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A CONTRIBUTION TO THE STABILITY THEORY OF DISTRIBUTED PARAMETER SYSTEMS

by

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CHAPTER I

INTRODUCTION

1.1. Objective Statement

Generally speaking, one of the most important and fundamental problems associated with the studying of dynamical systems is the investigation of stability, since a given dynamical system is seldom proven to be useful unless it is stable. Much attention has been confined to systems whose dynamic behavior is governed by ordinary differential equations[1-5]. However, there are many systems, for example, those systems which include transmission lines, wave guides, vibrating strings or membrances, torsional shafts, and heat exchangers, etc., in which the spatial configuration must be taken into account. Such systems, in which the spatial configuration constitutes an important factor, are often called distributed parameter systems, and their dynamic behavior is represented by partial differential equations instead of ordinary differential equations. For stability analysis, the second method of Liapunov is applied primarily to systems having finite degrees of freedom described by ordinary differential equations. The theory has been extended to distributed parameter systems by Zubov[6] and to systems having infinite degrees of freedom by Massera[7]. Movchan[8] Slobodkin[9], Wang[10] and Knop and Wilkes[11] discussed stability of elastic systems in the frame-work of Liapunov. Kostandian[12] and Rakhmatullina[13] considered the stability of solutions of the nonlinear heat conduction equation. Wang[14] considered the stability problems for particular classes of distributed parameter systems with feedback control. A variety of questions on the stability of partial differential equations are also examined by Eckhaus[15]. Recently, Hale and Infante[16] analized some stability questions in a format which is applicable to certain partial differential equations. However, problems that still remain open for further investigation are the stability of distributed parameter systems with periodic coefficients and the extension of existing results on bounded input bounded output stability of lumped systems to distributed parameter systems.

One of the main objectives of the present work is to develop methods for analyzing the stability of distributed parameter systems which can be described by a general partial differential equation of the type

$$\frac{\partial u(x,t)}{\partial t} = A(x,t,\frac{\partial}{\partial x})\ u(x,t) \qquad (1\text{-}1)$$

where $u(x,t)$ is a n-vector which is a function of the coordinate x (scalar variable) and time t, $A(x,t,\frac{\partial}{\partial x})$ is a linear polynomial operator in $\frac{\partial}{\partial x}$ of degree p with varying coefficients in x and t. This equation is of general interest in the description of many engineering problems and includes such important cases as the wave and the diffusion equations.

In chapter II, we develop a state space representation for distributed parameter systems governed by the nth order linear homogeneous partial differential equation. This approach is further extended to the general nth order linear nonhomogeneous partial differential equations.

Chapter III is devoted to the stability of linear periodic distributed parameter systems. Attention is first focused on systems where the operator A is a function of a polynomial in $\frac{\partial}{\partial x}$ with periodic coefficients depending on t only. The Floquet theory[17] is formulated for distributed parameter systems by using the Fourier transform technique. A theorem similar to the conventional Floquet theorem is stated for systems described by (1-1) with $A(t,\frac{\partial}{\partial x})$ periodic in t with period ω, i.e.,

$$A(t+\omega,\frac{\partial}{\partial x}) = A(t,\frac{\partial}{\partial x}). \tag{1-2}$$

A fundamental matrix operator solution $\Phi(t,\frac{\partial}{\partial x})$ exists such that

$$\Phi(t,\frac{\partial}{\partial x}) = P(t,\frac{\partial}{\partial x})\, e^{tR(\frac{\partial}{\partial x})} \tag{1-3}$$

where $P(t,\frac{\partial}{\partial x})$ is a linear operator with polynomial in $\frac{\partial}{\partial x}$ and its coefficients are periodic in t with period ω, i.e.,

$$P(t,\frac{\partial}{\partial x}) = P(t+\omega,\frac{\partial}{\partial x}) \tag{1-4}$$

and $R(\frac{\partial}{\partial x})$ is a linear operator in polynomial in $\frac{\partial}{\partial x}$ and is independent of t. Since $P(t,\frac{\partial}{\partial x})$ is periodic bounded linear operator, it is sufficient to examine the spectrum $\sigma[R(\frac{\partial}{\partial x})]$ of the linear operator $R(\frac{\partial}{\partial x})$ to determine whether the system is stable. The system is asymptotically stable if the maximum of the spectrum of $R(\frac{\partial}{\partial x})$ is negative, i.e.,

$$\max \; \sigma[R(\frac{\partial}{\partial x})] < 0. \tag{1-5}$$

If on the contrary,

$$\max \; \sigma[R(\frac{\partial}{\partial x})] > 0, \tag{1-6}$$

then the system is unstable.

By using this idea and an appropriate variable transformation, this stability analysis has been further extended to the case where the operator $A(t,x,\frac{\partial}{\partial x})$ has coefficients varying periodically both in t and x with periods, respectively, ω and β, i.e.,

$$A(t+\omega, x+\beta, \frac{\partial}{\partial x}) = A(t,x,\frac{\partial}{\partial x}) \tag{1-7}$$

This is illustrated by several examples from the diffusion process and wave propagation.

One of the most general and powerful approaches in the theory of stability is the so-called "second method" of Liapunov. It answers question of stability of differential equations without explicit knowledge of the solutions. Because of this property of the second method, it is sometimes called the "direct method". Chapter IV, begins by consideration of the absolute stability of a linear, constant coefficient, distributed parameter system described by (1-1) with $A(t,x,\frac{\partial}{\partial x}) = A(\frac{\partial}{\partial x})$. This serves as a basis for the stability analysis of nonlinear distributed parameter control systems.

The nonlinear distributed parameter system with control, which we consider, is described by a set of modified Lurie type equations of the form:

$$\frac{\partial u}{\partial t} = A(\frac{\partial}{\partial x})u - b\xi$$

$$\xi = \phi(\sigma) \tag{1-8}$$

$$\sigma = c'u \quad ,$$

where x, t and u are defined as before, $A(\frac{\partial}{\partial x})$ is the constant coefficient matrix operator mentioned above, b and c are constant n-vectors, the scalar variable ξ denotes the error, the scalar variable σ the output and the nonlinear function $\phi(\sigma)$ the control. For example, the equation that governs the motion of a string with controlled load may be put into the form of (1-8). By using a Liapunov functional approach similar to that of Lurie, conditions on the parameters, $\phi(\sigma)$ and the boundaries which are guarantee absolute stability of the null solution of (1-8) are derived.

In chapter V, we look at the stability of distributed parameter systems in presence of inputs. The bounded input-bounded output stability problem is studied for the linear control system

$$\frac{\partial u}{\partial t} = A\left(\frac{\partial}{\partial x}\right)u + gf \tag{1-9}$$

as well as the nonlinear control system described by (1-8) with an additional forcing term v:

$$\begin{aligned} \frac{\partial u}{\partial t} &= A\left(\frac{\partial}{\partial x}\right)u + b\xi \\ \xi &= v - \ell\sigma \\ \sigma &= c'u \end{aligned} \tag{1-10}$$

where u, t, $A(\frac{\partial}{\partial x})$, b, ξ, c, σ again are defined as before, v is the input (scalar) function. Bounded input bounded output stability is demonstrated by properly constructing a Liapunov functional V for (1-9) and (1-10) whose total time derivative satisfies

$$\dot{V} \leq - rV + sV^{1/2} \tag{1-11}$$

where r and s are positive constants. This differential inequality implies a bounded response and thus the bounded input produces a bounded output.

The stability theorem under persistent disturbances due to Malkin[1] is extended to a general distributed parameter system described by

$$\frac{\partial u(X,t)}{\partial t} = U(X,\frac{\partial}{\partial X},u,t) + R(X,\frac{\partial}{\partial X},u,t) \quad (1\text{-}12)$$

where $R(X,\frac{\partial}{\partial X},u,t)$ is a n-vector - the forcing function. Roughly speaking, our result establishes that provided the unforced system corresponding to (1-12) is asymptotically stable and under appropriate boundary conditions, the norm of the state vector is bounded if the norms of the initial (distributed) states and the input vector are suitable bounded.

What we have just stated is the inverse of the bounded input bounded output stability problem.

1.2. General Background

Some of the relevant definitions and theorems such as definitions for various degree of stability, Liapunov's stability theorem and the conventional Floquet theory will be briefly stated in this section.

(i) Notations and Definitions

Consider a distributed parameter system defined on a spatial domain $\Omega \subseteq R^m$, whose state at any time t can be generally specified by a real-valued vector function $u(X,t)$ - an element of some function space $\Gamma(\Omega)$ with a specified metric $\rho(u,u')$ at any time t. Let $X = (x_1, x_2, \ldots, x_m)$ be the spatial coordinate vector, R^m a real m-dimensional Euclidean space, Ω an open connected set, $\Gamma_i(\Omega)$, i =1, 2, ..., n, a set of function spaces defined on Ω, and $\Gamma(\Omega) = \Gamma_1(\Omega)\Gamma_2(\Omega)\ldots\Gamma_n(\Omega)$ a state function space. The system motion starting from any specified initial state $u(X,t_o)$ at time t_o is defined by

$\Phi(t,t_o)u(X,t_o)$, where $\Phi(t,t_o)$ is a continuous operator on $\Gamma(\Omega)$ defined in the interval t_o and t; it maps $\Gamma(\Omega)$ into itself. If the set of the operators $\Phi(t,t_o)$ obey the following conditions

$$\Phi(t_1,t)\Phi(t,t_o) = \Phi(t_1,t_o)$$

$$\Phi(t,t) = I, \qquad 0 \leq t_o \leq t < +\infty \tag{1-13}$$

where I is the identity operator, then it is said that $\Phi(t,t_o)$ has the properties of a semi-group. A particular motion resulting from a corresponding initial condition $u(X,t_o)$ determines a specific trajectory Λ in $\Gamma(\Omega)$ for all $t \geq t_o$. $\Lambda_{inv} \subseteq \Gamma(\Omega)$ is said to be an invariant set, if $u(X,t_o) \in \Lambda_{inv}$ implies that its corresponding trajectory also lies in Λ_{inv}. The invariant set Λ_{inv} usually consists of one or more trajectories of the dynamical system. The distance between a particular state u and the invariant set Λ_{inv} is defined as

$$\rho(u,\Lambda_{inv}) = \inf_{u' \in \Lambda_{inv}} (u,u'), \tag{1-14}$$

and the distance of a particular motion Λ_s from Λ_{inv} is defined by

$$\rho(\Lambda_s,\Lambda_{inv}) = \sup_{u \in \Lambda_s} \rho(u,\Lambda_{inv}). \tag{1-15}$$

Precise definition of stability can now be stated in the sense of Liapunov[2,3,6,14] as follows:

Definition: An invariant set Λ_{inv} of a dynamical system is said to be stable with respect to the metric defined in $\Gamma(\Omega)$ whenever for every t_o and $\varepsilon > 0$, there exists a real number $\delta(t_o,\varepsilon) > 0$ such that

$$\rho[u(X,t_o), \Lambda_{inv}] < \delta(t_o,\varepsilon)$$

implies

$$\rho(\Lambda_s, \Lambda_{inv}) < \varepsilon \qquad \text{for all } t \geq t_o,$$

where Λ_s denotes the corresponding trajectory of the system with respect to the initial state $u(X,t_o)$. If, in addition to the above conditions $\rho(\Lambda_s, \Lambda_{inv}) \to 0$ as $t \to \infty$, then the invariant set Λ_{inv} is said to be asymptotically stable. Furthermore, if Λ_{inv} is asymptotically stable for all $u(X,t_o) \in \Gamma(\Omega)$, then the invariant set Λ_{inv} is said to be globally asymptotically stable or asymptotically stable in the large.

In stability analysis, an equilibrium state is being considered. This is actually the special case where Λ_{inv} is an invariant set consisting of only an equilibrium state or the null state. It is sometimes convenient to reformulate the system equation to transfer the equilibrium state u_e to the null state by means of a displacement in the state function space. Therefore, without losing generality, the null state may be investigated for stability.

(ii) Floquet Theory

Consider the nth order linear differential equation with periodic coefficients in the vector form

$$\dot{x} = A(t)x \tag{1-16}$$

where x is a n-vector, A(t) is a n by n matrix and is periodic in time with period ω, i.e., $A(t+\omega) = A(t)$. The Floquet theory states that a fundamental matrix solution exists for (1-16), and is given by

$$\Phi(t) = P(t)\ e^{tR} \tag{1-17}$$

such that P(t) is apperiodic nonsingular matrix with period ω, and R is a constant matrix.

Let P(t) is a periodic nonsingular transformation with bounded coefficients of period such that

$$y = P(t)x\ , \tag{1-18}$$

using (1-17), (1-15) becomes

$$\dot{y} = Ry$$

where

$$R = P\ P^{-1} + PAP^{-1}\ . \tag{1-19}$$

According to a theorem by Liapunov (2), there exists a non-singular transformation P(t) with periodic coefficients with period ω such that R is a constant matrix. This means that a linear system with periodic coefficients is reducible to a linear system with constant coefficients through a nonsingular transformation P(t). Since P(t) is periodic and nonsingular, for the stability of system (1-16), it is only necessary to examine the eigenvalues of R. The system is asymptotically stable if the real part of all the eigenvalues are negative. The system becomes unstable if one of the eigenvalues has positive real part.

(iii) Stability Theorem.

(a) Liapunov's Theorem on Stability. Consider

$$\dot{x} = f(x,t), \qquad f(0,t) = 0 \tag{1-20}$$

as a n-vector system, where f(x,t) is continuous and satisfies a Lipschitz condition in the state space for all x and t. An important stability theorem by Liapunov[1] states that the system described by (1-20) is stable if there exists a function V(x,t) which is continuous together with its first partial derivatives such that

$$W_1(x) \geq V(x,t) \geq W_2(x) \geq 0, \quad x \neq 0 \tag{1-21}$$

$$V(0,t) = 0, \quad t \geq 0 \tag{1-22}$$

and

$$\dot{V}(x,t) \geq 0, \quad t \geq 0, \tag{1-23}$$

where $W_1(x)$ and $W_2(x)$ are positive definite. If in addition to (1-21) and (1-22)

$$-\dot{V}(x,t) \geq W_3(x) > 0, \tag{1-24}$$

where $W_3(x)$ is again positive definite, then the system is asymptotically stable.

(b) Zubov's Theorem on Stability. Liapunov's theorem can be extended to distributed parameter systems. The main idea is to select a Liapunov functional V which is a mapping from the state function space $\Gamma(\Omega)$ into a set of real numbers R instead of Liapunov function to give some estimate of the distance of the system state u from a specific invariant set in $\Gamma(\Omega)$. In order to study the stability of invariant sets of dynamical systems, and of systems in a metric space, the following important generalized Liapunov's theorem due to Zubov [6] is applicable.

Theorem: An invariant set Λ_{inv} of a distributed parameter system defined on $\Gamma(\Omega)$ is stable if and only if there exists a real Liapunov functional V having the following properties:

(i) $V(t,u)$ is defined for all t and u for $u \in S(\Lambda_{inv},r)$, $S(\Lambda_{inv},r)$ is a certain neighborhood: the set of all u such that the distance between the state u and the invariant set is less than r.

(ii) For every $c_1 > 0$, there is a $c_2 > 0$ such that $V(t,u) > c_2$ for all $\rho(u,\Lambda_{inv}) > c_1$ and $t \geq 0$.

(iii) $V(t,u) \to 0$ uniformly for all $t \geq 0$ as $u \to \Lambda_{inv}$.

(iv) The supremum of the function $V(t,u')$ for all $u' \in \Lambda_s$ is non-increasing for all $t \geq t_o$.

If in addition to the conditions (i) to (iv), the function defined in (iv) approaches zero as $t \to \infty$ for any $u \in S(\Lambda_{inv}, \delta)$ where $S(\Lambda_{inv}, \delta)$ is a certain neighborhood of Λ_{inv}.

CHAPTER II

STATE VARIABLE REPRESENTATION FOR DISTRIBUTED PARAMETER SYSTEMS

2.1. Introduction

In this chapter, the matrix techniques and state space concepts of ordinary differential equations are extended to the distributed parameter systems described by linear partial differential equations. The similarity between the theory of a system described by ordinary differential equation and that of a system governed by a partial differential equation is indicated.

By defining the state variables

$$x_1 = x,\ x_2 = \dot{x},\ \ldots,\ x_n = \frac{d^{n-1}x}{dt^{n-1}}\ , \qquad (2\text{-}1)$$

a nth order linear differential equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x = 0 \qquad (2\text{-}2)$$

has the matrix form

$$\dot{x} = Ax \qquad (2\text{-}3)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad (2\text{-}4)$$

and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 \\ -\frac{a_o}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & . & . & . & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{bmatrix} . \qquad (2-5)$$

Assuming A to be a constant n by n matrix with n distinct eigenvalues, the general solution of (2-3) is[18]

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} p_2 + \ldots + c_n e^{\lambda_n t} p_n \qquad (2-6)$$

or in matrix form[17]

$$x = e^{tA}\, x(0) = Pe^{tJ} P^{-1} x(0) \qquad (2-7)$$

where c_1, c_2, ..., c_n are arbitrary constants, λ_i distinct eigenvalues of A, p_i the eigenvectors associated with the eigenvalues λ_i, J the diagonal matrix similar to A, and P a nonsingular constant matrix such that

$$P = (p_1\ p_2\ \ldots\ p_n)\,. \qquad (2-8)$$

If A has repeated eigenvalues, J can in general be reduced to its Jordan canonical form

$$
J = \begin{bmatrix}
J_o & 0 & \cdot & \cdot & \cdot & 0 \\
0 & J_1 & \cdot & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & \cdot & \cdot & J_s
\end{bmatrix} \qquad (2\text{-}9)
$$

where J_o is a diagonal matrix with diagonal $\lambda_1, \lambda_2, \ldots, \lambda_m$ -- m distinct eigenvalues of A, and

$$
J_i = \begin{bmatrix}
\lambda_{m+i} & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & \lambda_{m+i} & 1 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & \lambda_{m+i} & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \lambda_{m+i} & 1 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & \lambda_{m+i}
\end{bmatrix} \qquad (2\text{-}10)
$$

$(i = 1, 2, \ldots, s)$

a r_i by r_i square matrix, r_i the multiplicity of λ_{m+i} (i.e., r_i = number of times λ_i is repeated) and $n = m + \sum_{i=1}^{s} r_i$. It follows that

$$e^{tJ} = \begin{bmatrix} e^{tJ_o} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{tJ_1} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & e^{tJ_s} \end{bmatrix}, \qquad (2\text{-}11)$$

where

$$e^{tJ_o} = \begin{bmatrix} e^{t\lambda_1} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & e^{t\lambda_2} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & e^{t\lambda_m} \end{bmatrix} \qquad (2\text{-}12)$$

and

$$e^{tJ_o} = \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdot & \cdot & \cdot & \frac{t^{r_i-1}}{(r_i-1)!} \\ 0 & 1 & t & \cdot & \cdot & \cdot & \frac{t^{r_i-2}}{(r_i-2)!} \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \frac{t^{r_i-3}}{(r_i-3)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad (2\text{-}13)$$

$(i = 1, 2, \ldots, s)$

2.2. Linear Homogeneous Partial Differential Equation

A linear homogeneous constant coefficient partial differential equation can be written as

$$a_n \frac{\partial^n u}{\partial t^n} + a_{n-1} \frac{\partial^n u}{\partial t^{n-1} \partial x} + a_{n-2} \frac{\partial^n u}{\partial t^{n-2} \partial x^2} + \cdots + a_1 \frac{\partial^n u}{\partial t \partial x^{n-1}} + a_o \frac{\partial^n u}{\partial x^n} = 0 , \tag{2-14}$$

where u is a scalar function of x and t, and a_i's are constants. If λ_i are distinct and satisfy the characteristic equation

$$a_n \lambda_i^n + a_{n-1} \lambda_i^{n-1} + \ldots + a_1 \lambda_i + a_o = 0, \tag{2-15}$$

the general solution will be given by

$$u = \sum_{i=1}^{n} \Phi_i \ (x + \lambda_i t) \tag{2-16}$$

where Φ_i are arbitrary functions.

We now introduce the state variables for the system (2-14) by defining

$$\begin{aligned} u_1 &= \frac{\partial^{n-1} u}{\partial x^{n-1}} \\ u_2 &= \frac{\partial^{n-1} u}{\partial t \partial x^{n-2}} \\ &\cdot \\ &\cdot \\ &\cdot \\ u_{n-1} &= \frac{\partial^{n-1} u}{\partial t^{n-2} \partial x} \\ u_n &= \frac{\partial^{n-1} u}{\partial t^{n-1}} \quad . \end{aligned} \tag{2-17}$$

(2-14) can then be written as

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{\partial u_2}{\partial x} \\
\frac{\partial u_2}{\partial t} &= \frac{\partial u_3}{\partial x} \\
&\vdots \\
\frac{\partial u_{n-1}}{\partial t} &= \frac{\partial u_n}{\partial x} \\
\frac{\partial u_n}{\partial t} &= -\frac{a_o}{a_n}\frac{\partial u_1}{\partial x} - \frac{a_1}{a_n}\frac{\partial u_2}{\partial x} - \dots - \frac{a_{n-1}}{a_n}\frac{\partial u_n}{\partial x} \ .
\end{aligned}
\tag{2-18}
$$

This can be put into the vector form

$$\frac{\partial u}{\partial t} = A \frac{\partial u}{\partial x} \tag{2-19}$$

with

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \tag{2-20}$$

and

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\
-\frac{a_o}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \cdot & \cdot & \cdot & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n}
\end{bmatrix}. \tag{2-21}
$$

Then a general linear homogeneous partial differential equation can be easily put into the vector form of (2-19) with the state variables defined in (2-17).

Note that the state variable representation expressed by (2-19) to (2-21) is valid when the parameter a_i, $i=1, 2,\ldots,n$, appearing in (2-14), are functions of x and t.

Returning to the constant coefficient case, a solution of (2-19) may be formally obtained as,

$$u = \phi(x + \lambda t)p \tag{2-22}$$

where p is a nonzero constant vector. Substituting (2-22) into (2-19),

$$\phi'(x + \lambda t)p = A\phi'(x + \lambda t)p$$

or

$$(A - \lambda I)p = 0. \tag{2-23}$$

Hence (2-22) is a solution of (2-14) precisely when p is an eigenvector of A associated with the eigenvalue λ. If A has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the corresponding eigenvectors $p_1, p_2, \ldots, p_m$ are linearly independent and form a basis for the space. Therefore the linear combination

$$u(x,t) = \phi_1(x+\lambda_1 t)p_1 + \phi_2(x+\lambda_2 t)p_2 + \ldots + \phi_n(x+\lambda_n t)p_n \quad (2\text{-}24)$$

is the general solution to (2-19).

The n arbitrary functions ϕ_i can be determined by the n given boundary conditions.

Let us denote by F the linear operator $A\frac{\partial}{\partial x}$ in the product space $H = L_2^N(\Omega) = L_2(\Omega)\ldots\ldots L_2(\Omega)$ (n times), where Ω is a subset of the real line. The boundary of Ω will be denoted by $\delta\Omega$.

We define the domain of F

$D(F) = \{u(x,t) \in H: \quad t \in [0,\infty)$; u absolutely conditions on Ω, u(x,t) satisfies a homogeneous boundary condition $c'u(x,t) = 0 \quad \forall\ x \in \delta\Omega$, where the components of the n-vector c may depend on both x and t$\}$. (2-25)

Let us impose the following two restrictions on the operator F

(1) D(F) is dense in H,

(2) there is a real number k such that

$$\| R(\lambda,F) \| \equiv (\lambda I-F)^{-1} \leq \frac{1}{\lambda - k} \quad (2\text{-}26)$$

for all $\lambda > k$; where I = identity operator.

Then by the Hille and Yoshida theorem, F is the infinitesimal generator of semigroup $\{\Phi(t,t')\}$ where

$$\Phi(t,t') = e^{(t-t')F} = e^{(t-t')A\frac{\partial}{\partial x}} \qquad (2\text{-}27)$$

and a solution of (2-19) satisfying the homogeneous boundary conditions (2-25) and corresponding to an initial function u(x,0) may be expressed as

$$u(x,t) = e^{tA\frac{\partial}{\partial x}}u(x,0) \;, \quad x \in \Omega \,, \quad t \in [0,\infty) \qquad (2\text{-}28)$$

Let J be the diagonal matrix similar to A, obtained by means of the nonsingular matrix P

$$J = P^{-1}A\ P \;. \qquad (2\text{-}29)$$

Then the fundamental matrix operator solution $\Phi(t,t')$ takes the form

$$\Phi(t,t') = e^{(t-t')A\frac{\partial}{\partial x}} = e^{(t-t')PJP^{-1}\frac{\partial}{\partial x}}$$

$$= Pe^{(t-t')J\frac{\partial}{\partial x_p}P^{-1}} \;, \qquad (2\text{-}30)$$

and the general solution (2-28),

$$u(x,t) = Pe^{(t-t')J\frac{\partial}{\partial x_p}P^{-1}}\,u(x,0) \;. \qquad (2\text{-}31)$$

In general, if A has repeated eigenvalues, (2-31) will againnbe the general solution provided that J and e^{tJ} are defined by the equations (2-9) to (2-13).

Notice the similarities between the nth order ordinary differential equation and the nth order homogeneous partial differential equation. The vector equations (2-3) and (2-19), their general solutions (2-6) or (2-7) and (2-24) or (2-31) are similar to each other.

Example 2-1. Consider the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} .$$

Using the state variable technique developed in the above section, and defining

$$u_1 = \frac{\partial u}{\partial x}$$

$$u_2 = \frac{\partial u}{\partial t} ,$$

the wave equation will then become

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial x}$$

$$\frac{\partial u_2}{\partial t} = c^2 \frac{\partial u_1}{\partial x}$$

or

$$\frac{\partial}{\partial t} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ c^2 & 0 \end{bmatrix} .$$

The characteristic equation of A is

$$|A - \lambda I| = 0 .$$

This implies $\lambda = \pm c$. The eigenvector associated with $\lambda_1 = + c$ is

$$p_1 = \begin{bmatrix} 1 \\ c \end{bmatrix} ,$$

and p_2 associated with $\lambda_2 = -c$ is

$$p_2 = \begin{bmatrix} 1 \\ -c \end{bmatrix} .$$

Hence,

$$P = \begin{bmatrix} 1 & 1 \\ c & -c \end{bmatrix} , \qquad |P| = -2c ,$$

$$P^{-1} = \frac{1}{-2c} \begin{bmatrix} -c & -1 \\ -c & 1 \end{bmatrix} .$$

$$J = P^{-1} A \, P = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} .$$

In terms of (2-24), the general solution is

$$u(x,t) = \phi_1(x+\lambda_1 t)p_1 + \phi_2(x+\lambda_2 t)p_2$$

$$= \begin{bmatrix} \phi_1(x+ct) + \phi_2(x-ct) \\ c\phi_1(x+ct) - c\phi_2(x-ct) \end{bmatrix}$$

The boundary conditions when determine, the functions $\phi_i(x,t)$ i = 1, 2, ..., n and the formulation of equations (2-27) throygh (2-31) can be applied without difficultyes.The way in which boundary conditions are taken into account is illustrated in the subsequent chapters.

2-3. Linear Nonhomogeneous Partial Differential Equation.

The state variable representation developed in the last section, will not in general be applicable to a n^{th}-oder linear system described by

$$a_{n,o}\frac{\partial^n u}{\partial t^n} + a_{n-1,1}\frac{\partial^n u}{\partial t^{n-1}\partial x} + a_{n-2,2}\frac{\partial^n u}{\partial t^{n-2}\partial x^2} + \cdots + a_{o,n}\frac{\partial^n u}{\partial x^n}$$

$$+ a_{n-1,o}\frac{\partial^{n-1} u}{\partial t^{n-1}} + a_{n-2,1}\frac{\partial^{n-1} u}{\partial t^{n-2}\partial x} + \cdots + a_{o,n-1}\frac{\partial^{n-1} u}{\partial x^{n-1}}$$

$$+ \cdots$$

$$+ a_{2,o}\frac{\partial^2 u}{\partial t^2} + a_{1,1}\frac{\partial^2 u}{\partial t\partial x} + a_{o,2}\frac{\partial^2 u}{\partial x^2}$$

$$+ a_{1,o}\frac{\partial u}{\partial t} + a_{o,1}\frac{\partial u}{\partial x}$$

$$+ a_{o,o}u = 0 .$$

(2-32)

A new state variable representation technique will be developed to transform (2-32) into the vector form

$$\frac{\partial u}{\partial t} = A\frac{\partial u}{\partial x} + Bu \quad , \qquad (3-33)$$

where u is a n-vector, A and B are n by n matrices. Consider the simple second order vector equation

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial x} + b_{11}u_1 \tag{2-34a}$$

$$\frac{\partial u_2}{\partial t} = -a_1\frac{\partial u_1}{\partial x} - a_2\frac{\partial u_2}{\partial x} + b_{21}u_1 + b_{22}u_2 \tag{2-34b}$$

Partially differentiating (2-34b) with respect to x, we obtain

$$\frac{\partial^2 u_2}{\partial t\partial x} = -a_1\frac{\partial^2 u_1}{\partial x^2} - a_2\frac{\partial^2 u_2}{\partial x^2} + b_{21}\frac{\partial u_1}{\partial x} + b_{22}\frac{\partial u_2}{\partial x} \tag{2-35}$$

The left hand side can be obtained by partial differentiating (2-34a) with respect to t

$$\frac{\partial^2 u_1}{\partial t^2} - b_{11}\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_2}{\partial t\partial x} \quad , \tag{2-36}$$

Equating (2-35) and (2-36),

$$\frac{\partial u_1}{\partial t^2} - b_{11}\frac{\partial u_1}{\partial t} = -a_1\frac{\partial u_1}{\partial x^2} - a_2\frac{\partial u_2}{\partial x^2} + b_{21}\frac{\partial u_1}{\partial x} + b_{22}\frac{\partial u_2}{\partial x} \tag{2-37}$$

$\frac{\partial^2 u_2}{\partial x^2}$ and $\frac{\partial u_1}{\partial x}$ can again be obtained from (2-34a) as

$$\frac{\partial u_2}{\partial x^2} = \frac{\partial^2 u_1}{\partial t\partial x} - b_{11}\frac{\partial u_1}{\partial x}$$

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_1}{\partial t} - b_{11}u_1 \quad . \tag{2-38}$$

Substituting (2-38) in (2-37), we get

$$\frac{\partial^2 u_1}{\partial t^2} + a_2\frac{\partial^2 u_1}{\partial t\partial x} + a_1\frac{\partial^2 u_1}{\partial x^2} - (b_{11}+ b_{22})\frac{\partial u_1}{\partial t} + (a_2 b_{11} + b_{21})\frac{\partial u_1}{\partial x} + b_{11}b_{22}u_1 = 0 \, . \tag{2-39}$$

This simply means that a constant coefficient system described by (2-32) for n = 2, can be transformed into the vector form (2-33) with

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix}, \tag{2-40}$$

The coefficients are related by

$$\begin{aligned} a_{2,0} &= 1 \\ a_{1,1} &= a_2 \\ a_{0,2} &= a_1 \\ a_{1,0} &= -(b_{11} + b_{22}) \\ a_{0,1} &= -(a_2 b_{11} + b_{21}) \\ a_{0,0} &= b_{11}b_{22} \quad . \end{aligned} \tag{2-41}$$

For the third order system, n = 3,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{2-42}$$

the relations among the coefficients being

$$\begin{aligned}
a_{3,0} &= 1 \\
a_{2,1} &= a_3 \\
a_{1,2} &= a_2 \\
a_{0,3} &= a_1 \\
a_{2,0} &= -(b_{11} + b_{22} + b_{33}) \\
a_{1,1} &= -a_3(b_{11} + b_{22}) + b_{21} + b_{32} \\
a_{0,2} &= -(a_2b_{11} + a_3b_{21} + b_{31}) \\
a_{1,0} &= (b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33}) \\
a_{0,1} &= a_3b_{22}b_{11} + b_{11}b_{32} + b_{21}b_{33} \\
a_{0,0} &= -b_{11}b_{22}b_{33} \quad .
\end{aligned} \tag{2-43}$$

For a general n^{th}-order system,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & . & . & . & -a_{n-1} & -a_n \end{bmatrix} \tag{2-44}$$

and

$$B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & . & . & . & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & . & . & . & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ b_{n1} & b_{n2} & b_{n3} & b_{n4} & . & . & . & b_{nn-1} & b_{nn} \end{bmatrix} . \tag{2-45}$$

By the method of induction, the following relations among coefficients are obtained.

$$
\begin{aligned}
a_{n0} &= 1 \\
a_{n-1,1} &= a_n \\
a_{n-2,2} &= a_{n-1} \\
&\vdots \\
a_{2,n-2} &= a_3 \\
a_{1,n-1} &= a_2 \\
a_{0n} &= a_1
\end{aligned}
$$

$$
\begin{aligned}
a_{n-1,0} &= - \sum_{i=1}^{n} b_{ii} \\
a_{n-2,1} &= - \sum_{i=1}^{n-1} a_n b_{ii} + b_{i+1,i} \\
a_{n-3,2} &= - \sum_{i=1}^{n-2} a_{n-1} b_{ii} + a_n b_{i+1,i} + b_{i+2,i} \\
&\vdots \\
a_{1,n-2} &= - \sum_{i=1}^{2} (a_3 b_{ii} + a_4 b_{i+1,i} + a_5 b_{i+2,i} + \cdots \\
&\qquad + a_{n-1} b_{i+n-3,i} + b_{i+n-2,i}) \\
a_{0,\, n-1} &= - \sum_{i=1}^{1} (a_2 b_{ii} + a_3 b_{i+1,i} + a_4 b_{i+2,i} + \cdots \\
&\qquad + a_{n-1} b_{i+n-2,i} + b_{i+n-1,i})
\end{aligned}
$$

$$a_{n-2,0} = \tfrac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} b_{ii} b_{jj}$$

$$\begin{aligned} a_{n-3,1} = a_n\Big(\tfrac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} b_{ii} b_{jj}\Big) &+ b_{32} b_{11} + b_{33} b_{21} \\ &+ b_{43}(b_{11} + b_{22}) + b_{44}(b_{32} + b_{21}) + \cdots \\ &+ b_{n,n-1}(b_{11} + b_{22} + \cdots + b_{n-2,n-2}) \\ &+ b_{nn}(b_{n-1,n-2} + b_{n-2,n-3} + \cdots + b_{32} + b_{21}) \end{aligned}$$

$$\begin{aligned} a_{n-4,2} = a_{n-1}\Big(\tfrac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n-2} b_{ii} b_{jj}\Big) &+ a_n\big[b_{32} b_{11} + b_{33} b_{21} \\ &+ b_{43}(b_{32} + b_{21}) + \cdots + b_{n-1,n-2}(b_{11} \\ &+ b_{22} + \cdots + b_{n-3,n-3})\big] \\ &+ b_{n-1,n-1}(b_{n-2,n-3} + b_{n-3,n-4} + \cdots \\ &+ b_{32} + b_{21}) \end{aligned}$$

$$\vdots$$

$$\begin{aligned} a_{0,n-2} = a_3\Big(\sum_{i=1}^{2} b_{ii}\Big) &+ a_4(b_{32} b_{11} + b_{33} b_{21}) + \cdots \\ &+ a_n(b_{n-1,2} b_{11} + b_{n-1,3} b_{21} + \cdots \\ &+ b_{n-1,n-1} b_{n-2,1}) \end{aligned}$$

$$
\begin{aligned}
a_{n-3,0} &= -\frac{1}{3}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n} b_{ii}b_{jj}b_{kk}\\
a_{n-4,1} &= -\Big\{a_n\Big(\frac{1}{3}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n-1} b_{ii}b_{jj}b_{kk}\Big) + b_{43}(b_{11}b_{22})\\
&\quad + b_{44}(b_{32}b_{11}+b_{33}b_{21}) + b_{54}(b_{11}b_{22}+b_{22}b_{33}+b_{33}b_{11})\\
&\quad + b_{55}\big[(b_{32}b_{11}+b_{33}b_{21})+b_{54}(b_{11}+b_{22})+b_{44}(b_{21}+b_{32})\big] + \cdots\\
&\quad + b_{n,n-1}\Big(\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n-2} b_{ii}b_{jj}\Big)+b_{nn}\big[b_{32}b_{11}+b_{33}b_{21}+b_{44}(b_{21}+b_{32})\\
&\quad + \cdots + b_{n-1,n-2}(b_{11}+b_{22}+\cdots+b_{n-3,n-3})\\
&\quad + b_{n-1,n-1}(b_{21}+b_{32}+\cdots+b_{n-2,n-3})\big]\Big\}
\end{aligned}
$$

$\cdot$

$\cdot$

$\cdot$

$$
\begin{aligned}
a_{0,n-3} &= -\Big\{a_4b_{11}b_{22}b_{33}+a_5\big[b_{43}(b_{11}b_{22})+b_{44}(b_{32}b_{11}+b_{33}b_{21})\big]\\
&\quad +a_6\big[b_{53}(b_{11}b_{22})+b_{54}(b_{32}b_{11}+b_{33}b_{21})+b_{55}(b_{42}b_{11}\\
&\quad +b_{43}b_{21}+b_{44}b_{32})\big]+a_7\big[b_{63}b_{11}b_{22}+b_{64}(b_{32}b_{11}+b_{33}b_{21})\\
&\quad +b_{65}(b_{42}b_{11}+b_{43}b_{21}+b_{44}b_{31})+b_{66}(b_{52}b_{11}+b_{53}b_{21}\\
&\quad +b_{54}b_{31}+b_{55}b_{41})\big]\\
&\quad + \cdots\\
&\quad +a_n\big[b_{n-1,3}b_{11}b_{22}+b_{n-1,4}(b_{32}b_{11}+b_{33}b_{21})\\
&\quad +b_{n-1,5}(b_{42}b_{11}+b_{43}b_{21}+b_{44}b_{31})+\cdots\\
&\quad +b_{n-1,n-1}(b_{n-2,2}b_{11}+b_{n-2,3}b_{21}+\cdots\\
&\quad +b_{n-2,n-2}b_{n-3,1})\big]\Big\}
\end{aligned}
$$

$\cdot$

$\cdot$

$\cdot$

$$\left[\begin{array}{l} a_{1,0} = (-1)^{n-1}\left[\dfrac{1}{(n-1)}\displaystyle\sum_{\substack{i,j,\cdots v=1\\ i\neq j\neq\cdots\neq v}}^{n} \overbrace{b_{ii}b_{jj}\cdots b_{vv}}^{n-1}\right] \\[2ex] a_{0,1} = (-1)^{n-1}\left\{a_n\left[\dfrac{1}{(n-1)}\displaystyle\sum_{\substack{i,j,\cdots v=1\\ i\neq j\neq\cdots\neq v}}^{n-1} \overbrace{b_{ii}b_{jj}\cdots b_{vv}}^{n-1}\right]\right. \\ \qquad + b_{n,n-1}(b_{11}b_{22}\cdots b_{n-2,n-2}) \end{array}\right.$$

$$+ b_{nn}\left[\begin{array}{l} b_{n-1,n-2}b_{n-3,n-3}b_{n-4,n-4}\cdots b_{22}b_{11} \\ \quad + \\ b_{n-1,n-1}\left[\begin{array}{l} b_{n-2,n-3}b_{n-4,n-4}\cdots b_{22}b_{11} \\ \quad + \\ b_{n-2,n-2}\left[\begin{array}{l} b_{n-3,n-4}\ b_{n-5,n-5}\cdots b_{22}b_{11} \\ \quad + \\ b_{n-3,n-3}\left[\begin{array}{l} \cdots \\ \quad + \\ \cdots b_{44}\left[\begin{array}{l} b_{32}\{b_{11} \\ \\ b_{33}\{b_{21} \end{array}\right. \end{array}\right. \end{array}\right. \end{array}\right. \end{array}\right.\left.\vphantom{\begin{array}{l}a\\a\\a\end{array}}\right\}$$

$$\left[\ a_{0,0} = (-1)^{n}(b_{11}b_{22}\cdots b_{nn})\right.$$

(2-46)

Generalizing these results, we are led to the following representation for a general linear distributed system with

$$\frac{\partial u(t,x_1,x_2,\cdots,x_n)}{\partial t} = \sum_{i=1}^{n}\left[A_i\frac{\partial u(t,x_1,x_2,\cdots x_n)}{\partial x_i} + B_i u(t,x_1,x_2,\cdots,x_n)\right]$$

(2-4[illegible])

where u is a function of time t and n-dimensional space coordinates x_1, x_2, $\cdots$, x_n .

Example 2-2. Consider the transmission line problem. The voltage e(x,t) in a line can be shown to satisfy the equation

$$\frac{\partial^2 e}{\partial t^2} + \frac{1}{LC}\frac{\partial^2 e}{\partial x^2} + \frac{RC + GL}{LC}\frac{\partial e}{\partial t} + \frac{RG}{LC} e = 0 \quad .$$

With the technique just developed in the last section, this can be written in the vector form

$$\frac{\partial u}{\partial t} = A \frac{\partial u}{\partial x} + Bu$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \text{and} \qquad u_1 = e \quad .$$

A and B are found to be

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{LC} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{R}{L} & 0 \\ 0 & -\frac{G}{C} \end{bmatrix} \quad .$$

CHAPTER III

STABILITY OF DISTRIBUTED PARAMETER SYSTEMS WITH PERIODIC COEFFICIENTS

3.1. Introduction

In the present chapter, the main concern will be with distributed parameter systems whose dynamic performance is governed by the partial differential equation of the form

$$\frac{\partial u(x,t)}{\partial t} = A(x,t,\frac{\partial}{\partial x})\ u(x,t), \tag{3-1}$$

where $u(x,t)$ is an n-vector which is a function of the coordinate x and time t, $A(x,t,\frac{\partial}{\partial x})$ is a polynomial in $\frac{\partial}{\partial x}$ with varying coefficients in x and t.

To begin with (section 3-1 and 3-3), Floquet theory will be extended to systems described by (3-1) with $A(t,\frac{\partial}{\partial x})$ periodic in t with period ω. In section 3-3, we will allow the coefficients of A to depend on, and be periodiciin, both t and x.

Consider the simple first order wave equation

$$\frac{\partial u(x,t)}{\partial t} = a(t)\ \frac{\partial u(x,t)}{\partial x} \quad . \tag{3-2}$$

Using separation of variables and assuming $u(x,t) = T(x)X(x)$, where T and X are chosen to satisfy appropriate boundary conditions, the general solution of (3-2) is obtained as

$$u(x,t) = \sum_{n=0}^{\infty} e^{\lambda_n \int_0^t a(t)dt}\, T(0)e^{\lambda_n x}X(0) = \sum_{n=0}^{\infty} e^{\lambda_n \int_0^t a(t)dt}\, u_n(x,0). \tag{3-3}$$

It is found that (3-3) is identical to

$$u(x,t) = e^{(\int_0^t a(t)dt)\frac{\partial}{\partial x}}\ u(x,0) \tag{3-4}$$

$$u(x,0) = \sum_{n=0}^{\infty} u_n(x,0) \quad . \tag{3-5}$$

Now, if $a(t) = a + \cos t$, the general solution (3-4) has the form

$$u(x,t) = e^{(\sin t)\frac{\partial}{\partial x}} \; e^{t(a\frac{\partial}{\partial x})} \; u(x,0) \tag{3-6}$$

or it can be said that for (3-2), there exists a fundamental matrix operator solution

$$\Phi(t,\tfrac{\partial}{\partial x}) = P(t,\tfrac{\partial}{\partial x}) \; e^{tR(\frac{\partial}{\partial x})} \tag{3-7}$$

such that $P(t+\omega, \frac{\partial}{\partial x})$, and $R(\frac{\partial}{\partial x})$ is a linear operator independent of t. For the example just considered,

$$P(t,\tfrac{\partial}{\partial x}) = e^{(\sin t)\frac{\partial}{\partial x}}$$

and
$$R(\tfrac{\partial}{\partial x}) = a \frac{\partial}{\partial x} \quad .$$

3.2. Linear Systems with Periodic Time Varying Coefficients

Consider a distributed parameter system described by the general partial differential equation of the form

$$\frac{\partial u(x,t)}{\partial t} = A(t,\tfrac{\partial}{\partial x}) \; u(x,t) \tag{3-8}$$

defined for $t > 0$ and $x \in \Omega$, where Ω is a subset of the line $-\infty < x < \infty$, and $\underline{u}(x,t) \in L_2^N(\Omega)$. $A(t,\frac{\partial}{\partial x})$ is a linear differential operator in $\frac{\partial}{\partial x}$ with coefficients that are periodic in t. The domain $D(A)$ of A consists of the functions $u(x,t) \in L_2^N(\Omega)$ which satisfy a set of homogeneous boundary conditions, and have a Fourier transform w.r.t. x. We assume that D(A) is dense in $L_2^N(\Omega)$ and that A is the infinitesimal generator of semigroup $\{\Phi(t,t')\}$. For obvious reasons, we will denote $\Phi(t,0)$ by $\psi(t,\frac{\partial}{\partial x})$.

We begin formally taking the Fourier transform of both sides of (3-8), thereby getting

$$\frac{\partial \bar{u}(\xi,t)}{\partial t} = A(t,i\xi)\ \bar{u}(\xi,t), \tag{3-9}$$

where

$$\bar{u}(\xi,t) \equiv \mathcal{F}\{u(x,t)\} \equiv \int_{-\infty}^{\infty} u(x,t)e^{-i\xi x}\,dx. \tag{3-10}$$

Since $\bar{u}(\xi,t)$ is independent of the coordinate variable x, the partial derivative may be replaced by the total derivative. Therefore,

$$\frac{d\bar{u}(\xi,t)}{dt} = A(t,\ i\xi)\bar{u}(\xi,t) \tag{3-11}$$

(3-11) is an ordinary differential equation with periodic coefficients of period ω, and its fundamental matrix solution by Floquet theory[17] is given by

$$\psi(t,i\xi) = P(t,i\xi)\ e^{tR(i\xi)}\ , \tag{3-12}$$

where

$$P(t+\omega,i\xi) = P(t,i\xi)\ , \tag{3-13}$$

and

$$e^{tR(i\xi)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} R^n(i\xi)\ , \tag{3-14}$$

and $R(i\xi)$ is independent of t.

Hence, given an initial function $\bar{u}(\xi,0)=\mathcal{F}\{u(x,0)\}$, where $u(x,0) \in D(A)$, the general solution $\bar{u}(\xi,t)$ can be written as

$$\bar{u}(\xi,t) = P(t,i\xi)\ e^{tR(i\xi)}\ \bar{u}(\xi,0)\ . \tag{3-15}$$

Using (3-14), (3-15) becomes

$$\bar{u}(\xi,t) = P(t,i\xi)[I + tR(i\xi) + \frac{t^2}{2!}R^2(i\xi) + \ldots]\bar{u}(\xi,0) \quad (3\text{-}16)$$

Taking the inverse Fourier transform of both sides of (3-16)

$$u(x,t) = P(t,\frac{\partial}{\partial x})[I + tR(\frac{\partial}{\partial x}) + \frac{t^2}{2}R^2(\frac{\partial}{\partial x}) + \ldots]u(x,0)$$

or $$u(x,t) = P(t,\frac{\partial}{\partial x})e^{tR(\frac{\partial}{\partial x})}\,u(x,0) \quad (3\text{-}17)$$

where $$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\xi,t)e^{i\xi x}\,d\xi \quad . \quad (3\text{-}18)$$

From the general solution (3-17), our first result is

Theorem (1). If $\psi(t,\frac{\partial}{\partial x})$ is the semigroup operator resulting from the infinitesimal generator of $A(t,\frac{\partial}{\partial x})$ in (3-8), then there exists a linear matrix operator $P(t,\frac{\partial}{\partial x})$, periodic in t with period ω, and a linear operator $R(\frac{\partial}{\partial x})$ which is independent of t, such that

$$\psi(t,\frac{\partial}{\partial x}) = P(t,\frac{\partial}{\partial x})\; e^{tR(\frac{\partial}{\partial x})} \quad (3\text{-}19)$$

It is now clear that to investigate the stability question, it is sufficient to examine the stability of the null solution of the system

$$\frac{\partial v(x,t)}{\partial t} = R(\frac{\partial}{\partial x})\; v(x,t) \quad , \quad (3\text{-}20)$$

since the general solution of (3-20) is given by

$$v(x,t) = e^{tR(\frac{\partial}{\partial x})}\; v(x,t) \quad . \quad (3\text{-}21)$$

This leads to the following theorem:

Theorem (2). For the system given in (3-8), there exists a nondingular transformation $P(t,\frac{\partial}{\partial x})$ with bounded coefficients of period ω such that

$$v(x,t) = P(t,\frac{\partial}{\partial x})\ u(x,t) \tag{3-22}$$

$$\frac{\partial v(t,x)}{\partial t} = R(\frac{\partial}{\partial x})\ v(x,t) \tag{3-23}$$

where

$$R(\frac{\partial}{\partial x}) = \frac{\partial P(t,\frac{\partial}{\partial x})}{\partial t} P^{-1}(t,\frac{\partial}{\partial x}) + P(t,\frac{\partial}{\partial x}) A(t,\frac{\partial}{\partial x}) P^{-1}(t,\frac{\partial}{\partial x}) \tag{3-24}$$

and $R(\frac{\partial}{\partial x})$ is a time independent operator.

Proof:

From Theorem (1), it is found that for a system given by (3-8), there exists a semigroup operator

$$\Psi(t,\frac{\partial}{\partial x}) = P(t,\frac{\partial}{\partial x})\ e^{tR(\frac{\partial}{\partial x})}$$

or

$$P(t,\frac{\partial}{\partial x}) = \Psi(t,\frac{\partial}{\partial x})\ e^{-tR(\frac{\partial}{\partial x})} \quad . \tag{3-25}$$

Now, making the coordinate transformation

$$v(x,t) = P(t,\frac{\partial}{\partial x})\ u(x,t) \tag{3-26}$$

or

$$u(x,t) = P^{-1}(t,\frac{\partial}{\partial x})\ v(x,t), \tag{3-27}$$

with (3-25), (3-26) becomes

$$v(x,t) = P(t,\frac{\partial}{\partial x})\ e^{-tR(\frac{\partial}{\partial x})}\ u(x,t). \tag{3-28}$$

Partially differentiating (3-28) with respect to t

$$\frac{\partial v(x,t)}{\partial t} = (A + PAP^{-1} - \Psi Re^{-tR}P^{-1})v(x,t). \tag{3-29}$$

It is necessary to show that the right hand side of

(3-29) is indeed Rv where R is defined by (3-24). Assuming the equality sign holds,

$$A + PAP^{-1} - \psi Re^{-tR} P^{-1} = R = \frac{\partial P}{\partial t} P^{-1} + PAP^{-1}$$

$$A - \psi Re^{-tR} P^{-1} = \frac{\partial P}{\partial t} P^{-1}$$

or

$$\frac{\partial P}{\partial t} = AP - \psi Re^{-tR} \quad . \tag{3-30}$$

Now,

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t} [\psi e^{-tR}] = \frac{\partial \psi}{\partial t} e^{-tR} - \psi Re^{-tR}$$

$$= A\psi e^{-tR} - \psi Re^{-tR}$$

$$= AP - \psi Re^{-tR}$$

which is indeed the right hand side of (3-30). This completes the proof.

Theorem (2) simply means that a linear distributed parameter system with periodic coefficients, as in (3-8), is reducible to a linear distributed parameter system with constant coefficients through a nonsingular transformation $P(t,\frac{\partial}{\partial x})$ with bounded coefficients of period ω. Hence, in order to study the stability question, it is only necessary to examine the spectrum of the linear operator $R(\frac{\partial}{\partial x})$. The stability theorem can now be stated as follow:

<u>Theorem (3)</u>. The null solution of $\frac{\partial u(x,t)}{\partial t} = A(t,\frac{\partial}{\partial x})u(x,t)$ with $A(t+\omega,\frac{\partial}{\partial x}) = A(t,\frac{\partial}{\partial x})$ is asymptotically stable in the large if the spectrum of $R(\frac{\partial}{\partial x}) \equiv \sigma[R(\frac{\partial}{\partial x})]$ is negative, i.e.,

$$\mathrm{Re}\,\sigma\,[R(\tfrac{\partial}{\partial x})] \leq -k, \quad k = \text{constant} > 0.$$

Furthermore, if $\| P(t,\frac{\partial}{\partial x}) \| \leq M$, its solution satisfies the inequality

$$\| u(x,t) \| \leq M\, e^{-kt} \, \| u(x,0) \| \quad .$$

<u>Proof</u>:

It follows directly from the Hille and Yosida theorem[19], when applicable to (3-23) that

$$\| e^{tR} \| \leq e^{-kt} \qquad \text{for all } t \geq 0 \; .$$

The general solution of u(x,t) can be written as

$$u(x,t) = P(t,\tfrac{\partial}{\partial x}) \; e^{tR(\frac{\partial}{\partial x})} \; u(x,0)$$

and its norm must satisfy

$$\| u(x,t) \| \leq \| P(t,\tfrac{\partial}{\partial x}) \| \, \| e^{tR(\frac{\partial}{\partial x})} \| \, \| u(x,0) \| \quad ,$$

where the norm of a linear operator A is defined by

$$\begin{aligned} \| A \| &= \sup\{ \| Ax \| \; ; \; \| x \| = 1 \} \\ &= \sup\{ \| Ax \| / \| x \| \; ; \; \| x \| \neq 0 \}, \quad x \in D(A) \; . \end{aligned}$$

Using the fact that $\| P(t,\frac{\partial}{\partial x}) \| \leq M$,

$$\| u(x,t) \| \leq M\, e^{-kt} \| u(x,0) \| \quad .$$

This completes the proof.

Theorem (1) to (3) can be easily extended to the n-dimensional system described by

$$\frac{\partial u(x_1,x_2,\cdots,x_n,t)}{\partial t} = A(t,\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n})\ u(x_1,x_2,\cdots,x_n,t) \tag{3-31}$$

with $$A(t+\omega,\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n}) = A(t,\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n}) \tag{3-32}$$

By defining the multiple Fourier transform

$$\bar{u}(\xi_1,\xi_2,\cdots,\xi_n,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1,x_2,\cdots,x_n,t)\,e^{-i(\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n)}dx_1\cdots dx_n \tag{3-33}$$

the general solution of (3-31) is given by

$$u(x_1,x_2,\cdots,x_n,t) = P(t,\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n})\,e^{tR(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n})}u(x_1,x_2,\cdots,x_n,0) \tag{3-34}$$

The null solution of (3-31) is asymptotically stable if $\max\sigma[R(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n})]$ is negative. If $\max\sigma[R(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n})]$ is positive then the null solution is unstable.

Example 4-1. The voltage $e(x,t)$ and the current $i(x,t)$, along the transmission line with periodic varying inductance and capacitance, are governed by

$$\begin{aligned} -\frac{\partial e(x,t)}{\partial x} &= Ri(x,t) + \frac{\partial[L(t)i(x,t)]}{\partial t} \\ -\frac{\partial i(x,t)}{\partial x} &= Ge(x,t) + \frac{\partial[C(t)e(x,t)]}{\partial t} \end{aligned} \tag{3-35}$$

It is more appropriate to consider the capacitance charge $u_1 = Ce$ and the inductor flux $u_2 = Li$ as the state variables in the present case. In terms of u_1 and u_2 , (3-35) can now be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{G}{C} & \frac{1}{LL}\frac{\partial}{\partial x} \\ -\frac{1}{C}\frac{\partial}{\partial x} & -\frac{R}{L} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{3-36}$$

If L and vary in the same way

$$L(t) = L_o/a(t) \quad , \quad C(t) = C_o/a(t) \tag{3-37}$$

where L_o and C_o are constants. (3-36) then takes the form

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a(t)\begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}\frac{\partial}{\partial x} \\ -\frac{1}{C_o}\frac{\partial}{\partial x} & -\frac{R}{L_o} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3-38}$$

Taking the Fourier transform on both side of (3-38)

$$\frac{d}{dt}\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = a(t)\begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}(i\xi) \\ -\frac{1}{C_o}(i\xi) & -\frac{R}{L_o} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} . \tag{3-39}$$

Since the operator $A(t,i\xi)$ and $\int_0^t A(\lambda,i\xi)d\lambda$ commute. $A(t,i\xi)$ is defined as

$$A(t,i\xi) = a(t)\begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}(i\xi) \\ -\frac{1}{C_o}(i\xi) & -\frac{R}{L_o} \end{bmatrix} . \tag{3-40}$$

The general solution to (3-39) is given by[17,20]

$$\begin{bmatrix} \bar{u}_1(\xi,t) \\ \\ \bar{u}_2(\xi,t) \end{bmatrix} = e^{\begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}(i\xi) \\ -\frac{1}{C_o}(i\xi) & -\frac{R}{L_o} \end{bmatrix} \int_0^t a(\lambda)\,d\lambda} \begin{bmatrix} \bar{u}_1(\xi,0) \\ \\ \bar{u}_2(\xi,0) \end{bmatrix} \qquad (3\text{-}41)$$

Assuming $\quad a(t) = a + \cos t$ (3-42)

where a is a constant, and taking the inverse Fourier transform, the general solution for u(x,t) is obtained as

$$\begin{bmatrix} u_1(x,t) \\ \\ u_2(x,t) \end{bmatrix} = e^{\begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{C_o}\frac{\partial}{\partial x} \\ -\frac{1}{C_o}\frac{\partial}{\partial x} & -\frac{R}{L_o} \end{bmatrix} \cos t} \; e^{ta \begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}\frac{\partial}{\partial x} \\ -\frac{1}{C_o}\frac{\partial}{\partial x} & -\frac{R}{L_o} \end{bmatrix}} \begin{bmatrix} u_1(x,t) \\ \\ u_2(x,t) \end{bmatrix}$$

(3-43)

(3-43) is in the form of (3-17) with

$$P(t,\tfrac{\partial}{\partial x}) = e^{\begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}\frac{\partial}{\partial x} \\ -\frac{1}{C_o} & -\frac{R}{L_o} \end{bmatrix} \cos t} \qquad (3\text{-}44)$$

and

$$R(\tfrac{\partial}{\partial x}) = a \begin{bmatrix} -\frac{G}{C_o} & -\frac{1}{L_o}\frac{\partial}{\partial x} \\ \\ -\frac{1}{C_o}\frac{\partial}{\partial x} & -\frac{R}{L_o} \end{bmatrix} . \qquad (3\text{-}45)$$

In order to study the stability problem of this system, it is necessary to find the spectrum of $R(\frac{\partial}{\partial x})$ according to Theorem (3). The characteristic equation of $R(\frac{\partial}{\partial x})$ is

$$R(\frac{\partial}{\partial x})\ u = \lambda u \ . \tag{3-46}$$

This reduces to

$$\frac{\partial^2 u_1}{\partial x^2} = - \alpha^2 u_1 \tag{3-47}$$

where $$- \alpha^2 = L_o C_o (\lambda/a + R/L_o)(\lambda/a + G/L_\theta) \ . \tag{3-48}$$

Note that (3-47) is now an ordinary differential equation. The typical boundary conditions are assumed to be

$$e(0,t) = e(L,t) = 0 \ , \tag{3-49}$$

i.e., the transmission line is grounded at $x = 0$ and $x = L$.

The general solution for u_1 is given by

$$u_1 = A \cos \alpha x + B \sin \alpha x \ . \tag{3-50}$$

In order to satisfy the boundary condition at $x = 0$, A must be zero. From the second boundary condition at $x = L$,

$$\sin \alpha L = 0,$$

therefore, $$\alpha L = n\pi$$

or $$\alpha = n\pi/L \ , \quad n = 1, 2, 3, \cdots \ . \tag{3-51}$$

λ/a can be found from

$$- L_oC_o(\lambda/a + R/L_o)(\lambda/a + G/C_o) = (n\pi/L)^2$$

to be

$$\lambda/a = -1/2(R/L_o + G/C_o) \pm \left\{[1/2(\frac{R}{L_o} + \frac{G}{C_o})]^2 - (\frac{RG}{L_oC_o}\ \frac{n^2\pi^2L_oC_o}{L^2})\right\}^{1/2}. \tag{3-53}$$

If R, G, L_o and C_o are positive constants, the real part of the spectrum will be negative if and only if a is positive. Hence the null solution of u_1 will be asymptotically stable according to Theorem (3).

3.3. Linear Systems with Periodic Time and Space Varying Coefficients.

Let us now consider distributed parameter systems with coefficients varying both with t and x. Such systems can generally be described by

$$\frac{\partial u}{\partial t} = A(t,x,\frac{\partial}{\partial x})\ u \tag{3-53}$$

where $A(t,x,\frac{\partial}{\partial x})$ is again a linear differential operator in $\frac{\partial}{\partial x}$ with coefficients varying both in t and x with period ω and β respectively, i.e.,

$$A(t+\omega,x+\beta,\frac{\partial}{\partial x}) = A(t,x,\frac{\partial}{\partial x}) \quad . \tag{3-54}$$

Remarks similar to the one made in the proceding section apply here.

We assume that $A(t,x,\frac{\partial}{\partial x})$ can be split into two parts either as a product or as a sum. First, consider the case where the coefficients in $A(t,x,\frac{\partial}{\partial x})$ can be separated into products of functions in t and x. Before going into the general formulation, a time invariant system is analyzed to illustratedthe technique.

Consider a second degree, first order equation

$$\frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2} + b(x)\frac{\partial u}{\partial x} \quad . \tag{3-55}$$

Since $A(x,\frac{\partial}{\partial x}) = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$ is independent of t, the general solution is given by

$$u(x,t) = e^{t[a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}]} u(x,0) . \tag{3-56}$$

Assuming $\quad a(x) \geq 0 \;, \quad b(x) \geq 0$ (3-57)

and introducing the new variables

$$x_1(x) = \int_0^x \int_0^{\lambda_2} \frac{1}{a(\lambda_1)} d\lambda_1 d\lambda_2$$

and

$$x_2(x) = \int_0^x \frac{1}{b(\lambda_1)} d\lambda_1 \quad , \tag{3-58}$$

since a(x) and b(x) are greater than zero, there a one-to-one correspondence between $x_1(x)$ and x, and likewise between $x_2(x)$ and x. Differentiating $x_1(x)$ twice and $x_2(x)$ once with respect to x,

$$a(x)\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x_1^2}$$

$$b(x)\frac{\partial}{\partial x} = \frac{\partial}{\partial x_2} \tag{3-59}$$

Considering u(x,t) now as a function of x_1, x_2 and t, with (3-59), (3-55) becomes

$$\frac{\partial u(t,x_1,x_2)}{\partial t} = \frac{\partial^2 u(t,x_1,x_2)}{\partial x_1^{\,2}} \quad \frac{\partial u(t,x_1,x_2)}{\partial x_2} \tag{3-60}$$

To solve (3-60), take Fourier transform with respect to x_1 and x_2 respectively and solve the resulting equation,

$$\bar{u}(t,\xi_1,\xi_2) = e^{t[(i\xi_1)^2 + (i\xi_2)]} \bar{u}(0,\xi_1,\xi_2) \tag{3-61}$$

Taking the inverse Fourier transform to (3-61)

$$u(t,x_1,x_2) = e^{t\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2}\right)} u(0,x_1,x_2). \tag{3-62}$$

Notice that $u(t,x_1,x_2)$ is nothing but u(x,t), and

$$\frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial}{\partial x_2} = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

therefore (3-62) can be written as

$$u(x,t) = e^{t\left[a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}\right]} u(x,0) \tag{3-63}$$

which is identical to the general solution (3-56)

This technique can be easily extended to the case where each element of $A(t,x,\frac{\partial}{\partial x})$ has the general form

$$a_{ij}(t,x,\frac{\partial}{\partial x}) = \sum_{k=0}^{p} a'_{ijk}(t)\; a''_{ijk}(x)\frac{\partial^k}{\partial x^k}$$

where $a'_{ijk}(t+\omega) = a'_{ijk}(t)$, $a''_{ijk}(x+\beta) = a''_{ijk}(x)$ (3-64)

$a''_{ijo}(x) = 1$, and $a''_{ijk}(x) > 0$ $i,j = 1,2,\cdots,n$

The main idea is to transform the system given by (3-53) and (3-64) into a system with coefficients in time only. Defining

$$x_{ijk}(x) = \int_0^x \int_0^{\lambda_k} \cdots \int_0^{\lambda_2} \frac{1}{a''_{ijk}(\lambda_1)} \, d\lambda_1 \, d\lambda_2 \cdots d\lambda_k \quad (3\text{-}65)$$

Since $a''_{ijk}(x) > 0$, there is a one-to-one correspondence between $x_{ijk}(x)$ and x. Differentiating $x_{ijk}(x)$ k times,

$$\frac{\partial^k x_{ijk}(x)}{\partial x^k} = \frac{1}{a''_{ijk}(x)}$$

or

$$a''_{ijk}(x)\frac{\partial^k}{\partial x^k} = \frac{\partial^k}{\partial x^k_{ijk}} \quad . \quad (3\text{-}66)$$

Now considering $u(x,t)$ as a function of x_{ijk}, $i,j = 1,2,\cdots,n$, $k = 1,2,\cdots,p$, the system under consideration will become

$$\frac{\partial u(t,x_{ijk}\text{:all } ijk)}{\partial t} = A(t,\frac{\partial}{\partial x_{ijk}}\text{:all } ijk)\; u(t,x_{ijk}\text{:all } ijk)$$

(3-67)

Assuming $u(x,t)$ has a Fourier transform, then $u(t,x_{ijk}\text{:all } ijk)$ will also have a multiple Fourier transform defined by

$$\bar{u}(t,\xi_{ijk}\text{:all } ijk) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} u(t,x_{ijk}\text{:all } ijk) \; \prod_{ijk} dx_{ijk}$$

(3-68)

since $x_{ijk} \rightarrow \infty$ as $x \rightarrow \infty$. As before, a multiple Fourier transform technique may be used to obtain the general solution

$$u(t,x_{ijk}:\text{all } ijk) = P(t,\frac{\partial}{\partial x_{ijk}}:ijk)\ e^{tR(\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)}\ u(0,\frac{\partial}{\partial x_{ijk}}:\text{all } ijk) \tag{3-69}$$

Again, there exists a semigroup operator for the system such that

$$\psi(t,\frac{\partial}{\partial x_{ijk}}:\text{all } ijk) = P(t,\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)\ e^{tR(\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)} \tag{3-70}$$

where $P(t+\omega,\frac{\partial}{\partial x_{ijk}}:\text{all } ijk) = P(t,\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)$

and $R(\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)$ is an operator independent of t. Given any initial function $u(o,x_{ijk}:\text{all } ijk)$, the general solution is given by (3-69). The stability theorem (3) can be extended to the system just considered.

Theorem (4). The system given by (3-53) and (3-64) is asymptotically stable if the maximum spectrum of $R(\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)$ (R defined in (3-69)) — $\max\sigma\,[R(\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)]$, is negative. If $\max\sigma\,[R(\frac{\partial}{\partial x_{ijk}}:\text{all } ijk)]$ is positive, then the system is unstable.

Next, we consider the case where

$$A(t,x,\frac{\partial}{\partial x}) = A_1(t,\frac{\partial}{\partial x}) + A_2(x,\frac{\partial}{\partial x}) \tag{3-71}$$

(3-53) can now be written as

$$\frac{\partial u}{\partial t} = A_1(t,\frac{\partial}{\partial x})u + A_2(x,\frac{\partial}{\partial x})u \tag{3-72}$$

and condition (3-54) has the form

$$\begin{aligned} A_1(t+\omega,\frac{\partial}{\partial x}) &= A_1(t,\frac{\partial}{\partial x}) \\ A_2(x+\beta,\frac{\partial}{\partial x}) &= A_2(x,\frac{\partial}{\partial x}) \end{aligned} \quad . \tag{3-73}$$

An asymptotic stability theorem for the system described by (3-72) can now be stated as follows:

Theorem (5). If A $(t,\frac{\partial}{\partial x})$ and A $(x,\frac{\partial}{\partial x})$ varying periodically in t and x with periods ω and β respectively, and $\|A_2(x,\frac{\partial}{\partial x})\| \leq a$ for some $a > 0$. Moreover, if all the solutions of the system $\frac{\partial u}{\partial t}=A_1(t,\frac{\partial}{\partial x})u$ are asymptotically stable, i.e., the maximum spectrum of $R(\frac{\partial}{\partial x})$ — $\max \sigma\,[R(\frac{\partial}{\partial x})] \leq -k < 0$, then the system (3-72) is asymptotically stable provided $k > aM$, where $\|P(t,\frac{\partial}{\partial x})\| \leq M$, and $P(t,\frac{\partial}{\partial x})$, $R(\frac{\partial}{\partial x})$ are related to the solution of $\frac{\partial u}{\partial t} = A_1(t,\frac{\partial}{\partial x})u$

$$u(x,t) = P(t,\frac{\partial}{\partial x})\; e^{tR(\frac{\partial}{\partial x})}\; u(x,0).$$

The method of proof is quite similar to that used by Cesari[22] for a result on systems described by linear time varying ordinary differential equation. Before going into the proof, an important lemma[21,22] is stated.

Lemma: If x(t) is a continuous function of t and y(t) is also a function of t integrable in a finite interval and

$$y(t) \geq 0$$
$$z(t) \geq 0; \quad 0 \leq t < \infty ,$$

moreover, if for some nonnegative constant C,

$$y(t) \leq C + \int_0^t y(\lambda)\ z(\lambda) d\lambda\ ,\ t \geq 0$$

then y(t) also satisfies the inequality

$$y(t) \leq C\ e^{\int_o^t z(\lambda)\ d\lambda} \qquad t \geq 0\ .$$

Proof: In proving the theorem, first treat $A_2(x,\frac{\partial}{\partial x})u$ as a forcing function, and consider the homogeneous system

$$\frac{\partial v}{\partial t} = A_1(t,\frac{\partial}{\partial x})v \tag{3-73}$$

From Theorem (1), one knows that there exists a semi-group operator for system (3-73)

$$\psi(t,\frac{\partial}{\partial x}) = P(t,\frac{\partial}{\partial x})\ e^{tR(\frac{\partial}{\partial x})}$$

then the general solution to (3-73) by considering $A_2(x,\frac{\partial}{\partial x})$ as a forcing function is given by

$$u(x,t) = \psi(t,\frac{\partial}{\partial x})u(x,0) + \int_0^t \psi(t-\lambda,\frac{\partial}{\partial x})A_2(x,\frac{\partial}{\partial x})u(x,\lambda)d\lambda \tag{3-74}$$

Since the system $\frac{\partial u}{\partial t} = A_1(t,\frac{\partial}{\partial x})u$ is asymptotically stable $\sigma[R(\frac{\partial}{\partial x})] < -k < 0$.

Therefore, $\left\| \psi(t,\frac{\partial}{\partial x}) \right\| \leq M\ e^{-kt}$

where $\left\| P(t,\frac{\partial}{\partial x}) \right\| \leq M$

and $\left\| e^{tR(\frac{\partial}{\partial x})} \right\| \leq e^{-kt}$.

If the norm of the initial function

$$\|u(x,0)\| \leq N < \infty$$

and
$$\|A(x,\tfrac{\partial}{\partial x})\| \leq a < \infty ,$$

then
$$\|u(x,t)\| e^{kt} \leq MNe^{-kt} + \int_0^t aMe^{-k(t-\lambda)} \; u(x,\lambda) \; d\lambda ,$$

or
$$\|u(x,t)\| e^{kt} \leq MN + \int_0^t aM \; u(x,\lambda) \; e^{k\lambda} \; d\lambda .$$

By using the lemma, this becomes

$$\|u(x,t)\| \; e^{kt} \leq MNe^{\int_0^t aM \, d\lambda}$$

$$\|u(x,t)\| \leq MN \; e^{-(k-aM)t} \quad .$$

Hence, if $k > aM$,

$$u(x,t) \to 0 \quad \text{as } t \to \infty .$$

This completes the proof.

Example 3-2. The flow of electricity in a long cable or transmission line with neglegible leakage conductance and inductance. Equations governed the transmission line with $L = G = 0$ can be written from Example 3-1 as

$$\frac{\partial v}{\partial x} = - Ri$$
$$\tag{3-75}$$
$$\frac{\partial i}{\partial x} = - C\frac{\partial e}{\partial t}$$

Now, if R is a function of t, and $R(t+\omega) = R(t)$, and C depends only on x, and $C(x) > 0$. The voltage along the line is governed by

$$\frac{\partial v}{\partial t} = \frac{1}{R(r)C(x)} \frac{\partial^2 v}{\partial x^2} \quad . \tag{3-76}$$

Defining $$x_{112}(x) = \int_0^x \int_0^{\lambda_2} C(\lambda_1)\, d\lambda_1\, d\lambda_2 \tag{3-77}$$

(3-76) can now be written as

$$\frac{\partial v}{\partial t} = \frac{1}{R(t)} \frac{\partial^2 v}{\partial x_{112}^2} \tag{3-78}$$

where e is now a function of t and x_{112} . By using a Fourier transform technique, with $1/R(t) = a^2 + \cos t$, the general solution is found to be

$$v(t, x_{112}) = e^{(a^2 t + \sin t)\frac{\partial^2}{\partial x_{112}^2}} v(0, x_{112}) \tag{3-79}$$

In this case

$$P(t, \frac{\partial}{\partial x_{112}}) = e^{(\sin t)\frac{\partial^2}{\partial x_{112}^2}}$$

and $$R(\frac{\partial}{\partial x_{112}}) = a^2 \frac{\partial^2}{\partial x_{112}^2} \quad .$$

For stability, it is necessary to examine the spectrum of $R(\frac{\partial}{\partial x_{112}})$. Hence

$$[R(\frac{\partial}{\partial x_{112}}) - \lambda I]v = 0$$

or $$a^2 \frac{\partial^2 v}{\partial x_{112}^2} = \lambda v \ .$$

If the transmission line is grounded at x = 0 and x = L, then v = 0 at x = 0 and L. This implies v = 0 at

$x_{112} = 0$ and L_{112} . With these boundary conditions,

$$\lambda = -\frac{n^2\pi^2a^2}{L_{112}^2} \qquad n = 1, 2, 3, \cdots$$

where $L_{112} = \int_0^L \int_0^{\lambda_2} C(\lambda_1)\, d\lambda_1\, d\lambda_2$.

Therefore the system is asymptotically stable.

CHAPTER IV

ABSOLUTE STABILITY OF DISTRIBUTED PARAMETER SYSTEMS

4.1. Introduction

In this chapter, the absolute stability is considered for the linear distributed parameter systems by using the second method of Liapunov, the result is further extended to the nonlinear distributed parameter systems with control described by a set of the modified Lurie type equations.

4.4. Stability of Linear Systems

In order to apply Theorem (2-3) to a practical problem, it is necessary to establish upper bounds for the real part of the spectrum of $R(\frac{\partial}{\partial x})$ or for a system with constant coefficient

$$\frac{\partial u}{\partial t} = A(\frac{\partial}{\partial x})\ u$$

the spectrum of $A(\frac{\partial}{\partial x})$. Sometimes this may be difficult to evaluate. The Liapunov's direct method enables one to find the sufficient condition for stability without explicit knowledge of the solution.

Consider an unforced linear system described by

$$\frac{\partial u(x,t)}{\partial t} = A\ \frac{\partial u(x,t)}{\partial x} + Bu(x,t), \tag{4-1}$$

where us is an n-vector, A and B are nxn constant matrices. Let the state space $\Gamma(\Omega) = L_2^{\,n}(\Omega)$ with a norm defined by

$$\|u(x,t)\| = [\int_\Omega u'u\ d\Omega]^{1/2}. \tag{4-2}$$

Select a suitable positive definite functional, the integral of a positive definite quadratic form, for system (4-1)

$$V = \int_0^L u'Pu \, dx \ , \tag{4-3}$$

where P is a positive definite matrix, and henceforth the ' on a vector matrix denotes its transpose. From now on, we will use the convention $M > 0$ to denote that a matrix M is positive definite. Positive and negative semidefinite and negative definite matrices are denoted in the same way.

Since

$$\underline{\lambda}\int_0^L \|u\|^2 dx \leq V \leq \overline{\lambda}\int_0^L \|u\|^2 dx, \tag{4-4}$$

where $\underline{\lambda}$ and $\overline{\lambda}$ are the minimum and the maximum eigenvalues of P respectively. The conditions (i) - (iii) of Zubov's theorem are automatically satisfied. A sufficient condition for stability is that the total derivative of V with respect to t

$$\frac{dV}{dt} < 0, \qquad t \geq 0 \tag{4-5}$$

or

$$\frac{dV}{dt} = \int_0^L [(u'PA\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} A''Pu) + u'(PB + B'P)u]dx \quad 0 \tag{4-6}$$

If

$$PA = A'P \tag{4-7}$$

$$\frac{dV}{dt} = \int_0^L [\frac{\partial}{\partial x}(u'PAu) - u'Qu]dx \tag{4-8}$$

where

$$- Q = PB + B'P \ . \tag{4-9}$$

The first term in (4-8) becomes zero if u(x,t) vanishes on the boundary, therefore,

$$\frac{dV}{dt} = - \int_0^L u'Qu \, dx \quad . \tag{4-10}$$

By imposing the boundary condition

$$u(0,t) = u(L,t) = 0 \qquad \text{for all } t,$$

and assuming a stable B, for the null solution of u to be asymptotically stable in the sense of norm (4-2), it is sufficient to select a symmetric $Q > 0$, to give a symmetric $P > 0$ such that $PA = A'P$. Since P is symmetric, condition (4-7) simply means that (PA) is symmetric. Now, if P is a diagonal matrix, it is sufficient to make PA symmetric by assuming a symmetric A. If A has n distinct eigenvalues, a nonsingular transformation

$$u = F\,v \tag{4-11}$$

can be employed to give a diagonal $A = FAF^{-1}$. An example will be given to illustrate the technique.

<u>Example 4-1</u>. Consider a transmission line problem, the voltage e and current i satisfy the equation

$$\begin{aligned} \frac{\partial e}{\partial t} &= - \frac{1}{C}\frac{\partial i}{\partial x} - \frac{G}{C}\,e \\ \frac{\partial i}{\partial t} &= - \frac{1}{L}\frac{\partial e}{\partial x} - \frac{R}{L}\,i \end{aligned} \tag{4-12}$$

This can be put into the vector form (4-1) with

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e \\ i \end{bmatrix}, \; A = \begin{bmatrix} 0 & -\frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -\frac{G}{C} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix} \tag{4-13}$$

B is stable if R, L, C, G are positive. A suitable Liapunov functional V is assumed to be

$$V = \int_0^L u'Pu \; dx \quad ,$$

then
$$\dot{V} = \int_0^L \left[\left(u'PA \frac{\partial u}{\partial x} + \frac{\partial u'}{\partial x} A'Pu \right) - u'Qu \right] dx \; .$$

Select a Q positive

$$Q = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix} > 0 \; ,$$

P can be found from $- Q = PB + B'P$ to be

$$P = \begin{bmatrix} \frac{C}{2} & 0 \\ 0 & \frac{L}{2} \end{bmatrix} > 0$$

such that $PA = A'P$, therefore,

$$\dot{V} = \int_0^L \left[\frac{\partial}{\partial x} (u'PA\; u) - u'Q\; u \right] dx$$

$$= u'PA\; u \Big|_0^L - \int_0^L u'Q\; u\; dx$$

or
$$\dot{V} = - \int_0^L u'Qu \; dx \; < \; 0$$

if the first term vanishes. Now,

$$u'PA\; u \Big|_0^L = - e(L)i(L) + e(0)i(0)$$

This is identically zero if

(i) both ends opened, $i(L) = i(0) = 0$,

(ii) both ends shorted, $e(L) = e(0) = 0$,

(iii) one end opened and the other end shorted, $i(L) = e(0) = 0$ or $e(L) = i(0) = 0$.

Thus shows that the null solution is asymptotically stable.

Example 4-2. Consider some variable $\phi(x,t)$ satisfying the equation

$$(\frac{\partial}{\partial t} + c_1\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x})\phi + (\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})\phi = 0. \qquad (4\text{-}14)$$

By using the state variable technique developed in section 2.2 and defining

$$u_1 = \frac{\partial}{\partial x}, \qquad u_2 = \frac{\partial}{\partial t} \quad ,$$

then (4-14) has the vector form

$$\frac{\partial u}{\partial t} = A\frac{\partial u}{\partial x} + Bu \qquad (4\text{-}15)$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -c_1c_2 & -(c_1+c_2) \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ -\lambda a & -\lambda \end{bmatrix}.$$

Employing the transformation $u = Fv$, (4-15) becomes

$$\frac{\partial v}{\partial t} = \tilde{A}\frac{\partial v}{\partial x} + \tilde{B}v$$

where $\qquad \tilde{A} = F^{-1}AF$

and $\qquad \tilde{B} = F^{-1}BF$.

The characteristic equation of matrix A is

$$|A - \lambda I| = 0$$

$$\lambda_1 = -c_1 \quad \text{and} \quad \lambda_2 = -c_2 \ .$$

F is found to be

$$F = \begin{bmatrix} 1 & 1 \\ -c_1 & -c_2 \end{bmatrix}$$

and

$$F^{-1} = \frac{1}{c_1-c_2} \begin{bmatrix} -c_2 & -1 \\ c_1 & 1 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} -c_1 & 0 \\ 0 & -c_2 \end{bmatrix} \text{ and } \tilde{B} = \frac{\lambda}{c_1-c_2} \begin{bmatrix} a-c_1 & a-c_2 \\ c_1-a & c_2-a \end{bmatrix} \ .$$

Let us consider the Liapunov functional

$$V = \int_0^L v'Pv \, dx$$

with

$$\dot{V} = v'PAv\Big|_0^L - \int_0^L v'Qv \, dx \ .$$

If the imposed boundary conditions are such that u vanishes on the boundary,

$$\dot{V} = - \int_0^L v'Qv \, dx$$

where

$$- Q = PB + B'P \ .$$

A necessary condition for $Q \geq 0$ is that $B \geq 0$, from Silvester's theorem[24]

$$\frac{\lambda(a - c_1)}{c_1 - c_2} < 0$$

and $$\frac{\lambda(c_2 - a)}{c_1 - c_2} < 0 \quad .$$

This implies

$$\lambda > 0$$

and $$c_1 > a > c_2 \quad .$$

A Q $\geq$ 0 will give a P > 0, hence for $\lambda > 0$, the solution is stable when a lies between c_1 and c_2 . This is the same result as that obtained by Whitham[24] by a different procedure.

The method developed here can be extended to the more general class of systems described by

$$\frac{\partial u}{\partial t} = A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial u}{\partial x} + Cu \tag{4-16}$$

where u is a n-vector and A, B and C are constant matrices. By selecting a Liapunov functional V in the form of (4-3)

$$V = \int_0^L u'Pu \, dx \;, \tag{4-17}$$

the total time derivative of V for the system (4-16) is given by

$$\dot{V} = \int_0^L [(u'PA\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u'}{\partial x^2}A'Pu)$$

$$+ (u'PB \frac{\partial u}{\partial x} + \frac{\partial u'}{\partial x} B'Pu)$$

$$+ (u'PCu + u'C'Pu)]dx \tag{4-18}$$

Now, if $$PB = B'P \tag{4-19}$$

$$\text{then,} \quad \dot{V} = \int_0^L [2u'PA \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(u'PBu) - u'Qu]dx$$

$$= \int_0^L [2u'PA \frac{\partial^2 u}{\partial x^{22}} - u'Qu]dx + u'PBu\Big|_0^L \tag{4-20}$$

where
$$- Q = PC + C'P \ . \tag{4-21}$$

At this point, we impose the restriction that

$u(x,t) = 0$ on the boundary, i.e., for $x=0$ and $x=L$. (4-22)

(4-22)

Hence
$$u'PBu\Big|_0^L = 0. \tag{4-23}$$

If in addition, Q is such that P, resulting from the solution of (4-21), is positive definite and

$$2u'PA \frac{\partial^2 u}{\partial x^2} - u'Qu < 0 \quad \forall\ x \in (0,L) \tag{4-24}$$

then $V > 0$ and $\dot{V} < 0$ (according to (4-21)) and hence the system is aymptotically stable.

Note that the above restrictions do not require -Q to be negative definite. In fact, in the example 4-3 below, we discuss a case in which -Q is actually positive and the system is still asymptotically stable provided the condition indicated above are satisfied.

Another set of sufficient conditions for asymptotic stability may be formulated as follows.

Assume that (4-22) holds. Then integrating (4-20) by parts, we get

$$\dot{V} = 2u'PA\frac{\partial u}{\partial x}\Big|_0^L - 2\int_0^L \frac{\partial u'}{\partial x}PA \frac{\partial u}{\partial x}\, dx - \int_0^L u'Qu\ dx \ . \tag{4-25}$$

The first term in the right side of (4-25) vanishes by virtue of (4-22).

Let $\qquad Q > 0 \; . \qquad\qquad$ (4-26)

Then by a well known theorem[1] P, resulting from the solution of (4-21), is positive definite if C is stable. (C stable = Real parts of eigenvalues of C are negative). If in addition

$$PA > 0 \; , \qquad\qquad (4\text{-}27)$$

then $\dot{V}$ in (4-25) is negative definite. Hence the system is asymptotically stable.

Summarizing our two sets of conditions:

<u>Proposition 1</u>. Let P and Q be related by (4-21) and assume that (4-22) is true. If there is a Q such that (4-24) holds, then the null solution of (4-16) is asymptotically stable.

<u>Proposition 2</u>. Assume that (4-21) and (4-22) hold. C is stable and there is a positive definite Q such that for P resulting from (4-21), (4-27) is true, then the null solution of (4-16) is asymptotically stable.

<u>Example 4-3</u>. Consider the following partial differential equation which appears in nuclear reactor physics.

$$\frac{\partial c_A}{\partial t} = D_{AB} \frac{\partial^2 c_A}{\partial x^2} + Kc_A \qquad\qquad (4\text{-}28)$$

with

$$c_A(0,t) = c_A(L,t) = 0 \; , \qquad\qquad (4\text{-}29)$$

where c_A is a scalar function of x and t, D_{AB} and K are constants greater than zero. In this case

$$u = c_A \;, \; A = D_{AB} \;, \; B = 0, \text{ and } C = K \;. \qquad (4\text{-}30)$$

Our boundary conditions on c_A coincide with condition (4-22). Select

$$Q = -\frac{2K}{D_{AB}} \;, \qquad (4\text{-}31)$$

then by (4-21)

$$P = 1/D_{AB} \;. \qquad (4\text{-}32)$$

For this Q, (4-24) takes the form

$$c_A \frac{\partial^2 c_A}{\partial x^2} + \frac{K}{D_{AB}} c_A{}^2 < 0, \qquad (4\text{-}33)$$

or

$$\frac{\partial^2 c_A}{\partial x^2} + \frac{K}{D_{AB}} c_A < 0, \quad c_A > 0 \qquad (4\text{-}34)$$

$$\frac{\partial^2 c_A}{\partial x^2} + \frac{K}{D_{AB}} c_A > 0, \quad c_A < 0 \qquad (4\text{-}35)$$

It is clear that our boundary conditions together with the differential inequalities (4-34) and (4-35) imply that c_A is of the form

$$c_A = \left(\sum_{n=1}^{\infty} \alpha_n(t) \sin \frac{n\pi}{L} x\right), \; n=1,2,3,\cdots. \qquad (4\text{-}36)$$

By the above representation we are constructing a domain of functions which is dense in H and satisfies the conditions of the Hille-Yosida theorem stated in Chapter II.

Now, if $c_A > 0$, we need

$$\sum_{n=1}^{\infty} \left\{ \left[-\left(\frac{n\pi}{L}\right)^2 + \frac{K}{D_{AB}}\right] \alpha_n(t) \sin \frac{n\pi}{L} x \right\} < 0 \qquad (4\text{-}37)$$

In order for this inequality to hold

$$\frac{K}{D_{AB}} < \left(\frac{\pi}{L}\right)^2 \tag{4-38}$$

or

$$L < \pi\left(\frac{D_{AB}}{K}\right)^{1/2} . \tag{4-39}$$

Similarly, we can prove that this condition is need to establish (4-35).

Thus (4-39) is then our condition for the asymptotic stability of (4-28) and (4-29).

4.3. Stability of Nonlinear Systems

Consider a nonlinear distributed parameter system governed by a vector partial differential equation differing from a linear equation with constant coefficients (4-1) only by an additively-entering nonlinear function, its argument is any one or a linear combination of the state variables.

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= A\,\frac{\partial u(x,t)}{\partial x} + Bu(x,t) + b\xi \\ \xi &= -\,\phi(\sigma) \\ \sigma &= c'u(x,t) \end{aligned} \tag{4-40}$$

and the nonlinearity satisfies the inequality

$$0 \le \frac{\phi(\sigma)}{\sigma} < \infty \tag{4-41}$$

where x, t, u, A abd B are defined as before, b and c are constant n-vector, σ denotes the output and the nonlinear function $\phi(\sigma)$ the control. This is the extended Lurie type of equation. ((4-40) can be rewritten as

$$\frac{\partial u(x,t)}{\partial t} = A\,\frac{\partial u(x,t)}{\partial x} + Bu(x,t) - b\phi(\sigma)$$

$$\frac{\partial \sigma}{\partial t} = a_o'\,\frac{\partial u(x,t)}{\partial x} + b_o'u(x,t) - \rho_o\phi(\sigma) \qquad (4\text{-}42)$$

where $\quad a_o = A'c ,$

$$b_o = B'c , \qquad (4\text{-}43)$$

and $\quad \rho_o = c'b .$

It is of interest to examine the stability of the system in terms of known parameters. Consider a Liapunov functional

$$V = \int_0^L [u'Pu + \int_0^\sigma \phi(\sigma)\,d\sigma]\,dx \qquad (4\text{-}44)$$

where $P > 0$ and $P' = P$. This is a positive definite function, it vanishes only when $u = \sigma = 0$. The total derivative of V is then

$$\begin{aligned}\frac{dV}{dt} &= \int_0^L [(\frac{\partial u'}{\partial x}\,A'Pu + u'PA\,\frac{\partial u}{\partial x}) \\ &\quad + u'(PB + B'P)u \\ &\quad + \phi(\sigma)(-b'Pu - u'Pb + b_o'u) - \rho_o\phi^2(\sigma) \\ &\quad + \phi(\sigma)a_o'\frac{\partial u}{\partial x}]\,dx \\ &= \int_0^L \{(\frac{\partial u'}{\partial x}\,A'Pu + u'PA\,\frac{\partial u}{\partial x}) + \phi(\sigma)a_o'\,\frac{\partial u}{\partial x} \\ &\quad - [u'Qu + 2\phi(\sigma)u'd_o + \rho_o\phi^2(\sigma)]\ dx \end{aligned} \qquad (4\text{-}45)$$

where $$d_o = Pb - b_\theta/2 \ . \qquad (4\text{-}46)$$

Select a $Q > 0$ to give a $P > 0$ such that $PA = A'P$, the integral

$$\int_0^L [(\frac{\partial u'}{\partial x} A'Pu + u'PA \frac{\partial u}{\partial x}) + \phi(\sigma) a_o' \frac{\partial u}{\partial x}] dx$$

$$= u'PAu \Big|_0^L + \int_{u(0,t)}^{u(L,t)} \phi(\sigma) c'A \, du$$

$$= 0 \qquad (4\text{-}47)$$

if u vanishes on the boundary. Therefore,

$$\frac{dV}{dt} = - \int_0^L [u'Qu + 2\phi(\sigma)u'd_o + \rho_o \phi^2(\sigma)] dx \ . \qquad (4\text{-}48)$$

The integrand is a negative definite quadratic form in u and $\phi(\sigma)$ if

$$\rho_o > d_o'Q^{-1}d \ . \qquad (4\text{-}49)$$

Hence the nonlinear system given in (4-40) and (4-41) is asymptotically stable if

(i) the linear system without control is asymptotically stable, and

(ii) $\rho_\theta > d_o'Q^{-1}d$.

CHAPTER V

BOUNDED INPUT BOUNDED OUTPUT STABILITY

5.1. Introduction

A physical system is said to be stable in the sense of boundedness if and only if every bounded input produces a bounded output. This is certainly different from the concept of stability due to Liapunov which deals with the local phenomena about a particular motion. The stability of distributed parameter systems will be discussed by the application of the second method of Liapunov. The stability theorem due to Malkin is extended to distributed systems. The bounded input bounded output stability is demonstrated by explicitly constructing a Liapunov functional V for the forced distributed system. The total time derivative of V satisfies the inequality

$$\dot{V} \leq -rV + sV^{1/2}$$

where r and s are positive constants. Since this inequality implies a bounded response as time approaches infinity, therefore every bounded input will produce a bounded output. Stability in this sense is demonstrated for an asymptotically stable constant coefficient system and for a forced nonlinear system of the modified Lurie type.

5.2. Linear Systems

The system under consideration can be described by

$$\frac{\partial u(x,t)}{\partial t} = A\frac{\partial u(x,t)}{\partial x} + Bu(x,t) + bf(x,t) \tag{5-1}$$

where f(x,t) is a scalar forcing function. We will assume that the boundary conditions given in the preceding chapter hold.

Theorem 5-1. If the unforced system is asymptotically stable, then a bounded input $(|f(x,t)| < M)$ produces a bounded output.

Proof: According to Proposition 2 in the preceding chapter, for a positive definite Q, the solution of

$$-Q = PB + B'P$$

for P yields a P which is positive definite such that

$$PA = A'P$$

and

$$u'PAu\Big|_0^L = 0$$

if u vanishes on the boundary.

Consider $$V = \int_0^L u'Pu\,dx\ , \tag{5-2}$$

then for the forced system

$$\dot{V} = \int_0^L [\frac{\partial}{\partial x}(u'PAu) - u'Qu + 2fb'Pu]dx\ . \tag{5-3}$$

The first term inside the integral is zero by the assumption, therefore

$$\dot{V} = \int_0^L (-u'Qu + 2fb'Pu)dx\ .$$

Let $\underline{\lambda}$ be the minimum eigenvalue of P,

$\overline{\lambda}$ be the maximum eigenvalue of P,

$\underline{\eta}$ be the minimum eigenvalue of Q,

$\overline{\eta}$ be the maximum eigenvalue of Q.

Then it is clear that

$$\underline{\lambda} \int_0^L \|u\|^2 \, dx \leq V \leq \overline{\lambda} \int_0^L \|u\|^2 \, dx \tag{5-4}$$

and
$$\underline{\eta} \int_0^L \|u\|^2 \, dx \leq \int_0^L u'Qu \, dx \leq \overline{\eta} \int_0^L \|u\|^2 \, dx \, . \tag{5-5}$$

Thus,

$$\dot{V} \leq -\underline{\eta} \int_0^L \|u\|^2 \, dx + \int_0^L 2M|b'Pu| \, dx \, . \tag{5-6}$$

Using (5-4), (5-6) becomes

$$\dot{V} \leq -\frac{\underline{\eta}}{\overline{\lambda}} V + 2M \int_0^L |b'Pu| \, dx \, . \tag{5-7}$$

Letting $u = \rho w$, then with $\|w\| = 1$,

$$\int_0^L |b'Pu| \, dx = \int_0^L |b'Pw| \, dx = \int_0^L |b'Pw| \, dx$$

$$\leq \rho \int_0^L \sup_{\|w\|=1} |b'Pw| \, dx \, . \tag{5-8}$$

Now,
$$V^{1/2} = \left(\int_0^L u'Pu \, dx\right)^{1/2} = \left(\int_0^L \rho^2 w'Pw \, dx\right)^{1/2}$$

$$= \rho \left(\int_0^L w'Pw \, dx\right)^{1/2} \tag{5-9}$$

or
$$V^{1/2} \geq \rho \left(\int_0^L \inf_{\|w\|=1} w'Pw \, dx\right)^{1/2} . \tag{5-10}$$

Thus,
$$\int_0^L |b'Pu| \, dx \leq \frac{1}{c} V^{1/2} \tag{5-11}$$

where c is a non-zero constant such that

$$c \leq \frac{\left[\int_0^L \inf_{\|w\|=1} (w'Pw)\, dx\right]^{1/2}}{\int_0^L \sup_{\|w\|=1} (b'Pw)\, dx} \quad . \tag{5-12}$$

Thus for $\dot{V}$,

$$\dot{V} \leq -\frac{\eta}{\bar{\lambda}} V + \frac{2M}{c} V^{1/2} \quad . \tag{5-13}$$

This inequality implies a bounded response[25] which yields

$$V^{1/2}(t) \leq \frac{2\bar{\lambda}}{\eta} \frac{M}{c} + \left[V^{1/2}(t_o) - \frac{2\bar{\lambda}}{\eta} \frac{M}{c}\right] e^{-\frac{\eta}{2\bar{\lambda}}(t-t_o)}$$

hence showing that V and hence u is bounded for all t. It is worth noting that both terms in (5-13) are necessary to establish the result, and that the method is easily extended to nonlinear systems.

5.3. Nonlinear Systems

The method developed in the last section can also be used on a class of forced system problems of the modified Lurie type (4-16)

$$\frac{\partial u(x,t)}{\partial t} = A \frac{\partial u(x,t)}{\partial x} + Bu(x,t) + b$$

$$\xi = f - \ell(\sigma) \tag{5-15}$$

$$\sigma = c'u(x,t)$$

where f(x,t) is the forcing function which is assumed to be bounded by M, and the nonlinearity

$$\phi(\sigma) = \ell(\sigma)\sigma \; . \tag{5-16}$$

The unforced system (f=0) is assumed to be asymptotically stable for any constant ℓ in $0 \leq \underline{\ell} < \ell \leq \overline{\ell}$. Let there exist a Liapunov functional for the unforced system

$$V = \int_0^L [u'Pu + \beta \int_0^{\sigma} \ell(\sigma)\sigma \, d\sigma] dx \tag{5-17}$$

then,
$$\dot{V} = -\int_0^L u'\hat{Q}u \; dx + \int_0^L [(\frac{\partial u'}{\partial x} A'Pu + u'PA\frac{\partial u}{\partial x}) + (\beta \ell c'uc'A\frac{\partial u}{\partial x})] dx \tag{5-18}$$

where
$$\hat{Q} = Q + \ell(cz' + zc') + \beta \ell^2 c'bcc' \tag{5-19}$$

$$-Q = B'P + PB \tag{5-20}$$

and
$$z = Pb - \frac{\beta}{2} B'c \; . \tag{5-21}$$

If u vanishes on the boundary, choosing a $Q > 0$ to give a $P > 0$ such that $PA = A'P$, the second term in (5-18) vanishes identically. Since the unforced system is assumed to be asymptotically stable, Q is positive definite. Now,

$$\int_0^L u'(P + \frac{\beta}{2}\underline{\ell}cc')u \; dx \leq V \leq \int_0^L u'(P + \frac{\beta}{2}\overline{\ell}cc')u \; dx \quad \beta \geq 0 \tag{5-22}$$

or
$$\int_0^L u'(P + \frac{\beta}{2}\overline{\ell}cc')u \; dx \leq V \leq \int_0^L u'(P + \frac{\beta}{2}\underline{\ell}cc')u \; dx \quad \beta \leq 0 \tag{5-23}$$

It is thus clear the left-hand expressions in (5-22) and (5-23) are positive definite since the system with $f = 0$ is asymptotically stable for any constant ℓ in $\underline{\ell} \leq \ell \leq \bar{\ell}$.

Hence,

$$\underline{\lambda} \int_0^L \|u\|^2 dx \leq V \leq \bar{\lambda} \int_0^L \|u\|^2 \, dx \tag{5-24}$$

where

$$\underline{\lambda} = \begin{cases} \text{minimum eigenvalue of } P + \frac{\beta}{2}\underline{\ell}cc' & \beta \; z \geq 0 \\ \text{minimum eigenvalue of } P + \frac{\beta}{2}\bar{\ell}cc' & \beta \leq 0 \end{cases} \tag{5-25}$$

$$\bar{\lambda} = \begin{cases} \text{maximum eigenvalue of } P + \frac{\beta}{2}\bar{\ell}cc' & \beta \geq 0 \\ \text{maximum eigenvalue of } P + \frac{\beta}{2}\underline{\ell}cc' & \beta \leq 0 \, . \end{cases} \tag{5-26}$$

Now, for the forced system, same V is assumed as in (5-17), the total time derivative of V is given by

$$\dot{V} = \int_0^L [(\frac{\partial u'}{\partial x} A'Pu + u'PA \frac{\partial u}{\partial x}) + (\beta \ell c'uc'A \frac{\partial u}{\partial x})$$

$$- u'\hat{Q}u + 2fb'(P + \frac{\beta}{2} \ell cc')u]dx \, . \tag{5-27}$$

The first two terms vanish identically by our boundary conditions, therefore

$$V = \int_0^L [- u'\hat{Q}u + 2fb'(P + \frac{\beta}{2} \ell cc')u]dx \, . \tag{5-28}$$

Again, let

$\underline{\eta}$ be the minimum eigenvalue of Q,

and $\bar{\eta}$ be the maximum eigenvalue of Q.

Then it is clear that

$$\dot{V} \leq -\underline{\eta}\int_0^L \|u\|^2\, dx + \int_0^L 2M|b'\tilde{P}u|\,dx \tag{5-29}$$

where $\tilde{P} = P + \frac{\beta}{2} cc'$, (5-30)

with (5-24), $\dot{V}$ also satisfies

$$\dot{V} \leq -\frac{\underline{\eta}}{\bar{\lambda}} V + 2M\int_0^L |b'\tilde{P}u|\,dx\ . \tag{5-31}$$

Following the same derivation under last section, $\dot{V}$ is satisfied by the inequality

$$\dot{V} \leq -\frac{\underline{\eta}}{\bar{\lambda}} V + \frac{2M}{c} V^{1/2} \tag{5-32}$$

where

$$c \leq \frac{\left[\int_0^L \inf_{\|w\|=1} (w'\tilde{P}w)\ dx\right]^{1/2}}{\int_0^L \sup_{\|w\|=1} (b'\tilde{P}w)\ dx}\ . \tag{5-33}$$

Again showing that V and hence u is bounded for all t, the similar theorem is obtained.

Theorem (5-2) For the system given by (5-15), if the unforced system is asymptotically stable, then a bounded input produces a bounded output.

One is able to explicitly determine the bounds as in (5-14) by solving the relevant equations above for P and Q and hence $\underline{\lambda}$, $\bar{\lambda}$, $\underline{\eta}$, $\bar{\eta}$ and c.

5.4. Extended Malkin's Theorem

In a practical situation, it is sometimes more desirable to have a better knowledge of the stability property besides the fact that a bounded input produces a bounded output.

Given any prescribed bound on the output (state), one is interested in estimating the maximum allowable range of the input norm that will keep the output norm within the prescribed bound. The method developed here represents an extension of a theorem due to Malkin[1] to distributed parameter systems.

Consider the general system which can be represented by

$$\frac{\partial u(X,t)}{\partial t} = U(x,\frac{\partial}{\partial X},u,t) \tag{5-34}$$

where $u(X,t)$ is a n-vector, and X is a m-dimensional spatial coordinate vector. Let be an open connected subset od m-dimensional Euclidean space X^m. Assume (5-34) has a trivial solution $u \equiv 0$, which is an invariant set of the system, and let the forced system be a modification of (5-34),

$$\frac{\partial u(X,t)}{\partial t} = U(X,\frac{\partial}{\partial X},u,t) + F(X,\frac{\partial}{\partial X},u,t) \tag{5-35}$$

where $F(X,\frac{\partial}{\partial X},u,t)$ is a forcing function which is assumed bounded.

Definition. The origin is stable in the sense of Malkin, whenever for any $0 < \varepsilon < A$, there exists two numbers $\mu(\varepsilon) > 0$, and $\nu(\varepsilon) > 0$ such that if

$$\|u(X,0)\| < \mu(\varepsilon),$$

$$F(X,\frac{\partial}{\partial X},u,t) < \nu(\varepsilon) \quad \text{for all } \|u\| < \varepsilon \text{ and } t \geq 0,$$

then $\|u(X,t)\| < \varepsilon$ for all $t \geq 0$.

Theorem 5-3. Let the origin of u be an asymptotic invariant set of (5-34) with a Liapunov functional

$$V = \int_{\Omega} W \, d\Omega = \int_{\Omega} u'Pu \, d\Omega , \tag{5-36}$$

the region of attraction is $\Gamma(A)$. In addition, there is a $M > 0$ such that in $\Gamma(A)$

$$\left| \frac{\delta W}{\delta u_i} \right| \leq M, \quad i = 1, 2, \cdots , n, \quad t \geq 0 . \tag{5-37}$$

Then the origin of (5-35) is stable in the sense of Malkin.

Proof:

Let the time derivative of V along the trajectories of (5-34) be denoted by $\dot{V}$ and $\dot{V}_f$ bedthe time derivative of V along (5-35). Hence

$$\dot{V} = \int_{\Omega} \left(\frac{\delta W}{\delta u}\right) U \, d\Omega \tag{5-38}$$

and $$\dot{V}_f = \int_{\Omega} \left(\frac{\delta W}{\delta u}\right) (U + F) d\Omega . \tag{5-39}$$

Let $\int_{\Omega} \eta \, d\Omega$ be the positive lower bound for $- \dot{V}$, i.e.,

$$\int_{\Omega} \eta \, d\Omega = \inf_{\mu < \|u\| < \varepsilon} (-\dot{V}) . \tag{5-40}$$

$\dot{V}_f$ can then be reduced to

$$\begin{aligned} \dot{V}_f &\leq - \int_{\Omega} \eta \, d\Omega + \int_{\Omega} \frac{\delta W}{\delta u} F \, d\Omega \\ &\leq \int_{\Omega} (-\eta + nMF) d\Omega \\ &\leq \int_{\Omega} (-\eta + nM) d\Omega. \end{aligned} \tag{5-41}$$

Let k be a constant such that

$$0 < k < 1$$

and set $$\nu \equiv \frac{k\eta}{nM} .$$

Then (5-41) becomes

$$\dot{V}_f \leq \int_\Omega (-\eta + nM\frac{k\eta}{nM})\, d\Omega$$

$$= -(1 - k)\int_\Omega \eta \, d\Omega < 0 ,$$

hence showing that V is decreasing along every trajectory of the forced system (5-35), therefore no trajectory of (5-35) starting in the sphere $S(\mu)$ can reach $S(\varepsilon)$.

CHAPTER VI

CONCLUSIONS

In formulating the stability problem for distributed parameter system, it is customary to approximate the distributed mathematical model by a lumped parameter system by some truncation method. Although, this approach is reasonable from a practical standpoint, it often leads to unsatisfactory stability information. The result thus obtained is an approximate one, it is neither necessary nor sufficient. Therefore from the analytical point of view, it is desirable to have such information directly from a distributed model in the form of partial differential equations. In this work, the problem of stability of distributed parameter system is analyzed in the framework of partial differential equations without resorting to their approximation by ordinary differential equations. Stability conditions are derived for distributed parameter systems with periodic coefficients. Sufficient conditions are derived via the second method of Liapunov for particular classes of distributed parameter systems. The bounded input bounded output stability are also demonstrated in the framework of the Liapunov theory. In applying the second method of Liapunov to sidtributed parameter systems, the main difficulty lies in the fact that there are no general and systematic procedure for finding a suitable Liapunov functional applied to systems. Sucessful manipulation techniques have been developed for only a relatively few cases. As should be evident, the proof of the bounded input bounded output stability of distributed parameter systems resides in the fact that one is able to obtain a Liapunov functional V such that the total time derivative along the trajectories of the forced system can be writtem as

$$\dot{V} \leq -rV + sV^{1/2} \tag{6-1}$$

where r and s are constants greater than zero, and it is clear that V has bounded solutions as t approaches infinity. There is no implication of bounded input bounded output stability if a suitable Liapunov functional satisfies (6-1) cannot be found.

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