SMOOTH BIORTHOGONAL WAVELETS FOR APPLICATIONS IN IMAGE COMPRESSION*

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ABSTRACT

In this paper we introduce a new family of smooth, symmetric biorthogonal wavelet basis. The new wavelets are a generalization of the Cohen, Daubechies and Feauveau (CDF) biorthogonal wavelet systems. Smoothness is controlled independently in the analysis and synthesis bank and is achieved by optimization of the discrete finite variation (DFV) measure recently introduced for orthogonal wavelet design. The DFV measure dispenses with a measure of differentiability (for smoothness) which requires a large number of vanishing wavelet moments (e.g., Hölder and Sobolev exponents) in favor of a smoothness measure that uses the fact that only a finite depth of the filter bank tree is involved in most practical applications. Image compression examples applying the new filters using the embedded wavelet zerotree (EZW) compression algorithm due to Shapiro shows that the new basis functions performs better when compared to the classical CDF 7/9 wavelet basis.

1. INTRODUCTION

Significant attention [1–7] has been given to the potential importance of wavelet smoothness versus regularity (e.g., vanishing wavelet moments) for wavelet based image compression. Although the choice of filter and hence the associated properties of the corresponding wavelet basis is sensitive to the compression algorithm considered (e.g., the quantization and bit allocation scheme) some form of smoothness is deemed to be of importance. This paper assumes the zerotree based compression algorithm as first proposed by Shapiro [8].

The paper presents a new class of biorthogonal wavelet basis (filters). The new class is based on a generalization of the well known biorthogonal Cohen, Daubechies and Feauveau (CDF) wavelet system [9–11]. In particular, we compare performance of the new basis to the classical CDF 7/9 wavelet basis (see Figure 2) successfully applied in a number of zerotree based image compression algorithms. The new design is based on optimizing a measure of smoothness on the iterated filter bank tree and takes in to account that only a finite number of filter bank stages (levels) are involved in any application. The measure, discrete finite variation (DFV) [5, 4] is related the Hölder measure of smoothness (in the limit) but does not require that: (1) the wavelets have a minimal number of vanishing moments to achieve smoothness of the functions or (2) underlying continuous basis functions have continuous higher order derivatives. For a given support (e.g., length filters) the new measure gives rise to a whole family of "smooth" orthogonal [5, 4] and biorthogonal wavelets.

1.1. Biorthogonal wavelet systems

Figure 1 shows a one level filter bank associated with a biorthogonal wavelet expansion. By iterating on the lowpass output (e.g., the h_0 branch) a multiscale wavelet expansion can be obtained. Associated with the analysis filter h_0 and synthesis filter g_0 are the scaling function $\phi(x)$ and the dual $\widetilde{\phi}(x)$ respectively defined by

$$\phi(x) = \sqrt{2}\sum_n h_0(n)\phi(2x-n) \tag{1}$$

$$\widetilde{\phi}(x) = \sqrt{2}\sum_n g_0(n)\widetilde{\phi}(2x-n). \tag{2}$$

Equations (1) and (2) converges to compactly supported basis functions if both

$$\sum_n h_0(n) = \sqrt{2} \tag{3}$$

and

$$\sum_n g_0(n) = \sqrt{2} \tag{4}$$

are satisfied. Associated with the scaling function $\phi(x)$ and its dual $\widetilde{\phi}(x)$ are the wavelets $\psi(x)$ and $\widetilde{\psi}(x)$ defined by

$$\psi(x) = \sqrt{2}\sum_n h_1(n)\phi(2x-n) \tag{5}$$

$$\widetilde{\psi}(x) = \sqrt{2}\sum_n g_1(n)\widetilde{\phi}(2x-n). \tag{6}$$

The system is then said to be biorthogonal if the following three conditions holds:

$$\int_{\mathbb{R}} \phi(x)\widetilde{\phi}(x-k)dx = \delta(k) \tag{7a}$$

$$\int_{\mathbb{R}} \phi(x)\widetilde{\psi}(x-k)dx = 0 \tag{7b}$$

$$\int_{\mathbb{R}} \widetilde{\phi}(x)\psi(x-k)dx = 0. \tag{7c}$$

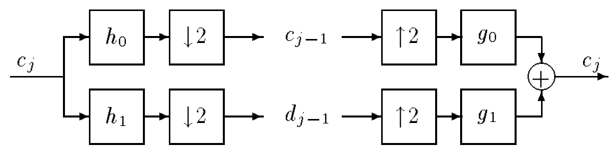

Figure 1: One level biorthogonal filter bank for implementing the biorthogonal wavelet analysis and synthesis.

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Equations (7a)-(7c) are equivalent to the following two conditions on the scaling and wavelet filters and their duals

$$\sum_n h_0(n) g_0(n-2l) = \delta(l) \tag{8a}$$

$$\sum_n h_0(n) g_1(n-2l) = 0 \tag{8b}$$

$$\sum_n g_0(n) h_1(n-2l) = 0 \tag{8c}$$

Hence, using the conditions in (8a)-(8c) a nonlinear constrained optimization problem for designing biorthogonal wavelets can be posed over the free filter parameters. Furthermore, by imposing that the filters be symmetric, linear phase solutions can be obtained. In fact, the CDF 7/9 wavelet system is the solution to this optimization problem if all free parameters are used for setting a maximal number of moments of the wavelets (ψ and $\widetilde{\psi}$) to zero.

2. DISCRETE FINITE VARIATION

Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x_i)$ for $i = 0, \ldots, m-1$ be m equally spaced samples of f. Now define the first order discrete difference operator $D_m : \mathbb{R}^m \to \mathbb{R}^{m-1}$ as

$$D_m f(x_i) = f(x_i) - f(x_{i-1}) \quad 1 < i \leq m \tag{9}$$

then the nth order discrete difference operator $D_m^n : \mathbb{R}^m \to \mathbb{R}^{m-n}$ is defined by

$$D_m^n f(x_i) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x_{i-j}) \quad n < i \leq m. \tag{10}$$

A measure of smoothness (related to both bounded variation as well as Hölder in the limit), *discrete finite variation* is then defined as follows:

Definition 1 (Discrete finite variation). *Let J be the number of stages of the iterated filter bank and let $\phi^J(x_i)$, supported on $[0, N-1]$, be the length L sequence of samples of the scaling function such that $0 = x_0 < x_1 < \cdots < x_{L-1} = N-1$ with $x_i - x_{i-1} = \Delta x$ for all $i = 1 \ldots, L-1$. Then the discrete finite variation of order n is defined as*

$$\mathcal{V}_n = \left| D_L^n \phi^J(x_i) \right| \tag{11}$$

The above measure is easily generalized to the biorthogonal case by applying the above definition to both the analysis scaling function ϕ as well as the synthesis scaling function $\widetilde{\phi}$. Using this, a general nonlinear constrained optimization problem is formulated for which the cost function is a linear combination of the DFV measures of each of the two scaling functions.

2.1. Design example

The class of finite scale smooth biorthogonal wavelets are obtained by solving the following problem.

$$\min_{h_0, g_0} \begin{bmatrix} \left| D_L^n \phi^J(x_i) \right| & \left| D_L^n \widetilde{\phi}^J(x_i) \right| \end{bmatrix} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \tag{12}$$

subject to

a) Linear constraints (e.g., (3) and (4))
b) Bi-linear constraints (e.g., (8a)-(8c))

For given lengths filters a range of solutions can be obtained by varying α and the number of vanishing moments desired of the wavelets associated with the analysis wavelet and it dual the synthesis wavelet.

In Figure 3 we show a particular solution of the 7/9 wavelet system optimized according to (12) and requiring that both the analysis and synthesis wavelet have 2 vanishing moments. For comparison we have also included plots of the roots for the corresponding lowpass filters and it is interesting to notice as was the case for the orthogonal solution [4, 5] that the optimal DFV solution has zeros on the unit circle in the stop band.

2.2. Image compression example

In Figure 4 we show the face of the image "Barbara" compressed 30:1 using both the CDF 7/9 and the optimal DFV 7/9 solution. The compressed image using the DFV 7/9 wavelet system results in a 0.25dB improvement in peak signal to noise ratio (PSNR) over the CDF 7/9 wavelet system. We also include a rate distortion curve for the same image comparing the two wavelet systems.

3. SUMMARY

Results of applying the new filters show in Figure 3 for image compression (see Figure 4) using the zerotree compression [8] algorithm shows improvements in PSNR compared to the well known CDF 7/9 wavelet system. It is visually hard to choose which of the two wavelet systems is the better (the difference in PSNR is only 0.25dB at 30:1 compression) of the two compressed images and an extensive study would be required. Furthermore, computing rate distortion curves on 10 different images indicates that the new filters achieves equal or better PSNR for compression ratios in the range of 8:1 to 80:1 for gray scale images.

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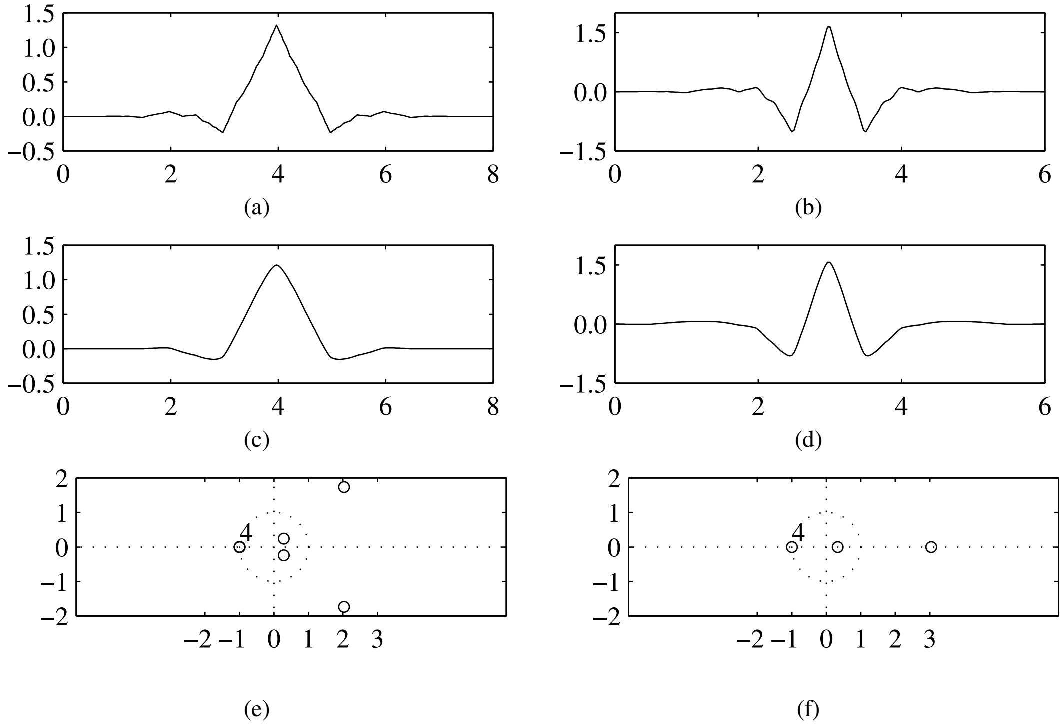

Figure 2: CDF (biorthogonal) 7/9 wavelet system. (a) Scaling function. (b) Wavelet function. (c) Dual scaling function. (d) Dual wavelet function. (e) Roots of analysis scaling filter. (f) Roots of synthesis scaling filter.

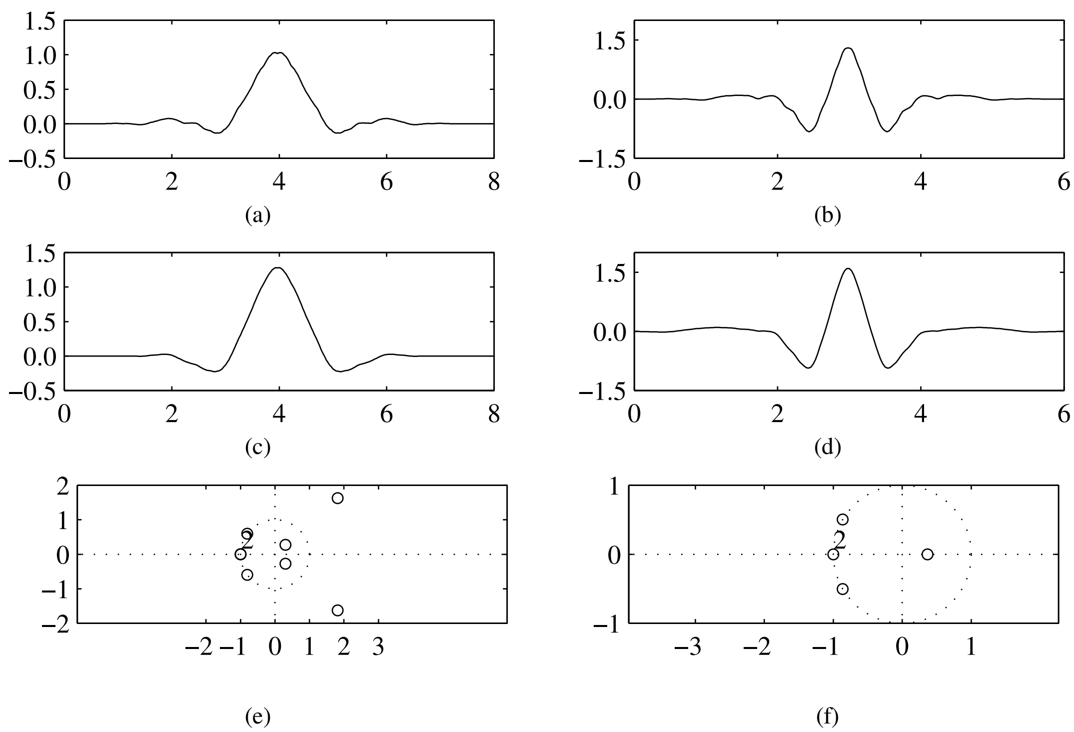

Figure 3: Optimal DFV (biorthogonal) 7/9 wavelet system. (a) Scaling function. (b) Wavelet function. (c) Dual scaling function. (d) Dual wavelet function. (e) Roots of analysis scaling filter. (f) Roots of synthesis scaling filter.

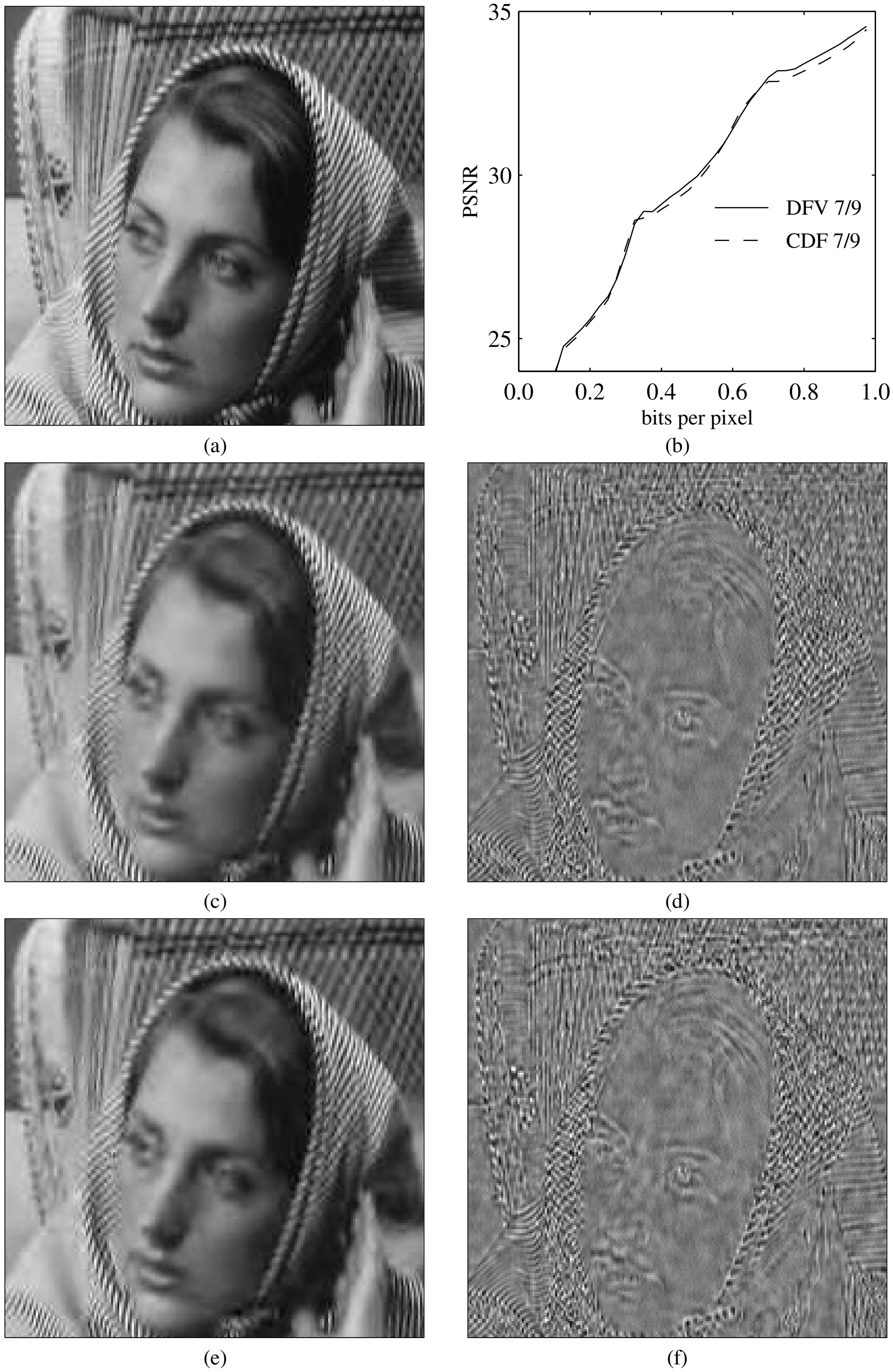

Figure 4: (a) Original "Barbara." (b) Rate distortion. (c) 30:1 using CDF 7/9 (PSNR = 27.58dB). (d) Error image for CDF 7/9 (e) 30:1 using optimal DFV 7/9 (PSNR = 27.84dB). (f) Error image for optimal DFV 7/9.