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ANALYTIC APPROXIMATION OF GALERKIN'S PROCEDURE FOR COMPUTING FORCED OSCILLATIONS OF NONLINEAR SYSTEMS

by

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CHAPTER I

INTRODUCTION

1.0 Definition of the Problem

The basic topic under consideration in this thesis concerns the problem of determining periodic (or steady state) responses of physical systems which can be described by nonlinear ordinary differential equations. In mathematical terms, it is desired to find all periodic functions, of period 2π, which satisfy the equation

$$\dot{x} = f(x,t) \tag{1}$$

where $\dot{x} = dx/dt$, x and f are n-dimensional vectors, and $f(x,t)$ is periodic (of period 2π) in the scalar variable t. (Note that the above definition excludes such periodic solutions as subharmonic oscillations and oscillations which are not harmonically related to the excitation.)

Unlike the case of linear, time-invariant ordinary differential equations, where the function $f(x,t)$ in (1) is of the form

$$f(x,t) = Ax + g(t) , \tag{2}$$

and A is an $n \times n$ constant matrix, there is no theory available which yields exact[1] solutions of the general nonlinear system. Consequently, since the physical world cannot always be adequately described by linear equations, there has been considerable effort devoted to the development of methods which generate approximations to the exact solutions of the

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[1] The phrases "exact solution" and "true solution" are redundant; however, the phrase "approximate solution" (which denotes any function of time that in some sense approximates a solution of (1)) is well established in the literature and hence the redundant modifiers will be freely used to distinguish the solution from its approximations.

system (1). These approximate methods have been of many different types, including graphical methods (phase plane techniques) which are valuable in analyzing second order systems, simulation using the analog computer, numerical integration methods using modern high speed digital computers, and various analytical techniques. Of the analytical methods, the most widely known are Poincare's method of small perturbations (which was used in the late nineteenth century by astronomers to analyze the motion of the planets), the method of successive approximations (with its many variations), and the method of harmonic balance. A concise presentation of these various analytical methods is given in the first chapter of Hayashi's very interesting book [5]. There exists considerable confusion in the literature regarding the proper nomenclature for the various methods. Variations such as Galerkin's method, the Ritz method, and equivalent linearization are either identical with, or very closely related to, the harmonic balance method -- depending on the precise definitions used; the same basic idea, when applied to nonlinear automatic control systems, is known as the describing function method. This thesis will be concerned with the method of harmonic balance (as defined by Hayashi [5]) or equivalently with Galerkin's method (as defined by Urabe [8]), and the two terms will be used interchangably. To avoid confusion, the method will be defined and illustrated in Chapter II.

The harmonic balance method (and its variants) has been widely utilized in practical investigations of nonlinear systems. It has been shown in many instances to be surprisingly accurate, considering the basic simplicity of the ideas involved. Unfortunately, it has also been demonstrated that the method can be quite inaccurate;[2] an example will be

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[2] See section 7, Chapter 3 of reference [11].

given in Chapter II which illustrates the variations in accuracy which can occur.

A well-known difficulty in the application of the harmonic balance method is the lack of knowledge about the accuracy of the resulting approximate solution. For example, in applying the first order method,[3] in which the approximating solution is a pure sinusoid, it has long been recognized that the source of error in the method is the presence of higher harmonics in the true solution; however, the first order method provides no estimate of the magnitudes of these higher harmonics. Therefore, one has no idea of the magnitude of the terms which have been neglected by the method, and hence no indication of its accuracy.

This problem has attracted the attention of many researchers. Approaches to the problem have been of two distinct types: the development of error bounds (together with existence conditions), and the development of error estimates and/or corrections for improving the approximate solution. The research to be presented in this thesis is of the latter approach; specifically, two methods for generating corrections to the first order harmonic balance solution(s) will be derived in Chapters III and IV. The meaning of the term <u>correction</u> as used in this thesis (and of the <u>improvement</u> in the approximation which it suggests) deserves clarification. It is <u>not</u> meant that under choice of an appropraite norm, the distance between the approximate solution and the true solution will always be reduced by application of the correction methods. What <u>is</u> meant is that provided certain quantities (to be subsequently specified) are sufficiently small, then the above distance will generally be reduced (independent of the particular choice of norm) because additional information

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[3] The concept of the <u>order</u> of the harmonic balance method will be presented in Chapter II.

about the true solution is being utilized. A simple analogy which illustrates the above statement is the approximation of a scalar function $g(t)$ by a truncated Taylor series about a point t_o:

$$g(t) \doteq g(t_o) + g'(t_o) \cdot (t - t_o) . \tag{3}$$

If we now include an additional term in the series

$$g\,t \doteq g(t_o) + g'(t_o) \cdot (t - t_o) + \tfrac{1}{2} g''(t_o) \cdot (t - t_o)^2, \tag{4}$$

then provided $(t-t_o)$ is sufficiently small, the second approximation will generally be better than the first.

In a similar sense, the m^{th} order Galerkin solutions are <u>corrections</u> to the first order Galerkin solutions (or alternatively, the m^{th} order procedure may be viewed as a means of estimating the error of the first order procedure). Provided all harmonic components above the m^{th} are sufficiently small, the m^{th} order solutions will generally be better approximations to the true solution. The two results to be presented in Chapter III and Chapter IV may be viewed as analytic <u>approximations</u> to the higher order Galerkin procedures. The approximations upon which the results are based are generalizations of the truncated Taylor series in (3). These methods will generally improve the first order Galerkin solution provided all harmonics above the first are sufficiently small.

The question of "how small is sufficiently small" as used in the preceding sentence has not been considered in this thesis, because this same question has yet to be answered for the (exact) m^{th} order harmonic balance procedure. We will return to this subject briefly in the con-

cluding chapter. Before discussing further the objectives and results of this thesis (and the results of two other researchers who have also pursued the estimation and correction approach), a brief summary of the research which has been directed toward the development of error bounds and existence conditions will be presented.

1.1 Error Bounds and Existence Conditions for Harmonic Balance Solutions

Most of the research related to the problem of inaccuracies in the harmonic balance method (or its variants) has been in the area of error bounds and/or existence conditions. The harmonic balance method (as well as most all other analytical methods of nonlinear analysis) is basically heuristic in origin, and efforts to provide it a firm mathematical foundation have been generally frustrating. R.W. Bass [1] has considered the mathematical legitimacy of equivalent linearization, and has proposed a criterion for determining its applicability. Unfortunately, as Bass himself points out, his results are quite difficult to apply in practice. The questions of existence are quite complicated. There may exist no periodic solution to system (1), or alternatively there may exist one or more periodic solutions (the interesting phenomenon of jump-resonance is a well-known consequence of the latter occurrence). The harmonic balance method may generate multiple approximate solutions when in fact there is only one true solution, and vice versa. Lamberto Cesari [2] has considered the question of the existence of an exact solution in some neighborhood of a harmonic balance solution and has exhibited an algorithm which may establish existence, and (in the affirmative case) may provide an error bound. The indecisiveness implied in the above statement is a consequence of the manner in which the results are stated: there is no constructive procedure for verifying whether or not Cesari's conditions for existence can be satisfied.

Minoru Urabe [8] has determined conditions under which the existence of an exact solution implies the existence of a Galerkin solution of sufficiently high order (and vice versa), together with an error bound. Urabe, in a subsequent paper co-authored with Reiter [9], presents a numerical method for approximating (as closely as desired) the high order Galerkin solutions necessary for applying his existence conditions. The numerical method of Urabe and Reiter is significantly related to the analytic methods of Chapters III and IV, and hence will be discussed further in the next section and in later chapters.

J.M. Holtzman [6] uses a formulation similar to that used by both Cesari and Urabe, and he develops sufficient existence conditions which can be stated quite concisely; a practical difficulty is that his conditions are stated in a manner which is reminiscent of Liapunov's stability theorems, in the sense that a function must be found which satisfies certain conditions -- and unfortunately no constructive procedure for obtaining the function is given. Both I.W. Sandberg [7], and Garber/Rozenvasser [4] have established results which can be interpreted graphically; however, in the case of the latter researchers, a fundamental difficulty in the application of some of their results has been recently noted by G.W. Duncan and R.A. Johnson [3]. Sandberg's results are relatively easy to apply and are of an intuitively pleasing form. The restrictions he imposes upon the form of the nonlinearity $f(x,t)$ are somewhat strange from a physical standpoint, however.

In general, all of the above approaches have in common the fact that they were derived from ideas of contraction mappings and various fixed-point theorems from the theory of functional analysis. They differ primarily in their restrictions imposed upon the form of the nonlinearity

$f(x,t)$ and upon their choice of a Banach space[4] in which to frame their results.

In the remainder of this thesis, we will leave the above questions of existence (and the related methods for bounding the error), and consider instead the problem of improving the approximation generated by the first order harmonic balance method, assuming the existence of both a true solution and a Galerkin solution.

1.2 Objective of This Thesis

The objective of the research leading to this thesis was to answer the following question: assuming the existence of a periodic solution to the system (1), and given the first order harmonic balance (or Galerkin) approximation to this solution, how can an improved approximation be obtained -- subject to the constraint that the analytic effort required be comparable to that of the first order harmonic balance procedure? The reason behind the imposed constraint is that although in principle[5] one can obtain an approximate solution of arbitrarily small error by employing a Galerkin procedure of sufficiently high order, in general only the first order procedure can actually be applied in practice because of the very rapid increase in analytic computational difficulty with increasing order of approximation. The analytic effort required for both the first and second order Galerkin method will be illustrated in Chapter II.

Urabe and Reiter [9] have studied methods of numerically approximating high order Galerkin solutions. As will be seen in the next chapter,

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[4] See Holtzman [6] for a concise discussion of these differences.

[5] Assuming that the true periodic solution is an isolated solution. (See [8])

the difficulty in applying the general m^{th} order harmonic balance method arises from the necessity of analytically expanding the function

$$f[\sum_{j=0}^{m} (x_{jc} \cos jt + x_{js} \sin jt), t] \tag{5}$$

in a Fourier series. Urabe and Reiter approximate the integrals necessary for determining the Fourier coefficients of (5) with finite sums, and then use Newton's method to obtain an approximation to the m^{th} order Galerkin solution. In Chapter IV of this thesis, an iterative procedure will be presented which is shown to be related to a Newton procedure; the method is somewhat similar to that of Urabe-Reiter, except that their method is, from the outset, a numerical procedure, whereas the iterative method of Chapter IV is primarily an analytical method. The only numerical aspect to the latter method is that an algebraic system

$$Az = b \tag{6}$$

must be solved at each stage of the iteration, where A is a square matrix of dimension no smaller than 4X4. The elements of A and b in (6) are all evaluated analytically, before the iteration begins; in the method of Urabe and Reiter, one also solves a system similar to (6), but in that case, the elements of A and b are computed at each stage of the iteration by numerical integration. The approximate solutions obtained by the two methods are different because their sources of error are different: in the method of Urabe-Reiter, the source of error is in the numerical integration; in the iterative method of Chapter IV, the source of error lies in the truncation of a Taylor's series expansion.

An advantage of the method of Urabe and Reiter is that by increasing indefinitely the accuracy of the numerical integration (at the cost

of increased computer usage), their approximate solutions can be made to approach the m^{th} order Galerkin solution as closely as desired. Such a capability does not exist in the method of Chapter IV, because to include any additional Taylor series terms would require an impractical amount of (analytic) computational effort. However, the method of Chapter IV has the advantage of being an analytical, rather than a numerical, method. For example, suppose that the required analytic evaluations of A and b have been performed for a third order scalar system

$$\dddot{x} = f(\ddot{x},\dot{x}, x,t)$$

where the function f has the form

$$f(\ddot{x},\dot{x},x,t) = \sum_{j=1}^{p} \alpha_j \ddot{x}^j + \sum_{j=1}^{q} \beta_j \dot{x}^j + \sum_{j=1}^{r} \gamma_j x^j + \sum_{j=1}^{s} \eta \cos jt \ , \qquad (7)$$

and p, q, r, and s are small integers. The important point is that once the original analytic work has been done, then for each iteration (and for each new choice of the parameters α_j, β_j, γ_j, and η_j) no additional analytic computation is required. It is only necessary to numerically solve the linear algebraic system. Consequently, if it is desired to investigate the system (7) for a large number of different parameter values, the method of Chapter IV can result in considerable savings in computer usage.

One other approach is pertinent to this thesis: Hayashi ([5], section 1.5) proposes that the third[6] order Galerkin solution be approximated by a linearization of the third order equations about the first order solution.

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[6] We consider the third (rather than second) order method because attention will be restricted in Chapter III to nonlinearities f(x,t) which are odd in x (in which case the second order method is degenerate).

A set of approximations will be presented in Chapter III which allows the linearized equations of Hayashi to be obtained with less effort than is required by his method.

1.3 The Nature of the Results Obtained

Before beginning the detailed derivation and illustration of the results in the remainder of the thesis, in this section the qualitative aspects of the two methods will be briefly discussed. Hayashi approximates the Fourier expansion of

$$f[A \cos t + B \sin t + C \cos 3t + D \sin 3t, t] \tag{8}$$

which is required in the third order Galerkin method with the Fourier expansion of

$$f[(A+\epsilon\delta A) \cos t + (B+\epsilon\delta B) \sin t + \epsilon\delta C \cos 3t + \epsilon\delta D \sin 3t, t] , \tag{9}$$

where all terms of order greater than unity in ϵ are ignored as the expansion progresses. The meaning of the phrase "progression of the expansion" deserves some comment at this point. In introducing the topic of this thesis in section 1.0, no restrictions were imposed on the form of the nonlinearity $f(x,t)$ in (1); for practical reasons, we will be concerned with polynomial nonlinearities, i.e., attention will be restricted to the class of functions $f(x,t)$ in which each of the n components f_i of the vector f are polynomials in the n components x_i of the state vector x. Specifically, the reason for this restriction is that in practice it is very difficult, if not impossible, to <u>analytically</u> expand expression (5) in a Fourier series for nonlinearities other than polynomials.[7] An exception in the special case of $m = 1$ and $f(x,t)$

[7] And, as will be demonstrated in Chapter II, even in the polynomial case it is not practical to analytically expand (3) for $m > 1$, except for very low order polynomials.

piecewise-linear in x : such evaluations have been performed in connection with the describing function method of automatic control theory.

When $f(x,t)$ is polynomial in x, the Fourier expansion of (5) is obtained not by evaluating the integrals which define the Fourier coefficients, but rather by direct algebraic expansion using trigonometric identities; this expansion process will be clearly illustrated in Chapter II.

We further restrict the nonlinearity $f(x,t)$ to be continuous in t. This results in no loss of generality in practice, however. For example, if the m^{th} order Galerkin method is used to analyze the steady state response of a nonlinear system excited with a squarewave, the excitation must be approximated with a finite number of terms from its Fourier series (and only the first m terms affect the Galerkin solution).

By ignoring all terms in the expansion of (5) of order greater than unity in ϵ, Hayashi obtains an approximation to the third order Galerkin solution in which only the first order effects (in the sense of a truncated Taylor series) of the third harmonics are taken into account. In Chapter III, a set of approximations is presented which (for odd nonlinearities) allows Hayashi's equations to be obtained by consideration only of the first order Galerkin results. If the Fourier components of

$$f[A \cos t + B \sin t, t] \tag{10}$$

are known analytic functions of A and B, then it will be shown how Hayashi's equations can be obtained by the substitution of various simple algebraic expressions (such as $A \pm C$, $A \pm D$, $B \pm C$, and $B \pm D$) for the parameters A and B in (10). A saving of analytic effort is thereby obtained by eliminating the necessity of expanding $f(x,t)$ when x is a sum of more than two sinusoids.

The second result to be presented in this thesis (Chapter IV) is an iterative method which can use the first order Galerkin solution as a starting point. The method possesses three significant features: first, its accuracy for oscillations[8] whose harmonic content decreases slowly with increasing frequency can be comparable to the third (or higher) order harmonic balance method, even though the analytical effort necessary to apply the method is comparable to that required for the first order harmonic balance procedure. Secondly, the method generates estimates of as many of the harmonics of the oscillation as desired, with no increase in required analytic effort; the only price one pays for additional harmonic information is a larger linear algebraic system which must be solved numerically at each stage of the iteration. Finally, in contrast to the general m^{th} order Galerkin procedure, the method produces an indication of its own accuracy by estimating the magnitude of the terms which have been neglected in the approximations used.

[8] We have implicitly assumed that the dominant Fourier component in the oscillation is the fundamental; the approximations upon which the iterative method is based are not physically meaningful if this assumption is not satisfied.

CHAPTER II

THE HARMONIC BALANCE METHOD (GALERKIN'S PROCEDURE)

2.0 Introduction

In this chapter, the harmonic balance method [5] will be discussed. First, the basic motivation behind the procedure will be presented, and two simple examples will be given to illustrate the ideas. The general m^{th} order harmonic balance method will then be precisely defined. Finally, the first and third[1] order procedures will be applied in detail to a specific example which will be used later to illustrate the results of the thesis.

2.1 Motivation Behind the Method

Consider the first order vector differential equation

$$\dot{x} = f(x,t) \tag{1}$$

where x and f are n-dimensional vectors, $\dot{x} = dx/dt$, and f is periodic (of period 2π)[2] in the scalar variable t. Hereafter, in the interest of brevity the term "periodic" will always imply periodicity in t of period 2π. The Galerkin procedure is well defined for an extremely broad class of systems (1). The functions $f(x,t)$ are restricted only to the extent that the time function

$$y(t) \triangleq f[\bar{x}(t,t)] ,$$

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[1] In the example chosen, all even harmonics are identically zero and hence the second order procedure is degenerate.

[2] Restriction of the period to 2π of course results in no loss of generality.

(where $\bar{x}(t)$ is periodic and possesses a finite number of Fourier components) be expandable in a Fourier series. However, in the two results to be presented in Chapters III and IV, attention is restricted to the class of systems (1) in which $f(x,t)$ is polynomial in x and continuous in t. As was explained in Chapter I, these restrictions are based on practical considerations and are more restrictive than is theoretically necessary.

Given any periodic differentiable function $\bar{x}(t)$, the functions $f(\bar{x}(t),t)$ and $\dot{\bar{x}}(t)$ are also periodic, as is their difference

$$\epsilon(t) \triangleq \epsilon'(\bar{x}(t),t) \triangleq \dot{\bar{x}} - f(\bar{x}(t),t) . \tag{2}$$

In the above definition, $\epsilon(t)$ is referred to as the equation error or the residual. The functional notation $\epsilon'(\bar{x},t)$ will be used in lieu of $\epsilon(t)$ when it is desired to emphasize the dependence of the equation error upon the periodic function $\bar{x}(t)$. If $\hat{x}(t)$ is a periodic function, and if

$$\epsilon'(\hat{x}(t),t) = 0, \tag{3}$$

then clearly $\hat{x}(t)$ is an exact solution of (1). Assuming a periodic solution $\hat{x}(t)$ to (1) exists, a reasonable way of obtaining an approximation to $\hat{x}(t)$ is to assume some functional form for $\bar{x}(t)$,

$$\bar{x}(t) = \bar{x}(t, a_1, a_2, \ldots, a_k) \tag{4}$$

and then choose the unknown parameters in (4) above such that $\epsilon'(\bar{x}(t, a_1, a_2, \ldots, a_k),t)$ is minimized in some sense. In particular, the basic idea behind harmonic balance is to assume for $\bar{x}(t)$ the form

$$\bar{x}(t) = \sum_{j=0}^{m} (x_{jc} \cos jt + x_{js} \sin jt) \tag{5}$$

and then choose the parameters x_{jc}, x_{js}, $j=0,1,\ldots,m$ so that as many as possible of the low-frequency components of the equation error are zero. This procedure will be precisely defined in the next section; however, to avoid obscuring the simplicity of the basic idea of harmonic balance with the mathematical details, consider the following simple illustration of the method: given the scalar equation

$$\ddot{x} + \alpha x + \beta x^3 = C \cos t \ , \tag{6}$$

assume for $\bar{x}(t)$ the form[3]

$$\bar{x}(t) = A \cos t \ , \tag{7}$$

where A is an unknown parameter whose value is to be determined by the harmonic balance method. The equation error is

$$\epsilon(\bar{x}, t) = \ddot{\bar{x}} + \alpha\bar{x} + \beta\bar{x}^3 - C \cos t \ . \tag{8}$$

Substituting (7) into (8), we have

$$\epsilon(t) = [-A \cos t] + \alpha[A \cos t] + \beta[A \cos t]^3 - C \cos t \ . \tag{9}$$

Using the trigonometric identity

$$\cos^3 t = \frac{3}{4} \cos t + \frac{1}{4} \cos 3t \ , \tag{10}$$

we can expand (9) in a Fourier series:

$$\epsilon(t) = [(\alpha-1)A + \frac{3}{4} \beta A^3 - C]\cos t + [\frac{1}{4} \beta A^3]\cos 3t \ . \tag{11}$$

The principle of harmonic balance asserts that the parameter A should

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[3] A sin t term is not included in (7) because equation (6) is known to be conservative; if a sin term were included, the procedure itself would subsequently specify that its coefficient be zero.

be chosen so that as many as possible of the low-order harmonics of $\epsilon(t)$ be zero. Hence, we choose A such that

$$(\alpha-1)A + \frac{3}{4}\beta A^3 - C = 0 \ . \tag{12}$$

Equation (12) is referred to the <u>determining</u> or <u>bifurcation</u> equation. Notice that since (12) is a nonlinear algebraic equation, multiple roots are possible. This allows the harmonic balance procedure to approximate the multiple steady state solutions which are responsible for the jump resonance phenomena frequently observed in nonlinear systems.

In addition to systems described by equation (1) (which are commonly referred to as driven, or forced, systems), the harmonic balance procedure is also applicable to autonomous systems of the form

$$\dot{x} = f(x) \tag{13}$$

where the function f is not an explicit function of the scalar t, and the period of oscillation is unknown. A simple example is the Van-der-Pol equation

$$\ddot{x} - \delta(1-x^2)\dot{x} + \eta x = 0 \ . \tag{14}$$

Assume for $\bar{x}(t)$ the form

$$\bar{x}(t) = A \cos(\omega t) \ . \tag{15}$$

A sine component is omitted since in this case it is clear that if $x(t)$ is a solution of (14), then so also is $x(t + \tau)$, where τ is arbitrary. The equation error is

$$\epsilon(t) = [-\omega^2 A \cos(\omega t)] + \delta[1-(A \cos^2(\omega t)] \cdot [-\omega A \sin(\omega t)] + \eta[A \cos(\omega t)] \ . \tag{16}$$

Again by the use of several trigonometric identities the equation error is expanded in a Fourier series to obtain

$$\epsilon(t) = [(\eta-\omega^2)A] \cos(\omega t) + \delta\omega A[\tfrac{1}{4}A^2-1]\sin(\omega t) + \tfrac{1}{4}\delta\omega A^3 \sin(3\omega t) . \tag{17}$$

The determining equations are therefore

$$\begin{aligned} &(\eta - \omega^2)A = 0 \\ &\delta\omega A(\tfrac{1}{4}A^2 - 1) = 0 \end{aligned} \tag{18}$$

and hence the harmonic balance solution is

$$\bar{x}(t) = 2 \cos(\sqrt{\eta}\, t) . \tag{19}$$

2.2 Definition of the General m^{th} order Harmonic Balance, or Galerkin, Procedure

Stated concisely, the m^{th} order harmonic balance approximation to periodic solutions of system (1) is a trigonometric series of order m,

$$x_m(t) = \sum_{j=0}^{m} (x_{jc} \cos jt + x_{js} \sin jt) , \tag{20}$$

which exactly satisfies the related system

$$\begin{aligned} \dot{x} = \frac{1}{\pi} \sum_{j=0}^{m} \{&(\cos jt) \int_0^{2\pi} f(x,s) \cos js \, ds \\ &+ (\sin jt) \int_0^{2\pi} f(x,s) \sin js \, ds\} . \end{aligned} \tag{21}$$

The relationship of equation (21) to equation (1) is clearer if equation (1) is rewritten as

$$\begin{aligned} \dot{x} = \frac{1}{\pi} \sum_{j=0}^{\infty} \{&(\cos jt) \int_0^{2\pi} f(x,s) \cos js \, ds \\ &+ (\sin jt) \int_0^{2\pi} f(x,s) \sin js \, ds\} ; \end{aligned} \tag{1'}$$

equations (1') and (21) are identical except for the upper limits on the summations. Solutions of (21) are obtained as follows: for notational convenience,

the Fourier components of $f(x_m(t),t)$ will be denoted with braces as

$$\{f(x_m,t)\}_{jc} = \frac{1}{\pi}\int_0^{2\pi} f[x_m(s),s] \cdot \cos js \, ds \qquad j=0,1,\ldots$$
$$\{f(x_m,t)\}_{js} = \frac{1}{\pi}\int_0^{2\pi} f[x_m(s),s] \cdot \sin js \, ds \; . \tag{22}$$

Note that although x_m has no harmonics of order greater than m, the function $f(x_m,t)$ in general possesses harmonics of all orders. Therefore $f(x_m(t),t)$ is written

$$f(x_m(t),t) = \sum_{j=0}^{\infty} (\{f(x_m,t)\}_{js} \cos jt + \{f(x_m,t)\}_{js} \sin jt) \; . \tag{23}$$

From (20),

$$\dot{x}_m(t) = \sum_{j=1}^{m} (jx_{js} \cos jt - jx_{jc} \sin jt). \tag{24}$$

Therefore the equation error is

$$\begin{aligned}
\epsilon(t) &\triangleq \dot{x}_m - f(x_m) \\
&= \sum_{j=1}^{m} (jx_{js} \cos jt - jx_{jc} \sin jt) - \sum_{j=0}^{\infty} (\{f(x_m,t)\}_{jc} \cos jt \\
&\qquad + \{f(x_m,t)\}_{js} \sin jt) \\
&= \sum_{j=0}^{m} [(jx_{js} - \{f(x_m,t)\}_{jc}) \cos jt + (-jx_{jc} - \{f(x_m,t)\}_{js}) \sin jt] \\
&\quad + \sum_{j=m+1}^{\infty} (\{f(x_m,t)\}_{jc} \cos jt + \{f(x_m,t)\}_{js} \sin jt) \; .
\end{aligned} \tag{25}$$

Hence if the $(2m + 1)$ parameters x_{oc}, x_{jc}, x_{js}, $j=1,2,\ldots,m$ can be chosen such that the $(2m+1)$ determining equations

$$\begin{aligned} \{f(x_{\underline{n}}(t),t)\}_{0c} &= 0 \\ jx_{js} - \{f(x_{\underline{n}}(t),t)\}_{jc} &= 0 \qquad j=1,2,\ldots,m \\ -jx_{jc} - \{f(x_{\underline{n}}(t),t)\}_{js} &= 0 \end{aligned} \tag{26}$$

are satisfied, then $x_{\underline{n}}(t)$ is a solution of (21). In this thesis, the $(2m + 1)$ roots of (26) will be referred to as the Galerkin estimates of the exact Fourier coefficient of the oscillation. It should be noted that the analytic effort required to obtain equations (26) for a particular system lies in expanding the function

$$f[\sum_{j=0}^{m} (x_{jc} \cos jt + x_{js} \sin jt), t] \tag{27}$$

in a Fourier series; as will be clearly demonstrated in the following sections, the difficulty increases very rapidly with increasing m. Of course, it is also true that as m increases, the number of (coupled) nonlinear algebraic equations (26) which must be solved increases. This is not a significant problem in practice, however, since even for small m the equations must generally be solved numerically, and consequently larger values of m merely require more computer time for solution; the actual limiting factor in the application of the method for $m > 1$ is the necessity of expanding (27). It is for this reason that the use of the Galerkin method as a practical tool has been largely limited to the first order procedure.

2.3 Application of the First Order Procedure

In this section, the first order harmonic balance procedure will be applied to an example; the same system will be used in Chapters III and V to illustrate the results of the thesis. Consider the second order differential equation

$$\ddot{x} + \alpha\dot{x} + \beta x + \gamma x^3 = \eta \cos(\omega t) \ . \tag{28}$$

Equation (28) is the classical Duffing equation. There were two reasons for choosing this particular system to illustrate the thesis results. First of all, the equation has been widely studied; hence the computer programs which were written to generate the numerical results reported in later chapters could be checked against known results in the literature. Secondly, it was felt that the ideas to be presented later would be clarified if a comparison between the first and third order Galerkin results could be made; however, the third order procedure is impractical to apply except for very low order polynomial nonlinearities. This latter restriction, together with the practical importance of odd nonlinearities, naturally led to the choice of the cubic term in (28).

For a first order harmonic balance solution, the approximation is of the form

$$\bar{x}(t) = x_{1c} \cos \omega t + x_{1s} \sin \omega t \ . \tag{29}$$

To apply the procedure, we must expand

$$\epsilon'(\bar{x}(t),t) = \ddot{\bar{x}} + \alpha\dot{\bar{x}} + \beta\bar{x} + \gamma\bar{x}^3 - \eta \cos \omega t \tag{30}$$

in a Fourier series. Of course, the only term in (30) which causes any analytic difficulty is the nonlinear one; because the required analytic effort for both the first and third order procedures will be an important consideration in later chapters, the necessary manipulations are shown in detail below. For reasons which will become apparent in Chapter IV, the expansion is performed in a more general manner than is actually necessary for the present example; instead of expanding the expression

$$(\bar{x})^3 \triangleq \prod_{i=1}^{3} (A \cos \omega t + B \sin \omega t), \text{ where } A=x_{1c} \text{ and } B=x_{1s} \ , \tag{31}$$

we instead expand[4]

$$\prod_{i=1}^{3} (A_i \cos \omega_i t + B_i \sin \omega_i t) \ . \tag{32}$$

The expansion of expression (32) and other similar expressions will be repeatedly referred to in Chapters IV and II. Hence it will be convenient to define a function h by

$$\prod_{i=1}^{k} (A_i \cos \omega_i t + B_i \sin \omega_i t) \triangleq h(A_1,B_1,\omega_1;A_2,B_2,\omega_2;\ldots;A_k,B_k,\omega_k)$$
$$\triangleq h(A_i,B_i,\omega_i:i=1,2,\ldots,k) \ .$$

Each of the two methods (and combinations thereof) employed above for denoting the arguments of the function h will prove useful.

Proceeding with the expansion of (32), we have

$$\begin{aligned} h &= (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)(A_2 \cos \omega_2 t + B_2 \sin \omega_2 t)(A_3 \cos \omega_3 t + B_3 \sin \omega_3 t) \\ &= (A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)[A_2A_3 \cos \omega_2 t \cdot \cos \omega_3 t \\ &\quad + A_3B_2 \cos \omega_3 t \sin \omega_2 t + A_2B_3 \cos \omega_2 t \sin \omega_3 t \\ &\quad + B_2B_3 \sin \omega_2 t \sin \omega_3 t] \ . \end{aligned} \tag{32'}$$

We now make use of the trigonometric identities for products of sinusoids to obtain

$$\begin{aligned} h = \tfrac{1}{2}(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t)[&(A_2A_3 + B_2B_3) \cos (\omega_2-\omega_3)t \\ &+ (A_2A_3 - B_2B_3) \cos (\omega_2 + \omega_3)t + (A_2B_3 - A_3B_2) \sin (\omega_3-\omega_2)t \\ &+ (A_2B_3 + A_3B_2) \sin (\omega_3 + \omega_2)t] \end{aligned}$$

---

[4] It is important to note that the number of trigonometric multiplications required for expansions (31) and (32) is the same in each case; (32) is more difficult to evaluate only to the extent that more "bookkeeping" is necessary.

or

$$
\begin{aligned}
h = \frac{1}{4} \; & [A_1(A_2A_3+B_2B_3) \cos \omega_1 t \cos (\omega_3-\omega_2)t \\
& + B_1(A_2A_3+B_2B_3) \sin \omega_1 t \cos (\omega_3-\omega_2)t \\
& + A_1(A_2A_3-B_2B_3) \cos \omega_1 t \cos (\omega_2+\omega_3)t \\
& + B_1(A_2A_3-B_2B_3) \sin \omega_1 t \cos (\omega_2+\omega_3)t \\
& + A_1(A_2B_3-A_3B_2) \cos \omega_1 t \sin (\omega_3-\omega_2)t \\
& + B_1(A_2B_3-A_3B_2) \sin \omega_1 t \sin (\omega_3-\omega_2)t \\
& + A_1(A_2B_3+A_3B_2) \cos \omega_1 t \sin (\omega_3+\omega_2)t \\
& + B_1(A_2B_3+A_3B_2) \sin \omega_1 t \sin (\omega_3+\omega_2)t] \; .
\end{aligned}
\tag{32''}
$$

Again, using the trigonometric identities, we finally obtain the desired result:

$$
\begin{aligned}
& h(A_1,B_1,\omega_1;A_2,B_2,\omega_2;A_3,B_3,\omega_3) = \\
& \frac{1}{4} \{[A_1(A_2A_3+B_2B_3) + B_1(A_2B_3-A_3B_2)] \cos [\omega_1-(\omega_3-\omega_2)]t \\
& + [A_1(A_2A_3+B_2B_3) - B_1(A_2B_3-A_3B_2)] \cos [\omega_1+(\omega_3-\omega_2)]t \\
& + [A_1(A_2A_3-B_2B_3) + B_1(A_2B_3+A_3B_2)] \cos [\omega_1-(\omega_3+\omega_2)]t \\
& + [A_1(A_2A_3-B_2B_3) - B_1(A_2B_3+A_3B_2)] \cos [\omega_1+(\omega_3+\omega_2)]t \\
& + [B_1(A_2A_3+B_2B_3) - A_1(A_2B_3-A_3B_2)] \sin [\omega_1-(\omega_3-\omega_2)]t \\
& + [B_1(A_2A_3+B_2B_3) + A_1(A_2B_3-A_3B_2)] \sin [\omega_1+(\omega_3-\omega_2)]t \\
& + [B_1(A_2A_3-B_2B_3) - A_1(A_2B_3+A_3B_2)] \sin [\omega_1-(\omega_3+\omega_2)]t \\
& + [B_1(A_2A_3-B_2B_3) + A_1(A_2B_3+A_3B_2)] \sin [\omega_1+(\omega_3+\omega_2)]t\} \; .
\end{aligned}
\tag{33}
$$

Equation (33) will be extensively used in Chapter V in applying the iterative method (to be presented in Chapter IV) to an example. For the present, we are interested in the first order Galerkin method, and hence equation (33) is specialized for expression (31) by the substitutions

$$
\begin{aligned}
A_1 &= A_2 = A_3 = x_{1c} \\
B_1 &= B_2 = B_3 = x_{1s} \\
\omega_1 &= \omega_2 = \omega_3 = \omega \; .
\end{aligned}
\tag{34}
$$

Equation (33) then becomes

$$\begin{aligned}(\bar{x})^3 &= [\tfrac{3}{4} x_{1c}^3 + \tfrac{3}{4} x_{1c}x_{1s}^2] \cos \omega t \\ &+ [\tfrac{3}{4} x_{1s}^3 + \tfrac{3}{4} x_{1c}^2 x_{1s}] \sin \omega t \\ &+ [\tfrac{1}{4} x_{1c}^3 - \tfrac{3}{4} x_{1c}x_{1s}^2] \cos 3\omega t \\ &+ [-\tfrac{1}{4} x_{1s}^3 + \tfrac{3}{4} x_{1c}^2 x_{1s}] \sin 3\omega t \ .\end{aligned} \tag{35}$$

The Fourier expansions for the linear terms in the equation error (30) can be obtained by inspection:

$$\{\ddot{\bar{x}} + \alpha\dot{\bar{x}} + \beta\bar{x} - \eta \cos \omega t\}_{1c} = -\omega^2 x_{1c} + \alpha\omega x_{1s} + \beta x_{1c} - \eta$$

and (35′)

$$\{\ddot{\bar{x}} + \alpha\dot{\bar{x}} + \beta\bar{x} - \eta \cos \omega t\}_{1s} = -\omega^2 x_{1s} - \alpha\omega x_{1c} + \beta x_{1s} \ .$$

Therefore, from (30),(35), and (35′), the first order Galerkin determining equations are

$$\begin{aligned}&[(\beta-\omega^2)x_{1c} + \alpha\omega x_{1s} - \eta + \gamma(\tfrac{3}{4} x_{1c}^3 + \tfrac{3}{4} x_{1c}x_{1s}^2)] = 0 \\ &[(\beta-\omega^2)x_{1s} - \alpha\omega x_{1c} + \gamma(\tfrac{3}{4} x_{1s}^3 + \tfrac{3}{4} x_{1c}^2 x_{1s})] = 0 \ .\end{aligned} \tag{36}$$

2.4 Numerical Results

In spite of the fact that the first order harmonic balance method essentially ignores all harmonics higher than the fundamental, the accuracy of the approximation can nevertheless be quite good. As one would expect, the greater the harmonic content of the actual solution, the poorer is the first order harmonic balance approximation. To illustrate the variations in accuracy which can be encountered, the results obtained by solving the determining equations (36) for the specific system

$$\ddot{x} + 0.3\dot{x} + 10x - 0.1x^3 = 8 \cos 0.9t \tag{37}$$

are compared below with the exact solutions obtained by numerical integration. The particular values of the parameters in (37) were chosen so that the harmonic content of the oscillations would not be negligible in every case. (The same parameter set will be used in Chapter V, except that the driving frequency ω will be varied over the interval [1,4].)

There are three periodic solutions to equation (37). The true solutions, through the 5^{th} harmonic, are followed in each case by the corresponding first order Galerkin solution.

$$x_1(t) = -9.6 \cos .9t + 7.4 \sin .9t - 1.2 \cos 3(.9)t - 2.4 \sin 3(.9)t + 0.7 \cos 5(.9)t - 0.16 \sin 5(.9)t + \ldots$$

$$\bar{x}_1(t) = -10.6 \cos .9t + 4.4 \sin .9t$$

$$x_2(t) = 9.3 \cos .9t + 6.6 \sin .9t + .57 \cos 3(.9)t - 2.5 \sin 3(.9)t - 0.66 \cos 5(.9)t + 0.09 \sin 5(.9)t + \ldots \qquad (38)$$

$$\bar{x}_2(t) = 9.9 \cos .9t + 3.8 \sin .9t$$

$$x_3(t) = 0.87526 \cos .9t + 0.025890 \sin .9t + 0.0057 \cos 3(.9)t + 0.0024 \sin 3(.9)t - 0.00003 \cos 5(.9)t - 0.00001 \sin 5(.9)t + \ldots$$

$$\bar{x}_3(t) = 0.87523 \cos .9t + 0.025876 \sin .9t \quad .$$

Notice that $x_3(t)$, which is very nearly sinusoidal, is approximated quite closely by the harmonic balance solution; however, in the case of $x_1(t)$ and $x_2(t)$ (each of which has much higher relative harmonic content than $x_3(t)$), the approximation is not nearly so accurate. Unfortunately, when one applies the first order procedure to a practical problem, the

method does not generate estimates of the higher harmonics components, and hence there is no indication whatsoever of the accuracy of the approximate solution. Applying the third order harmonic balance procedure[5] would generate estimates of the first and third harmonics; however, as will be demonstrated in the next section, harmonic balance procedures of order higher than the first are generally too difficult to be used as a practical tool for obtaining periodic solutions to nonlinear systems.

2.5 Application of Higher-Order Harmonic Balance

Suppose one wishes to improve the approximation generated by the first order results in Section 2.4 by applying the third order method. This would require the expansion not of

$$(\bar{x}(t))^3 = (x_{1c} \cos t + x_{1s} \sin t)^3 \tag{39}$$

as in the first order method, but of

$$(\bar{x}(t))^3 = (x_{1c} \cos t + x_{1s} \sin t + x_{3c} \cos 3t + x_{3s} \sin 3t)^3. \tag{40}$$

In Section 2.4, expression (39) required the evaluation of 8 trigonometric multiplications; expression (40) requires 64 such multiplications. For a low-order polynomial nonlinearity such as $f(x) = x^3$, evaluation of (40) is certainly not a practical impossibility, though it is somewhat tedious. What we would like to be able to do, however, is to approximate any arbitrary nonlinearity with a power series

$$f(x) = \sum_{j=1}^{p} \alpha_j x^j , \tag{41}$$

---

[5] In this example, there are no even order harmonics (because the nonlinearity is odd), and hence the next level of approximation above the first is the third.

with p large enough to allow fairly irregular nonlinearities to be simulated. Suppose we let $p = 7$. The first order procedure will then require 2^7 or 128 trigonometric multiplications; though not a simple task, this is not an unreasonable amount of effort to expend, considering the generality of the result. However, the evaluation of the third order procedure for $p = 7$ would require 4^7 or 16,384 trigonometric multiplications, which is hardly a practical task. The results to be presented in the remainder of the thesis are intended to answer the following question: how can the accuracy of the first order harmonic balance solution be improved, subject to the constraint that the analytic effort required for the improvement be comparable to that of the first order procedure?

CHAPTER III

A SIMPLIFIED METHOD OF OBTAINING THE LINEARIZED DETERMINING EQUATIONS OF HAYASHI

3.0 Introduction

In this chapter a method will be presented which, for odd nonlinearities, results in Hayashi's approximation [5] to the third order[1] Galerkin solution, but which requires less effort to apply than his method. The motivation behind the results will first be explained, and a simple example to illustrate the ideas will be presented. In section 3.2, the general results will be derived. The method will then be applied to several examples, and numerical results reported. Finally, some limitations on Hayashi's approximation will be discussed.

3.1 Motivation Behind the Results and an Observation

Recall that in Chapter II it was demonstrated that the practical difficulty in applying Galerkin's method for orders greater than the first is that expressions of the form

$$f[\sum_{j=0}^{m} (x_{jc} \cos jt + x_{js} \sin jt),\ t] \ , \ m > 1 \qquad (1)$$

must be analytically expanded in a Fourier series. The vector function $f(x,t)$ is polynomial in the vector x, continuous in the scalar t, and periodic (in t) of period 2π. In particular, for $f(x,t)$ odd in x, the third order Galerkin method requires expansion of

$$f[x_{1c} \cos t + x_{1s} \sin t + x_{3c} \cos 3t + x_{3s} \sin 3t,\ t] \ . \qquad (2)$$

---

[1] Hayashi refers to the third order solution as the second order solution, when $f(x,t)$ is odd, reflecting the number of harmonic oscillations present in the argument of f; we will conform to the definition in Chapter II.

As was emphasized at the end of the last chapter, analytical evaluation of (2) is simply a practical impossibility except in the case of very low order polynomials. In his book [5], Hayashi suggests that instead of expanding (2) in a Fourier series, one can instead expand the expression

$$f[(x_{1c}^{*} + \epsilon\delta_{1c}) \cos t + (x_{1s}^{*} + \epsilon\delta_{1s}) \sin t + \epsilon\delta_{3c} \cos 3t + \epsilon\delta_{3s} \sin 3t, t], \tag{3}$$

where x_{1c}^{*} and x_{1s}^{*} are first order Galerkin solutions; as the expansion progresses, all terms of order greater than unity in ϵ are to be ignored, thus reducing the amount of computational effort required. The resulting equations may be viewed as the third order determining equations linearized about the first order solutions. Provided the terms in ϵ are sufficiently small, the solution of the linearized equations will be close (in some sense) to the solution of the determining equations. More will be said about the limitations implied by the preceding statement in the final section of this chapter; for the present, the objective is to obtain Hayashi's linearized equations without having to expand (3). Specifically, the expansion of $f(x,t)$, when the argument x consists of a sum of more than two sinusoids, is to be avoided. This last statement essentially constitutes the motivation behind both of the results to be presented in this thesis.

Suppose that for a particular nonlinearity $f(x,t)$, the first order harmonic balance procedure has been carried out, so that the following information is available:

$$f[A \cos t + B \sin t, t] = \sum_{j=1,3,5,\ldots}^{k_1} (g_j(A,B) \cos jt + h_j(A,B) \sin jt), \tag{4}$$

where g_j and h_j are known (polynomial) functions of the parameters

A and B . The fact that the summation in (4) terminates (as indicated by the upper limit k_1) is a result of the restriction of $f(x,t)$ to be polynomial in x . In the braces notation introduced in Chapter II, the algebraic functions are denoted as

$$g_j(A,B) = \{f[A \cos t + B \sin t, t]\}_{jc}$$

$$h_j(A,B) = \{f[A \cos t + B \sin t, t]\}_{js} \ . \tag{5}$$

If we could perform the third order Galerkin method, then we would obtain

$$f[A \cos t + B \sin t + C \cos 3t + D \sin 3t, t] = \sum_{j=1,3,5,\ldots}^{k_2} (g_j'(A,B,C,D) \cos jt + h_j'(A,B,C,D) \sin jt), \tag{6}$$

where g_j' and h_j' are (vector) functions which reduce to g_j and h_j when $C = D = 0$. If we could obtain the functions g_j' and h_j' , then the third order Galerkin solution could be computed exactly. Since this is not generally practical, our objective is to approximate the functions g_j' and h_j' in some manner. In the braces notation, we want to approximate the functions

$$g_j'(A,B,C,D) = \{f[A \cos t + B \sin t + C \cos 3t + D \sin 3t, t]\}_{jc}$$

$$h_j'(A,B,C,D) = \{f[A \cos t + B \sin t + C \cos 3t + D \sin 3t, t]\}_{js} \ , \tag{7}$$

$$j = 1,3 \ .$$

In Hayashi's approximation, all terms in g_j' and h_j' of order greater than unity in C and D are omitted.

The objective of this chapter is to obtain Hayashi's approximations

to g_j' and h_j' by manipulating the known algebraic functions g_j and h_j in some manner. To clarify this idea, consider a simple example. In Chapter II, the first order Galerkin procedure was applied to the conservative system

$$\ddot{x} + \alpha x + \beta x^3 = C \cos t \ , \tag{8}$$

and it was found that

$$Nf[A \cos t, t] = \beta(A \cos t)^3 = \frac{A^3}{4} \beta[3 \cos t + \cos 3t] \ , \tag{9}$$

where the notation $Nf(x,t)$ is used to denote the nonlinear terms of $f(x,t)$; this notation will be frequently used throughout this thesis when the inclusion of the linear terms of $f(x,t)$ would be a needless complication.

From (9), we have that

$$\begin{aligned} \{Nf[A \cos t,t]\}_{1c} &= g_1(A) = \frac{3}{4} \beta A^3 \\ \text{and} \qquad \{Nf[A \cos t,t]\}_{3c} &= g_3(A) = \frac{1}{4} \beta A^3 \ . \end{aligned} \tag{10}$$

In this simple example, the third order Galerkin procedure is not difficult to perform. The result is

$$\begin{aligned} Nf[A \cos t + C \cos 3t, t] &= \beta(A \cos t + C \cos 3t)^3 \\ &= \beta[\tfrac{3}{4} A^3 + \tfrac{3}{4} A^2 C + \tfrac{3}{2} AC^2]\cos t + \beta[\tfrac{1}{4} A^3 + \tfrac{3}{2} A^2 C + \tfrac{3}{4} C^3]\cos 3t \\ &+ \beta[\tfrac{3}{4} A^2 C + \tfrac{3}{4} AC^2]\cos 5t + \beta[\tfrac{3}{4} AC^2]\cos 7t + \beta[\tfrac{1}{4} C^3]\cos 9t \ . \end{aligned} \tag{11}$$

If all terms in (11) of order greater than unity in C are ignored, we have Hayashi's approximation

$$\begin{aligned} Nf(\bar{x},t) \doteq\ & \beta[\tfrac{3}{4} A^3 + \tfrac{3}{4} A^2 C] \cos t + \beta[\tfrac{1}{4} A^3 + \tfrac{3}{2} A^2 C] \cos 3t \\ & + \beta[\tfrac{3}{4} A^2 C] \cos 5t \ . \end{aligned} \tag{12}$$

Throughout this thesis, the symbol "$\doteq$" will be used to denote a first order approximation in the above sense.

From (10) and (12), note that

$$g_1(A) = \frac{3}{4}\beta A^3 \quad \text{and} \quad g_1'(A,C) = \frac{3}{4}\beta A^3 + \frac{3}{4}\beta A^2 C \tag{13}$$

Thus in this simple example, we see that the first order effect of including the third harmonic in the argument of $f(x,t)$ is the additional term

$$\frac{3}{4}\beta A^2 C \tag{14}$$

in g_1'. (Of course, the presence of the third harmonic in the argument also generates additional terms in g_3'; however, in this example attention will be restricted to the effect on the fundamental component, g_1'.) The question is, can the additional term (14) be obtained directly from the functions $g_j(A)$ without having to expand (11)? The answer, for this specific example, is yes: observe that if we substitute the algebraic expression $(A+C)$ for the amplitude parameter A in $g_3(A)$, we have

$$g_3(A+C) \triangleq \{Nf[(A+C)\cos t]\}_{3c} = \frac{1}{4}\beta(A+C)^3 \doteq \frac{1}{4}\beta[A^3 + 3A^2C], \tag{15}$$

where in the expression of $(A+C)^3$ we have ignored terms of order greater than unity in C. From (13) and (15), we have that

$$g_1'(A,C) \doteq g_1(A) + g_3(A+C) - g_3(A) \tag{16}$$

or, in braces notation,

$$\{f[A\cos t + C\cos 3t,\ t]\}_{1c} \doteq \{f[A\cos t,\ t]\}_{1c} + \{f[(A+C)\cos t, t]\}_{3c} - \{f[A\cos t,\ t]\}_{3c} \tag{17}$$

where the left- and right-hand sides of the above expressions are identical except for terms of order greater than unity in C. What is important to note is that each of the three terms on the right-hand side can be evaluated directly from the first order Galerkin results, i.e., it is only necessary to know the functions $g_1(A)$ and $g_3(A)$. If it can be shown that approximations of the above type exist for any odd polynomial function, then it should be clear that we will have a very easily applied general technique for obtaining Hayashi's equations. Fortunately, as will be demonstrated in the next section, (16) is not a relationship which happens to be true only for the cubic nonlinearity, but rather a general property of all polynomial nonlinearities and their Fourier coefficients. In order that this approach be generally useful, however, the approximation (16) must be extended to the case in which both sines and cosines appear in the argument x of $f(x,t)$, since otherwise only undamped systems could be handled. In addition, a similar approximation is required for the third harmonics. To be specific, in order to obtain Hayashi's equations it is necessary to obtain approximations for the following four (vector) functions:

$$g_1'(A,B,C,D) = \{f[A\cos t + B\sin t + C\cos 3t + D\sin 3t,\ t]\}_{1c} \quad (18a)$$

$$h_1'(A,B,C,D) = \{f[A\cos t + B\sin t + C\cos 3t + D\sin 3t,\ t]\}_{1s} \quad (18b)$$

$$g_3'(A,B,C,D) = \{f[A\cos t + B\sin t + C\cos 3t + D\sin 3t,\ t]\}_{3c} \quad (18c)$$

$$h_3'(A,B,C,D) = \{f[A\cos t + B\sin t + C\cos 3t + D\sin 3t,\ t]\}_{3s}. \quad (18d)$$

The objective is to approximate each of the above functions with a sum of as few terms as possible, each term being of the form

$$g_j(r,s) \quad \text{or} \quad h_j(r,s), \quad j = 1, 3, 5, \ldots \quad (19)$$

where the g_j and h_j are known from the first order Galerkin analysis, and the arguments r and s in (19) are simple algebraic expressions of the form $A \pm C$, $A \pm D$, $B \pm C$, and $B \pm D$.

The derivation of four approximations which fulfill the above objectives will be presented in the next section. It should be obvious how the objectives as stated above can be generalized to general polynomial functions; however, the results of the next section are restricted to odd nonlinearities. The reader may notice that the derivations themselves do not depend upon the polynomial form of $f(x,t)$, nor upon its being odd. The fact that we are concerned with odd functions is indicated only by our choice (18) of the functions to be approximated. The restriction that $f(x,t)$ be polynomial in x results from the practical considerations discussed in Chapter I; polynomials are smoother than is actually required in the derivations.

3.2 Derivation of the Approximations

Consider first the approximation of the function (18a). Our approach will be to obtain an equation of the form

$$g_1'(A,B,dC,dD) = g_1'(A,B,0,0) + dg_1'(A,B,dC,dD) , \qquad (20)$$

where $dg_1(A,B,dC,dD)$ is the differential change in g_1 resulting from differential changes in C and D about the point $C = D = 0$, with A and B constant. Note that the first term on the right-hand side of (20) is just $g_1(A,B)$. To accomplish our objectives, we need to obtain an analytic expression for $dg_1'(A,B,dC,dD)$ in terms of the first order Galerkin functions g_j and h_j. From our definition of dg_1', we have

$$dg_1'(A,B,dC,dD) = \frac{\partial}{\partial C} g_1'(A,B,C,D)\Big|_{C=D=0} \cdot dC + \frac{\partial}{\partial D} g_1'(A,B,C,D)\Big|_{C=D=0} \cdot dD \qquad (21)$$

The meaning of the partial derivatives in (21) is the usual one: if the j^{th} component of the vector g_1' is denoted $g_1'^{(j)}$, and similarly for the components of the vectors C and D, then

$$\frac{\partial g_1'}{\partial C} = \begin{bmatrix} \frac{\partial g_1'^{(1)}}{\partial C^{(1)}} & \cdots & \frac{\partial g_1'^{(1)}}{\partial C^{(n)}} \\ \vdots & & \\ \frac{\partial g_1'^{(n)}}{\partial C^{(1)}} & \cdots & \frac{\partial g_1'^{(n)}}{\partial C^{(n)}} \end{bmatrix} \tag{22}$$

and similarly for $\frac{\partial g_1'}{\partial D}$. These matrices are commonly referred to as the Jacobian matrices of g_1' with respect to C and D. From the definition of $g_1'(A,B,C,D)$, (21) is

$$dg_1'(A,B,dC,dD) =$$

$$= \frac{\partial}{\partial C}\left(\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f[A\cos t + B\sin t + C\cos 3t + D\sin 3t, t]dt\right)\Big|_{C=D=0} \cdot dC$$

$$+ \frac{\partial}{\partial D}\left(\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f[A\cos t + B\sin t + C\cos 3t + D\sin 3t, t]dt\right)\Big|_{C=D=0} \cdot dD \,. \tag{23}$$

Since $f(x,t)$ is polynomial in x and continuous in t, the order of integration and differentiation in (23) may be interchanged, with the result that

$$dg_1'(A,B,dC,dD) =$$

$$= \left[\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot \frac{\partial}{\partial C} f\,(A\cos t + B\sin t + C\cos 3t + D\sin 3t, t)\Big|_{C=D=0} dt\right] \cdot dC$$

$$+ \left[\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot \frac{\partial}{\partial D} f\,(A\cos t + B\sin t + C\cos 3t + D\sin 3t, t)\Big|_{C=D=0} dt\right] \cdot dD \tag{24}$$

If we denote by f_x the Jacobian $f(x,t)$ with respect to x, evaluated at $x = A \cos t + B \sin t$, then (24) becomes

$$dg_1' = \left[\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f_x \cdot \frac{\partial}{\partial C}(A \cos t + B \sin t + C \cos 3t + D \sin 3t)\Big|_{C=D=0} dt\right]\cdot dC$$

$$+ \left[\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f_x \cdot \frac{\partial}{\partial D}(A \cos t + B \sin t + C \cos 3t + D \sin 3t)\Big|_{C=D=0} dt\right]\cdot dD \qquad (25a)$$

$$= \left[\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f_x \cdot \cos 3t\, dt\right] \cdot dC + \left[\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f_x \cdot \sin 3t\, dt\right] \cdot dD \qquad (25b)$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f_x \cdot (dC \cos 3t + dD \sin 3t)dt\ . \qquad (25c)$$

Recall that one objective is to express $dg_1'(A,B,C,D)$ in terms of the functions

$$g_j(x,y) \quad \text{and} \quad h_j(x,y)\ ,\ j = 1, 3, 5, \ldots$$

where the arguments x and y are algebraic functions (as simple in form as possible) of A, B, C, and D. This objective can be accomplished as follows: first, (25c) is rearranged to give

$$dg_1' = \left[\frac{1}{\pi}\int_0^{2\pi} (\cos 3t \cdot f_x \cdot \cos t)dt\right] \cdot dC +$$

$$+ \left[\frac{1}{\pi}\int_0^{2\pi} (\sin 3t \cdot f_x \cdot \cos t)dt\right] \cdot dD\ . \qquad (26)$$

Now observe that the matrix $I \cdot \cos t$, where I is the identity matrix, is equivalent to the expression

$$\frac{\partial}{\partial \xi}\left[(A + \xi) \cos t + B \sin t\right]\ .$$

Equation (26) can thus be written

$$dg_1' = \left[\frac{1}{\pi}\int_0^{2\pi} \cos 3t \cdot f_x \cdot \frac{\partial}{\partial C}\left[(A+C) \cos t + B \sin t\right]dt\right] \cdot dC$$

$$+ \left[\frac{1}{\pi}\int_0^{2\pi} \sin 3t \cdot f_x \cdot \frac{\partial}{\partial D}\left[(A+D) \cos t + B \sin t\right]dt\right] \cdot dD\ . \qquad (27)$$

We now make use of the fact that from the definition of f_x (as defined immediately before equation (25a)),

$$f_x \cdot \frac{\partial}{\partial C}\,[(A+C)\cos t + B \sin t] = \frac{\partial}{\partial C}\,(f[(A+C)\cos t + B \sin t])\Big|_{C=0}$$

and

$$f_x \cdot \frac{\partial}{\partial D}\,[(A+D)\cos t + B \sin t] = \frac{\partial}{\partial D}\,(f[(A+D)\cos t + B \sin t])\Big|_{D=0}.$$

Hence, we can write (27) as

$$\begin{aligned} dg_1' = &\left[\frac{1}{\pi}\int_0^{2\pi} \cos 3t\, \frac{\partial}{\partial C} f[(A+C)\cos t + B \sin t]\Big|_{C=0} dt\right] \cdot dC \\ &+ \left[\frac{1}{\pi}\int_0^{2\pi} \sin 3t\, \frac{\partial}{\partial D} f[(A+D)\cos t + B \sin t]\Big|_{D=0} dt\right] \cdot dD\,. \end{aligned} \tag{28}$$

Interchanging the order of integration and differentiation in (28) gives

$$\begin{aligned} dg_1' = &\frac{\partial}{\partial C}\left(\frac{1}{\pi}\int_0^{2\pi} \cos 3t \cdot f[(A+C)\cos t + B \sin t]dt\right)\Big|_{C=0} \cdot dC \\ &+ \frac{\partial}{\partial D}\left(\frac{1}{\pi}\int_0^{2\pi} \sin 3t \cdot f[(A+D)\cos t + B \sin t]dt\right)\Big|_{D=0} \cdot dD\,. \end{aligned} \tag{29}$$

The first term on the right-hand side of (29) is not a function of D. Therefore, it can be expressed as

$$\frac{\partial}{\partial C}\left(\frac{1}{\pi}\int_0^{2\pi} \cos 3t \cdot f[(A+C)\cos t + B \sin t]dt\right)\Big|_{C=D=0} \cdot dC = dg_3(A+C,B) \tag{30}$$

where the differential change dg_3 is understood to result from differential changes in C and D about the point $C=D=0$, with A and B constant. The second term can likewise be written

$$\frac{\partial}{\partial D}\left(\frac{1}{\pi}\int_0^{2\pi} \sin 3t \cdot f[(A+D)\cos t + B \sin t]dt\right)\Big|_{C=D=0} \cdot dD = dh_3(A+D,B). \tag{31}$$

Combining (20), (29), (30) and (31), we have

$$g_1'(A,B,dC,dD) = g_1(A,B) + dg_3(A+dC,B) + dh_3(A+dD,B). \qquad (33)$$

We can now use the differential relationship (33) to approximate the effect of finite variations of C and D about the point $C=D=0$:

$$g_1'(A,B,C,D) \doteq g_1(A,B) + g_3(A+C,B) - g_3(A,B) + h_3(A+D,B) - h_3(A,B). \qquad (34)$$

Equation (33) asserts that the approximation (34) is correct to first order in the variables C and D.

In applying approximation (34) in practice, it is not necessary to actually compute the third and fifth terms; for example, to compute the expression

$$g_3(A+C,B) - g_3(A,B), \qquad (35)$$

one simply computes the first term, $g_3(A+C,B)$, and then deletes all terms not containing the variable C as a factor. (In addition, all terms of order greater than unity in C and D can, of course, be deleted.) To emphasize this point, we define the operator Δ by

$$\Delta g_3(A+C,B) \triangleq g_3(A+C,B) - g_3(A,B). \qquad (36)$$

Using this notation, approximation (34) becomes

$$\boxed{g_1'(A,B,C,D) \doteq g_1(A,B) + \Delta g_3(A+C,B) + \Delta h_3(A+D,B)} \qquad (37)$$

Once again, it will be emphasized that the value of approximation (37) is due to the following two facts: the expression on the left-hand side of (37) is generally impractical to evaluate analytically, whereas the three terms on the right-hand side can easily be obtained by making simple substitutions into the first order harmonic balance results.

The derivation of the approximation for expression (18b) is very similar to that for (18a) and will consequently be omitted. The differential relation analogous to (33) is

$$h_1'(A,B,dC,dD) = h_1(A,B) + dg_3(A,B+dC) + dh_3(A,B+dD) \qquad (38)$$

and the corresponding approximation is

$$\boxed{h_1'(A,B,C,D) \doteq h_1(A,B) + \Delta g_3(A,B+C) + \Delta h_3(A,B+D)} \qquad (39)$$

To derive the first order approximation for (18c), we proceed in exactly the same manner as for (18a) to obtain

$$dg_3'(A,BdC,dD) = \frac{1}{\pi}\int_0^{2\pi} \cos 3t \cdot f_x \cdot (dC \cos 3t + dD \sin 3t)dt. \qquad (40)$$

We then make use of the two trigonometric identities

$$\begin{aligned} \cos 3t \cdot \cos 3t &\equiv (\cos t - \cos 3t + \cos 5t)(\cos t) \\ \cos 3t \cdot \sin 3t &\equiv (\cos t + \cos 3t + \cos 5t)(\sin t) \end{aligned} \qquad (41)$$

to obtain

$$\begin{aligned} dg_3' = &\frac{1}{\pi}\int_0^{2\pi} \cos t \cdot f_x \cdot (dC \cos t + dD \sin t)\, dt \\ &- \frac{1}{\pi}\int_0^{2\pi} \cos 3t \cdot f_x \cdot (dC \cos t - dD \sin t)\, dt \\ &+ \frac{1}{\pi}\int_0^{2\pi} \cos 5t \cdot f_x \cdot (dC \cos t + dD \sin t)\, dt \ . \end{aligned} \qquad (42)$$

Proceeding from (42) in a manner similar to that used to obtain (33) from (25c), the result is that

$$\begin{aligned} g_3'(A,B,dC,dD) = &\ g_3(A,B) + dg_1(A+dC,B+dD) \\ &+ dg_3(A-dC,B+dD) + dg_5(A+dC,B+dD) \ , \end{aligned} \qquad (43)$$

and, therefore, the first order approximation for (18c) is

$$\boxed{g_3'(A,B,C,D) \doteq g_3(A,B) + \Delta g_1(A+C,B+D) + \Delta g_3(A-C,B+D) + \Delta g_5(A+C,B+D)} \tag{44}$$

The derivation of the approximation for (18d) is similar to the above except that instead of the trigonometric identities (41), the identities

$$\begin{aligned} \sin 3t \cdot \cos 3t &\equiv (\sin t - \sin 3t + \sin 5t) \cos t \\ \sin 3t \cdot \sin 3t &\equiv (\sin t + \sin 3t + \sin 5t) \sin t \end{aligned} \tag{45}$$

are required. The differential relation analogous to (43) is

$$\begin{aligned} h_3'(A,B,dC,dD) = h_3(A,B) &+ dh_1(A+dC,B+dD) \\ &+ dh_3(A-dC,B+dD) + dh_5(A+dC,B+dD) , \end{aligned} \tag{46}$$

and, therefore, the approximation for (18d) is

$$\boxed{h_3'(A,B,C,D) \doteq h_3(A,B) + \Delta h_1(A+C,B+D) + \Delta h_3(A-C,B+D) + \Delta h_5(A+C,B+D).} \tag{47}$$

In the next section, the above results will be illustrated with a simple example.

3.3 An example

Hayashi [5] applied his method to the second order scalar system

$$\ddot{x} + 0.2\dot{x} + x^3 = 0.3 \cos t . \tag{48}$$

A typical sample of his numerical results will be included at the end of this section.

In Chapter II, the first order Galerkin procedure was applied to a system with a cubic nonlinearity, and it was found that

$$(A\cos t + B\sin t)^3 = \frac{3}{4}[A^3+AB^2]\cos t + \frac{3}{4}[B^3+A^2B]\sin t + \frac{1}{4}[A^3-3AB^2]\cos 3t + \frac{1}{4}[-B^3+3A^2B]\sin 3t \,. \tag{49}$$

Using the notation of the last section, from (49) we obtain

$$g_1(A,B) = \frac{3}{4}[A^3+AB^2]\;;\; h_1(A,B) = \frac{3}{4}[B^3+A^2B]$$
$$g_3(A,B) = \frac{1}{4}[A^3-3AB^2];\; h_3(A,B) = \frac{1}{4}[-B^3+3A^2B]\,. \tag{50}$$

Now, the application of the second order Galerkin method requires that the expression

$$(A\cos t + B\sin t + C\cos 3t + D\sin 3t)^3 \tag{51}$$

be expanded in a Fourier series -- or, using the notation of the last section, it is required that the four functions

$$g_1'(A,B,C,D),\; h_1'(A,B,C,D),\; g_3'(A,B,C,D), \text{ and } h_3'(A,B,C,D) \tag{52}$$

be known in analytic form. For the cubic nonlinearity, it is not prohibitively difficult to obtain these functions exactly, and it will be instructive to observe the effect on the Fourier components of adding third harmonics to the argument of $f(\bar{x},t)$. The result of expanding (51) is shown in its entirety below and on the next page.

$$\begin{aligned}(A\cos t + B\sin t + C\cos 3t + D\sin 3t)^3 = \\ &\tfrac{3}{4}[A^3 + AB^2 + (A^2-B^2)C + 2ABD + 2A(C^2+D^2)]\cdot\cos t \\ +\;&\tfrac{3}{4}[B^3 + A^2B + (A^2-B^2)D - 2ABC + 2B(C^2+D^2)]\cdot\sin t \\ +\;&\tfrac{1}{4}[A^3 - 3AB^2 + 6(A^2+B^2)C + 3C^3 + 3CD^2]\cdot\cos 3t \\ +\;&\tfrac{1}{4}[-B^3 + 3A^2B + 6(A^2+B^2)D + 3D^3 + 3C^2D]\cdot\sin 3t \\ +\;&\tfrac{3}{4}[(A^2-B^2)C - 2ABD + A(C^2-D^2) + 2BCD]\cdot\cos 5t\end{aligned} \tag{53}$$

$$+\frac{3}{4}[A^2-B^2)D + 2ABC - B(C^2-D^2) + 2ACD] \cdot \sin 5t$$

$$+\frac{3}{4}[A(C^2-D^2) - 2BCD] \cdot \cos 7t$$

$$+\frac{3}{4}[B(C^2-D^2) + 2ACD] \cdot \sin 7t \qquad \text{(53 cont'd)}$$

$$+\frac{1}{4}[C^3 - 3CD^2] \cdot \cos 9t$$

$$+\frac{1}{4}[-D^3 + 3C^2D] \cdot \sin 9t \quad .$$

By comparing (53) with (49), one obtains an appreciation of the complication caused by the inclusion of additional harmonics in the argument of $f(x,t)$. Hayashi approximates the above expansion by ignoring all terms in (53) of order greater than unity in C and D , as well as all harmonics above the third. By performing the above deletions as the expansion progresses, a savings in analytic effort is obtained; however, this process is not quite so straightforward as might be expected: considerable care must be exercised when deleting higher harmonics during the expansion to insure that they would not subsequently contribute to the desired result by heterodyning with other high harmonics.

From (53), we have that the first order approximations to the functions (52) are

$$g_1'(A,B,C,D) \doteq \frac{3}{4}[A^3 + AB^2 + (A^2-B^2)C + 2ABD] \qquad (54a)$$

$$h_1'(A,B,C,D) \doteq \frac{3}{4}[B^3 + A^2B + (A^2-B^2)D - 2ABD] \qquad (54b)$$

$$g_3'(A,B,C,D) \doteq \frac{1}{4}[A^3 - 3AB^2 + 6(A^2+B^2)C] \qquad (54c)$$

$$h_3'(A,B,C,D) \doteq \frac{1}{4}[-B^3 + 3A^2B + 6(A^2+B^2)D] \qquad (54d)$$

The reader should compare the above expressions with the first order Galerkin results in (50).

To clearly illustrate how the results of the last section are used to obtain the above approximations, the necessary computations will be

shown explicitly for (54a). Applying (37) and using (50), we have

$$\begin{aligned} g_1'(A,B,C,D) &\doteq g_1(A,B) + \Delta g_3(A+C,B) + \Delta h_3(A+D,B) \\ &= \frac{3}{4}\,[A^3+AB^2] + \frac{1}{4}\,\Delta[(A+C)^3 - 3(A+C)B^2] \\ &+ \frac{1}{4}\,\Delta[-B^3 + 3(A+D)^2B] \\ &\doteq \frac{3}{4}\,[A^3 + AB^2 + (A^2-B^2)C + 2ABD]\ , \end{aligned} \tag{55}$$

which agrees with (54a). The other three results, i.e., (39), (44), and (47) in the previous section, are applied in the same manner to generate (54b), (54c), and (54d) .

Numerical results obtained by solving the linearized determining equations generated from approximations (54) are reported by Hayashi [5]. Of the three periodic solutions to example (48), only one will be included here. The true solution, through the third harmonic, is

$$\hat{x}(t) = 0.6864 \cos t + 0.9841 \sin t - 0.0597 \cos 3t + 0.0214 \sin 3t. \tag{56}$$

The corresponding first order Galerkin solution is

$$x_1(t) = 0.703 \cos t + 1.012 \sin t\ , \tag{57}$$

and the linearized second order Galerkin solution is

$$\bar{x}(t) = 0.684 \cos t + 0.988 \sin t - 0.061 \cos 3t + 0.021 \sin 3t\ . \tag{58}$$

The third harmonic components of the solution (56) above are fairly small relative to the fundamental components, and the improvement in the first order Galerkin solution is quite good, expecially considering the ease of obtaining the approximations. However, as is evident from the nature of the approximations, it is to be expected that the accuracy observed above will deteriorate with increasing harmonic content of the

oscillation. Some numerical results will be reported in the next section, and this chapter will then be concluded with some brief comments regarding the relationship of this method to the iterative method to be presented in the next chapter.

3.4 Error of the Linearization

To provide an indication of the magnitude of error which is typically encountered for various nonlinearities, the following numerical results were obtained: the first and third harmonics of the time functions

$$f_n(\bar{x}) = (A \cos t + B \sin t + C \cos 3t + D \sin 3t)^n \ , \quad n = 3,5,7, \text{ and } 9 \tag{59}$$

were numerically computed, with the values of the parameters arbitrarily selected as

$$\begin{aligned} A &= 1.0 & C &= 0.05 \\ B &= 0.6 & D &= 0.02 \ . \end{aligned} \tag{60}$$

In addition, the first, third, and fifth harmonics of the time functions

$$\begin{aligned} &[(A+C) \cos t + B \sin t]^n \quad [A \cos t + (B+C) \sin t]^n \\ &[(A+D) \cos t + B \sin t]^n \quad [A \cos t + (B+D) \sin t]^n \\ &[(A\pm C) \cos t + (B+D) \sin t]^n \quad [A \cos t + B \sin t]^n \\ &n = 3,5,7, \text{ and } 9 \end{aligned} \tag{61}$$

were numerically computed, so that the approximations (37),(39),(44), and (47) could be evaluated and compared with the exact values from (59).

The results are shown in the table below. In order to emphasize the error of the approximations, the entries reported are changes in the harmonic components of (42) about their respective values for C=D=0. In each

case, the results of the approximations are shown directly beneath the exact values.

	$\Delta g_1'$	$\Delta h_1'$	$\Delta g_3'$	$\Delta h_3'$	
x^3:	0.0464	-0.0328	0.1021	0.0408	←(exact)
	0.0437	-0.0337	0.1058	0.0412	←(approximate)
x^5:	0.1038	-0.0412	0.1757	0.0732	
	0.0938	-0.0433	0.1867	0.0706	
x^7:	0.1873	-0.0378	0.2692	0.1279	
	0.1611	-0.0415	0.2922	0.1186	
x^9:	0.3115	-0.0222	0.3919	0.2168	
	0.2550	-0.0287	0.4330	0.1942	

(62)

To clarify the table above, note that the first entry was obtained by numerically computing

$$\Delta g_1'(1.0,\ 0.6,\ 0.05,\ 0.02) =$$

$$\frac{1}{\pi}\int_0^{2\pi} (1.0\cos t + 0.6\sin t + 0.05\cos 3t + 0.02\sin 3t)^3 \cos t\, dt$$

$$-\frac{1}{\pi}\int_0^{2\pi} (1.0\cos t + 0.6\sin t)^3 \cos t\, dt = 0.0464 \qquad (63)$$

and

$$\Delta g_3(1.0 + 0.05, 0.6) + \Delta h_3(1.0 + 0.02, 0.6) =$$

$$\frac{1}{\pi}\int_0^{2\pi} (1.05\cos t + 0.6\sin t)^3 \cos 3t\, dt$$

$$-\frac{1}{\pi}\int_0^{2\pi} (1.0\cos t + 0.6\sin t)^3 \cos 3t\, dt$$

$$+\frac{1}{\pi}\int_0^{2\pi} (1.02\cos t + 0.6\sin t)^3 \sin 3t\, dt$$

$$-\frac{1}{\pi}\int_0^{2\pi} (1.0\cos t + 0.6\sin t)^3 \sin 3t\, dt = 0.0437\ . \qquad (64)$$

3.5 Conclusions

In this chapter, four approximations have been presented which (for odd nonlinearities) allow Hayashi's linearized third order Galerkin solution to be obtained easily from the results of the first order Galerkin analysis. An example was given which demonstrates that the linearized solution can significantly improve the first order Galerkin solution. However, from the nature of the approximations used, it is obvious that if the third harmonic amplitudes C and D are not sufficiently small, then the terms which we have ignored may not be negligible, and the accuracy of the result may be quite poor. Some improvement in accuracy may be obtained by iterating the linearized equations; the iterative method to be presented in the next chapter, although derived in a completely different manner, can be regarded as a generalization of this idea. More will be said about the relationship of the two methods in Chapter IV.

CHAPTER IV

AN ITERATIVE METHOD

4.0 Introduction

In this chapter, an iterative procedure will be presented which generates an improved approximate solution using the first order Galerkin solution as a starting point. The result may be viewed as an analytic approximation to arbitrarily high order Galerkin methods. As was stated in Chapter I, the process possesses the following significant features: first, its accuracy for oscillations with rich harmonic content (but with a dominant fundamental component) is comparable to the accuracy of third (or higher) order harmonic balance, whereas the analytic effort necessary to apply the method is comparable to that required for the first order procedure. Secondly, the method is capable of generating estimates of as many of the harmonics of the oscillation as desired, with no increase in analytical effort; the only price one pays for additional harmonic information is a larger linear algebraic system which must be numerically solved at each stage of the iteration. Finally, in contrast to the general m^{th} order Galerkin procedure (and the linearized equations of Hayashi), the method produces an indication of its own accuracy by estimating the magnitudes of the terms which have been neglected in the approximations used.

In the next section, the problem of generating an improved approximate solution will be formulated in terms of a linear algebraic system in matrix form. In section 4.2, the matrix formulation will be related to the general m^{th} order Galerkin procedure (and to the method of Urabe-Reiter) by showing that the procedure is equivalent to the Newton method for finding the solutions of a vector equation $F(x) = 0$. In section

4.3, some approximations within the matrix formulation will be made so that the procedure can be practically applied, i.e., so as to satisfy the imposed constraint that the required analytic effort be comparable to the first order harmonic balance procedure. The resulting process will again be related to a Newton procedure. To facilitate understanding of the application of the method (which will be illustrated in detail in the next chapter), in section 4.4 the results of the foregoing sections will be specialized for the case of second order scalar systems. In the final section, the accuracy of the iterative method (and the nature of its relationship with the Galerkin method) will be discussed.

4.1 The Matrix Formulation

Recall again the vector nonlinear system,

$$\dot{x} = f(x,t) \tag{1}$$

which was defined previously. We will continue to impose the restriction that $f(x,t)$ be polynomial in the state vector x and continuous in the scalar t ; however, in contrast to Chapter III, it will no longer be required that $f(x,t)$ be odd in x . We assume that there exists an exact periodic solution which will be denoted $\hat{x}(t)$, and for some integer p let

$$\bar{x}(t) = \sum_{j=0}^{p} (x_{jc} \cos jt + x_{js} \sin jt) \tag{2}$$

be any approximate periodic solution. We will continue to use the term "periodic" to imply periodicity in t of period 2π . The solution error, $\delta(t)$, is defined as

$$\delta(t) \triangleq \hat{x}(t) - \bar{x}(t) = \sum_{j=0}^{\infty} (\delta_{jc} \cos jt + \delta_{js} \sin jt) . \tag{3}$$

Then the exact solution $\hat{x}(t)$ may be written

$$\hat{x}(t) = \sum_{j=0}^{\infty} [(x_{jc} + \delta_{jc}) \cos jt + (x_{js} + \delta_{js}) \sin jt] , \tag{4}$$

where it has been implied that $x_{jc} = x_{js} = 0 \quad \forall\, j > p$. Recall the definition of equation error as

$$\epsilon(t) = \epsilon'(\bar{x},t) = \dot{\bar{x}} - f(\bar{x},t) . \tag{5}$$

Then from (1), (3), and (5) we obtain

$$\dot{\delta}(t) = \dot{\hat{x}} - \dot{\bar{x}} = f(\hat{x},t) - [f(\bar{x},t) + \epsilon'(\bar{x},t)] . \tag{6}$$

Since $\hat{x}(t)$ and $\bar{x}(t)$ are each assumed periodic, their difference $\delta(t)$ is also periodic, as is its derivative. Therefore, $\dot{\delta}(t)$ can be expanded in a Fourier series

$$\begin{aligned}\dot{\delta}(t) = \sum_{j=1}^{\infty} [&\{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{jc} \cos jt \\ &+ \{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{js} \sin jt] ,\end{aligned} \tag{7}$$

where the previously introduced braces notation is used to denote the Fourier coefficients. Integrating (7) gives*

$$\begin{aligned}\delta(t) = \delta_{oc} + \sum_{j=1}^{\infty} (1/j)[&\{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{jc} \sin jt \\ &- \{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{js} \cos jt] .\end{aligned} \tag{8}$$

Furthermore, the assumed periodicity of $\hat{x}(t)$ and $\bar{x}(t)$ implies that their derivatives must have zero average value. Consequently, it is required that

$$\{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{oc} = 0 . \tag{9}$$

---

* The convergence of (3) and (6), as well as the validity of term-by-term integration of (7), is guaranteed by the quasi-differentiability of $\hat{x}(t)$. (See [14], sections 20.4 and 20.5.)

In addition to (9), we have from equation (8) the system of equations

$$\begin{aligned} \delta_{js} &= (1/j)\{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{jc} \\ \delta_{jc} &= -(1/j)\{f(\hat{x},t) - f(\bar{x},t) - \epsilon'(\bar{x},t)\}_{js} \end{aligned} \quad (j=1,2,3,\ldots) \ . \qquad (10)$$

Note that once the function $\bar{x}(t)$ is specified, equations (9) and (10) are functions only of the unknown parameters δ_{jc}, δ_{js}, $j=0,1,\ldots$. If this infinite system of nonlinear algebraic equations could be solved for the infinite number of unknowns, then from equation (4) it is clear that the exact solution $\hat{x}(t)$ would be obtained. Therefore, if we can obtain sufficiently accurate approximate solutions, $\bar{\delta}_{jc}$, $\bar{\delta}_{js}$ $(j=0,1,\ldots p)$, to equations (9) and (10), then the periodic function

$$y(t) = \sum_{j=0}^{p} [(x_{jc} + \bar{\delta}_{jc}) \cos jt + (x_{js} + \bar{\delta}_{js}) \sin jt] \qquad (11)$$

will be an improved approximation to $\hat{x}(t)$. If the variables $\bar{\delta}_{jc}$, $\bar{\delta}_{js}$ approach the true error variables δ_{jc} , δ_{js} as the periodic function $\bar{x}(t)$ approaches $\hat{x}(t)$, then the iterative process defined by

$$\begin{aligned} \bar{x}^{(n+1)} &\triangleq \sum_{j=0}^{p} (x_{jc}^{(n+1)} \cos jt + x_{js}^{(n+1)} \sin jt) \\ &= \sum_{j=0}^{p} [(x_{jc}^{(n)} + \bar{\delta}_{jc}^{(n)}) \cos jt + (x_{js}^{(n)} + \bar{\delta}_{js}^{(n)}) \sin jt] \ , \end{aligned} \qquad (12)$$

(where $\bar{\delta}_{jc}^{(n)}$, $\bar{\delta}_{js}^{(n)}$, $(j=0,1,\ldots,p)$ are approximate solutions to equations (9) and (10) with $\bar{x}(t) = \bar{x}^{(n)}(t)$) may converge to $\hat{x}(t)$.

The question to be considered in the remainder of this chapter is the following: how can approximations to the error variables δ_{js} and δ_{jc} be obtained subject to our imposed constraint on analytic effort?

Note that equations (9) and (10) cannot be solved exactly for two reasons: first, the number of equations is not finite; secondly, the argument $\hat{x}$ of the nonlinearity $f(\hat{x},t)$ has an infinite number of terms in its Fourier series. It is obviously the latter condition which is the prime source of difficulty. As has been previously demonstrated, we are generally unable (for practical reasons) to analytically expand $f(x,t)$ in a Fourier series if $x(t)$ contains more than two sinusoidal terms.

The first approximation which will be made is to expand the expression

$$f(\hat{x},t) - f(\bar{x},t)$$

in a Taylor series about $\bar{x}$, and then neglect all terms of order greater than unity in $(\hat{x} - \bar{x})$:

$$f(\hat{x},t) - f(\bar{x},t) \doteq f_x(\bar{x},t) \cdot (\hat{x}-\bar{x}) \tag{13}$$

where f_x is the Jacobian of the vector function $f(x,t)$ with respect to the vector x, evaluated at $x = \bar{x}(t)$. Given a periodic function $\bar{x}(t)$, $f_x(\bar{x},t)$ is an $n \times n$ matrix whose elements are known functions of time. From definitions (2) and (4), approximation (13) can be written

$$f(\hat{x},t) - f(\bar{x},t) \doteq f_x(\bar{x},t) \cdot \sum_{k=0}^{\infty} (\delta_{kc} \cos kt + \delta_{ks} \sin kt) . \tag{14}$$

In the above form, the j^{th} cosine component in (9) and (10) is approximately given[1] by

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[1] Interchange of the limiting processes in (15) is valid since $f_x(x,t)$ is, by assumption, continuous in its arguments, and consequently the integrand is continuous in t. (See [14], section 20.5.)

$$\{f(\hat{x},t)-f(\bar{x},t)\}_{jc} \doteq \frac{1}{\pi}\int_0^{2\pi} f_x(\bar{x},t)\Big[\sum_{k=0}^{\infty}(\delta_{kc}\cos kt + \delta_{ks}\sin kt)\Big]\cos jt\,dt$$

$$= \sum_{k=0}^{\infty}\Big[\frac{1}{\pi}\int_0^{2\pi} f_x(\bar{x},t)\cdot\cos kt\cos jt\,dt\Big]\cdot\delta_{kc}$$

$$+ \sum_{k=0}^{\infty}\Big[\frac{1}{\pi}\int_0^{2\pi} f_x(\bar{x},t)\cdot\sin kt\cos jt\,dt\Big]\cdot\delta_{ks} \qquad (15)$$

$$= \sum_{k=0}^{\infty}\left[\{f_x(\bar{x},t)\cos kt\}_{jc}\delta_{kc} + \{f_x(\bar{x},t)\sin kt\}_{jc}\delta_{ks}\right]$$

and similarly

$$\{f(\hat{x},t)-f(\bar{x},t)\}_{js} \doteq \sum_{k=0}^{\infty}\left[\{f_x(\bar{x},t)\cos kt\}_{js}\delta_{kc}\right.$$

$$\left. + \{f_x(\bar{x},t)\sin kt\}_{js}\delta_{ks}\right]. \qquad (16)$$

If the approximations (15) and (16) are substituted into equations (9) and (10), the result may be written in the matrix form

$$(A_1 + A_2)z \triangleq Az = b \qquad (17)$$

where

$$A_1 = \begin{bmatrix} \{f_x\}_{0c} & \{f_x\cos t\}_{0c} & \{f_x\sin t\}_{0c} & \{f_x\cos 2t\}_{0c} & \cdots \\ \{f_x\}_{1c} & \{f_x\cos t\}_{1c} & \{f_x\sin t\}_{1c} & \{f_x\cos 2t\}_{1c} & \cdots \\ \{f_x\}_{1s} & \{f_x\cos t\}_{1s} & \{f_x\sin t\}_{1s} & \{f_x\cos 2t\}_{1s} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\ 0 & 0 & -I & 0 & 0 & 0 & 0\cdots \\ 0 & I & 0 & 0 & 0 & 0 & 0\cdots \\ 0 & 0 & 0 & 0 & -2I & 0 & 0\cdots \\ 0 & 0 & 0 & 2I & 0 & 0 & 0\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -3I\cdots \\ 0 & 0 & 0 & 0 & 0 & 3I & 0\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$b = \begin{bmatrix} \{\epsilon'(\bar{x},t)\}_{0c} \\ \{\epsilon'(\bar{x},t)\}_{1c} \\ \{\epsilon'(\bar{x},t)\}_{1s} \\ \{\epsilon'(\bar{x},t)\}_{2c} \\ \vdots \end{bmatrix} \qquad z = \begin{bmatrix} \bar{\delta}_{0c} \\ \bar{\delta}_{1c} \\ \bar{\delta}_{1s} \\ \bar{\delta}_{2c} \\ \bar{\delta}_{2s} \\ \vdots \end{bmatrix}$$

In A_1, the Jacobian $f_x(\bar{x},t)$ has been simply denoted by f_x. In A_2 the symbols I and 0 denote the $n \times n$ identity and zero matrices, respectively.

Before concluding this section, several observations should be made about the preceding formulation. First, note that the only approximation which has been made to obtain (17) is (13), which approaches equality as $\bar{x}$ approaches $\hat{x}$. Recall that the Fourier components of $\bar{x}(t)$ are denoted as x_{jc}, x_{js}, $(j=0,1,2,\ldots)$. Define the vector X as

$$X = (x_{0c}\ x_{1c}\ x_{1s}\ x_{2c}\ \ldots\ )^T , \tag{18}$$

where the superscript "T" denotes the transpose of the row vector. Then the matrix A and the vector b in (17) can be considered to be functions only of X. We formally define the following iterative process:

$$X_{n+1}^{(p)} = X_n^{(p)} + A_{(p)}^{-1}(X_n^{(p)}) \cdot b^{(p)}(X_n^{(p)}) , \tag{19}$$

where the superscript p indicates that the infinite dimensional vectors and matrices of (17) and (18) are truncated at some finite (but arbitrary) dimension p. The reader should perhaps be reminded that the <u>elements</u> of the matrices and vectors in (19) are themselves n dimensional matrices and vectors, respectively.

In the next section, it will be shown that (19) is equivalent to the Newton technique for finding the solutions of a vector equation

$F(x) = 0$

The final observation which should be noted is that the process (19), with A and b as in (17), violates our imposed constraint on analytic effort: the expansion of

$$\varepsilon'\left(\sum_{j=0}^{p} (x_{jc} \cos jt + x_{js} \sin jt), t\right) \tag{20}$$

in a Fourier series, which is required for the computation of the b vector in (19), is precisely what makes the p^{th} order Galerkin procedure analytically impractical to apply. In addition, analytical computation of the elements of the A matrix is almost as difficult as evaluating the b vector. In section 4.3, we will resolve this difficulty by making use of another approximation.

4.2 The Matrix Formulation as a Newton Procedure

Given a vector function $F(x)$, of a vector variable x (where F and x are of the same dimension), the Newton method is an iterative procedure for finding solutions of the equation

$$F(x) = 0 . \tag{21}$$

Starting with an initial point x_0, the algorithm is

$$x_{n+1} = x_n - F_x^{-1}(x_n) \cdot F(x_n) \ . \tag{22}$$

This algorithm can be used to solve the Galerkin determining equations. From the definition of equation error $\epsilon'(x,t)$ and the system equations (1), it is clear that if for some function $y(t)$,

$$\epsilon'(y(t),t) = 0 \ ,$$

Then $y(t)$ is a solution of (1). The function $y(t)$ can be written

$$y(t) = \sum_{j=0}^{\infty} (y_{jc} \cos jt + y_{js} \sin jt) \ . \tag{23}$$

Define the vector Y_p as

$$Y_p = (y_{0c} \ y_{1c} \ y_{1s} \ y_{2c} \ y_{2s} \ \cdots \ y_{ps})^T \ , \tag{24}$$

and the vector Function $F_p(Y_p)$ as

$$F_p(Y_p) = \begin{bmatrix} \{\epsilon'(\sum_{j=0}^{p} (y_{jc} \cos jt + y_{js} \sin jt), t)\}_{0c} \\ \{\epsilon'(\sum_{j=0}^{p} (y_{jc} \cos jt + y_{js} \sin jt), t)\}_{1c} \\ \{\epsilon'(\sum_{j=0}^{p} (y_{jc} \cos jt + y_{js} \sin jt), t)\}_{1s} \\ \vdots \\ \{\epsilon'(\sum_{j=0}^{p} (y_{jc} \cos jt + y_{js} \sin jt), t)\}_{ps} \end{bmatrix} \ . \tag{25}$$

To apply the Newton algorithm (22), the Jacobian of F_p must be evaluated. From (24) and (25),

$$\frac{\partial F_p(Y_p)}{\partial Y_p} = \begin{bmatrix} \frac{\partial\{\epsilon'\}_{0c}}{\partial y_{0c}} & \frac{\partial\{\epsilon'\}_{0c}}{\partial y_{1c}} & \frac{\partial\{\epsilon'\}_{0c}}{\partial y_{1s}} & \cdots & \frac{\partial\{\epsilon'\}_{0c}}{\partial y_{ps}} \\ \frac{\partial\{\epsilon'\}_{1c}}{\partial y_{0c}} & & & & \vdots \\ \vdots & & & & \vdots \\ \frac{\partial\{\epsilon'\}_{ps}}{\partial y_{0c}} & \cdots & \cdots & \cdots & \frac{\partial\{\epsilon'\}_{ps}}{\partial y_{ps}} \end{bmatrix} \tag{26}$$

Note that each of the elements of the Jacobian matrix (26) is itself an $n \times n$ matrix.

The Newton process,

$$Y_p^{(n+1)} = Y_p^{(n)} - \left[\frac{\partial F_p(Y_p^{(n)})}{\partial Y_p}\right]^{-1} \cdot F_p(Y_p^{(n)})$$

is the procedure which Urabe and Reiter utilize [9]. At each stage of the iteration, they evaluate the elements of both the Jacobian matrix (26) and the vector F_p (25) using techniques of numerical integration.

To compare the above Newton process with the iterative method defined in equation (19), we must evaluate the elements of the Jacobian matrix analytically. We will consider separately the two terms which comprise the equation error. Denote the function of time obtained by truncating the Fourier expansion of $y(t)$ after the p^{th} harmonic by $y_p(t)$:

$$y_p(t) = \sum_{i=0}^{p} (y_{ic} \cos it + y_{is} \sin it) .$$

We now decompose the vector function $F_p(Y_p)$ into a sum of two terms, $F_p(Y_p) = F_p^{(1)}(Y_p) + F_p^{(2)}(Y_p)$, where

$$F_p^{(1)}(Y_p) \triangleq (\{\dot{y}_p(t)\}_{0c} \{\dot{y}_p(t)\}_{1c} \{\dot{y}_p(t)\}_{1s} \cdots \{\dot{y}_p(t)\}_{ps})^T \quad (27a)$$

and

$$F_p^{(2)}(Y_p) \triangleq -(\{f(y_p(t),t)\}_{0c} \cdot \cdot \cdot \{f(y_p(t),t)\}_{ps})^T . \quad (27b)$$

To evaluate the Jacobian of $F_p^{(1)}$ with respect to Y_p , consider first its terms of the form

$$\begin{aligned} \frac{\partial\{\dot{y}_p(t)\}_{jc}}{\partial y_{ks}} &= \frac{\partial}{\partial y_{ks}} \{\frac{d}{dt} \sum_{i=0}^{p} (y_{ic} \cos it + y_{is} \sin it)\}_{jc} \\ &= \frac{\partial}{\partial y_{ks}} \{ \sum_{i=1}^{p} (-iy_{ic} \sin it + iy_{is} \cos it\}_{jc} \\ &= \frac{\partial}{\partial y_{ks}} (jy_{js}) \\ &= jI\delta_{kj} \quad , \end{aligned} \quad (28)$$

where $\delta_{kj} = \begin{cases} 1, & k=j \\ 0, & k\neq j \end{cases}$, and I is the $n\times n$ identity matrix. In a similar way, it can be shown that

$$\frac{\partial\{\dot{y}_p(t)\}_{js}}{\partial y_{kc}} = -jI\delta_{kj} \tag{29}$$

and

$$\frac{\partial\{\dot{y}_p(t)\}_{js}}{\partial y_{ks}} = \frac{\partial\{\dot{y}_p(t)\}_{jc}}{\partial y_{kc}} = 0 \quad , \tag{30}$$

where 0 is the $n\times n$ zero matrix. Combining equations (28), (29), and (30) we have

$$\frac{\partial F_p^{(1)}(Y_p)}{\partial Y_p} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & 0 & & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 2I & & 0 & 0 \\ 0 & 0 & 0 & -2I & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & & 0 & pI \\ 0 & 0 & 0 & 0 & 0 & \cdots & -pI & 0 \end{bmatrix} \tag{31}$$

To evaluate the Jacobian of $F_p^{(2)}$ with respect to Y_p , consider first its terms of the form

$$\begin{aligned} -\frac{\partial}{\partial y_{ks}}\{f(y_p(t),t)\}_{jc} &= \\ &= -\frac{\partial}{\partial y_{ks}}\left(\frac{1}{\pi}\int_0^{2\pi} f(y_p(t),t)\cdot\cos jt dt\right) \\ &\overset{2}{=} -\frac{1}{\pi}\int_0^{2\pi}\left[\frac{\partial}{\partial y_{ks}} f(y_p(t),t)\right]\cos jt dt \\ &= -\frac{1}{\pi}\int_0^{2\pi}\frac{\partial}{\partial y_{ks}} f\left[\sum_{i=0}^{p}(y_{ic}\cos it + y_{is}\sin it),t\right]\cos jt\, dt \\ &= -\frac{1}{\pi}\int_0^{2\pi} f_x(y_p(t),t)\frac{\partial}{\partial y_{ks}}\left(\sum_{i=0}^{p}(y_{ic}\cos it + y_{is}\sin it)\right)\cdot\cos jt dt \\ &= -\frac{1}{\pi}\int_0^{2\pi} f_x(y_p(t),t)\cdot\sin kt\cdot\cos jt\, dt \\ &\triangleq -\{f_x(y_p(t),t)\cdot\sin kt\}_{jc} \quad . \end{aligned} \tag{32}$$

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[2] Interchange of integration and differentiation is valid provided f and $\partial f/\partial y_{ks}$ are continuous; our previous restrictions on $f(x,t)$ clearly fulfill this requirement.

In a similar way, it can be shown that

$$-\frac{\partial}{\partial y_{ks}}\{f(y_p(t),t)\}_{js} = -\{f_x(y_p(t),t)\cdot \sin kt\}_{js}\ ,$$

$$-\frac{\partial}{\partial y_{kc}}\{f(y_p(t),t)\}_{jc} = -\{f_x(y_p(t),t)\cdot \cos kt\}_{jc}\ , \qquad (33)$$

and

$$-\frac{\partial}{\partial y_{kc}}\{f(y_p(t),t)\}_{js} = -\{f_x(y_p(t),t)\cdot \cos kt\}_{js}\ .$$

Combining equations (32) and (33), we have

$$\frac{\partial F_p^{(2)}(Y_p)}{\partial Y_p} = -\begin{bmatrix} \{f_x\}_{0c} & \{f_x\cos t\}_{0c} & \{f_x\sin t\}_{0c} & \{f_x\cos 2t\}_{0c} & \cdots & \{f_x\sin pt\}_{0c} \\ \{f_x\}_{1c} & & & & & \vdots \\ \{f_x\}_{1s} & & & & & \vdots \\ \vdots & & & & & \vdots \\ \{f_x\}_{ps} & \cdots & \cdots & \cdots & \cdots & \{f_x\sin pt\}_{ps} \end{bmatrix} \qquad (34)$$

If the matrices (31) and (34) are substituted into the Newton algorithm (22), then the result is just the iterative process (19) of the previous section.

Kantorović [12] has established sufficient conditions that guarantee quadratic convergence of the Newton procedure from a given starting point. The statement of this theorem, specialized for our particular case, is rather lengthy and hence will be relegated to an appendix. For the present, it should simply be noted that provided certain conditions are satisfied, the iterative process (19) converges quadratically to the p^{th} order Galerkin solution. In addition, if $p=2$, and if the starting point is taken as the first order Galerkin solution, then the <u>first step</u> of the iterative process is equivalent to solving the linearized equations of Hayashi. This suggests that one might improve Hayashi's linearized solution by iteratively solving his equations.

4.3 An Approximation to the Matrix Formulation

At the end of section 4.1 it was observed that the matrix formulation (17) cannot be practically evaluated. Specifically, to obtain the elements of the A matrix and b vector requires the Fourier expansions of

$$f_x\Big(\sum_{j=0}^{p}(x_{jc}\cos jt + x_{js}\sin jt),t\Big)$$

and (35)

$$f\Big(\sum_{j=0}^{p}(x_{jc}\cos jt + x_{js}\sin jt),t\Big)$$

respectively. We have imposed the constraint that no more than two sinusoidal terms be allowed as the first argument of $f(x,t)$. Define $x_o(t)$ to be the fundamental component of $\bar{x}(t)$, i.e.,

$$x_o(t) = x_{oc} + x_{1c}\cos t + x_{1s}\sin t\ . \tag{36}$$

In a similar manner as was done in deriving the matrix formulation of section 4.1, we expand the functions $f(\bar{x},t)$ and $f_x(\bar{x},t)$ about $x_o(t)$, and ignore terms of order greater than unity in $(\bar{x}-x_o)$. For $f(\bar{x}(t),t)$,

$$\begin{aligned} f(\bar{x}(t),t) &\doteq f(x_o(t),t) + f_x(x_o(t),t)\cdot(\bar{x}(t) - x_o(t)) \\ &= f(x_o(t),t) + f_x(x_o(t),t)\cdot\sum_{j=2}^{p}(x_{jc}\cos jt + x_{js}\sin jt)\ . \end{aligned} \tag{37}$$

Likewise, the Jacobian matrix $f_x(\bar{x}(t),t)$ is approximated by expanding each of its elements about $x_o(t)$. Denote the scalar elements of the vector x_{kc} by

$$x_{kc} = (x_{kc}^{(1)}\ \ x_{kc}^{(2)}\ .\ .\ .\ x_{kc}^{(n)})^T\ ,$$

and similarly for x_{ks}. We then define the scalars

$$\delta x_j = \sum_{k=2}^{p} (x_{kc}^{(j)} \cos kt + x_{ks}^{(j)} \sin kt) \ , \ j=1,2,\ldots,n \ .$$

δx_j is simply the j^{th} element of $(\bar{x}(t) - x_0(t))$. Finally, if we denote the scalar elements of the vectors f and x as

$$f(x,t) \triangleq (f_1(x_1,x_2,\ldots,x_n,t),\ldots,f_n(x_1,x_2,\ldots,x_n,t))^T \ ,$$

then the expansion of the Jacobian matrix $f_x(\bar{x},t)$ about $f(x_0(t),t)$ can be written as

$$f_x(\bar{x}(t),t) \doteq f_x(x_0(t),t) + \begin{bmatrix} \sum_{j=1}^{n} \frac{\partial^2 f_1}{\partial x_1 \partial x_j} \delta x_j & \cdots & \sum_{j=1}^{n} \frac{\partial^2 f_1}{\partial x_n \partial x_j} \delta x_j \\ \vdots & & \\ \sum_{j=1}^{n} \frac{\partial^2 f_n}{\partial x_1 \partial x_j} \delta x_j & \cdots & \sum_{j=1}^{n} \frac{\partial^2 f_n}{\partial x_n \partial x_j} \delta x_j \end{bmatrix} \ . \qquad (38)$$

Note that only the first approximation (37) has any effect on the accuracy of the result. The quality of the second approximation (38) affects only the convergence properties of the procedure; it does not affect the point of convengence. In the next section, the iterative procedure will be specialized to second order scalar differential systems. The nature of the approximations (37) and (38) should then become clearer. Before concluding this section, however, it will be shown how the iterative procedure (19) -- with approximations (37) and (38) used to evaluate the A matrix and b vector -- is related to a Newton process.

In the preceding section, the Newton method was used to find the values of $x_{0c}, x_{1c}, x_{1s}, \ldots, x_{ps}$ for which the first p Fourier coefficients of

$$\epsilon'(\bar{x}(t),t) = \epsilon'(\sum_{j=0}^{p} (x_{jc} \cos jt + x_{js} \sin jt),t) \qquad (39)$$

were zero. These roots clearly correspond to the p^{th} order Galerkin solution. Suppose instead we apply the method to the related problem of

finding x_{0c}, x_{1c}, x_{1s},...,x_{ps} so that the first p Fourier coefficients of

$$\begin{aligned} &\epsilon'(x_0(t),t) + \epsilon_x'(x_0) \cdot (\bar{x}(t) - x_0(t)) \triangleq \\ &\quad \epsilon'(x_{0c} + x_{1c} \cos t + x_{1s} \sin t, t) \\ &\quad + \epsilon_x'(x_{0c} + x_{1c} \cos t + x_{1s} \sin t, t) \cdot \sum_{j=2}^{p} (x_{jc} \cos jt + x_{js} \sin jt) \end{aligned} \tag{40}$$

are zero. Note that (40) is a Taylor series expansion of (39) about $x_{0c} + x_{1c} \cos t + x_{1s} \sin t$, with all terms of order greater than unity in $(\bar{x}(t) - x_0(t))$ ignored.

Analogous to the definitions (25) through (27b), we define

$$X_p = (x_{0c} \quad x_{1c} \quad x_{1s} \quad x_{2c} \quad \cdots \quad x_{ps})^T ,$$

$$G_p(X_p) = \begin{bmatrix} \{\epsilon'(x_0,t) + \epsilon_x'(x_0) \cdot (\bar{x} - x_0)\}_{0c} \\ \{\epsilon'(x_0,t) + \epsilon_x'(x_0) \cdot (\bar{x} - x_0)\}_{1c} \\ \{\epsilon'(x_0,t) + \epsilon_x'(x_0) \cdot (\bar{x} - x_0)\}_{1s} \\ \vdots \\ \{\epsilon'(x_0,t) + \epsilon_x'(x_0) \cdot (\bar{x} - x_0)\}_{ps} \end{bmatrix} ,$$

and

$$G_p(X_p) = G_p^{(1)}(X_p) + G_p^{(2)}(X_p) ,$$

where $G_p^{(1)}$ corresponds to the time derivative term in $\epsilon'(x,t)$, and $G_p^{(2)}$ corresponds to the nonlinear term $f(x,t)$. Since differentiation is a linear operation, we have that

$$\frac{\partial G_p^{(1)}(X_p)}{\partial X_p} = \frac{\partial F_p^{(1)}(Y_p)}{\partial Y_p} .$$

To obtain the Jacobian $\frac{\partial G_p^{(2)}}{\partial X_p}$, consider terms of the form

$$\frac{\partial}{\partial x_{jc}} \{f(x_o,t) + f_x(x_o,t) \cdot (\bar{x}-x_o)\}_{kc} =$$

$$= \frac{\partial}{\partial x_{jc}} \{f(x_o,t)\}_{kc} + \frac{\partial}{\partial x_{jc}} \{f_x(x_o,t) \cdot (\bar{x}-x_o)\}_{kc} \qquad (41)$$

$$= \{\frac{\partial}{\partial x_{jc}} f(x_o,t)\}_{kc} + \{\frac{\partial}{\partial x_{jc}} f_x(x_o,t) \cdot (\sum_{i=2}^{p} (x_{ic} \cos it + x_{is} \sin it)\}_{kc} \cdot$$

Now, consider two cases: $j \leq 1$ and $j > 1$. If $j > 1$, then the first term on the right-hand side of (41) is zero, and hence the result is

$$\frac{\partial}{\partial x_{jc}} \{f(x_o,t) + f_x(x_o,t) \cdot (\bar{x}-x_o)\}_{kc} =$$

$$= \{f_x(x_o,t) \frac{\partial}{\partial x_{jc}} \sum_{i=2}^{p} (x_{ic} \cos it + x_{is} \sin it)\}_{kc} \qquad (42)$$

$$= \{f_x(x_o,t) \cdot \cos jt\}_{kc} .$$

Alternatively, if $j \leq 1$,

$$\{\frac{\partial}{\partial x_{jc}} f(x_o,t)\}_{kc} + \{\frac{\partial}{\partial x_{jc}} f_x(x_o,t) \cdot (\bar{x}-x_o)\}_{kc} =$$

$$= \{f_x(x_o,t) \cdot \cos jt\}_{kc} + \{\frac{\partial}{\partial x_{jc}} f_x(x_o,t) \cdot (\bar{x}-x_o)\}_{kc} \qquad (43)$$

The remaining elements

$$\frac{\partial}{\partial x_{js}} \{f(x_o(t),t) + f_x(x_o(t),t) \cdot (\bar{x}(t) - x_o(t))\}_{kc}$$

$$\frac{\partial}{\partial x_{jc}} \{f(x_o(t),t) + f_x(x_o(t),t) \cdot (\bar{x}(t) - x_o(t))\}_{ks}$$

and

$$\frac{\partial}{\partial x_{js}} \{f(x_o(t),t) + f_x(x_o(t),t) \cdot (\bar{x}(t) - x_o(t))\}_{ks}$$

are evaluated in the same fashion. Comparing (42) and (43) with (38), one sees that the first three columns (corresponding to the case $j \leq 1$) of $\partial G_p / \partial X_p$ in the Newton method are equivalent ot those of the matrix A_1 (with approximation (38)); however, the remaining (p-3) columns of Newton's Jacobian matrix are not equivalent to those of A_1: they are

approximations to A_1 obtained by ignoring the matrix of second partial derivatives in eqn. (38).

We have found in the above investigation that the iterative method of this thesis is closely related to, though not the equivalent of, a Newton procedure.[3] Their convergence properties are not identical because of the presence of second order partial derivatives in A_1 which are not contained in the Jacobian of the Newton method. However, because we are concerned with oscillations in which the fundamental harmonic is dominant, we would expect the convergence of the two processes to be quite similar since the first three columns of the two matrices are identical. A quadratic rate of convergence has in fact been observed in all numerical examples investigated. (The application of the method to an example is illustrated in Chapter V.)

Since we have the capability of deleting (if we so desire) the second order partial derivatives in all columns of A_1 beyond the third, the results of Kantorovic (see [10] and [12]) can be used to investigate questions of convergence (see Appendix). However, it should be mentioned that the application of the theorem requires a considerable amount of computational effort, and in practice the algorithm is frequently used without benefit of guaranteed convergence.

4.4 A Special Case: Second Order Scalar Systems

Because of their importance in practice, and because their simpler form should help to clarify the approximations made in the last section, the iterative method will now be specialized to the case of second

[3] If, as suggested in the last section, the linearized third order equations of Hayashi are solved iteratively, the resulting process is equivalent to the Newton method.

order[4] differential systems of the form

$$\ddot{x} = f(x,\dot{x},t) \ , \tag{44}$$

where f and x are now scalars, and f is polynomial in both x and $\dot{x}$ (and continuous in t).

So that the results will conform exactly with those needed for the practical applications in the next chapter, f will be further restricted to be odd in its arguments x and $\dot{x}$. This latter restriction simply means that all even harmonics are assumed at the outset to be zero.

Instead of expansion (7) in section 4.1, we have

$$\ddot{\delta}(t) = \sum_{j=1,3,\ldots}^{\infty} [\{f(\hat{x},\dot{\hat{x}},t) - f(\bar{x},\dot{\bar{x}},t) - \epsilon'(\bar{x},\dot{\bar{x}},t)\}_{jc} \cos jt + \{f(\hat{x},\dot{\hat{x}},t) - f(\bar{x},\dot{\bar{x}},t) - \epsilon'(\bar{x},\dot{\bar{x}},t)\}_{js} \sin jt] \ . \tag{45}$$

After integrating twice (with each constant of integration taken as zero because of the assumption that $f(x,\dot{x},t)$ is odd in x and $\dot{x}$), we have the following equations which are analagous to (10):

$$\begin{aligned} \delta_{jc} &= -(1/j)^2\{f(\hat{x},\dot{\hat{x}},t) - f(\bar{x},\dot{\bar{x}},t) - \epsilon'(\bar{x},\dot{\bar{x}},t)\}_{jc} \\ \delta_{js} &= -(1/j)^2\{f(\hat{x},\dot{\hat{x}},t) - f(\bar{x},\dot{\bar{x}},t) - \epsilon'(\bar{x},\dot{\bar{x}},t)\}_{js} \ . \end{aligned} \qquad j=1,3,\ldots \tag{46}$$

If, in (46), the following Taylor series approximation (analogous to (13)) is made,

$$f(\hat{x},\dot{\hat{x}},t) - f(\bar{x},\dot{\bar{x}},t) \doteq f_x(\bar{x},\dot{\bar{x}},t)\cdot(\hat{x}-\bar{x}) + f_{\dot{x}}(\bar{x},\dot{\bar{x}},t)\cdot(\dot{\hat{x}}-\dot{\bar{x}}) \ , \tag{47}$$

then we obtain a matrix formulation $(A_1 + A_2)z = b$, where

---

[4] The generalization of the derivation in this section to n^{th} order (scalar) differential systems is straightforward and will not be presented.

$$A_1 = \begin{bmatrix} \{f_x\cos t - f_{\dot{x}}\sin t\}_{1c} & \{f_x\sin t + f_{\dot{x}}\cos t\}_{1c} & \{f_x\cos 3t - 3f_{\dot{x}}\sin 3t\}_{1c}\cdots \\ \{f_x\cos t - f_{\dot{x}}\sin t\}_{1s} & \{f_x\sin t + f_{\dot{x}}\cos t\}_{1s} & \{f_x\cos 3t - 3f_{\dot{x}}\sin 3t\}_{1s}\cdots \\ \{f_x\cos t - f_{\dot{x}}\sin t\}_{3c} & & \vdots \\ \{f_x\cos t - f_{\dot{x}}\sin t\}_{3s} & & \vdots \\ \vdots & & \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 9 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 9 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 25 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 25 & \\ \vdots & & & & & \ddots & \end{bmatrix} \qquad (48)$$

$$b = \begin{bmatrix} \{\epsilon'(\bar{x},\dot{\bar{x}},t)\}_{1c} \\ \{\epsilon'(\bar{x},\dot{\bar{x}},t)\}_{1s} \\ \{\epsilon'(\bar{x},\dot{\bar{x}},t)\}_{3c} \\ \{\epsilon'(\bar{x},\dot{\bar{x}},t)\}_{3s} \\ \vdots \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} \bar{\delta}_{1c} \\ \bar{\delta}_{1s} \\ \bar{\delta}_{3c} \\ \bar{\delta}_{3s} \\ \vdots \end{bmatrix}$$

In a similar manner as was done for the vector system in (37) and (38), we expand the functions $f(\bar{x},\dot{\bar{x}},t)$, $f_x(\bar{x},\dot{\bar{x}},t)$ and $f_{\dot{x}}(\bar{x},\dot{\bar{x}},t)$ about $x_o(t)$, and retain only the first order terms of the expansion:

$$f(\bar{x},\dot{\bar{x}},t) \doteq f(x_o,\dot{x}_o,t)+f_x(x_o,\dot{x}_o,t)\cdot(\bar{x}(t)-x_o(t))+f_{\dot{x}}(x_o,\dot{x}_o,t)\cdot(\dot{\bar{x}}(t)-\dot{x}_o(t))$$

$$f_x(\bar{x},\dot{\bar{x}},t) \doteq f_x(x_o,\dot{x}_o t)+f_{xx}(x_o,\dot{x}_o,t)\cdot(\bar{x}(t)-x_o(t)) + f_{x\dot{x}}(x_o,\dot{x}_o,t)\cdot(\dot{\bar{x}}(t)-\dot{x}_o(t))$$

$$f_{\dot{x}}(\bar{x},\dot{\bar{x}},t) \doteq f_{\dot{x}}(x_o,\dot{x}_o,t)+f_{\dot{x}x}(x_o,\dot{x}_o,t)\cdot(\bar{x}(t)-x_o(t)) + f_{\dot{x}\dot{x}}(x_o,\dot{x}_o,t)\cdot(\dot{\bar{x}}(t)-\dot{x}_o(t)) \; . \qquad (49)$$

With the approximations (49) substituted into the matrix system (48), we have the desired result.

It can now be seen how the A matrix and b vector can be evaluated from knowledge of the first order Galerkin results. Suppose the nonlinearity has the form

$$f(x,\dot{x},t) = \sum_{k=1}^{m_1} \alpha_k x^k + \sum_{k=1}^{m_2} \beta_k \dot{x}^k + \sum_{k=1}^{m_3} \sum_{\ell=1}^{m_4} \gamma_{k\ell} \dot{x}^k x^\ell \qquad (50)$$

where we have ignored the dependence of f upon t .

If, in expanding $f(x_{1c} \cos t + x_{1s} \sin t, t)$ in a Fourier series, one performs the expansion in the general manner of section 2.3 (Chapter II), then each of the elements in A and b is obtained merely from direct substitution into this first order result. For example, consider the particular term $\dot{x}^3 x^5$ from (50). Instead of expanding

$$g(A,B) = (B \cos t - A \sin t)^3 (A \cos t + B \sin t)^5 \qquad (51)$$

we instead expand the more general expression

$$h(A_i, B_i, \omega_i : i=1,2,\ldots,8) = \prod_{i=1}^{8} (A_i \cos \omega_i t + B_i \sin \omega_i t). \qquad (52)$$

Once this is done, each of the elements in (48) can be obtained by merely substituting various values of A_i, B_i, and ω_i into (52). For example, with $f(x,\dot{x}) = \dot{x}^3 x^5$, then

$$f_x(x,\dot{x}) = 5\dot{x}^3 x^4$$

and therefore

$$f_x \cos \omega t = 5 \cdot h(x_{1s}, -x_{1c}, 1 : i=1,2,3 \; ; \; x_{1c}, x_{1s}, 1 : i=1,2,3,4 \; ; \; 1,0,\omega) .$$

In a similar manner, the term

$$f_{\dot{x}}(x,\dot{x}) \sin \omega t = 3\dot{x}^2 x^5 \sin \omega t$$

is evaluated by

$f_x \sin \omega t = 3 \cdot h(x_{1s}, -x_{1c}, 1{:}i=1,2 \; ; \; x_{1c}, x_{1s}, 1{:}i=1,2,3,4,5 \; ; \; 0,1,\omega)$.

In the next chapter, all of the techniques necessary for applying the method to a particular system will be illustrated in detail. Before concluding this chapter, however, the question of the accuracy of the iterative method and its relationship to harmonic balance will be discussed.

4.5 Accuracy of the Iterative Method and the Nature of its Relationship to the Galerkin Procedure

For simplicity, in this section we will consider the nonlinearity to be a scalar function of a scalar variable; the more general n-vector case is not qualitatively different. In particular, consider the function

$$f(x) = x^n \tag{53}$$

where n is an integer. If we expand

$$f\left[\sum_{j=0}^{p} (x_{jc} \cos jt + x_{js} \sin jt)\right] = f\left[\sum_{j=0}^{p} x_j \cos (jt + \varphi_j)\right] \tag{54}$$

in a Fourier series, the various coefficients will consist of weighted[5] sums of terms of the form

$$\begin{array}{llll} x_1^n & & & \\ x_1^{n-1}x_2 & x_1^{n-2}x_2^2 & \cdots\cdots\cdots\cdots & x_2^n \\ x_1^{n-1}x_3 & x_1^{n-2}x_3^2 & x^{n-2}x_2x_3 \cdots & x_3^n \\ x_1^{n-1}x_4 & x_1^{n-2}x_4^2 & \cdots\cdots\cdots\cdots & x_4^n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1}x_p & x_1^{n-2}x_p^2 & \cdots\cdots\cdots\cdots & x_p^n \; . \end{array} \tag{55}$$

To better compare the relative magnitudes of the terms in (55), we normalize

---

[5] The weighting factors of the various terms in (55) are determined by the structure of the particular nonlinearity (the value of n in (53)), and by the phase angles φ_j .

each term by dividing by x_1^n , with the result

$$\begin{array}{c|l} 1 & \\ x_2/x_1 & (x_2/x_1)^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (x_2/x_1)^n \\ \hline x_3/x_1 & (x_3/x_1)^2 \quad (x_3/x_1)(x_2/x_1) \ \ldots\ (x_3/x_1)^n \\ x_4/x_1 & (x_4/x_1)^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (x_4/x_1)^n \\ \vdots & \vdots \qquad\qquad\qquad\qquad\qquad\qquad\quad \vdots \\ x_p/x_1 & (x_p/x_1)^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (x_p/x_1)^n \end{array} \tag{56}$$

The first order Galerkin method neglects all terms above in comparison with unity; the second order Galerkin method neglects all terms below the second row (below the dotted horizontal line). By considering the terms which were neglected in the truncated Taylor series approximations of section 4.3, it is clear that the iterative method of this chapter ignores all terms to the right of the dotted vertical line.

From the above discussion, it is obvious that the approximations made in the iterative procedure are qualitatively different from those made in the harmonic balance method. The second order harmonic balance method takes into account the exact effect of the first and second harmonics, but completely ignores the third and higher harmonics; the third order method completely accounts for the first three harmonics, but totally ignores the fourth and higher, etc.[6] The iterative method of this thesis accounts for the first order effects of all of the harmonics. The source of error in the iterative method is that the higher order effects of all harmonics other than the first are neglected.

There are two important consequences of the above comments which should be noted. First, it is possible to see from (56) that the iterative procedure can be more, or less, accurate than second (or higher) order harmonic balance, depending upon the harmonic content of the oscil-

[6] The other well-known analytic methods [5] (the perturbation method and the method of successive approximations) are also approximations of this type.

lation. For example, when the oscillation has a large second harmonic component (e.g., $x_2/x_1 > 0.5$) and all its other harmonics are relatively small (e.g., $x_j/x_1 < 0.01$: $j=3,4,5,\ldots$), then the second order Galerkin method will generally be the more accurate, since the terms

$$(x_2/x_1)^k \ , \ k=2,3,\ldots,n \tag{57}$$

which are neglected by the iterative method will in such a case be more significant than the terms

$$(x_j/x_1)^1 \ , \ j=3,4,5,\ldots \tag{58}$$

which are neglected by the second order Galerkin method. Alternatively, when the harmonic content of the oscillation is more evenly distributed (consistent with our previously stated requirement that the fundamental component be dominant), then the iterative method will generally be more accurate than the second (or higher) order harmonic balance method, even though the second-order procedure is much more difficult (if not practically impossible) to apply. For example, consider a squarewave oscillation; the magnitude of its j^{th} harmonic component is proportional to $1/j$, $j=1,3,5,\ldots$. In this case, the terms neglected by the third order harmonic balance method are of the order of

$$1/5 \ , \ 1/7 \ , \ 1/9 \ , \ \ldots \ ,$$

whereas the terms neglected by the iterative method are of the order of

$$(1/3)^2 = 1/9 \ , \ (1/3)^3 = 1/27 \ , \ (1/3)^4 = 1/81 \ , \ \ldots \ .$$

Hence in this case, the first order effect of the ninth harmonic is potentially as important as the higher order effects of the third harmonic.

Before concluding this chapter, an additional consequence of the qualitative difference in the approximations should be noted. In the case of the second order harmonic balance method, the accuracy of the method depends upon the magnitude of the third and higher harmonics, yet the procedure does not generate any estimate of these harmonics. Hence in applying the method, one has no indication of the magnitude of the terms which have been neglected. On the other hand, the iterative method of this chapter estimates harmonics of arbitrarily high order, and therefore the magnitude of the terms neglected in the truncated Taylor series approximations can be estimated.

CHAPTER V

APPLICATION OF THE ITERATIVE METHOD TO AN EXAMPLE

5.0 Introduction

In this chapter, it will be shown how the iterative method of Chapter IV is applied in practice; in particular, techniques for evaluating the elements of the A matrix and b vector in a general manner (so that the dimension of the algebraic system can be subsequently chosen arbitrarily) will be exhibited. The particular nonlinear system which was selected to illustrate the application of the method is the classical Duffing equation

$$\ddot{x} + \alpha\dot{x} + \beta x + \gamma x^3 = \eta \cos(\omega t) \ . \tag{1}$$

The improvement over the first order Galerkin solution which can be obtained using the iterative method will be illustrated with numerical results for the special case

$$\ddot{x} + 0.3\dot{x} + 10x - 0.1x^3 = 8 \cos(\omega t) \ , \tag{2}$$

for the forcing frequency ω in the range [1,4]. The particular values of the parameters in (1) were chosen so that the resulting oscillations would have significant harmonic content.

5.1 Evaluation of the A Matrix

For the system (1), the function $f(x,\dot{x},t)$ is

$$f(x,\dot{x},t) = -\alpha\dot{x} - \beta x - \gamma x^3 + \eta \cos \omega t \tag{3}$$

and the nonlinear portion is

$$Nf(x,\dot{x},t) = -\gamma x^3 \ . \tag{4}$$

To apply the iterative procedure, we need the Fourier expansion of

$$\prod_{i=1}^{3} (A_i \cos \omega_i t + B_i \sin \omega_i t) \ , \tag{5}$$

which was computed in section 4 of Chapter II, Equation (II-33). In this and the following section, it will be shown how the various elements in equation (48) of the last chapter are evaluated by making appropriate substitutions into expression (II-33). Consider first the evaluation of the matrix A_1 . To avoid confusion, the various component terms will be considered separately and hence we make the following definitions:

$$A_1^{(1)} = \begin{bmatrix} \{Nf_x(x_o) \cdot \cos t\}_{1c} & \{Nf_x(x_o) \cdot \sin t\}_{1c} \ \ldots \\ \{Nf_x(x_o) \cdot \cos t\}_{1s} & \vdots \\ \{Nf_x(x_o) \cdot \cos t\}_{3c} & \\ \vdots & \end{bmatrix}$$

$$A_1^{(2)} = \begin{bmatrix} \{Nf_{xx}(x_o) \cdot (\bar{x} - x_o) \cdot \cos t\}_{1c} \ \ldots \\ \vdots \end{bmatrix}$$

$$A_1^{(3)} = \begin{bmatrix} \{(I-N)f_x(x_o) \cdot \cos t\}_{1c} & \{(I-N)f_x(x_o) \cdot \sin t\}_{1c} \ \ldots \\ \vdots & \vdots \end{bmatrix}$$

$$A_1^{(4)} = \begin{bmatrix} \{-f_{\dot{x}}(x_o) \cdot \sin t\}_{1c} & \{f_{\dot{x}}(x_o) \cdot \cos t\}_{1c} \ \ldots \\ \vdots & \vdots \end{bmatrix} \ .$$

The notation $(I-N)f$ in the matrix $A_1^{(3)}$ is used to denote the linear

terms in the function $f(x,\dot{x},t)$. Note that because of the particular form of the nonlinearity in this example, the matrices corresponding to the derivatives $Nf_{x\dot{x}}$, $Nf_{\dot{x}x}$, and $Nf_{\dot{x}\dot{x}}$ are zero.

Consider first the matrix $A_1^{(1)}$. To evaluate the <u>odd</u> columns of $A_1^{(1)}$, observe that

$$\begin{aligned} Nf_x(x_o) \cdot \cos kt &= \alpha \cdot \frac{d}{dx}(x^3)\Big|_{x=x_o} \cdot \cos kt = 3\alpha x_o^2 \cdot \cos kt \\ &= 3\alpha h(1,0,k;\, x_{1c},\, x_{1s},\, 1;\, x_{1c},\, x_{1s},\, 1) , \end{aligned} \tag{6}$$

where the function $h(\cdot)$ is defined in expression (II-33) of Chapter II. The notation used in (6) is simply meant to indicate that the parameters of expression (II-33) should be given the following values:

$$\begin{aligned} &A_1 = 1 && B_1 = 0 && \omega_1 = k \\ &A_2 = A_3 = x_{1c} && B_2 = B_3 = x_{1s} && \omega_2 = \omega_3 = 1 . \end{aligned} \tag{7}$$

The result of substituting (7) into (II-33) is

$$\begin{aligned} Nf_x(x_o) \cdot \cos kt = \frac{3}{4}\alpha \,[&2(x_{1c}^2 + x_{1s}^2)\cos kt \\ &+ (x_{1c}^2 - x_{1s}^2)\cos (k-2)t \\ &+ (x_{1c}^2 - x_{1s}^2)\cos (k+2)t \\ &- 2x_{1c}x_{1s}\sin (k-2)t \\ &+ 2x_{1c}x_{1s}\sin (k+2)t] . \end{aligned} \tag{8}$$

Similarly, to evaluate the <u>even</u> columns of $A_1^{(1)}$, we need the Fourier components of $f_x(x_o) \cdot \sin kt$. Hence, we can make the substitution

$$\begin{aligned} &A_1 = 0 && B_1 = 1 && \omega_1 = k \\ &A_2 = A_3 = x_{1c} && B_2 = B_3 = x_{1s} && \omega_2 = \omega_3 = 1 \end{aligned} \tag{9}$$

to get

$$\begin{aligned} Nf_x(x_o) \cdot \sin kt = \frac{3}{4}\alpha\, [&2(x_{1c}^2 + x_{1s}^2) \sin kt \\ &+ (x_{1c}^2 - x_{1s}^2) \sin (k-2)t \\ &+ (x_{1c}^2 - x_{1s}^2) \sin (k+2)t \\ &+ 2x_{1c}x_{1s} \cos (k-2)t \\ &- 2x_{1c}x_{1s} \cos (k+2)t] \ . \end{aligned} \tag{10}$$

The elements of the matrix $A_1^{(1)}$ can now be written down, column by column, from inspection of (8) and (10); to save space, in (8) and (10) we let

$$\begin{aligned} 2(x_{1c}^2 + x_{1s}^2) &= a \\ x_{1c}^2 - x_{1s}^2 &= b \\ 2x_{1c}x_{1s} &= c \ . \end{aligned} \tag{11}$$

Then the matrix $A_1^{(1)}$ is

$$\frac{3}{4}\alpha \begin{bmatrix} a+b & c & b & c & 0 & 0 & 0 & 0 & \cdots \\ c & a-b & -c & b & 0 & 0 & 0 & 0 & \cdots \\ b & -c & a & 0 & b & c & 0 & 0 & \cdots \\ c & b & 0 & a & -c & b & 0 & 0 & \cdots \\ 0 & 0 & b & -c & a & 0 & b & c & \cdots \\ 0 & 0 & c & b & 0 & a & -c & b & \cdots \\ 0 & 0 & 0 & 0 & b & -c & a & 0 & \cdots \\ 0 & 0 & 0 & 0 & c & b & 0 & a & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & b & -c & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & c & b & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{12}$$

Recall that it was stated earlier that the analytic effort required in applying the iterative method does not increase as the dimension of the matrix system is increased; it should now be clear why this is the case: the structure of the matrix (12) is quite redundant, as is emphasized by the dotted rectangles. The first two columns are exceptions to this uniform structure because of the "folding over" of negative frequencies which occurs below the third harmonic.

Note that (12) is symmetric; this is a general property of both $A_1^{(1)}$ and $A_1^{(2)}$, since

$$\{f_x \cos kt\}_{js} = \frac{1}{\pi}\int_0^{2\pi} (f_x \cdot \cos kt) \sin jt \, dt = \{f_x \sin jt\}_{kc} , \quad (13)$$

etc. However, the elements of $A_1^{(4)}$ (in which the partials are with respect to $\dot{x}$) are not arranged in this symmetrical manner.

Next, we need to evaluate the matrix $A_1^{(2)}$. Since

$$Nf_{xx}(x_o) \cdot (\bar{x}-x_o) = \alpha \frac{d^2}{dx^2} x^3 \Big|_{x=x_o} \cdot (\bar{x}-x_o)$$

$$= 6\alpha\, x_o(t) \cdot \sum_{j=3,5,\ldots}^{p} (x_{jc} \cos jt + x_{js} \sin jt) \quad (14)$$

and so

$$Nf_{xx}(x_o) \cdot (\bar{x}-x_o) \cdot \cos \omega t = 6\alpha \sum_{j=3,5,\ldots}^{p} h(1,0,k;x_{1c},x_{1s},1;x_{jc},x_{js},j) . \quad (15)$$

In other words, we need to evaluate expression (II-33) for the substitutions

$$\begin{array}{lll} A_1 = 1 & B_1 = 0 & \omega_1 = k \\ A_2 = x_{1c} & B_2 = x_{1s} & \omega_2 = 1 \\ A_3 = x_{jc} & B_3 = x_{js} & \omega_3 = j \ . \end{array} \quad (16)$$

The result is

$$
\begin{aligned}
Nf_{xx}(x_o) \cdot (\bar{x}-x_o) \cdot \cos kt = \\
\frac{3}{2}\alpha \sum_{j=3,5,\ldots}^{p} & [(x_{1c}x_{jc} + x_{1s}x_{js}) \cos [k - (j-1)]t \\
& + (x_{1c}x_{jc} + x_{1s}x_{js}) \cos [k + (j-1)]t \\
& + (x_{1c}x_{jc} - x_{1s}x_{js}) \cos [k - (j+1)]t \\
& + (x_{1c}x_{jc} - x_{1s}x_{js}) \cos [k + (j+1)]t \\
& - (x_{1c}x_{js} - x_{1s}x_{jc}) \sin [k - (j-1)]t \\
& + (x_{1c}x_{js} - x_{1s}x_{jc}) \sin [k + (j-1)]t \\
& - (x_{1c}x_{js} + x_{1s}x_{jc}) \sin [k - (j+1)]t \\
& + (x_{1c}x_{js} + x_{1s}x_{jc}) \sin [k + (j+1)]t] \, .
\end{aligned}
\tag{17}
$$

Similarly, for $Nf_{xx}(x_o) \cdot (\bar{x}-x_o) \cdot \sin kt$ we make the substitutions

$$
\begin{aligned}
A_1 &= 0 & B_1 &= 1 & \omega_1 &= k \\
A_2 &= x_{1c} & B_2 &= x_{1s} & \omega_2 &= 1 \\
A_3 &= x_{jc} & B_3 &= x_{js} & \omega_3 &= j
\end{aligned}
$$

with the result

$$
\begin{aligned}
Nf_{xx}(x_o) \cdot (\bar{x} - x_o) \cdot \sin kt = \\
\frac{3}{2}\alpha \sum_{j=3,5,\ldots}^{p} & [(x_{1c}x_{jc} + x_{1s}x_{js}) \sin [k - (j-1)]t \\
& + (x_{1c}x_{jc} + x_{1s}x_{js}) \sin [k + (j-1)]t \\
& + (x_{1c}x_{jc} - x_{1s}x_{js}) \sin [k - (j+1)]t \\
& + (x_{1c}x_{jc} - x_{1s}x_{js}) \sin [k + (j+1)]t \\
& + (x_{1c}x_{js} - x_{1s}x_{jc}) \cos [k - (j-1)]t \\
& - (x_{1c}x_{js} - x_{1s}x_{jc}) \cos [k + (j-1)]t \\
& + (x_{1c}x_{js} + x_{1s}x_{jc}) \cos [k - (j+1)]t \\
& + (x_{1c}x_{js} + x_{1s}x_{jc}) \cos [k + (j+1)]t] \, .
\end{aligned}
\tag{19}
$$

Because the matrix $A_1^{(2)}$ is symmetrical, we need evaluate only the elements on and below the diagonal, i.e., we need expressions for the following Fourier coefficients:

$$\{Nf_{xx}(x_o) \cdot (\bar{x} - x_o) \cdot \cos kt\}_{(k+m)c} \qquad (20a)$$

$$\{Nf_{xx}(x_o) \cdot (\bar{x} - x_o) \cdot \cos kt\}_{(k+m)s} \qquad \begin{matrix} m=0,2,4,\ldots \\ k=1,3,5,\ldots \end{matrix} \qquad (20b)$$

and

$$\{Nf_{xx}(x_o) \cdot (\bar{x} - x_o) \cdot \sin kt\}_{(k+m)c} \qquad (20c)$$

$$\{Nf_{xx}(x_o) \cdot (\bar{x} - x_o) \cdot \sin kt\}_{(k+m)s} \; . \qquad (20d)$$

These expressions are obtained by collecting the appropriate terms from (17) and (19). For example, consider the first expression in (20): we need to collect all $(k+m)^{th}$ order cosine components in (17). In the first sum[1] of (17),

$$\sum_{j=3,5,\ldots}^{p} (x_{1c}x_{jc} + x_{1s}x_{js}) \cos [k - (j-1)]t \; , \qquad (21)$$

we select the term for which

$$k - (j-1) = -(k+m) \quad \text{or} \quad j = 2k + m + 1 \; . \qquad (22)$$

Hence, the portion of expression 20a due to the first sum of cosines in (17) is

$$x_{1c}x_{(2k+m+1)c} + x_{1s}x_{(2k+m+1)s} \; . \qquad (23)$$

From the second sum in (17), i.e.,

$$\sum_{j=3,5,\ldots}^{p} (x_{1c}x_{jc} + x_{1s}x_{js}) \cos [k + (j-1)]t \; , \qquad (24)$$

we select the term for which

---

[1] In writing expression (21), we have implicitly rearranged the order of summation in (17).

$$k + (j-1) = k + m \qquad \text{or} \qquad j = m + 1 \tag{25}$$

so the sum (24) contributes the terms

$$x_{1c}x_{(m+1)c} + x_{1s}x_{(m+1)s} \cdot \tag{26}$$

In a similar way, the contributions due to the third and fourth cosine sums in (17) are

$$x_{1c}x_{(2k+m-1)c} - x_{1s}x_{(2k+m-1)s} \tag{27}$$

and

$$x_{1c}x_{(m-1)c} - x_{1s}x_{(m-1)s} \cdot \tag{28}$$

Combining (23), (26), (27) and (28) we have that for m = 0,2,4,... and k = 1,3,5,...

$$\begin{aligned} \{Nf_{xx}(x_o) \cdot (\bar{x}-x_o) \cdot \cos kt\}_{(k+m)c} = \\ (x_{1c}x_{(m-1)c} - x_{1s}x_{(m-1)s}) \\ + (x_{1c}x_{(m+1)c} + x_{1s}x_{(m+1)s}) \\ + (x_{1c}x_{(2k+m-1)c} - x_{1s}x_{(2k+m-1)s}) \\ + (x_{1c}x_{(2k+m+1)c} + x_{1s}x_{(2k+m+1)s}) \cdot \end{aligned} \tag{29}$$

Implicit in the preceding manipulations is the fact that the subscripts of the second variable in each term is a member of the class

$$(3, 5, 7, 9, \ldots) , \tag{30}$$

and therefore one must use care in applying (29) to delete any terms not in this class. For example, when m=0 the first four terms in (29) are

$$(x_{1c}x_{-1c} - x_{1s}x_{-1s}) + (x_{1c}x_{1c} + x_{1s}x_{1s}) ;$$

these terms should be ignored because they are not present in the original sum (17).

The results of this "sorting" process for the remaining three expressions in (20) are as follows:

$$\begin{aligned}\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\cos kt\}_{(k+n)\,s} = \\ (x_{1c}x_{(n-1)\,s} + x_{1s}x_{(n-1)\,c}) \\ + (x_{1c}x_{(n+1)\,s} - x_{1s}x_{(n+1)\,c}) \\ + (x_{1c}x_{(2k+n-1)\,s} + x_{1s}x_{(2k+n-1)\,c}) \\ + (x_{1c}x_{(2k+n+1)\,s} - x_{1s}x_{(2k+n+1)\,c})\end{aligned} \tag{31}$$

$$\begin{aligned}\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\sin kt\}_{(k+n)\,c} = \\ (x_{1c}x_{(n-1)\,s} + x_{1s}x_{(n-1)\,c}) \\ - (x_{1c}x_{(n+1)\,s} - x_{1s}x_{(n+1)\,c}) \\ + (x_{1c}x_{(2k+n-1)\,s} + x_{1s}x_{(2k+n-1)\,c}) \\ + (x_{1c}x_{(2k+n+1)\,s} - x_{1s}x_{(2k+n+1)\,c})\end{aligned} \tag{32}$$

and

$$\begin{aligned}\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\sin kt\}_{(k+n)\,s} = \\ (x_{1c}x_{(n-1)\,c} - x_{1s}x_{(n-1)\,s}) \\ + (x_{1c}x_{(n+1)\,c} + x_{1s}x_{(n+1)\,s}) \\ - (x_{1c}x_{(2k+n-1)\,c} - x_{1s}x_{(2k+n-1)\,s}) \\ - (x_{1c}x_{(2k+n+1)\,c} + x_{1s}x_{(2k+n+1)\,s})\,.\end{aligned} \tag{33}$$

The lower triangular elements of $A_1^{(2)}$ are defined by the formulas (29), (31), (32), and (33). If we represent the elements of $A_1^{(2)}$ as a_{ij}, then

$$
\begin{aligned}
&\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\cos kt\}_{(k+m)c} = a_{k+m,k}\\
&\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\cos kt\}_{(k+m)s} = a_{k+m+1,k}\\
&\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\sin kt\}_{(k+m)c} = a_{k+m-1,k+1}\\
&\{Nf_{xx}(x_o)\cdot(\bar{x}-x_o)\cdot\sin kt\}_{(k+m)s} = a_{k+m,k+1}\\
&(\text{for } k=1,3,5,\ldots \text{ and } m=0,2,4,\ldots)\ .
\end{aligned}
\tag{34}
$$

The remaining elements are defined by the symmetry of $A_1^{(2)}$.

The evaluation of the remaining matrices, $A_1^{(3)}$ and $A_1^{(4)}$ is trivial since they correspond to the linear terms in $f(x,\dot{x},t)$. From eqn. (3),

$$(I-N)f_x(x_o,\dot{x}_o,t) = -\beta$$

and (35)

$$(I-N)f_{\dot{x}}(x_o,\dot{x}_o,t) = -\alpha$$

Hence the matrices $A_1^{(3)}$ and $A_1^{(4)}$ are simply

$$
A_1^{(3)} = -\beta\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots\\
0 & 1 & 0 & 0 & \cdots\\
0 & 0 & 1 & 0 & \cdots\\
0 & 0 & 0 & 1 & \cdots\\
\vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\qquad
A_1^{(4)} = -\alpha\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots\\
-1 & 0 & 0 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 3 & 0 & 0 & \cdots\\
0 & 0 & -3 & 0 & 0 & 0 & \cdots\\
0 & 0 & 0 & 0 & 0 & 5 & \cdots\\
0 & 0 & 0 & 0 & -5 & 0 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\tag{36}
$$

5.2 Evaluation of the b Vector

The vector b is evaluated in the same manner as was A . Again we consider separately the various component terms of the vector and define

$$b_1 = -\begin{bmatrix} \{Nf(x_o,\dot{x}_o,t)\}_{1c} \\ \{Nf(x_o,\dot{x}_o,t)\}_{1s} \\ \{Nf(x_o,\dot{x}_o,t)\}_{3c} \\ \vdots \end{bmatrix} \quad ; \quad b_2 = -\begin{bmatrix} \{Nf_x(x_o,\dot{x}_o,t)\cdot[\bar{x}(t) - x_o(t)]\}_{1c} \\ \{Nf_x(x_o,\dot{x}_o,t)\cdot[\bar{x}(t) - x_o(t)]\}_{1s} \\ \{Nf_x(x_o,\dot{x}_o,t)\cdot[\bar{x}(t) - x_o(t)]\}_{3c} \\ \vdots \end{bmatrix}$$

$$b_3 = -\begin{bmatrix} \{(I-N)f(\bar{x}(t),\dot{\bar{x}}(t),t)\}_{1c} \\ \{(I-N)f(\bar{x}(t),\dot{\bar{x}}(t),t)\}_{1s} \\ \{(I-N)f(\bar{x}(t),\dot{\bar{x}}(t),t)\}_{3c} \\ \vdots \end{bmatrix} \quad ; \quad b_4 = \begin{bmatrix} \{\ddot{\bar{x}}\}_{1c} \\ \{\ddot{\bar{x}}\}_{1s} \\ \{\ddot{\bar{x}}\}_{3c} \\ \{\ddot{\bar{x}}\}_{3s} \\ \vdots \end{bmatrix} . \tag{37}$$

The first component, b_1 , is evaluated by observing that

$$Nf(x_o,\dot{x}_o,t) = -\alpha x_o(t)^3 = -\alpha h(x_{1c},x_{1s},1 : i=1,2,3) \tag{38}$$

i.e., that the parameters in expression (II-33) are

$$\begin{aligned} A_1 &= A_2 = A_3 = x_{1c} \\ B_1 &= B_2 = B_3 = x_{1s} \\ \omega_1 &= \omega_2 = \omega_3 = 1 . \end{aligned} \tag{39}$$

The result of substituting (39) into (II-33) is

$$\begin{aligned} Nf(x_o,\dot{x}_o,t) = \frac{3}{4}\gamma[&(x_{1c}^3 + x_{1c}x_{1s}^2)\cos t \\ &+ (x_{1s}^3 + x_{1s}x_{1c}^2)\sin t \\ &+ (\tfrac{1}{3}x_{1c}^3 - x_{1c}x_{1s}^2)\cos 3t \\ &+ (-\tfrac{1}{3}x_{1s}^3 + x_{1c}^2x_{1s})\sin 3t] . \end{aligned} \tag{40}$$

(Note that this is the expression which was required for the application of the first order Galerkin method in chapter II.) The component b_1 is therefore

$$b_1 = \frac{3}{4}\gamma \begin{bmatrix} x_{1c}^3 + x_{1c}x_{1s}^2 \\ x_{1s}^3 + x_{1s}x_{1c}^2 \\ \frac{1}{3}x_{1c}^3 - x_{1c}x_{1s}^2 \\ -\frac{1}{3}x_{1s}^3 + x_{1c}^2 x_{1s} \\ \vdots \end{bmatrix} . \tag{41}$$

To evaluate b_2 , note that

$$Nf_x(x_o,\dot{x}_o,t) \cdot [\bar{x}(t)-x_o(t)] = -3\alpha x_o^2(t) \cdot \sum_{j=3,5,\ldots} (x_{jc} \cos jt + x_{js} \sin jt) \tag{42}$$

$$= -3\alpha \sum_{j=3,5,\ldots} h(x_{jc},x_{js},1;x_{1c},x_{1s},1;x_{1c},x_{1s},1)$$

In other words, we need to evaluate (II-33) for the parameter values

$$\begin{aligned} A_1 &= x_{jc} & B_1 &= x_{js} & \omega_1 &= j \\ A_2 &= A_3 = x_{1c} & B_2 &= B_3 = x_{1s} & \omega_2 &= \omega_3 = 1 . \end{aligned} \tag{43}$$

The result of substituting (43) into (II-33) is

$$\begin{aligned} &Nf_x(x_o,\dot{x}_o,t) \cdot [\bar{x}(t) - x_o(t)] = \\ &\quad -\frac{3}{4}\gamma \sum_{j=3,5,7\ldots}^{p} [(2x_{jc}[x_{1c}^2 + x_{1s}^2]) \cos jt \\ &\qquad + (x_{jc}(x_{1c}^2-x_{1s}^2)+2x_{js}(x_{1c}x_{1s})) \cos (j-2)t \\ &\qquad + (x_{jc}(x_{1c}^2-x_{1s}^2)-2x_{js}(x_{1c}x_{1s})) \cos (j+2)t \\ &\qquad + (2x_{js}(x_{1c}^2+x_{1s}^2)) \sin jt \\ &\qquad + (x_{js}(x_{1c}^2-x_{1s}^2)-2x_{jc}(x_{1c}x_{1s})) \sin (j-2)t \\ &\qquad + (x_{js}(x_{1c}^2-x_{1s}^2)+2x_{jc}(x_{1c}x_{1s})) \sin (j+2)t] \end{aligned} \tag{44}$$

and so for $k = 1,3,5,\ldots$

$$\begin{aligned} &\{Nf_x(x_o,\dot{x}_o,t) \cdot [\bar{x}(t) - x_o(t)]\}_{kc} = \\ &\quad 2x_{kc}[x_{1c}^2 + x_{1s}^2] \\ &\quad + x_{(k+2)c}[x_{1c}^2 - x_{1s}^2] + 2x_{(k+2)s}x_{1c}x_{1s} \\ &\quad + x_{(k-2)c}[x_{1c}^2 - x_{1s}^2] + 2x_{(k-2)s}x_{1c}x_{1s} \end{aligned} \tag{45}$$

where subscripts of the first variable in each term above are constrained to be in the set

$$(3,5,7,9,\ldots) \; . \tag{46}$$

Similarly,

$$\begin{aligned}\{Nf_x(x_o,\dot{x}_o,t) \cdot [\bar{x}(t) - x_o(t)]\}_{ks} = \\ 2x_{ks}(x_{1c}^2 + x_{1s}^2) \\ + x_{(k+2)s}(x_{1c}^2 - x_{1s}^2) - 2x_{(k+2)c}x_{1c}x_{1s} \\ + x_{(k-2)s}(x_{1c}^2 - x_{1s}^2) + 2x_{(k-2)c}x_{1c}x_{1s} \; .\end{aligned} \tag{47}$$

To save space, define

$$\begin{aligned} a &= 2(x_{1c}^2 + x_{1s}^2) \\ b &= x_{1c}^2 - x_{1s}^2 \\ c &= 2x_{1c}x_{1s} \; . \end{aligned}$$

Then

$$b_2 = \frac{3}{4}\gamma \begin{bmatrix} bx_{3c} + cx_{3s} \\ -cx_{3c} + bx_{3s} \\ ax_{3c} + bx_{5c} + cx_{5s} \\ ax_{3s} - cx_{5c} + bx_{5s} \\ bx_{3c} - cx_{3s} + ax_{5c} + bx_{7c} + cx_{7s} \\ cx_{3c} + bx_{3s} + ax_{5s} + bx_{7s} - cx_{7c} \\ \vdots \end{bmatrix} \; . \tag{48}$$

The remaining two vectors correspond to the linear terms in $\epsilon'(\bar{x}(t),t)$ and their evaluation is trivial. To evaluate b_3,

$$
\begin{aligned}
(I-N)f(\bar{x}(t), \dot{\bar{x}}(t), t) &= -\alpha\dot{\bar{x}}(t) - \beta\bar{x}(t) + \eta\cos t \\
&= -\alpha\sum_{j=1}^{p}(-jx_{jc}\sin jt + jx_{js}\cos jt) \\
&\quad -\beta\sum_{j=0}^{p}(x_{jc}\cos jt + x_{js}\sin jt) \\
&\quad +\eta\cos t
\end{aligned}
\tag{49}
$$

so that

$$
b_3 = -\begin{bmatrix} -\alpha x_{1s} - \beta x_{1c} + \eta \\ +\alpha x_{1c} - \beta x_{1s} \\ -3\alpha x_{3s} - \beta x_{3c} \\ +3\alpha x_{3c} - \beta x_{3s} \\ \vdots \\ \vdots \end{bmatrix} . \tag{50}
$$

And for b_4,

$$
\ddot{\bar{x}}(t) = -\sum_{j=1}^{p} j^2(x_{jc}\cos jt + x_{js}\sin jt) \tag{51}
$$

so that

$$
b_4 = -\begin{bmatrix} x_{1c} \\ x_{1s} \\ 9x_{3c} \\ 9x_{3s} \\ 25x_{5c} \\ 25x_{5s} \\ \vdots \end{bmatrix} . \tag{52}
$$

This completes the evaluation of all the terms needed to apply the method of the last chapter. Collecting the various components which have been evaluated separately in this section, the complete matrix system is

$$[A_1^{(1)} + A_1^{(2)} + A_1^{(3)} + A_1^{(4)} + A_2]z = b_1 + b_2 + b_3 + b_4 \ , \quad (53)$$

where the matrix A_2 was defined in Chapter IV, eqn. (48).

5.3 An Additional Correction Term for the b Vector

Before reporting the numerical results obtained for a particular example in the next section, a correction term for the b vector will be evaluated which for the simple system of this chapter results in an improved approximation with little increase in effort. If in the Taylor series expansion of the equation error we attempt to retain one additional term

$$\begin{aligned} \epsilon'(\bar{x},t) \doteq \ddot{\bar{x}} - f(\bar{x},\dot{\bar{x}},t) = \ddot{\bar{x}} - f(x_0,\dot{x}_0,t) - f_x(x_0)\cdot(\bar{x}-x_0) \\ - f_{\dot{x}}(\dot{x}_0)\cdot(\dot{\bar{x}}-\dot{x}_0) - \underline{\tfrac{1}{2}f_{xx}(x_0)\cdot(\bar{x}-x_0)^2} \ , \end{aligned} \quad (54)$$

this requires the expansion of

$$[\sum_{j=1}^{p}(x_{jc}\cos jt + x_{js}\sin jt)]^2 \ , \quad (55)$$

which would cause a considerable increase in the amount of analytic effort necessary to apply the iterative method. In addition, it is clear that such a correction in the general n-dimensional case would severely complicate the method. For this reason, this correction term was not discussed in the presentation of the general method in Chapter III. In this simple scalar case, however, we can rather easily account for the second order effects of the third harmonics by substituting (for the underlined term) in (54) the correction term

$$-\tfrac{1}{2}f_{xx}(x_0)\cdot(x_{3c}\cos 3t + x_{3s}\sin 3t)^2 \ . \quad (56)$$

This is easy to evaluate in this case since for

$Nf(x) = -\gamma x^3$, $f_{xx}(x_o) = -6\gamma x_o$, and so

$$-\tfrac{1}{2} f_{xx}(x_o) \cdot (x_{3c} \cos 3t + x_{3s} \sin 3t)^2 = + 3\gamma\, h(x_{1c},x_{1s},1;\ x_{3c},x_{3s},3;\ x_{3c},x_{3s},3) \tag{57}$$

and hence we can use the same direct substitution technique into (II-33) which was used for the other components of A and b . Substituting the values

$$\begin{aligned} A_1 &= x_{1c} & B_1 &= x_{1s} & \omega_1 &= 1 \\ A_2 &= A_3 = x_{3c} & B_2 &= B_3 = x_3 s & \omega_2 &= \omega_3 = 3 \end{aligned} \tag{58}$$

into expression (II-33) results in

$$\begin{aligned} -\tfrac{1}{2} f_{xx}(x_o) \cdot (x_{3c} \cos 3t + x_{3s} \sin 3t)^2 = \\ + \frac{3}{4}\gamma[&2x_{1c}(x_{3c}^2+x_{3s}^2) \cos t \\ &+ (x_{1c}(x_{3c}^2+x_{3s}^2) + x_{1s}(2x_{3c}x_{3s})) \cos 5t \\ &+ (x_{1c}(x_{3c}^2-x_{3s}^2) - x_{1s}(2x_{3c}x_{3s})) \cos 7t \\ &+ 2x_{1s}(x_{3c}^2+x_{3s}^2) \sin t \\ &- (x_{1s}(x_{3c}^2-x_{3s}^2) - x_{1c}(2x_{3c}x_{3s})) \sin 5t \\ &+ (x_{1s}(x_{3c}^2-x_{3s}^2) + x_{1c}(2x_{3c}x_{3s})) \sin 7t]. \end{aligned} \tag{59}$$

To save space, we define the parameters

$$\begin{aligned} a &= 2(x_{3c}^2 + x_{3s}^2) \\ b &= x_{3c}^2 - x_{3s}^2 \\ c &= 2x_{3c}x_{3s} \quad . \end{aligned} \tag{60}$$

Then the additional component to be added to the b vector is

$$b_5 = \frac{3}{4}\gamma \begin{bmatrix} ax_{1c} \\ ax_{1s} \\ 0 \\ 0 \\ bx_{1c} + cx_{1s} \\ -bx_{1s} + cx_{1c} \\ bx_{1c} - cx_{1s} \\ bx_{1s} + cx_{1c} \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad . \tag{61}$$

It should be noted that the inclusion of the above correction vector can be expected to result in an improvement only for those oscillations having large third harmonic content relative to the fifth and higher harmonics.

5.4 Numerical Results

In this section, numerical results obtained by applying the iterative method to the specific example

$$\ddot{x} + 0.3\dot{x} + 10x - 0.1x^3 = 8 \cos(\omega t) \tag{62}$$

(for values of the forcing frequency in the interval $1 < \omega < 4$) will be reported. The results of the iterative method will be compared with both the first order harmonic balance results and with the exact solutions as obtained by numerical integration.

The bulk of the results will be presented in tabular and graphical form; for clarity, however, the results for one particular value of ω

($\omega = 1.5$) are shown explicitly below. The expansions are complete through the fifth harmonic; it was observed during the numerical iteration that increasing the matrix dimension to the 7^{th} harmonic and higher had negligible effect on the accuracy.

The first order Galerkin solutions to (62) with $\omega = 1.5$ were obtained by numerically solving the cubic determining equations (II-36).[2] At the above driving frequency, the steady state solution of (62) is not unique: there are three distinct periodic functions of time which satisfy (62). To conserve space, numerical results for only one of the three will be reported here. The chosen Galerkin solution is

$$x(t) = -8.49 \cos 1.5t + 6.27 \sin 1.5t \quad . \tag{63}$$

Using (63) as a starting point for the iterative procedure, the resulting approximate solution is

$$\bar{x}(t) = -7.9763 \cos 1.5t + 7.5753 \sin 1.5t - 0.829 \cos 3(1.5)t - 0.875 \sin 3(1.5)t + 0.11317 \cos 5(1.5)t - 0.1026 \sin 5(1.5)t + \ldots \tag{64}$$

and the true solution is

$$\hat{x}(t) = -7.9751 \cos 1.5t + 7.5610 \sin 1.5t - 0.835 \cos 3(1.5)t - 0.885 \sin 3(1.5)t + 0.11327 \cos 5(1.5)t - 0.1017 \sin 5(1.5)t + \ldots \tag{65}$$

The improvement in the accuracy of the fundamental components in the above example is typical of that which has been observed in all other examples studied: percentage error is reduced by factors of between 10^2 and 10^3.

---

[2] These two coupled cubic equations can be reduced to one cubic equation in the magnitude $\sqrt{x_{1c}^2 + x_{1s}^2}$; see Hayashi [5], p. 29.

Since it is necessary in the iterative method to perform the iterations numerically (i.e., using a digital computer), it is appropriate to compare the time required for the iteration method with the time required to perform the numerical integration. In the above example, the oscillation was observed to be unstable; consequently, the solution obtained by numerical integration shown in (65) was very difficult to obtain. The result (64) of the iterative method was used as a first "guess," and a rather sophisticated technique [13] for predicting new initial conditions based on initial and final values (over one period) was necessary. It was not possible to obtain the true solution for this example starting from initial conditions calculated from the Galerkin solution (63), because the instability of the oscillation was such that the integration procedure diverged in a mere fraction of the period. Under such circumstances, the iterative method (or some other means of approximating high order Galerkin solutions such as that of Urabe-Reiter [9]) obviously possesses a great practical advantage over direct numerical integration of the differential equations. Even in the case of oscillations which were stable, the time required for the integration procedure was, at best, an order of magnitude greater than that required for the iterative method.

Comparisons between the results of the iterative method, the first order Galerkin solution, and the true solution for the system (62) with various forcing frequencies are shown in the plots on the next page, and in Table 1 on page 21. The plots are of the fundamental components x_{1c} and x_{1s} ; the error of the iterative solution is too small to be seen on these graphs. The non-uniqueness of steady state responses in nonlinear systems is clearly evident.

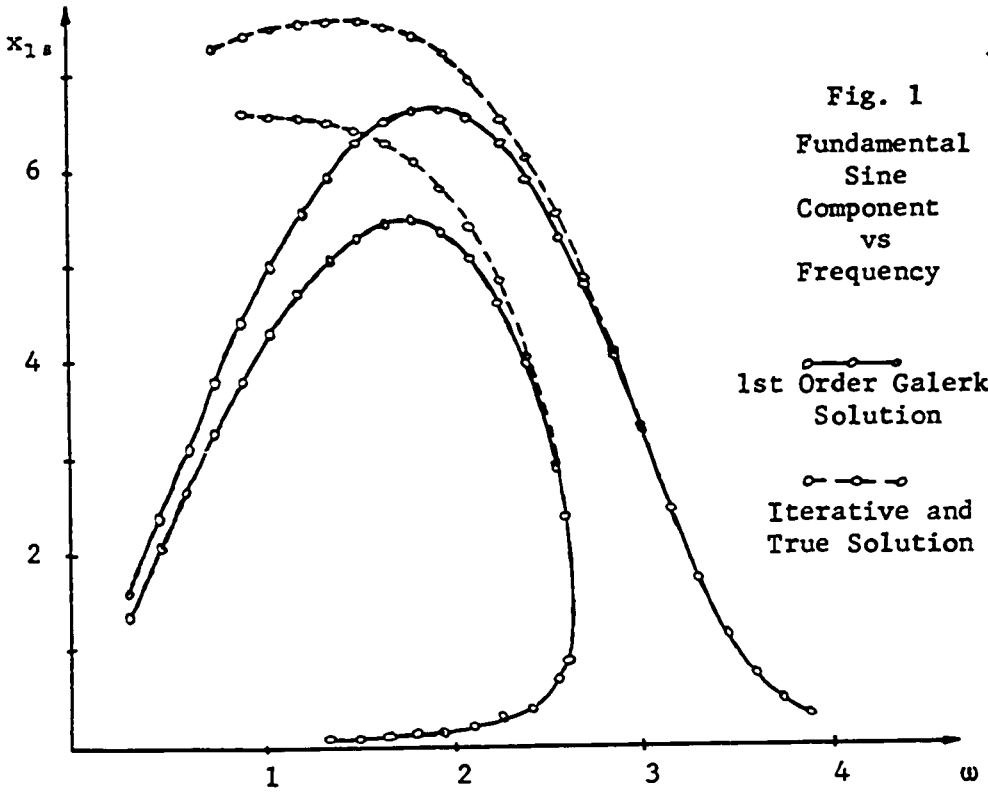

Fig. 1

Fundamental Sine Component vs Frequency

1st Order Galerkin Solution

Iterative and True Solution

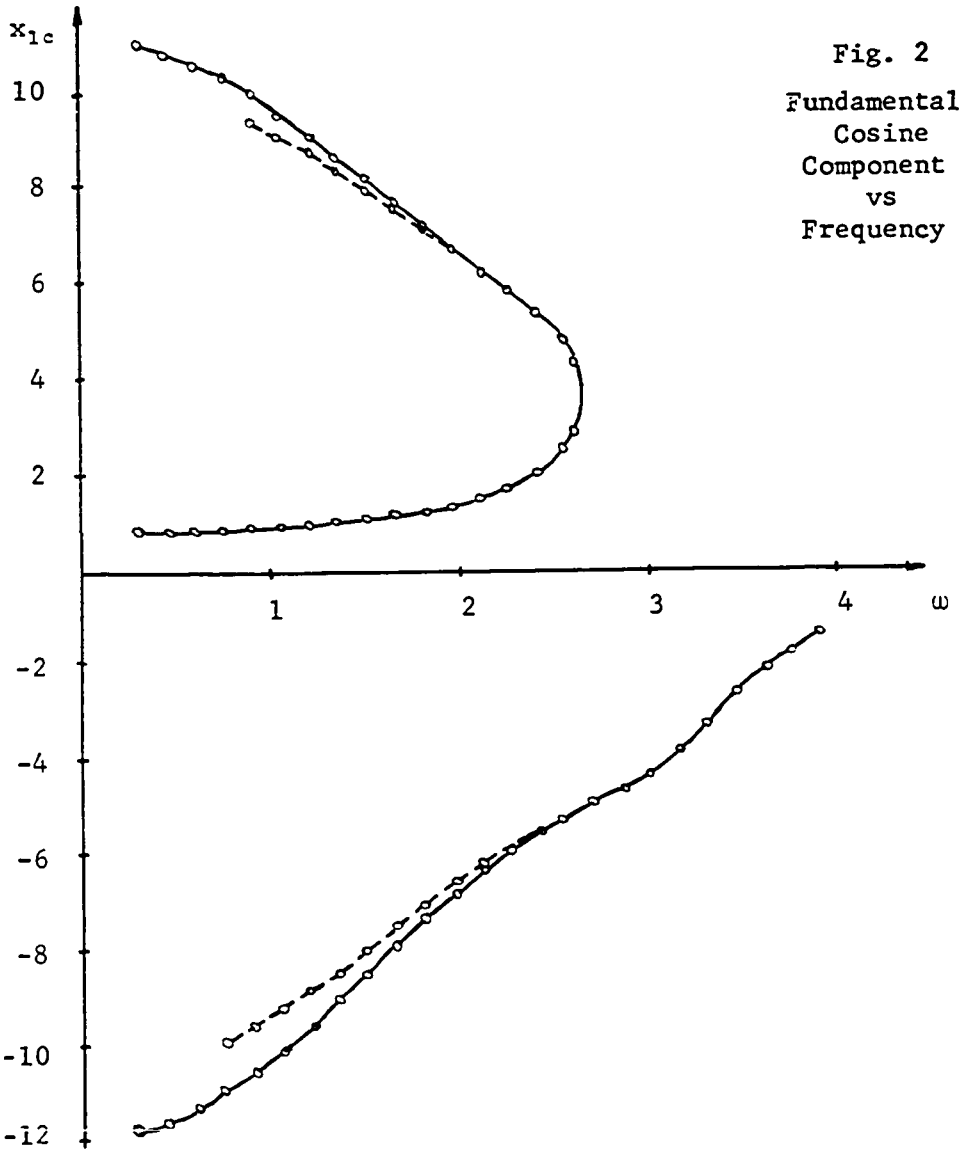

Fig. 2

Fundamental Cosine Component vs Frequency

TABLE 1

COMPARISON OF THE FIRST ORDER GALERKIN SOLUTION, THE ITERATIVE SOLUTION, AND THE TRUE SOLUTION FOR THE SYSTEM

$$\ddot{x} + 0.3\dot{x} + 10x - 0.1x^3 = 8 \cos(\omega t)$$

FORCING FREQUENCY	1st HARMONIC COMPONENTS		3rd HARMONIC COMPONENTS		5th HARMONIC COMPONENTS	
	x_{1c}	x_{1s}	x_{3c}	x_{3s}	x_{5c}	x_{5s}
$\omega = 1.5$						
GALERKIN	-8.49	6.2697				
ITERATIVE	-7.9751	7.5610	-0.835	-0.885	0.11327	-0.1017
EXACT	-7.9763	7.5753	-0.829	-0.875	0.11317	-0.1026
$\omega = 1.8$						
GALERKIN	7.14	5.45				
ITERATIVE	7.0340	6.089	0.3144	-0.551	-0.0430	-0.0155
EXACT	7.0341	6.093	0.3143	-0.550	-0.0431	-0.0156
$\omega = 2.4$						
GALERKIN	-5.545	5.90				
ITERATIVE	-5.4698	6.1098	-0.22292	-0.14540	0.003872	-0.008146
EXACT	-5.4697	6.1102	-0.22288	-0.14534	0.003870	-0.008138
$\omega = 2.85$						
GALERKIN	-4.6380	4.065				
ITERATIVE	-4.636939	4.11394	-0.051882	-0.069164	0.0010566	-0.0006408
EXACT	-4.636940	4.11396	-0.051883	-0.069165	0.0010564	-0.0006407

In Table 1, numerical results (through the fifth harmonic) are shown for four different values of the forcing frequency. (The first case is a repetition of the solutions shown explicitly in equations (63), (64), (65); this redundancy should help clarify the meaning of the entries in Table 1.)

5.5 Conclusions

In this chapter, it was explicitly shown how the elements of the matrix A and the vector b are obtained for the Duffing equation. The evaluations were performed in such a way that the dimension of the linear algebraic system can be chosen arbitrarily at each stage of the numerical iteration. Although in this chapter these techniques are specifically applied to the cubic nonlinearity, they are directly applicable to the general case as well.

As demonstrated in Table 1 and in (63) and (64), the improvement obtained using the iterative method can be quite significant.

CHAPTER VI

CONCLUSIONS

6.1 Final Comments on the Nature of the Thesis Results

In this thesis, we have been concerned with the problem of determining steady state oscillations resulting from excitation of the same period. Two methods have been presented which, for polynomial nonlinearities, generate corrections to the first order Galerkin solution. The methods may be viewed as analytical approximations to the higher order Galerkin procedures. The value of these methods lies in the fact that the m^{th} order Galerkin determining equations are generally impractical to obtain analytically if $m > 1$, and hence the ability to approximate them is an important practical consideration.

The first of the two methods consists of a set of approximations which, for odd nonlinearities, allows Hayashi's [5] linearized third order Galerkin equations to be obtained directly from the first order Galerkin results. A savings in analytic effort is achieved as a result of the fact that only first order Galerkin equations are required.

The second of the two methods is an iterative procedure which is more general than the above method in that it approximates the m^{th} order Galerkin solution for arbitrary m. A significant distinction of this method relative to other analytic methods is the fact that the first order effects of essentially every harmonic component are taken into account (yet the analytic effort necessary to apply the procedure is comparable with the first order Galerkin method). As illustrated in the example of Chapter V, the accuracy of the iterative method can be quite impressive.

6.2 Suggested Extensions

There are several directions in which the methods of this thesis might be profitably extended. Hayashi [5] makes extensive use of the harmonic balance procedure in investigations of super- and sub-harmonic oscillations, and of spontaneous oscillations in unforced (autonomous) systems. Combinations of the above types of behavior (e.g., almost periodic oscillations and the entrainment of frequency [5]) in which non-harmonically related oscillations exist simultaneously can also be studied by the Galerkin method. Although in this thesis we have been concerned with oscillations of the same period as the excitation, the basic ideas behind the methods may be extendible to the above phenomena.

Attention has also been restricted in this thesis to polynomial nonlinearities; the methods of Chapters III and IV may be useful in the analysis of systems with other types of nonlinearities. In the case of piecewise linear characteristics (whose derivatives fail to exist at a finite number of points), the method of Chapter IV can be applied, at least formally, if impulses are allowed. Whether the truncated Taylor series approximations are meaningful in such a case is not clear.

As a final comment, it should be noted that the restriction to systems with polynomial nonlinearities is not a severe limitation in practice. Polynomials of reasonably low order can be used to simulate a wide variety of nonlinear characteristics and hence the applicability of the thesis results is actually quite broad.

APPENDIX

CONVERGENCE OF NEWTON'S METHOD

In this appendix, it will be shown how Kantorović's theorem [12] on the convergence of Newton's method (as presented by Rall [10]) can be applied to the particular algebraic equations

$$F_p(Y_p) = 0 \quad \text{and} \quad G_p(X_p) = 0 \tag{1}$$

defined in Chapter IV. Recall that G_p, X_p, F_p, and Y_p are p-dimensional vectors whose elements are themselves n-dimensional vectors. The statement of Kantorović's theorem will be significantly simplified if we consider the above vectors to have $p \cdot n = q$ scalar elements, rather than p n-dimensional elements. By so doing, we can frame the theorem in the well-known vector space R_∞^q, which utilizes the easily computed Chebychev norm. Let us define the q-dimensional vector $\bar{G}$ (whose elements $\bar{G}_i$, $i=1,2,\ldots,q$, are scalars) in the following manner: if we denote the k^{th} element of G_p as a_k (a_k is an n-vector), and the i^{th} element of a_k as a_k^i (a_k^i is a scalar), then

$$a_k^i = \bar{G}_{(k-1)n+i} \quad , \quad \begin{matrix} k=1,2,\ldots,p \\ i=1,2,\ldots,n \end{matrix} \quad . \tag{2}$$

Thus the vector G_p is a partition of the vector $\bar{G}$ into p n-dimensional segments. In a similar manner, we define q-vectors $\bar{X}$, $\bar{F}$, and $\bar{Y}$, corresponding to the p-vectors X_p, F_p, and Y_p.

We can now use Rall's statement of Kantorović's sufficient conditions for the convergence of

$$\bar{X}^{(n+1)} = \bar{X}^{(n)} - \left[\frac{\partial \bar{G}(\bar{X}^{(n)})}{\partial \bar{X}}\right]^{-1} \cdot \bar{G}(\bar{X}^{(n)}) \; . \tag{3}$$

we first assume that the starting point $X^{(o)}$ is such that $[\partial\overline{G}(\overline{X}^{(o)})/\partial\overline{X}]^{-1}$ exists. We can then use (3) to compute $\overline{X}^{(1)}$, from which we determine two constants which satisfy

$$\eta_o \geq \max_{(i)} |\overline{X}_i^{(1)} - \overline{X}_i^{(o)}| \tag{4}$$

and

$$B_o \geq \max_{(i)} \sum_{j=1}^{q} |h_{ij}(\overline{X}^{(o)})| , \tag{5}$$

where the h_{ij}, $i,j=1,2,\ldots,q$ are the elements of the inverse of the Jacobian:

$$H(\overline{X}) \triangleq [\frac{\partial\overline{G}(\overline{X})}{\partial X}]^{-1} . \tag{6}$$

we define the <u>closed ball</u> $\overline{U}(\overline{X}^{(o)},r)$, of center $\overline{X}^{(o)}$ and radius r, as

$$\overline{U}(\overline{X}^{(o)},r) \triangleq \{\overline{X} \in R_\infty^q : \max_{(i)} |\overline{X}_i - \overline{X}_i^{(o)}| < r\} . \tag{7}$$

Finally, we define the scalar function $\alpha(\overline{X})$ as

$$\alpha(\overline{X}) \triangleq \max_{(i)} \sum_{j=1}^{q} \sum_{k=1}^{q} |g_{ijk}(\overline{X})| , \text{ where} \tag{8}$$

$$g_{ijk} \triangleq \frac{\partial^2\overline{G}_i(\overline{X})}{\partial\overline{X}_k\partial\overline{X}_j} , \quad i,j,k = 1,2,\ldots,q . \tag{9}$$

In terms of the above definitions, Kantorović's theorem states that if

$$\alpha(\overline{X}) \leq K$$

in some closed ball $\overline{U}(\overline{X}^{(o)},r)$ and

$$h_o \triangleq B_o\eta_oK \leq \tfrac{1}{2} ,$$

then the Newton sequence (3), starting from the initial point $\overline{X}^{(o)}$,

will converge to a solution $\bar{X}^*$ of

$$\bar{G}(\bar{X}) = 0$$

which exists in $\bar{U}(\bar{X}^{(o)}, r)$, provided that

$$r \geq \frac{1 - \sqrt{1 - 2\, h_o}}{h_o}\, \eta_o \ .$$

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