An improved multifractal formalism and self-affine measures

Summary of Ph. D. thesis by Rolf Riedi[1], Sept. 7, 1992

It has been recognized that most fractals K in nature are actually composed of an infinite set of interwoven subfractals. This structure becomes apparent when a particular measure μ of total mass 1 supported by K is considered: To every *singularity exponent* α belongs a set C_α of all points of K, for which the measure of the balls with radius ρ roughly scales as ρ^α for $\rho \to 0$. These sets are usually fractals, giving μ the name *multifractal*.

The complexity of the geometry of C_α is measured by the *spectrum* $f(\alpha)$ which can be thought of as representing the box dimension of C_α. However the singularities of μ may also be measured through the *generalized dimensions* d_q.

Spectrum and generalized dimensions are very helpful when comparing multifractals appearing in nature with analytically treatable measures. One central fact of the multifractal formalism is the close relation between d_q and $f(\alpha)$: the convex $\tau(q) = (1-q)d_q$ is the Legendre transform of the concave $f(\alpha)$. This allows to reduce the somewhat tedious, if not impossible computation of $f(\alpha)$ to the simpler one of d_q.

Though widely used, the various definitions of $f(\alpha)$ and d_q differ only slightly. A mathematically precise definition as well as the important relation $\tau(q) = \sup f(\alpha) - q\alpha$ can be found in [F]: But unfortunately this concept turns out to be *unsatisfactory* for two reasons.

First of all Falconers $f(\alpha)$ is defined through a double limes, which usually does not exist for great α. Secondly the generalized dimensions usually take the irrelevant value $d_q = \infty$ for negative q. More concretely, for as simple multifractals as the middle third Cantor measure half of the spectrum is lost and proposition 17.2 in [F] concerning the Legendre relation cannot be applied.

The concept we propose meets the two mentioned problems by a simple improvement. Instead of $B = \prod[l_k\delta, (l_k+1)\delta[$ taken from a grid G_δ of size δ, we use a kind of parallel body $B_1 := \prod[(l_k-1)\delta, (l_k+2)\delta[$. This renders a measurement of the singularities of μ which depends more regularly on δ:

$$T(q) := \limsup_{\delta\downarrow 0} \frac{\log \sum \mu(B_1)^q}{-\log\delta} \qquad D_q := \frac{T(q)}{1-q}.$$

This method will especially lead to regular spectra for self-similar measures: Adopting the definition of Falconer [F] let $N_\delta(\alpha)$ denote the number of boxes $B \in G_\delta$ with $\mu(B) \neq 0$ and $\mu((B)_1) \geq \delta^\alpha$ and set

$$F(\alpha) := \lim_{\varepsilon\downarrow 0} \limsup_{\delta\downarrow 0} \frac{\log(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon))}{-\log\delta}.$$

Note that $F(\alpha)$ always exists. However, we will call $F(\alpha)$ resp. $T(q)$ *δ-regular*, whenever the respective $\limsup_{\delta\to 0} = \liminf_{\delta\to 0}$. The superiority of our concept is reflected in the following facts, for which we provide rigorous proofs:

1present adress: ETH Zuerich, Zentrum HG G35.5, 8092 Zuerich, Switzerland

- D_q is invariant under bijective mappings Φ, which are lipschitzian together with their inverse. Moreover its computation may be performed by considering any sequence δ_n with $\delta_n \geq \delta_{n+1} \geq \nu\delta_n$ for some constant ν. Essentially the same holds for $F(\alpha)$ as well.

- $D_q = d_q$ for positive q and $F(\alpha)=f(\alpha)$ in the rising part.

- The Legendre connection holds. Especially, the function $T(q) := (1-q)D_q$ is convex, and, assuming that T is differentiable at q then

$$F(-T'(q)) = T(q) - qT'(q). \tag{1}$$

If T is in addition continuously differentiable and strictly convex near q, then F is continuous and strictly concave near $\alpha = -T'(q)$. On the other hand if T is differentiable and δ-regular on all of $\mathbb{R}$, then $F(\alpha)$ is δ-regular on all of $\mathbb{R}$ too.
Thus there is a lot to derive from D_q about the spectrum without assuming anything about it.

- The new concept allows to give a rigorous proof of the well-known implicit formula

$$\sum_{i=1}^{r} p_i^q \lambda_i^{T(q)} = 1 \tag{2}$$

for the generalized dimensions of self-similar measures with ratios $\lambda_1, \ldots, \lambda_r$ (see below). Moreover T is in fact δ-regular for all q. Concerning the proof of (2) we are so far only aware of arguments [HP] valid for positive q and for similarities respecting a particular grid, as it is the case for the middle third Cantor measure. Finally we provide an explicit formula for $F(\alpha)$ in the case $r = 2$:

$$F(\alpha) = \frac{c_2 \log(-c_2) + (c_1 - c_2)\log(c_1 - c_2) - c_1 \log(c_1)}{\log\lambda_1 \log p_2 - \log\lambda_2 \log p_1} \quad (\alpha \in]D_\infty, D_{-\infty}[).$$

where we set $c_i = \log p_i - \alpha \log \lambda_i$ and where we assumed without loss of generality that $D_\infty = \log p_1 / \log \lambda_1 < \log p_2 / \log \lambda_2 = D_{-\infty}$.

- Since we use the measure of B_1 instead of the one of B, a related result of Collet et al [CLP] can be transformed to read as: For totally disconnected self-similar multifractals on $\mathbb{R}$, such as the middle third Cantor measure, define C_α to be the set of all points x for which

$$\lim_{\substack{|E|\to 0 \\ x\in \mathrm{int}(E)}} \frac{\log \mu(E)}{\log |E|} = \alpha.$$

Then the Hausdorff dimension of C_α equals $F(\alpha)$. This is important for the intuitive interpretation of the spectrum.

Our approach assumes nothing about the spectrum, neither about its concavity or differentiability nor about the existence of any particular limes . To support this attitude we provide two examples presented in figure 1.
Finally we apply the new concept to certain self-affine multifractals of the plane. One way to set up the construction of a multifractal is to use a set of contractions $(w_1, \ldots, w_r)$ of $\mathbb{R}^d$ with $\mathrm{Lip}(w_i) < 1$ and a set of positive numbers p_i with $p_1 + \ldots + p_r = 1$. Then there is a unique invariant measure μ, which may be explicitely constructed using the fixpoint-lemma [Hut]:

$$\mu = \sum_{i=1}^{r} p_i \cdot w_{i*}\mu \qquad \text{and} \qquad \mathrm{supp}(\mu) = K = \bigcup_{i=1}^{r} w_i(K).$$

The measure μ is called *self-similar*, when $|w_i(x) - w_i(y)| = \lambda_i \cdot |x - y|\ \forall x \forall y \in \mathbb{R}^d$ $(i = 1, \ldots, r)$ and when the open set condition is satisfied, i.e. the sets $w_i(O)$ are mutually disjoint and contained in O for some bounded open set O. The spectrum of μ in this case is given by (2) and (1).
Now take $w_i(x^{(1)}, x^{(2)}) = (\vartheta_i \lambda_i x^{(1)} + u_i, \zeta_i \nu_i x^{(2)} + v_i)$, where ϑ_i and ζ_i are from $\{-1, +1\}$ and λ_i, ν_i from $]0, 1[$. When the open set condition is satisfied by a polyederon O with angles greater or equal to $\pi/4$, then we call μ *self-affine multifractal*, for short SAMF. The generalized dimensions of μ are in this case determined by the projections onto the respective axes. So let $\mu^{(k)} := \pi^{(k)}{}_*\mu$, where $\pi^{(k)}(x^{(1)}, x^{(2)}) = x^{(k)}$, and denote the generalized dimensions of $\mu^{(k)}$ by $D_q{}^{(k)}$. Furthermore to deal with negative q even more control on the measure is required. When $\pi^{(k)}(w_i(O)) \cap \pi^{(k)}(w_j(O)) \neq \emptyset$ implies that $\pi^{(k)} \circ w_i = \pi^{(k)} \circ w_j$, then the sets $V_{ij} := w_i(w_j(O))$ are arranged in rows and columns. Now if in addition every such row or column contains at least one V_{ij} which does not touch the boundary of O, we call μ a *centered self-affine multifractal*, C-SAMF for short.
Now define the 'characteristic functions'

$$\chi(a, \gamma, q) := \sum_{i=1}^{r} p_i{}^q \lambda_i{}^a \nu_i{}^{\gamma - a} \quad \text{and} \quad \psi(b, \gamma, q) := \sum_{i=1}^{r} p_i{}^q \nu_i{}^b \lambda_i{}^{\gamma - b}.$$

For fixed q there are unique numbers γ^+ and γ^- satisfying $\chi(T^{(1)}(q), \gamma^+, q) = 1$, resp. $\psi(T^{(2)}(q), \gamma^-, q) = 1$. Provided $\lambda_i > \nu_i$ and $\lambda_j < \nu_j$ for some i and j, there is a unique pair (a_0, γ_0) satisfying the simultaneous equations $\chi(a_0, \gamma_0, q) = 1$, $\frac{\partial}{\partial a}\chi(a_0, \gamma_0, q) = 0$. Otherwise set $\gamma_0 := -\infty$ for all q. Define

$$\Gamma^+(q) := \begin{cases} \gamma^+ & \text{if } \frac{\partial}{\partial a}\chi(T^{(1)}, \gamma^+, q) \geq 0, \\ \gamma_0 & \text{otherwise}, \end{cases} \qquad \Gamma^-(q) := \begin{cases} \gamma^- & \text{if } \frac{\partial}{\partial b}\psi(T^{(2)}, \gamma^-, q) \geq 0, \\ \gamma_0 & \text{otherwise}. \end{cases}$$

Theorem 1 *Let μ be a C-SAMF. Then T is δ-regular on all of $\mathbb{R}$ and*

$$\boxed{T(q) = \max\left(\Gamma^+(q), \Gamma^-(q)\right).}$$

The assertion holds also for arbitrary SAMF provided that $q \geq 0$ and that $T^{(1)}(q)$ and $T^{(2)}(q)$ are δ-regular.

Note that Γ^+ resp. Γ^- are continuously differentiable near q provided $T^{(1)}(q)$ resp. $T^{(2)}(q)$ are. This may import in (1). Moreover, assuming only that $T^{(1)}$ and $T^{(2)}$ are C^1 near 1, we have

$$D_1 := \lim_{q\to 1} D_q = -T'(1) = \limsup_{\delta\downarrow 0} \frac{1}{\log(\delta)} \frac{\sum \mu((B)_1)\log\mu((B)_1)}{\sum \mu((B)_1)}.$$

For applications it may be useful to know (writing $a = T^{(1)}(q)$, $b = T^{(2)}(q)$ for short)

$$\gamma^+ \leq a+b \Leftrightarrow \gamma^- \leq a+b \Leftrightarrow T(q) = \max(\gamma^+,\gamma^-) \Leftrightarrow T(q) \leq a+b.$$

This occurs certainly for $q=0$, since $T(0)$ equals the box dimension of K. So it is easy to compare $T(0)$ with the almost sure box- and Hausdorff dimension Δ of the self-affine set K given in [F88], for which we provide the formula: $\Delta = \max(\Delta^+,\Delta^-)$, where

$$\begin{cases} \sum_{i=1}^{r} \lambda_i^{\Delta^+} & \text{if } \sum\lambda_i \leq 1, \\ \sum_{i=1}^{r} \lambda_i\nu_i^{\Delta^+-1} & \text{otherwise,} \end{cases} \quad \text{and} \quad \begin{cases} \sum_{i=1}^{r} \nu_i^{\Delta^-} & \text{if } \sum\nu_i \leq 1, \\ \sum_{i=1}^{r} \nu_i\lambda_i^{\Delta^--1} & \text{otherwise,} \end{cases}$$

One special kind of C-SAMF are a generalization of the so-called *Sierpiński carpets* [Mu, Bed], defined by the property $\lambda_i = \lambda > \nu = \nu_i$ for $i = 1,\ldots,r$. See figure 2 for an example. Since $\max(\Gamma^+(q),\Gamma^-(q)) \equiv \gamma^+(q)$ in this case, T is C^∞ and so is $F(\alpha)$ by (1). Moreover F is δ-regular. Denoting the sum of all p_j with $w_i(O)$ in the i-th column by p_i^+ we have

$$T(q) = \Big(\frac{1}{\log\nu} - \frac{1}{\log\lambda}\Big)\log\Big(\sum_{i=1}^{s}(p_i^+)^q\Big) - \frac{1}{\log\nu}\log\Big(\sum_{i=1}^{r} {p_i}^q\Big).$$

We add two explicitely solvable examples. For the 'circular' C-SAMF (see figure 3)

$$\begin{array}{rclrcl} w_1(x,y) &=& (x/2+1/4, y/4) & w_2(x,y) &=& (x/4+3/4, y/2+1/4) \\ w_3(x,y) &=& (x/2+1/4, y/4+3/4) & w_4(x,y) &=& (x/4, y/2+1/4) \end{array}$$

with $p_1 = \ldots = p_4 = 1/4$ one finds $T^{(1)}(q) = T^{(2)}(q) \equiv 1-q$ and

$$T(q) = \begin{cases} \gamma^+ & = 3-2q-\log(\sqrt{1+2^{4-q}}-1)/\log 2 & \text{if } q \leq 1 \\ \gamma_0 & = 4/3\cdot(1-q) & \text{otherwise.} \end{cases}$$

So T is C^1 but not C^2. However, $F(\alpha)$ is δ-regular.
Consider the maps

$$\begin{array}{rclrcl} w_1(x,y) &=& (x/2-1/2, y/4) & w_2(x,y) &=& (x/2, y/2-1/2) \\ w_3(x,y) &=& (x/2+1/2, y/4) & w_4(x,y) &=& (x/2, y/2+1/2) \end{array}$$

with the open set $O = \{(x,y) : |x|+|y| < 1\}$. Choosing $p_i = 1/4$ we have

$$\gamma^+ = \begin{cases} 2-q-\log\left(\sqrt{1+2^{2+q}}-1\right)/\log 2 & \text{if } q \geq -1, \\ 1-2q-\log\left(\sqrt{3}-1\right)/\log 2 & \text{otherwise.} \end{cases}$$

Since $\lambda_i \geq \nu_i$ for all i, $\max(\Gamma^+,\Gamma^-) \equiv \gamma^+$. But this 'rosette' is only a SAMF. So $T(q) \leq \gamma^+(q)$ with equality only for $q \geq 0$. See figure 4.
The explicit formulas above enable one to study the set of the most probable resp. most rarefied points, resulting in interesting insights concerning the geometric properties of self-affine measures.

References

[Bed] Tim Bedford *Ph. D. Thesis* Warwick Univ. (UK) (1984)

[CLP] P. Collet, J. Lebovitc et A. Porcio *The Dimension Spectrum of Some Dynamical Systems* J. Stat. Phys.: **47** (1987) pp 609–644

[F88] Kenneth J. Falconer *The Hausdorff dimension of self-affine fractals* Math. Proc. Camb. Phil. Soc.: **103** (1988) pp 339–350

[F] Kenneth J. Falconer *Fractal Geometry* John Wiley and Sons, New York (1990)

[HP] H. Hentschel & I. Procaccia *The Infinite number of Generalized Dimensions of Fractals and Strange Attractors* Physica D: **8** (1983) pp 435–444

[Hut] John Hutchinson *Fractals and Selfsimilarities* Indiana Univ. Math. J.: **30** (1981) pp 713–747

[Mu] Curt McMullen *The Hausdorff Dimension of general Sierpiński Carpets* Nagoya Math. J.:**96** (1984) pp1–9

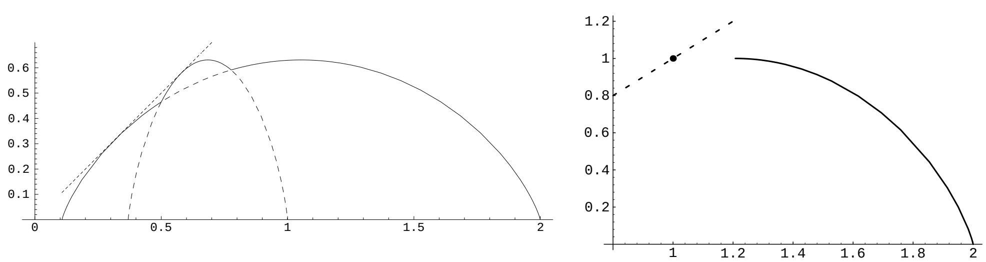

Figure 1: On the right: The linear combination $\mu := \mu_1 + \mu_2$ of two multifractals with disjoint support may possess a nonconcave spectrum, since the latter is the maximum of the spectra of μ_1 and μ_2. The dashed parts show the internal bisector of the axes and the spectra of μ_1 and μ_2. On the right: The spectrum of the graph of the fractal interpolation function through the points $(0, 0)$, $(1/2, 1/4)$ and $(1, 1)$. The dashed part shows the internal bisector of the axes.

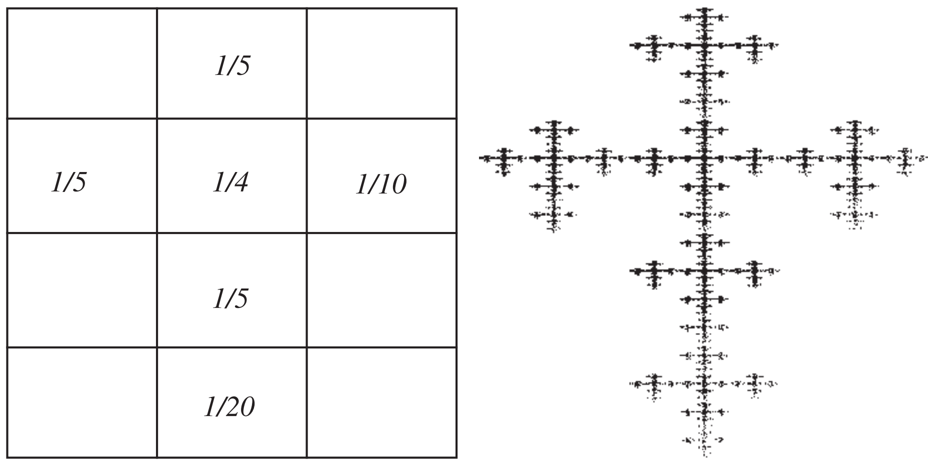

Figure 2: The construction of a Sierpiński carpet. The labeled rectangles show the images $w_i([0,1]^2)$ and the assigned probabilities.

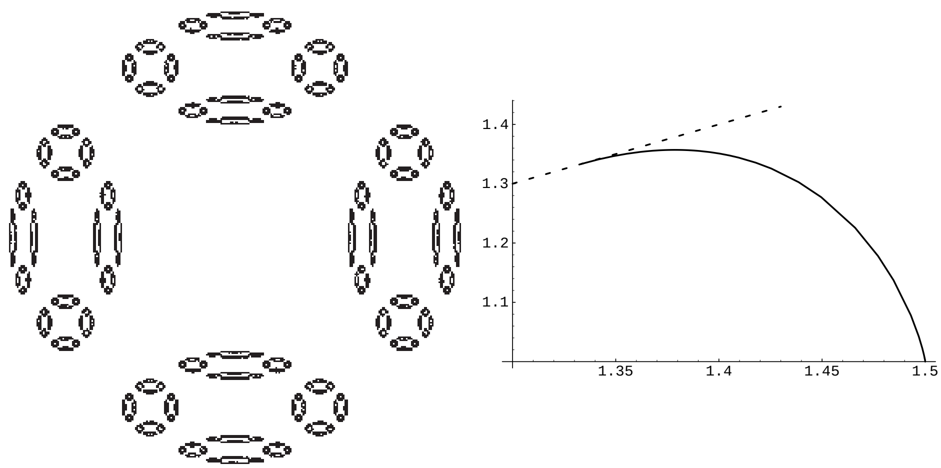

Figure 3: The 'circular' C-SAMF and its spectrum.

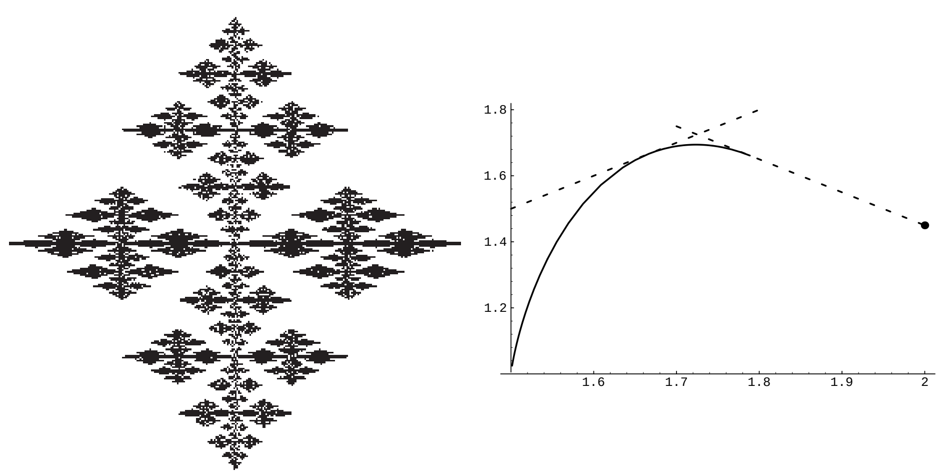

Figure 4: On the left the 'rosette'. On the right the Legendre transform of a formaly deduced $T(q)/(1-q)$, which equals the spectrum of the 'rosette'-multifractal at least in the rising part.