CONSTRAINED FIR FILTER DESIGN FOR 2-BAND FILTER BANKS AND ORTHONORMAL WAVELETS

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October 25, 1994

ABSTRACT

2-band paraunitary FIR filter banks can be used to generate a multiresolution analysis with compactly supported orthonormal (ON) wavelets. The filter design problem is formulated and solved (a) as a constrained L_∞ optimization problem and (b) as a constrained L_2 optimization problem which allows arbitrary compromises between an L_2 and an L_∞ approach with both of them as special cases. Additional flatness constraints can also be easily included. The L_2 and the L_∞ design are based on the Kuhn-Tucker (KT) conditions and the alternation theorem, respectively. Therefore, optimality of the solution is guaranteed. The method (a) is a simpler alternative to a known method. The method (b) solves a more general problem than the approaches known in the literature including all of them as special cases.

1. INTRODUCTION

As is well-known [2] 2-band paraunitary FIR filter banks can be used to generate a multiresolution analysis with compactly supported orthonormal (ON) wavelets. The real-valued FIR filter bank shown in Fig. 1 must satisfy the condition $H_1(z) = -z^{-N} H_0(-z^{-1})$ (N = filter degree) to have the paraunitary property. Therefore, one can concentrate on designing only one filter, e.g., the lowpass filter

$$H_0(z) = \sum_{k=0}^{N} h_0(k) z^{-k}. \tag{1}$$

Using

$$\tilde{H}(z) = H_0(z^{-1}) H_0(z) = \sum_{k=-N}^{N} h(k) z^{k}, \tag{2}$$

the paraunitary condition becomes

$$\tilde{H}(z) + \tilde{H}(-z) = 2c \tag{3}$$

which means $\tilde{H}(z)$ is a halfband filter with $h(k) = h(-k)$. Furthermore, Eq. (2) shows that $\tilde{H}(e^{j\Omega})$ must be nonnegative.

*This research was supported by Alexander von Humboldt foundation and by AFOSR under grant F49620-1-0006 funded by DARPA.

†This research was supported by DARPA under an NDSEG fellowship

‡This research was supported by AFOSR under grant F49620-1-0006 funded by DARPA

§This research was supported by BNR

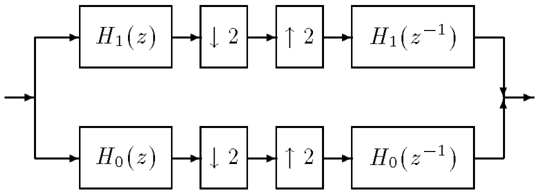

Figure 1. Paraunitary 2-band FIR filter bank

There are several ways of finding a set of filter coefficients $h_0(k)$ meeting these conditions which, together with $\sum_{k=0}^{N} h_0(k) = \sqrt{2c}$, uniquely determine a compactly supported ON wavelet basis. If the corresponding wavelet function is required to be regular, one usually computes $\tilde{H}(z)$ as a half-band maximally flat (at $z = 1$ and $z = -1$) filter [2, 6]. These filters are known to have poor frequency selectivity compared to an L_2 or L_∞ design which is often an important property in applications. On the other hand the latter filter design gives rise to wavelets with poor regularity. Obviously, there is a tradeoff between regularity and selectivity.

The L_∞ design problem can be solved by applying the Remez algorithm to the coefficients $h(k)$ — the autocorrelation of $h_0(k)$ — and spectral factorization for computing $h_0(k)$ [10]. To increase the regularity one can include additional flatness at $z = 1$ using linear programming with explicit flatness constraints. This is expensive in terms of storage and computation time [14]. These disadvantages can be overcome by a modified Remez algorithm where the positivity constraint is met by iteratively adapting the desired function [14]. The flatness constraint can be included by a relatively complicated parametrization.

There are several approaches for solving the L_2 design problem. One approach tries to minimize the stopband energy in terms of the so-called lattice coefficients [16]. This leads to a nonlinear optimization problem with the well-known difficulties of finding an initial solution and characterizing the optimal solution. The advantage of this approach is that all the necessary conditions (2), (3) mentioned above are automatically met by the lattice parameterization.

Alternatively one can try to minimize the stopband energy using the filter coefficients $h_0(k)$ and imposing the constraints

$$h(2k) = \delta_{0k}, \tag{4}$$

where δ_{0k} is the Kronecker symbol. Eq. (4) is equivalent to the above constraints (2), (3). This quadratic optimization problem with nonlinear constraints can be solved using a general purpose constrained optimization method [11] or the method of Lagrange multipliers [12]. Again there is the problem of finding an initial solution and characterization of the optimal solution.

We present two different methods for the design of paraunitary FIR 2-band filter banks that do not have the above disadvantages. In both methods we guarantee the halfband property by using the formulation

$$H(\Omega) = c + \sum_{k=1}^{J} b_k \cos[(2k-1)\Omega], \quad J = \frac{N+1}{2}. \tag{5}$$

where $H(\Omega) = \tilde{H}(\mathrm{e}^{\mathrm{j}\Omega})$ is used for convenience. Thus, it is sufficient to only consider the stopband B.

Problem: The problem we like to solve is to minimize $\|H(\Omega)\|$, where $\|\cdot\|$ denotes either the L_2 or the L_∞ norm over the stopband B, subject to the (in)equality constraints

$$\begin{aligned} B_1(\mathbf{b}, \Omega) &= \frac{d^m}{d\Omega^m} H(\Omega)|_{\Omega=\pi} = 0, & (6) \\ B_2(\mathbf{b}, \Omega) &= H(\Omega) \geq 0, \quad \Omega \in B. & (7) \end{aligned}$$

The vector $\mathbf{b}$ contains the approximation parameters b_k. Usually one requires Eq. (6) to hold for $m = 0, 2, \ldots, M$ which guarantees flatness of $H(\Omega)$ up to order $M+1$ at $\Omega = 0$ and $\Omega = \pi$. The inequality constraints (7) explicitely state the nonnegativity condition mentioned above.

The desired filter coefficients can be easily computed using the spectral factorization approach of [7].

2. DESCRIPTION OF THE PROCEDURE

2.1. L_∞ Problem

The first method solves the L_∞ problem for $\tilde{H}(z)$ by a modified Remez procedure. We apply an idea of Grenez [4] to guarantee the nonnegativity of $H(\Omega)$. In contrast to [14] the desired function does not change during the algorithm which leads to a cleaner formulation. We give a simple way to include the flatness constraints by appending explicit equations thereby avoiding the complicated parametrizations of [14]. However, the parametrization of [14] can be used in our algorithm too.

Let us first exclude the equality constraints (6) and only discuss the nonnegativity constraints (7). In [4] Grenez presents a method for constrained (by upper and lower bound tolerances) Chebyshev design for general linear phase FIR filters. He shows that the algorithm converges to the optimum solution. This idea can be adapted to our problem since the inequality constraints (7) can be interpreted as a lower bound tolerance.

The algorithm is very similar to the classical Remez algorithm [13] with the major difference that here the error has to alternate between 0 and δ (the levelled error) instead of $-\delta$ and δ. We describe a single exchange version of that algorithm.

Initialization: Find an initial solution such that $H(\Omega)$ alternates at least $J+1$ times over B. The set of these extremal frequencies is the initial reference set.

Exchange: Find that frequency Ω_i that exceeds the actual tolerance scheme $0, \delta$ the most and include it in the actual reference set. Determine a frequency to leave the actual reference set such that the alternation property is preserved.

Interpolation: Compute new coefficients b_k and a new levelled error δ such that $H(\Omega)$ alternatingly interpolates δ and 0 over the reference set.

To include the flatness constraints (6) this algorithm has to be slightly modified. First notice that a desired flatness up to order $M+1$ requires $M/2$ parameters. To accomplish this one has to reduce the number of reference points by $M/2$. Furthermore, the search for the maximum violation of the tolerance scheme has to be done with the exclusion of the frequency π.

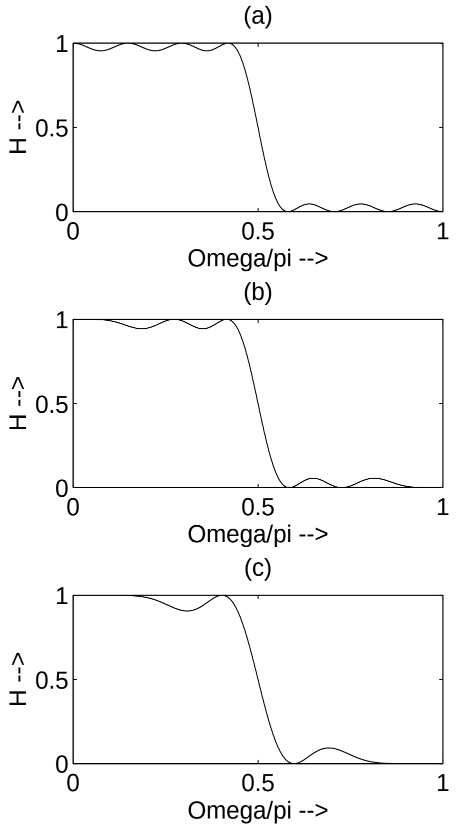

Figure 2. L_∞ design with $J = 7$ and $\Omega_s = 1137\pi/2048$. (a) $M = 0$; (b) $M = 2$; (c) $M = 4$.

Alternatively, the flatness constraints may be included similarly to the approach in [14]. There, a relatively complicated parametrization avoids the explicit inclusion of (6) and thus the numerical problems arising when dealing with higher degree of flatness. It is not clear, however, that numerical problems do not ocur elsewhere in the process of computing the filter coefficients.

Notice that we could also use frequency dependent weighting or lower bounds and a desired function larger than 0. To summarize, we present an alternative, simpler, and somewhat more general method than does [14] with similar computational and storage load.

Example 1: We give three examples for the L_∞ design with $J = 7$, stopband frequency $\Omega_s = 1137\pi/2048$ and $M = 0, 2, 4$ respectively. As can be seen in Figure 2 the ripple size increases with increasing order of flatness. In contrast to Daubechies filters, however, we have control about the selectivity of the resulting filter.

2.2. L_2 Problem

Our second method solves a constrained L_2 problem for $H(\Omega)$. To our knowledge there is no solution to this problem in the generality with which we solve it. The nonnegativity constraint is included using the ideas of [1, 9, 8, 15] for constrained filter design. Specifically we refer to [15] where a constrained L_2 approximation over the whole frequency range $[0, \pi]$ without any transition bands is considered. Notice that the lower bound constraints in (7) can be easily augmented by upper bound constraints, thus prescribing the Chebyshev error $\|H(\Omega)\|_\infty$. As is shown in [15] it is sufficient to prescribe the upper and lower bound tolerances for the local extrema. This approach has the following interesting characteristics. (1) There is no 'don't care' region (which corresponds to the fact that one usually does care if the frequency response is not monotonic in the transition band) and consequently the minimization of the error is performed over the whole frequency range. (2) Arbitrarily small tolerances can be prescribed leading to extra ripple solutions. (3) There is nothing like a stopband frequency. Instead, the transition band adjusts itself according to the tightness of the constraints.

As can be seen from our problem formulation above, we have a quadratic optimization problem subject to linear equality (6) and inequality (7) constraints. We use the necessary and sufficient Kuhn-Tucker optimality conditions[1] to construct an iterative algorithm. It basically consists of a series of *equality* constrained problems which are solved by the method of Lagrange multipliers. Each of these steps has the following form:

1. Solve the linear system of equations

$$\nabla_{\mathbf{b}}\{\|H(\Omega)\|_2^2 + \boldsymbol{\mu}^T \mathbf{B}(\mathbf{b}, \Omega)\} = \mathbf{0} \qquad (8)$$

$$\nabla_{\boldsymbol{\mu}}\{\|H(\Omega)\|_2^2 + \boldsymbol{\mu}^T \mathbf{B}(\mathbf{b}, \Omega)\} = \mathbf{B}(\mathbf{b}, \Omega) = \mathbf{0} \qquad (9)$$

   for $\mathbf{b}$ and $\boldsymbol{\mu}$ with the superscript T denoting transposition. μ are the Lagrange multipliers corresponding to the constraints $\mathbf{B}(\mathbf{b}, \Omega) = \mathbf{0}$. These constraints include all the equality constraints (6) $(\Omega = \pi)$ and all those constraints (7) with equality signs where the actual frequency response has a local extremum which violates the constraints.

2. Check whether any of the Lagrange multipliers corresponding to the constraints (7) are negative. If this is the case remove the constraint corresponding to the most negative one and repeat the first step. Else stop.

With this solution a new frequency response is computed and the corresponding set of local extrema violating the constraints is determined (similar to the Remez algorithm). Since this algorithm takes the Kuhn-Tucker optimality conditions into account convergence of the algorithm is tantamount to optimality.

Example 2: We give two examples for the L_2 design with $J = 5$ and only lower bound inequality constraints. Figure 3 shows the result for $M = 0$ (flatness up to order 1) and $M = 4$ (flatness up to order 5) respectively. The comparison with Daubechies filters of length 4 (same order of flatness as in Figure 3(b)) and length 10 (same filter degree as our filter $h_0(k)$) clearly shows that one can considerably improve the selectivity of the Daubechies filters by increasing the filter degree while keeping the order of flatness or by giving up some flatness constraints and keeping the degree.

[1] The Kuhn-Tucker conditions generalize the idea of Lagrange multipliers for equality constraints to inequality constraints. The basic difference to the method of Lagrange multipliers is that the multipliers have to be nonnegative.

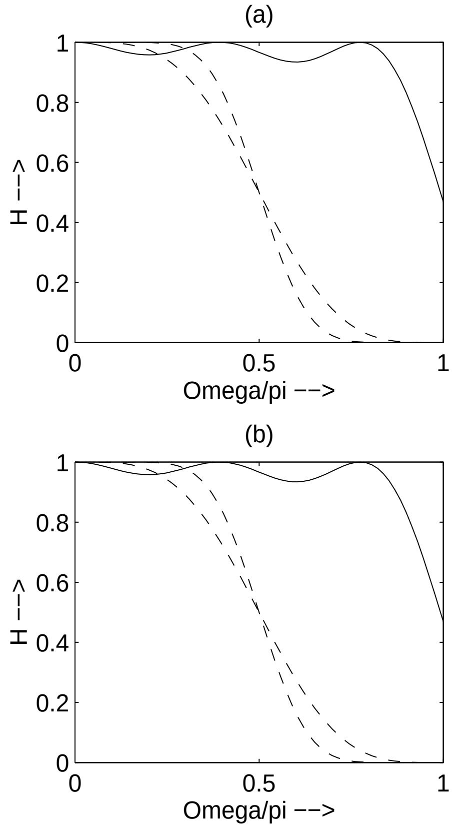

Figure 3. L_2 design for $J = 5$ and lower bound inequality constraints only. The dashed lines correspond to the Daubechies filter of length 4 and 10 respectively. (a) $M = 0$; (b) $M = 4$.

Example 3: Here we use the same design parameters as in the previous example but additionally impose the upper bound constraint $H(\Omega) \leq 0.04, \Omega \in B$. The resulting frequency responses are depicted in Figure 4 and exhibit an equiripple behaviour. Note that an arbitrarily small upper bound could be prescribed due to the fact that we do not specify a stopband frequency. Note also that arbitrary tradeoffs between the L_2 and these equiripple solutions can be attained by continously decreasing the upper bound constraint. Again there is an obvious tradeoff between flatness of the filters (Daubechies filters) and selectivity $(M = 0)$.

3. DISCUSSION AND SUMMARY

As the examples above show we have presented two very flexible and versatile algorithms for the design of paraunitary 2 band filter banks. The algorithm for an L_∞ design essentially solves the same problem as does [14]. Our formulation, however seems to be more straightforward and gives rise to a slightly more general formulation.

The second algorithm finds a constrained L_2 solution for $\tilde{H}(z)$, which corresponds to an L_4 solution for $H_0(z)$. We are currently investigating whether the results of this design can be used as initial solutions for the algorithms based on an L_2 approximation for $H_0(z)$. As mentioned above these approaches employ a nonlinear optimization approach and an L_4 design should lie relatively close to the L_2 solution. The second algorithm allows a wide range of speci-

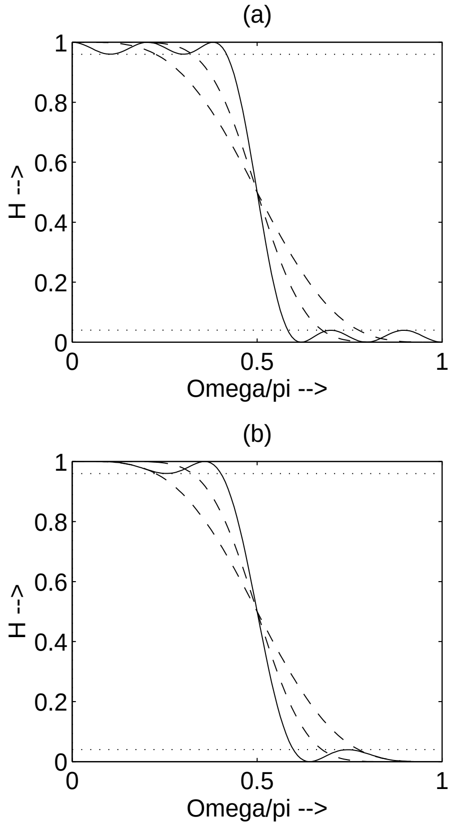

Figure 4. L_2 design for $J = 5$ and upper (0.04) as well as lower bound inequality constraints. The dashed lines correspond to the Daubechies filter of length 4 and 10 respectively. (a) $M = 0$; (b) $M = 4$.

fications. It is possible to include arbitrary linear (in the coefficients b_k) equality constraints, like the flatness constraints, and also arbitrary (possibly frequency dependent) upper bounds. As a special case we get equiripple filters. There are tradeoffs between the L_2 design, the L_∞ design and the order of flatness possible. Also, we may use arbitrary (possibly frequency dependent) weighting of the L_2 error or prescribe a stopband frequency thus giving up the possibility of prescribing arbitrarily small tolerances.

Another interesting question is the impact of the different designs on the scaling and wavelet functions and especially on their degree of regularity. Furthermore, it might be advantageous to use these filters for applications such as wavelet denoising of SAR images, enhancement of decompressed images or image compression [3, 5]. We will discuss these problems in more detail and give more examples in a planned journal version. Matlab m-files are available from the authors.

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