ADVANCES IN APPLIED MATHEMATICS **19**, 332–354 (1997)
ARTICLE NO. AM970550

Inversion Formula for Continuous Multifractals

Rudolf H. Riedi[1]

Projet Fractales, INRIA Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex, France

and

Benoit B. Mandelbrot[2]

Mathematics Department, Yale University, New Haven CT 06520-8283, USA

Received November 2, 1996; accepted December 30, 1996

Abstract

In a previous paper [MR] the authors introduced the inverse measure $\mu^\dagger$ of a probability measure μ on $[0,1]$. It was argued that the respective multifractal spectra are linked by the 'inversion formula' $f^\dagger(\alpha) = \alpha f(1/\alpha)$. Here, the statements of [MR] are put in more mathematical terms and proofs are given for the inversion formula in the case of continuous measures. Thereby, f may stand for the Hausdorff spectrum, the packing spectrum, or the coarse grained spectrum. With a closer look at the special case of self-similar measures we offer a motivation of the inversion formula as well as a discussion of possible generalizations. Doing so we find a natural extension of the scope of the notion 'self-similar' and a failure of the usual multifractal formalism. (Facsimile for personal use) ©1997 Academic Press

1 Introduction

Let μ be a probability measure on $[0,1]$ with its integral function $M(t) = \mu([0,t])$. Then, M is increasing and right-continuous. The differential of the inverse function $M^\dagger$ of M, defined as follows, is a probability measure denoted by $\mu^\dagger$:

$$\mu^\dagger([0,\theta]) := M^\dagger(\theta) := \begin{cases} \inf\{t : M(t) > \theta\} & \text{if } \theta < 1 \\ 1 & \text{if } \theta = 1. \end{cases}$$

[1]E-mail: Rolf.Riedi@inria.fr, riedi@rice.edu
[2]fractal@watson.ibm.com

We call $\mu^\dagger$ the *inverse measure* of μ. As $M^\dagger$ is increasing and right-continuous, $\mu^\dagger$ is again a probability measure.

We are interested in the relation between the spectra of μ and $\mu^\dagger$ and possible implications of such a connection. In [MR] it was argued that the respective spectra should be related by the so-called inversion formula

$$f^\dagger(\alpha) = \alpha f(1/\alpha). \tag{1}$$

The practical use of such a formula is most evident when dealing with left-sided spectra [M90, MEH, RM1] since it allows us to transform the infinite range $[\alpha_{\min}, \infty]$ of Hölder exponents of a left-sided spectrum into the finite range $[0, 1/\alpha_{\min}]$ of a right-sided spectrum.

A further application of the inversion formula is to self-similar measures which reveals telling details on the multifractal formalism. Recall that a compactly supported measure μ is called *self-similar* iff

$$\mu = \sum_{i=0}^{u-1} p_i \mu(w_i^{-1}(\cdot)), \tag{2}$$

where $w_0, \ldots, w_{u-1}$ are similarity maps of $\mathbb{R}^d$ with *contraction ratios* $r_i \in (0,1)$, and where the *probabilities* $p_i > 0$ satisfy $p_0 + \ldots + p_{u-1} = 1$. As Hutchinson [H] showed, such measures exist and are unique even under the weaker condition that the w_i are contractions.

Provided a condition on possible overlap in (2) holds, it can be shown [AP, R1, CM, F2, O] that all reasonable definitions of the multifractal spectrum of μ coincide. In particular, all spectra equal the Legendre transform $\beta^*(\alpha) := \inf_q(q\alpha - \beta(q))$ of β which is implicitly defined by

$$\sum_{i=0}^{u-1} {p_i}^q {r_i}^{-\beta(q)} = 1. \tag{3}$$

It is easy enough to verify the inversion formula (1) for self-similar measures with support $[0,1]$: In this case we have $r_0 + \ldots + r_{u-1} = 1$ due to $[0,1] = \cup_i w_i([0,1])$. A moments thought shows that the inverse measure $\mu^\dagger$ is self-similar with ratios $r_i^\dagger = p_i$, and probabilities $p_i^\dagger = r_i$, whence $q = -\beta^\dagger(q^\dagger)$, $q^\dagger = -\beta(q)$. Now, (1) follows immediately from $f(\alpha) = \inf_q(q\alpha - \beta(q))$.

Section 2 is devoted to the inversion formula in the case where μ and $\mu^\dagger$ are continuous. We introduce the *fine multifractal spectra* f_{H} and f_{P} in 2.1 and prove (1) for f_{H} and f_{P} in 2.2. In 2.3 we comment on the 'degenerated' Hölder exponents 0 and ∞. In 2.4, finally, we turn to the *coarse grained spectrum* f_{G} and the *Legendre spectrum* f_{L}, comparing them to the fine multifractal spectra and establishing (1) for f_{G}.

Revisiting the self-similar measures in Section 3 we leave the realm of continuous measures by showing that self-similarity can be naturally extended to discontinuous measures.

Doing so we find a class of invariant measures for which the multifractal formalism does not hold, which means that not all spectra coincide. This is a consequence of the fact that (1) fails here for f_G, while [RM2] establishes (1) for f_H and f_P also in the case of discontinuous measures.

Discussing possible generalizations, we compare discontinuous self-similar measures with equilibrium measures and comment on the second multifractal phenomenon found with discontinuous self-similar measures: there are 'right-sided' multifractal spectra with a tangent through the origin of slope strictly smaller than 1. This slope is directly related to the particular way of renormalizing mass in an iterative construction of discontinuous self-similar measures.

2 The inversion formula

2.1 Preliminaries

Let M be the distribution function of an arbitrary probability measure on $[0,1]$ as in Section 1. In this section, an assumption will often appear which can be stated in several equivalent ways:

- M is continuous and strictly increasing.
- $M : [0,1] \mapsto [0,1]$ is onto and one-to-one with inverse $M^\dagger$.
- μ and $\mu^\dagger$ are both continuous.
- μ is continuous and no interval of positive length has zero μ measure.

Given a number $\alpha \geq 0$, the set K_α is defined by

$$K_\alpha := \left\{ t \in [0,1] \, : \, \alpha(t) := \lim_{I \to \{t\}} \frac{\log \mu(I)}{\log |I|} \text{ exists and equals } \alpha. \right\}.$$

The limit $\alpha(t)$, if it exists, is called *Hölder exponent* of μ at t. Here, $I \to \{t\}$ means that I may run through any sequence $(I_k)_{k \in \mathbb{N}}$ of intervals such that $t \in I_k$ for all $k \in \mathbb{N}$ and such that $|I_k| \to 0$ as $k \to \infty$.

Definition 1 *The two fine multifractal spectra are the Hausdorff spectrum and the packing spectrum which are given by*

$$f_H(\alpha) = \dim(K_\alpha) \qquad \text{and} \qquad f_P(\alpha) = \mathrm{Dim}(K_\alpha)$$

respectively, where dim *and* Dim *denote the Hausdorff and the packing dimension respectively.*

For completeness, we recall the definitions of the dimensions dim and Dim. Denoting by $\eta^\gamma(E)$ the *γ-dimensional Hausdorff measure* of a set E, i.e.

$$\eta^\gamma(E) = \sup_{\delta\to 0} \eta^\gamma_\delta(E), \qquad \eta^\gamma_\delta(E) = \inf\Big\{ \sum_{\mathbb{N}} |I_k|^\gamma \,:\, E \subset \cup_{\mathbb{N}} I_k \text{ and } |I_k| \le \delta \Big\},$$

the *Hausdorff dimension* is defined as

$$\dim(E) = \inf\{\gamma \ge 0 \,:\, \eta^\gamma(E) = 0\} = \sup\{\gamma \ge 0 \,:\, \eta^\gamma(E) = \infty\}.$$

Following Tricot [Tr] one defines the *γ-dimensional packing pre-measure* by

$$\hat\pi^\gamma(E) = \inf_{\delta\to 0} \hat\pi^\gamma_\delta(E), \qquad \hat\pi^\gamma_\delta(E) = \sup\Big\{ \sum_{\mathbb{N}} |I_k|^\gamma \,:\, \{I_k\}_{\mathbb{N}} \text{ is a } \delta\text{-packing of } E \Big\}.$$

Here, a δ-packing $\{I_k\}_{\mathbb{N}}$ of E is a collection of mutually disjoint, open balls, i.e. intervals, each of length less or equal to δ and each intersecting E. Then the *γ-dimensional packing measure* is given by

$$\pi^\gamma(E) := \inf\left\{ \sum_n \hat\pi^\gamma(E_n) \,:\, E \subset \textstyle\bigcup_n E_n \right\}$$

(the sets E_n are arbitrary here) and the *packing dimension* by

$$\mathrm{Dim}(E) = \inf\{\gamma \ge 0 \,:\, \pi^\gamma(E) = 0\} = \sup\{\gamma \ge 0 \,:\, \pi^\gamma(E) = \infty\}.$$

In [MR], the inversion formula was established heuristically by a counting argument, covering K_α by $N(\varepsilon,\alpha) \simeq \varepsilon^{-f(\alpha)}$ intervals of size ε. As it was argued, M maps these ε-intervals to $N(\varepsilon,\alpha)$ intervals, each of length approximately equal to $\varepsilon^\dagger := \varepsilon^\alpha$, covering the set $K^\dagger_{\alpha^\dagger}$ of points θ with $\mu^\dagger$-Hölder exponent $\alpha^\dagger = 1/\alpha$. Thus, $N(\varepsilon,\alpha)$ should behave as $\simeq (\varepsilon^\dagger)^{-f^\dagger(1/\alpha)}$ from which the inversion formula was deduced.

This proof will become rigorous for f_{H} and f_{P} by considering coverings of K_α by arbitrary sets I. A proof for f_{G}, however, cannot follow the same lines because the coarse graining approach f_{G} estimates Hölder exponents of intervals for which a precise relation corresponding to $\alpha^\dagger = 1/\alpha$ (lemma 4) is not available.

The first step in the proof is to establish that the operation $\mu \mapsto \mu^\dagger$ is inverse to itself. This holds, though $M^\dagger$ is not everywhere inverse to M.

Lemma 2 *Fix a t from $[0,1)$. If $M(t') > M(t)$ whenever $1 \ge t' > t$, then*

$$M^\dagger(M(t)) = t.$$

Otherwise, $M^\dagger(M(t)) > t$.

Proof
By definition we have

$$M^\dagger(M(t)) = \inf\{t' : M(t') > M(t)\} \geq \sup\{t' : M(t') = M(t)\} \geq t.$$

This proves the inequality. Consider a sequence $t_n \searrow t$. If $M(t_n) > M(t)$ for all n we conclude $M^\dagger(M(t)) \leq \inf_n t_n = t$. $\diamondsuit$

Proposition 3 *We have $\mu^{\dagger\dagger} = \mu$, in other words $M^{\dagger\dagger} = M$.*

Proof
Take $t < 1$ and let $\theta := M(t)$. Recall that $M^{\dagger\dagger}(t) = \inf\{\theta' : M^\dagger(\theta') > t\}$.

Assume first that $M^{\dagger\dagger}(t) < \theta$. Then, we find $\theta' < \theta$ with $M^\dagger(\theta') > t$. Take $t' > t$ with $M^\dagger(\theta') > t'$. The definition of $M^\dagger$ implies $M(t') \leq \theta' < \theta = M(t)$, a contradiction to monotony.

Assume now that $M^{\dagger\dagger}(t) > \theta$. Then, we find $\theta' > \theta$ with $M^\dagger(\theta') \leq t$. Take $t' > t$. The definition of $M^\dagger$ implies $M(t') > \theta'$. Letting $t' \searrow t$ yields $M(t+) \geq \theta' > \theta$, a contradiction to right-continuity. $\diamondsuit$

Lemma 4 *Assume that M is onto and one-to-one, or equivalently, that μ is continuous and non-vanishing. Then,*

$$t \in K_\alpha \Leftrightarrow M(t) \in K^\dagger_{1/\alpha}.$$

Proof
Consider any interval $I^\dagger$ containing $\theta := M(t)$ and let $I := M^{-1}(I^\dagger)$. Since $I \searrow \{t\}$ iff $I^\dagger \searrow \{\theta\}$ and since

$$\frac{\log \mu^\dagger(I^\dagger)}{\log |I^\dagger|} = \frac{\log |I|}{\log \mu(I)}$$

the claim follows. $\diamondsuit$

2.2 Hausdorff and Packing spectrum

Because the operation $\mu \mapsto \mu^\dagger$ is inverse to itself (proposition 3), estimates in one direction only are sufficient. Therefore, we set

$$F_\alpha \quad = \quad \left\{ t \in [0,1] \, : \, \limsup_{I \to \{t\}} \frac{\log \mu(I)}{\log |I|} \leq \alpha \right\}$$

$$\begin{aligned} G_\alpha &= \left\{ t \in [0,1] \,:\, \liminf_{I \to \{t\}} \frac{\log \mu(I)}{\log |I|} \geq \alpha \right\} \\ K^\dagger_{\alpha^\dagger} &= \left\{ \theta \in [0,1] \,:\, \lim_{I^\dagger \to \{\theta\}} \frac{\log \mu^\dagger(I^\dagger)}{\log |I^\dagger|} = \alpha^\dagger \right\} \end{aligned}$$

and similarly for $F^\dagger_{\alpha^\dagger}$ and $G^\dagger_{\alpha^\dagger}$.

Proposition 5 *For any μ and any subset A of G_α one has*

$$\dim(A) \geq \alpha \cdot \dim(M(A)),$$

provided $0 < \alpha < \infty$.

Proof
Fix $\alpha' < \alpha$ and let

$$A_m = \left\{ t \in A \,:\, \mu(I) \leq |I|^{\alpha'} \text{ if } t \in I \text{ and } |I| \leq 1/m \right\}. \tag{4}$$

Since A is a subset of G_α we have

$$A = \bigcup_{m \geq 1} A_m.$$

Note, that for any interval I

$$|M(I)| \leq \mu(I) \tag{5}$$

even if M is not continuous. More precisely, if a is the left boundary point of an interval I, then $|M(I)| = \mu(I \backslash \{a\})$ since M is right continuous. Thus, we have equality in (5) iff I is left open or $\mu(\{a\}) = 0$.

Let $\{I_j\}_j$ be a covering of A_m by intervals of length less than $1/n$ $(n > m)$ and assume that all I_j intersect A_m. We have

$$|M(I_j)| \leq \mu(I_j) \leq |I_j|^{\alpha'} \leq (1/n)^{\alpha'}.$$

Consequently, $\{M(I_j)\}_j$ forms a covering of $M(A_m)$ by intervals of length less than $\delta_n := (1/n)^{\alpha'}$ and we find

$$\eta^{\gamma/\alpha'}_{\delta_n}\Big(M(A_m)\Big) \leq \sum |M(I_j)|^{\gamma/\alpha'} \leq \sum |I_j|^\gamma.$$

It is clear that the same estimate must hold also for arbitrary covers of A_m. Thus,

$$\eta^{\gamma/\alpha'}_{\delta_n}\Big(M(A_m)\Big) \leq \eta^\gamma_{1/n}(A_m) \leq \eta^\gamma(A_m) \leq \eta^\gamma(A)$$

which proves that $\dim(M(A_m)) \leq \dim(A)/\alpha'$. Recalling the σ-stability of Hausdorff dimension $\dim(M(A)) = \sup_m \dim(M(A_m))$, the claim follows by letting $\alpha' \nearrow \alpha$. $\diamondsuit$

Proposition 6 *Assume that μ is continuous and non-vanishing. Then*

$$\mathrm{Dim}(A) \leq \alpha \cdot \mathrm{Dim}(M(A))$$

for any subset A of F_α, provided $0 \leq \alpha < \infty$.

Proof
In its basic structure this proof is very similar to proposition 5. Note, that $\alpha = 0$ is allowed here. Fix $\alpha' > \alpha$ and let

$$A_m = \Big\{ t \in A \,:\, \mu(I) \geq |I|^{\alpha'} \text{ if } t \in I \text{ and } |I| \leq 1/m \Big\}.$$

Since A is a subset of F_α we have

$$A = \bigcup_{m \geq 1} A_m.$$

Fix m for the moment and let E_k denote an arbitrary subset of A_m. Consider a $1/n$-packing $\{I_j\}_j$ of E_k which is a collection of mutually disjoint, open intervals, each of length less or equal $1/n$ and each intersecting E_k. Since M and $M^\dagger = M^{-1}$ are continuous, the collection of all $I_j^\dagger := M(I_j)$ provides a packing of $M(E_k)$. The central estimate is here

$$|I_j^\dagger| = \mu(I_j) \geq |I_j|^{\alpha'}$$

which follows since we have equality in (5). To get the obvious argumentation started, we need an upper estimate of the length of $I_j^\dagger$. Again, we use the continuity of M; more precisely, its uniform continuity. Choose $\delta > 0$. Then there is n such that $|I| \leq 1/n$ implies $|M(I)| \leq \delta$.

In summary, $\{I_j^\dagger\}_j$ is a δ-packing of $M(E_k)$. This allows us to estimate the γ-dimensional packing pre-measure $\hat{\pi}$:

$$\hat{\pi}_\delta^\gamma\Big(M(E_k)\Big) \geq \sum |M(I_j)|^\gamma \geq \sum |I_j|^{\gamma\alpha'}.$$

Since $\{I_j\}_j$ is an arbitrary $1/n$-packing it follows that

$$\hat{\pi}_\delta^\gamma\Big(M(E_k)\Big) \geq \hat{\pi}_\delta^{\gamma\alpha'}\Big(E_k\Big) \geq \hat{\pi}^{\gamma\alpha'}\Big(E_k\Big)$$

and letting $\delta \searrow 0$, we obtain $\hat{\pi}^\gamma\Big(M(E_k)\Big) \geq \hat{\pi}^{\gamma\alpha'}\Big(E_k\Big)$.

In order to estimate the packing measure of $M(A_m)$, consider a countable cover, say $M(A_m) \subset \cup_k E_k^\dagger$. Then the sets $E_k := M^{-1}(E_k^\dagger \cap M(A_m))$ form a cover of A_m. Since $E_k \subset A_m$ for all k, the previous reasoning applies, and by the definition of the packing measure,

$$\sum_k \hat{\pi}^\gamma\Big(E_k^\dagger\Big) \geq \sum_k \hat{\pi}^\gamma\Big(E_k^\dagger \cap M(A_m)\Big) \geq \sum_k \hat{\pi}^{\gamma\alpha'}\Big(E_k\Big) \geq \pi^{\gamma\alpha'}\Big(A_m\Big).$$

Taking the infimum over all possible covers $\{E_k^\dagger\}$ of $M(A_m)$ we get, due to $A \supset A_m$,

$$\pi^\gamma\Big(M(A)\Big) \geq \pi^\gamma\Big(M(A_m)\Big) \geq \pi^{\gamma\alpha'}\Big(A_m\Big).$$

This proves that $\alpha' \cdot \mathrm{Dim}(M(A)) \geq \mathrm{Dim}(A_m)$. Finally, the σ-stability of the packing dimension, i.e. $\mathrm{Dim}(A) = \sup_m \mathrm{Dim}(A_m)$ yields the claim when letting $\alpha' \searrow \alpha$. $\diamondsuit$

Corollary 7 (Inversion formula) *Assume that M is onto and one-to-one, i.e. μ is continuous and nonvanishing. For any subset A of K_α we have*

$$\dim(A) = \alpha \cdot \dim(M(A)), \qquad \textit{and} \qquad \mathrm{Dim}(A) = \alpha \cdot \mathrm{Dim}(M(A)),$$

provided $0 < \alpha < \infty$.

This corollary implies, in particular,

$$f_{\mathrm{H}}^\dagger(\alpha^\dagger) = \dim(K_{\alpha^\dagger}^\dagger) = \dim(M(K_{1/\alpha^\dagger})) = \alpha^\dagger \dim(K_{1/\alpha^\dagger}) = \alpha^\dagger f_{\mathrm{H}}^\dagger(1/\alpha^\dagger)$$

and similar for f_{P}.

Proof
Note first that $M(A) \subset K_{1/\alpha}^\dagger$ by lemma 4, and that $M^\dagger(M(A)) = A$ by lemma 2. Applying proposition 5 once to μ and $A \subset K_\alpha \subset G_\alpha$, and once to $\mu^\dagger$ and $M(A) \subset K_{1/\alpha}^\dagger \subset G_{1/\alpha}^\dagger$ yields $\dim(A) \geq \alpha\dim(M(A)) \geq \dim(M^\dagger(M(A))) = \dim(A)$. The argument for the packing dimension is the same. $\diamondsuit$

Remark 8 Proposition 5 could be used to establish the inversion formula in general if it were not for a generalization of lemma 4 which appears to be cumbersome. In the context of [RM2] this generalization will be achieved more naturally.

Remark 9 In the definition of $K_\alpha, F_\alpha \ldots$ all the intervals are considered. In certain situations, however, it is convenient to restrict attention to a family $\mathcal{J}$ of intervals. If so, the sets $K_{\alpha^\dagger}^\dagger$, $F_{\alpha^\dagger}^\dagger$, and $G_{\alpha^\dagger}^\dagger$ have to be defined using the images by M of the intervals in $\mathcal{J}$, and the definitions of dimensions on t- and θ-axis have to be modified accordingly in order for the inversion formula to remain valid.

2.3 Hölder exponents 0 and ∞

As will be demonstrated with self-similar measures, it becomes natural to consider also the degenerate Hölder exponents 0 and ∞ when dealing with measures which can have atoms and gaps. It is worthwhile noting that these values $\alpha = 0$ and ∞ can occur not only in the trivial places where M is constant or discontinuous, but also as non–trivial limits. As an example we refer to the left sided multifractal presented in [MEH, RM1] some of which are continuous and non–vanishing and have Hölder exponent ∞ (Lebesgue) almost everywhere [RM1, Example 1].

The sets of Hölder exponent 0 or ∞ have to be treated seperately, since most of the results of the preceding section do not apply. Only the following corollary of proposition 6 is available:

Corollary 10 *Assume that μ is continuous and nonvanishing. Set*

$$K_0 := \left\{ t \in [0,1] \, : \, \lim_{I \to \{t\}} \frac{\log \mu(I)}{\log |I|} = 0 \text{ and } \mu(I) \to 0 \text{ iff } I \to \{t\} \right\}$$

Then,

$$\dim(K_0) = \mathrm{Dim}(K_0) = 0.$$

The points with Hölder exponent 0 which are not included in K_0 are the atoms. Being countable they always form a set of Hausdorff and packing dimension 0.

The corresponding inversion result would be that $M(K_0)$ has dimension 1. This is not true in general, however, as K_0 may be empty. Nevertheless, this phenomen occurs—as we just mentioned—with left sided infinitely self-similar multifractals, at least if one restricts the eligable intervals I in the definition of K_α, F_α, G_α, $\dim(\cdot)$ and $\mathrm{Dim}(\cdot)$ to the ones which occur naturally in the construction of the measure. (See the remark 9 at the end of subsection 2.2.) This fact, i.e. $\mathrm{Dim}(K_0) = 0$ and $\dim(M(K_0)) = 1$, reflects the fact that M is not Hölder continuous of any order, though it is continuous.

2.4 The Coarse grained spectrum

In applications, f_{H} and f_{P} are often hard, if not impossible to calculate, and one might prefer to work with the spectra f_{G} and f_{L} obtained by a coarse graining approach instead. We start by giving definitions and by comparing the new notions with the fine multifractal spectra. Then we collect some results from [R1, R2] which are used to show that the inversion formula (1) holds also for f_{G} in the case of continuous and non–vanishing μ. As follows from section 3 this formula fails, though, for discontinuous self-similar measures.

The *coarse grained spectrum* $f_{\mathrm{G}}(\alpha)$ is defined by

$$f_{\mathrm{G}}(\alpha) := \lim_{\varepsilon \to 0} \limsup_{\delta \to 0} \frac{\log N_\delta(\alpha, \varepsilon)}{\log 1/\delta}$$

where N_δ denotes the number of 'intervals of size δ with coarse Hölder exponent $\alpha(B) = \log \mu(B)/\log |B|$ roughly equal to α'. As was described earlier in [PR, R1], the straightforward or naive way of counting gives poor results in theory as well as in numerical application. Among the various possible improvements [R2], we favor the following for its simplicity: Let H_δ be the set of all intervals $B = [l\delta, (l+1)\delta)$ with integer l and with $\mu(B) \neq 0$, and let $B_1 := [(l-1)\delta, (l+2)\delta)$. Then

$$N_\delta(\alpha, \varepsilon) = \#\{B \in H_\delta \,:\, |B_1|^{\alpha+\varepsilon} \leq \mu(B_1) < |B_1|^{\alpha-\varepsilon}\}.$$

Though tempting it is *wrong* to interpret f_{G} as the box dimension of K_α. The truth is that K_α has the same box dimension as its topological closure which is, in the case of self-simliar measures, equal to the whole support of the measure. In fact, letting $K_{\alpha,\alpha'} := G_\alpha \cap F_{\alpha'}$ and setting

$$A_m := \{t \in K_{\alpha-\varepsilon,\alpha+\varepsilon} \,:\, |I|^{\alpha+2\varepsilon} \leq \mu(I) < |I|^{\alpha-2\varepsilon} \text{ if } t \in I \text{ and } |I| \leq 1/m\}$$

we find

$$\#\{B \in H_\delta \,:\, B \cap A_m \neq \emptyset\} \leq N_\delta(\alpha, 2\varepsilon), \tag{6}$$

provided $3\delta < 1/m$. Denoting the box dimension of a bounded set E by $\Delta(E)$, we have

$$\Delta(A_m) := \limsup_{\delta\to 0} \frac{\log \#\{B \in H_\delta \,:\, B \cap A_m \neq \emptyset\}}{\log 1/\delta} \leq \limsup_{\delta\to 0} \frac{\log N_\delta(\alpha, 2\varepsilon)}{\log 1/\delta}.$$

It is well known that $\dim(E) \leq \mathrm{Dim}(E) \leq \Delta(E)$ (see Tricot or Falconer [Tr, F1]). Together with $K_\alpha \subset K_{\alpha-\varepsilon,\alpha+\varepsilon} \subset \cup_m A_m$ and $\mathrm{Dim}(\cup_m A_m) = \sup_m \mathrm{Dim}(A_m)$, one concludes $f_{\mathrm{H}}(\alpha) \leq f_{\mathrm{P}}(\alpha) \leq f_{\mathrm{G}}(\alpha)$. If the box dimension was σ-stable like Hausdorff and packing dimension, one could argue $\Delta(\cup_m A_m) = \sup_m \Delta(A_m) \leq f_{\mathrm{G}}(\alpha)$. This is obviously not true for self-similar measures where $\cup_m A_m = \mathrm{supp}(\mu)$.

Lemma 11

$$f_{\mathrm{H}}(\alpha) \leq f_{\mathrm{P}}(\alpha) \leq f_{\mathrm{G}}(\alpha).$$

The spectrum f_{G} is related to the *partition function* $\tau(q)$

$$\tau(q) := \liminf_{\delta\to 0} \frac{\log \sum_{B\in H_\delta} \mu(B_1)^q}{\log \delta}$$

through the Legendre transform [R1]

$$\tau(q) = \inf_{\alpha\in\mathbb{R}} \Big(q\alpha - f_{\mathrm{G}}(\alpha)\Big). \tag{7}$$

This relation holds also in the much more general context of Choquet capacities (see Levy-Vehel and Vojak [LV, Thm 3]). The tentative inversion formula (1) translates to:

$$q^\dagger = -\tau \qquad\qquad \tau^\dagger = -q. \tag{8}$$

Most evidently it holds for self-similar measures (compare (3) and (10)). In general, however, (8) may fail, as is the case with discontinuous self-similar measures.

It is natural to introduce the Legendre transform of $\tau(q)$ as a further multifractal spectrum:

$$f_{\mathrm{L}}(\alpha) := \tau^*(\alpha) = \inf_{q \in \mathbb{R}} \Big(q\alpha - \tau(q) \Big).$$

An equivalent form of (7) is to say that f_{L} is the concave hull of f_{G}. Consequently:

Lemma 12 *For all* α

$$f_{\mathrm{G}}(\alpha) \leq f_{\mathrm{L}}(\alpha),$$

with equality in points of strict concavity. Moreover [R2],

$$\begin{array}{rcll} f_{\mathrm{G}}(\alpha^+) & = & q\alpha^+ - \tau(q) & \qquad (q > 0) \\ f_{\mathrm{G}}(\alpha^-) & = & q\alpha^- - \tau(q) & \qquad (q < 0) \end{array} \tag{9}$$

where $\alpha^+ := \tau'(q+)$ *and* $\alpha^- := \tau'(q-)$ *denote the one-sided derivatives of* $\tau(q)$*.*

We say that the *multifractal formalism* holds for a given measure μ if the inequalities in lemmata 11 and 12 can be replaced by equalities. To establish this formalism under various assumptions has been a point of major interest in multifractal analysis [AP, O, R1]. In general, however, the estimate (6) can clearly be sharp, meaning that an interval B can show a coarse Hölder exponent α although it contains no point t with $\alpha(t) = \alpha$.

The most simple example of this kind is the absolutely continuous measure μ with density $\phi(t) = t$ on $[0, 1]$, i.e. $M(t) = t^2/2$. Here, $\alpha(t) = 1$ for $0 < t \leq 1$ and $\alpha(0) = 2$, hence $f_{\mathrm{H}}(1) = 1$, $f_{\mathrm{H}}(2) = 0$ and K_α is empty otherwise. A direct calculation shows, on the other hand, that $f_{\mathrm{G}}(\alpha) = 2 - \alpha$ for $1 \leq \alpha \leq 2$. What seems to be a paradox is readily explained: while $\log \mu(I)/\log |I|$ tends to 1 for all $t > 0$ in the limit, a coarse graining on any 'pre-asymptotic' level $\delta > 0$ will show a non–trivial distribution of Hölder exponents. The striking difference between f_{H} and f_{G} in this example expresses the strong non-uniformity of the convergence of the Hölder exponents $\alpha(t)$. Further examples of this kind are found with the inverse measures of self-similar measures which are presented in section 3.

Consider now a continuous, non–vanishing measure μ and its inverse measure $\mu^\dagger$. In order to compute $f_{\mathrm{G}}^\dagger$ one divides the θ-axis into intervals of equal lengths. Since M and $M^\dagger$ are continuous this translates into dividing the t-axis into intervals of equal μ-measure. (Note that this is not true for discontinuous measures μ.) This kind of partitioning of the t-axis is exactly the procedure used when computing the so-called *fixed mass spectrum* f_{FM} of μ. As is shown in [R2], f_{FM} is related to f_{G} by the formula

$$f_{\mathrm{G}}(\alpha) = \alpha f_{\mathrm{FM}}(1/\alpha)$$

where f_{G} is strictly concave. We conclude:

Proposition 13 *Let μ be continuous and non–vanishing. Then the inversion formula holds for f_{G} in points α where it is strictly concave.*

Corollary 14 *Assume that μ is continuous and non–vanishing with strictly concave f_{G}. Then the multifractal formalism $f_{\mathrm{H}} = f_{\mathrm{G}}$ holds either for both, μ and $\mu^{\dagger}$, or for neither.*

3 Self-similar measures

Let μ be a self-similar measure as in (2):

$$\mu(E) = \sum_{i=0}^{u-1} p_i \mu(w_i^{-1}(E)).$$

As condition on possible overlap we will assume that $(0,1)$ satisfies the *open set condition* which means that $w_i((0,1))$ are mutually disjoint subsets of $(0,1)$. It is then easy to see that the unit interval $[0,1]$ is divided into u subintervals V_i $(i = 0, \ldots, u-1)$, length and mass of which are the r_i and p_i fraction of their 'parent interval' $[0,1]$. The same applies to the subintervals, and iteratively ad infinitum. More precisely, for all $n \in \mathbb{N}$ the mass of μ is located on u^n intervals $V_{\varepsilon_1 \ldots \varepsilon_n}$ of length $r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_n}$ and mass $p_{\varepsilon_1} \cdot \ldots \cdot p_{\varepsilon_n}$. Define the convex function $\beta(q)$ as in (3):

$$\sum_{i=0}^{u-1} {p_i}^q {r_i}^{-\beta(q)} = 1.$$

Then, the following holds (see sections 2.2 and 2.4 for notation): The partition function $\tau(q)$ equals $\beta(q)$, and the multifractal spectra all coincide. In summary,

$$f_{\mathrm{H}}(\alpha) = f_{\mathrm{P}}(\alpha) = f_{\mathrm{G}}(\alpha) = f_{\mathrm{L}}(\alpha) = \beta^*(\alpha) = \tau^*(\alpha) = \begin{cases} q\beta'(q) - \beta(q) & \text{for } \alpha = \beta'(q) \\ -\infty & \text{otherwise.} \end{cases} \tag{10}$$

First results in this direction are found with Kahane & Peyrière [KP], Cawley & Mauldin [CM], Falconer [F2], Olsen [O], and Riedi [R1]. In the stated form, (10) is a special case of the result by Arbeiter and Patzschke [AP].

3.1 The inverse of self-similar measures: Continuous case.

Here, we assume that the support of μ, denoted $\mathrm{supp}(\mu)$, is all of $[0,1]$. As a self-similar set ($\mathrm{supp}(\mu) = \cup_i w_i(\mathrm{supp}(\mu))$) it must have dimension $D = -\tau(0) = -\beta(0)$ [H]. But $D = 1$ here, which is equivalent with $\sum r_i = 1$.

In this case, the inverse measure $\mu^{\dagger}$ is obtained simply by exchanging the ratios $r_0, \ldots, r_{u-1}$ and the probabilities $p_0, \ldots, p_{u-1}$. In other words, $\mu^{\dagger}$ is self-similar with probability vector

$(r_0, \ldots, r_{u-1})$ and with the unique linear maps $w_i^\dagger$ which have the same orientation as w_i and for which $w_i^\dagger([0,1]) = [p_0 + \ldots + p_{i-1}, p_0 + \ldots + p_i]$. Since (3) establishes a one-to-one relation between β and q, we obtain $\beta^\dagger = -q$, $q^\dagger = -\beta$. Applying (10) to μ and $\mu^\dagger$ this yields (8) immediately, and the inversion formula (1) follows for all spectra by writing

$$\beta^*(\alpha) = \inf_q(q\alpha - \beta(q)) = \alpha \inf_q(q - \beta/\alpha) = \alpha \inf_q(q^\dagger/\alpha - \beta^\dagger) = \alpha \cdot (\beta^\dagger)^*(1/\alpha).$$

Proposition 15 *For self-similar measures supported on* $[0,1]$ *the inversion formula (1) holds for all four spectra* f_{H}, f_{P}, f_{G}, *and* f_{L}.

3.2 Discontinuous self-similar measures

In this case supp(μ) has dimension $D = -\tau(0) < 1$, consequently $\sum r_i < 1$. Consider

$$[0,1] \setminus \bigcup_{i=0}^{r-1} w_i([0,1]).$$

This set has at the most $u+1$ components which are open intervals. It is obvious how to define maps w_j $(j = u, \ldots, v-1)$ such that $(0,1)$ is still an open set and such that $r_0 + \ldots + r_{v-1} = 1$. We assign the probabilities $p_j = 0$ $(j = u, \ldots, v-1)$ to the new maps and define $w_i^\dagger$ as before. Then $\mu^\dagger$ is invariant under $w_0^\dagger, \ldots, w_{v-1}^\dagger$ with probability vector $(p_0^\dagger, \ldots, p_{v-1}^\dagger) = (r_0, \ldots, r_{v-1})$. As we will show in an example, the newly added maps $w_u^\dagger, \ldots, w_{v-1}^\dagger$ are constant functions and create the atoms of which $\mu^\dagger$ consists. With this procedure we have actually performed the step toward generalized self-similar measures which may include vanishing probabilities and/or vanishing contraction ratios, hence, toward *discontinuous self-similar multifractals.*

Example 1 Consider a Cantor measure μ_C, i.e. a self-similar measure with $u = 2$, $w_0(t) = r_0 t$, $w_1(t) = r_1 t + 1 - r_1$, where we assume $r_0 + r_1 < 1$, and $p_0 = p_1 = 1/2$. Then, the inverse measure $\mu_C^\dagger$ is invariant under the maps $w_0^\dagger(\theta) = \theta/2$, $w_2^\dagger(\theta) = 1/2$ and $w_1^\dagger(\theta) = \theta/2 + 1/2$ with probabilities $p_0^\dagger = r_0$, $p_1^\dagger = r_1$ and $p_2^\dagger = 1 - r_0 - r_1$. By invariance of $\mu_C^\dagger$ or directly from the definition of $M^\dagger$ it follows that $w_2^\dagger$ creates an atom at $\theta = 1/2$ of mass $p_2^\dagger$ corresponding to the gap $(r_0, 1 - r_1)$ in the support of μ_C. Iterating, we find that other atoms are present, corresponding to the gaps of supp(μ_C) at the various scales. Moreover, since the length of these gaps adds up to 1, so must the masses of the atoms and $\mu_C^\dagger$ is purely atomic. ○

An analysis of the Hölder exponents of $\mu^\dagger$ starts with the simple observation that the Hölder exponent 0 is assumed in the atoms. In other words, $\alpha^\dagger(\theta) = 0$ $\mu^\dagger$-almost surely. Alternatively, in the language of the specialist, $D_1 := -(\tau^\dagger)'(1) = 0$. Assuming that the inversion formula (1) is valid in general, it is also easy to determine the Hölder exponents

$\alpha^\dagger(\theta) \neq 0$. Instead of giving a general proof of (1), though, we would like to give an intuition of the singular behavior of $\mu^\dagger$ in points other than atoms.

To this end, one has to consider a measure $\mu_s^\dagger$ which concentrates on a suitable subset of nonatomic points. (We use the letter s instead of $q^\dagger$ for ease of notation.) This 'zooming in', however, is only useful for f_{H} and f_{P}: since they are defined pointwise they provide a 'local' analysis. It has no implication on f_{G}, which is defined in 'global' terms. The reader familiar with the usual arguments in this context (see e.g. [CM, R1]) will not be surprised that this measure $\mu_s^\dagger$ is closely related to the inverse measure of μ_q, the measure which concentrates on the points of μ-Hölder exponent $\alpha_q = \beta'(q)$. The value of q being fixed, μ_q is a self-similar measure like μ itself, with the only difference that its probabilities in (2) are ${p_i}^q {r_i}^{-\beta}$ rather than just p_i.

Translating this to $\mu^\dagger$, fix a real number s and let $\mu_s^\dagger$ be the self-similar measure invariant under $w_0^\dagger, \ldots, w_{u-1}^\dagger$ and with the probabilities

$$\overline{p}_i^\dagger := (p_i^\dagger)^s (r_i^\dagger)^{-\gamma} = {r_i}^s {p_i}^{-\gamma} \qquad (i = 0, \ldots, u-1).$$

Here, γ has to be chosen such that the new probabilities $\overline{p}_i^\dagger$ sum up to 1, i.e.

$$\sum_{i=0}^{u-1} (p_i^\dagger)^s (r_i^\dagger)^{-\gamma} = \sum_{i=0}^{u-1} {r_i}^s {p_i}^{-\gamma} = 1.$$

(We use the letter γ instead of $\beta^\dagger$ for ease of notion.) With the convention $0^x := 0$ for all x, the definition of γ generalizes (3). By (3) we find the same simple relation between the auxiliary functions of μ and $\mu^\dagger$ as in the continuous case:

$$\gamma(-\beta(q)) = -q.$$

Note, that we disregard the additional maps since we want to avoid atoms. This has the further advantage of providing a natural encoding of nonatomic points θ by infinite sequences of intervals $V^\dagger_{\varepsilon_1 \ldots \varepsilon_n}$ which are non-degenerate, i.e. of length $r^\dagger_{\varepsilon_1} \cdot \ldots \cdot r^\dagger_{\varepsilon_n} > 0$. (In the simple case of the inverse Cantor distribution, where $u = 2$ and $r_0^\dagger = r_1^\dagger = 1/2$, this is exactly the binary representation of θ.) Following the usual arguments [R1, CM] one writes the Hölder exponents $\alpha^\dagger(t)$ of $\mu^\dagger$ as

$$\alpha^\dagger(t) = \lim_{n\to\infty} \frac{\log p^\dagger_{\varepsilon_1} \cdot \ldots \cdot p^\dagger_{\varepsilon_n}}{\log r^\dagger_{\varepsilon_1} \cdot \ldots \cdot r^\dagger_{\varepsilon_n}} = \lim_{n\to\infty} \frac{(1/n)\sum_{k=1}^n \log p^\dagger_{\varepsilon_k}}{(1/n)\sum_{k=1}^n \log r^\dagger_{\varepsilon_k}}.$$

Clearly, the Hölder exponents $\alpha[\mu_s^\dagger](t)$ of $\mu_s^\dagger$ can be written in a similar fashion, replacing $p_i^\dagger$ by $\overline{p}_i^\dagger$.

The Law of Large Numbers implies now that for $\mu_s^\dagger$-almost all t

$$\alpha^\dagger(t) = \frac{\mathbb{E}_s \log p_i^\dagger}{\mathbb{E}_s \log r_i^\dagger} = \frac{\sum_{i=0}^{u-1} \overline{p}_i^\dagger \log p_i^\dagger}{\sum_{i=0}^{u-1} \overline{p}_i^\dagger \log r_i^\dagger} = \gamma'(s) =: \alpha_s^\dagger$$

and, simultaneously,

$$\alpha[\mu_s^\dagger](t) = \frac{\mathbb{E}_s \log \overline{p}_i^\dagger}{\mathbb{E}_s \log r_i^\dagger} = \frac{\sum_{i=0}^{u-1} \overline{p}_i^\dagger \log \overline{p}_i^\dagger}{\sum_{i=0}^{u-1} \overline{p}_i^\dagger \log r_i^\dagger} = s \cdot \gamma'(s) - \gamma = \gamma^*(\alpha_s^\dagger).$$

Fixing $\alpha^\dagger = \alpha_s^\dagger = \gamma'(s)$ for ease of notation, the first property implies that $K_{\alpha^\dagger}^\dagger$ has full $\mu_s^\dagger$-measure. The second property means that $\mu_s^\dagger$ is equivalent to the $\gamma^*(\alpha^\dagger)$-dimensional Hausdorff measure restricted to $K_{\alpha^\dagger}^\dagger$, allowing the estimate $\dim(K_{\alpha^\dagger}^\dagger) \geq \gamma^*(\alpha^\dagger)$. A completely rigorous argument, which is beyond the scope of this paper, is contained in [RM2]. It applies the main result of [AP] to $\mu_s^\dagger$. Finally, the usual covering methods [F2, lem 4.3] [RM1, prop 4] [RM2, thm 16] yield the upper bound for $\dim(K_{\alpha^\dagger}^\dagger)$. In summary,

Proposition 16 *The inversion formula for discontinuous self-similar measures holds for* f_{H} *and* f_{P}*:*

$$f_{\mathrm{H}}^\dagger(\alpha^\dagger) = f_{\mathrm{P}}^\dagger(\alpha^\dagger) = \gamma^*(\alpha^\dagger) = \alpha^\dagger \beta^*(1/\alpha^\dagger) = \alpha^\dagger f_{\mathrm{H}}(1/\alpha^\dagger),$$

A special role is played by the zero of γ, i.e. $\gamma(D) = 0$ where

$$\sum_{i=0}^{u-1} r_i^D = 1.$$

To the contrary with β where $\beta(1) = 0$, the zero of γ will be strictly less than 1. This is, of course, just another way of expressing that the support of μ has dimension D less than 1. Again in other words, while $\mu_1 = \mu$ none of the $\mu_s^\dagger$ will coincide with $\mu^\dagger$: A self-similar measure constructed with the probabilities $p_i^\dagger$ would 'die out'. To obtain a non-trivial distribution using $p_i^\dagger$, the mass of the intervals $V_{\varepsilon_1 \ldots \varepsilon_n}^\dagger$ had to be normalized on each level n. This could be achieved in the way it is done with equilibrium measures of dynamical systems (compare subsection 3.6) or by 'putting mass aside in atoms' as it is done with discontinuous self-similar measures. Let us be more specific.

For the Cantor distribution, e.g. the mass of $\mu_C^\dagger$ at a given level n is distributed as atoms in the dyadic points of order n and in the intermediate open intervals. The evolution of the mass in these intervals follows the rules of a multiplicative process with probabilities $p_0^\dagger$ and $p_1^\dagger$.

This has immediate and important consequences for the partition function $\tau^\dagger$: For $s > D$, the contribution coming from these 'intermediate' intervals is overwhelmed by the constant contribution of the atoms; the contrary is true for $s < D$.

Proposition 17 *For the Cantor measure* μ_C *(Example 1) and its inverse measure* $\mu_C^\dagger$ *we have*

$$\tau^\dagger(s) = \begin{cases} -\log_2({r_0}^s + {r_1}^s) & \text{for } s \leq D \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Comparing this with (3) and (10) it becomes apparent that the inversion formula (8) holds exactly in the region $s \le D$, i.e. $q \le 0$.

Proof

First note, that it is sufficient to consider grids H_n of size $\delta = 1/2^n$ [R1]. The support of $\mu_C^\dagger$ is all of $[0,1]$, so, all intervals $[(l-1)/2^n, (l+2)/2^n)$ contribute. Consider a dyadic point θ of order n, i.e. $\theta = .\varepsilon_1 \ldots \varepsilon_n$ in dyadic representation. For $\theta \neq 0$ we have

$$\mu_C^\dagger((\theta, \theta + 1/2^n)) = r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_n} \qquad \text{and} \qquad \mu_C^\dagger(\{\theta\}) = r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_{k-1}} \cdot r_2 \tag{12}$$

where $k = \max\{l \le n : \varepsilon_l = 1\}$. We may call k the *minimal dyadic order* of θ since $\theta = .\varepsilon_1 \ldots \varepsilon_k$ is the shortest possible dyadic representation of θ. From this it becomes clear that the atoms at the left boundary point dominate the measure of the intervals from H_n. Writing such intervals as $[\theta, \theta + 2^{-n})$ with θ as in the preceding text, we find

$$\begin{aligned} \sum_{\theta = .\varepsilon_1 \ldots \varepsilon_n} \mu_C^\dagger\Big([\theta, \theta + 2^{-n})\Big)^s \;&=\; \xi_n \sum_{k=0}^{n-1} \sum_{\varepsilon_1 \ldots \varepsilon_k \in \{0,1\}^k} (r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_k})^s \\ = \xi_n \sum_{k=0}^{n-1} \Big({r_0}^s + {r_1}^s\Big)^k \;&=\; \xi'_n \Big(1 - ({r_0}^s + {r_1}^s)^n\Big), \end{aligned}$$

where the error terms ξ_n and ξ'_n are bounded independently of n, i.e. ξ_n lies between r_2^s and $(r_2 + \max_i r_i)^s$, and $\xi'_n = (1 - {r_0}^s - {r_1}^s)\xi_n$. Finally, we stress that we do not have to pass to the enlarged intervals B_1 since $\mu_C^\dagger$ is supported on an interval. Instead of giving a general proof, we provide a short argument adapted to this case.

First, it follows by induction that among two *neighboring* atoms the one with the smaller 'minimal dyadic order' has the larger mass. Using this fact and denoting by θ the dyadic point with largest mass in B_1 one obtains that $\mu_C^\dagger([\theta, \theta + 2^{-n})) \le \mu_C^\dagger(B_1) \le 3\xi''_n \mu_C^\dagger(\{\theta\}) \le 3\xi''_n \mu_C^\dagger([\theta, \theta + 2^{-n}))$ with ξ''_n bounded as $\xi_n^{1/s}$. Estimating for all B in this way, one obtains a new sum where none of the θ of order n will contribute, but all of order $\le n-1$ contribute at least once and at the most three times. Hence, $\sum_{B \in H_n} \mu_C^\dagger(B_1)^s = \xi'''_n (1 - ({r_0}^s + {r_1}^s)^{n-1})$ with bounded ξ'''_n. This completes the proof. $\diamondsuit$

Proposition 18 *For the coarse grained spectrum $f_{\mathrm{G}}^\dagger$ of the inverse measure $\mu_C^\dagger$ of the Cantor distribution μ_C we find*

$$f_{\mathrm{G}}^\dagger(\alpha) = f_{\mathrm{L}}^\dagger(\alpha) = \begin{cases} D \cdot \alpha & \text{for } 0 \le \alpha \le \gamma'(D) \\ f_{\mathrm{H}}^\dagger(\alpha) & \text{for } \alpha = \gamma'(s) \text{ and } s < D \end{cases}$$

Proof

Take an arbitrary number $\nu \in (0,1]$. We will show that a lower bound on $f_{\mathrm{G}}^\dagger(\alpha)$ is found in the Legendre transform of $\nu \cdot \gamma$, i.e. in $\nu \cdot f_{\mathrm{H}}^\dagger(\alpha/\nu)$. Proposition 17 yields $\sup_{\nu \le 1} \nu f_{\mathrm{H}}^\dagger(\alpha/\nu) = f_{\mathrm{L}}^\dagger(\alpha)$, whence $f_{\mathrm{G}} \ge f_{\mathrm{L}}$, and the claim follows from lemma 12.

Again, we can restrict our attention to $\delta = 1/2^n$ [R1, R2] and to non–enlarged dyadic intervals B (see the preceding proof). From (12) we get the distribution of Hölder exponents immediately. Looking at those θ with $k = \lfloor \nu n \rfloor$, the largest integer smaller than νn, we derive the necessary estimate.

To do so, however, we will need a large deviation result of Ellis-Gärtner [E]. Define random variables $X_n = \log \mu_C^\dagger(B)$, where B is chosen randomly, i.e., each with probability $1/2^k$, out of those intervals from $H_{1/2^n}$ with left boundary point θ being dyadic of minimal order $k = \lfloor \nu n \rfloor$. First, we need the moment generating function of X_n. By (12),

$$c_n(s) := \mathbb{E}[\exp(sX_n)] = \xi_n 2^{-k} \cdot \sum_{\varepsilon_1 \ldots \varepsilon_{k-1} \in \{0,1\}^{k-1}} (r_{\varepsilon_1} \cdot \ldots \cdot r_{\varepsilon_{k-1}})^s,$$

where ξ_n is bounded. Letting $a_n := n \log 2$ we find that

$$c(s) := \lim_{n\to\infty} \frac{1}{a_n} \log c_n(s) = \lim_{n\to\infty} \frac{k-1}{n} \log_2({r_0}^s + {r_1}^s) - \frac{k}{n} = -\nu\gamma(s) - \nu$$

This being a convex and differentiable function, Ellis' theorem II.2 [E] applies. Denote by $P_n(U)$ the probability that $(1/a_n)X_n$ lies in U for a randomly picked B. If U is open, then

$$-I(U) \le \liminf_{n\to\infty} \frac{\log P_n(U)}{a_n},$$

where $I(U) := \inf\{I(\alpha) : \alpha \in U\}$ and $I(\alpha) = \sup_s(s\alpha - c(s))$. Choosing $U = (-\alpha - \varepsilon, -\alpha + \varepsilon)$ we have $P_n(U) \le 2^{-k} N_{\delta_n}(\alpha, \varepsilon)$ since $(1/a_n)X_n$ is the coarse Hölder exponent of B. Noting that

$$I(\alpha) = -\inf_s(c(s) - s\alpha) = -\nu \inf_s(s(-\alpha/\nu) - \gamma(s) - 1) = -\nu(f_{\mathrm{H}}^\dagger(-\alpha/\nu) - 1),$$

we obtain

$$\nu \sup\{f_{\mathrm{H}}^\dagger(\alpha'/\nu) : \alpha - \varepsilon < \alpha' < \alpha + \varepsilon\} \le \liminf_{n\to\infty} \frac{\log N_{\delta_n}(\alpha, \varepsilon)}{a_n}.$$

By continuity the left hand side tends to $\nu f_{\mathrm{H}}^\dagger(\alpha/\nu)$ as $\varepsilon \to 0$ from which $f_{\mathrm{G}}^\dagger(\alpha) \ge \nu f_{\mathrm{H}}^\dagger(\alpha/\nu)$. The proof is complete.

Using techniques introduced in [R2], in particular the so-called semi-spectra, one can use $f_{\mathrm{G}} \le f_{\mathrm{L}}$ and the estimate of $\liminf_{\delta\to 0}$ given previously to show that the $\limsup_{\delta\to 0}$ is actually a limit. $\diamondsuit$

Nothing is special about μ_C in propositions 17 and 18: appart from technical details the same proofs work for general self-similar measures as is shown in [RM2].

3.3 Impact on the multifractal formalism

A *weak form* of the so-called *multifractal formalism* is said to hold if

$$f_{\mathrm{G}} = f_{\mathrm{L}}.$$

(Compare lemma 12.) Examples to which the formalism applies are the 'classical' self-similar measures [AP, O, R1], as well as the discontinuous ones as we just saw for $\mu_C^\dagger$ and as is shown in general in [RM2]. The linear part we found with the spectrum $f_{\mathrm{G}}^\dagger$ of $\mu_C^\dagger$ is a consequence of the presence of a whole hierarchy of atoms which produces a non-trivial range of 'frequently occuring' coarse Hölder exponents.

The more important *strong form* of the multifractal formalism states that

$$f_{\mathrm{H}} = f_{\mathrm{G}}.$$

(Compare lemma 11.) This property has been shown to hold for quite general constructions of (random) self-similar measures (see Arbeiter and Patzschke, Olsen, and Lau [AP, O, L] and also Kahane and Peyrière, Cawley and Mauldin, and Falconer [KP, CM, F2]) as well as in the context of dynamical systems (see Rand, and Pesin and Weiss [R, PW] and also Brown, Michon and Peyrère, as well as Collet, Lebovitc and Porcio [BMP, CLP]).

For $\mu_C^\dagger$, however, we find

$$f_{\mathrm{H}}^\dagger = f_{\mathrm{P}}^\dagger \neq f_{\mathrm{G}}^\dagger = f_{\mathrm{L}}^\dagger.$$

The difference between fine multifractal spectra and coarse grained spectrum expresses, therefore, the strong dependence of the convergence rate of $\log \mu^\dagger(I)/\log|I| \to \alpha^\dagger(\theta)$ on θ. Yet, $f_{\mathrm{G}}^\dagger$ is the concave hull of $f_{\mathrm{H}}^\dagger$. This fact confirms our point of view which is to include all points of $[0,1]$, and, hence, also the vanishing Hölder exponents in the fine multifractal spectra. Otherwise, a convincing connection between $f_{\mathrm{G}}^\dagger$ and $f_{\mathrm{H}}^\dagger$ would not exist.

3.4 Conservative random case

The random self-similar measures Φ considered in [M74, KP, AP, F2, O] are obtained by randomizing the usual multiplicative process as follows. Take a codespace $\{0,\ldots,u-1\}^{\mathbb{N}}$. To each finite sequence $\mathbf{i} \in \cup_n\{0,\ldots,u-1\}^n$ assign independent random variables $r_{\mathbf{i}}$ and $p_{\mathbf{i}}$ such that $r_{i_1\ldots i_n}$ and $p_{i_1\ldots i_n}$ are of equal distribution as r_{i_n} and p_{i_n}, respectively, and such that $\sum p_i = 1$ almost surely. When assuming in addition that $\sum r_i = 1$ almost surely there is no difficulty in understanding the construction of a random self-similar measure generalizing (2). The inverse random measure $\Phi^\dagger$ is obtained simply by exchanging the random variables r_i and p_i. Doing so, corresponding realizations will indeed be inverse to each other.

Thus, provided the open set condition holds, the results of [AP, F2, O] imply the inversion formula (1) for the fine multifractal spectra f_{H} and f_{P}. Note, that we have $f_{\mathrm{H}} = f_{\mathrm{P}} = \max(f_{\mathrm{L}}, 0)$. Using large deviation principles [R2] shows that a properly defined f_{G} satisfies $f_{\mathrm{G}}(\alpha) = f_{\mathrm{L}}(\alpha)$ for all α. This yields the inversion formula for the coarse graining approach.

In [M89] negative values $f_{\mathrm{G}}(\alpha) < 0$ have been called *negative dimension* for reasons of analogy. One should keep in mind, however, that $f_{\mathrm{G}}(\alpha)$ is not a dimension in the strict sense (compare subsection 2.4). If negative, $f_{\mathrm{G}}(\alpha)$ cannot be a 'counting function' either. The correct interpretation is rather as follows: the probability that the coarse Hölder exponent $\log \mu(I)/\log |I| \simeq \alpha$ for a random measure μ and a randomly picked interval I from the δ–grid is roughly equal to $\delta^{1-f_G(\alpha)}$. Since there are only δ^{-1} such intervals one has to sample μ itself $\delta^{f_G(\alpha)}$ times in order to 'observe' the Hölder exponent α.

3.5 Higher embedding dimension

A generalization to self-similar measures in d-dimensional Euclidean space is possible in special cases. In order to carry out a construction analogous to the one dimensional case, one will assume in a first case that the measure is supported on the unit d-cube $[0,1]^d$. Then it is straightforward to define an 'inverse' measure on the θ-line, making the natural choice $p_i^\dagger = r_i^d$, $r_i^\dagger = p_i$. An adapted form of the inversion formula will hold due to (3), when adding the term d at the right places.

There is a freedom in choosing the order of the maps $w_i^\dagger$. In addition, the inverse measure will live on the interval $[0,1]$. This reflects the fact that the spectra of self-similar measures depend in fact very little on the geometry of the construction, i.e., only on the *numbers* r_i and p_i, and on respecting a separation condition.

This comes to its extreme when the measure lives on a fractal set of dimension D. One may then construct an 'inverse' self-similar measure using $p_i^\dagger = r_i^D$ (destroying the usual inversion formula) or by adding maps with zero probability as in subsection 3.2. It has to be assumed, then, that the extended family produces a tiling of the space. (See Strichartz [S, thm. 5.2] and references therein. More general cases might become treatable when considering infinite systems of maps. See Mauldin and Urbanski [MU] and [RM1].) In any case, it is not clear how to interpret the inverse measure.

A generalization to vector-valued self-similar measures [FO] in d-dimensional Euclidean space might appear more natural. Again, a procedure is only clear in very special cases and similar problems as mentioned arise. A duality as desired between two vector-valued self-similar measures can be found, e.g., in the following situation. In the notation of [FO] assume that $[0,1]^d$ is self-similar under the maps S_i $(i = 0, \ldots, u-1)$ as well as under $S_i^\dagger$ $(i = 0, \ldots, v-1)$. Let $T_i(x) := (r_i^\dagger)^d \cdot x$ and $T_i^\dagger(x) := r_i^d \cdot x$. Then, the inversion formula holds due to the results of Falconer and O'Neil [FO], again provided that the term d has been added at the right places.

3.6 Equilibrium measures

A natural generalization of the notion of self-similar measures are the equilibrium measures which appear in the theory of dynamical systems. In a typical situation on the line, one will consider a conformal mapping g which maps some disjoint intervals $I_i \subset [0,1]$ onto $[0,1]$ such that $-\log|g'|$ is negative and Hölder continuous. The invariant measure in question will then live on the repeller of g, more precisely, it will be the equilibrium measure of another Hölder continuous function ϕ. This scheme reduces to the self-similar case if g is such that the w_i are its inverse branches and if ϕ takes the constant value $\log p_i$ on I_i.

The multifractal formalism $f_{\mathrm{H}}(\alpha) = f_{\mathrm{L}}(\alpha)$ has been established for Cookie-cutters by Rand [R], and for equilibrium measures of certain Moran constructions by Pesin and Weiss [PW]. Set $\psi = \exp(\phi - P\{\phi\})$ with P denoting the pressure function, and let β be (uniquely) defined through

$$P\{q\log\psi - \beta(-\log|g'|)\} = 0.$$

Then, τ equals β, and the spectra of μ collapse with the Legendre transform β^*. Note, that the definition of β reduces to the usual one (3) in the self-similar case.

Reciprocal equilibrium measures: It is tempting to produce new measures analogously to self-similar measures, i.e., to exchange the roles of 'geometry' $-\log|g'|$ and 'mass' ϕ, and to compare this procedure with the inversion. Assume, therefore, that $\phi = -\log|h'|$ for some function h with properties analogous to g. Denote the h-invariant equilibrium measure corresponding to $\overline{\phi} := -\log|g'|$ by $\overline{\mu}$.

First, the fine multifractal spectra of $\mu^\dagger$ can be obtained through the inversion formula [RM2], hence by taking the Legendre transform of the inverse β^{-1}. In analogy with (11), especially since gaps are present, we conjecture that the partition function of $\mu^\dagger$ equals $\min\{\beta^{-1}, 0\}$.

Secondly, being an equilibrium measure, $\overline{\mu}$ has its fine multifractal spectra equal to $\overline{\beta}^*$ where, as before, $P\{t\log\overline{\psi} - \overline{\beta}(-\log|h'|)) = 0$ with $\overline{\psi} = \exp(\overline{\phi} - P\{\overline{\phi}\})$. Though very closely related, the spectra of $\mu^\dagger$ and $\overline{\mu}$ are very well distinguished, i.e. $\overline{\beta} \neq \beta^{-1}$, unless $P\{\phi\}$ and $P\{\overline{\phi}\}$ vanish. But this is the degenerate case when $\overline{\mu}$ and μ are supported on all of $[0,1]$.

Special feature of the spectra: One particular difference between the spectra of $\mu^\dagger$ and $\overline{\mu}$ is the *slope of their tangent through the origin.* Recall that this slope is the zero of $\overline{\beta}$ and β^{-1} respectively. With the continuous $\overline{\mu}$, this slope is 1 and its spectra must touch the bisector since $\overline{\tau}(1) = \overline{\beta}(1) = 0$. For $\mu^\dagger$, on the other hand, the slope of the tangent through the origin is strictly less than 1 since $\beta^\dagger(D) = 0$, $D = -\beta(0)$ being the dimension of the support of μ.

This fact reflects the fundamentally different way of dealing with the fact of 'loosing mass' when approximating the measure iteratively. With $\mu^\dagger$, loss of mass in the generating process is compensated by producing atoms (compare (12)). The contrary is true with $\overline{\mu}$ which is 'renormalized' in each step by a factor e^{-P} in order to prevent it from dying out or exploding (compare [R, p 389]). (For equilibrium measures, the sets corresponding to the intervals $V_{\varepsilon_1\ldots\varepsilon_n}$ in (12) are obtained iteratively as the components of the sets $h^{-n}([0,1])$.) This renormalization by e^{-P} brings a shift in the Hölder exponents which causes the distinct yet closely related shape of the spectra of $\mu^\dagger$ and $\overline{\mu}$.

It is this different way of compensating mass which causes the failure of the multifractal formalism for the inverse measure $\mu^\dagger$.

Acknowledgment.

We are very grateful to Professor Jacques Peyrière for supplying a first version of proposition 5 at an earlier time. RHR gratefully acknowledges financial support by the ONR, grant N00014-90.J.1026.

References

[AP] M. ARBEITER AND N. PATZSCHKE, Self-Similar Random Multifractals, *Math. Nachr.* **181** (1996) pp 5–42

[BMP] G. BROWN, G. MICHON AND J. PEYRIÈRE, On the Multifractal Analysis of Measures, *J. Stat. Phys.* **66** (1992) pp 775–790

[CM] R. CAWLEY AND R. D. MAULDIN, Multifractal Decompositions of Moran Fractals, *Adv. in Math.* **92** (1992) pp 196–236

[CLP] P. COLLET, J. LEBOVITC AND A. PORCIO, The Dimension Spectrum of Some Dynamical Systems, *J. Stat. Phys.* **47** (1987) pp 609–644.

[E] R. ELLIS, Large Deviations for a General Class of Random Vectors, *Ann. Prob.* **12** (1984) pp 1–12

[F1] K. J. FALCONER, 'Fractal Geometry: Mathematical Foundations and Applications' John Wiley and Sons, New York (1990)

[F2] K. J. FALCONER, The Multifractal Spectrum of Statistically Self-Similar Measures, *J. Theor. Prob.* **7** (1994) pp 681–702

[FO] K. J. FALCONER AND T. C. O'NEIL, Vector-valued multifractal measures, *Proc. R. Soc. Lond. A* **452** (1996) pp 1433-1457

[H] J. HUTCHINSON, Fractals and Self Similarity, *Indiana Univ. Math. J.* **30** (1981) pp 713–747

[KP] J.-P. Kahane and J. Peyrière, Sur Certaines Martingales de Benoit Mandelbrot, *Adv. Math.* **22** (1976) pp131–145

[L] K.-S. Lau and S.-M. Ngai, Multifractal Measures and a Weak Separation Condition, *Adv. Math.* accepted for publication (1995).

[LV] J. Levy-Vehel and R. Vojak, Multifractal analysis of Choquet capacities: Preliminary Results, *Adv. Appl. Math.* accepted for publication (1995).

[M] B. B. Mandelbrot, "The Fractal Geometry of Nature", New York, Freeman (1982).

[MEH] B. B Mandelbrot, C. J. G. Evertsz and Y. Hayakawa, Exactly Self-Similar Left Sided Multifractal Measures, *Physical Review A* **42** 8 (1990) pp 4528–4536

[M74] B. B. Mandelbrot, Intermittent turbulence in self similar cascades : divergence of high moments and dimension of the carrier *J. Fluid. Mech.* **62** (1974) p 331

[M89] B. B Mandelbrot, A Class of Multifractal Measures with Negative (Latent) Values for the 'Dimension' $f(\alpha)$, *Fractals' Physical Origin and Properties* edited by Luciano Pietronero, New York Plenum, 1989, pp 3–29

[M90] B. B Mandelbrot, New 'Anomalous' Multiplicative Multifractals: Left-Sided $f(\alpha)$ and the Modelling of DLA, *Physica* **A 168**, 1990, pp 95–111

[MR] B. B Mandelbrot and R. H. Riedi, Inverse Measures, the Inversion formula and Discontinuous Multifractals, *Adv. Appl. Math.* **18** (1997) pp 50–58.

[MU] R. D. Mauldin and M. Urbanski, Dimension and Measures in Infinite Iterated Function Systems, *Proc. London Math. Soc.* **73** (1996) 105–154

[O] L. Olsen, Random geometrically graph directed self-similar multifractals, Pitman Research Notes, Math. Series **307** (1994)

[PW] Y. Pesin and H. Weiss, A Multifractal Analysis of Equilibrium Measures for Conformal Expanding Maps and Moran-like Geometric Constructions, *J. Stat. Phys.* **86** (1997) 233–275

[PR] R. Pastor-Satorras and R. H. Riedi, Numerical Estimates of Generalized Dimensions $D(q)$ for Negative q, *J. Phys. A: Math. Gen.* **29** (1996) L391–L398

[P] J. Peyrière, *Personal Communication*, (1995)

[R] D. A. Rand, The Singularity Spectrum $f(\alpha)$ for Cookie-Cutters, *Erg. Th. Dyn. Sys.* **9** (1989) pp 527–541

[R1] R. H. Riedi, An Improved Multifractal Formalism and Self-Similar Measures, *J. Math. Anal. Appl.* **189** (1995) pp 462–490

[R2] R. H. RIEDI, Seven Definitions of Multifractal Scaling Exponents, *forthcoming*

[RM1] R. H. RIEDI AND B. B MANDELBROT, Multifractal Formalism for Infinite Multinomial Measures, *Adv. Appl. Math.* **16** (1995) pp 132–150

[RM2] R. H. RIEDI AND B. B MANDELBROT, Exceptions to the Multifractal Formalism for Discontinuous Measures, *Math. Proc. Cambr. Phil. Soc.* **123** (1998) 133-157.

[S] R. S. STRICHARTZ, Wavelets and Self-Affine Tilings, *Constr. Approx.* **9** (1993) pp 327–346.

[Tr] C. TRICOT, Two Definitions of Fractal Dimension, *Math. Proc. Cambr. Phil. Soc.* **91** (1982) pp 57–74.