ab^3 + \frac{16}{3}a^3b < 9b^4 + 72a^2b^2$. Note that as $c > 2$, we have $b > 4c$, so the left hand side is bounded above by $\frac{65}{12}b^4 + \frac{148}{9}a^2b^2$, and it is clear that the desired inequality holds. Thus $|\xi/\eta| < \frac{1}{3}$, and $l_{a,b}$ intersects the stable manifold transversally.

Thus all the results for the case $a > b$ either hold for the case $b > a$ or are slightly different, as noted above, but the main theorems still all hold.

In conclusion, in the study of quasicrystals, the off-diagonal Fibonacci operator is of natural interest. Now we have upper and lower bounds for the Hausdorff and lower box counting dimensions of the spectrum of the operator, and we have seen that equality can be achieved between the two dimensions. There are other aspects of the model to consider, and there is already further interest in this operator among physicists (e.g. [18]). Thus perhaps more questions about this operator will be answered in the future.

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