M-BAND MULTIWAVELET SYSTEMS

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ABSTRACT

In this paper we investigate multiwavelet systems whose scaling functions have disjoint support. We demonstrate that, with the exception of a trivial case, this property may not be attained by two band multiwavelet systems. We show that to enjoy this property, it is indeed necessary to invoke M-band multiwavelet systems. This indicates the existence of tilings of the time-frequency plane that may be obtained with M-band multiwavelet systems but not with two band multiwavelet systems. Hence M-band multiwavelet systems are inherently more powerful than two band multiwavelet systems and deserve a thorough investigation. Finally, we derive K^{th} order balancing conditions for M-band multiwavelet systems. These conditions will enable M-band multiwavelet systems to be used for practical digital signal processing applications.

1. INTRODUCTION

Two band wavelet systems are powerful because they may be designed to possess certain useful properties. Unfortunately, a given two band wavelet system may not possess all such useful properties simultaneously, and we may be forced to choose a subset of the desired properties for a particular application [1].

On the other hand, two band multiwavelets have attracted considerable attention due to their ability to simultaneously offer a larger subset of useful properties than is possible with wavelet systems. In particular, multiwavelet systems can simultaneously offer real-valued basis functions with compact support, specified approximation order, orthogonality and symmetry [2, 3, 4, 5, 6, 7, 8]. Since two band multiwavelets operate upon vectors, they may be designed to take advantage of structure along a single time series as well as structure along different time series [9]. A significant disadvantage that accompanies two band multiwavelet systems is the urgent need to prefilter data due to spectral imbalance in the multifilter banks [10, 11, 12]. Lebrun et al. [13] and Selesnick [14] circumvent the prefiltering problem by designing "balanced" multiwavelets.

M-Band wavelet systems have also been the focus of several recent investigations [15, 16, 17, 18]. Noteworthy advantages of M-band wavelet systems over two band wavelet systems are their richer parameter space leading to a greater variety of compactly supported wavelets that are practically implementable, and their ability to more rapidly achieve a given frequency resolution as a function of decomposition scale, providing greater freedom and flexibility in choosing time-frequency tilings [19].

In this paper, we demonstrate that, except for a trivial case, multiwavelet systems whose scaling functions have disjoint support must be M-band multiwavelet systems with $M > 2$. This implies the existence of a time-frequency tiling which exists for M-band multiwavelet systems but not for two band multiwavelet systems. Hence it is probable that M-band multiwavelet systems are inherently more powerful than two band multiwavelet systems. In the final section, we prove that Selesnick's K^{th} order balancing conditions [14] may be extended to M-band multiwavelet systems. These conditions will render M-band multiwavelet systems to be of practical use in digital signal processing applications.

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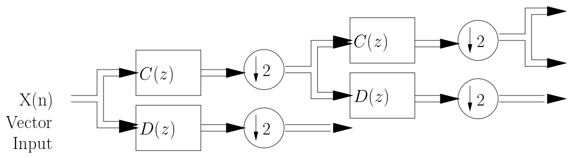

Figure 1: Analysis bank for two band multiwavelet system

2. PRELIMINARIES

First, we formally define the support set of a measurable function, f, from the real $\mathcal{R}$ to the complex $\mathcal{C}$ as done by So et al. in [20]. A point $t \in \mathcal{R}$ is called a *support point* of f if the measure of the intersection $\{x : f(x) \neq 0\} \cap (t-\epsilon, t+\epsilon)$ is not zero for any $\epsilon > 0$. Then the support set of f, denoted by $\text{supp}(f)$ is defined as the convex hull of the set of support points of F. For $F = [f_1, \ldots, f_r]$, a vector of functions, we define

$$\text{supp}(F) = \text{conv}\left(\cup_{i=1}^{r} \text{supp}(f_i)\right)$$

where "conv" refers to the convex hull of a set.

In a two band multiwavelet system (figure 1), translates of a scaling vector $\Phi(t) = [\phi_1(t), \ldots, \phi_r(t)]^T$ are used to span a subspace of $\mathcal{L}^2(\mathcal{R})$, where r is a positive integer and T refers to a matrix transpose operator. $\phi_1(t), \ldots, \phi_r(t)$ are the component multiscaling functions.

The matrix dilation equation is

$$\Phi(t) = \sqrt{2} \sum_n C_n \Phi(2t - n), \tag{1}$$

where C_n are $r \times r$ matrices referred to as the lowpass matrix filter coefficients. The scaling vector, $\Phi(t)$ is obtained by solving the matrix dilation equation (1). We shall view the sequence C_n as an r-input r-output matrix scaling filter for which we adopt the following notation:

$$[C_n]_{i,j} = h_i(nr + j), \tag{2}$$

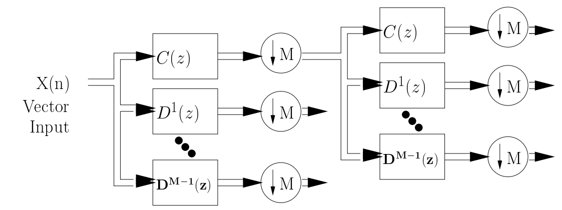

Figure 2: Analysis bank for M-band multiwavelet system

where $[C_n]_{i,j}$ is the $(i,j)^{\text{th}}$ entry of the matrix C_n. Let us define

$$C(z) = \sum_{n=0}^{N} C_n z^n \tag{3}$$

to be the lowpass *matrix symbol* associated with the matrix scaling coefficients, C_n. $D(z)$ is the associated highpass matrix symbol in figure (1).

Figure (2) shows the analysis bank of an M-band multiwavelet system. The matrix dilation equation associated with this system is

$$\Phi(t) = \sqrt{M}\sum_n C_n \Phi(Mt-n). \tag{4}$$

$D^l(z), 1 \leq l \leq (M-1)$ are the bandpass matrix symbols associated with the lowpass matrix symbol $C(z)$. The bandpass multiwavelets associated with the system are given by

$$\Psi^l(t) = \sqrt{M}\sum_n D_n^l \Phi(Mt-n), \quad 1 \leq i \leq (M-1). \tag{5}$$

In the synthesis bank, we use the notation,

$$[A_n]_{i,j} = f_i(nr+j)$$

to represent the lowpass matrix filter coefficients and its entries.

3. THE DISJOINT SUPPORT CRITERION

Given a scaling vector, $\Phi(t) = [\phi_1(t), \ldots, \phi_r(t)]^T$, the disjoint support property of the scaling functions requires

$$\cap_{i\in\{1,\ldots,r\},k\in\mathcal{Z}} \ \operatorname{supp}(\phi_i(t-k)) = \emptyset. \tag{6}$$

We shall obtain the disjoint support property by requiring the support sets of the scaling functions to be determined by

$$\operatorname{supp}(\phi_i(t)) = [N_i, N_{i+1}], \quad 1 \leq i \leq r, \tag{7}$$

where $N_i \in [0,1], N_i < N_{i+1}, N_1 = 0$ and $N_{r+1} = 1$. Equation (7) represents the disjoint support criterion.

4. OBTAINING DISJOINT SCALING FUNCTIONS

Assume that a two band multiwavelet system meets the disjoint support criterion. Notice that in the dilation equation (1),

$$\operatorname{supp}(\Phi(t)) = [0,1] \tag{8}$$

and

$$\operatorname{supp}(\Phi(2t-n)) = [n/2, (n+1)/2] \tag{9}$$

Hence in order to satisfy the disjoint support criterion we must have $0 \leq n \leq 1$. It is also evident that only the first row of C_0 must have all non-zero entries while all other rows must have zero entries. This is to ensure that only $\phi_1(t)$ has support $[0, N_2] = [0, 0.5]$. Similarly, the second row of C_1 must have all non-zero entries, while all other entries must be zero. This guarantees that $\phi_2(t)$ has support $[N_2, N_3] = [0.5, 1]$. None of $\phi_3(t), \ldots, \phi_r(t)$ may satisfy the disjoint support criterion, since $N_3 = 1$. It is clear that two band multiwavelet systems can only fulfill the disjoint support criterion if they have two matrix filter coefficients with four non-zero entries and the following structure:

$$C_0 = \begin{bmatrix} h_1(1) & h_1(2) \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ h_2(3) & h_2(4) \end{bmatrix}$$

In order to obtain non-trivial instances of the disjoint support property, we must turn to M-band multiwavelet systems with $M = r$. Applying the same reasoning to the M-band multiwavelet dilation equation (4), we observe that

$$\operatorname{supp}(\Phi(rt-n)) = \left[\frac{n}{r}, \frac{n+1}{r}\right], \tag{10}$$

any M-band multiwavelet system possessing the disjoint support property must have r matrix coefficients $C_0, \ldots, C_{r-1}$. Matrix coefficient C_i will have a non-zero $(i+1)^{th}$ row while all other entries are zero. The structure of the M-band multiwavelet system that satisfies the disjoint support criterion is now revealed to be

$$[C_n]_{i,j} = \begin{cases} 0, & n+1 \neq i, \\ h_i(nr+j), & \text{otherwise.} \end{cases} \tag{11}$$

The coefficients may also be written as

$$C_0 = \begin{bmatrix} h_1(1) & \ldots & h_1(r) \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 0 & \ldots & 0 \\ h_2(1) & \ldots & h_2(r) \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix},$$
$$\vdots$$
$$C_{r-1} = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ h_r(1) & \ldots & h_r(r) \end{bmatrix}.$$

The supports of each of the scaling functions are determined by:

$$N_i = \frac{i-1}{r}, \quad 1 \leq i \leq r. \tag{12}$$

It is interesting to note that the above reasoning may be easily extended to equation (5) in order to obtain bandpass multiwavelets with disjoint support. In particular, the multiwavelets in the l^{th} band will satisfy

$$\operatorname{supp}(\psi_i^l(t)) = [N_i, N_{i+1}], \quad 1 \leq i \leq r, \tag{13}$$

where $N_i \in [0,1], \ \ N_i < N_{i+1}, \ \ N_1 = 0$ and $N_{r+1} = 1$ if and only if the bandpass matrix symbol $D^l(z)$ is determined by matrix coefficients $D_n^l, \ \ 0 \leq n \leq r-1$, with the following structure:

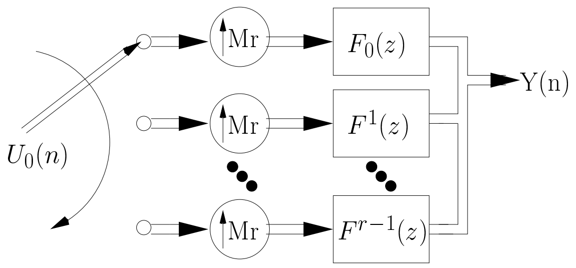

Figure 3: Time-varying Lowpass Synthesis Subband for M-band multiwavelet system

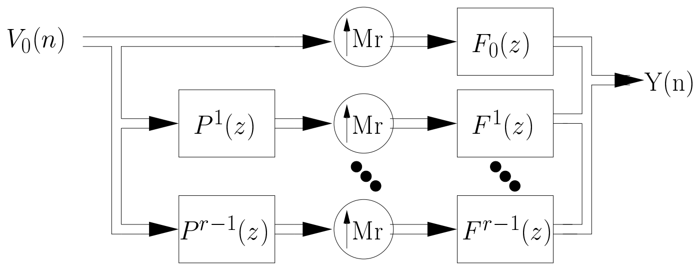

Figure 4: Output Derived From Polyphase Component of Lowpass Subband

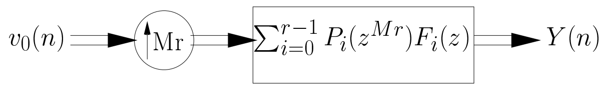

Figure 5: Output Derived From Polyphase Component of Lowpass Subband

$$D_0^l = \begin{bmatrix} h_1^l(1) & \ldots & h_1^l(r) \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix},$$

$$D_1^l = \begin{bmatrix} 0 & \ldots & 0 \\ h_2^l(1) & \ldots & h_2^l(r) \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix},$$

$$\vdots$$

$$D_{r-1}^l = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \\ h_r^l(1) & \ldots & h_r^l(r) \end{bmatrix}.$$

5. K^{TH} ORDER BALANCING CONDITIONS FOR M-BAND MULTIWAVELET SYSTEMS

We now extend Selesnick's balancing conditions [14] to M-band multiwavelet systems. Vetterli et al. [21] have shown that a multiwavelet system may be represented as a time-varying filter bank using interlacing/delacing operations in the analysis/synthesis banks respectively. Figure (3) depicts the time-varying lowpass synthesis subband for an M-band multiwavelet system. K^{th} order balancing requires that the discrete-time monomial $u_0(n) = n^K$ input to the lowpass synthesis subband should be reproduced exactly at the output if zeros are fed to the bandpass subbands. Let $v_i(n) = (rn+i)^K$ represent the polyphase components of $u_0(n)$. Writing $P_i(z) = V_i(z)/V_0(z)$, enables us to redraw figure (3) using the polyphase component $V_0(z)$ alone as input. Applying standard multirate operations, figure (4) may be represented equivalently by figure (5). This is the lowpass synthesis filter of an Mr-band wavelet system, and $v_0(n)$ is a monomial of degree K. This monomial will be preserved if $\sum_{i=0}^{r-1} P_i(z^{Mr})F_i(z)$ has a zero of order K at the Mr roots of unity, though not necessarily at $z = 1$ [16].

6. CONCLUSIONS AND FUTURE WORK

We defined the disjoint support property as applicable to the scaling functions in a multiwavelet system and used it to show the existence of a time-frequency tiling that is attainable with M-band multiwavelet systems but not with two band multiwavelet systems. Finally, we extended Selesnick's K^{th} order balancing condition to M-band multiwavelet systems, in order to allow the use of these systems in practical applications. Our preliminary results are encouraging since they illustrate the potential power of M-band multiwavelet systems. and justify further work channelled toward reaping the benefits of M-band multiwavelet systems.

7. ACKNOWLEDGEMENTS

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