Generalized Digital Butterworth Filter Design *

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Abstract

This paper presents a formula-based method for the design of IIR filters having more zeros than (nontrivial) poles. The filters are designed so that their square magnitude frequency responses are maximally-flat at $\omega = 0$ and at $\omega = \pi$ and are thereby generalizations of classical digital Butterworth filters. A main result of the paper is that, for a specified half-magnitude frequency and a specified number of zeros, there is only one valid way in which to split the zeros between $z = -1$ and the passband. Moreover, for a specified number of zeros and a specified half-magnitude frequency, the method directly determines the appropriate way to split the zeros between $z = -1$ and the passband. IIR filters having more zeros than poles are of interest, because often, to obtain a good trade-off between performance and the expense of implementation, just a few poles are best.

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1 Introduction

Probably the best known and most commonly used method for the design of IIR digital filters is the transformation of the classical analog filters (the Butterworth, Chebyshev I and II, and Elliptic filters) [8]. One advantage of this technique is the existence of formulas for these filters. Unfortunately, all such IIR filters have an equal number of poles and zeros. It is desirable to be able to design filters having more zeros than poles (away from the origin), for implementation purposes. This paper presents a method for the design of maximally-flat lowpass IIR filters having more zeros than poles and which possess a specified half-magnitude frequency. It is worth noting that not all the zeros are restricted to lie on the unit circle. The method consists of the use of a formula, a transformation of variables, and a spectral factorization. Note that no phase approximation is done; the approximation is in the magnitude squared - as are the classical IIR filter types.

Another main result of the paper is that for a specified number of zeros and a specified half-magnitude frequency, there is only one valid way to divide the number of zeros between $z = -1$ and the passband. This is in contrast to the classical digital Butterworth filter, for which all the zeros lie at $z = -1$, regardless of the position of the half-magnitude frequency in $(0, \pi)$. The formulas given below provide a direct way to determine the number of zeros that must lie at $z = -1$ and the number of zeros that must contribute to the passband.

Given a half-magnitude frequency ω_o, the filters produced by the formulas described below are optimal in the sense that the maximum number of derivatives at $\omega = 0$ and $\omega = \pi$ are set to zero, under the constraint that the filter possesses the half-magnitude frequency ω_o. The digital IIR filters obtained by transforming the classical Butterworth filters, and the FIR filters obtained by the use of Herrmann's formulas [1] are both special cases of the filters produced by the formulas given in this paper.

2 Notation

Let $B(z)/A(z)$ denote the transfer function of a digital filter. Its frequency response magnitude $M(\omega)$ is given by $|B(e^{j\omega})/A(e^{j\omega})|$. Throughout this paper, the degree of $B(z)$ will be denoted by $L + M$, where L is the number of zeros at $z = -1$ and M is the number of zeros that contribute to the passband. No filter in this paper has zeros on the unit circle other than at $z = -1$. The degree of $A(z)$ will be denoted by N.

The zeros at $z = -1$ produce a flat behavior in the frequency response magnitude at $\omega = \pi$, while the remaining zeros, together with the poles, are used to produce a flat behavior at $\omega = 0$.

Table 1: Notation.

Parameters	
$L+M$	total number of zeros
L	number of zeros at $z=-1$
M	number of zeros contributing to the passband
N	total number of poles
ω_o	half-magnitude frequency
Flatness	
$L+M+N$	total degrees of flatness
$M+N$	degree of flatness at $\omega = 0$
L	degree of flatness at $\omega = \pi$

The meaning of the parameters is shown in table 1. The half-magnitude frequency is that frequency at which the magnitude equals one half.

3 Examples

The classical digital Butterworth filters (defined by $L = N$ and $M = 0$) are special cases of the filters discussed in this paper. Figure 1 shows the frequency response, pole-zero plot, and group delay for a classical digital Butterworth filter of order 4 ($L = 4$, $M = 0$, $N = 4$). It has a half-magnitude frequency of 0.4585π.

The first generalization of the classical digital Butterworth filter described below permits L to be greater than N: $L > N$ with $M = 0$. Figure 2 shows the frequency response, pole-zero plot, and group delay for an IIR filter with $L = 6$, $M = 0$, $N = 4$. It was designed to have the same half-magnitude frequency as the previous example ($\omega_o = 0.4585\pi$). It turns out that when $L > N$, the restriction that M equal zero limits the range of achievable half-magnitude frequencies, as will be elaborated below. This motivates the second generalization.

In addition to permitting L to be greater than N, the second generalization of the classical digital Butterworth filter described below permits M to be greater than zero: $L \geq N$ and $M > 0$. Figure 3 shows the frequency response, pole-zero plot, and group delay for an IIR filter with $L = 16$, $M = 7$, $N = 4$. It was designed to have a half-magnitude frequency of 0.4585π.

As mentioned above, for a specified half-magnitude frequency ω_o and a specified number of zeros ($L + M$), there is only one correct way to split the zeros between $z = -1$ and the passband. To illustrate this property, it is helpful to construct a table that indicates the appropriate values for L, M and N. When $N = 4$ and $L + M$ equals $4, \ldots, 10$, table 2 gives the appropriate choice for L and M to achieve a desired half-magnitude frequency. As can be seen from the table, the intervals cover the interval (0,1) and do not overlap. This will be true in general, as long as the number of poles is at least one. Notice that in the case of the classical Butterworth filter ($L+M=N$), L equals N and M equals zero, regardless of the specified half-magnitude frequency. As will be explained below, these intervals can be unambiguously computed by inspecting the roots of an appropriate set of polynomials.

To illustrate the different trade-offs that can be achieved with the generalized Butterworth filters described in this paper, it is useful to examine a set of filters all of which have the same half-magnitude frequency and the same total number of poles and zeros ($L+M+N$). For example, when $L+M+N$ is fixed at 20 and the half-magnitude ω_o is fixed at 0.6π, the filters shown in figure 4 are obtained. The number of poles of the filters in this figure vary from 0 to 10 in steps of 2. It is also interesting to compare the slope of the magnitude $M(\omega)$ at the half-magnitude frequency – this indicates the sharpness of the transition band. This information is summarized in table 3. The third graph of the figure shows the negative reciprocal of the slope of $M(\omega)$ at ω_o. It turns out that when an odd number of poles between 1 and 9 is used, the resulting filter is very similar to the filter having one fewer pole. Notice from the table and the figure, that for this example, as the number of poles and zeros become more equal, the slope of the magnitude at ω_o becomes more negative and that the transition region becomes sharper. However, it is interesting to note that the improvement in magnitude is greatest when the number of poles is increased from 0 to 2. Also, notice the behavior of the group delay as the number of poles and zeros are varied. When the cost of implementing a filter with many poles is taken into consideration, the filters with 2 or 4 poles appear to attain a good trade-off between performance and implementation cost.

4 Discussion

Several authors have addressed the design and the advantages of IIR filters with an unequal number of (nontrivial) poles and zeros.

While [13, 17, 18] give formulas for IIR filters with Chebyshev stopbands having more zeros than poles, these methods require that all the zeros lie on the unit circle.

In [5] Martinez and Parks describe an exchange algorithm for the Chebyshev design of IIR filters in which all the zeros lie on the unit circle. Other variations on this are [2, 4, 11, 14]. In [2] Jackson improves the Martinez/Parks algorithm and notes that the use of just 2 poles "is often the most attractive compromise between computational complexity and other performance measures of interest."

Table 2: For the choice L, M, and N shown in the table, the interval of permissible half-magnitude frequencies ω_o is given by ω_{min} and ω_{max}. $L+M$ is the numerator degree (number of zeros) and N is the denominator degree (number of poles).

$L+M$	L	M	N	ω_{min}/π	ω_{max}/π
4	4	0	4	0	1
5	5	0	4	0	0.5349
	4	1	4	0.5349	1
6	6	0	4	0	0.4620
	5	1	4	0.4620	0.6017
	4	2	4	0.6017	1
7	7	0	4	0	0.4140
	6	1	4	0.4140	0.5299
	5	2	4	0.5299	0.6446
	4	3	4	0.6446	1
8	8	0	4	0	0.3788
	7	1	4	0.3788	0.4807
	6	2	4	0.4807	0.5754
	5	3	4	0.5754	0.6756
	4	4	4	0.6756	1
9	9	0	4	0	0.3515
	8	1	4	0.3515	0.4435
	7	2	4	0.4435	0.5266
	6	3	4	0.5266	0.6093
	5	4	4	0.6093	0.6996
	4	5	4	0.6996	1
10	10	0	4	0	0.3294
	9	1	4	0.3294	0.4141
	8	2	4	0.4141	0.4891
	7	3	4	0.4891	0.5615
	6	4	4	0.5615	0.6359
	5	5	4	0.6359	0.7188
	4	6	4	0.7188	1

Table 3: For the half-magnitude frequency $\omega_o = 0.6\pi$ and $L + M + N = 20$, the table shows the correct values of L and M, and the derivative of the magnitude at ω_o, for a fixed $L + M$ and N.

L	M	N	$M'(\omega_o)$
8	12	0	-1.4366
8	10	2	-2.5410
8	8	4	-3.1869
8	6	6	-3.6882
9	3	8	-3.8012
10	0	10	-3.9430

In [12] Saramäki discusses the trade-offs between numerator and denominator order and describes an iterative algorithm in which zeros are not constrained to lie on the unit circle for the design of filters having Chebyshev stopbands. In [11, 12], Saramäki finds that the classical Elliptic and Chebyshev filter types are seldom the best choice. However, to our knowledge, no formulas have been presented for the design of IIR filters in which zeros are not constrained to lie on the unit circle.

The design of FIR maximally-flat filters [1, 3, 19] is a special case of the design formulas given below. It is also likely that the generalized Butterworth filters described in this paper can be used with some of the filter design techniques that employ FIR maximally-flat filters, for example [20].

The ability to design the non-classical IIR filters described in this paper and the papers cited above, allows the user to trade-off between properties of FIR filters and properties of the four classical IIR filters. Note that one reason FIR filters are sometimes preferred over IIR filters is the relative ease with which they can be implemented. However, by using a low-order recursive section, a potentially better trade-off can be obtained.

5 Derivations

The approach described below provides formulas for two nonnegative polynomials $P(x)$ and $Q(x)$. Then, by (i) using a suitable transformation ($x = \frac{1}{2}(1 - \cos\omega)$ as in [1]) and (ii) taking a spectral factor, a stable IIR filter $B(z)/A(z)$ is obtained having a magnitude squared frequency response

$|M(\omega)|^2$ given by

$$|M(\omega)|^2 = \frac{P(\frac{1}{2} - \frac{1}{2}\cos\omega)}{Q(\frac{1}{2} - \frac{1}{2}\cos\omega)}.$$

Accordingly, $P(x)/Q(x)$ is designed to approximate a lowpass response over $x \in [0, 1]$. This results in a formula-based method for the design of generalized digital Butterworth filters. No iterations are required for finding $P(x)$ and $Q(x)$.

We begin by deriving the classical digital Butterworth filter. This establishes notation and makes clear the way in which the generalization uses the same ideas in its derivation.

5.1 Classical Digital Butterworth Filter

Let the degree of $P(x)$ be L and the degree of $Q(x)$ be L, and define the rational function $F(x) = P(x)/Q(x)$. To find $P(x)$ and $Q(x)$ so that $F(x)$ possesses the lowpass behavior shown in figure 1, we will require that $F(x)$ have L degrees of flatness at $x = 1$ and that $F(x) - 1$ have L degrees of flatness at $x = 0$.

In order to obtain L degrees of flatness at $x = 1$, $F(x)$ must have the following form:

$$F(x) = \frac{P(x)}{Q(x)} = \frac{(1-x)^L}{Q(x)}. \tag{1}$$

The L degrees of flatness is obtained by specifying a root of order L at $x = 1$. In order that $F(x) - 1$ have an L degree of flatness at $x = 0$, $F(x)$ must satisfy

$$F(x) - 1 = \frac{P(x) - Q(x)}{Q(x)} = -\frac{cx^L}{Q(x)} \tag{2}$$

where c is an appropriately chosen constant. Solving equations (1) and (2) for $Q(x)$ gives

$$Q(x) = (1-x)^L + cx^L \tag{3}$$

and

$$F(x) = \frac{(1-x)^L}{(1-x)^L + cx^L}. \tag{4}$$

Note that $|M(\pi/2)|^2 = F(1/2) = \frac{1}{1+c}$. Clearly, c should be chosen so that this value lies between 0 and 1. Therefore, c should be chosen to be greater than zero. Notice that when L is odd and c is chosen to be 1, the degree of $Q(x)$ is decreased by 1 because the leading terms of $Q(x)$ cancel – in this case the number of nontrivial poles becomes $L - 1$.

To choose c to achieve a specified half-magnitude frequency is straight-forward. Let ω_o denote the specified half-magnitude frequency (the frequency at which $|M(\omega_o)|$ equals $\frac{1}{2}$). The equation $|M(\omega_o)| = \frac{1}{2}$ becomes $F(x_o) = \frac{1}{4}$ where $x_o = \frac{1}{2}(1 - \cos\omega_o)$. Solving this equation for c, one obtains

$$c = 3\frac{(1-x_o)^L}{x_o^L}. \tag{5}$$

For the classical digital Butterworth filter, ω_o can be chosen to be any value in $(0, \pi)$.

5.2 First Generalization

The first generalization of the classical digital Butterworth filter has more zeros than poles and, as in the classical case, all the zeros lie at $x = 1$ ($z = -1$).

Let L denote the number of zeros at $x = 1$ and let N denote the number of poles with $L \geq N$. Then, as above,

$$F(x) = \frac{P(x)}{Q(x)} = \frac{(1-x)^L}{Q(x)} \tag{6}$$

where $Q(x)$ has degree N. But

$$F(x) - 1 = \frac{P(x) - Q(x)}{Q(x)} = -\frac{x^N U(x)}{Q(x)} \tag{7}$$

where $U(x)$ is a polynomial of degree at most $L - N$. (The degree of $x^N U(x)$ can not exceed the degree of $P(x) - Q(x)$). Solving equations (6) and (7) for $Q(x)$ gives

$$Q(x) = (1-x)^L + x^N U(x). \tag{8}$$

Since $Q(x)$ has degree N and since N is no greater than L, $Q(x)$ must equal the polynomial obtained by taking only the first $N+1$ coefficients of $(1-x)^L + x^N U(x)$. Notice that $U(x)$ can always be chosen so that the remaining coefficients of this polynomial are zero. Using the identity, $(1-x)^L = \sum_{i=0}^{L} \binom{L}{i}(-x)^i$, $Q(x)$ can be written as

$$Q(x) = \sum_{i=0}^{N} \binom{L}{i}(-x)^i + cx^N. \tag{9}$$

Introducing the notation $\mathcal{T}_N$ for polynomial truncation (discarding all terms beyond the N^{th} power), $Q(x)$ can be written as

$$Q(x) = \mathcal{T}_N\{(1-x)^L\} + cx^N \tag{10}$$

Table 4: Permissible ranges for c for the first generalization.

N even	$c \geq 0$
N odd	$c \geq \binom{L-1}{N}$

and $F(x)$ can be written as

$$F(x) = \frac{(1-x)^L}{\mathcal{T}_N\{(1-x)^L\} + cx^N}. \tag{11}$$

The constant term of $U(x)$, c, becomes the free parameter that, as in the classical case, must be chosen to lie within an appropriate range. The allowable ranges for c are given in table 4. When c is chosen to lie in the ranges shown in the table, then $0 < F(x) < 1$ for $x \in (0,1)$.

It turns out that when N is odd, the degree of the denominator can be reduced by 1 if c is chosen to be $\binom{L}{N}$.

To choose c to achieve a specified half-magnitude frequency is straight-forward. Let ω_o denote the specified half-magnitude frequency and let $x_o = \frac{1}{2}(1 - \cos\omega_o)$. Solving the equation $F(x_o) = \frac{1}{4}$ for c yields

$$c = \frac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}. \tag{12}$$

The value this expression gives for c may not lie in the appropriate range shown in table 4. If this is the case, then the specified half-magnitude frequency is too high for the current choice of L and N. It should be noted that, although the passband can be made arbitrarily narrow, it can not be made arbitrarily wide for a fixed L and N $(L > N)$.

The greatest half-magnitude frequency achievable for a fixed L and N can be found by setting c equal to the appropriate value shown in table 4 and solving equation (12) for x_o. It is seen that x_o is a root of an L degree polynomial:

$$\mathcal{T}_N\{(1-x)^L\} + cx^N - 4(1-x)^L = 0. \tag{13}$$

Consequently, x_o can be found by using a polynomial root finder. All roots of this polynomial which are not real and in the interval (0,1) can be discarded. As shown in the appendix, it turns out that there will be only one real root in (0,1). We also found that the real roots have the properties listed in table 5. The maximum achievable half-magnitude frequency is given by $\omega_o = \arccos(1 - 2x_o)$.

Table 5: The number and locations of the real roots of $\mathcal{T}_N\{(1-x)^L\}+cx^N-4(1-x)^L$ for $L > N > 0$.

	L even	L odd
N even $c = 0$	2 real roots: $x_1 \in (0,1), x_2 > 1$	1 real roots: $x_1 \in (0,1)$
N odd $c = \binom{L-1}{N}$	2 real roots: $x_1 \in (0,1), x_2 = 1$	3 real roots: $x_1 \in (0,1), x_2 = 1, x_3 > 1$

Example: For $L = 6$ and $N = 4$, the boundary value for c from table 4 is 0, so the polynomial equation (13) becomes:

$$\mathcal{T}_4\{(1-x)^6\} - 4(1-x)^6 = 0. \tag{14}$$

It roots are : $3.9476, 0.3798 \pm 1.1659\jmath, 0.4262 \pm 0.3245\jmath, 0.4404$. Therefore, for this choice of L and N, the interval of x_o is $(0, 0.4404]$ – the interval of ω_o is $(0, 0.4620\pi]$.

To obtain filters having wider passbands with the same number of zeros and (nontrivial) poles, it is necessary to move at least one zero from $x = 1$ ($z = -1$) to the passband. Maximally flat filters having passband zeros, the second generalization of the classical digital Butterworth filter, are discussed in the next section. (Passband zeros are so named here, because they contribute to the flatness of the frequency response at $\omega = 0$).

5.3 Second Generalization

The second generalization of the classical digital Butterworth filter uses additional zeros lying off the unit circle. These zeros are used here to obtain a higher degree of flatness at $\omega = 0$. Such a filter is shown in figure 3. The filters described below possess a degree of flatness of $M + N + 1$ at $\omega = 0$, and a degree of flatness of L at $\omega = \pi$.

Let the degree of $P(x)$ be $L + M$, and let the degree of $Q(x)$ be N, with $L > N$ and $M > 0$. The M zeros off the unit circle are used to increase the degree of flatness at $x = 0$. This leads to the following pair of equations:

$$F(x) = \frac{P(x)}{Q(x)} = \frac{(1-x)^L S(x)}{Q(x)} \tag{15}$$

$$F(x) - 1 = \frac{P(x) - Q(x)}{Q(x)} = -\frac{x^{M+N+1} U(x)}{Q(x)} \tag{16}$$

where $S(x)$ is a polynomial of degree at most M and where $U(x)$ is a polynomial of degree at most $L-N-1$. Solving equations (15) and (16) for $Q(x)$ gives

$$Q(x) = (1-x)^L S(x) + x^{M+N+1} U(x). \tag{17}$$

Following the same reasoning as above, since $Q(x)$ is a polynomial of degree N, $Q(x)$ must be the polynomial obtained by taking only the first $N+1$ coefficients of $(1-x)^L S(x) - x^{M+N+1}U(x)$. However, the coefficients of $S(x)$ and $U(x)$ must be determined so that the remaining higher power coefficients of this polynomial $Q(x)$ are zero. Since $U(x)$ can be chosen so that the last $L-N$ coefficients of $Q(x)$ are zero, it remains to choose the coefficients of $S(x)$ such that the coefficients of $Q(x)$ of powers $N+1$ thru $M+N$ are zero. Let $S(x) = s_0 + s_1 x + \cdots + s_M x^M$. Since $S(x)$ can be scaled by a constant without changing the function $F(x)$, there are M degrees of freedom. Note that there are M coefficients of $Q(x)$ that must be set equal to zero. This gives rise to a system of M linear equations. It is expected that these equations become ill conditioned, because they effectively involve the specification of many derivatives of a polynomial at a single point. A closed form solution for $S(x)$ was found to be given by the following:

$$S(x) = \sum_{k=0}^{M} \binom{M+N-k}{N} \binom{L-N+k-1}{k} x^k \tag{18}$$

where $\binom{n}{k}$ is a binomial coefficient. Using the polynomial $S(x)$ given by this expression, $P(x)$ is given by $(1-x)^L S(x)$. $Q(x)$ can be found by simply taking the first $N+1$ coefficients of $P(x)$. Using the polynomial truncation notation, one has

$$Q(x) = \mathcal{T}_N\{(1-x)^L S(x)\}. \tag{19}$$

It turns out that $(1-x)^L S(x)$ can be written as

$$(1-x)^L S(x) = \sum_{k=0}^{M+L} \binom{M+N-k}{M} \binom{M+L}{k} (-x)^k. \tag{20}$$

To evaluate $\binom{n}{k}$ for negative values of n we use the following convention [10]:

$$\binom{n+k-1}{k} = (-1)^k \binom{-n}{k} \tag{21}$$

for $k \geq 0$. In addition, note that $\binom{n}{k} = 0$ for $n \geq 0$, $k < 0$; and that $\binom{n}{k} = 0$ for $n \geq 0$, $k > n$.

In contrast to the classical digital Butterworth filter and its first generalization above, there is no extra degree of freedom in equations (18) and (19). The exact location of the half-magnitude frequency is dictated by the parameters L, M and N. Its location can be only approximately positioned. The filters given by (18) share this property with the maximally-flat FIR filters given in [1].

Fixing $L+M$ at 22 and $N=4$, the frequency response magnitudes of the filters obtained using equation (18) for $L=5,\ldots,21$, $M=22-L$ are shown in figure 5. In order to obtain filters having transition bands that lie between those shown in figure 5, a degree of flatness at either $x=0$ or $x=1$ can be given up and the extra degree of freedom can be used to more accurately position the location of the transition band.

When a degree of flatness is given up, the degree of flatness at $x=1$ becomes $M+N$, rather than $M+N+1$ as in (16). In this case, the numerator $P(x)$ and denominator $Q(x)$ of $F(x)$ are given by the following:

$$P(x) = (1-x)^L (R(x) + cT(x)) \tag{22}$$

$$Q(x) = \mathcal{T}_N\{P(x)\} \tag{23}$$

where

$$R(x) = \sum_{k=0}^{M-1} \binom{M+N-k-1}{N} \binom{L-N+k-1}{k} x^k \tag{24}$$

and

$$T(x) = x \sum_{k=0}^{M-1} \binom{M+N-k-2}{N-1} \binom{L-N+k}{k} x^k. \tag{25}$$

It turns out that $(1-x)^L R(x)$ can be written as

$$(1-x)^L R(x) = \sum_{k=0}^{M+L} \binom{M+N-k-1}{M-1} \binom{M+L-1}{k} (-x)^k \tag{26}$$

and that $(1-x)^L T(x)$ can be written as

$$(1-x)^L T(x) = -\sum_{k=0}^{M+L} \binom{M+N-k-1}{M-1} \binom{M+L-1}{k-1} (-x)^k. \tag{27}$$

The free parameter here is the constant c. If so desired, it can be chosen to more accurately position the location of the transition band (at the expense of a degree of flatness at $x=0$ or $x=1$). The variable c must be chosen to lie in the ranges shown in table 6. When N is even, the positive endpoint of this interval is that point beyond which $F(x)$ is no longer monotonic - and the negative endpoint of this interval is that point beyond which $F(x)$ is no longer nonnegative.

Table 6: Permissible ranges for c for the second generalization.

N even	$-1 \le c \le \dfrac{L-N}{M+N}$
N odd	$\dfrac{L-N}{N} \le c$

To choose c to achieve a specified half-magnitude frequency, note that $F(x)$ can be written as

$$F(x) = \frac{(1-x)^L R(x) + c(1-x)^L T(x)}{\mathcal{T}_N\{(1-x)^L R(x)\} + c\mathcal{T}_N\{(1-x)^L T(x)\}}. \tag{28}$$

As above, let ω_o denote the specified half-magnitude frequency and let $x_o = \frac{1}{2}(1-\cos\omega_o)$. Solving the equation $F(x_o) = \frac{1}{4}$ for c yields

$$c = \frac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}. \tag{29}$$

The value this expression gives for c may not lie in the appropriate range given by table 6. If this is the case, then the specified half-magnitude frequency is either too high or too low for the current choice of L, M and N – it is necessary to alter the distribution of zeros between $x = 1$ ($z = -1$) and the passband.

For fixed L, M, and N, the minimum and maximum permissible values of the half-magnitude frequency ω_o can be computed by (i) setting c to the values in table 6, (ii) solving equation (29) for x and (iii) using $\omega = \arccos(1-2x)$. When c is finite, it is seen that x is a root of the $L+M$ degree polynomial:

$$\mathcal{T}_N\{(1-x)^L(R(x)+cT(x))\} - 4(1-x)^L(R(x)+cT(x)) = 0. \tag{30}$$

Note that when N is odd, c can be chosen to be arbitrarily large. Letting c approach infinity, the expression for $F(x)$ in (28) approaches

$$F(x) = \frac{(1-x)^L T(x)}{\mathcal{T}_N\{(1-x)^L T(x)\}} \tag{31}$$

in which case the appropriate polynomial equation becomes:

$$\mathcal{T}_N\{(1-x)^L T(x)\} - 4(1-x)^L T(x) = 0. \tag{32}$$

Therefore, for both even and odd N, the range of achievable half-magnitude frequencies can be found by computing the roots of the appropriate pair of polynomials. In all the examples we

Table 7: The number and locations of the real roots of the polynomials used to compute the minimum and maximum permissible values of the half-magnitude frequency ω_o for a fixed L, M, and N, with $L > N > 1$, $M > 1$. 'e' denotes 'even', 'o' denotes 'odd'.

L	M	N	Num. and loc. of real roots (ω_{min})	Num. and loc. of real roots (ω_{max})
e	e	e	$2 : x_1 \in (0,1), x_2 < 0$	$2 : x_1 \in (0,1), x_2 > 1$
o	e	e	$1 : x_1 \in (0,1)$	$3 : x_1 \in (0,1), x_2 < 0, x_3 > 1$
e	o	e	$3 : x_1 \in (0,1), x_2 < 0, x_3 > 1$	$1 : x_1 \in (0,1)$
o	o	e	$2 : x_1 \in (0,1), x_2 > 1$	$2 : x_1 \in (0,1), x_2 < 0$
e	e	o	$4 : x_1 \in (0,1), x_2 = 0, x_3 < 0, x_4 > 1$	$2 : x_1 \in (0,1), x_2 = 1$
o	e	o	$3 : x_1 \in (0,1), x_2 = 0, x_3 > 1$	$3 : x_1 \in (0,1), x_2 = 1, x_3 < 0$
e	o	o	$3 : x_1 \in (0,1), x_2 = 0, x_3 < 0$	$3 : x_1 \in (0,1), x_2 = 1, x_3 > 1$
o	o	o	$2 : x_1 \in (0,1), x_2 = 0$	$4 : x_1 \in (0,1), x_2 = 1, x_3 < 0, x_4 > 1$

examined, each polynomial has exactly one real root in the interval (0,1). It is this root that is used to compute the interval permissible values of ω_o - because ω_o must lie in $(0, \pi)$. The number and the location of the real roots of the polynomials used to compute this minimum and maximum are given in table 7.

It turns out that for a specified half-magnitude frequency and for a specified number of total zeros $(L + M)$ and (nontrivial) poles (N), there is exactly one choice of L and M for which the specified half-magnitude is achievable. The appropriate L and M can be found systematically by finding the roots of the appropriate set of polynomials described in the preceding paragraphs. Also, because it is known that the root sought for each case is the real one in the interval (0,1), a general polynomial root finder is unnecessary - a more efficient program can be written that computes only the desired root.

6 Further Remarks

By using the formulas above, a program can be written that requires from the user the three parameters: the number of poles (N), the total number of zeros $(L + M)$, and the half-magnitude frequency (ω_o). By tabulating a table such as table 2, the appropriate way to split the number of zeros between $z = -1$ and the passband (L and M) can be determined. The corresponding formula can then be used to compute $F(x)$. After a transformation and spectral factorization, the filter coefficients are obtained. Such a program (in Matlab) is available from the authors and will be made available on the Internet. Table 8 gives a summary of the filter design formulas. Table 9 gives auxiliary polynomials.

Note that, if desired, a frequency other than the half-magnitude frequency can be specified. To specify a frequency ω_o for which $M(\omega) = M_o$ is possible for any M_o, $0 < M_o < 1$. The resulting design formulas differ only in that they contain slightly different constants.

6.1 Behavior for Odd N

Note that when N is odd, one of the poles must lie on the real line. When there are zeros that lie off the unit circle, in the passband ($M > 0$), it is expected that the pole lying on the real line does little to contribute to the performance of the frequency response. This is indeed true. In some situations, a pole and a zero will lie on the unit circle and, depending on the specified half-magnitude frequency, almost cancel. For this reason, it is expected that generalized digital Butterworth filters having and odd number of poles with passband zeros will be of little interest - they have been presented in this paper for completeness.

6.2 A Note on Implementation

Note that the numerator can be implemented as a cascade of two FIR sections, one of which has all its zeros at $z = -1$. It can be beneficial to implement these sections separately, because the FIR section with all its zeros at $z = -1$ can be implemented without the use of multiplications. Since the number of zeros at $z = -1$ depends on the specified half-magnitude frequency, the number of multiplications is dependent on this parameter. A lowpass filter with a relatively narrow passband will have more zeros at $z = -1$ than will a lowpass filter with a relatively wide passband - and it will thus appear to require fewer multiplications. Note that in the design of odd length linear phase FIR filters, this can be overcome by implementing a wideband lowpass filter as a sum of narrowband filter and a pure delay. It is unclear if an analogous technique exists for the minimum phase IIR filters described in this paper.

6.3 Butterworth Filters Having More Poles Than Zeros

Although the focus of this paper has been on digital Butterworth filters having more zeros than poles, it should be noted that the design of Butterworth filters having more poles than zeros can also be easily carried out. For such filters, all the zeros will always lie at $z = -1$, and the formula for $F(x)$ is very similar to that of (4). If L is the number of (nontrivial) zeros and N is the number of poles ($N > L$), $F(x)$ is given by

$$F(x) = \frac{(1-x)^L}{(1-x)^L + cx^N}. \tag{33}$$

To choose c to achieve a desired half-magnitude frequency ω_o, the following formula can be used:

$$c = 3\frac{(1-x_o)^L}{x_o^N} \tag{34}$$

where $x_o = \frac{1}{2}(1-\cos\omega_o)$. Like the classical case, ω_o can be chosen to be any value in $(0,\pi)$.

6.4 FIR Butterworth Filters

The design of FIR digital Butterworth filters having a specified half-magnitude frequency can also be carried out using the formulas given above. But in the FIR case, the formulas of Herrmann [1] can be used, even for half-magnitude frequencies that lie between those of the maximally-flat filters described by Herrmann: Recall that the formulas described by Herrmann produce a discrete set of filters, and consequently only a discrete set of half-magnitude frequencies can be obtained for a fixed filter length. However, by appropriately averaging two neighboring Herrmann filters, maximally-flat filters between the two filters can be obtained. Therefore, Herrmann's formulas can be used to obtain maximally-flat FIR filters having specified half-magnitude frequencies exactly. As above, the use of this technique means giving up a single degree of flatness at either $\omega = 0$ or $\omega = \pi$ in order to achieve the specified half-magnitude frequency exactly. In the FIR case, the filters obtained from the formulas in this paper are the same as the filters that can be obtained using Herrmann's formulas.

One difference between the FIR and IIR cases is that in the FIR case there is a maximum and minimum achievable passband width. In other words, the half-magnitude frequency can not be made arbitrarily close to 0 or π. This is one situation in which the use of a single pole could be useful – by using a single pole, the specified half-magnitude frequency can be chosen to be any value in $(0,\pi)$.

7 Conclusion

The design of generalized classical digital Butterworth filters can be carried out without the need to solve ill conditioned equations. By using appropriate formulas and a transformation, and by taking a spectral factor, maximally flat IIR filters having more zeros than (nontrivial) poles can be easily designed - and without the restriction that all zeros lie on the unit circle. In addition, for a fixed number of zeros and a fixed number of (nontrivial) poles, the formulas above give a direct way of finding the correct way to split the number of zeros between $z = -1$ and the passband.

The maximally-flat FIR filters described by Herrmann [1] and the classical Butterworth filter are special cases of the filters given by the formulas described in this paper.

Table 8: The expression for $F(x)$ gives the magnitude squared function in the x domain in terms of a constant c. When c is chosen according to the expression given in the table, $F(x_o)$ equals $1/4$.

$F(x)$	c
$\dfrac{(1-x)^L}{(1-x)^L + cx^L}$	$3\dfrac{(1-x_o)^L}{x_o^L}$
$\dfrac{(1-x)^L}{\mathcal{T}_N\{(1-x)^L\} + cx^N}$	$\dfrac{4(1-x_o)^L - \mathcal{T}_N\{(1-x)^L\}(x_o)}{x_o^N}$
$\dfrac{(1-x)^L(R(x)+cT(x))}{\mathcal{T}_N\{(1-x)^L(R(x)+cT(x))\}}$	$\dfrac{4(1-x_o)^L R(x_o) - \mathcal{T}_N\{(1-x)^L R(x)\}(x_o)}{\mathcal{T}_N\{(1-x)^L T(x)\}(x_o) - 4(1-x_o)^L T(x_o)}$
$\dfrac{(1-x)^L S(x)}{\mathcal{T}_N\{(1-x)^L S(x)\}}$	—

Table 9: Auxiliary polynomials

$S(x) = \sum_{k=0}^{M} \binom{M+N-k}{N} \binom{L-N+k-1}{k} x^k$
$R(x) = \sum_{k=0}^{M-1} \binom{M+N-k-1}{N} \binom{L-N+k-1}{k} x^k$
$T(x) = x \sum_{k=0}^{M-1} \binom{M+N-k-2}{N-1} \binom{L-N+k}{k} x^k$

A On the Proofs for the Formulas

In this section the statements regarding the first generalization are proven. It will be useful to note a few properties. The first is a recursive relation on the truncation of the polynomial $(1-x)^L$:

$$\mathcal{T}_N\{(1-x)^L\} = (1-x)\mathcal{T}_{N-1}\{(1-x)^{L-1}\} + (-1)^N \binom{L-1}{N} x^N. \tag{35}$$

The next property concerns the derivative of a truncated polynomial: For any polynomial $G(x)$,

$$\frac{\partial \mathcal{T}_N\{G(x)\}}{\partial x} = \mathcal{T}_{N-1}\left\{\frac{\partial G(x)}{\partial x}\right\}. \tag{36}$$

A third property will also be very useful:

Identity: For $L \geq N \geq 0$

$$\mathcal{T}_N\{(1-x)^L\} = (1-x)^L - (-x)^{N+1} \sum_{i=0}^{L-N-1} \binom{L-i-1}{N} (1-x)^i. \tag{37}$$

Proof: Rewrite (37) as

$$(-1)^N \left[\mathcal{T}_N\{(1-x)^L\} - (1-x)^L\right] = x^{N+1} \sum_{i=0}^{L-N-1} \binom{L-i-1}{N} (1-x)^i. \tag{38}$$

Using the binomial theorem and collecting like terms, the left hand side of (38) can be written as

$$\text{LHS} = x^{N+1} \sum_{k=0}^{L-N-1} \binom{L}{N+k+1} (-x)^k.$$

The right hand side of (38) can be rewritten in the following way:

$$\begin{aligned} \text{RHS} &= x^{N+1} \sum_{i=0}^{L-N-1} \binom{L-i-1}{N} (1-x)^i & (39)\\ &= x^{N+1} \sum_{i=0}^{L-N-1} \binom{L-i-1}{N} \sum_{k=0}^{i} \binom{i}{k} (-x)^k & (40)\\ &= x^{N+1} \sum_{k=0}^{L-N-1} \sum_{i=k}^{L-N-1} \binom{L-i-1}{N} \binom{i}{k} (-x)^k. & (41) \end{aligned}$$

To show that the left and right hand sides are equal, it remains to show that corresponding coefficients of like powers of x are equal: For $0 \leq k \leq L-N-1$ and $L > N$, we need

$$\binom{L}{N+k+1} = \sum_{i=k}^{L-N-1} \binom{L-i-1}{N} \binom{i}{k} \tag{42}$$

which follows from a well-known binomial identity [16]. $\square$

A.1 The First Generalization

We begin by showing that if c is chosen according to table 4 and $F(x)$ is given by equation (11) for $L > N$, then $0 < F(x) < 1$ for $x \in (0,1)$. Since the numerator of $F(x)$ is positive over (0,1), it will be necessary to show that the denominator $Q(x)$, given by

$$Q(x) = \mathcal{T}_N\{(1-x)^L\} + cx^N, \tag{43}$$

is also positive over (0,1). To show that $F(x) < 1$ over (0,1), note that $F(x) < 1$ is equivalent to $F(x) - 1 < 0$ or, using equation (8), $U(x) > 0$ over (0,1). Note that $U(x)$ is chosen to cancel the coefficients of $(1-x)^L$ for powers $N+1$ thru L so that $Q(x)$ has degree N. That is, $U(x)$ is given by $c + [\mathcal{T}_N\{(1-x)^L\} - (1-x)^L]/(x^N)$. To summarize this paragraph, we will show that $0 < F(x) < 1$ for $x \in (0,1)$ by showing that $Q(x)$ and $U(x)$ are both positive over (0,1). We will consider the two cases, N even and N odd, separately.

$Q(x) > 0$, N even:

To show that $Q(x)$ is positive over (0,1) when $c > 0$ and N is even, note that (37) gives

$$\mathcal{T}_N\{(1-x)^L\} = (1-x)^L + x^{N+1} \sum_{i=0}^{L-N-1} \binom{L-i-1}{N} (1-x)^i \tag{44}$$

for even N. Since $x^k(1-x)^i$ is positive over (0,1), and since each of the binomial coefficients in the sum is positive, this shows that $\mathcal{T}_N\{(1-x)^L\}$ is positive over (0,1) for even N. Therefore, when $c > 0$ and N is even, we have $Q(x) > 0$ over (0,1).

$U(x) > 0$, N even:

To show that $U(x)$ is positive over (0,1) when $c > 0$ and N is even, write $U(x)$ as

$$U(x) = c + [\mathcal{T}_N\{(1-x)^L\} - (1-x)^L]/(x^N). \tag{45}$$

We can use (44) to write

$$U(x) = c + x \sum_{i=0}^{L-N-1} \binom{L-i-1}{N} (1-x)^i \tag{46}$$

for even N. Again, since the rightmost term is a positively weighted sum of polynomials positive over (0,1), we have the result $U(x) > 0$ over (0,1) when $c > 0$ and N is even.

$Q(x) > 0$, N odd:

To show that $Q(x)$ is positive over (0,1) when $c > \binom{L-1}{N}$ and N is odd, note that (35) gives

$$\mathcal{T}_N\{(1-x)^L\} + \binom{L-1}{N} x^N = (1-x)\mathcal{T}_{N-1}\{(1-x)^{L-1}\} \tag{47}$$

for odd N. Using (44) it follows that

$$\mathcal{T}_N\{(1-x)^L\} + \binom{L-1}{N} x^N = \tag{48}$$

$$(1-x)\left[(1-x)^{L-1} + x^N \sum_{i=0}^{L-N-1} \binom{L-i-2}{N-1}(1-x)^i\right] \tag{49}$$

for odd N. Since the right hand side is a product of two polynomials positive over (0,1), we have the result $Q(x) > 0$ over (0,1) when $c > \binom{L-1}{N}$ and N is odd.

<u>$U(x) > 0$, N odd:</u>

To show that $U(x)$ is positive over (0,1) when $c > \binom{L-1}{N}$ and N is odd, write $U(x)$ as

$$U(x) = c + [\mathcal{T}_N\{(1-x)^L\} - (1-x)^L]/(x^N). \tag{50}$$

We can use (48) to write

$$U(x) = c - \binom{L-1}{N} + (1-x) \sum_{i=0}^{L-N-1} \binom{L-i-2}{N-1}(1-x)^i \tag{51}$$

for odd N. Again, since the rightmost term is positive over (0,1) we have the result $U(x) > 0$ over (0,1) when $c > \binom{L-1}{N}$ and N is odd.

We will next prove that the polynomial of equation (13) has exactly one real root in (0,1) for $L > N$ when (i) N is even and c is chosen to equal 0 or (ii) N is odd and c is chosen to equal $\binom{L-1}{N}$.

<u>Exactly one real root in (0,1), N even:</u>

Let us define $G_e(x)$ as $G_e(x) = \mathcal{T}_N\{(1-x)^L\} - 4(1-x)^L$. Take N to be even for the following discussion. To show that $G_e(x)$ has exactly one real root in (0,1), first note that $G_e(0) = -3$ and $G_e(1) = \binom{L-1}{N}$. Therefore, $G_e(x)$ must have at least one root in (0,1). Next, use (36) to write the derivative of $G_e(x)$ as

$$G'_e(x) = 4L(1-x)^{L-1} - L\mathcal{T}_{N-1}\{(1-x)^{L-1}\}. \tag{52}$$

Using (37), $G'_e(x)$ can be written as

$$G'_e(x) = 3L(1-x)^{L-1} + Lx^N \sum_{i=0}^{L-N-1} \binom{L-i-2}{N}(1-x)^i \tag{53}$$

for even N. Therefore, $G'_e(x)$ is positive over (0,1) – and thus $G_e(x)$ can have no more than one root in (0,1). Therefore, $G_e(x)$ has exactly one root in (0,1) for even N.

<u>Exactly one real root in (0,1), N odd:</u>

Let us define $G_o(x)$ as $G_o(x) = \mathcal{T}_N\{(1-x)^L\} + \binom{L-1}{N}x^N - 4(1-x)^L$. Use equation (35) to write $G_o(x)$ as

$$G_o(x) = (1-x)\mathcal{T}_{N-1}\{(1-x)^{L-1}\} - 4(1-x)^L \tag{54}$$

for odd N. It will be convenient to show that $G_o(x)$ has exactly one root in (0,1) for odd N by showing that $G_o(x)/(1-x)$ has exactly one root in (0,1) for odd N. Note that

$$\frac{G_o(x)}{1-x} = \mathcal{T}_{N-1}\{(1-x)^{L-1}\} - 4(1-x)^{L-1}. \tag{55}$$

Compare (55) with $G_e(x)$ above. Because $G_e(x)$ has exactly one root in (0,1) for even N, $G_o(x)$ has exactly one root in (0,1) for odd N.

A.2 The Second Generalization

The proofs for the second generalization are similar to those for the first generalization - and will not be given. However, some of the relevant identities are listed below, and it is shown that $0 < F(x)$ when N is even and c is in the interval stated in table 6.

The following two identities show that when N is even, the truncated polynomials $\mathcal{T}_N\{(1-x)^L S(x)\}$ and $\mathcal{T}_N\{(1-x)^L R(x)\}$ are positive over (0,1).

Identity: For $L \geq N \geq 0$, $M \geq 0$,

$$\mathcal{T}_N\{(1-x)^L S(x)\} = \tag{56}$$
$$(1-x)^L S(x) + (-1)^N x^{M+N+1} \sum_{i=0}^{L-N-1} \binom{M+i}{M}\binom{L-i-1}{N}(1-x)^i$$

where $S(x)$ is given in (18).

Identity: For $L \geq N > 0$, $M > 0$,

$$\mathcal{T}_N\{(1-x)^L R(x)\} = \tag{57}$$
$$(1-x)^L R(x) + (-1)^N x^{M+N} \sum_{i=0}^{L-N} \binom{M+i-1}{M-1}\binom{L-i-1}{N}(1-x)^i$$

where $R(x)$ is given in (24).

The following identity shows that when N is odd, the polynomial $\mathcal{T}_N\{(1-x)^L T(x)\} - (1-x)^L T(x)$ is positive over (0,1).

Identity: For $L \geq N > 0$, $M > 0$,

$$\mathcal{T}_N\{(1-x)^L T(x)\} = \tag{58}$$
$$(1-x)^L T(x) - (-1)^N x^{M+N} \sum_{i=0}^{L-N} \binom{M+i-1}{M-1}\binom{L-i-1}{N-1}(1-x)^i$$

where $T(x)$ is given in (25).

Note that $R(x)$ and $T(x)$ have positive coefficients, thus they are positive for all positive x. In the formulas above, the term $R(x) + cT(x)$ appears. The following two identities can be used in conjunction with the previous two identities to support the required bound on c stated above in table 6.

Identity: For $L \geq N > 0$, $M > 0$,

$$R(x) - T(x) = (1-x) \sum_{i=0}^{M-1} \binom{M+N-i-1}{N} \binom{L-N+i}{i} x^i. \tag{59}$$

Therefore, $R(x) - T(x)$ is positive for all x in (0,1). Consequently, $R(x) + cT(x)$ is positive for all $c > -1$ (since $R(x) + cT(x) = R(x) - T(x) + (1+c)T(x)$).

Identity: For $L \geq N > 0$, $M > 0$,

$$\begin{aligned} &\mathcal{T}_N\left\{(1-x)^L(R(x) + \tfrac{L-N}{M+N}T(x))\right\} = \\ &\quad (1-x)^L\left(R(x) + \tfrac{L-N}{M+N}T(x)\right) + \\ &\quad (-1)^N \tfrac{M}{M+N} x^{M+N+1} \sum_{i=0}^{L-N-1} \binom{M+i}{M} \binom{L-i-1}{N} (1-x)^i. \end{aligned} \tag{60}$$

Using this identity, it can be shown that when N is even and $-1 < c < \frac{L-N}{M+N}$, the function $\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\}$ is positive over (0,1). For even N:

$$\begin{aligned} &\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))\} \\ &= \mathcal{T}_N\{(1-x)^L(R(x) + \tfrac{L-N}{M+N}T(x))\} + (c - \tfrac{L-N}{M+N})\mathcal{T}_N\{(1-x)^L T(x)\} \\ &= (1-x)^L(R(x) + \tfrac{L-N}{M+N}T(x)) + Y_1(x) + (c - \tfrac{L-N}{M+N})((1-x)^L T(x) + Y_2(x)) \\ &= (1-x)^L(R(x) + cT(x)) + Y_1(x) + (\tfrac{L-N}{M+N} - c)Y_2(x) \end{aligned} \tag{61}$$

where $Y_1(x)$ is the right hand side of (60) and $Y_2(x)$ is the right most term of (58). Since $Y_1(x)$ and $Y_2(x)$ are both positive over (0,1), and since $R(x) + cT(x)$ is positive over (0,1) for $c > -1$, the expression above shows that $\mathcal{T}_N\{(1-x)^L(R(x) + cT(x))$ is positive over (0,1) for even N and $-1 < c < \frac{L-N}{M+N}$. It follows that $0 < F(x)$ for even N and $-1 < c < \frac{L-N}{M+N}$.

B Connection to a Series of Gauss

The polynomials $R(x)$, $T(x)$, and $S(x)$ given above, are special cases of the Gauss hypergeometric series [6], $F(a, b; c; z)$, given by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{62}$$

where the pochhammer symbol $(a)_k$ denotes $(a) \cdot (a+1) \cdot (a+2) \cdots (a+k-1)$. Note that when a or b is a negative integer, the $F(a, b; c; z)$ becomes a polynomial. The polynomials $R(x)$, $T(x)$, and $S(x)$ can be written as

$$S(x) = \tfrac{(M+1)_N}{N!} F(-M, L-N; -M-N; x) \tag{63}$$

$$R(x) = \tfrac{(M)_N}{N!} F(1-M, L-N; 1-M-N; x) \tag{64}$$

$$T(x) = \tfrac{(M)_{N-1}}{(N-1)!} F(2-M, L-N-1; 2-M-N; x). \tag{65}$$

Although the Gauss hypergeometric series is generally not defined for $c < 0$, because the case which arise here is a polynomial, and because c is more negative than a, the series is well defined.

There are many recurrence formulas for this function. Using these recurrence formulas, one can write recursion formulas for $R(x)$, $S(x)$, and $T(x)$. For example: if $S_0(x) = 1$, $S_1(x) = (L-N)/(M+N)x + 1$, then $S_{M-1}(x)$ computed by the recursive formula

$$S_k(x) = \frac{1}{M+N-k} \left(((k+L-N)x + M + N - 2k) S_{k-1}(x) + k(1-x) S_{k-2}(x)\right) \tag{66}$$

equals $S(x)$ given above.

These relationships may facilitate the computation of the roots of the polynomials, as suggested in [7, 15]. There is also a differential equation the solution of which is the Gauss hypergeometric series, see [6].

References

[1] O. Herrmann. On the approximation problem in nonrecursive digital filter design. *IEEE Trans. on Circuit Theory*, 18(3):411–413, May 1971. Also in [9].

[2] L. B. Jackson. An improved Martinez/Parks algorithm for IIR design with unequal numbers of poles and zeros. *IEEE Trans. on Signal Processing*, 42(5):1234–1238, May 1994.

[3] J. F. Kaiser. Design subroutine (MXFLAT) for symmetric FIR low pass digital filters with maximally-flat pass and stop bands. In IEEE ASSP Soc. Digital Signal Processing Committee, editor, *Programs for Digital Signal Processing*, chapter 5.3, pages 5.3–1 – 5.3–6. IEEE Press, 1979.

[4] J. Liang and R. J. P. De Figueiredo. An efficient iterative algorithm for designing optimal recursive digital filters. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 31(5):1110–1120, October 1983.

[5] H. G. Martinez and T. W. Parks. Design of recursive digital filters with optimum magnitude and attenuation poles on the unit circle. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 26(2):150–156, April 1978.

[6] F. Oberhettinger. Hypergeometric functions. In M. Abromowitz and I. A. Stegun, editors, *Handbook of Mathematical Functions*, chapter 15. Dover, 1970.

[7] H. J. Orchard. The roots of maximally flat-delay polynomials. *IEEE Trans. on Circuit Theory*, 12(???):452–454, September 1965.

[8] T. W. Parks and C. S. Burrus. *Digital Filter Design.* John Wiley and Sons, 1987.

[9] L. R. Rabiner and C. M. Rader, editors. *Digital Signal Processing.* IEEE Press, 1972.

[10] John Riordan. *Combinatorial Identities.* John-Wiley and Sons, 1968.

[11] T. Saramäki. Design of optimum recursive digital filters with zeros on the unit circle. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 31(2):450–458, April 1983.

[12] T. Saramäki. Design of digital filters with maximally flat passband and equiripple stopband magnitude. *Circuit Theory and Applications*, 13(2):269–286, April 1985.

[13] T. Saramäki and Y. Neuvo. IIR filters with maximally flat passband and equiripple stpoband. In *Europ. Conf. on Circuit Theory and Design*, volume ECCTD 80, Pt. 2, pages 468–473, 1980.

[14] K. Shenoi and B. Agrawal. On the design of recursive low-pass digital filters. *IEEE Trans. on Acoust., Speech, Signal Proc.*, 28(1):79–84, February 1980.

[15] J. P. Thiran. Recursive digital filters with maximally flat group delay. *IEEE Trans. on Circuit Theory*, 18(6):659–664, November 1971.

[16] A. Tucker. *Applied Combinatorics.* John-Wiley and Sons, 1995.

[17] R. Unbehauen. Recursive digital low-pass filters with maximally flat group delay and Chebyshev stop-band attenuation. In *Proc. IEEE Int. Symp. Circuits and Systems*, pages 593–596, 1980.

[18] R. Unbehauen. On the design of recursive digital low-pass filters with maximally flat pass-band and Chebyshev stop-band attenuation. In *Proc. IEEE Int. Symp. Circuits and Systems*, pages 528–531, 1981.

[19] P. P. Vaidyanathan. On maximally-flat linear-phase FIR filters. *IEEE Trans. on Circuits and Systems*, 31(9):830–832, September 1984.

[20] P. P. Vaidyanathan. Efficient and multiplierless design of FIR filters with very sharp cutoff via maximally flat building blocks. *IEEE Trans. on Circuits and Systems*, 32(3):236–244, March 1985.

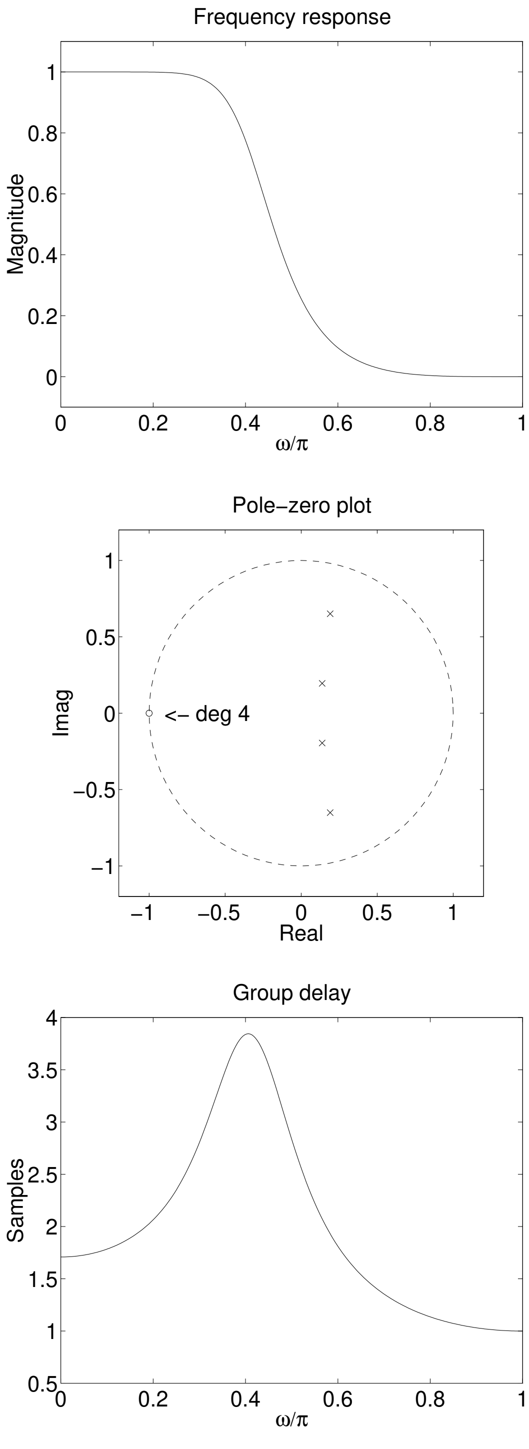

Figure 1: $L = 4$, $M = 0$, $N = 4$.

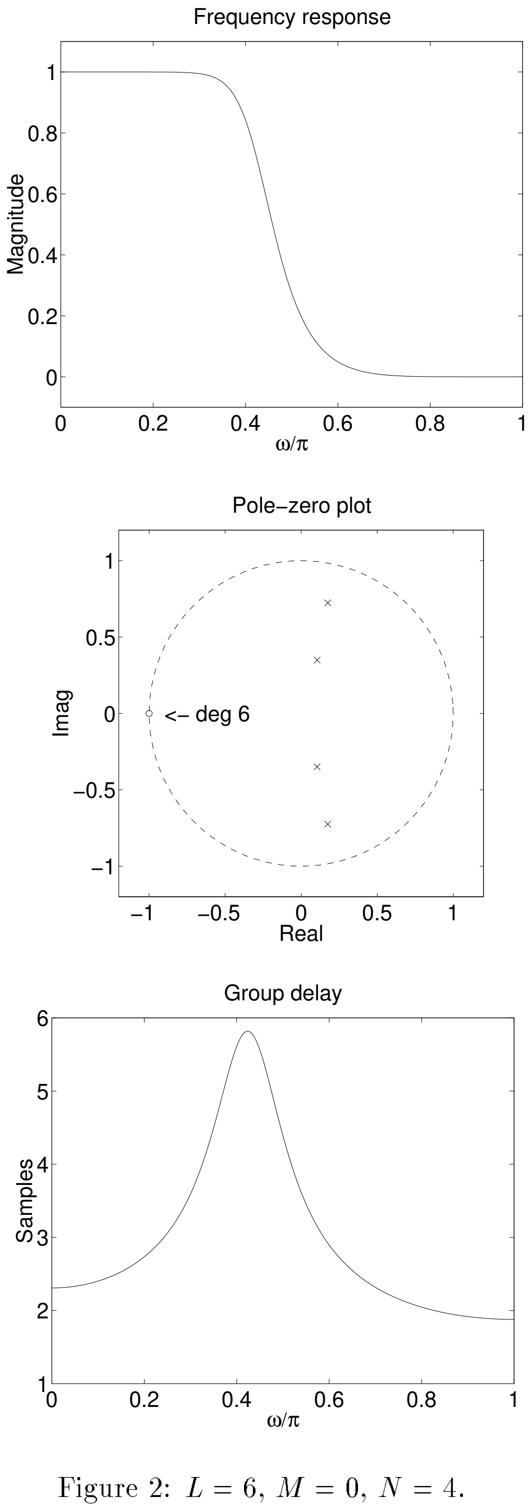

Figure 2: $L = 6$, $M = 0$, $N = 4$.

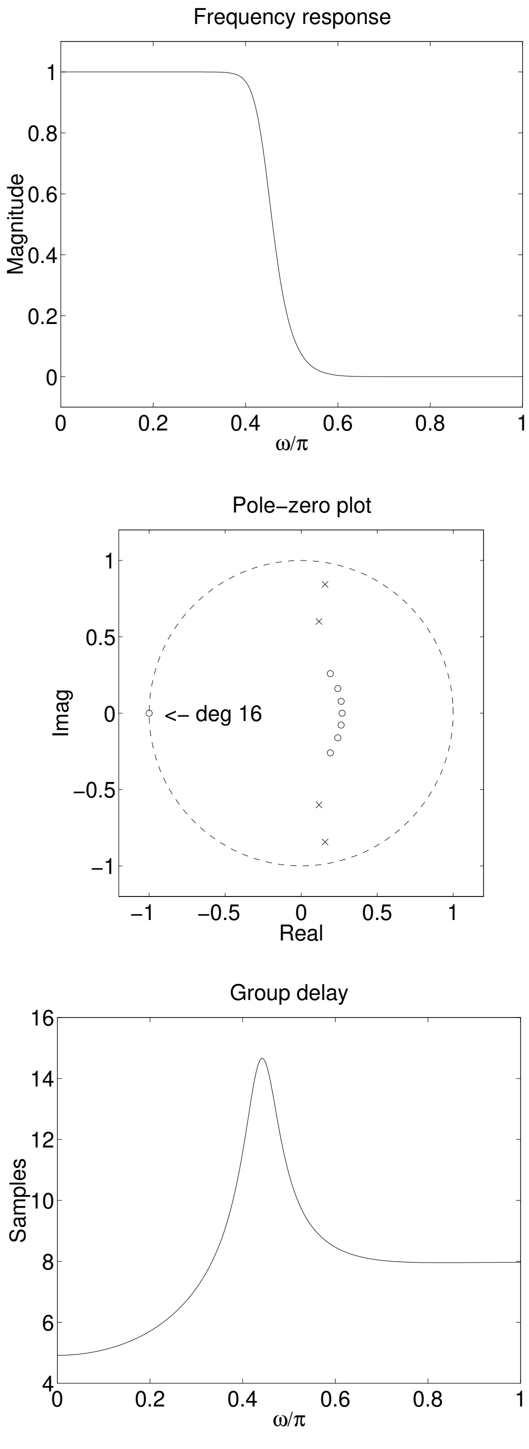

Figure 3: $L = 16$, $M = 7$, $N = 4$.

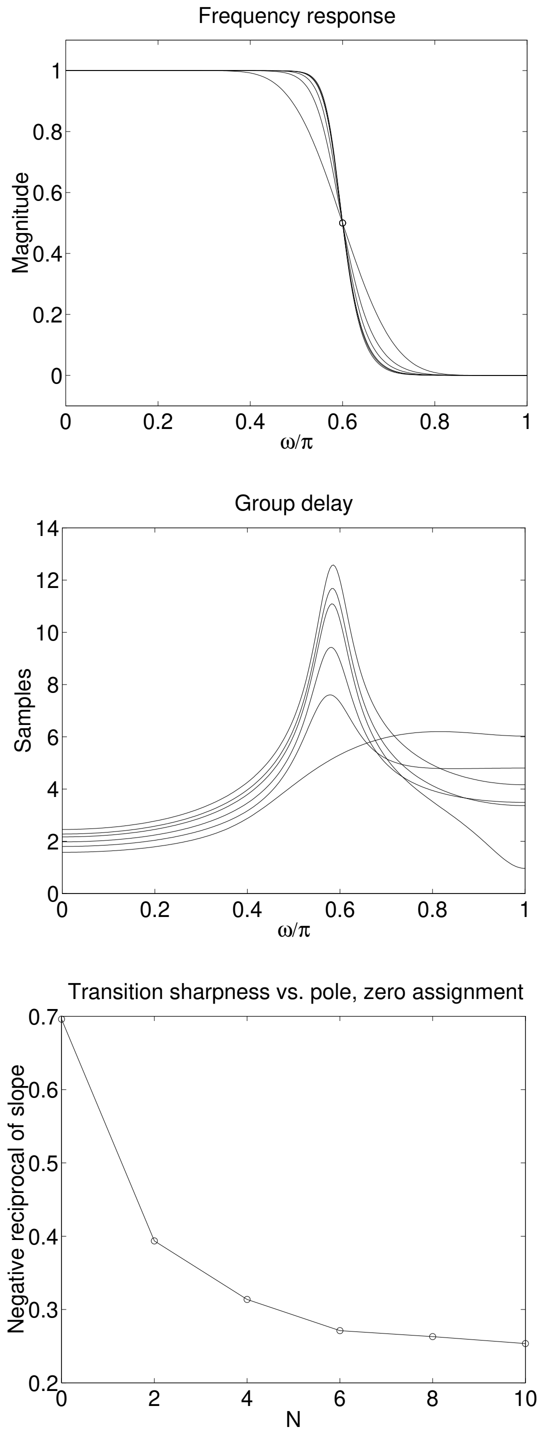

Figure 4: Shown in the figures are the filters having half-magnitude frequency $\omega_o = 0.6$, and $N = 0:2:10$. For all filters shown, $L + M + N = 20$. $N = 10$ corresponds to the filter having the steepest transition and the most peaked group delay.

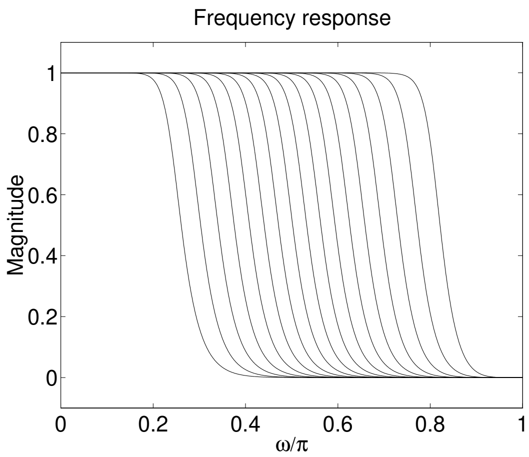

Figure 5: $L = 5, \ldots, 21$, $M = 22 - L$, $N = 4$. The widest band filter corresponds to $L = 5$.