A Domain Decomposition Method for the Acoustic Wave Equation Allowing for Discontinuous Coefficients and Grid Change by Using Hybrid Finite Element Formulation

Alain Bamberger
Roland Glowinski
Quang Huy Tran

January 1994

TR94-05

A Domain Decomposition Method for the Acoustic Wave Equation allowing for Discontinuous Coefficients and Grid Change by using Hybrid Finite Element Formulation

Alain Bamberger[†], Roland Glowinski[*] and Quang Huy Tran[†] [‡]

January 27, 1994

Abstract

A domain decomposition technique is proposed for the computation of the acoustic wave equation, in which the bulk modulus and density fields are allowed to be discontinuous at the interfaces. Inside each subdomain, the method presented coincides with the second order finite difference schemes traditionally used in geophysical modelling. However, the possibility of assigning to each subdomain its own space-step makes numerical simulations much less expensive.

Another interest of the method lies in the fact that its hybrid variational formulation naturally leads to exact equations for gridpoints on the interfaces. Transposing Babuška-Brezzi's formalism on mixed and hybrid finite elements provides a suitable functional framework for this domain decomposition formulation and shows that the inf-sup condition remains the basic requirement for convergence to occur.

Keywords. Acoustic wave equation, domain decomposition, hybrid finite element.

AMS(MOS) subject classification: 35L05, 65M12, 65M55.

---

[*]Dept. of Mathematics, University of Houston, 4800 Calhoun Rd, Houston, TX 77204-3476, USA.

[†]Institut Français du Pétrole, DIMA-DER, B.P. 311, 92506 Rueil-Malmaison Cedex, France.

[‡]*Presently at*: Dept. of C&AM, Rice University, P.O. Box 1892, Houston, TX 77251-1892, USA.

1. Introduction

1.1 Geophysical context

Over the past twenty years, explicit finite difference methods have been the standard techniques for the simulation of acoustic wave propagation. Kelly et al. [11] first advocated their use to produce synthetic seismograms. In the present state of the art, their implementation on supercomputers not only enables geophysicists to create huge synthetic databases [18], but also supplies scientists with a convenient tool for any iterative inverse process or study on acoustic waves [17]. Although there have been developed a lot of variants such as higher-order differencing [5] and Fourier spectral methods [12], the two most classical schemes remain the so-called *2-2 and 2-4 finite difference schemes.* We will have the opportunity to recall them in greater details.

The starting point of the present work comes from the following remark. Both 2-2 and 2-4 methods involve a uniform grid, the spacing of which is determined by the lowest velocity and the cut-off frequency of the source wavelet. Generally speaking, the space-step must be chosen so that the shortest wavelength contains at least some fixed number of gridpoints [1]. This condition is meant to ensure accuracy for the numerical solution. Nevertheless, when applied to cases such as layered-media with strongly contrasting velocities, this accuracy condition results in a considerable loss of efficiency: since all layers must be as finely sampled as the one corresponding to the lowest velocity, a significant amount of extra gridpoints have to be updated at every time-step. As will be shown later on, this turns out to be particularly annoying in media with a thin water layer on the top.

Ideally, it should be possible to discretize layers with higher velocities by coarser grids, within the requirement of the accuracy condition that should be considered separately for each layer. This physically makes sense, but could be achieved only on the condition that an appropriate scheme is devised so as to compute gridpoints on interfaces. Besides, such a scheme would have to be exact if accuracy is to be preserved. In this respect, simple-mindedly designed tricks such as interpolation or extrapolation often fall short of our requirement. By means of domain decomposition techniques, however, the equations for updating gridpoints on interfaces can be found in a rigorous way.

The primary purpose of this paper is to show how domain decomposition ideas can successfully cope with the problem of grid change for the acoustic wave. As a matter of fact, there exist some previous works by Meza and Symes [15] and Lions [14] on domain decomposition methods for the wave equation. Their methods are inspired from the Schwarz alternating procedure and hence involves overlapping computational subdomains. Despite their advantages of simplicity, overlapping methods do not appear to us as a suitable strategy for the wave problem. On one hand, it somehow seems awkward that the computational subdomains do not square with the physical layers. On the other hand, overlapping methods do not minimize the computational amount, which is the key issue in geophysical modelling.

We wish to propose a nonoverlapping method based on the introduction of auxiliary Lagrange multipliers at the interfaces. This approach bears a formal resemblance to the hybrid finite element methods proposed by Raviart and Thomas [16] for elliptic problems. We will lay out firm foundations to our method by generalizing Babuška-Brezzi's theory of mixed and hybrid finite elements to this peculiar hyperbolic case. Such an attempt has never been made before, to the best of our knowledge.

Prior to going through the nitty-gritty of the method, it is necessary for us to state the problem in a more precise and quantitative fashion.

1.2 Modelling background

Let Ω be a bounded open domain of $\mathbf{R}^2$, the boundary of which is $\partial\Omega$. Any point of Ω is denoted by $x = (x_1, x_2)$, while t represents the time. For $T > 0$, consider the **C**lassical **P**roblem

(CP) GIVEN

$$\rho,\ K,\ f,\ \tilde{u}_0,\ \text{and}\ \tilde{u}_0'\ \text{smooth enough on}\ \Omega.$$

FIND

$$u \in C^2([0,T] \times \Omega, \mathbf{R}) \quad \text{such that}$$

- the acoustic wave equation is satisfied in $\Omega \times]0,T[$

$$\frac{1}{K(x)}\,\ddot{u}(x,t) - \nabla.\left(\frac{1}{\rho(x)}\nabla u\right)(x,t) = f(x,t)\,, \tag{1.1}$$

- the boundary condition is satisfied on $\partial\Omega \times]0,T[$

$$\frac{1}{\sqrt{K(x)\rho(x)}}\,\dot{u}(x,t) + \frac{1}{\rho(x)}\nabla u(x,t)\,.\,n(x) = 0\,, \tag{1.2}$$

- the initial conditions are satisfied at $t = 0$

$$u(x,0) = \tilde{u}_0(x) \quad \text{and} \quad \dot{u}(x,0) = \tilde{u}_0'(x)\,. \tag{1.3}$$

In the above equations K represents the bulk modulus, ρ the density, f the source wavelet, and u the unknown pressure. Equation (1.2) is the first order absorbing boundary condition [8] to which it is for the moment advisable not to pay much attention. Furthermore, it is convenient to define the velocity c and the acoustic impedance σ as

$$c = \sqrt{\frac{K}{\rho}} \qquad \text{and} \qquad \sigma = \sqrt{K\rho}\,. \tag{1.4}$$

Let us temporarily assume, for simplicity, that the density ρ is uniform, so that Eq. (1.1) boils down to

$$\frac{1}{c^2(x)}\,\ddot{u}(x,t) - \Delta u(x,t) = g(x,t)\,, \tag{1.5}$$

where Δ denotes the Laplacian and $g = \rho f$. Suppose Ω is rectangular, so it can be divided into squares of side h. Likewise, the interval $[0,T]$ is cut into pieces of length Δt. At time-step n and on each vertex (i,j) of the mesh, an approximation $u_{i,j}^n$ to $u(ih, jh, n\Delta t)$ is sought. This can be done via any of the following schemes:

1. the 2-2 finite difference scheme (**P1**)

$$u_{i,j}^{n+1} = 2u_{i,j}^n - u_{i,j}^{n-1} + \left(\frac{c_{i,j}\Delta t}{h}\right)^2 \Delta_{i,j}^{2,5} u^n + \Delta t^2 g_{i,j}^n\,, \tag{1.6}$$

where $\Delta_{i,j}^{2,5}$ is the five-point second order discrete Laplacian (within a factor h^2)

$$\Delta_{i,j}^{2,5} u = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} .$$

2. the 2-2 finite difference scheme (**Q1**)

$$u_{i,j}^{n+1} = 2u_{i,j}^{n} - u_{i,j}^{n-1} + \frac{1}{3}\left(\frac{c_{i,j}\Delta t}{h}\right)^2 \Delta_{i,j}^{2,9} u^n + \Delta t^2 g_{i,j}^n , \tag{1.7}$$

where $\Delta_{i,j}^{2,9}$ is the nine-point second order discrete Laplacian (within a factor $3h^2$)

$$\Delta_{i,j}^{2,9} u = (u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1})+(u_{i+1,j+1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i-1,j-1})-8u_{i,j}.$$

The second order schemes (**P1**) and (**Q1**) can be derived [2] from the finite element method in which (i) the basis functions are defined over regular triangular or rectangular meshes and (ii) mass lumping is performed. In practice (**Q1**) is not of great interest since it requires up to nine points per node for the same order of accuracy.

Usually, the right-hand side of (1.5) takes the tensor form $g(x,t) = D(x)S(t)$. Let $f_{\max}$ be the cut-off frequency of the signal S, and define the shortest wavelength $\lambda_{\min} = c_{\min}/f_{\max}$. Theoretical analyses and experimental studies [1, 2, 11] show that in order to be sufficiently accurate, the following condition must be satisfied:

$$\frac{\lambda_{\min}}{h} \geq q , \tag{1.8}$$

where the number of gridpoints per shortest wavelength q also depends on the propagation time. In most real-life simulations $q = 10$ for (**P1**) and (**Q1**).

Now, consider the situation depicted in Fig. 1. We have to deal with two layers, the velocities of which are respectively $c_1 = 1500$m/s and $c_2 = 3000$m/s. The maximal frequency of the excitation source is $f_{\max} = 75$Hz, which gives $\lambda_{\min} = 20$m. When (**P1**) is employed, according to (1.8), the space-step should be at most $h = 2$m. Obviously, this value is imposed to us by the slower layer alone. It would be economically desirable to sample the latter twice as finely as the other layer, as shown in Fig. 2. Once the grids are settled, (**P1**) can be applied inside each subdomain. At the interface, a natural idea would be: (a) resort to the coarse

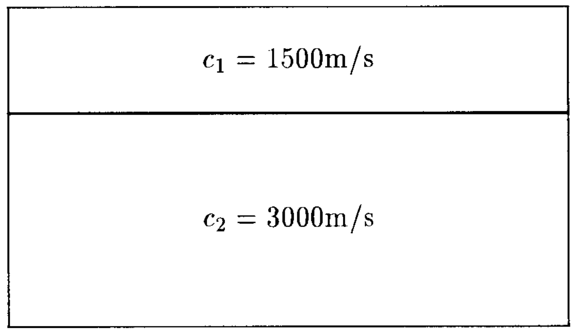

Figure 1: A typical two-layer medium

five-point stencil to update even-numbered gridpoints; then (b) take the arithmetic mean of two neighbor even-numbered points to get the new value of an odd-numbered point. Such a strategy is recommended by Jastram and Behle [10] for the fourth order scheme. However, it is not derived from any sound principle, and besides, its reliability has never been proven.

1.3 Outline

This paper is organized as follows. First, from the variational formulation of (**CP**) the main ideas of the domain decomposition method are sketched out. This intuitive presentation is next rigorously justified by a theoretical framework. Afterwards, semi- and full-discretizations are considered with a view to deriving error estimates. Hints on the practical implementation of the method are given. Finally, numerical results are presented and commented on.

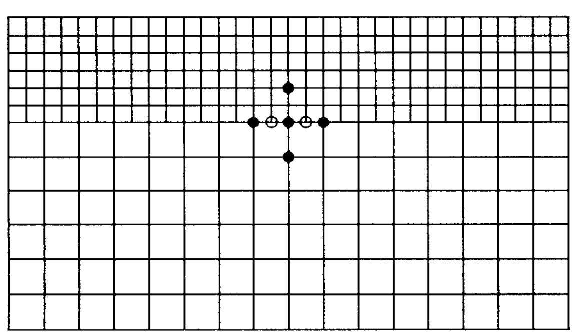

Figure 2: Filled circles represent the coarse five-point stencil for even-numbered gridpoints. Plain circles correspond to odd-numbered gridpoints on the interface.

2. Intuitive ideas of the method

This section is aimed at expounding the basic ideas of the method. All calculations are formal, but they will be given a sense in the next section.

2.1 Notations and settings

Since generalization to multi-layered media is straightforward, we can restrict ourselves to the case of two open subdomains Ω_I and Ω_{II}, the interface of which is the open set Γ defined by

$$\Omega_I \cap \Omega_{II} = \emptyset\,, \quad \overline{\Omega_I} \cup \overline{\Omega_{II}} = \overline{\Omega} \quad \text{and} \quad \partial\Omega_I \cap \partial\Omega_{II} = \bar{\Gamma}\,. \tag{2.1}$$

If w is a real-valued function defined over Ω, $w_{|i}$ denotes its restriction to Ω_i for $i \in \{I, II\}$. Let V_i be a space of functions defined over Ω_i, the generic element of which is v_i. Usually, $V_i = H^1(\Omega_i)$. In domain decomposition, the solution u to the classical problem is sought for as a mapping from $[0,T]$ to $V_I \times V_{II}$. Thus, u is identified with a couple (u_I, u_{II}) where u_i maps $[0,T]$ to V_i. For conveniency, the notation $u_i(t)$ will be used for the function $x \in \Omega_i \mapsto u_i(x,t) \in \mathbf{R}$.

Regardless of whether or not domain decomposition is applied, let V be a space of functions defined over Ω, the generic element of which is v. Usually, $V = H^1(\Omega)$. It is the space to which the solution u is presumed to map $[0,T]$, in the context of weak formulation without domain decomposition. As before, the notation $u(t)$ will be used for the function $x \in \Omega \mapsto u(x,t) \in \mathbf{R}$.

Assume that V can be assimilated to a strict subspace of $V_I \times V_{II}$ via a linear constraint. For instance, if γ_Γ designates the trace operator on Γ, it is well-known [3, 16] that

$$H^1(\Omega) \approx \left\{ v = (v_I, v_{II}) \in H^1(\Omega_I) \times H^1(\Omega_{II}) \;\middle|\; \gamma_\Gamma v_I = \gamma_\Gamma v_{II} \right\}\,. \tag{2.2}$$

2.2 Variational formulation

Suppose $(u_I, u_{II}) : [0,T] \mapsto V_I \times V_{II}$ is a solution to the wave problem. Let us multiply both sides of (1.1) by a test function $v_i \in V_i$. Next, integrate over Ω_i by using Green's theorem to transform the integrals. Substitute condition (1.2) into the boundary integrals that appear in the course of calculations. Finally, sum over $i \in \{I, II\}$ to get

$$\begin{aligned}\sum_{i=I}^{II} \int_{\Omega_i} \frac{1}{K_{|i}}\, \ddot{u}_i(t) v_i \, dx \;+\; \sum_{i=I}^{II} \int_{\Omega_i} \frac{1}{\rho_{|i}}\, \nabla u_i(t) . \nabla v_i \, dx \;+\; \sum_{i=I}^{II} \int_{\partial_i} \frac{1}{\sigma_{|i}}\, \dot{u}_i(t) v_i \, d\xi \\ = \int_\Gamma \sum_{i=I}^{II} \frac{1}{\rho_{|i}}\, \nabla u_i(t) \,.\, n_i\, v_i \, d\xi \;+\; \sum_{i=I}^{II} \int_{\Omega_i} f_{|i}(t) v_i \, dx\end{aligned} \tag{2.3}$$

where n_i stands for the exterior normal of Ω_i and $\partial_i = \partial\Omega_i \setminus \bar{\Gamma} = \partial\Omega \cap \partial\Omega_i$. Needless to say, the integral relation (2.3) is to be satisfied for every couple $(v_I, v_{II}) \in (V_I, V_{II})$.

Now, regardless of whether or not domain decomposition is applied, it is possible to do the same calculations by taking a test function $v \in V$ and by integrating over Ω. This yields

$$\int_\Omega \frac{1}{K}\, \ddot{u}(t) v \, dx + \int_\Omega \frac{1}{\rho}\, \nabla u(t) . \nabla v \, dx + \int_{\partial\Omega} \frac{1}{\sigma}\, \dot{u}(t) v \, d\xi = \int_\Omega f(t) v \, dx \,, \tag{2.4}$$

where $u : [0, T] \mapsto V$ is the solution to (**CP**). This is utterly equivalent to

$$\begin{aligned}\sum_{i=I}^{II} \int_{\Omega_i} \frac{1}{K_{|i}}\, \ddot{u}_{|i}(t) v_{|i} \, dx \;+ \sum_{i=I}^{II} \int_{\Omega_i} \frac{1}{\rho_{|i}}\, \nabla u_{|i}(t) . \nabla v_{|i} \, dx \;+ \sum_{i=I}^{II} \int_{\partial_i} \frac{1}{\sigma_{|i}}\, \dot{u}_{|i}(t) v_{|i} \, d\xi \\ = \sum_{i=I}^{II} \int_{\Omega_i} f_{|i}(t) v_{|i} \, dx\end{aligned} \tag{2.5}$$

The solution obtained by the domain decomposition formulation (2.3) is correct if and only if $u_i = u_{|i}$. This seemingly naive requirement has two tremendous consequences.

1. For the mapping (u_I, u_{II}) to have range in V, it is necessary to impose constraints on the u_i's. Consider $V_i = H^1(\Omega_i)$ and $V = H^1(\Omega)$. The matching condition (2.2) can be rewritten variationally as

$$\forall\, t \in [0, T] \,, \quad \forall\, \mu \in \Lambda \,, \quad \int_\Gamma u_{II}(t)\, \mu \, d\xi = \int_\Gamma u_I(t)\, \mu \, d\xi \,,$$

   in which the space Λ of test functions defined over Γ remains to be precised.

2. If the test function $(v_I, v_{II}) \in V_I \times V_{II}$ in (2.3) happens to belong to $H^1(\Omega)$, then (2.3) should give us (2.5). This is tantamount to requiring

$$\gamma_\Gamma v_I = \gamma_\Gamma v_{II} \quad \Longrightarrow \quad \int_\Gamma \left(\frac{1}{\rho_{|I}} \nabla u_I . n_I\, v_I + \frac{1}{\rho_{|II}} \nabla u_{II} . n_{II}\, v_{II} \right) d\xi = 0 \,,$$

   as can be seen by comparing (2.3) and (2.5). Introduce then the *co-normal derivative*

$$\lambda(t) = \frac{1}{\rho_{|II}} \nabla u_{II}(t) \,.\, n_{II} = -\frac{1}{\rho_{|I}} \nabla u_I(t) \,.\, n_I \;\in \Lambda \tag{2.6}$$

as an element of Λ. It will play the role of a Lagrange multiplier at the interface.

We are thus led to the following **F**ormal **D**omain decomposition formulation, from which numerical schemes will be deduced.

(FD) GIVEN

$$\rho,\ K,\ \sigma = \sqrt{K\rho},\ f,\ \tilde{u}_0,\ \text{and}\ \tilde{u}_0'\ \text{smooth enough on}\ \Omega.$$

FIND

$$(u_I, u_{II}) : [0,T] \mapsto V_I \times V_{II} \quad \text{and} \quad \lambda : [0,T] \mapsto \Lambda \ \text{such that}$$

- $\forall\, v_i \in V_i$, the following integral relation holds

$$\begin{aligned} \int_{\Omega_i} \frac{1}{K_{|i}} \ddot{u}_i(t) v_i\, dx \ + \int_{\Omega_i} \frac{1}{\rho_{|i}} \nabla u_i(t).\nabla v_i\, dx \ &+ \int_{\partial_i} \frac{1}{\sigma_{|i}} \dot{u}_i(t) v_i\, d\xi \\ = \int_\Gamma (-1)^i v_i \lambda(t)\, d\xi \qquad &+ \int_{\Omega_i} f_{|i}(t) v_i\, dx \end{aligned} \tag{2.7}$$

- $\forall\, \mu \in \Lambda$, the following continuity condition holds

$$\int_\Gamma (u_{II} - u_I)(t)\, \mu\, d\xi \ = \ 0 \tag{2.8}$$

- the initial conditions are satisfied

$$\forall i \in \{I, II\}, \quad u_i(0) = \tilde{u}_{0|i} \quad \text{and} \quad \dot{u}_i(0) = \tilde{u}'_{0|i}. \tag{2.9}$$

This formulation calls for several remarks.

REMARK 2.1 Although no summation sign over i appears in (2.7), the latter is indeed equivalent to (2.3), to the extent that (v_I, v_{II}) is allowed to vary freely in $V_I \times V_{II}$. □

REMARK 2.2 The same space Λ is used for two distinct purposes, which are (a) testing the equality of traces on Γ as indicated by (2.8) and (b) describing the co-normal derivative of u across Γ as shown by (2.6). Shedding light on the spaces V_i and Λ is an absolute necessity; this will be taken up in the next section. □

REMARK 2.3 Since the calculations are purely formal, the integrals involving λ and μ may in fact represent duality products. In the elliptic case, it is classical [13] to define the co-normal derivative of u as an element of $H^{-1/2}(\partial\Omega)$ if $u \in H^1(\Omega)$ and $\nabla.\left(\frac{1}{\rho}\nabla u\right) \in L^2(\Omega)$. □

REMARK 2.4 We will also have to precise what *smooth enough* means for K and ρ. □

2.3 Time and space discretization

Let $N \in \mathbf{N}$ and divide the interval $[0,T]$ into $N \geq 1$ pieces of length $\Delta t = T/N$. Let V_i^N be a finite-dimensional subspace of $V_i = H^1(\Omega_i)$. The dimension of V_i^N is connected to the space-step h_i of the mesh defined on Ω_i. The superscript N reminds that V_i^N may have to be chosen in accordance with Δt in order for some stability condition is satisfied. Let also $\Lambda^N \subset L^2(\Gamma)$ be a finite-dimensional subspace of Λ.

For $n \in \{1, 2, ..., N-1\}$, consider the sequence of discrete problems

$(\mathbf{G}^n)$ GIVEN

$$f^n \in L^2(\Omega), \quad u_i^n,\, u_i^{n-1} \in V_i^N, \quad \text{and } K,\ \rho,\ \sigma = \sqrt{K\rho} \text{ smooth enough} \tag{2.10}$$

FIND

$$u_i^{n+1} \in V_i^N \quad \text{and} \quad \lambda^n \in \Lambda^N \quad \text{such that} \tag{2.11}$$

- $\forall\, v_i \in V_i^N$, the following discrete integral relation holds

$$\begin{aligned}\int_{\Omega_i} \frac{1}{K_{|i}}\, \delta_2^n u_i\, v_i\, dx \;+\; \int_{\Omega_i} \frac{1}{\rho_{|i}}\, \nabla u_i^n . \nabla v_i\, dx \;+\; \int_{\partial_i} \frac{1}{\sigma_{|i}}\, \delta_1^n u_i\, v_i\, d\xi \\ = \int_\Gamma (-1)^i v_i\, \lambda^n\, d\xi \;+\; \int_{\Omega_i} f_{|i}^n\, v_i\, dx\end{aligned} \tag{2.12}$$

  where the discrete derivation operators δ_2 and δ_1 are defined by

$$\delta_2^n u_i \;=\; \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} \quad \text{and} \quad \delta_1^n u_i \;=\; \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t}.$$

- $\forall\, \mu \in \Lambda^N$, the following discrete continuity condition holds

$$\int_\Gamma (\bar{\delta}_0^n u_{II} - \bar{\delta}_0^n u_I)\, \mu\, d\xi \;=\; 0 \quad \text{where} \quad \bar{\delta}_0^n u_i \;=\; \frac{u_i^{n+1} + 2u_i^n + u_i^{n-1}}{4}. \tag{2.13}$$

In this discrete formulation, the time discretization is inspired from finite difference methods, while the space discretization via the subspaces V_i^N stems from finite element procedures.

REMARK 2.5 Condition (2.13) is the discrete version of (2.8). The rationale of $\bar{\delta}_0^n u_i$ is to express (2.8) at time $n\Delta t$ while keeping u_i^{n+1} involved to generate additional equations. □

REMARK 2.6 By doing so, we have as many equations as unknowns. Equation (2.12) written for $i \in \{I, II\}$ and for v_i in a basis of V_i^N produces $\dim V_I^N + \dim V_{II}^N$ scalar equalities. Equation (2.13) written for μ in a basis of Λ^N produces $\dim \Lambda^N$ scalar equalities. On the other hand, the number of unknowns amounts exactly to $\dim V_I^N + \dim V_{II}^N + \dim \Lambda^N$. □

REMARK 2.7 For $n = 0$, the greatest care must be devoted to taking the initial conditions (2.9) into account. It is natural to specify

$$u_i^0 = (\tilde{u}_{0|i})^N ,$$

where $(\tilde{u}_{0|i})^N$ is the projection [in a certain sense, $H^1(\Omega_i)$ mostly] of $\tilde{u}_{0|i}$ on V_i^N. As far as the initial value for the time derivative of u is concerned, it can be taken into account by introducing a fictitious u_i^{-1} such that

$$\delta_1^0 u_i = \frac{u_i^1 - u_i^{-1}}{2\Delta t} = (\tilde{u}'_{0|i})^N$$

where $(\tilde{u}'_{0|i})^N$ is the projection [in a certain sense, $L^2(\Omega_i)$ mostly] of $\tilde{u}'_{0|i}$ on V_i^N, and then by eliminating u_i^{-1} with the help of (2.12). This enables us to consider $(\mathbf{G}^0)$ in a similar fashion to $(\mathbf{G}^n)$ and to solve it with u_I^1, u_{II}^1 and λ^0 as unknowns. □

2.4 Matrix interpretation

Interesting insights into this domain decomposition formulation can be obtained by looking at the problem from the standpoint of matrix computations. Let

$$p_i^N = \dim V_i^N \quad \text{and} \quad q^N = \dim \Lambda^N .$$

Consider $(\widehat{v}_i^k)$ with $k \in \{1, 2, ..., p_i^N\}$ a basis of V_i^N. Likewise choose $(\widehat{\mu}^j)$ with $j \in \{1, 2, ..., q^N\}$ to be a basis of Λ^N. Introduce

◇ $\mathbf{D}_i^N$ the $p_i^N \times q^N$-matrix representing the cross products

$$(\mathbf{D}_i^N)_{k,j} = \int_\Gamma \widehat{\mu}^j \, \widehat{v}_i^k \, d\xi \tag{2.14}$$

$\diamond$ $\mathbf{M}_i^N$ the $p_i^N \times p_i^N$-matrix representing the elementary products

$$(\mathbf{M}_i^N)_{k,l} = \int_{\Omega_i} \frac{1}{K_{|i}} \widehat{v}_i^k \widehat{v}_i^l \, dx + \frac{\Delta t}{2} \int_{\partial_i} \frac{1}{\sigma_{|i}} \widehat{v}_i^k \widehat{v}_i^l \, d\xi \tag{2.15}$$

If $\mathbf{u}_i^{n+1}$ is the column vector representing the decomposition of u_i^{n+1} in $(\widehat{v}_i^k)$ and if $\boldsymbol{\lambda}^n$ is the column vector representing that of λ^n in $(\widehat{\mu}^j)$, then for $n \geq 1$, problem $(\mathbf{G}^n$ can be expressed by the linear system

$$\left\{ \begin{aligned} \mathbf{M}_I^N \mathbf{u}_I^{n+1} &= -\Delta t^2 \, \mathbf{D}_I^N \boldsymbol{\lambda}^n + \mathbf{r}_I^n \\ \mathbf{M}_{II}^N \mathbf{u}_{II}^{n+1} &= \Delta t^2 \mathbf{D}_{II}^N \boldsymbol{\lambda}^n + \mathbf{r}_{II}^n \\ \mathbf{g}^n &= {}^t\mathbf{D}_{II}^N \mathbf{u}_{II}^{n+1} - {}^t\mathbf{D}_I^N \mathbf{u}_I^{n+1} \end{aligned} \right. \tag{2.16}$$

in which $\mathbf{r}_I^n$, $\mathbf{r}_{II}^n$ and $\mathbf{g}^n$ can be explicited in terms of the data. The linear system corresponding to $(\mathbf{G}^0)$ is of about the same form. By eliminating $\mathbf{u}_I^{n+1}$ and $\mathbf{u}_{II}^{n+1}$ from the first two equations with the last one, it follows that for $n \in \{1, ..., N-1\}$,

$$\Delta t^2 \left[{}^t\mathbf{D}_I^N (\mathbf{M}_I^N)^{-1} \mathbf{D}_I^N + {}^t\mathbf{D}_{II}^N (\mathbf{M}_{II}^N)^{-1} \mathbf{D}_{II}^N \right] \boldsymbol{\lambda}^n = \mathbf{h}^n , \tag{2.17}$$

where the right-hand side $\mathbf{h}^n$ depends on $\mathbf{r}_I^n$, $\mathbf{r}_{II}^n$ and $\mathbf{g}^n$. The matrix

$$\mathbf{S}^N = {}^t\mathbf{D}_I^N (\mathbf{M}_I^N)^{-1} \mathbf{D}_I^N + {}^t\mathbf{D}_{II}^N (\mathbf{M}_{II}^N)^{-1} \mathbf{D}_{II}^N \tag{2.18}$$

does not change with the time-step n, although it does depend on the discretization level N. It will be referred to as the *Stecklov-Poincaré matrix*. For $\mathbf{S}^N$ to be well-defined and for the system (2.17) to be well-posed, a few technical requirements are necessary:

1. $\mathbf{M}_i^N$ must be invertible. From definition (2.15), it is easily seen that $\mathbf{M}_i^N$ is symmetric. Moreover, it is positive because K and ρ are positive. Definiteness is ensured for $\mathbf{M}_i^N$ as soon as K and ρ have upper-bounds. Then, $(\mathbf{M}_i^N)^{-1}$ exists and is also positive definite.

2. $\mathbf{S}^N$ must be invertible. From definition (2.18), it is easily seen that $\mathbf{S}^N$ is symmetric. Moreover, it is positive because

$$\left(\mathbf{S}^N \boldsymbol{\lambda}, \boldsymbol{\lambda}\right) = \sum_{i=I}^{II} \left({}^t\mathbf{D}_i^N (\mathbf{M}_i^N)^{-1} \mathbf{D}_i^N \boldsymbol{\lambda}, \boldsymbol{\lambda}\right) = \sum_{i=I}^{II} \left((\mathbf{M}_i^N)^{-1} \mathbf{D}_i^N \boldsymbol{\lambda}, \mathbf{D}_i^N \boldsymbol{\lambda}\right)$$

and because $(\mathbf{M}_i^N)^{-1}$ is positive definite. Definiteness is ensured for $\mathbf{S}^N$ as soon as

$$\boldsymbol{\lambda} \neq \mathbf{0} \quad \Longrightarrow \quad \exists\, i \in \{I, II\} \mid \mathbf{D}_i^N \boldsymbol{\lambda} \neq \mathbf{0}\,. \tag{2.19}$$

The above condition deserves some further developments. Intrinsically, it means that if $\lambda \in \Lambda^N$ and $\lambda \neq 0$, then λ should not be simultaneously orthogonal to V_I^N and V_{II}^N, the concept of orthogonality being associated with the matrices $\mathbf{D}_I^N$ and $\mathbf{D}_{II}^N$. A careful examination of (2.14) reveals that only the traces on Γ of the basis functions $(\widehat{v}_i^k)$ are involved in the definition of $\mathbf{D}_i^N$. For $i \in \{I, II\}$, introduce the space of traces on Γ of V_i^N

$$\gamma_\Gamma V_i^N \;=\; \{\, w_i \in L^2(\Gamma) \mid \exists\, v_i \in V_i^N\,,\;\; w_i \;=\; \gamma_\Gamma v_i \,\}\,, \tag{2.20}$$

and its orthogonal space

$$(\gamma_\Gamma V_i^N)^\perp \;=\; \{\, \eta \in L^2(\Gamma) \mid \forall\, w_i \in \gamma_\Gamma V_i^N\,,\;\; \int_\Gamma \eta w_i \, d\xi \;=\; 0 \,\}\,. \tag{2.21}$$

Clearly, $\mathbf{D}_i^N \boldsymbol{\lambda} = \mathbf{0}$ if and only if $\lambda \in \left(\gamma_\Gamma V_i^N\right)^\perp$. Thence, condition (2.19) is equivalent to

$$\Lambda^N \,\cap\, [\,(\gamma_\Gamma V_I^N)^\perp \,\cap\, (\gamma_\Gamma V_{II}^N)^\perp\,] \;=\; \{0\}\,,$$

where Λ^N and $\gamma_\Gamma V_i^N$ are to be regarded as subspaces of $L^2(\Gamma)$. In a more compact form,

$$\Lambda^N \,\cap\, \left(\gamma_\Gamma V_I^N + \gamma_\Gamma V_{II}^N\right)^\perp \;=\; \{0\}\,. \tag{2.22}$$

REMARK 2.8 An immediate consequence of (2.22) is that

$$\dim \Lambda^N \;\leq\; \dim \left(\gamma_\Gamma V_I^N + \gamma_\Gamma V_{II}^N\right)\,. \tag{2.23}$$

In other words, the multipliers should not be oversampled relatively to $\gamma_\Gamma V_I^N + \gamma_\Gamma V_{II}^N$. □

REMARK 2.9 In practice, mass lumping is applied to $\mathbf{M}_i^N$. This leads us back to finite difference schemes in the strict interior of each subdomain, if regular grids are used. □

REMARK 2.10 Once the multiplier λ^n is known, the pressures u_I^{n+1} and u_{II}^{n+1} can be updated in parallel on different processors. □

3. Functional framework for the continuous problem

At this stage, it is time to build up a functional framework for the problem at hand. A good framework not only gives a sense to the calculations presented thus far, but also provides us with optimal existence and uniqueness results. The wave problem without domain decomposition is first addressed. Its interpretation will turn out to be of great interest for the variational formulation with domain decomposition.

3.1 Preliminaries

Let $d \in \mathbf{N}$ be the space dimension considered. In this paper, $d = 2$.

Sobolev spaces

Let $\Omega \subset \mathbf{R}^d$ be a bounded open domain, and v a real-valued function defined over Ω. If α is an n-index, then $\partial^\alpha v$ denotes the α-derivative of v taken in the sense of distributions. For $m \in \mathbf{N}$,

$$H^m(\Omega) = \{ v \in L^2(\Omega) \mid \partial^\alpha v \in L^2(\Omega), \forall |\alpha| \le m \}$$

is a vector space, equipped with the norm and seminorm

$$\|v\|^2_{m,\Omega} = \sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha v|^2 \, dx \qquad |v|^2_{m,\Omega} = \sum_{|\alpha| = m} \int_\Omega |\partial^\alpha v|^2 \, dx \, .$$

It is a Hilbert space for the norm $\|.\|_{m,\Omega}$.

Throughout this paper, Ω is assumed regular enough [13] for the trace to exist. We denote by $H^{1/2}(\partial\Omega)$ the space of traces on $\partial\Omega$ of all functions $v \in H^1(\Omega)$. The trace operator γ_0 from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$ is continuous and has a continuous right inverse. The norm

$$\|w\|_{1/2,\partial\Omega} = \inf_{\gamma_0 v = w} \|v\|_{1,\Omega}$$

is associated with $H^{1/2}(\partial\Omega)$, whose dual space $H^{-1/2}(\partial\Omega)$ is provided with the norm

$$\|v^\star\|_{-1/2,\partial\Omega} = \sup_{v \ne 0} \frac{\langle v^\star, v\rangle}{\|v\|_{1/2,\partial\Omega}} \, .$$

Dual spaces on a portion of the boundary

If S is an open subset of $\partial\Omega$, the trace on S of $v \in H^1(\Omega)$ is defined by $\gamma_S v = \gamma_0 v_{|S}$. The space $H^{1/2}(S)$ of traces on S of all functions $v \in H^1(\Omega)$ is equipped with the norm

$$\|w\|_{1/2,S} = \inf_{\gamma_S v = w} \|v\|_{1,\Omega}\,.$$

We define $H^{-1/2}(S)$ as the dual space of $H^{1/2}(S)$. Contrary to our intuition, $H^{-1/2}(S)$ has no direct connection whatsoever with $H^{-1/2}(\partial\Omega)$. This technical detail, the inconvenience of which will be pointed out in a moment, originates from the following

Lemma 3.1 *Let S be a strict portion of $\partial\Omega$ and $w \in H^{1/2}(S)$. Then, the extension by zero of w, defined over $\partial\Omega$ by $v_{|S} = w$ and $v_{|\partial\Omega\setminus\bar{S}} = 0$, does not belong to $H^{1/2}(\partial\Omega)$ in general.*

PROOF See [13]. ◁

This somewhat pathological property of $H^{1/2}(S)$ makes it impossible for us to identify a given element of $H^{-1/2}(\partial\Omega)$ with an element of $H^{-1/2}(S)$ by "setting the trial function to zero outside S." Nonetheless, a little change makes such an identification permissible. Consider

$$H_{00}^{1/2}(S) = \{\, \gamma_S v\,,\ v \in H^1(\Omega) \text{ and } \gamma_{\partial\Omega\setminus\bar{S}} v = 0 \,\}\,, \tag{3.1}$$

equipped with the quotient norm. $H_{00}^{1/2}(S)$ is a subspace of $H^{1/2}(S)$. By virtue of definition (3.1), the extension by zero of any element in $H_{00}^{1/2}(S)$ is now an element of $H^{1/2}(\partial\Omega)$. So,

$$H^{-1/2}(\partial\Omega) \subset [H_{00}^{1/2}(S)]' \qquad \text{and also} \qquad H^{-1/2}(S) \subset [H_{00}^{1/2}(S)]'\,.$$

Sum of dual forms defined over different portions

Let S be an open subset of $\partial\Omega$ and S_1 and S_2 two open subsets of S such that

$$S_1 \cap S_2 = \emptyset \quad \text{and} \quad \overline{S} = \overline{S_1} \cup \overline{S_2}\,.$$

Given any $\mu_1 \in H^{-1/2}(S_1)$ and any $\mu_2 \in H^{-1/2}(S_2)$, it is possible to define

$$\forall\, w \in H^{1/2}(S)\,, \quad \langle \mu, w\rangle = \langle \mu_1, w_{|S_1}\rangle + \langle \mu_2, w_{|S_2}\rangle$$

It can be easily checked that $\mu \in H^{-1/2}(S)$, and $\mu = \mu_j$ in $[H_{00}^{1/2}(S_j)]'$ for $j \in \{1,2\}$. To emphasize the fact that μ has been defined as a special sum of two dual forms, we will write

$$\mu = \mu_1 \oplus \mu_2\,. \tag{3.2}$$

Normal trace

Let $\Omega_T =]0,T[\times\Omega$. For any function $q : \Omega_T \mapsto \mathbf{R}^{d+1}$, $\nabla.q$ denotes its divergence taken in the sense of distributions. The space

$$H(\mathrm{div};\Omega_T) = \{ q \in [L^2(\Omega)]^{d+1} \mid \nabla\cdot q \in L^2(\Omega_T) \},$$

provided with the norm

$$\|q\|^2_{\mathrm{div},\Omega_T} = |q|^2_{0,\Omega_T} + |\nabla\cdot q|^2_{0,\Omega_T},$$

is a Hilbert space. Over $H(\mathrm{div};\Omega_T)$, the normal trace on $\partial\Omega_T$ can be defined thanks to

Lemma 3.2 *Each function $q \in H(\mathrm{div};\Omega_T)$ can be assigned an element of $H^{-1/2}(\partial\Omega_T)$, denoted by $q.n$, such that*

- *For $q \in \mathcal{D}(\overline{\Omega_T})$, $q.n$ coincides with the normal trace taken in the classical sense.*
- *Green's formula can be extended to the general case by*

$$\forall\, w \in H^{1/2}(\partial\Omega_T), \qquad \langle\, q.n,\, w \,\rangle = \int_{\Omega_T} v\,\nabla.q\,dx + \int_{\Omega_T} q.\nabla v\,dx \tag{3.3}$$

  where $v \in H^1(\Omega_T)$ and $\gamma_0 v = w$.

- *The mapping $q \in H(\mathrm{div};\Omega_T) \mapsto q.n \in H^{-1/2}(\partial\Omega_T)$ is continuous and on-to.*

Proof See [3, 4, 6]. $\lhd$

In the first statement of this lemma, $\mathcal{D}(\overline{\Omega_T})$ is the set of restrictions to Ω_T of indefinitely differentiable and compact-supported functions defined over $\mathbf{R}^{d+1}$. If $q \in \mathcal{D}(\overline{\Omega_T})$, then $q.n$ clearly exists in the classical sense as a real-valued function over $\partial\Omega_T$, and $q.n \in L^2(\partial\Omega_T)$. Identifying $L^2(\partial\Omega_T)$ with a subspace of $H^{-1/2}(\partial\Omega_T)$ gives a sense to the short-cut expression "the element $q.n$ coincides with the normal trace taken in the classical sense."

Functions with range in a Hilbert space

Let X be a Hilbert space, and $T > 0$. The set of square integrable functions from $[0,T]$ to X is designated by $L^2(0,T;\,X)$. Let us provide it with the norm

$$\|u\|^2_{L^2(X)} = \int_0^T \|u(t)\|^2_X\,dt\,.$$

If $u \in L^2(0,T;X)$, the derivative of u in the sense of distributions is defined as the linear mapping $\dot{u} : \mathcal{D}(]\,0,T\,[) \mapsto X$ whose explicit expression is

$$\forall \varphi \in \mathcal{D}(]\,0,T\,[), \qquad [\,\dot{u},\varphi\,] = -\int_0^T u(t)\dot{\varphi}(t)\,dt\,. \tag{3.4}$$

Here $\mathcal{D}(]\,0,T\,[)$ is the space of indefinitely differentiable and compact-supported real-valued functions defined over $]\,0,T\,[$. It can be shown that $\dot{u}$ defined by (3.4) is actually a distribution. The reader is referred to [6] for greater details.

Let Y be another Hilbert space with $X \subset Y$, and let $u \in L^2(0,T;X)$. We say that $\dot{u} \in L^2(0,T;Y)$ if there exists a function $v \in L^2(0,T;Y)$ such that

$$\forall \varphi \in \mathcal{D}(]\,0,T\,[), \qquad [\,\dot{u},\varphi\,] = \int_0^T v(t)\varphi(t)\,dt\,. \tag{3.5}$$

The distribution $\dot{u}$ is then identified with the function v. Note that in (3.4), the result of the integral is an element of X, while in (3.5), the integral converges in Y. Thus, the fact that (3.5) should hold for all $\varphi \in \mathcal{D}(]\,0,T\,[)$ is worth pondering over.

The way $\dot{u}$ has been defined does not rely on the L^2 character of the mapping $t \mapsto u(t)$. If $L^2(0,T;X)$ were replaced by $C^0(0,T;X)$, the space of continuous functions with range in X, provided with the norm

$$\|u\|_{C^0(X)} = \sup_{t\in[0,T]} \|u(t)\|_X\,,$$

it could be equally possible to define $\dot{u}$ by (3.4) since the integral of the right-hand side still exists. We could then look for functions $u \in C^0(0,T;X)$ such that $\dot{u} \in C^0(0,T;Y)$.

3.2 One-domain variational formulation

We are first going to set a framework to the standard acoustic wave problem, with some ulterior motives on domain decomposition. This will help clarify ideas and will lay grounds to upcoming discussions on domain decomposition.

For the sake of notation conveniency, let us introduce some bilinear forms. Let

$$\begin{aligned} &\diamond \quad c_0(v,w) = \int_\Omega \frac{1}{K}\, vw\, dx && \text{for } v,\, w \in L^2(\Omega) \\ &\diamond \quad a_0(v,w) = \int_\Omega \frac{1}{\rho} \nabla v \nabla w\, dx && \text{for } v,\, w \in H^1(\Omega) \\ &\diamond \quad b_0(v,w) = \int_{\partial\Omega} \frac{1}{\sigma}\, vw\, d\xi && \text{for } v,\, w \in L^2(\partial\Omega) \end{aligned} \tag{3.6}$$

The bilinear forms a_0, b_0 and c_0 are well-defined thanks to

Hypothesis 3.1 *Throughout this paper, it is assumed that*

1. *The data K and ρ are measurable, bounded above and below over Ω, i.e. there exist constants $K_{\min}$, $K_{\max}$, $\rho_{\min}$ and $\rho_{\max}$ such that for all $x \in \Omega$*

$$0 < K_{\min} \leq K(x) \leq K_{\max} \quad \textit{and} \quad 0 < \rho_{\min} \leq \rho(x) \leq \rho_{\max}\,.$$

2. *The data σ can be extended by continuity to a function $\sigma_{|\partial\Omega}$ that is measurable, bounded above and below over $\partial\Omega$.*

REMARK 3.1 Condition (2) is much less demanding than $\sigma \in H^1(\Omega)$. To see this, consider a decomposition of Ω into Ω_I and Ω_{II} as described by (2.1). Take $\sigma_{|I} = 1$ and $\sigma_{|II} = 2$. Obviously, the discontinuity of σ along the interface Γ prevents it from being in $H^1(\Omega)$. However, $\sigma_{|\partial\Omega}$ does exist as $\sigma_{|\partial\Omega\cap\partial\Omega_I} = 1$ and $\sigma_{|\partial\Omega\cap\partial\Omega_{II}} = 2$. □

Under the assumptions of Hypothesis 3.1, it is not difficult to see that

- c_0 is $\|.\|_{0,\Omega}$-continuous and $\|.\|_{0,\Omega}$-coercive.
- a_0 is $\|.\|_{1,\Omega}$-continuous and $\|.\|_{1,\Omega}$-coercive relatively to $\|.\|_{0,\Omega}$, namely

$$\forall v \in H^1(\Omega), \quad a(v,v) \geq \frac{1}{\rho_{\max}} |v|^2_{1,\Omega} = \frac{1}{\rho_{\max}} \left(\|v\|^2_{1,\Omega} - \|v\|^2_{0,\Omega} \right). \tag{3.7}$$

- b_0 is $\|.\|_{0,\partial\Omega}$-continuous and $\|.\|_{0,\partial\Omega}$-coercive.

Consider now, at last, the following variational problem.

(T) GIVEN

$$K,\ \rho,\ \text{ and } \sigma = \sqrt{K\rho} \text{ satisfying Hypothesis 3.1}, \tag{3.8}$$

$$f \in L^2(0,T;\ L^2(\Omega)), \tag{3.9}$$

$$\tilde{u}_0 \in H^1(\Omega) \quad \text{and} \quad \tilde{u}_0' \in L^2(\Omega). \tag{3.10}$$

FIND

$$u \in C^0(0,T;\ H^1(\Omega)) \text{ such that} \tag{3.11}$$

- in addition to (3.11)

$$\dot{u} \in C^0(0,T;\ L^2(\Omega)) \quad \text{and} \quad \dot{\widehat{\gamma_0 u}} \in L^2(0,T;\ L^2(\partial\Omega)), \tag{3.12}$$

- for all $v \in H^1(\Omega)$, the integral relation

$$\frac{d}{dt}\, c_0(\dot{u}(t), v) + a_0(u(t), v) + b_0(\dot{\widehat{\gamma_0 u}}(t), \gamma_0 v) = (f(t), v)_{L^2(\Omega)} \tag{3.13}$$

holds in the sense of scalar distributions over $]\,0,T\,[$,

- the initial conditions are satisfied

$$u(0) = \tilde{u}_0 \quad \text{and} \quad \dot{u}(0) = \tilde{u}_0'. \tag{3.14}$$

This formulation calls for a few remarks.

REMARK 3.2 The initial condtions (3.14) make sense thanks to requirements (3.11) and (3.12) and to condition (3.10). □

REMARK 3.3 $\dot{\widehat{\gamma_0 u}}$ is the derivative of $\gamma_0 u$ in the sense of distributions over $]\,0,T\,[$ with range in $H^{1/2}(\partial\Omega)$. It should not be confused with $\gamma_0 \dot{u}$, which does not necessarily exist. □

The continuity and coercivity properties of a_0, b_0 and c_0 give rise to

Theorem 3.1 *Problem* **(T)** *has a unique solution u, which depends continuously on the data, i.e. there exists a constant C such that*

$$\|u\|^2_{C^0(H^1(\Omega))} + \|\dot{u}\|^2_{C^0(L^2(\Omega))} + \|\dot{\widehat{\gamma_0 u}}\|^2_{L^2(L^2(\partial\Omega))} \le C\left(\|f\|^2_{L^2(L^2(\Omega))} + \|\tilde{u}_0\|^2_{H^1(\Omega)} + \|\tilde{u}_0'\|^2_{L^2(\Omega)}\right).$$

PROOF A proof is given in [6] for a general class of hyperbolic problems. Some alterations are necessary to take into account the presence of $\dot{\widehat{\gamma_0 u}} \in L^2(0,T;\, L^2(\partial\Omega))$. ◁

The question that now arises is: to which extent is u still a solution to the classical problem **(CP)** ?. Our next task is thus to find an appropriate interpretation to **(T)**.

Proposition 3.1 *Consider*

— *the cylinder* $\Omega_T =]\,0,T\,[\times \Omega$ *as an open bounded domain of* $\mathbf{R}^{d+1}$ *;*

— *the surface* $S_T =]\,0,T\,[\times \partial\Omega$ *as an open subset of* $\partial\Omega_T$*, the boundary of* Ω_T*.*

Let u be the unique solution to **(T)***. Then, u satisfies*

- $\dfrac{1}{K}\,\ddot{u} - \nabla.\left(\dfrac{1}{\rho}\nabla u\right) = f$ *in the sense of* $\mathcal{D}'(\Omega_T)$*, scalar distributions over* Ω_T*;*
- $\dfrac{1}{\sigma}\,\dot{\widehat{\gamma_0 u}} + \dfrac{1}{\rho}\nabla u \,.\, n = 0$ *in the sense of* $[H^{1/2}_{00}(S_T)]'$*, dual forms of* $H^{1/2}_{00}(S_T)$*.*

PROOF Adaptation of ideas from [6] is straightforward. ◁

3.3 Two-domain hybrid formulation

Consider again a two-domain decomposition of Ω into Ω_I, Ω_{II} and Γ as described by (2.1). Before getting into the formulation in itself, we need to introduce some specific materials.

Vector spaces

The Cartesian product nature of spaces being reminded by underlined names, let

- $\underline{V} = H^1(\Omega_I) \times H^1(\Omega_{II})$, equipped with the norm and seminorm

$$\|v\|^2_{\underline{V}} = \|v_I\|^2_{1,\Omega_I} + \|v_{II}\|^2_{1,\Omega_{II}} \quad \text{and} \quad |v|^2_{\underline{V}} = |v_I|^2_{1,\Omega_I} + |v_{II}|^2_{1,\Omega_{II}}\,,$$

- $\underline{H} = L^2(\Omega_I) \times L^2(\Omega_{II})$, equipped with the norm

$$\|v\|^2_{\underline{H}} = \|v_I\|^2_{0,\Omega_I} + \|v_{II}\|^2_{0,\Omega_{II}}\,,$$

- $\underline{Z} = L^2(\partial_I) \times L^2(\partial_{II})$, where $\partial_i = \partial\Omega_i \setminus \bar{\Gamma}$, equipped with the norm

$$\|w\|^2_{\underline{Z}} = \|w_I\|^2_{0,\partial_I} + \|w_{II}\|^2_{0,\partial_{II}}\,,$$

- $\Lambda = H^{-1/2}(\Gamma)$, equipped with the norm

$$\|\mu\|_\Lambda = \sup_{s \in H^{1/2}(\Gamma)} \frac{\langle \mu, s \rangle}{\|s\|_{1/2,\Gamma}}.$$

Bilinear forms

Introduce the following bilinear forms defined over these product spaces.

$$\begin{aligned}
&\diamond \quad \underline{c}(v,w) = \sum_{i=I}^{II} \int_{\Omega_i} \frac{1}{K_{|i}} v_i w_i \, dx && \text{for } v, w \in \underline{H} \\
&\diamond \quad \underline{a}(v,w) = \sum_{i=I}^{II} \int_{\Omega_i} \frac{1}{\rho_{|i}} \nabla v_i \nabla w_i \, dx && \text{for } v, w \in \underline{V} \\
&\diamond \quad \underline{b}(v,w) = \sum_{i=I}^{II} \int_{\partial_i} \frac{1}{\sigma_{|i}} v_i w_i \, d\xi && \text{for } v, w \in \underline{Z}
\end{aligned} \tag{3.15}$$

Under the assumptions of Hypothesis 3.1, the bilinear forms $\underline{a}$, $\underline{b}$ and $\underline{c}$ enjoy continuity and coercivity properties similar to those of a_0, b_0 and c_0. Now, the novelty is

$$\diamond \quad \underline{d}(v,\mu) = \sum_{i=I}^{II} (-1)^i \langle \mu, \gamma_\Gamma v_i \rangle \qquad \text{for } v \in \underline{Z},\ \mu \in \Lambda \tag{3.16}$$

The role of this cross bilinear form is to reflect the difference of traces on Γ.

Basic properties of the cross bilinear form

Obviously, the bilinear form $\underline{d}$ is continuous with respect to v and μ measured by $\|v\|_{\underline{V}}$ and $\|\mu\|_\Lambda$. This allows us to define the mapping

$$\begin{aligned}
\underline{D} \ : \ \underline{V} \ &\longmapsto \ \Lambda' \ \ (\text{dual of } \Lambda) \\
v \ &\longmapsto \ [\, \mu \in \Lambda \mapsto \underline{d}(v,\mu) \in \mathbf{R} \,]
\end{aligned} \tag{3.17}$$

where Λ' can be identified to $H^{1/2}(\Gamma)$, and its transpose

$$\begin{aligned}
{}^t\underline{D} \ : \ \Lambda \ &\longmapsto \ \underline{V}' \ \ (\text{dual of } \underline{V}) \\
\mu \ &\longmapsto \ [\, v \in \underline{V} \mapsto \underline{d}(v,\mu) \in \mathbf{R} \,]
\end{aligned} \tag{3.18}$$

To begin with, observe that

$$\operatorname{Ker} \underline{D} = H^1(\Omega) \quad \text{and} \quad \operatorname{Ker} {}^t\underline{D} = \{0\}. \tag{3.19}$$

Next, introduce

$$\begin{aligned}
(\operatorname{Ker} \underline{D})^0 &= \{\, v^\star \in \underline{V}' \mid \forall\, v \in \operatorname{Ker} \underline{D}, \quad \langle v^\star, v \rangle = 0 \,\} \\
(\operatorname{Ker} {}^t\underline{D})^0 &= \{\, \mu^\star \in \Lambda' \mid \forall\, \mu \in \operatorname{Ker} {}^t\underline{D}, \ \langle \mu^\star, \mu \rangle = 0 \,\}
\end{aligned}$$

Lemma 3.3 *$\underline{d}$ and $\underline{D}$ enjoy the following equivalent properties:*

a. $\operatorname{Im} {}^t\underline{D} = (\operatorname{Ker} \underline{D})^0$

b. ${}^t\underline{D}$ *admits a continuous lifting from* $(\operatorname{Ker} \underline{D})^0 \subset \underline{V}'$ *to* Λ

c. $\exists\, k_0 > 0 \mid \forall \mu \in \Lambda, \quad \sup_{v \in \underline{V}} \dfrac{\underline{d}(v,\mu)}{\|v\|_{\underline{V}}} \geq k_0 \|\mu\|_\Lambda .$

Proof The equivalence of a, b and c comes from functional analysis. See [3] for a more exhaustive list of equivalent statements. As for the proof of c, we can proceed as follows.

Let $\mu \in \Lambda$, fixed for the moment. Then,

$$\sup_{v \in \underline{V}} \frac{\underline{d}(v,\mu)}{\|v\|_{\underline{V}}} \geq \sup_{v=(0,v_{II})} \frac{\underline{d}(v,\mu)}{\|v\|_{\underline{V}}} = \sup_{v_{II} \in V_{II}} \frac{\langle \mu, \gamma_\Gamma v_{II} \rangle}{\|v_{II}\|_{1,\Omega_{II}}} .$$

But, there exists a continuous lifting $\gamma_{\Gamma,II}^{-1} \,:\, w \in H^{1/2}(\Gamma) \mapsto v_{II}(w) \in H^1(\Omega_{II})$ such that

$$\|v_{II}(w)\|_{1,\Omega_{II}} \leq C \|w\|_{1/2,\Gamma} .$$

This is a property of the trace operator. As a result,

$$\sup_{v \in \underline{V}} \frac{\underline{d}(v,\mu)}{\|v\|_{\underline{V}}} \geq \sup_{v_{II} \in \gamma_{\Gamma,II}^{-1}(H^{1/2}(\Gamma))} \frac{\langle \mu, \gamma_\Gamma v_{II} \rangle}{\|\gamma_\Gamma v_{II}\|_{1/2,\Gamma}} \frac{\|\gamma_\Gamma v_{II}\|_{1/2,\Gamma}}{\|v_{II}\|_{1,\Omega_{II}}} \geq \frac{1}{C} \|\mu\|_\Lambda$$

This argument being valid for all $\mu \in \Lambda$, we obtain c with $k_0 = 1/C$. ◁

Property c is also known as the *continuous inf-sup condition.* It is the key condition for existence of the multiplier λ in the variational formulation with domain decomposition

(G) GIVEN

the data specified by (3.8), (3.9) and (3.10)

FIND

$$u = (u_I, u_{II}) \in C^0(0,T;\underline{V}) \text{ and } \lambda \in \mathcal{D}'(]\,0,T\,[;\,\Lambda) \text{ such that} \tag{3.20}$$

- in addition to (3.20)

$$\dot{u} \in C^0(0,T;\underline{H}) \quad \text{and} \quad \dot{\widehat{\gamma_\partial u}} \in L^2(0,T;\underline{Z}) ; \tag{3.21}$$

- $\forall\, v = (v_I, v_{II}) \in \underline{V}\,,\ \forall\, \varphi \in \mathcal{D}(]\,0,T\,[)$, the following integral relation holds

$$\begin{aligned} -\,\underline{c}([\dot{u},\dot{\varphi}],\, v) \quad &+\ \underline{a}([u,\varphi],\, v) \quad +\ \underline{b}([\dot{\widehat{\gamma_\partial u}},\varphi],\, \gamma_\partial v) \\ &= \quad ([f,\varphi],\, v) \quad +\ \underline{d}(v,\, [\lambda,\varphi]) \end{aligned} \tag{3.22}$$

where $[.,.]$ denotes the duality product of vector-valued distributions;

- $\forall\, \mu \in \Lambda\,,\ \forall\, t \in]\,0,T\,[$, the following continuity condition holds

$$\underline{d}(u(t),\mu) \;=\; 0\,; \tag{3.23}$$

- $\forall\, i \in \{I, II\}$, the initial conditions are satisfied

$$u_i(0) = \tilde{u}_{0|i} \quad \text{and} \quad \dot{u}_i(0) = \tilde{u}'_{0|i}\,. \tag{3.24}$$

REMARK 3.4 In (3.21) and (3.22), $\gamma_\partial u$ denotes $(\gamma_{\partial_I} u_I,\, \gamma_{\partial_{II}} u_{II}) \in H^{1/2}(\partial_I) \times H^{1/2}(\partial_{II})$, whose derivative $\dot{\widehat{\gamma_\partial u}}$ is taken in the sense of distributions. □

REMARK 3.5 Condition (3.23) is equivalent to $u(t) \in \operatorname{Ker} \underline{D}\,,\ \forall\, t \in]\,0,T\,[$. □

Theorem 3.2 *Problem* $(\mathbf{G})$ *has a unique solution* (u,λ)*, the first component of which solves problem* $(\mathbf{T})$*.*

PROOF Suppose (u,λ) is a solution to $(\mathbf{G})$. According to Remark 3.5, $u \in C^0(0,T;\, \operatorname{Ker} \underline{D})$ where $\operatorname{Ker} \underline{D} = H^1(\Omega)$. Evidently, u satisfies (3.12). Besides, if the test function v of (3.22) is taken in $\operatorname{Ker} \underline{D}$, then $\underline{d}(v,\, [\lambda,\varphi])$ vanishes and we end up with (3.13). Of course, u also meets (3.14) because of (3.24). Therefore, u must solve $(\mathbf{T})$. Now, Theorem 3.1 ensures existence and uniqueness for u.

If (u,λ_1) and (u,λ_2) are two solutions to $(\mathbf{G})$, then by applying (3.22),

$$\forall\, \varphi \in \mathcal{D}'(]\,0,T\,[)\,, \quad [\lambda_2 - \lambda_1, \varphi] \in \operatorname{Ker} {}^t\underline{D}\,.$$

Since $\operatorname{Ker} {}^t\underline{D} = \{0\}$, we must have $\lambda_1 = \lambda_2$ in $\mathcal{D}'(]\,0,T\,[;\,\Lambda)$. Uniqueness for λ is thus proven.

Let $\varphi \in \mathcal{D}(]\,0,T\,[)$. Put (3.22) under the form

$$\forall\, v \in \underline{V}\,, \quad \underline{d}(v,\, [\lambda.\,\varphi]) \;=\; \langle L_\varphi, v\rangle$$

where $L_\varphi \in (\mathrm{Ker}\,\underline{D})^0 = \mathrm{Im}\,{}^t\underline{D} \subset \underline{V}'$ and furthermore, there exists $C > 0$ such that

$$\|L_\varphi\|^2_{\underline{V}'} \leq C\,(\,\|u\|^2_{C^0(\underline{V})} + \|\dot{u}\|^2_{C^0(\underline{H})} + \|\dot{\widehat{\gamma\partial u}}\|^2_{L^2(\underline{Z})} + \|f\|^2_{L^2(\underline{H})}\,)\,(\,\|\varphi\|^2_0 + \|\dot{\varphi}\|^2_0\,)\,. \tag{3.25}$$

By virtue of the continuous lifting property b of Lemma 3.3, there exists $\lambda_\varphi \in \Lambda$ such that ${}^t\underline{D}\lambda_\varphi = L_\varphi$ and $\|\lambda_\varphi\|_{\underline{V}'} \leq \dfrac{1}{k_0}\|L_\varphi\|_{\underline{V}'}$.

The latter inequality, together with (3.25), shows that the mapping $\lambda : \varphi \mapsto [\lambda, \varphi] = \lambda_\varphi$ is continuous with respect to the topology of $\mathcal{D}(]\,0,T\,[)$. Hence, $\lambda \in \mathcal{D}'(]\,0,T\,[;\,\Lambda)$. $\lhd$

Does (u, λ) depend continuously on the data? The question cannot be answered right now since $\mathcal{D}'(]\,0,T\,[;\,\Lambda)$ is not a normed space. We first have to agree on how λ is to be measured. To begin with, recall that $\mathcal{D}(]\,0,T\,[)$ is a dense subspace in the Sobolev space

$$H^1_0(]\,0,T\,[) \;=\; \{\psi \in H^1(]\,0,T\,[) \;\mid\; \psi(0) = \psi(T) = 0\}\,,$$

equipped with the H^1-norm $\|\psi\|^2_1 \;=\; \|\psi\|^2_0 \,+\, \|\dot{\psi}\|^2_0$. Next,

Definition 3.1 *We introduce*

$$H^{-1}_t(\Lambda) \;=\; \mathcal{L}\,(H^1_0(]\,0,T\,[)\,;\,\Lambda)\,,$$

the space of linear continuous mappings from $H^1_0(]\,0,T\,[)$ to Λ, equipped with the norm

$$\|\nu\|_{H^{-1}_t(\Lambda)} \;=\; \sup_{\psi\neq 0}\, \frac{\|\,\langle \nu, \psi\rangle\,\|_\Lambda}{\|\psi\|_1}\,.$$

As can be noticed from the proof of Theorem 3.2, the mapping $\lambda : \mathcal{D}(]\,0,T\,[) \mapsto \Lambda$ is, in reality, continuous with respect to $\|.\|_1$ and $\|.\|_\Lambda$. By density, it can be extended to a linear mapping from $H^1_0(]\,0,T\,[)$ to Λ which has the same "norm." Once the extension has been performed, it makes sense to speak about $\|\lambda\|_{H^{-1}_t(\Lambda)}$.

Proposition 3.2 *Let $\underline{W}$ be the space of all functions $u \in C^0(0,T;\underline{V})$ that satisfy the additional regularity condition (3.21), provided with*

$$\|u\|^2_{\underline{W}} \;=\; \|u\|^2_{C^0(\underline{V})} \;+\; \|\dot{u}\|^2_{C^0(\underline{H})} \;+\; \|\dot{\widehat{\gamma\partial u}}\|^2_{L^2(\underline{Z})}\,.$$

Then, the solution (u, λ) to (**G**) *depends continuously on the data in the sense that there exists a constant C such that*

$$\|u\|^2_{\underline{W}} + \|\lambda\|^2_{H_t^{-1}(\Lambda)} \leq C\left(\|f\|^2_{L^2(\underline{H})} + \|\tilde{u}_0\|^2_{H^1(\Omega)} + \|\tilde{u}_0'\|^2_{L^2(\Omega)} \right).$$

Proof The estimate for u comes from Theorem 3.1. As far as λ is concerned, we have seen in the proof of Theorem 3.2 that

$$\forall \varphi \in \mathcal{D}(]\,0, T\,[), \quad \|\,[\lambda, \varphi]\,\|^2_\Lambda \leq C\left(\|u\|^2_{\underline{W}} + \|f\|^2_{L^2(\underline{H})} \right) \|\varphi\|^2_{\mathrm{I}}.$$

The extension of λ from $\mathcal{D}'(]\,0, T\,[; \Lambda)$ to $H_t^{-1}(\Lambda)$ satisfies

$$\|\lambda\|^2_{H_t^{-1}(\Lambda)} \leq C\left(\|u\|^2_{\underline{W}} + \|f\|^2_{L^2(\underline{H})} \right),$$

from which the estimate proposed for λ is readily deduced. ◁

Before closing the case of the continuous problem, let us interpret (**G**).

Proposition 3.3 *For $i \in \{I, II\}$, consider*

— *the cylinder* $(\Omega_i)_T =]\,0, T\,[\times \Omega_i$ *as an open bounded domain of* $\mathbf{R}^{d+1}$ *;*

— *the surface* $(S_i)_T =]\,0, T\,[\times \partial\Omega_i$ *as an open subset of* $(\partial\Omega_i)_T$ *;*

— *the surface* $(\partial_i)_T =]\,0, T\,[\times \partial_i$ *as an open subset of* $(S_i)_T$ *.*

Consider also $\Gamma_T =]\,0, T\,[\times \Gamma$. Let (u, λ) with $u = (u_I, u_{II})$ be the solution to (**G**). *Then,*

- $\displaystyle \frac{1}{K_{|i}}\,\ddot{u}_i - \nabla.\left(\frac{1}{\rho_{|i}}\nabla u_i\right) = f_{|i}$ *in the sense of* $\mathcal{D}'((\Omega_i)_T)$
- $\displaystyle -\frac{1}{\sigma_{|i}}\,\gamma_{\partial_i}\widehat{\dot{u}_i} \oplus (-1)^i\,\lambda = \frac{1}{\rho_{|i}}\nabla u_i\,.\,n_i$ *in the sense of* $[H^{1/2}_{00}((S_i)_T)]'$

where $\oplus$, defined by (3.2), operates on $-\dfrac{1}{\sigma_{|i}}\,\gamma_{\partial_i}\widehat{\dot{u}_i}$, regarded as a dual form of $H^{1/2}((\partial_i)_T)$, and on $(-1)^i\,\lambda$, regarded as a dual form of

$$H_0^{1/2}(\Gamma_T) = \textit{closure in } H^{1/2}(\Gamma_T) \textit{ of } \{\,\varphi \otimes w_\Gamma\,,\ \varphi \in \mathcal{D}(]0, T[)\,,\, w_\Gamma \in H^{1/2}(\Gamma)\,\}.$$

Proof Proceed as in a standard [6] evolution problem. Realize that $\lambda \in \mathcal{D}'(]\,0, T\,[; \Lambda)$ can be interpreted as a dual form of $H_0^{1/2}(\Gamma_T)$, thanks to its $\|\varphi\|_{\mathrm{I}}$-continuity: define $\ll \lambda, \Psi \gg = \langle\, [\lambda, \varphi], w_\Gamma \,\rangle$ for $\Psi = \varphi \otimes w_\Gamma$ first, then extend $\ll \lambda, . \gg$ by density to $H_0^{1/2}(\Gamma_T)$. ◁

4. Semi-discrete approximation

In this section, our attention is focused on the space approximation of the continuous problem. Error estimates will be derived, which will highlight some features of paramount importance for the analysis of the fully-discrete approximation.

4.1 Discretization spaces

Let $\underline{V}^h \subset \underline{V}$ be a finite-dimensional subspace of $\underline{V}$. In practice, $\underline{V}^h = V_I^{h_I} \times V_{II}^{h_{II}}$ with $V_i^{h_i} \subset V_i$. The index $h = (h_I, h_{II})$ refers to the characteristic sizes of the meshes on which these spaces are built.

Let $\Lambda^h \subset L^2(\Gamma)$ be a finite-dimensional subspace of Λ. The index $h = h_\Gamma$ refers to the size of the mesh from which Λ^h is derived. We have seen, and will see again, that Λ^h cannot be chosen independently of $\underline{V}^h$. We have required $\Lambda^h \subset L^2(\Gamma)$ for computational conveniency.

The cross bilinear form $\underline{d}$ is continuous on $\underline{V}^h \times \Lambda^h$, regarded as a subspace of $\underline{V} \times \Lambda$. This allows us to define the mapping

$$\begin{aligned} \underline{D}^h \ : \ \underline{V}^h \ &\longmapsto \ (\Lambda^h)' \ \text{(dual of } \Lambda^h) \\ v_h \ &\longmapsto \ [\, \mu_h \in \Lambda^h \mapsto \underline{d}(v_h, \mu_h) \in \mathbf{R} \,] \end{aligned} \tag{4.1}$$

and its transpose

$$\begin{aligned} {}^t\underline{D}^h \ : \ \Lambda^h \ &\longmapsto \ (\underline{V}^h)' \ \text{(dual of } \underline{V}^h) \\ \mu_h \ &\longmapsto \ [\, v_h \in \underline{V}^h \mapsto \underline{d}(v_h, \mu_h) \in \mathbf{R} \,] \end{aligned} \tag{4.2}$$

In general, $\operatorname{Ker} \underline{D}^h \not\subset \operatorname{Ker} \underline{D}$ and $\operatorname{Ker} {}^t\underline{D}^h \neq \{0\}$. This might cause trouble for the uniqueness of λ_h (approximating λ). Fortunately enough, $\operatorname{Ker} {}^t\underline{D}^h = \{0\}$ can be guaranteed under

Hypothesis 4.1 *The discretization spaces $\underline{V}^h$ and Λ^h are chosen in such a way that*

$$\Lambda^h \cap \left(\gamma_\Gamma V_I^{h_I} + \gamma_\Gamma V_{II}^{h_{II}}\right)^\perp = \{0\},$$

where the concept of orthogonality $\perp$ is taken in the sense of $L^2(\Gamma)$.

It can be shown from the definition of $\underline{d}$ that Hypothesis 4.1, which is none other than (2.22), actually secures $\operatorname{Ker} {}^t\underline{D}^h = \{0\}$.

4.2 Semi-discrete problem

The semi-discret problem can be formulated as

$(\mathbf{G}_h)$ GIVEN

the data specified by (3.8), (3.9) and (3.10)

FIND

$$u_h \in C^0(0,T;\underline{V}^h) \text{ and } \lambda_h \in \mathcal{D}'(]\,0,T\,[;\Lambda^h) \text{ such that} \tag{4.3}$$

- in addition to (4.3), posing $\underline{Z}^h = \gamma_{\partial_I} V_I^{h_I} \times \gamma_{\partial_{II}} V_{II}^{h_{II}}$ $(\subset \underline{Z})$,

$$\dot{u}_h \in C^0(0,T;\underline{V}^h) \quad \text{and} \quad \dot{\widehat{\gamma_\partial u_h}} \in L^2(0,T;\underline{Z}^h)\,; \tag{4.4}$$

- $\forall\, v_h \in \underline{V}^h$, $\forall\, \varphi \in \mathcal{D}(]\,0,T\,[)$, the following integral relation holds

$$\begin{aligned} -\,\underline{c}([\dot{u}_h,\dot{\varphi}],v_h) \;&+\; \underline{a}([u_h,\varphi],v_h) \;+\; \underline{b}([\dot{\widehat{\gamma_\partial u_h}},\varphi],\gamma_\partial v_h) \\ =\; ([f,\varphi],v_h) \;&+\; \underline{d}(v_h,[\lambda_h,\varphi]) \end{aligned} \tag{4.5}$$

where $[.,.]$ denotes the duality product of vector-valued distributions;

- $\forall\, \mu_h \in \Lambda^h$, $\forall\, t \in]\,0,T\,[$, the following continuity condition holds

$$\underline{d}(u_h(t),\mu_h) \;=\; 0\,; \tag{4.6}$$

- $\forall\, i \in \{I, II\}$, the initial conditions are satisfied

$$u_h(0) = \tilde{u}_{0,h} \quad \text{and} \quad \dot{u}_h(0) = \tilde{u}'_{0,h}\,. \tag{4.7}$$

REMARK 4.1 Condition (4.6) is equivalent to $u_h(t) \in \operatorname{Ker} \underline{D}^h$, $\forall\, t \in]\,0,T\,[$. Since $\operatorname{Ker}\underline{D}^h$ is finite dimensional (and so closed), condition (4.4) implies that $\dot{u}_h(t) \in \operatorname{Ker}\underline{D}^h$. □

REMARK 4.2 In the initial conditions (4.7), $\tilde{u}_{0,h}$ is the $\|.\|_{\underline{V}}$-projection of $\tilde{u}_0$ on $\operatorname{Ker}\underline{D}^h$, while $\tilde{u}'_{0,h}$ is the $\|.\|_{\underline{H}}$-projection of $\tilde{u}'_0$ on $\operatorname{Ker}\underline{D}^h$. □

The existence and uniqueness result for $(\mathbf{G}_h)$ is given by

Theorem 4.1 *Problem $(\mathbf{G}_h)$ has a unique solution (u_h, λ_h). Moreover, there exist a constant*

C, independent of h, and a constant $\kappa_h > 0$, in general dependent on h, such that

$$\begin{aligned} \|u_h\|^2_{\underline{W}} &\leq C\,(\,\|f\|^2_{L^2(\underline{H})} + \|\tilde{u}_{0,h}\|^2_{\underline{V}} + \|\tilde{u}'_{0,h}\|^2_{\underline{H}}\,) \\ \|\lambda_h\|^2_{H_t^{-1}(\Lambda)} &\leq \frac{1}{\kappa_h^2}\,(\,\|f\|^2_{L^2(\underline{H})} + \|u_h\|^2_{\underline{W}}\,) \end{aligned}$$

where we recall that $\|u\|^2_{\underline{W}} = \|u\|^2_{C^0(\underline{V})} + \|\dot{u}\|^2_{C^0(\underline{H})} + \|\dot{\widehat{\gamma_\partial u}}\|^2_{L^2(\underline{Z})}$.

Proof Suppose (u_h, λ_h) is a solution to $(\mathbf{G}_h)$. According to Remark 4.1, we have

$$u_h \in C^0(0,T;\,\mathrm{Ker}\,\underline{D}^h) \quad \text{and} \quad \dot{u}_h \in C^0(0,T;\,\mathrm{Ker}\,\underline{D}^h)\,.$$

– *Existence and uniqueness of u_h*

If the test function v_h of (4.5) is taken in $\mathrm{Ker}\,\underline{D}^h$, then $\underline{d}(v_h,\,[\lambda_h,\varphi]) = 0$ and we end up with

$$\underline{c}\,(\ddot{u}_h,\,v_h) + \underline{a}\,(u_h,\,v_h) + \underline{b}\,(\dot{\widehat{\gamma_\partial u_h}},\,\gamma_\partial v_h) = (f,\,v_h) \tag{4.8}$$

in the sense of scalar distributions over $]\,0,T\,[$. In the finite dimensional space $\mathrm{Ker}\,\underline{D}^h$, take a basis $\mathcal{B}_h$ in which $u_h(t)$ is represented by $\mathbf{u}_h(t)$ for all t. In (4.8), set $v_h \in \mathcal{B}_h$. We obtain a second order differential system, that can be reduced to the first order differential system

$$\frac{d}{dt}\begin{pmatrix} \mathbf{u}_h \\ \overline{\mathbf{u}}_h \end{pmatrix} + \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{M}_h^{-1}\mathbf{K}_h & \mathbf{M}_h^{-1}\mathbf{B}_h \end{pmatrix}\begin{pmatrix} \mathbf{u}_h \\ \overline{\mathbf{u}}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{M}_h^{-1}\,\mathbf{f}_h \end{pmatrix},$$

in which

$\mathbf{M}_h$: mass matrix representing $\underline{c}$ over $\mathrm{Ker}\,\underline{D}^h$ in $\mathcal{B}_h$

$\mathbf{K}_h$: stiffness matrix representing $\underline{a}$ over $\mathrm{Ker}\,\underline{D}^h$ in $\mathcal{B}_h$

$\mathbf{B}_h$: boundary matrix representing $\underline{b}$ over $\mathrm{Ker}\,\underline{D}^h$ in $\mathcal{B}_h$

$\mathbf{f}_h(t)$: vector representing $L^2(\Omega)$-products of $f(t)$ with $\mathcal{B}_h$'s elements

Note that $\mathbf{M}_h$ is invertible because $\underline{c}$ is coercive. As far as the initial conditions are concerned,

$$(\,\mathbf{u}_h(0),\,\overline{\mathbf{u}}_h(0)\,) = (\,\tilde{\mathbf{u}}_{0,h},\,\tilde{\mathbf{u}}'_{0,h}\,) \in \mathrm{Ker}\,\underline{D}^h \times \mathrm{Ker}\,\underline{D}^h\,.$$

It is well-known [6] that (i) there exists a unique solution to such a system and (ii) the regularity of the solution depends on that of the right-hand side. Here, $\mathbf{f}_h \in L^2(0,T;\,\mathbf{R}^{\dim[\mathrm{Ker}\underline{D}_h]})$ because

$f \in L^2(0,T;\, \underline{H})$. It follows that $(\mathbf{u}_h, \overline{\mathbf{u}}_h)$ is continuous with respect to t. In other words, after recombination, u_h satisfies (4.3) and (4.4).

– *Continuous dependence of u_h on the data*

It also follows from $\mathbf{f}_h \in L^2(0,T;\, \mathbf{R}^{\dim[\mathrm{Ker}\underline{D}_h]})$ that $\frac{d}{dt}(\mathbf{u}_h, \overline{\mathbf{u}}_h)$ is L^2 with respect to t. This means, via the linear combination, that $\ddot{u}_h \in L^2(0,T;\, \mathrm{Ker}\,\underline{D}^h)$. Hence, we can carry out the very classical energy calculations [6] by putting $v_h = \dot{u}_h \in \mathrm{Ker}\,\underline{D}^h$ in (4.8). [The next steps are: integrate over $t \in [0,T]$, find a lower bound to the semi-discrete energy, and after a few tricks, use Gronwall's lemma to conclude.] The final result is that there exists C, independent of h, such that

$$\|u_h\|^2_{\underline{W}} \le C\,(\, \|f\|^2_{L^2(\underline{H})} + \|\tilde{u}_{0,h}\|^2_{\underline{V}} + \|\tilde{u}'_{0,h}\|^2_{\underline{H}}\,)\,.$$

– *Existence and uniqueness of λ_h*

If (u_h, λ_h^1) and (u_h, λ_h^2) are two solutions to $(\mathbf{G}_h)$, then the side-by-side difference of (4.5) yields

$$\forall\, \varphi \in \mathcal{D}'(]\,0,T\,[)\,, \quad [\lambda_h^2 - \lambda_h^1, \varphi] \in \mathrm{Ker}\,{}^t\underline{D}^h\,.$$

Since $\mathrm{Ker}\,{}^t\underline{D}^h = \{0\}$ (thanks to Hypothesis 4.1), we must have $\lambda_h^1 = \lambda_h^2$ in $\mathcal{D}'(]\,0,T\,[;\, \Lambda^h)$.

Let $\varphi \in \mathcal{D}(]\,0,T\,[)$. Put (4.5) under the form

$$\forall\, v_h \in \underline{V}^h\,, \quad \underline{d}(v_h, [\lambda_h, \varphi]) = \langle L_\varphi^h, v_h \rangle$$

where $L_\varphi^h \in (\mathrm{Ker}\,\underline{D}^h)^0 \subset (\underline{V}^h)'$. Furthermore, there exists $C' > 0$, independent of h, such that

$$\|L_\varphi^h\|^2_{(\underline{V}^h)'} \le C'\,(\, \|u_h\|^2_{\underline{W}} + \|f\|^2_{L^2(\underline{H})}\,)\, \|\varphi\|_1^2\,. \tag{4.9}$$

The finite dimensions of $\underline{V}^h$ and Λ^h imply that $(\mathrm{Ker}\,\underline{D}^h)^0 = \mathrm{Im}\,{}^t\underline{D}^h$ and that there exists k_h, the largest constant that satisfies

$$\forall\, \mu_h \in \Lambda^h\,, \quad \sup_{v_h \in \underline{V}^h} \frac{\underline{d}(v_h, \mu_h)}{\|v_h\|_{\underline{V}}} \ge k_h\, \|\mu_h\|_\Lambda\,. \tag{4.10}$$

Note that $k_h > 0$ because $\mathrm{Ker}\,{}^t\underline{D}^h = \{0\}$. In fact, (4.10) expresses that there is a continuous lifting from $(\mathrm{Ker}\,\underline{D}^h)^0$ to Λ^h. Consequently, there exists $\lambda_\varphi^h \in \Lambda^h$ such that ${}^t\underline{D}^h \lambda_\varphi^h = L_\varphi^h$ and $\|\lambda_\varphi^h\|_{\underline{V}'} \le \frac{1}{k_h} \|L_\varphi^h\|_{(\underline{V}^h)'}$. The latter inequality, together with (4.9), shows that the mapping

$\lambda_h \,:\, \varphi \mapsto [\lambda_h, \varphi] \,=\, \lambda_\varphi^h$ is continuous with respect to the topology of $\mathcal{D}(]\,0,T\,[)$. Hence, $\lambda_h \in \mathcal{D}'(]\,0,T\,[;\, \Lambda^h)$.

– *Continuous dependence of λ_h on the data*

On the other hand, since $\mathcal{D}(]0,T[)$ is dense in $H_0^1(]\,0,T\,[)$ and

$$\forall\, \varphi \in \mathcal{D}(]\,0,T\,[)\,, \quad \|\, [\lambda_h, \varphi]\,\|_\Lambda^2 \,\leq\, \frac{C'}{k_h^2}\,(\,\|u_h\|_{\underline{W}}^2 + \|f\|_{L^2(\underline{H})}^2\,)\,\|\varphi\|_1^2\,,$$

it is possible to extend $\lambda_h : \mathcal{D}(]\,0,T\,[) \mapsto \Lambda^h$ to an element of $H_t^{-1}(\Lambda^h) \,=\, \mathcal{L}(H_0^1(]\,0,T\,[)\,;\, \Lambda^h)$, regarded as a subspace of $H_t^{-1}(\Lambda)$, that has the same norm, i.e.

$$\|\lambda_h\|_{H_t^{-1}(\Lambda)}^2 \,\leq\, \frac{C'}{k_h^2}\,(\,\|u_h\|_{\underline{W}}^2 + \|f\|_{L^2(\underline{H})}^2\,)\,,$$

which completes the proof with $\kappa_h^2 \,=\, \dfrac{C'}{k_h^2}$. ◁

It might happen that $k_h \to 0$ as $h \to 0$. In such a case, not only the bound on λ_h will go to infinity, but also the error estimates (addressed in the next subsection) will be much worsened. To avoid this unfavorable situation, we put forward

Hypothesis 4.2 *The discretization spaces $\underline{V}^h$ and Λ^h are chosen in such a way that*

$$\exists\, k_0 > 0 \;\mid\; \forall\, h > 0\,, \quad \forall\, \mu_h \in \Lambda^h\,, \quad \sup_{v_h \in \underline{V}^h} \frac{\underline{d}(v_h, \mu_h)}{\|v_h\|_{\underline{V}}} \,\geq\, k_0\,\|\mu_h\|_\Lambda\,.$$

Put another way, the constants k_h defined in (4.10) are always greater than some $k_0 > 0$.

Hypothesis 4.2 is referred to as the *discrete inf-sup condition* [3, 4], or sometimes *uniform continuous lifting* property. Anyhow, the constant κ_h of Theorem 4.1 no longer depends on h.

4.3 Error estimates

We wish to find some upper-bounds for the errors

$$\epsilon_h \,=\, u - u_h \;\in \underline{W} \qquad \text{and} \qquad \zeta_h \,=\, \lambda - \lambda_h \;\in H_t^{-1}(\Lambda)$$

where the normed spaces $\underline{W}$ and $H_t^{-1}(\Lambda)$ were introduced earlier. Putting $v \,=\, v_h$ in (3.22) and subtracting to (4.5) shows that the errors ϵ_h and ζ_h satisfy

$$-\;\underline{c}([\dot{\epsilon}_h, \dot{\varphi}],\, v_h) \,+\, \underline{a}([\epsilon_h, \varphi],\, v_h) \,+\, \underline{b}([\dot{\widehat{\gamma_\partial \epsilon_h}}, \varphi],\, \gamma_\partial v_h) \,=\, \underline{d}(v_h,\, [\zeta_h, \varphi])\,, \tag{4.11}$$

$\forall v_h \in \underline{V}^h$ and $\forall \varphi \in \mathcal{D}(]0,T[)$. Putting $\mu = \mu_h$ in (3.23) and subtracting to (4.6) results in

$$\underline{d}(\epsilon_h(t), \mu_h) = 0, \tag{4.12}$$

$\forall \mu_h \in \Lambda^h$ and $\forall t \in]0,T[$. From (4.11), it can be shown that

Proposition 4.1 *There exists a constant C, independent of h, such that*

$$\|\zeta_h\|_{H_t^{-1}(\Lambda)} \leq \frac{C}{k_0} \|\epsilon_h\|_{\underline{W}}.$$

In other words, an estimate for ζ_h is known as soon as an estimate for ϵ_h is avalaible.

PROOF For $\varphi \in \mathcal{D}(]0,T[)$, we have ${}^t\underline{D}^h[\zeta_h, \varphi] = L_\varphi^h$ where $L_\varphi^h \in (\underline{V}^h)'$ is defined by the left-hand side of (4.11) and is bounded by

$$\|L_\varphi^h\|_{(\underline{V}^h)'} \leq C\|\epsilon_h\|_{\underline{W}} \|\varphi\|_1 \qquad \text{with } C \text{ independent of } h.$$

By the uniform continuous lifting property,

$$\|[\zeta_h, \varphi]\|_\Lambda \leq \frac{C}{k_0} \|\epsilon_h\|_{\underline{W}} \|\varphi\|_1,$$

so that by extension from $\mathcal{D}'(]0,T[;\Lambda)$ to $H_t^{-1}(\Lambda)$, we obtain the desired estimate. $\triangleleft$

Our task is now to find an estimate for ϵ_h. In conformity with the philosophy of finite element error analysis, we will try to bound ϵ_h by some projection errors ϵ_h^P and ζ_h^P. Let us elaborate a little on these projection errors.

Definition 4.1 *For $v \in \underline{V}$, let $\pi_h v$ be the projection of v on* $\operatorname{Ker} \underline{D}^h$ *in the sense of the weighted norm $\|.\|_{\underline{H}}^2 + \underline{a}(.,.)$. The difference $\epsilon_h^P(v) = v - \pi_h v$ is termed projection error. If $v_h \in \operatorname{Ker} \underline{D}^h$ is supposed to approximate v, the difference $\epsilon_h^A(v) = \pi_h v - v_h$ is called approximation error.*

Let u be the pressure component of the solution to $(\mathbf{G}_h)$. Define $(\pi_h u)(t) = \pi_h(u(t))$ for $t \in [0,T]$. The errors $\epsilon_h^P(u)$ and $\epsilon_h^A(u)$ are similarly defined and is designated by ϵ_h^P and ϵ_h^A.

REMARK 4.3 π_h is continuous from $\underline{V}$ onto $\operatorname{Ker} \underline{D}^h$. Therefore, $u \in C^0(0,T;\underline{V})$ implies $\epsilon_h^P \in C^0(0,T;\underline{V})$. Also, $u_h \in C^0(0,T;\operatorname{Ker} \underline{D}^h)$ implies $\epsilon_h^A \in C^0(0,T;\operatorname{Ker} \underline{D}^h)$. $\square$

Definition 4.2 *For $\mu \in L^2(\Gamma)$, let $\Xi_h \mu$ be the projection of μ on $\Lambda^h \subset L^2(\Gamma)$ in the sense of the $L^2(\Gamma)$-norm. The difference $\zeta_h^P(\mu) = \mu - \Xi_h \mu$ is termed projection error.*

Let λ be the multiplier component of the solution to $(\mathbf{G}_h)$, and assume $\lambda : \varphi \mapsto [\lambda, \varphi]$ has range in $L^2(\Gamma) \subset \Lambda$. Define $\Xi_h \lambda : \varphi \mapsto [\Xi_h \lambda, \varphi] = \Xi_h [\lambda, \varphi]$ for $\varphi \in \mathcal{D}(]0, T[)$. The projection error $\zeta_h^P(\lambda)$ is similarly defined and is designated by ζ_h^P.

Remark 4.4 Ξ_h is continuous from $L^2(\Gamma)$ onto Λ^h, both regarded as subspaces of Λ. So, $\lambda \in H_t^{-1}(L^2(\Gamma))$ implies $\zeta_h^P \in H_t^{-1}(L^2(\Gamma))$. □

What we have in mind is to bound ϵ_h by some combination of ϵ_h^P and ζ_h^P. These depend on the discretization spaces alone, and not on the particular form of the problem at hand. Finding error estimates for projection errors is a matter for interpolation theory, to which we have thus "passed the buck." At some point in the calculations, however, the derivatives $\dot{\epsilon}_h^P$ and $\ddot{\epsilon}_h^P$ will appear, for which the existence of estimates relies on that of $\pi_h \dot{u}$ and $\pi_h \ddot{u}$. For π_h to operate on $\dot{u}$ and $\ddot{u}$, the solution u should be more regular than required in the formulation.

Hypothesis 4.3 *Let (u, λ) be the solution to $(\mathbf{G})$. For the purpose of error estimates, we assume*

1. *There exists an integer $r \geq 2$ such that $u \in L^\infty(0, T; H^r(\Omega))$, $\dot{u} \in L^\infty(0, T; H^{r-1}(\Omega))$, and $\ddot{u} \in L^2(0, T; H^{r-2}(\Omega))$;*
2. *There exists an integer $l \geq 1$ such that $\lambda \in L^\infty(0, T; H^l(\Gamma))$ and $\dot{\lambda} \in L^2(0, T; H^{l-1}(\Gamma))$.*

From now on, we can safely speak of $\zeta_h^P \in L^2(0, T; \Lambda)$ and the like

$$\epsilon_h^P \in C^0(0, T; \underline{V}), \qquad \dot{\epsilon}_h^P \in C^0(0, T; \underline{V}), \qquad \ddot{\epsilon}_h^P \in L^2(0, T; \underline{H}),$$
$$\gamma_\partial \epsilon_h^P \in C^0(0, T; \underline{Z}), \quad \gamma_\partial \dot{\epsilon}_h^P \in C^0(0, T; \underline{Z}), \quad \gamma_\partial \ddot{\epsilon}_h^P \in L^2(0, T; \underline{Z}).$$

We denote

$$\begin{aligned} |||\, \epsilon_h^P \,|||^2 &= \|\epsilon_h^P\|^2_{C^0(\underline{V})} + \|\ddot{\epsilon}_h^P\|^2_{L^2(\underline{H})} \\ |||\, \gamma_\partial \epsilon_h^P \,|||^2 &= \|\gamma_\partial \dot{\epsilon}_h^P\|^2_{C^0(\underline{Z})} + \|\gamma_\partial \ddot{\epsilon}_h^P\|^2_{L^2(\underline{Z})} \end{aligned} \tag{4.13}$$

for short-hand conveniency.

According to Hypothesis 4.3, λ is an element of $L^\infty(0,T;\, H^l(\Gamma))$, i.e. a function. As a result, its projection $\Xi_h \lambda$ is a function, and so $\zeta_h^P \in L^\infty(0,T;\, \Lambda)$. Likewise, $\dot{\zeta}_h^P \in L^2(0,T;\, \Lambda)$. This allows us to introduce

$$|||\, \zeta_h^P \,|||^2 \;=\; \|\, \zeta_h^P \,\|^2_{L^\infty(\Lambda)} \;+\; \|\, \dot{\zeta}_h^P \,\|^2_{L^2(\Lambda)} \,. \tag{4.14}$$

Theorem 4.2 *There exists a constant C, independent of h, such that*

$$\|\, \epsilon_h^A \,\|^2_{\underline{W}} \;\leq\; C\, \Big(\, \|\, \epsilon_h^A(0) \,\|^2_{\underline{V}} + \|\, \dot{\epsilon}_h^A(0) \,\|^2_{\underline{H}} + |||\, \epsilon_h^P \,|||^2 + |||\, \gamma_\partial \epsilon_h^P \,|||^2 + |||\, \zeta_h^P \,|||^2 \,\Big)$$

where $\|.\|_{\underline{W}}$ was defined in Proposition 3.2 as $\|\epsilon_h^A\|^2_{\underline{W}} \;=\; \|\epsilon_h^A\|^2_{C^0(\underline{V})} + \|\dot{\epsilon}_h^A\|^2_{C^0(\underline{H})} + \|\gamma_\partial \dot{\epsilon}_h^A\|^2_{L^2(\underline{Z})}$.

Proof Splitting ϵ_h into $\epsilon_h = \epsilon_h^P + \epsilon_h^A$ and setting $v_h = \dot{\epsilon}_h^A(t) \in \underline{V}^h$ in (4.11), we get

$$\begin{aligned} -\,\underline{c}([\dot{\epsilon}_h^A, \dot{\varphi}],\, \dot{\epsilon}_h^A) \,+\, \underline{a}([\epsilon_h^A, \varphi],\, \dot{\epsilon}_h^A) \,+\, \underline{b}([\gamma_\partial \dot{\epsilon}_h^A, \varphi],\, \gamma_\partial \dot{\epsilon}_h^A) \;&=\; \underline{c}([\dot{\epsilon}_h^P, \dot{\varphi}]),\, \dot{\epsilon}_h^A) \\ -\,\underline{a}([\epsilon_h^P, \varphi],\, \dot{\epsilon}_h^A) \,-\, \underline{b}([\gamma_\partial \dot{\epsilon}_h^P, \varphi],\, \gamma_\partial \dot{\epsilon}_h^A) \;&+\; \underline{d}(\dot{\epsilon}_h^A,\, [\zeta_h, \varphi]) \end{aligned} \tag{4.15}$$

Let us apply some transformations. On one hand, since $\dot{\epsilon}_h^A \in \operatorname{Ker} \underline{D}^h$ and $[\zeta_h - \zeta_h^P, \varphi] \in \Lambda^h$ for all $\varphi \in \mathcal{D}(]\,0,T\,[)$, we have $\underline{d}(\dot{\epsilon}_h^A,\, [\zeta_h, \varphi]) \;=\; \underline{d}(\dot{\epsilon}_h^A,\, [\zeta_h^P, \varphi])$. On the other hand, as a projection error, ϵ_h^P can be characterized in the weak sense by

$$\forall\, v_h \in \operatorname{Ker} \underline{D}^h\,,\;\; \forall\, \varphi \in \mathcal{D}(]\,0,T\,[)\,, \qquad ([\epsilon_h^P, \varphi],\, v_h) \,+\, \underline{a}([\epsilon_h^P, \varphi],\, v_h) \;=\; 0$$

Thence, in view of Hypothesis 4.3, (4.15) becomes

$$\begin{aligned} \underline{c}(\ddot{\epsilon}_h^A,\, \dot{\epsilon}_h^A) \,+\, \underline{a}(\epsilon_h^A,\, \dot{\epsilon}_h^A) \,+\, \underline{b}(\gamma_\partial \dot{\epsilon}_h^A,\, \gamma_\partial \dot{\epsilon}_h^A) \;&=\; -\underline{c}(\ddot{\epsilon}_h^P,\, \dot{\epsilon}_h^A) \,+\, (\epsilon_h^P,\, \dot{\epsilon}_h^A) \\ &\quad\; -\,\underline{b}(\gamma_\partial \dot{\epsilon}_h^P,\, \gamma_\partial \dot{\epsilon}_h^A) \,+\, \underline{d}(\dot{\epsilon}_h^A,\, \zeta_h^P) \end{aligned} \tag{4.16}$$

in the sense of functions (ζ_h^P is a function, but ζ_h is not necessarily so).

At this stage, we resort once again to the classical energy calculations, as suggested by Dupont [7] for the continuous case. In essence, after integrating (4.16) over $t \in [0,T]$ and using integration by parts, we can manage to place ϵ_h^A only where a $\underline{V}$-bound exists and $\dot{\epsilon}_h^A$ only where a $\underline{H}$-bound exists. Although the details are a little intricate, there is no difficulty in transposing the calculations presented in [7] to our problem. $\lhd$

By triangular inequality, a similar $\underline{W}$-estimate for ϵ_h can be worked out in terms of $\epsilon_h^A(0)$, ϵ_h^P, and ζ_h^P. In the right-hand side of this estimate, however, the only component for which we have an immediate upper-bound is $|||\,\zeta_h^P\,|||$. Indeed, it is well-known from finite element analysis [4] that if $\mu \in H^l(\Gamma)$ and if the mesh on the interface Γ meets some uniformity criterion, then there exist C, independent of h_Γ, such that

$$\|\mu - \Xi_h\mu\|_\Lambda \;\leq\; C\, h_\Gamma^l\, |\mu|_{l,\Lambda}\,. \tag{4.17}$$

By continuity of the projection operator Ξ_h, we obtain $|||\,\zeta_h^P\,||| \;\leq\; C'\, h_\Gamma^{l-1}\, |||\,\lambda\,|||_l$.

The remaining terms in the upper-bound of the error estimate raise two major issues concerning the control of ϵ_h.

1. The initial data must be "well" approximated, i.e. in such a manner that $\|\,\epsilon_h^A(0)\,\|_{\underline{V}}^2 + \|\,\dot{\epsilon}_h^A(0)\,\|_{\underline{H}}^2$ —the error brought about by discretizing the initial data— is bounded by a suitable power of $h_I^2 + h_{II}^2$. Fortunately, in practical modelling, it occurs very often that $\tilde{u}_0 = \tilde{u}_0' = 0$, so that $\tilde{u}_{0,h} = \tilde{u}_{0,h}' = 0$, and $\epsilon_h^A(0) = \dot{\epsilon}_h^A(0) = 0$.

2. The spaces $\underline{V}^h$ and Λ^h must be "consistently" chosen, i.e. in such a manner that $|||\,\epsilon_h^P\,|||^2$ —the error induced by projecting on $\operatorname{Ker}\underline{D}^h$— is bounded by a suitable power of $h_I^2 + h_{II}^2 + h_\Gamma^2$. The main difficulty lies in the fact that π_h has range in $\operatorname{Ker}\underline{D}^h$ instead of $\underline{V}^h$ (otherwise, estimates in terms of h^r are available from finite element literature). For the moment, we are not in a position to assess the error due to the weighted projection π_h on $\operatorname{Ker}\underline{D}^h$. It is very likely that an additional hypothesis would have to be assumed between $\underline{V}^h$ and Λ^h.

REMARK 4.5 There is a third issue in error estimates that is worth mentioning, even though it goes far beyond the scope of this paper. By definition, $\|.\|_{\underline{W}}$ involves an H^1-norm, which is known not to be optimal in the Galerkin context: the power of h can be increased by one if the error is measured in the L^2-norm. The reader is referred to [3, 4, 7, 16] for a more comprehensive treatment of this aspect (the Aubin-Nitsche technique). □

5. Fully-discrete approximation

The purpose of this section is to demonstrate that the error corresponding to the fully-discrete scheme proposed in Section 2 is second order in time. To achieve this goal, we will have to devise a judicious way of comparing the fully-discrete solution, represented by the sequence (u_h^n, λ_h^n), $n \in \{0, 1, \cdots, N\}$, with the continuous solution (u, λ) studied in Section 3.

5.1 Fully-discrete norms

Let $N = T/\Delta t$. Following the spirit of finite difference analysis, we define

Definition 5.1 *Let $\tilde{W}^N$ be the space of all finite sequences $s = (s^n)_{n\in\{0,1,\cdots,N\}}$, taking values in $\underline{V}$. As a norm over $\tilde{W}^N$, we consider*

$$\|s\|^2_{\tilde{W}} = \sup_{0\le n\le N-1} \left\| \frac{s^{n+1} - s^n}{\Delta t} \right\|^2_{\underline{H}} + \sup_{0\le n\le N-1} \left\| \frac{s^{n+1} + s^n}{2} \right\|^2_{\underline{V}}$$

This norm is the discrete counterpart of $\|\dot{s}\|^2_{C^0(\underline{H})} + \|s\|^2_{C^0(\underline{V})}$. Its dependence on N should not come as a surprise to the reader accustomed to finite difference methods. It is by means of $\|.\|_{\tilde{W}}$ that the error associated with u will be measured. Similarly, introduce

Definition 5.2 *Let $\tilde{H}_0^N$ be the space of all finite sequences $\varphi = (\varphi^n)_{n\in\{0,1,\cdots,N\}}$, taking values in $\mathbf{R}$ such that $\varphi^0 = \varphi^1 = \varphi^{N-1} = \varphi^N = 0$. As a norm over $\tilde{H}_0^N$, we define*

$$\|\varphi\|^2_{\tilde{1}} = \sum_{n=0}^{N-1} \left(\left| \frac{\varphi^{n+1} - \varphi^n}{\Delta t} \right|^2 + \left| \frac{\varphi^{n+1} + \varphi^n}{2} \right|^2 \right) \Delta t\,.$$

The space $\tilde{H}_0^N$ can be seen as the discrete counterpart of $\mathcal{D}(]\,0, T\,[)$, while $\|.\|_{\tilde{1}}$ appears as the discrete version of $\|.\|_1$ defined in Section 3. Using the elements of $\tilde{H}_0^N$ as test sequences, it is possible to figure out a discrete equivalent of $H_t^{-1}(\Lambda)$.

Definition 5.3 *Let $\tilde{D}^N$ be the space of all finite sequences $\mu = (\mu^n)_{n\in\{0,1,\cdots,N\}}$, taking values in Λ. Suppose that $\underline{V}^h$ depends on N. Then, a norm over $\tilde{D}^N$ can be defined as the smallest constant $k_N > 0$ such that*

$$\forall\, \varphi = (\varphi^n)_{n\in\{0,1,\cdots,N\}} \in \tilde{H}_0^N\,, \quad \forall\, v_h \in \underline{V}^h\,, \quad \left| \sum_{n=0}^{N} \underline{d}\,(v_h, \mu^n)\, \varphi^n \right| \le k_N\, \|v_h\|_{\underline{V}}\, \|\varphi\|_{\tilde{1}}\,.$$

The existence of $\|\mu\|_{\tilde{D}}$ follows from the finite dimension of $\tilde{H}_0^N$, which also accounts for its dependence on N. This is the norm by means of which we will be measuring the error associated with λ_h. We need a technical lemma before going into the heart of the matter.

Lemma 5.1 *For all $v = (v^n)_{n\in\{0,1,\cdots,N\}} \in \tilde{W}^N$, for all $\varphi = (\varphi^n)_{n\in\{0,1,\cdots,N\}} \in \tilde{H}_0^N$, and for all $w \in \underline{V}$, posing $\phi^n = \varphi^n \otimes w$, we have the average equality*

$$\sum_{n=1}^{N-1} \underline{a}(v^n, \phi^n) = \sum_{n=0}^{N-1} \underline{a}\left(\frac{v^{n+1}+v^n}{2}, \frac{\phi^{n+1}+\phi^n}{2}\right) + \frac{\Delta t^2}{4} \sum_{n=0}^{N-1} \underline{a}\left(\frac{v^{n+1}-v^n}{\Delta t}, \frac{\phi^{n+1}-\phi^n}{\Delta t}\right) \tag{5.1}$$

and the integration by parts identities

$$\sum_{n=1}^{N-1} \underline{b}\left(\gamma\partial\frac{v^{n+1}-v^{n-1}}{2\Delta t}, \gamma\partial\phi^n\right) = -\sum_{n=0}^{N-1} \underline{b}\left(\gamma\partial\frac{v^{n+1}+v^n}{2}, \gamma\partial\frac{\phi^{n+1}-\phi^n}{\Delta t}\right) \tag{5.2}$$

$$\sum_{n=1}^{N-1} \underline{c}\left(\frac{v^{n+1}-2v^n+v^{n-1}}{\Delta t^2}, \phi^n\right) = -\sum_{n=0}^{N-1} \underline{c}\left(\frac{v^{n+1}-v^n}{\Delta t}, \frac{\phi^{n+1}-\phi^n}{\Delta t}\right) \tag{5.3}$$

PROOF The proof is fairly elementary. It suffices to carry out the discrete summations explicitly. The assumption $\varphi^0 = \varphi^N = 0$ turns out to be capital for the three equalities. $\varphi^1 = \varphi^{N-1} = 0$ is needed for (5.3) only. $\triangleleft$

5.2 Fully-discrete problem

The fully-discrete problem, namely sequence $(\mathbf{G}_n)_{n\in\{0,1,\cdots,N-1\}}$ of stationary problems, was already formulated in Section 2. Notwithstanding, let us state it once again with the bilinear-form notations in order to specify the discretization spaces.

$(\mathbf{G}_h^n)$ GIVEN

$$f^n \in L^2(\Omega), \quad u_h^n, u_h^{n-1} \in \underline{V}^h, \text{ and } K, \rho \text{ fulfilling Hypothesis 3.1} \tag{5.4}$$

FIND

$$u_h^{n+1} \in \underline{V}^h \text{ and } \lambda_h^n \in \Lambda^h \text{ such that} \tag{5.5}$$

- $\forall\, v_h \in \underline{V}^h$, the following discrete integral relation holds

$$\underline{c}(\delta_2^n u_h,\, v_h) + \underline{a}(u_h^n,\, v_h) + \underline{b}(\delta_1^n \gamma_\partial u_h,\, \gamma_\partial v_h) = \underline{d}(v_h,\, \lambda_h^n) + (f^n,\, v_h) \quad (5.6)$$

where the discrete derivation operators δ_2 and δ_1 are defined by

$$\delta_2^n u_h = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} \quad \text{and} \quad \delta_1^n w_h = \frac{w_h^{n+1} - w_h^{n-1}}{2\Delta t}.$$

- $\forall\, \mu_h \in \Lambda^h$, the following discrete continuity condition holds

$$\underline{d}(\bar{\delta}_0^n u_h,\, \mu_h) = 0 \quad \text{where} \quad \bar{\delta}_0^n u_h = \frac{u_h^{n+1} + 2u_h^n + u_h^{n-1}}{4}. \quad (5.7)$$

Remark 5.1 The finite dimensional spaces $\underline{V}^h$ and Λ^h are those introduced for the semi-discrete approximation. □

Remark 5.2 For $n = 0$, it is natural to impose $u_h^0 = \tilde{u}_{0,h}$. As far as the fictitious value for u_h^{-1} is concerned, we can proceed by setting

$$\frac{u_h^1 - u_h^{-1}}{2\Delta t} = \tilde{u}'_{0,h}\,, \quad (5.8)$$

where $\tilde{u}_{0,h}$ and $\tilde{u}'_{0,h}$ were defined in Section 4. Equations (5.8) and (5.6) written for $n = 0$ are combined together with a view to eliminating u_h^{-1}. □

Condition (5.7) is equivalent to $\bar{\delta}_0^n u_h \in \operatorname{Ker} \underline{D}^h$ for all $n \in \{0, 1, \cdots, N-1\}$. In addition, according to their definitions, $\tilde{u}_{0,h} \in \operatorname{Ker} \underline{D}^h$ and $\tilde{u}'_{0,h} \in \operatorname{Ker} \underline{D}^h$. Under those circumstances, it can then be shown that

Lemma 5.2 *If (u_h^{n+1}, λ_h^n) is a solution to $(\mathbf{G}_h^n)$ for $n \in \{0, 1, \cdots, N-1\}$, then $u_h^n \in \operatorname{Ker} \underline{D}^h$ for all $n \in \{0, 1, \cdots, N\}$.*

Proof The proof can be done by induction on n. For $n = 0$, $u_h^0 = \tilde{u}_{0,h} \in \operatorname{Ker} \underline{D}^h$. For $n = 1$, a little algebra leads to

$$u_h^1 = 2\,\bar{\delta}_0^0 u_h + \Delta t\, \delta_1^0 u_h - u_h^0\,,$$

which is a linear combination of elements of $\operatorname{Ker} \underline{D}^h$, especially because $\delta_1^0 u_h = \tilde{u}'_{0,h} \in \operatorname{Ker} \underline{D}^h$.

Suppose, for $n \geq 1$, that u_h^{n-1} and u_h^n belong to $\mathrm{Ker}\,\underline{D}^h$. Then

$$u_h^{n+1} = 4\,\bar{\delta}^n u_h - 2\,u_h^n - u_h^{n-1}$$

is evidently a linear combination of elements of $\mathrm{Ker}\,\underline{D}^h$, and thus belongs to $\mathrm{Ker}\,\underline{D}^h$. ◁

The following Proposition is concerned with existence and uniqueness result.

Proposition 5.1 *For all $n \in \{0, 1, ..., N-1\}$, there exists a unique solution (u_h^{n+1}, λ_h^n) to problem $(\mathbf{G}_h^n)$.*

Proof The proof is based on the fact that $\mathrm{Ker}\,\underline{D}^h$ is finite dimensional. Set $v_h \in \mathrm{Ker}\,\underline{D}^h$ in (5.6) so as to cancel out $\underline{d}(v_h, \lambda_h^n)$. Take any basis of $\mathrm{Ker}\,\underline{D}^h$. In this basis, the matrix representing the bilinear form $\underline{c}(.,.) + \dfrac{\Delta t}{2}\,\underline{b}(.,.)$ is symmetric positive definite, which entails existence and uniqueness for u_h^{n+1}. The uniqueness of λ_h^n follows from $\mathrm{Ker}\,{}^t\underline{D}^h = \{0\}$, while its existence is implied by the continuous lifting from $(\mathrm{Ker}\,\underline{D}^h)^0 = \mathrm{Im}\,{}^t\underline{D}^h$ to Λ^h. ◁

We wish to investigate about the stability of such a scheme. It is known, for Galerkin methods, that a stability condtion must be imposed between Δt and h. In the present instance, this would mean that $\underline{V}^h$ and possibly Λ^h must be linked to N. For reasons that will be clarified later [proof of Theorem 5.1], we are compelled to assume

Hypothesis 5.1 *$\underline{V}^h$ is chosen accordingly with $N = T/\Delta t$ in such a way that*

$$\exists\,\eta \in]0,1[\;\mid\; \forall\,\Delta t \geq 0\,,\;\; \forall\,v_h \in \underline{V}^h\,,\quad \frac{\Delta t^2}{4}\,\underline{a}(v_h, v_h) \;\leq\; (1-\eta)\,\underline{c}(v_h, v_h)\,.$$

We will refer to Hypothesis 5.1 as the *uniform stability condition.* Practically, since the order of magnitude of $\underline{a}(.,.)$ is about constant and that of $\underline{c}(.,.)$ is proportional to h^2 in 2-D, Hypothesis 5.1 means that

$$\forall\, i \in \{I, II\}\,,\quad \frac{(c_{|i})_{\max}\Delta t}{h_i} \leq (1-\eta)\,CFL\,. \tag{5.9}$$

where *CFL* represents the stability threshold, determined by the type of finite elements in use. A nice —and very useful— consequence of this is

Lemma 5.3 *Let $\chi_h > 0$ be the smallest constant for which*

$$\forall\, v_h \in \underline{V}^h, \quad \|v_h\|_{\underline{V}} \le \chi_h \|v_h\|_{\underline{H}}.$$

(χ_h exists because in a finite dimensional space, all norms are equivalent one another). If Hypothesis 5.1 holds, then the product $\chi_h \Delta t$ is bounded by a constant, independently of Δt.

Proof This result is a consequence of (5.9), since χ_h is readily seen (inverse assumption, see [4]) to be proportional to $1/\min(h_I, h_{II})$. ◁

Things are ripe now for the study of stability. Introduce the finite sequences

$$u_N = (u_h^n)_{n\in\{0,1,\cdots,N-1\}} \qquad \text{and} \qquad \lambda_N = (\lambda_h^n)_{n\in\{0,1,\cdots,N-1\}}.$$

Proposition 5.2 *For $\Delta t \to 0$, there exists a constant C, independent of Δt, such that*

$$\|\lambda_N\|_{\tilde{D}}^2 \le C\left(\|f\|_{L^2(\underline{H})}^2 + \|u_N\|_{\tilde{W}}^2\right).$$

Proof Let $w_h \in \underline{V}^h$ and $\varphi = (\varphi^n)_{n\in\{0,1,\cdots,N-1\}} \in \tilde{H}_0^N$. Pose $\phi_h^n = \varphi^n \otimes w_h$. In (5.6), put $v_h = \phi_h^n$, then sum over n. Apply Lemma 5.1 to transform the different sums. This yields

$$\begin{aligned}
\sum_{n=0}^{N} \underline{d}(\phi_h^n, \lambda_h^n) = & \sum_{n=0}^{N-1} \underline{a}\left(\frac{u_h^{n+1}+u_h^n}{2}, \frac{\phi_h^{n+1}+\phi_h^n}{2}\right) - \sum_{n=0}^{N-1} \underline{b}\left(\frac{u_h^{n+1}+u_h^n}{2}, \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right)\\
& - \sum_{n=0}^{N-1} \underline{c}\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right) + \frac{\Delta t^2}{4} \sum_{n=0}^{N-1} \underline{a}\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right)\\
& - \sum_{n=0}^{N-1} \left(\frac{f^{n+1}+f^n}{2}, \frac{\phi_h^{n+1}+\phi_h^n}{2}\right) - \frac{\Delta t^2}{4} \sum_{n=0}^{N-1} \left(\frac{f^{n+1}-f^n}{\Delta t}, \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right)
\end{aligned}$$

where we have omitted the sign γ_∂ in the arguments of the $\underline{b}$-sum. Using continuity properties and the Cauchy-Schwartz inequality, we can easily bound the first four sums in the right-hand side by $C\,\|u_N\|_{\tilde{W}}\,\|\varphi\|_{\tilde{1}}\,\|w_h\|_{\underline{V}}$. Note that for the fourth sum, which involves $\underline{a}(.,.)$, we have to resort to

$$\Delta t \left\|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right\|_{\underline{V}} \le \chi_h \Delta t \left\|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right\|_{\underline{H}}$$

and to invoke Lemma 5.3 in order to for $\|u_N\|_{\tilde{W}}$ to appear. The fifth sum is bounded without any difficulty by $C\,\|f\|_{L^2(\underline{H})}\,\|\varphi\|_{\tilde{1}}\,\|w_h\|_{\underline{V}}$, insofar as $f^n = f(n\Delta t)$ and $\Delta t \to 0$. As far as the last sum is concerned, it can be bounded quite "ruthlessly" by

$$\frac{\Delta t}{4}\sum_{n=0}^{N-1}\left(\|f^{n+1}\|_{\underline{H}}+\|f^{n}\|_{\underline{H}}\right)\left\|\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right\|_{\underline{V}} \;\leq\; \frac{C'\Delta t}{2}\,\|f\|_{L^2(\underline{H})}\,\|\varphi\|_{\tilde{1}}\,\|w_h\|_{\underline{V}},$$

on the grounds of $f^n = f(n\Delta t)$ and $\Delta t \to 0$. In short, there exists a constant C, independent of Δt, such that

$$\left|\sum_{n=0}^{N}\underline{d}(w_h,\,\lambda_h^n)\,\varphi^n\right| \;\leq\; C\left(\|u_N\|_{\tilde{W}}^2 + \|f\|_{L^2(\underline{H})}^2\right)^{1/2}\|\varphi\|_{\tilde{1}}\,\|w_h\|_{\underline{V}}$$

for all $\varphi \in \tilde{H}_0^N$ and $w_h \in \underline{V}^h$. This completes the proof. ◁

This above proposition shows that λ_N depends continuously on the data if u_N does. The latter question is elucidated in the upcoming theorem.

Theorem 5.1 *For $\Delta t \to 0$, there exists a constant C, independent of Δt, such that*

$$\|u_N\|_{\tilde{W}}^2 \;\leq\; C\left(\|f\|_{L^2(\underline{H})}^2 + \|\tilde{u}_{0,h}\|_{\underline{V}}^2 + E_u^{1/2}\right),$$

where $E_u^{1/2}$ is the discrete energy defined by setting $n=0$ in (5.10). If we furthermore assume that the first time-step occurs in such a way that $E_u^{1/2} \leq C'\left(\|\tilde{u}_{0,h}\|_{\underline{V}}^2 + \|\tilde{u}'_{0,h}\|_{\underline{H}}^2\right)$, then u_N depends continuously on the data.

Proof In the integral relation (5.6), for $n \in \{1,2,\cdots,N-1\}$, set

$$v_h \;=\; \delta_1^n u_h \;=\; \frac{u_h^{n+1}-u_h^n}{2\Delta t}.$$

The term containing $\underline{d}(.,.)$ vanishes because $\delta_1^n u_h \in \operatorname{Ker}\underline{D}^h$ after Lemma 5.2. Apply Lemma 5.1 to transform the remaining terms. The *discrete* energy $E_u^{n+1/2}$, defined as

$$\begin{aligned} 2\,E_u^{n+1/2} \;&=\; \underline{c}\left(\frac{u_h^{n+1}-u_h^n}{\Delta t},\,\frac{u_h^{n+1}-u_h^n}{\Delta t}\right) - \frac{\Delta t^2}{4}\,\underline{a}\left(\frac{u_h^{n+1}-u_h^n}{\Delta t},\,\frac{u_h^{n+1}-u_h^n}{\Delta t}\right) \\ &+\; \underline{a}\left(\frac{u_h^{n+1}+u_h^n}{2},\,\frac{u_h^{n+1}+u_h^n}{2}\right) \end{aligned} \tag{5.10}$$

enables us to recast the integral relation in the form

$$E_u^{n+1/2} - E_u^{n-1/2} \;=\; -\,\underline{b}(\gamma_\partial\delta_1^n u_h,\,\gamma_\partial\delta_1^n u_h) + (f^n,\,\delta_1^n u_h).$$

Summing over $m \in \{1, 2, \cdots, n\}$, we get

$$E_u^{n+1/2} \leq E_u^{1/2} + \sum_{m=1}^{n} \left(f^m, \frac{u_h^{m+1} - u_h^{m-1}}{2\Delta t} \right). \tag{5.11}$$

Hypothesis 5.1 makes it possible for us to find a positive lower-bound for $E_u^{n+1/2}$. Indeed,

$$\begin{aligned} E_u^{n+1/2} &\geq \eta\, \underline{c}\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, \frac{u_h^{n+1} - u_h^n}{\Delta t} \right) + \underline{a}\left(\frac{u_h^{n+1} + u_h^n}{2}, \frac{u_h^{n+1} + u_h^n}{2} \right) \\ &\geq \frac{\eta}{K_{\max}} \left\| \frac{u_h^{n+1} - u_h^n}{\Delta t} \right\|_{\underline{H}}^2 + \frac{1}{\rho_{\max}} \left(\left\| \frac{u_h^{n+1} + u_h^n}{2} \right\|_{\underline{V}}^2 - \left\| \frac{u_h^{n+1} + u_h^n}{2} \right\|_{\underline{H}}^2 \right) \end{aligned}$$

The rest of the proof is really classical [6, 7]. ◁

5.3 Error estimates

We wish to assess the difference between the fully-discrete solution (u_N, λ_N) and the continuous solution (u, λ). Assume that Hypothesis 4.3 holds, so that we are justified in defining

$$\begin{aligned} e_N &= (e^n)_{n \in \{0,1,\cdots,N\}} \quad \text{with} \quad e^n = u_h^n - u(n\Delta t) \\ q_N &= (q^n)_{n \in \{0,1,\cdots,N\}} \quad \text{with} \quad q^n = \lambda_h^n - \lambda(n\Delta t) \end{aligned} \tag{5.12}$$

We also need to introduce the notion of truncation errors, the definition of which is made possible by Hypothesis 4.3.

Definition 5.4 *For $n \in \{1, 2, \cdots, N-1\}$, the following quantities are called truncation errors*

$$\begin{aligned} \tau_2^n &= \frac{u((n+1)\Delta t) - 2u(n\Delta t) + u((n-1)\Delta t)}{\Delta t^2} - \ddot{u}(n\Delta t) \\ \tau_1^n &= \frac{u((n+1)\Delta t) - u((n-1)\Delta t)}{2\Delta t} - \dot{u}(n\Delta t) \end{aligned}$$

The finite sequences $(\tau_{2,N})$ and $(\tau_{1,N})$ can be constituted. Their values at $n = 0$ and $n = N$ are not involved in the definition of $\|.\|_{\tilde{H}}$ and $\|.\|_{\tilde{V}}$, and thus can be set to zero.

By comparing the integral relations (5.6) and (3.22), we obtain the equations satisfied by the errors, namely

$$\underline{c}(\delta_2^n e, v_h) + \underline{a}(e^n, v_h) + \underline{b}(\delta_1^n \gamma_\partial e, \gamma_\partial v_h) = \underline{d}(v_h, q^n) + \underline{c}(\tau_2^n, v_h) + \underline{b}(\gamma_\partial \tau_1^n, \gamma_\partial v_h) \tag{5.13}$$

for all $v_h \in \underline{V}^h$. Note, however, that a priori $e^n \notin \operatorname{Ker} \underline{D}^h$ and $q^n \notin \Lambda^h$.

Analogously to the semi-discrete approximation, we can start by estimating q_N in terms of e_N, measured by some discrete norms.

Proposition 5.3 *For $\Delta t \to 0$, there exists a constant C, independent of Δt, such that*

$$\| q_N \|_{\tilde{D}}^2 \leq C \left(\| e_N \|_{\tilde{W}}^2 + \| \tau_{2,N} \|_{\tilde{H}}^2 + \| \tau_{1,N} \|_{\tilde{V}}^2 \right) .$$

PROOF The proof is based on (5.13) and is utterly similar to that of Proposition 5.2. $\triangleleft$

Our objective is now to estimate e_N in terms of the truncation errors. As was the case for the semi-discrete approximation, we need to decompose e_N into the sum of a projection error and an approximation error.

Definition 5.5 *The projection error e_N^P and the approximation error e_N^A are defined as the finite sequences whose general terms are*

$$e_P^n = \pi_h u(n\Delta t) - u(n\Delta t) \qquad and \qquad e_A^n = u_h^n - \pi_h u(n\Delta t)$$

for $n \in \{0, 1, \cdots, N\}$. Likewise, the projection error q_N^P and the approximation error q_N^A are the finite sequences whose terms are

$$q_P^n = \Xi_h \lambda(n\Delta t) - \lambda(n\Delta t) \qquad and \qquad q_A^n = \lambda_h^n - \Xi_h \lambda(n\Delta t) .$$

In preparation for the upcoming Theorem, we would like to introduce some notations. We apologize for having to set up so many Definitions in this section, but in our opinion, this is the clearest way to present the error estimates.

Definition 5.6 *If $s_N = (s^n)_{n \in \{0,1,\cdots,N\}}$ takes values in some space S, we denote by $\delta_1 s_N$ and $\delta_2 s_N$ the finite sequences defined by $\delta_1 s^n = \delta_1^n s$ and $\delta_2 s^n = \delta_2^n s$ for $n \in \{1, 2, \cdots, N-1\}$. For $j \in \{1, 2\}$, the sequence $\delta_j s_N$ is measured by either*

$$\| \delta_j s_N \|_{\tilde{L}^2(S)}^2 = \sum_{n=1}^{N-1} \| \delta_j s^n \|_S^2 \, \Delta t \qquad or \qquad \| \delta_j s_N \|_{\tilde{L}^\infty(S)}^2 = \sup_{1 \leq n \leq N-1} \| \delta_j s^n \|_S^2 .$$

If $s_N = (s^n)_{n \in \{1,\cdots,N-1\}}$ is another finite sequence taking values S, we denote by $\delta_{1/2} s_N$ the finite sequence whose terms are indexed by half-integer ι and are equal to

$$\forall\, \iota = n + \frac{1}{2} \in \left\{ \frac{3}{2}, \frac{5}{2}, \cdots, N - \frac{3}{2} \right\} , \quad \delta_{1/2} s^\iota = \frac{s^{n+1} - s^n}{\Delta t} .$$

The newly defined sequence can be measured by the norm

$$\| \delta_{1/2} s_N \|^2_{\widehat{L}^2(S)} = \sum_{\iota=3/2}^{N-3/2} \| \delta_{1/2} s^\iota \|^2_S \, \Delta t \, .$$

Finally, for short-hand conveniency, we denote

$$\begin{aligned}
||| e_N^P |||^2 &= \| e_N^P \|^2_{\widehat{L}^\infty(\underline{V})} &+ \| \delta_2 e_N^P \|^2_{\widehat{L}^2(\underline{H})} \\
||| \gamma_\partial e_N^P |||^2 &= \| \gamma_\partial \delta_1 e_N^P \|^2_{\widehat{L}^\infty(\underline{Z})} &+ \| \gamma_\partial \delta_{1/2} \delta_1 e_N^P \|^2_{\widehat{L}^2(\underline{Z})} \\
||| \tau_N |||^2 &= \| \tau_{2,N} \|^2_{\widehat{L}^\infty(\underline{H})} &+ \| \gamma_\partial \delta_{1/2} \tau_{1,N} \|^2_{\widehat{L}^2(\underline{Z})} \\
||| q_N^P |||^2 &= \| q_h^P \|^2_{\widehat{L}^\infty(\Lambda)} &+ \| \delta_{1/2} q_N^P \|^2_{\widehat{L}^2(\Lambda)}
\end{aligned}$$

Theorem 5.2 *For $\Delta t \to 0$, there exists a constant C, independent of Δt, such that*

$$\| e_N^A \|^2_{\widetilde{W}} \le C \left(||| e_A^{1/2} |||^2 + ||| \tau_N |||^2 + ||| e_N^P |||^2 + ||| \gamma_\partial e_N^P |||^2 + ||| q_N^P |||^2 \right) ,$$

where the initial approximation error $||| e_A^{1/2} |||$ is defined by

$$||| e_A^{1/2} |||^2 = \left\| \frac{e_A^1 - e_A^0}{\Delta t} \right\|^2_{\underline{H}} + \left\| \frac{e_A^1 + e_A^0}{2} \right\|^2_{\underline{V}} .$$

Proof Inspiration can be drawn from the proofs of Theorem 4.2 and Theorem 5.1. The situation is very similar. Note that the stability condition is required. ◁

It is not difficult to see from the Definition 5.5 that $e_P^n = \epsilon_h^P(n\Delta t)$ and $q_P^n = \zeta_h^P(n\Delta t)$, where ϵ_h^P and ζ_h^P, projection errors of the semi-discrete case, have been proven to be sufficiently regular. Therefore, as Δt goes to zero, there exists a constant C', independent of Δt, such that

$$||| e_N^P |||^2 + ||| \gamma_\partial e_N^P |||^2 + ||| q_N^P |||^2 \le C' \left(||| \epsilon_h^P |||^2 + ||| \gamma_\partial \epsilon_h^P |||^2 + ||| \zeta_h^P |||^2 \right) .$$

Hopefully, these various projection errors can be "controlled" by some power of h, as was discussed at the end of Section 4. As for the truncation errors, it is well-known [7] that

$$||| \tau_N |||^2 \le C'' \Delta t^4 \left(\| u^{(4)} \|^2_{L^2(\underline{H})} + \| \gamma_\partial u^{(3)} \|^2_{L^2(\underline{Z})} \right)$$

as soon as $u^{(4)} \in L^2(0,T;\, \underline{H})$ and $\gamma_\partial u^{(3)} \in L^2(0,T;\, \underline{Z})$. There remains to be solved a technical subtelty regarding the initial approximation error, which systematically arises in finite difference analysis via the energy technique.

6. Example of practical implementation

In this section, hints are given on the implementation of the method proposed for two sub-domains, one of which being sampled twice as finely as the other. For simplicity, only **(P1)** mass-lumped finite element basis functions are considered. These lead in reality to the standard 2-2 finite difference scheme. A simpler version of the method will be worked out, which no longer involves the Lagrange multipliers representing the co-normal derivative.

6.1 Mesh

Consider again the example of grid change mentioned in the introduction. In Fig. 3, where the numbering of gridpoints is explained, the notations are different from those used so far. From now on, *lowercase* letters correspond to variables pertaining to the *fine* subdomain and

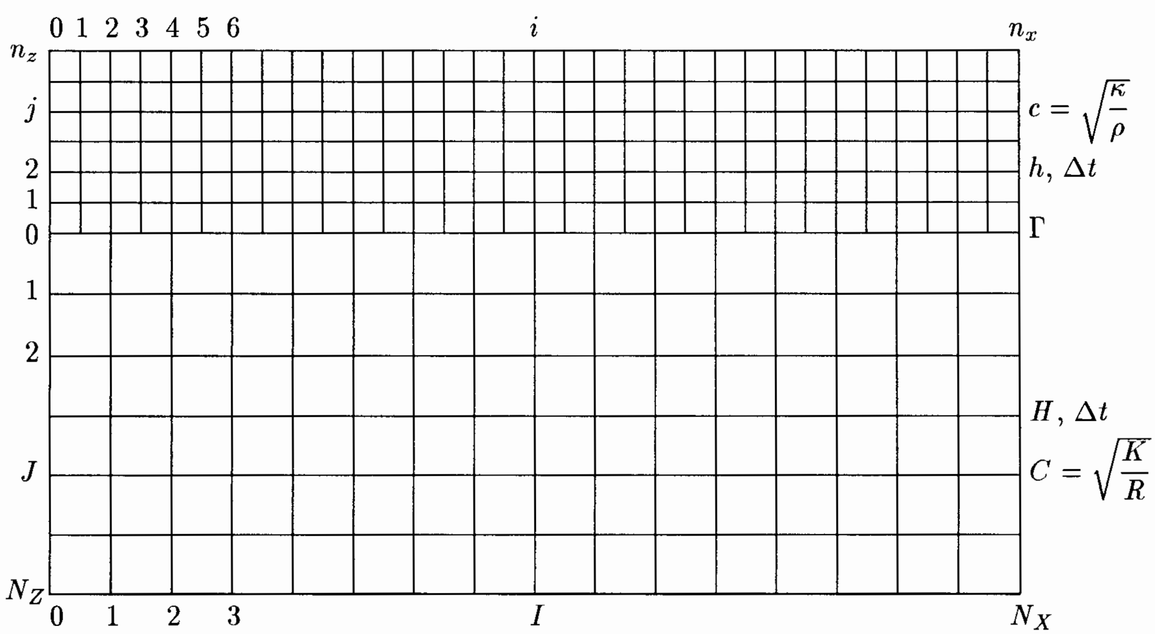

Figure 3: Numbering of gridpoints in the neighborhood of the interface. Capital letters correspond to the coarse subdomain. Note that $H = 2h$ implies $i = 2I$ and $n_x = 2N_X$.

UPPERCASE letters to those of the *COARSE* one. In particular,

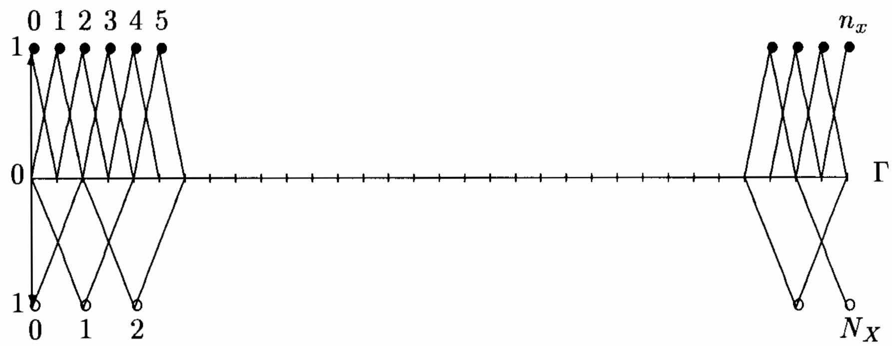

Figure 4: Basis functions $(w_i)_{i\in\{0,1,\ldots,n_x\}}$ of $\gamma_\Gamma \mathbf{v}^N$ and $(W_I)_{I\in\{0,1,\ldots,N_X\}}$ of $\gamma_\Gamma \mathbf{V}^N$

- The indices of fine subdomain are (i, j), while those of the coarse subdomain are (I, J). Here (i, j) must run twice as fast as (I, J). For clarity, the interface Γ has been taken as the horizontal line $z = Z = 0$, so that j is counted upward and J downward.
- The pressures at time-step n are denoted by u^n in the fine subdomain and U^n in the coarse one. The finite-dimensional spaces to which they belong are $\mathbf{v}^N$ and $\mathbf{V}^N$.
- The piecewise linear basis functions of $\gamma_\Gamma \mathbf{V}^N$ and $\gamma_\Gamma \mathbf{v}^N$ are denoted by $(w_i)_{i\in\{0,1,\ldots,n_x\}}$ and $(W_I)_{I\in\{0,1,\ldots,N_X\}}$. They are depicted in Fig. 4.

6.2 Data and source

For simplicity, we deliberately restrict ourselves to when

1. The bulk modulus and the density are constant inside each subdomain. Following the convention of Fig. 3, they are named respectively κ, K and ρ, R.
2. The initial data are zero, i.e. $\tilde{u}_0 = 0$ and $\tilde{u}'_0 = 0$. This assumption often holds true in real-life geophysical modeling.
3. The excitation is a point-source, the time-dependence of which is the so-called *Ricker function.* More specifically, $f(x, z, t) = \delta_0(x - x_S, z - z_S) \otimes \mathcal{R}(t)$ with

$$\mathcal{R}(t) = \left(1 - 2\pi^2(f_0 t - 1)^2\right) \exp\left[-\pi^2(f_0 t - 1)^2\right] , \tag{6.1}$$

where δ_0 is the Dirac function and f_0 is called the central frequency of the source wavelet. The Ricker wavelet stands out as one of the most popular sources among geophysicists, since its shape is similar to that of a real explosion. In the simulations presented, it will always be located inside the fine subdomain.

REMARK 6.1 As a point-source, $f \notin L^2(0,T;\, L^2(\Omega))$. We will explain how to numerically implement the Dirac function. □

REMARK 6.2 The above assumptions are designed to make formulae simpler, insofar as we do not have to worry about how geophysical parameters should be taken into account via numerical integration. □

6.3 Update formulae for interior regions, boundaries, and corners

In the integral relation (2.12), take v_i equal to a basis function of $\mathbf{v}^N$ or $\mathbf{V}^N$. Mass-lumping is performed, which leads to explicit formulae for updating interior and boundary points.

Based on the convention of Figure 3, the interior regions are defined by

- fine interior $\quad 1 \le i \le n_x - 1$ and $1 \le j \le n_z - 1$
- coarse interior $\quad 1 \le I \le N_X - 1$ and $1 \le J \le N_Z - 1$

Proposition 6.1 *Introduce the velocities*

$$c = \sqrt{\frac{\kappa}{\rho}} \qquad \text{and} \qquad C = \sqrt{\frac{K}{R}}\,.$$

Then, for the $(\mathbf{P1})$ *scheme, the update formulae for the two interior regions are*

$$\begin{aligned} u_{i,j}^{n+1} &= 2u_{i,j}^n - u_{i,j}^{n-1} + \left(\frac{c\Delta t}{h}\right)^2 [-4u_{i,j}^n + u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n] \\ U_{I,J}^{n+1} &= 2U_{I,J}^n - U_{I,J}^{n-1} + \left(\frac{C\Delta t}{H}\right)^2 [-4U_{I,J}^n + U_{I-1,J}^n + U_{I+1,J}^n + U_{I,J-1}^n + U_{I,J+1}^n] \end{aligned}$$

Furthermore, if the point-source is located at the node (i_S, j_S) *in the interior of the fine subdomain, then a source term equal to* $\left(\frac{c\Delta t}{h}\right)^2 \mathcal{R}(n\Delta t)$ *must be added to* u_{i_S,j_S}^{n+1} *previously computed.*

PROOF By mass-lumping, the mass matrix [associated with the bilinear form $\underline{c}(.,.)$] becomes diagonal, and what is more, the entries corresponding to interior points are equal to h^2 in the fine subdomain and H^2 in the coarse one. The calculation of the stiffness matrix is not difficult and is left to the reader. It provides an approximation of the Laplacian.

The presence of h^2 in the denominator of the source term is accounted for by the fact that the Dirac function is spread over a square cell of size h^2. $\lhd$

The update formulae are slightly different on the boundaries. To put it more accurately, the following subsets of gridpoints are called boundaries

- North $\quad j = n_z - 1$ and $1 \le i \le n_x - 1$
- South $\quad J = N_Z - 1$ and $1 \le I \le N_X - 1$
- fine West $\quad i = 1$ and $1 \le j \le n_z - 1$
- coarse West $\quad I = 1$ and $1 \le J \le N_Z - 1$

The East boundaries are defined similarly for $i = n_x - 1$ and $I = N_X - 1$.

Proposition 6.2 *Introduce*

$$d = 1 + \frac{c\Delta t}{h}, \quad e = 1 - \frac{c\Delta t}{h} \quad \textit{and} \quad D = 1 + \frac{C\Delta t}{H}, \quad E = 1 - \frac{C\Delta t}{H}. \tag{6.2}$$

Then, for the **(P1)** *scheme, the update formulae for the North and South boundaries are*

$$\begin{aligned} u^{n+1}_{i,n_z} &= \frac{1}{d}\left\{ 2u^n_{i,n_z} - e u^{n-1}_{i,n_z} + \left(\frac{c\Delta t}{h}\right)^2 \left[-4u^n_{i,n_z} + u^n_{i-1,n_z} + u^n_{i+1,n_z} + 2u^n_{i,n_z-1}\right] \right\} \\ U^{n+1}_{I,N_Z} &= \frac{1}{D}\left\{ 2U^n_{I,N_Z} - E U^{n-1}_{I,N_Z} + \left(\frac{C\Delta t}{H}\right)^2 \left[-4U^n_{I,N_Z} + U^n_{I-1,N_Z} + U^n_{I+1,N_Z} + 2U^n_{I,N_Z-1}\right] \right\} \end{aligned}$$

The formulae for the remaining boundaries are deduced by symmetrically changing the indices.

Proof By mass-lumping, the entries of the diagonal mass-matrix [associated with the bilinear form $\underline{c}(.,.)$] corresponding to boundary points are equal to $\frac{1}{2}h^2$ in the fine subdomain and $\frac{1}{2}H^2$ in the coarse one. Mass-lumping is also applied to the boundary mass-matrix [associated with the bilinear form $\underline{b}(.,.)$], thereby giving an additional term proportional to h in the fine subdomain and to H in the coarse one. The calculations are left to the readers. $\lhd$

An additional change in the update formulae is necessary for the corners. In fact, the following gridpoints are called corners

- North East $j = n_z$ and $i = n_x$
- North West $j = n_z$ and $i = 0$
- South East $J = N_Z$ and $I = N_X$
- South West $J = N_Z$ and $I = 0$

Proposition 6.3 *Introduce*

$$b = 1 + 2\frac{c\Delta t}{h}, \quad q = 1 - 2\frac{c\Delta t}{h} \quad \text{and} \quad B = 1 + 2\frac{C\Delta t}{H}, \quad Q = 1 - 2\frac{C\Delta t}{H}. \tag{6.3}$$

Then, for the (**P1**) *scheme, the update formulae for the North East and South East corners are*

$$\begin{aligned} u^{n+1}_{n_x,n_z} &= \frac{1}{b}\left\{ 2u^n_{n_x,n_z} - q u^{n-1}_{n_x,n_z} + \left(\frac{c\Delta t}{h}\right)^2 \left[-4u^n_{n_x,n_z} + 2u^n_{n_x-1,n_z} + 2u^n_{n_x,n_z-1}\right] \right\} \\ U^{n+1}_{N_X,N_Z} &= \frac{1}{B}\left\{ 2U^n_{N_X,N_Z} - Q U^{n-1}_{N_X,N_Z} + \left(\frac{C\Delta t}{H}\right)^2 \left[-4U^n_{N_X,N_Z} + 2U^n_{N_X-1,N_Z} + 2U^n_{N_X,N_Z-1}\right] \right\} \end{aligned}$$

The formulae for the remaining corners are deduced by symmetrically changing the indices.

The update formulae in the last three Propositions are valid as soon as the grid is regular in each subdomain, regardless of the ratio H/h. We are now giving the update formulae for the gridpoints on the interface. These will be valid only when $H = 2h$.

6.4 Linear system at the interface

The space Λ^N is taken to be the that of piecewise constant functions over Γ. The pieces are determined by the finer grid, and therefore its $\dim \Lambda^N = n_x + 1$. A basis of Λ^N is

- ∘ $i = 0$ $\mu_0(x) = 1$ for $x \in [\,0,\, h/2\,]$, 0 elsewhere;
- ∘ $1 \le i \le n_x - 1$ $\mu_i(x) = 1$ for $x \in [\,(i - 1/2)\,h,\, (i + 1/2)\,h\,]$, 0 elsewhere;
- ∘ $i = n_x$ $\mu_{n_x}(x) = 1$ for $x \in [\,(n_x - 1/2)\,h,\, n_x h\,]$, 0 elsewhere.

The following lemma shows that such a choice for Λ^N is likely to be a good one, to the extent that Hypothesis 4.1 is actually satisfied.

Lemma 6.1 *With the choice of Λ^N made above, we have* $\Lambda^N \cap (\gamma_\Gamma \mathbf{V}^N + \gamma_\Gamma \mathbf{v}^N)^\perp = \{0\}$.

PROOF From Fig. 4, it is obvious that $\gamma_\Gamma \mathbf{V}^N \subset \gamma_\Gamma \mathbf{v}^N$, so that $\gamma_\Gamma \mathbf{V}^N + \gamma_\Gamma \mathbf{v}^N = \gamma_\Gamma \mathbf{v}^N$. Since Λ^N and $\gamma_\Gamma \mathbf{v}^N$ have the same dimension, we can prove the lemma by considering the $n_x \times n_x$-matrix $\mathbf{d}$ of elementary products $(\mu_i, w_j)_{L^2(\Gamma)}$. Calculations yield

$$\mathbf{d} = \frac{h}{8}\begin{pmatrix} 3 & 1 & & & & \\ 1 & 6 & 1 & & & \\ & 1 & 6 & 1 & & \\ & & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\ & & & 1 & 6 & 1 \\ & & & & 1 & 3 \end{pmatrix}$$

This matrix is not singular because its diagonal is strictly dominant. ◁

At time-step n, let $\boldsymbol{\lambda}^n$ be the column vector representing the multiplier λ^n

$$\boldsymbol{\lambda}^n = {}^t(\lambda_0^n, \lambda_1^n, \ldots, \lambda_{n_x}^n) \quad \text{with} \quad \lambda^n = \sum_{i=0}^{n_x} \lambda_i^n \mu_i$$

The coefficient λ_i^n also represents the value of λ^n at the ith node on Γ.

Proposition 6.4 *For the interior segment of the interface Γ, i.e. for $1 \le i \le n_x - 1$ and $1 \le I \le N_X - 1$, let us introduce*

$$\begin{aligned} beg_i^n &= 2u_{i,0}^n - u_{i,0}^{n-1} + \left(\frac{c\Delta t}{h}\right)^2 [-4u_{i,0}^n + u_{i-1,0}^n + u_{i+1,0}^n + 2u_{i,1}^n] \\ BEG_I^n &= 2U_{I,0}^n - U_{I,0}^{n-1} + \left(\frac{C\Delta t}{H}\right)^2 [-4U_{I,0}^n + U_{I-1,0}^n + U_{I+1,0}^n + 2U_{I,1}^n] \end{aligned}$$

If the multipliers $(\lambda_i^n)_{i\in\{0,1,\ldots,n_x\}}$ were known, then the update formulae for the interior segment of Γ would be

$$\begin{aligned} u_{i,0}^{n+1} &= beg_i^n - \frac{\kappa}{4}\frac{\Delta t^2}{h}[6\lambda_i^n + (\lambda_{i+1}^n + \lambda_{i-1}^n)] \\ U_{I,0}^{n+1} &= BEG_I^n + \frac{K}{16}\frac{\Delta t^2}{H}[14\lambda_{2I}^n + 8(\lambda_{2I+1}^n + \lambda_{2I-1}^n) + (\lambda_{2I+2}^n + \lambda_{2I-2}^n)] \end{aligned} \tag{6.4}$$

PROOF In the integral relation (2.12), take v_i as the basis functions of $\mathbf{v}^N$ and $\mathbf{V}^N$ associated with the nodes on Γ. Calculations are left to the readers, who should be very careful. ◁

In these formulae, the quantities beg_i^n and BEG_I^n stand for the **beg**inning expressions of $u_{i,0}^{n+1}$ and $U_{I,0}^{n+1}$. At the ends of the interface Γ, the pressure values are similarly updated by

Proposition 6.5 *For the left end of the interface Γ, i.e. for $i = I = 0$, let us introduce*

$$
\begin{aligned}
\mathit{beg}_0^n &= \frac{1}{d}\left\{ 2u_{0,0}^n - e u_{0,0}^{n-1} + \left(\frac{c\Delta t}{h}\right)^2 \left[-4u_{0,0}^n + 2u_{1,0}^n + 2u_{0,1}^n\right] \right\} \\
\mathit{BEG}_0^n &= \frac{1}{D}\left\{ 2U_{0,0}^n - E U_{0,0}^{n-1} + \left(\frac{C\Delta t}{H}\right)^2 \left[-4U_{0,0}^n + 2U_{1,0}^n + 2U_{0,1}^n\right] \right\}
\end{aligned}
$$

If the multipliers $(\lambda_i^n)_{i\in\{0,1,2\}}$ were known, then the update formulae for this left end would be

$$
\begin{aligned}
u_{0,0}^{n+1} &= \mathit{beg}_0^n - \frac{\kappa}{2d}\frac{\Delta t^2}{h}\left[3\lambda_0^n + \lambda_1^n\right] \\
U_{0,0}^{n+1} &= \mathit{BEG}_0^n + \frac{K}{8D}\frac{\Delta t^2}{H}\left[7\lambda_0^n + 8\lambda_1^n + \lambda_2^n\right]
\end{aligned}
\tag{6.5}
$$

The update formulae for the right end of Γ are deduced by symmetrically changing the indices.

Let us now express the continuity condition (2.13) in terms of u^n, U^n and λ^n. This gives us two series of equations, depending on whether the index of the gridpoint considered is odd or even. Several intermediate quantities need to be introduced. Define

$$
\begin{aligned}
\mathit{EVEN}_I^n &= \frac{1}{16}U_{I-1,0}^n + \frac{7}{8}U_{I,0}^n + \frac{1}{16}U_{I+1,0}^n && \text{for } 1 \le I \le N_X - 1 \\
\mathit{ODD}_k^n &= \frac{1}{2}U_{I,0}^n + \frac{1}{2}U_{I+1,0}^n && \text{for } 1 \le k = 2I+1 \le n_x - 1 \\
\mathit{ave}_i^n &= \frac{1}{8}u_{i-1,0}^n + \frac{3}{4}u_{i,0}^n + \frac{1}{8}u_{i+1,0}^n && \text{for } 1 \le i \le n_x - 1
\end{aligned}
\tag{6.6}
$$

Expressing the continuity condition at an even-numbered gridpoint on Γ is equivalent to setting $\mu = \mu_{2I}$ in (2.13). By taking into account the fact that $H = 2h$ we obtain

$$
\mathit{EVEN}_I^{n+1} - \mathit{ave}_{2I}^{n+1} = -2\left(\mathit{EVEN}_I^n - \mathit{ave}_{2I}^n\right) - \left(\mathit{EVEN}_I^{n-1} - \mathit{ave}_{2I}^{n-1}\right) \tag{6.7}
$$

for $I \in \{1, \ldots, N_X - 1\}$. Express now the continuity condition at an odd-numbered gridpoint on Γ by setting $\mu = \mu_k$ with $k = 2I+1$ in (2.13). It follows that

$$
\mathit{ODD}_k^{n+1} - \mathit{ave}_k^{n+1} = -2(\mathit{ODD}_k^n - \mathit{ave}_k^n) - (\mathit{ODD}_k^{n-1} - \mathit{ave}_k^{n-1}) \tag{6.8}
$$

for odd $k \in \{1, \ldots, n_x - 1\}$. At the left end of the interface, by posing

$$
\mathit{EVEN}_0^n = \frac{7}{16}U_{0,0}^n + \frac{1}{16}U_{1,0}^n \quad \text{and} \quad \mathit{ave}_0^n = \frac{3}{8}u_{0,0}^n + \frac{1}{8}u_{1,0}^n\,, \tag{6.9}
$$

the continuity condition is expressed as

$$
\mathit{EVEN}_0^{n+1} - \mathit{ave}_0^{n+1} = -2(\mathit{EVEN}_0^n - \mathit{ave}_0^n) - (\mathit{EVEN}_0^{n-1} - \mathit{ave}_0^{n-1}) \tag{6.10}
$$

A similar relation holds for the right end of Γ.

The most painstaking task consists in eliminating the unknowns u^{n+1} and U^{n+1} by plugging the update formulae (6.4) and (6.5) into the continuity relations (6.7), (6.8) and (6.10). In the following theorem, note that for $4 \le i \le n_x - 4$, we have two alternating series of equations.

Theorem 6.1 *The multipliers* $(\lambda_i^n)_{i\in\{0,1,\dots,n_x\}}$ *must be solution to the linear system* $\mathbf{S}^N \boldsymbol{\lambda}^n = \boldsymbol{\tau}^n$. *The* $(n_x+1)\times(n_x+1)$*-matrix* $\mathbf{S}^N$ *is a banded matrix*

$$\mathbf{S}^N = \left(\begin{array}{ccccccccccccc}
\varepsilon_0 & \delta_1 & \gamma_2 & 8 & 1 & & & & & & & & \\
\delta_1 & \delta_0 & \varepsilon_1 & \beta_2 & 8 & 0 & & & & & & & \\
\gamma_2 & \varepsilon_1 & \gamma_0 & \gamma_1 & \alpha_2 & 8 & 1 & & & & & & \\
8 & \beta_2 & \gamma_1 & \beta_0 & \gamma_1 & \beta_2 & 8 & 0 & & & & & \\
1 & 8 & \alpha_2 & \gamma_1 & \alpha_0 & \gamma_1 & \alpha_2 & 8 & 1 & & & & \\
 & 0 & 8 & \beta_2 & \gamma_1 & \beta_0 & \gamma_1 & \beta_2 & 8 & 0 & & & \\
 & & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & & \\
 & & & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \\
 & & & & 1 & 8 & \alpha_2 & \gamma_1 & \alpha_0 & \gamma_1 & \alpha_2 & 8 & 1 \\
 & & & & & 0 & 8 & \beta_2 & \gamma_1 & \beta_0 & \gamma_1 & \beta_2 & 8 \\
 & & & & & & 1 & 8 & \alpha_2 & \gamma_1 & \gamma_0 & \varepsilon_1 & \gamma_2 \\
 & & & & & & & 0 & 8 & \beta_2 & \varepsilon_1 & \delta_0 & \delta_1 \\
 & & & & & & & & 1 & 8 & \gamma_2 & \delta_1 & \varepsilon_0
\end{array}\right) \tag{6.11}$$

with

$$\begin{array}{lll}
\alpha_0 = 198 + 608r & \beta_0 = 128 + 608r & \\
\gamma_0 = 197 + 608r + 2G & \delta_0 = 64 + 592r + 128G + 32rg & \varepsilon_0 = 1 + 16r + 98G + 288rg \\
\gamma_1 = 120 + 192r & \delta_1 = 8 + 96r + 112G + 96rg & \varepsilon_1 = 112 + 192r + 16G \\
\alpha_2 = 28 + 16r & \beta_2 = 64 + 16r & \gamma_2 = 14 + 16r + 14G
\end{array}$$

where

$$r = \frac{\kappa}{K}, \quad g = \frac{1}{d} \quad \textit{and} \quad G = \frac{1}{D}.$$

The right-hand side $\boldsymbol{\tau}^n$ is defined as

$$\forall i \in \{0, 1, ..., n_x\}, \quad \tau_i^n = \frac{256H}{K\Delta t^2}\left(avbeg_i^n - AVEBEG_i^n - 2AVEDIF_i^n - AVEDIF_i^{n-1}\right)$$

where the terms in the parentheses are defined as follows:

- *for $i = 0$*

$$\begin{aligned} avbeg_0^n &= \frac{3}{8}beg_0^n + \frac{1}{8}beg_1^n \\ AVEBEG_0^n &= \frac{7}{16}BEG_0^n + \frac{1}{16}BEG_1^n \\ AVEDIF_0^n &= EVEN_0^n - ave_0^n \end{aligned}$$

- *for $i = n_x$ analogous formulae*

- *for even $i \in \{1, ..., n_x - 1\}$ with $i = 2I$*

$$\begin{aligned} avbeg_i^n &= \frac{1}{8}beg_{i-1}^n + \frac{3}{4}beg_i^n + \frac{1}{8}beg_{i+1}^n \\ AVEBEG_i^n &= \frac{1}{16}BEG_{I-1}^n + \frac{7}{8}BEG_I^n + \frac{1}{16}BEG_{I+1}^n \\ AVEDIF_i^n &= EVEN_I^n - ave_i^n \end{aligned}$$

- *for odd $k \in \{1, ..., n_x - 1\}$ with $k = 2I + 1$*

$$\begin{aligned} avbeg_k^n &= \frac{1}{8}beg_{k-1}^n + \frac{3}{4}beg_k^n + \frac{1}{8}beg_{k+1}^n \\ AVEBEG_k^n &= \frac{1}{2}BEG_I^n + \frac{1}{2}BEG_{I+1}^n \\ AVEDIF_k^n &= ODD_k^n - ave_k^n \end{aligned}$$

In practice, the banded-matrix $\mathbf{S}^N$ is Cholesky factorized once for all as $\mathbf{S}^N = {}^t\mathbf{L}^N\mathbf{L}^N$. At each time step, after the right-hand side is computed, a library subroutine is called, which solves the linear system with the given banded triangular matrix $\mathbf{L}^N$. Once $\boldsymbol{\lambda}^n$ is known, formulae (6.4) and (6.5) are applied to update u^{n+1} and U^{n+1} on Γ.

REMARK 6.3 It could be interesting to check the discrete inf-sup condition on the mesh chosen for this example. Although we have not taken up this study, we refer the reader to [3] for greater details on the matrix form of the inf-sup condition. □

7. A significant simplification

The $h-2h$ grid change studied the previous section is one of the simplest examples that could be envisaged. Yet, its computer implementation looks much trickier than the plain finite difference scheme. From the computational standpoint, not only a 1-D array of multipliers has to be stored, but also a linear system has to be solved at every time-step. Even though the matrix of the system is banded, the multipliers cannot be determined "locally."

7.1 Elimination of multipliers

In return for an extra level of approximation, we can obtain a much handier algorithm. Let us temporarily come back to the notations used in $(\mathbf{G}_h^n)$. We know by Lemma 5.2 that $u_h^{n+1} \in \operatorname{Ker} \underline{D}^h$. Taking $v_h \in \operatorname{Ker} \underline{D}^h$ as a test function in (5.6) naturally cancels out $\underline{d}(v_h, \lambda_h^n)$. Therefore, $u_h^{n+1} \in \operatorname{Ker} \underline{D}^h$ necessarily satisfies

$$\forall\, v_h \in \operatorname{Ker} \underline{D}^h\,, \quad \underline{c}\,(\delta_2^n u_h,\, v_h) \,+\, \underline{a}\,(u_h^n,\, v_h) \,+\, \underline{b}\,(\delta_1^n \gamma_\partial u_h,\, \gamma_\partial v_h) \;=\; (f^n,\, v_h) \tag{7.1}$$

Conversely, if (7.1) holds, then by proceeding as in the proof of Proposition 5.1, a little algebra shows the existence of a $\lambda_h^n \in \Lambda^h$ such that (u_h^{n+1}, λ_h^n) is solution to $(\mathbf{G}_h^n)$.

The obvious interest of (7.1) lies in the fact that it does not contain any multiplier explicitly. The problem is that $\operatorname{Ker} \underline{D}^h$, which depends on $\underline{V}^h$ and Λ^h, does not lend itself to an easy interpretation in terms of node values of u_h. The idea consists then to replace $\operatorname{Ker} \underline{D}^h$ by another space $\underline{B}^h$, the characterization of which is purportedly more straightforward in terms of node values. We are thus led to the simplified method

$(\mathbf{B}_h^n)$ GIVEN

$$\underline{B}^h \text{ as a substitute space for } \operatorname{Ker} \underline{D}^h \tag{7.2}$$

$$f^n \;=\; f(n\Delta t) \in L^2(\Omega) \quad \text{and} \quad u_h^n,\, u_h^{n-1} \in \underline{B}^h \tag{7.3}$$

$$K,\, \rho \text{ fulfilling Hypothesis 3.1} \tag{7.4}$$

FIND

$$u_h^{n+1} \in \underline{B}^h \quad \text{such that} \tag{7.5}$$

- $\forall\, v_h \in \underline{B}^h$, the following discrete integral relation holds

$$\underline{c}(\delta_2^n u_h,\, v_h) + \underline{a}(u_h^n,\, v_h) + \underline{b}(\delta_1^n \gamma_\partial u_h,\, \gamma_\partial v_h) \;=\; (f^n,\, v_h) \tag{7.6}$$

Given the initial data $\tilde{u}_{0,h}$ and $\tilde{u}'_{0,h}$, there exists a unique solution $u_N = (u_h^n)_{n\in\{0,1,\cdots,N\}}$ to the sequence of problems $(\mathbf{B}_h^n)$. The question of stability can be addressed in a manner similar to Section 3. The most fundamental question arises as to how $\underline{B}^h$ should be selected. As was announced earlier, this is via mass-lumping performed on the continuity conditions. The next subsection supplies us with details on $\underline{B}^h$, as well as on the implementation of the new version for the $h-2h$ example.

7.2 Application to the previous example

Once again, we opt for the notations of Section 6. Let $U_{I,J}$ and $u_{i,j}$ denote node values of the pressure in the coarse and fine subdomains. Before saying what $\underline{B}^h$ could be, let us try to characterize $\operatorname{Ker}\underline{D}^h$ in terms of equations between $U_{I,0}$ and $u_{i,0}$. From the calculations of the previous section, it is not difficult to see that the function u_h determined by the node values $(u,\, U)$ belongs to $\operatorname{Ker}\underline{D}^h$ if and only if

$$\begin{aligned} \forall\, I \in \{0,1,\cdots,N_X\}\,, &\qquad EVEN_I \;=\; \textit{\ae}_{2I} \\ \forall\, k = 2I+1 \in \{1,3,\cdots,n_x-1\}\,, &\qquad ODD_k \;=\; \textit{\ae}_k \end{aligned}$$

where $EVEN_I$, ODD_k and $\textit{\ae}_i$ were defined by (6.6) and (6.9). To be more specific,

$$\begin{aligned} I = i = 0 \;: &\qquad 7U_{0,0} + U_{1,0} = 6u_{0,0} + 2u_{1,0} \\ 2 \le i = 2I \le n_x - 2 \;: &\qquad U_{I-1,0} + 14U_{I,0} + U_{I+1,0} = 2u_{i-1,0} + 12u_{i,0} + 2u_{i+1,0} \\ 1 \le k = 2I+1 \le n_x - 1 \;: &\qquad 4U_{I,0} + 4U_{I+1,0} = u_{k-1,0} + 6u_{k,0} + u_{k+1,0} \end{aligned} \tag{7.7}$$

By mass-lumping the U's at even-numbered gridpoint and the u's everywhere, we get

$$\begin{aligned} 0 \le i = 2I \le n_x \;: &\qquad U_{I,0} = u_{i,0} \\ 1 \le k = 2I+1 \le n_x - 1 \;: &\qquad U_{I,0} + U_{I+1,0} = 2u_{k,0} \end{aligned} \tag{7.8}$$

The equations (7.8) are extremely easy to implement. This is the reason why they will be used to define $\underline{B}^h$. In the present example, the characterization of $\underline{B}^h$ is strikingly straightforward. If the $U_{I,0}$'s are known, then the $u_{i,0}$'s can be deduced by (i) forcing the equality at even-numbered gridpoint and (ii) taking the arithmetic mean of two consecutive even-numbered gridpoints to obtain the odd-numbered gridpoint.

REMARK 7.1 The characterization (7.8) also implies that $\gamma_\Gamma U = \gamma_\Gamma u$, insofar as both of them are piecewise linear. So, $\underline{B}^h \subset H^1(\Omega)$ whereas $\operatorname{Ker} \underline{D}^h \not\subset H^1(\Omega)$. This tends to prove that $\underline{B}^h$ is even "better" than $\operatorname{Ker} \underline{D}^h$. □

REMARK 7.2 From the standpoint of finite elements, the $U_{I,0}$'s for $I \in \{0, 1, \cdots, N_X\}$ are degrees of freedom, while the $u_{i,0}$'s are not. The support of a basis function in $\underline{B}^h$, which corresponds to a gridpoint at the interface, is depicted in Fig. 5. □

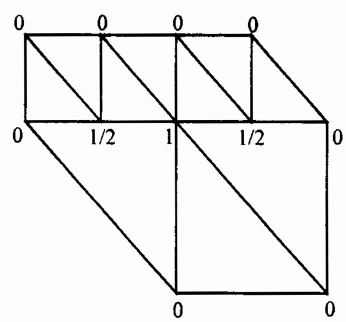

Figure 5: A (**P1**) basis function corresponding to an interface gridpoint $i = 2I$

The update formulae for the interior regions, boundaries and corners are unchanged compared to Section 5. In contrast, those for the interface gridpoints are a little simpler.

Lemma 7.1 *Let $u_h = (u, U) \in \underline{B}^h$ and let $v_{h,I}$ be the basis function of $\underline{B}^h$ corresponding to the interface gridpoint $i = 2I$. Denote by $\underline{c}^\star$ and $\underline{b}^\star$ the mass-lumped versions of $\underline{c}$ and $\underline{b}$. Then, for $I \in \{1, 2, \cdots, N-1\}$,*

$$\begin{aligned}
\underline{c}^\star(u_h, v_{h,I}) &= 2h^2 \left(\frac{1}{K} + \frac{1}{2\kappa} \right) U_{I,0} \\
\underline{a}(u_h, v_{h,I}) &= \frac{1}{2R} \left[4U_{I,0} - (2U_{I,1} + U_{I-1,0} + U_{I+1,0}) \right] \\
&+ \frac{1}{2\rho} \left[4u_{i,0} - (2u_{i,1} + u_{i-1,1} + u_{i+1,1}) \right]
\end{aligned}$$

and for $I = i = 0$

$$\begin{aligned}
\underline{c}^\star(u_h, v_{h,0}) &= h^2\left(\frac{1}{K} + \frac{1}{2\kappa}\right) U_{0,0} \\
\underline{b}^\star(u_h, v_{h,0}) &= h\left(\frac{1}{\Sigma} + \frac{1}{2\sigma}\right) U_{0,0} \\
\underline{a}(u_h, v_{h,0}) &= \frac{1}{R}\left[\,2U_{0,0} - (U_{0,1} + U_{1,0})\,\right] \\
&+ \frac{1}{\rho}\left[\,2u_{0,0} - (u_{0,1} + u_{1,1})\,\right]
\end{aligned}$$

with $\sigma = \sqrt{\kappa\rho}$ and $\Sigma = \sqrt{KR}$. The formulae for $I = N_X = 2n_x$ are deduced by symmetry.

PROOF It suffices to carry out the calculations carefully. Express the results in terms of

$$U_{I-1,0}\,,\ U_{I,0}\,,\ U_{I+1,0}\,,\ U_{I,1}\ \text{ and }\ u_{i-2,0}\,,\ u_{i-1,0}\,,\ u_{i,0}\,,\ u_{i+1,0}\,,\ u_{i+2}\,, u_{i-1,1}\,,\ u_{i,1}\,,\ u_{i+1,1}\,.$$

Next, simplify by taking into account (7.8) and $H = 2h$. ◁

These preliminary results being stated, the update formulae for interface gridpoints are given by

Proposition 7.1 *Let $\dfrac{1}{\widehat{k}} = 4\left(\dfrac{1}{K} + \dfrac{1}{2\kappa}\right)$. Then, for the interior segment, we have*

$$\begin{aligned}
U_{I,0}^{n+1} &= 2U_{I,0}^n - U_{I,0}^{n-1} + \frac{1}{\widehat{k}}\left(\frac{\Delta t}{h}\right)^2 \left[\,\frac{1}{R}(\,U_{I-1,0}^n + U_{I+1,0}^n + 2U_{I,1}^n - 4U_{I,0}^n\,)\right. \\
&\qquad \left. + \frac{1}{\rho}(\,u_{i-1,1}^n + u_{i+1,1}^n + 2u_{i,1}^n - 4u_{i,0}^n\,)\,\right]
\end{aligned}$$

Introduce

$$\widehat{c} = \widehat{k}\left(\frac{1}{\Sigma} + \frac{1}{2\sigma}\right), \quad \widehat{d} = 1 + \frac{\widehat{c}\Delta t}{h} \quad \text{and} \quad \widehat{e} = 1 - \frac{\widehat{c}\Delta t}{h}.$$

For the left-end of the interface, we have

$$\begin{aligned}
U_{0,0}^{n+1} &= \frac{1}{\widehat{d}}\left\{\,2U_{0,0}^n - \widehat{e}\,U_{0,0}^{n-1} + \frac{2}{\widehat{k}}\left(\frac{\Delta t}{h}\right)^2 \left[\,\frac{1}{R}(\,U_{1,0}^n + U_{0,1}^n - 2U_{0,0}^n\,)\right.\right. \\
&\qquad \left.\left. + \frac{1}{\rho}(\,u_{1,1}^n + u_{0,1}^n - 2u_{0,0}^n\,)\,\right]\,\right\}
\end{aligned}$$

The update formula for the right-end of the interface is deduced by symmetry.

PROOF Apply Lemma 7.1 and (7.6) in which $\underline{c}^\star$ and $\underline{b}^\star$ are substituted for $\underline{c}$ and $\underline{b}$. We have to assume that the source is not located on the interface, which seems reasonable. ◁

8. Numerical results

In order to illustrate the validity of the method $\mathbf{G}_h^n$ as well as its simplified version $\mathbf{B}_h^n$, we briefly present some numerical results obtained for the $h-2h$ example. Like geophysicists, we proceed by examining snapshots and trace recordings for various situations.

8.1 Snapshots of propagation

Three different models are considered. In the description below, the densities ρ and R are given in kg.m^{-3} ; the bulk moduli κ and K in kg.m^{-1}.s^{-2} ; the velocities c and C in **m.s**$^{-1}$; finally, the acoustic impedances σ and Σ in kg.m^{-2}.s^{-1}.

A– $\rho = R = 10^3$ and $\kappa = K = 2.25\,10^9$

This is in actuality a homogeneous medium with $c = C = \mathbf{1.5\,10^3}$ and $\sigma = \Sigma = 1.5\,10^6$. Its interest is to allow for a direct visualization of the effects caused by the grid change alone.

B– $\rho = R = 10^3$ and $\kappa = 2.25\,10^9$, $K = 9.0\,10^9$

The jump in the bulk moduli entails $c = \mathbf{1.5\,10^3}$, $C = \mathbf{3.0\,10^3}$ and $\sigma = 1.5\,10^6$, $\Sigma = 3.0\,10^6$. We want to see how well the physically predicted reflection and transmission waves are modeled by the grid change.

C– $\rho = 2.0\,10^3$, $R = 10^3$ and $\kappa = 4.5\,10^9$, $K = 9.0\,10^6$

We still have $c = \mathbf{1.5\,10^3}$, $C = \mathbf{3.0\,10^3}$ but this time $\sigma = \Sigma = 3.0\,10^6$. It is well-known in geophysics that when the acoustic impedances are equal, there is no reflection wave at normal incidence. This is what we wish to check.

The central frequency of the Ricker source is set to $f_0 = 30\,\mathrm{Hz}$. The source itself is always located in the finer subdomain, at 75 m above the interface. The numerical simulations are performed with $h = 1\,\mathrm{m}$ and $H = 2\,\mathrm{m}$, which corresponds to roughly 20 points per shortest wavelength. The time-step is deliberately taken as small as $\Delta t = 2.5\,10^{-4}\,\mathrm{s}$ so as to minimize the error due to time-discretization.

A *snapshot* is the image of the pressure field at a given time. For instance, Fig. 6 represents the reference solution in model **A** at $t = 1.5\,\mathrm{s}$. By *reference solution* we mean the numerical solution computed with the classical 2-2 finite difference method on a uniformly fine grid h. The wavefront appears to be circular, as is expected in a homogeneous medium.

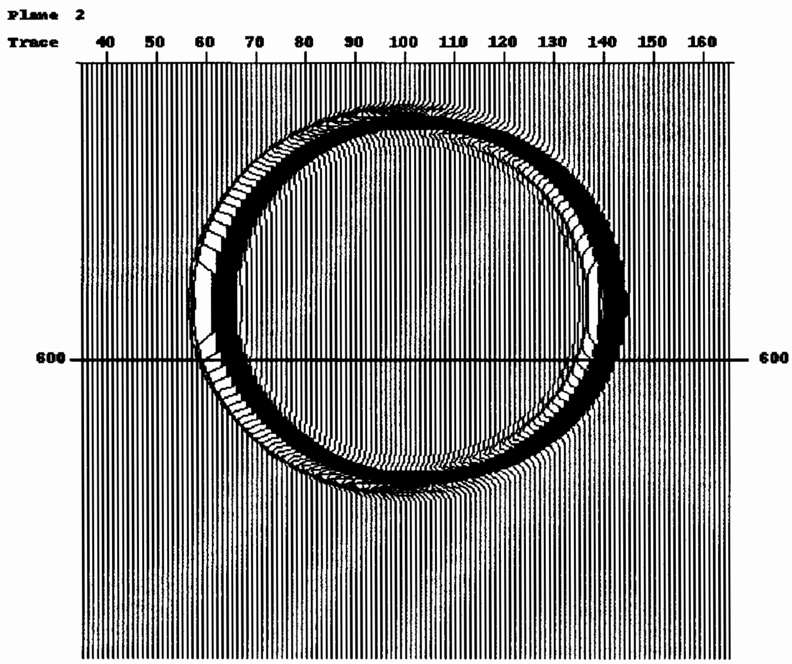

Figure 6: Reference solution at time $t = 1.5\,\mathrm{s}$ for model **A**. The thick line coincides with the interface. The numerical solutions (not represented) are pretty much the same.

The pressure snapshots computed with $\mathbf{G}_n^h$ and $\mathbf{B}_n^h$ at the same time look much alike. Their differences with the reference solution are displayed in Fig. 7, using a much smaller scale of course. In addition to the difference in the transmitted front, there is a reflected front whose amplitude is relatively small and whose origin could be attributed to the interface. The total relative error —measured by the discrete L^2-norm— is about 2.3%. This figure can be further decomposed into 2.0% from the transmitted front and 0.3% from the reflected front.

In model **B**, the reference solution itself is composed of a transmitted and a reflected wavefront, as illustrated by panel a of Fig. 8. The transmitted part is propagated at the higher

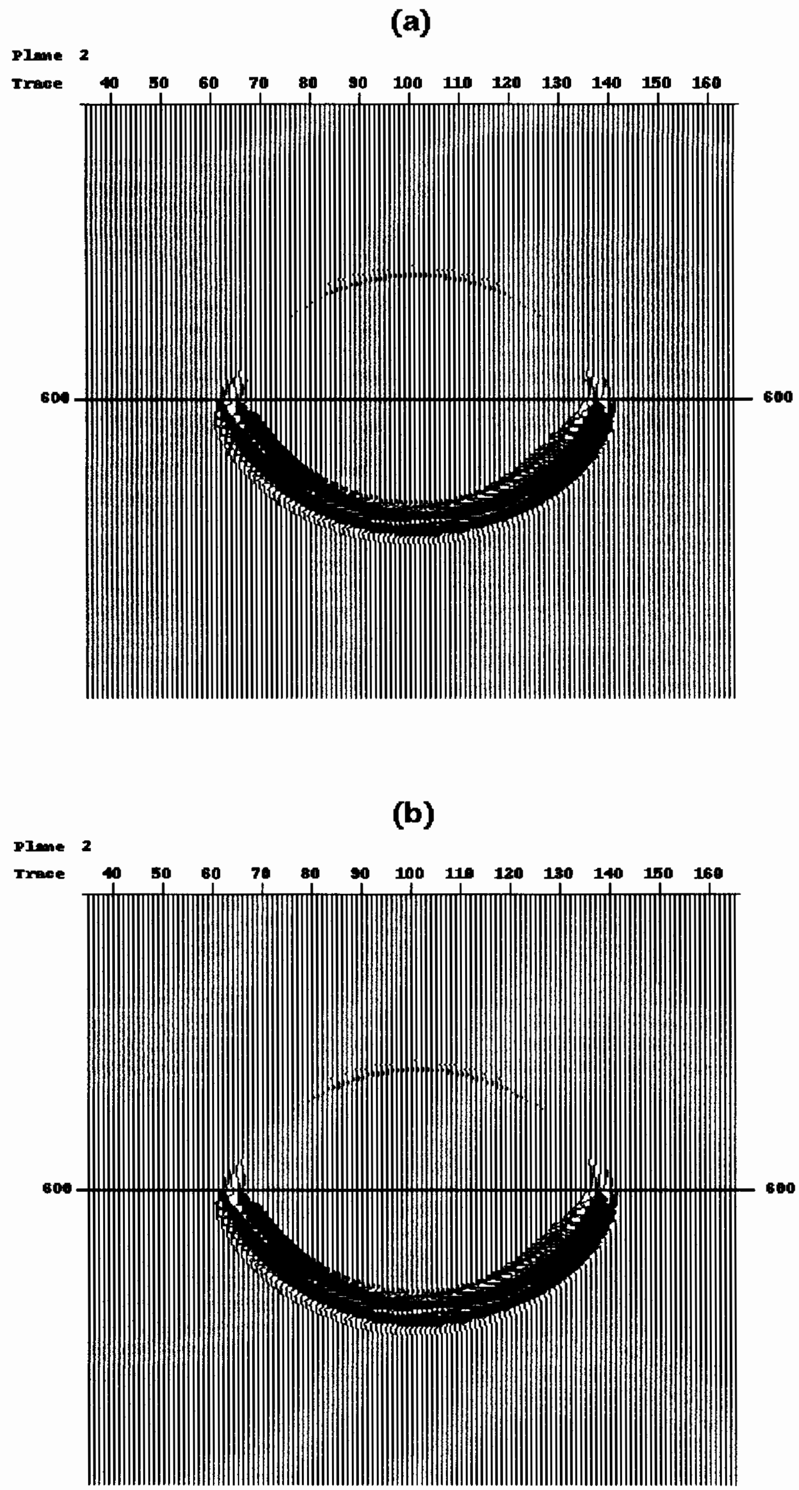

Figure 7: Errors between the previous reference solution and the numerical solutions computed with (a) the original scheme $\mathbf{G}_h^n$ or (b) the simplified scheme $\mathbf{B}_h^n$.

speed $C = 2c$. The difference between the $\mathbf{G}_h^n$-solution and the reference solution is plotted in Fig. 8*b*. The difference between the $\mathbf{B}_h^n$-solution and the reference solution looks very much alike, and this is why it will not be represented. In both instances, the error created by the grid changed is mainly concentrated in the transmitted front. Their relative order of magnitude —again measured by the discrete L^2-norm— amounts to 2.5%.

In model **C**, the reference solution behaves as predicted, i.e. there is no visible reflected waves at normal incidence. This is clearly evidenced in Fig. 9*a*. Barring from a slightly smaller amplitude, no noticeable difference with model **B** can be detected from the error snapshot in Fig. 9*b*. The L^2 relative error is approximately 2.1%.

8.2 Study of accuracy by trace recordings

A series of receivers are placed along the horizontal line $Z = 150\,\mathrm{m}$ inside the coarse subdomain. At each receiver location, we keep track of the development in time of the pressure. The result is called a *trace recording* or simply *trace*. For the study of accuracy, we are interested in the traces obtained at the same receiver locations, in the same physical model, but with a sequence of decreasing space-steps h and time-steps Δt such as $\Delta t/h = $ cte. For each value of h, we plot the difference between the $\mathbf{G}_h^n$- or $\mathbf{B}_h^n$-trace and the reference trace.

Figure 10*a* depicts this trace difference for the $\mathbf{G}_h^n$-method and three values of h. The model considered is **A**, and the receiver to which the traces correspond is located right below the source. We see that every time h is divided by 2, the error amplitude is roughly divided by 4. Since the reference solution has been computed with the same parameters h and Δt, this behavior suggests that the error introduced by the grid change is of second order.

A similar observation holds true for the $\mathbf{B}_h^n$-method, as shown by Fig. 10*b*. The trace differences obtained at other receiver locations exhibit the same second order convergence.

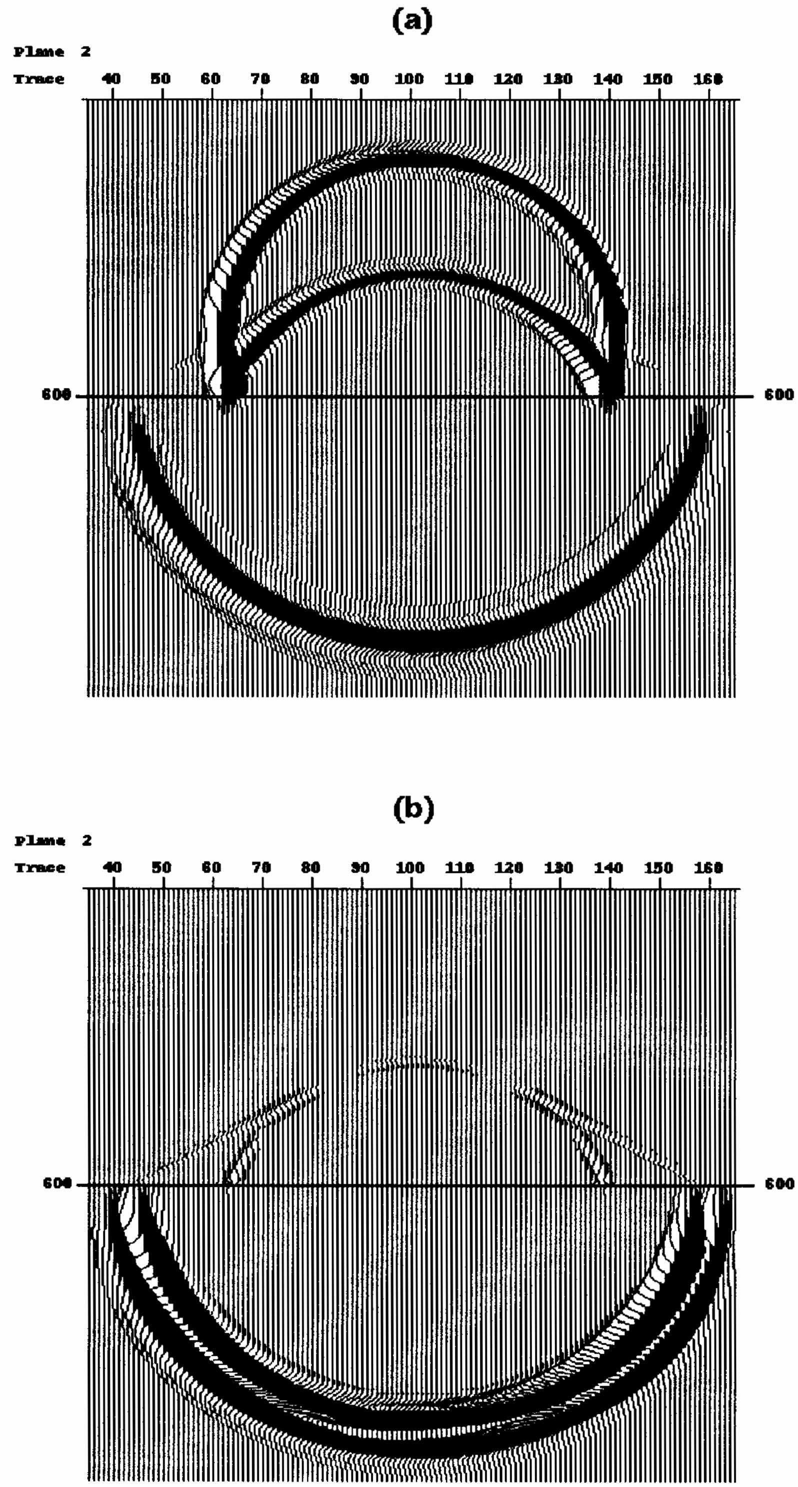

Figure 8: Reference solution (a) for model $\mathbf{B}$ at time $t = 1.5\,\mathrm{s}$ and (b) error between the numerical solutions and this reference solution (with a larger scaling factor).

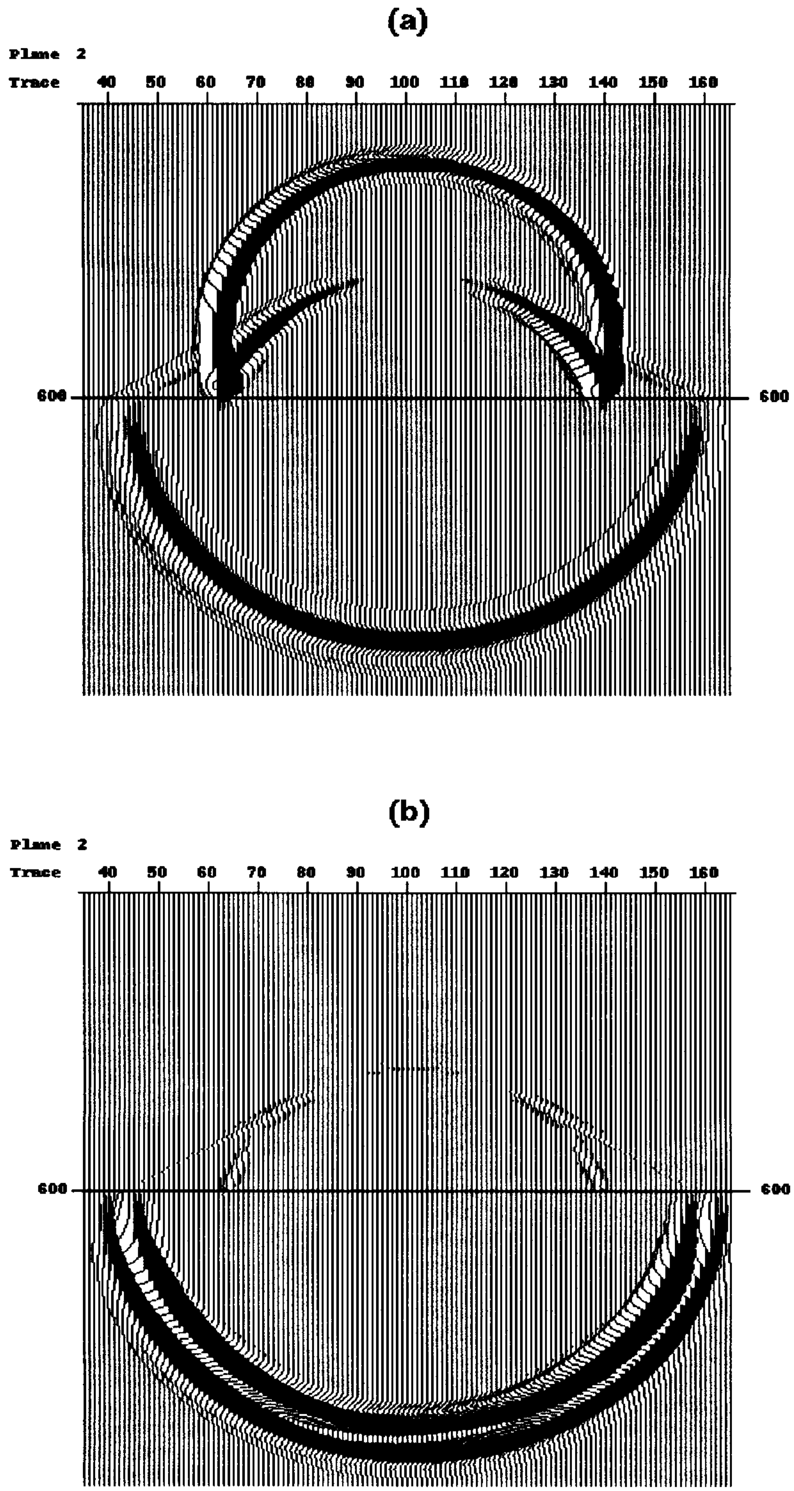

Figure 9: Reference solution (a) for model **C** at time $t = 1.5\,\mathrm{s}$ and (b) error between the numerical solutions and this reference solution (with a larger scaling factor).

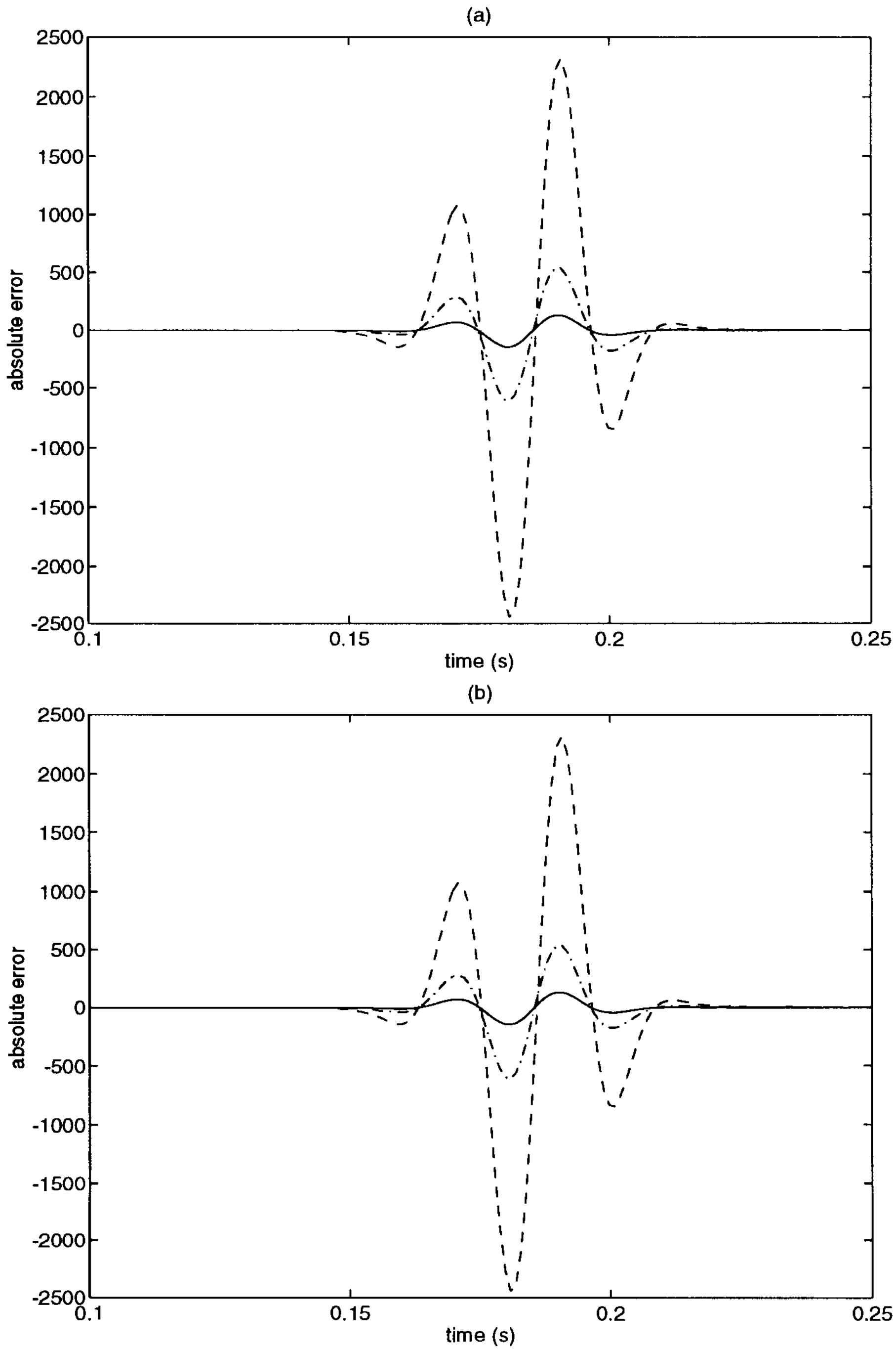

Figure 10: Trace difference between the reference solution and (a) the $\mathbf{W}_h^n$-solution or (b) the $\mathbf{G}_h^n$-solution for $h = 1.0\,\mathrm{m}$ (dashed line), $0.5\,\mathrm{m}$ (dash-dotted line) and $0.25\,\mathrm{m}$ (solid line).

9. Conclusion

The domain decomposition methods proposed in this paper are primarily aimed at meeting a practical need in wave propagation modelling. Their mathematical properties have been thoroughly investigated, and their implementations have been validated by the numerical simulations, which testified to a second order convergence.

Although the simplified version appears to be much more interesting than the original one from the computational point of view, it turns out that a prominent part of this paper has been devoted to the original method. The rationale of this seemingly paradoxical situation is that, in reality, we have tried to take advantage of this opportunity to generalize Babuška-Brezzi's formalism of mixed and hybrid finite element to the hyperbolic case. In the course of error estimates, however, the question remains as to how the projection error on Ker $\underline{D}^h$ could be controlled. We dare hope that this difficulty would be overcome soon, most probably by means of a new condition between the discretization spaces $\underline{V}^h$ and Λ^h.

Finally, we believe that these methods will be more widely used in the future by geophysicists, so long as they are restricted to second order methods at least. A fourth order domain decomposition method for the wave equation is currently under study.

Acknowledgments

The authors are grateful to Fadel Bendali and Azarab Laqua for their internship contribution to the coding of the schemes and the running of some numerical simulations. We also wish to thank Professor William W. Symes for his kindness in making it possible for us to finish writing this paper at Rice University.

References

[1] R. M. ALFORD, K. R. KELLY and D. M. BOORE, Accuracy of Finite-Difference Modeling of the Acousic Wave Equation, *Geophysics* **39**, 834-842 (1974).

[2] A. BAMBERGER, G. CHAVENT and P. LAILLY, *Étude de Schémas Numériques pour les Équations de l'Élastodynamique Linéaire*, Rapport de Recherche **41**, INRIA, Rocquencourt, 1980.

[3] F. BREZZI and M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[4] P. G. CIARLET and J. L. LIONS, eds., *Handbook of Numerical Analysis*, Vol. **II**: Finite Element Methods—Part 1, North-Holland, Amsterdam, 1989.

[5] M. A. DABLAIN, The Application of High-Order Differencing to the Scalar Wave Equation, *Geophysics* **51**, 54-66 (1986).

[6] R. DAUTRAY and J. L. LIONS, *Analyse Mathématique et Calcul Numérique pour les Sciences et Techniques*, Vol. **7**: Évolution—Semi-groupe, Variationnel, Masson, Paris, 1987.

[7] T. DUPONT, L^2-Estimates for Galerkin Methods for Second Order Hyperbolic Equations, *SIAM J. Numer. Anal.* **10**, 880-897 (1973).

[8] B. ENGQUIST and A. MAJDA, Absorbing Boundary Conditions for the Numerical Simulation of Waves, *Math. Comp.* **31**, 629-651 (1977).

[9] R. GLOWINSKI and M. F. WHEELER, Domain Decomposition and Mixed Finite Element Methods for Elliptic Problems, in *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, R. GLOWINSKI, G. H. GOLUB, G. MEURANT, and J. PÉRIAUX, eds., SIAM, Philadelphia, 1988.

[10] C. JASTRAM and A. BEHLE, Acoustic Modelling on a Grid of Vertically Varying Spacing, *Geophysical Prospecting* **40**, 157-169 (1992).

[11] K. R. KELLY, R. W. WARD, S. TREITEL and R. M. ALFORD, Synthetic Seismograms: A Finite-Difference Approach, *Geophysics* **41**, 2-27 (1976).

[12] D. KOSLOFF and E. BAYSAL, Forward Modeling by a Fourier Method, *Geophysics* **47**, 1402-1412 (1982).

[13] J. L. LIONS and E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. **I**, Springer-Verlag, Berlin, 1972.

[14] P. L. LIONS, On the Schwarz Alternating Method I, in *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, R. GLOWINSKI, G. H. GOLUB, G. MEURANT, and J. PÉRIAUX, eds., SIAM, Philadelphia, 1988.

[15] J. C. MEZA and W. W. SYMES, *Domain Decomposition Algorithms for Linear Hyperbolic Equations*, Technical Report 87-20, Rice University, Houston, Texas 77251-1892, 1987.

[16] P. A. RAVIART and J. M. THOMAS, Primal Hybrid Finite Element Method for Second Order Elliptic Equations, *Math. Comp.* **31**, 391-413 (1977).

[17] Q. H. TRAN, L. JANNAUD and P. ADLER, Coda Spectral Power and Apparent Attenuation Factor of Acoustic Waves in 3-D Random Media, *J. Acoust. Soc. Amer.* **94**, 2397-2407 (1993).

[18] R. VERSTEEG and G. GRAU, eds., *The Marmousi Experience*, Proceedings of the 52nd EAEG Workshop on Practical Aspects of Seismic Data Inversion, Copenhagen, 1990.