Order Number 9408686

Symbiotic transfer, arbitrage, and equilibrium

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Rice University, 1993

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RICE UNIVERSITY

SYMBIOTIC TRANSFER, ARBITRAGE, AND EQUILIBRIUM

BY

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE
DOCTOR OF PHILOSOPHY

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ABSTRACT

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By

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We lay a unified foundation for a theory of general equilibrium by proving the existence of an equilibrium for a grand model which covers all the well-known general equilibrium models under the convexity and continuity assumptions. The grand model allows an economy to have an extended list of commodities including assets which can be traded on unlimited short sales. The conceptual framework we develop for the existence problem is simple. Consider an economy consisting of two agents. If there were a commodity bundle which is always desirable to one agent and always undesirable to the other agent, the economy could not reach an equilibrium because they can increase their utility through an indefinite give-and-take process. What we need for the existence of an equilibrium is to exclude the presence of commodity bundles that can bring an economy into this state of "economic symbiosis." We proceed further by taking the Closedness Hypothesis that the utility possibility set is compact.

The finite dimensional findings do not hold for an economy with an infinite dimensional commodity space so that we investigate under what circumstances the Closedness Hypothesis holds. We develop sufficient conditions for the Closedness Hypothesis to hold and prove the existence of an equilibrium of an infinite dimensional economy under some spanning conditions on consumption sets.

ACKNOWLEDGEMENTS

I could not find the best words expressing thanks to my mother, wife, and son. At best I might say that I owe them what I have been, am, and will be. I hope my professional career established through their devoted life will help bear a blessed reincarnation of their sacrificed dreams during the rest of their life.

I would like to thank professors Marc Dudey, Richard Young, and William M. Taylor for discussions and encouragements. In particular, I cannot but express a deep gratitude to my advisor Marc Dudey for his invaluable support and suggestions. Discussions with professors Dagobert Brito and John Bryant are appreciated.

I thank my friends Jeong Ae, Jae Jung, Juan, and Joya for their suggestions. Special thanks go to the constant friendship with Guangsuk of Northwestern University who is doing research on the same theoretical issues.

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CHAPTER 1

Introduction

In the literature of general equilibrium theory, the main body of research was conducted on the analysis of competitive equilibrium in the markets with real goods whereas in the literature of asset pricing theory, most works were restricted to the markets with financial assets. Asset pricing models have developed into their theoretical underpinning the concept of arbitrage opportunity which cannot exist in equilibrium with frictionless financial markets. In fact, the arbitrage concept has been used implicitly in a general equilibrium model of the markets with real goods. The rationale is that in equilibrium no economic agent can purchase a bundle of goods at no cost that would increase the utility; otherwise, some agent could not accept the status quo. For example, the absence of an arbitrage opportunity with a monotonic economy is represented by the condition that a commodity bundle of nonnegative quantity should be nonnegatively priced. One aspect of asset pricing models that distinguish themselves from the classical general equilibrium model of an exchange economy is to allow the possibilities of trading negative quantities of assets, i.e. short sales. If a list of commodities encompasses assets like bonds and securities, it is not reasonable to restrict the consumption sets to the nonnegative orthant of a Euclidean space. More generally speaking, it is not acceptable to put some lower bound on the consumption sets of economies with an extended list of commodities involving assets in the face of unrestricted short sales. The presence of assets has an important impact on the existence of an equilibrium of an extended exchange economy. Therefore, the classical existence theorems can not be applied to an extended economy.

All the models of general equilibrium under the convexity and continuity assumptions postulate either the compactness of the set of feasible allocations or the existence of nonarbitrage prices common to all economic agents. Debreu (1962) is the most general model of the former category whereas Werner (1987) represents the latter one. The compactness postulation is related to the data of an economy but it lacks economic intuition because it serves only as a technical basis for applying a fixed point theorem or its variant. The second postulation is more acceptable but not clear as to where the overall nonarbitrage prices come from.

The question arises if there is any desirable alternative to these postulations without any additional cost. The answer is simple but general. Consider an economy consisting of two agents. If there were a commodity bundle which is always desirable to one agent and always undesirable to the other agent, the economy could not reach an equilibrium because they can increase their utility indefinitely through a give-and-take process. What we need for the existence of an equilibrium is to exclude the presence of commodity bundles that can bring an economy into this state of "economic symbiosis." However, it is not difficult to comprehend that such an economic symbiosis could not exist unless the economic behavior of agents differs substantially. The condition which frees an economy from the symbiotic state is well-expressed by the positive semi-independence of the recession cones of the sets of commodity bundles at least as good as the initial endowments. We propose the absence of economic symbiosis as a conceptual framework which gives coherence to the two seemingly unrelated postulations mentioned above. Indeed, we will show each postulation is looking at a different side of a coin.

Since an equilibrium consumption is preferred to the initial endowment, we can restrict a search for an equilibrium allocation to the set of rationally feasible allocations, i.e. ones which are feasible and at the same time, preferred to the endowments for all agents. We present the Trinity Principle that there exists an equivalence in a finite dimensional setting between the existence of overall nonarbitrage prices, the compactness of the rationally feasible set, and the positive semi-independence of the recession cones of the preferred sets for all agents. By applying the Trinity Principle, we lay a unified foundation for a theory of general equilibrium of an extended economy. Indeed, we build a grand model which covers both categories of general equilibrium models including Debreu (1962) and Werner (1987). The model is so general that the rationally feasible set for the economy may not be compact. Our strategy is to construct a new economy from the original economy in such a way that the rationally feasible set for the new economy is compact and to an equilibrium of the new economy there corresponds an equilibrium of the original economy in a canonical way. But we give an example which is not covered by the grand model and therefore, not by either Debreu (1962) or Werner (1987). It is possible to proceed further by taking the Closedness Hypothesis which is introduced by Mas-Colell (1986) since the hypothesis is satisfied with the new economy as well as the unconquered example. Under the Closedness Hypothesis, we prove the existence of an equilibrium of a finite dimensional economy when preferences are locally non-satiable. In addition, we prove the genericity of local uniqueness of equilibria for an extended economy under the regularity conditions.

The finite dimensional Trinity Principle is not extended to an economy with an infinite dimensional commodity space so that we need to investigate under what circumstances the Closedness Hypothesis holds. For an exchange economy in a positive cone of a normed linear space, we impose the normality condition on the positive cone to restore the boundedness of the feasible set. Similarly, for an extended economy we generalize both the finite dimensional positive semi-independence postulation and the normality condition of a positive cone. These conditions are sufficient for the Closedness Hypothesis to be fulfilled provided the bounded set of allocations is compact with respect to the topology on the possibility sets. As Cheng (1991) shows, infinite dimensional contingent claims markets does not allow the boundedness of the rationally feasible set under the expected utility hypothesis in general. Nevertheless, Cheng (1991) proves that the Closedness

Hypothesis is fulfilled when there exists a single commodity in all states of the world. Under natural conditions, we conjecture that the conclusion of Cheng holds for cases with a finite number of commodities by providing intermediate results. Based on these consequences, we prove the existence of an equilibrium of an infinite dimensional extended economy under some spanning conditions.

The approach to the existence proof we take in both the finite and the infinite dimensional settings is based on the Second Fundamental Theorem of welfare economics, which utilizes the one-to-one correspondence between the Pareto frontier of utility possibility sets and the simplex of the Euclidean space. This approach which was pioneered by Negishi (1960) and Arrow and Hahn (1971) has been used by Bewley (1969), Magill (1981) and Mas-Colell (1986) in the infinite dimensional setting. To obtain some insight into the Trinity Principle, we present an illustrative example. This example is simple but not completely covered by the existing literature.

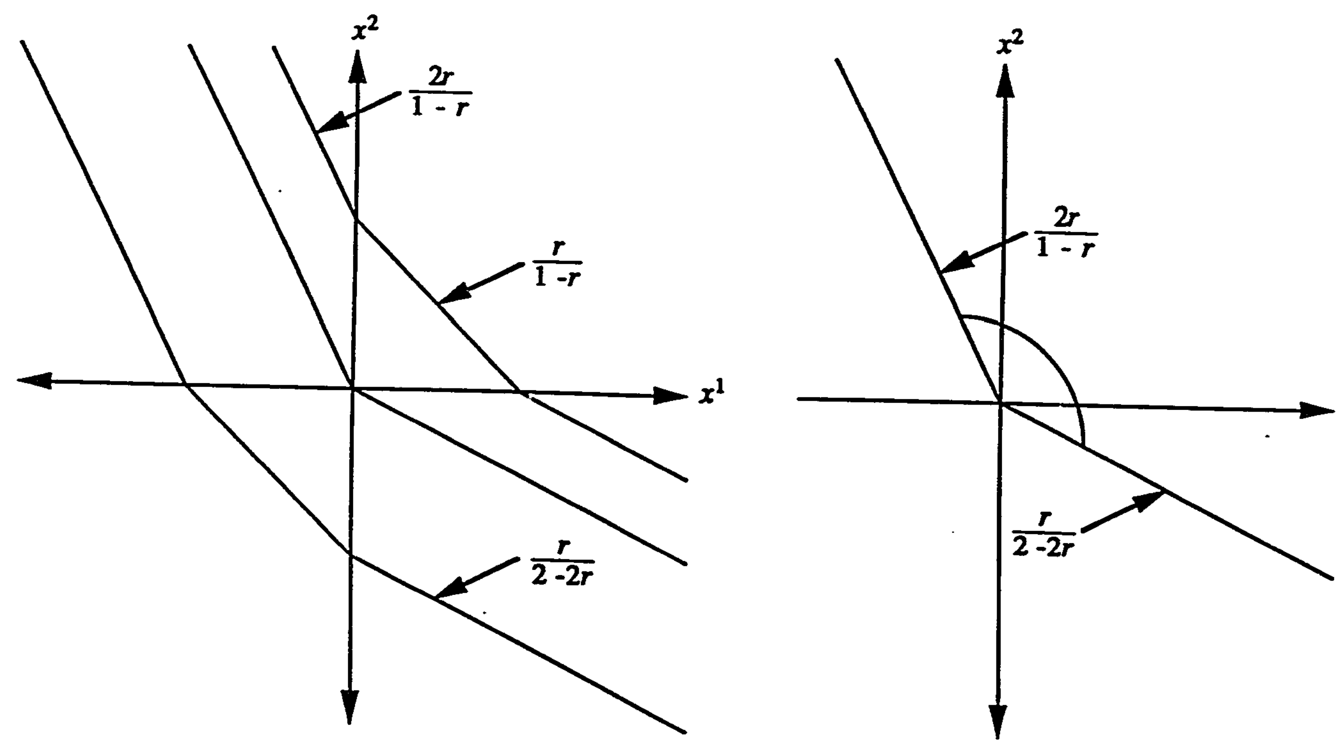
Consider an one-period economy in which two markets (j = 1, 2) for contingent claims to money are open at the outset and the claimed money is delivered to the holders at the end of the period. Suppose that uncertainty of the future consists of states s_1 and s_2 and that there are two agents (i = 1, 2) in the economy with probability beliefs r_1 and r_2 over state s_1 with $r_1 < r_2$, respectively. Let $w_1 = (1, 0)$ and $w_2 = (0, 1)$ be the endowments of each agent. Assume that the first claim delivers one unit of money to the holder if s_1 is realized at the end of the period and otherwise, nothing, and that the second claim delivers nothing if s_1 is realized in the end of the period and otherwise, one unit of money. Agent i is assumed to choose a portfolio of contingent claims (x^1, x^2) over R^2 to maximize the expected utility $U_i(x^1, x^2) = r_i u(x^1) + r_i u(x^2)$ for i = 1, 2, where u is a utility function of wealth which is defined by

$$u(x) = \begin{cases} x & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases}$$

The indifference curves and the recession cone of preferred sets are shown in Figure 1. We denote by p_j the price of the jth claim. Take the simple case with $r_1 = 0$ and $r_2 = 1$ in which the second consumer is interested in acquiring only the first claim whereas the first consumer in acquiring only the second option. It is easy to find a portfolio which brings the economy into an economic symbiosis and therefore prevents it from reaching an weak optimum. Thus the economy cannot have an equilibrium.

We can now embark a complete analysis of the example. The demand correspondences for the second claims are depicted in Figure 2 with respect to the relative price $p = p_1/p_2$. It is easy to check that an economy $w = \{(1, 0), (0, 1)\}$ has an equilibrium if and only if $\frac{2r_1}{1 - r_1} \geq \frac{r_2}{2 - 2r_2}$, i.e. $r_2 \leq \frac{4r_1}{1 + 3r_2}$. In case with $\frac{2r_1}{1 - r_1} < \frac{r_2}{2 - 2r_2}$, i.e. $r_2 > \frac{4r_1}{1 + 3r_2}$, the demand for one of the agents is not well-defined. For $p < \frac{r_2}{2 - 2r_2}$, agent 2 can increase his utility indefinitely through profitable arbitrage operations of buying any multiple of a bundle (v^1, v^2) with $v^1 > 0$ and $pv^1 + v^2 = 0$. Similarly, for $p > \frac{r_2}{2 - 2r_2}$, agent 1 can increase his utility indefinitely through profitable arbitrage operations of buying any multiple of a bundle (v^1, v^2) with $v^2 > 0$ and $pv^1 + v^2 = 0$.

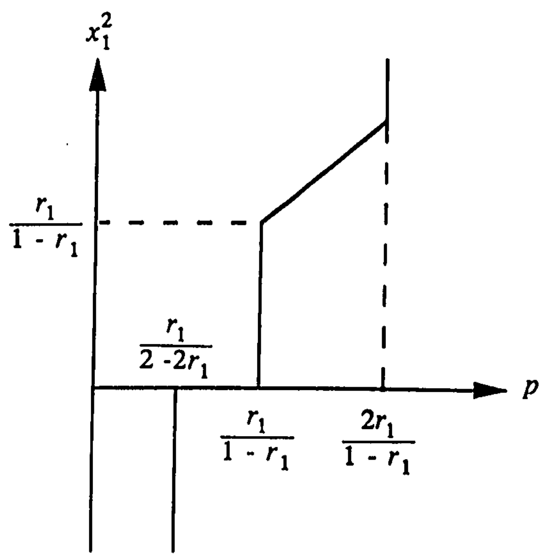
The above example demonstrates some informative properties that shed light on the problem of the existence of an equilibrium of extended economies. First, all preferred sets for an agent have the same recession cone. It is shown in Lemma A2 of the appendix that this property holds for any concave function. Second, the economy would have no weak optimum and therefore no equilibrium if agents disagree too much over the realizations of the future state of the world. We can consider a less extreme example than mentioned above. If $r_1 = 0.5$ and $r_2 > 0.8$, the economy has no equilibrium as well as no optimal allocation. That is only because the probabilistic dissidence over the contingencies leads to a substantial discrepancy in the consumption behavior, which brings about economic



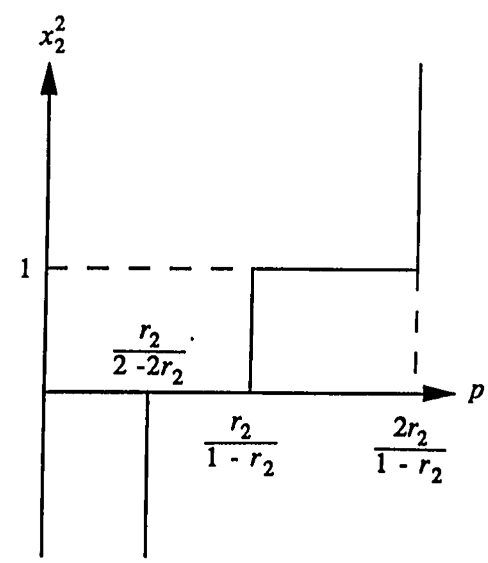
Indifference curves with the probability belief r

Recession cone of the level sets

Figure 1.



Demand correspondence for agent 1



Demand correspondence for agent 2

Figure 2

symbiosis; there exists a portfolio which one agent always likes to dispose of and the other agent always likes to take over. But this will not occur if the recession cones of preferred sets for both agents are positively semi-independent. It is easy to check that their recession cones are positively semi-independent if and only if $r_2 < \frac{4r_1}{1 + 3r_2}$. Moreover, the set of rationally feasible allocations is compact if and only if $r_2 < \frac{4r_1}{1 + 3r_2}$. We note the case with $r_2 = \frac{4r_1}{1 + 3r_2}$ is not covered by either Debreu (1962) or Werner (1987) because the nonarbitrage prices common to both agents does not exist. Nevertheless, we will demonstrate this case is satisfied with the Closedness Hypothesis.

We develop the idea of this example into a theoretical basis by which we can reinterpret and extend all the well-known cases under quite general conditions. Indeed, the Trinity Principle allows us to put the general equilibrium literature concerning a finite dimensional commodity space into a simple perspective. Debreu (1959, 1962) proves the existence of an equilibrium of a production economy via the well-chosen truncation in such a way that an equilibrium of the well-truncated economy or the limit of a proper sequence of equilibria for a sequence of well-truncated economies is in fact an equilibrium of the original economy. The truncation argument is made possible on the basis of the compactness of the set of feasible allocations. To verify the compactness argument is one of the major steps in proving the existence of an equilibrium. In order to guarantee the compactness of the feasible set, Debreu (1959, 1962) imposes on consumption sets and aggregate production sets the positive semi-independence of their recession cones or its variant condition. But as for extended economies, the positive semi-independence restrictions on the possibility sets are unwarranted in general because of the possibilities of unrestricted short sales. It seems to be natural to direct our concerns toward preferences in order to find the conditions relevant to the existence of an equilibrium of an extended economy. Hart (1974) integrates the asset pricing models pioneered by Lintner (1965) and Sharpe (1964) into the framework of general equilibrium theory and suggests necessary conditions and sufficient conditions for the existence of an equilibrium. The conditions used in Hart (1974), which are more or less technical are concerned with the properties of preferences and expectations of agents about the markets to be held in the future but they do not explicitly reflect the importance of arbitrage concept to its fullest extent. Werner (1987) provides a new approach to the general equilibrium model of an extended economy by articulating the role of arbitrage opportunities. But it leaves something to be desired in two respects. Werner gives no systematic treatment to the relationship between the data of an economy and the postulation that there exists a price system that admits no arbitrage opportunities for all agents. Secondly the preferences of agents are assumed to have the same recession cone for all preferred sets, which is not compatible with convex preferences in general.

In the context of regular economies, the model of Balasko (1988) is the same as ours except for his hypothesis that the preferences are bounded from below, which has the same effect on the existence problem as the lower boundedness of consumption sets. He took the hypothesis to be 'not essential but merely convenient.' But we show the absence of economic symbiosis is required to verify the properness of the projection mapping when preferences are not bounded from below.

This paper consists of six chapters. In the second chapter, we present The Trinity Principle and work out its applications. The third chapter is devoted to analyzing optimality in terms of cones generated by preferred sets and the price supportability. We suggest a condition which brings out the equivalence between optima and weak optima. The existence of a quasi-equilibrium of a finite dimensional economy is given under the Closedness Hypothesis in chapter 4. The genericity of local uniqueness of equilibria for an extended economy is proved in chapter 5. In the sixth chapter, we propose the conditions which bring about the boundedness of the feasible set of an infinite dimensional economy by extending the counterparts of well-established cases. Some spanning conditions are introduced which are necessary for the price supportability of optima for an extended economy in a normed linear space. In addition, we provide an appendix reviewing mathematical concepts and technical lemmas related to convex sets, recession cones and a topological vector space. The appendix is self-contained about the lemmas involving the recession cones used in this paper.

We introduce some mathematical notations and definitions. A set C in a topological vector space E is a *cone* if $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$. A cone C may not contain the origin. A convex cone C is *pointed* if $(C \cup \{0\}) \cap (-C \cup \{0\}) = \{0\}$. Let C_j be a nonempty cone in an affine space Y_j for each $j=1,\dots,d$. A set of cones $\{C_j\}$ is *positively semi-independent* if $x_j \in C_j \cup \{0\}$ for every j and $\sum_{j=1}^{d} x^j = 0$ implies $x_j = 0$ for all j. Let Γ be a nonempty convex subset of E. A vector $x \in E$ is called a *direction of recession* of Γ if $y + \lambda x \in \Gamma$ for all $y \in \Gamma$ and $\lambda \geq 0$. The set of all directions of recession of Γ is called the *recession cone* of Γ, denoted by $0^+\Gamma$. A set $L(\Gamma) = (-0^+\Gamma) \cap 0^+\Gamma$ is called the *linearity space* of Γ. The closure and the interior of a set Γ in E is denoted by cl Γ and int Γ, respectively. Let Y be an affine space in E which contains Γ. We denote by $\mathrm{int}_Y(\Gamma)$ the interior of Γ with respect to Y. If the underlying affine space Y is clear in the context, we omit the subscript Y. We denote by $\boldsymbol{C}$ the product $\Pi_{i=1}^{m} C_i$ of subsets C_i of a vector space and by a bold face $\boldsymbol{y}$ a generic element $(y_1,\dots, y_m)$ in $\boldsymbol{C}$ with y_i in C_i. Let S^{l-1} denote a unit sphere of R^l.

CHAPTER 2

Economic Symbiosis and Equilibrium

2.1 The Trinity Principle

We consider an extended exchange economy with l commodities consisting of securities as well as real goods, which is composed of m consumers indexed by i. Consumer i is assumed to have a preference relation $\leq_i$ on a consumption set X_i and the endowment w_i in X_i. In this paper we make the preferences of consumers fixed. An exchange economy is defined by an m-tuple of endowment vectors, $w = (w_1,..., w_m)$. We make the following assumptions for each i.

$\mathbf{a_1}$: The set X_i is a nonempty closed convex subset of R^l.

$\mathbf{a_2}$: The ordering $\leq_i$ on X_i is continuous, convex, and complete.

All assumptions are standard so that they are well-understood with no additional comment. We notice that no bound needs to be imposed on X_i in the condition $\mathbf{a_1}$ because a list of commodities may include securities. The unbounded consumption sets is relevant to a case with no restrictions on short sales of securities or contingent claims and the varied consumption sets among consumers is concerned with the possibilities of asymmetric participation in financial markets. It is well-known that under the conditions $\mathbf{a_1}$ and $\mathbf{a_2}$, the preferences admit a representation by a continuous quasi-concave utility function u_i. In the sequel, we will sometimes use the utility function u_i instead of $\leq_i$. Since $\leq_i$ is continuous and convex, the level set

$$P_i(x) = \{y \in R^l : u_i(x) \leq u_i(y)\}$$

is closed and convex for each $x \in X_i$. The recession cone $0^+P_i(x)$ contains a nondecreasing direction of utility because $y \in 0^+P_i(x)$ implies $u_i(x) \leq u_i(z + \lambda y)$ for all $\lambda \geq 0$ and z in X_i which leaves consumer i indifferent to x. In particular, the preferences $\leq_i$ are monotonic if $0^+P_i(x) = R^l_+$ for all x in X_i. If a commodity bundle y is such that $u_i(x) < u_i(x + \lambda y)$ for some $x \in X_i$ and all $\lambda > 0$, y is called a *desirable* direction for i at x and if y is desirable at all $x \in X_i$, y is called a *strongly desirable* direction. If y is desirable at x for i, we have $y \in 0^+P_i(x)$ by the corollary of Lemma A1 in Appendix. The recession cone $0^+P_i(x)$ is decomposed as follows;

$$0^+P_i(x) = L(P_i(x)) \cup K_i(x),$$

where $K_i(x)$ denotes the complement of the linearity space $L(P_i(x))$ in $0^+P_i(x)$. Obviously, $K_i(x)$ is a pointed convex cone for every x in X_i which does not contain the origin. We denote $0^+P_i(w_i)$, $L(P_i(w_i))$ and $K_i(w_i)$ by 0^+P_i, L_i and K_i, respectively. One interesting case is that all level sets $P_i(x)$ have the same recession cone for consumer i, which implies $K_i(x)$ is identified with K_i for all $x \in X_i$. In particular, Lemma A2 in Appendix shows that convex preferences represented by a concave utility function admit the same recession cone for all the level sets. Werner (1987) uses the following condition.

(1) The recession cones $0^+P_i(x)$ are the same for all $x \in X_i$ and K_i is not empty for all i.

In a pure asset pricing model, an arbitrage portfolio is any bundle of securities that gives an arbitrage opportunity to an investor and furthermore, monetary gain is supposed to be always beneficial to the investor. However, we should be careful about defining an arbitrage opportunity in a general equilibrium model because the choice of a commodity bundle directly affects the budget constraint as well as the utility level of a consumer. A bundle in $0^+P_i(x)$ is a candidate for an arbitrage opportunity in a general equilibrium model because the choice of bundles in $0^+P_i(x)$ does not directly decrease the utility. We will borrow an idea from Werner (1987) in defining a nonarbitrage price system. Let y be a vector in K_i. The vector y is a useful commodity bundle in the sense that it is a nondecreasing direction of utility for consumer i. Under the condition (1), K_i is the set of useful commodity bundles. A price system p admits no arbitrage opportunity for consumer i if every y in K_i has a positive value, i.e. $py > 0$. Let $S_i = \{p \in R^l : py > 0 \text{ for all } y \text{ in } K_i\}$ be the set of nonarbitrage prices for agent i and $S = \{p \in R^l : py > 0$ for all y in the union of all K_i's $\}$ the set of economy-wide nonarbitrage prices.

Werner (1987) add the following assumption for the existence of an equilibrium.

(2) The set S is not empty.

However, no relationship was explained between the condition (2) and the primitive factors of an economy. How are they related ? We present a proposition from which we can deduce a satisfactory answer to this question.

Proposition 1. (The Trinity Principle) Let $\{C_j\}$ be a set of d closed convex sets in R^l with the nonempty pointed cone $0^+C_j\backslash\{0\}$ and C_j^0 a set $\{p \in R^l : py > 0$ for all y in $0^+C_j\backslash\{0\}\}$ for each j. Then the following statements are equivalent.

(i) $\{0^+C_j\}$ is positively semi-independent.

(ii) The intersection C^0 of all C_j^0's is not empty.

(iii) A set $B(w) = \{x \in R^{dl} : \sum_{j=1}^{d} x_j = w$ and $x_j \in C_j$ for all $j\}$ is convex and compact for a vector w in R^l.

Proof) First, we will show the equivalence between (i) and (ii). Suppose that $\{0^+C_j\}$ is not positively semi-independent. Then there is a set of d vectors $x_j \in 0^+C_j$, not all zero, such that $\sum_{j=1}^{d} x_j = 0$. Without loss of generality, we may assume $x_1 \neq 0$. It implies $px_1 > 0$ for all $p \in C_1^0$. But we have $p\sum_{j=1}^{d} x_j = px_1 + \sum_{j=2}^{d} px_j = 0$, which implies there is $j \neq 1$ such that $px_j < 0$. Thus $p \notin C_j^0$, which leads to contradiction. Let C^+ denote the sum $\sum_{j=1}^{d} 0^+C_j$. It is clear that C^+ is a convex cone. We claim C^+ is pointed. Suppose that it is not pointed. Then there is a nonzero vector x such that both x and $-x$ are in C^+. Let x_j and z_j be vectors in 0^+C_j such that $x = \sum_{j=1}^{d} x_j$, and $-x = \sum_{j=1}^{d} z_j$. By definition, we obtain $\sum_{j=1}^{d} (x_j + z_j) = 0$. Since $\{0^+C_j\}$ is positively semi-independent, we have $(x_j + z_j) = 0$ for all j. By the pointedness of each 0^+C_j, we have $x_j = z_j = 0$ and therefore $x = 0$, which leads to a contradiction. Hence C^+ is a pointed convex cone. Since 0^+C_j contains the origin, each 0^+C_j is a subset of C^+. Then Lemma 3A in the appendix implies C_j^0 is not empty because each 0^+C_j is a subset of a pointed convex cone C^+.

Now prove the equivalence between (i) and (iii). The convexity and closedness of $B(w)$ are clear. Suppose $B(w)$ is not bounded. Pick a sequence (x_n) with $\sum_{j=1}^{d} \|x_{jn}\| \to \infty$ as $n \to \infty$ and put $\gamma_n = \sum_{j=1}^{d} \|x_{jn}\|$. Since (x_n) diverges to ∞, without loss of generality we may assume $\gamma_n > 0$ for all n. Put $\delta_{jn} = x_{jn}/\gamma_n$. Since $\sum_{j=1}^{d} \|\delta_{jn}\| = 1$ for all n, (δ_{jn}) is bounded for each j, implying it has a subsequence converging to a vector δ_j. By Lemma A1, the vector δ_j is in 0^+C_j. Since $\sum_{j=1}^{d} \|\delta_j\| = 1$, there exists an index j with a nonzero δ_j. On the other hand, we derive the following equalities from the fact that x_n is in $B(w)$

$$\sum_{j=1}^{d} \delta_j = \lim \sum_{j=1}^{d} \delta_{jn} = \lim \sum_{j=1}^{d} w/\gamma_n = 0.$$

It contradicts the positive semi-independence of $\{0^+C_j\}$. Conversely, suppose that there exists a set of vectors δ_j, not all zero, such that $\sum_{j=1}^{d} \delta_j = 0$ and $\delta_j \in 0^+C_j$ for all j. For any x in $B(w)$, we have $x_j + \lambda\delta_j \in C_j$ and $\sum_{j=1}^{d} (x_j + \lambda\delta_j) = w$ for all $\lambda > 0$, which leads to a contradiction. *Q.E.D.*

We note that the duality between (i) and (ii) in Proposition 1 still holds for a case where C_j is a closed convex set with a pointed $0^+C_j \backslash L(C_j)$ for all j. In many applications, we are concerned about the pointedness of $0^+C_j \backslash L(C_j)$. The economic application of the Trinity Principle is as follows: the set C_j can be considered the set of consumptions preferred to the endowment or the negative of the production set, C^0 the set of nonarbitrage prices for the economy, and the set $B(w)$ the feasible set with the total endowment w which contains consumptions preferred to the endowments for all consumers. The Principle has been partially incarnated in the literature of general equilibrium theory. By taking advantage of these interpretations, we can lay down a unified foundation for the theory of general equilibrium.

2.2. Classical Models

In the classical works including Debreu (1959, 1962) and Arrow and Hahn (1971), restrictions were imposed on consumption sets or production sets rather than on preferences in order to guarantee the compactness of the set of feasible allocations. Since Debreu (1959) is a special case of Debreu (1962), we focus on the latter work. Let $\{Y_f\}$ be a set of k production sets in R^l and Y their sum which is assumed to be closed. Here we confine our concern to the case where Y is convex. Let L_0 be the linearity space of $0^+(-Y)$ and $L_0^\perp$ the orthogonal complement of L_0 in R^l. Clearly, a set $L_0^\perp \cap Y$ has the pointed recession cone $L_0^\perp \cap 0^+Y$. Let X be the sum of m individual consumption sets X_i. Debreu (1962) assumes that 0^+X is pointed and that 0^+X and $0^+(-Y)$ are positively semi-independent. Lemma A3 of the appendix implies that 0^+X is pointed if and only if every 0^+X_i is pointed and $\{0^+X_i\}$ is positively semi-independent. Since 0^+X and $0^+(-Y)$ are positively semi-independent, so is a set $\{(0^+X_i), L_0^\perp \cap 0^+(-Y)\}$. By Proposition 1, the feasible set $A(w) = \{(x,y) \in X \times (L_0^\perp \cap Y) : \sum_{i=1}^m x_i = \sum_{i=1}^m w_i + y\}$ is compact. Since $P_i(x)$ is a closed convex subset of X_i for all x in X_i, $0^+P_i(x)$ is a subset of 0^+X_i for all i, leading to the positive semi-independence of $\{(0^+P_i(x_i)), 0^+(-Y)\backslash L_0\}$ for all x_i in X_i. Thus as a duality of the positive semi-independence of the recession cones, 'nonarbitrage prices' for all agents always exist and $A(w)$ is compact. Roughly speaking, Debreu (1962) sets the model under the weakest restrictions on the possibility sets in that the absence of nonarbitrage prices for the possibility sets may occur in light of the Trinity Principle. But we will go further than Debreu (1962) by applying arbitrage arguments for preferences rather than for the possibility sets.

Keeping in mind the impossibility of social free production, Arrow and Hahn (1971) assumes that if $\sum_{j=1}^{f} y_i \geq 0$ with y_i in Y_i, then $y = 0$. Under this assumption, they prove that the feasible production set $A_1(w) = \{y \in Y : \sum_{i=1}^{m} y_i \geq -w\}$ is compact via long argument. But under a more general assumption that $\sum_{j=1}^{f} v_i \geq 0$ with v_i in 0^+Y_i implies $v = 0$, we can easily prove the compactness of A_1 by the same argument as in the second part of the proof for Proposition 1. If consumptions are restricted to the nonnegative quantities, the feasible set $A_2(w) = \{(x,y) \in X \times Y : \sum_{i=1}^{m} x_i \leq \sum_{i=1}^{m}(w_i + y_i)\}$ is compact. By the Trinity Principle, we obtain the nonempty set S of nonarbitrage prices for the economy.

2.3. Models with Possibly Unbounded Sets

For extended economies, the positive semi-independence restrictions on the possibility sets are unwarranted in general. For example, no restrictions on short sales in contingent claims markets lead to the unbounded consumption sets. It seems to be natural to investigate the properties of preferences rather than of the possibility sets to ensure the existence of equilibrium in a general equilibrium model of extended economies. Based on the Trinity Principle, we will analyze the conditions (1) and (2) of Werner (1987) and generalize his results in our context. By the assumption (1), we have $0^+P_i = 0^+P_i(x)$ and $L_i = L(P_i(x))$ for all x in X_i. Denote by $L_i^{\perp}$ the orthogonal complement of L_i in R^l. Clearly, $P_i(x_i) \cap L_i^{\perp}$ has the recession cone $L_i^{\perp} \cap 0^+P_i$ for all x_i in X_i. Define a feasible set $A(w) = \{x \in X : \sum_{i=1}^{m} (w_i - x_i) = 0\}$ for an economy w in X. We know the condition (2) is equivalent to the positive semi-independence of $\{K_i\}$. Thus $\{L_i^{\perp} \cap 0^+P_i\}$ is positively semi-independent and by Proposition 1, a set $\Pi_m^{i=1}(P_i(w_i) \cap L_i^{\perp}) \cap A(w)$ is compact. It is possible to relax the conditions (1) and (2) in some direction by imposing the positive semi-independence condition only on the initial positions as follows;

(3) Each 0^+P_i is pointed with the nonempty K_i and $\{0^+P_i\}$ is positively semi-independent.

At the cost of excluding the case that 0^+P_i has the nontrivial linearity space, we have dropped the assumption that every preferred set has the same recession cone for each agent. We denote by $\Omega(w)$ the set of rationally feasible allocations,i.e. $\Omega(w) = \Pi_m^{i=1}P_i(w_i) \cap A(w)$. Under the conditions $\mathbf{a}_1$, $\mathbf{a}_2$ and (3), the triple equivalence between the positive semi-independence of $\{K_i\}$, the compactness of $\Omega(w)$ and the existence of no arbitrage prices for all agents is restored.

2.4. A Grand Model

We construct a model which includes as special cases both the examples in 2.2. and in 2.3. Consider a production economy which consists of k producers with production possibility sets Y_f along with the consumption sector which satisfies the conditions $\mathbf{a}_1$ and $\mathbf{a}_2$. We keep the notations used in (1.1) and (1.2). Let $\mathbb{E} = ((X_i, \leq_i, w_i), (\eta_{if}), Y)$ denote the production economy, where η_{if} is the share of profit of the jth producer owned by the ith consumer and Y is the total production set which is closed and convex. Let M_i be the intersection of $L(P_i(x))$'s for all x in X_i which is prefered to w_i, and $M_i^{\perp}$ the orthogonal complement of M_i in R^l. Consider an economy $\mathbb{E}^{\perp} = ((X_i \cap M_i^{\perp}, \leq_i^{\perp}, w_i^1), (\eta_{if}), Y \cap L_0^{\perp})$ which is the projection economy of $\mathbb{E}$ in the following sense. The set $Y \cap L_0^{\perp}$ is the projection of Y onto $L_0^{\perp}$ and $X_i \cap M_i^{\perp}$ the projection of X_i onto $M_i^{\perp}$ for each i. Finally, $\leq_i^{\perp}$ is the restriction of $\leq_i$ to $X_i \cap M_i^{\perp}$ and w_i^1 is the projection of w_i onto $M_i^{\perp}$, i.e. there exists a vector w_i^2 in M_i such that $w_i = w_i^1 + w_i^2$. Since $w_i^1 = w_i + (w_i^1 - w_i)$ and $w_i^1 - w_i \in$

$0^+P_i(w_i)$, we have $w_i^1 \in X_i \cap M_i^{\perp}$ for all i. Let $B_0 = 0^+(-Y)\backslash L_0$ and $B_i = 0^+P_i(w_i)\backslash M_i$ for each i. We pause to give the precise definition of economic symbiosis.

Definition 1. We say an economy $\mathbb{E}$ allows an *economic symbiosis* if a set of cones $\{(B_i), B_0\}$ is not positively semi-independent.

We make the following assumption.

(4) An economy $\mathbb{E}$ does not allow an economic symbiosis.

The condition (4) includes both Werner's two assumptions and the condition (3) as a special case. Since $0^+Y \cap L_0^{\perp}$ is the recession cone of $Y \cap L_0^{\perp}$ and $0^+P_i(w_i) \cap M_i^{\perp}$ is the recession cone of $P_i(w_i) \cap M_i^{\perp}$ for each i, a set of cones $\{(0^+P_i(w_i) \cap M_i^{\perp}), 0^+Y \cap L_0^{\perp}\}$ is positively semi-independent under the condition (4). By the Trinity Principle, the rationally feasible set of $\mathbb{E}^{\perp}$ is compact. Let $L^{\perp}(x)$ be the intersection of $\{\{L_i^{\perp}(x_i)\}, L_0^{\perp}\}$ where $L_i^{\perp}(x_i)$ denotes the orthogonal complement of $L(P_i(x_i))$ in R^l. We will postpone the proof of the existence of equilibrium of the projection economy until chapter 4. Under the condition (4), we propose that an equilibrium of the original economy is related to that of the projection economy in a canonical way.

Proposition 2. Assume $((x_i^1), y^*, p)$ is an equilibrium of the projection economy $\mathbb{E}^{\perp}$ with $p \in L^{\perp}(x) \cap S^{l-1}$ under the condition (4). Then $((x_i^*), y^*, p)$ is an equilibrium of the original economy $\mathbb{E}$ where $x_i^* = x_i^1 + w_i^2$ for all i.

Proof) (Feasibility) By definition, we obtain the equalities

$$\sum_{i=1}^{m} x_i^* - y^* = \sum_{i=1}^{m} (x_i^1 + w_i^2) - y^* = (\sum_{i=1}^{m} x_i^1 - y^*) + \sum_{i=1}^{m} w_i^2 = \sum_{i=1}^{m} (w_i^1 + w_i^2) = \sum_{i=1}^{m} w_i.$$

(Utility maximization) In equilibrium, we have $px_i^1 = pw_i^1 + \sum_{f=1}^{k} \eta_{if} p y_f^*$ for all i. Let z_i be a vector in X_i such that $z_i >_i x_i^1$ for all i. Then the projection z_i^1 of z_i onto $M_i^{\perp}$ is in $X_i \cap$

$M_i^{\perp}$ by the same reasoning as in verifying $w_i^1 \in X_i \cap M_i^{\perp}$ and it satisfies $u_i(z_i) = u_i(z_i^1)$ under the condition (4). Since $z_i^1 >_i x_i^1$ for all i, we obtain $pz_i^1 > px_i^1$ by the equilibrium conditions for the projection economy. On the other hand, we have $u_i(x_i^*) = u_i(x_i^1)$ and $p \in L_i^{\perp}(x_i) \cap S^{l-1}$, which leads to a relation $pz_i = pz_i^1 > px_i^1 = px_i^*$. In other words, $z_i >_i x_i^*$ implies $pz_i > pw_i^1 + \sum_{f=1}^{k} \eta_{if} py_f^*$ for all i.

(Profit maximization) Let y be a vector in Y. Then the projection y^1 of y onto $L_0^{\perp}$ is in $Y \cap L_0^{\perp}$. Suppose that $py > py^*$. Since $p \in L_0^{\perp}$ implies $py = py^1$, we have $py^1 > py^*$, contradicting the profit maximization for the projection economy. *Q.E.D.*

From now on, we will restrict our analysis to an exchange economy simply because the result can be directly extended to the case of a production economy. Thus the sets $A(w)$, $\Omega(w)$, and $L^{\perp}(x)$ are understood in the context of an exchange economy.

CHAPTER 3

The Positive Semi-Independence and Optimality

We will characterize the optima in terms of the cones generated by the upper level sets of preferences. For each point x in X_i, we define two sets

$$G_i(x) = \{v \in E : u_i(x + \alpha v) > u_i(x) \text{ for some } \alpha > 0\},$$

and

$$F_i(x) = \{v \in E : u_i(x + \alpha v) \geq u_i(x) \text{ for some } \alpha > 0\}.$$

We call each vector v in $G_i(x)$ a *locally desirable* direction for i at x. Both $G_i(x)$ and $F_i(x)$ turn out to be cones. It is easy to check that $P_i(x)$ is in a set $x + \text{cl } G_i(x)$. We add the following assumption for each i.

$\mathbf{a_3}$: For all $x \in X_i$ and $\varepsilon > 0$, there is some $z \in X_i$ such that $\|z - x\| < \varepsilon$ and $x <_i z$ (local non-satiation).

From now on, the condition $\mathbf{a_3}$ will be used along with $\mathbf{a_1}$ and $\mathbf{a_2}$. The following lemma explains the properties of these sets.

Lemma 1. (i) For some $\alpha > 0$, $u_i(x + \alpha v) > u_i(x)$ implies $u_i(x + \mu v) > u_i(x)$ for all $0 < \mu \leq \alpha$. (ii) Both $G_i(x)$ and $F_i(x)$ are convex cones for each i. Furthermore a set $G_i(x)$ is a pointed cone.

Proof) (i) It is clear that $u_i(x + \mu v) = u_i\{(1 - (\mu/\alpha))x + (\mu/\alpha)(x + \alpha v)\} > u_i(x)$ because of the convexity and local non-satiation of preferences.

(ii) We will prove the arguments only for $G_i(x)$ because the same reasoning applies to the case with $F_i(x)$. Let v be a locally desirable direction for i at x such that $u_i(x + \alpha v) > u_i(x)$ for some $\alpha > 0$. Then we have

$$u_i(x + \alpha v) = u_i\{x + (\alpha/\lambda)(\lambda v)\} > u_i(x) \text{ for all } \lambda > 0,$$

implying $G_i(x)$ is a cone. Pick two locally desirable directions v_1 and v_2 at x such that $u_i(x + \alpha_h v_h) > u_i(x)$, for some $\alpha_h > 0$, $h = 1, 2$. Let $\alpha = \max\{\alpha_1, \alpha_2\}$. By (i), we have $u_i(x + \alpha v_h) > u_i(x)$ for $h = 1, 2$. Without loss of generality, we may assume $u_i(x + \alpha v_1) \geq u_i(x + \alpha v_2)$. For any $0 \leq \mu \leq 1$, the convexity of preferences yields a relation

$$u_i\{x + \alpha(\mu v_1 + (1 - \mu)v_2)\} = u_i\{\mu(x + \alpha v_1) + (1 - \mu)(x + \alpha v_2)\} \geq u_i(x + \alpha v_2) > u_i(x),$$

implying $\mu v_1 + (1 - \mu)v_2 \in G_i(x)$. It remains to show the pointedness of $G_i(x)$. On the contrary, suppose $G_i(x)$ is not pointed. Then there is a locally desirable direction v in $G_i(x)$ such that $-v$ is a locally desirable direction at the same time. For some $\alpha > 0$, we have $u_i(x + \alpha v)) > u_i(x)$ and $u_i(x - \alpha v) > u_i(x)$. But $u_i(x) = u_i\{(1/2)(x + \alpha v) + (1/2)(x - \alpha v)\} > u_i(x)$, which is impossible. *Q.E.D.*

The following proposition shows that Pareto optimality is nicely characterized in terms of the cones introduced above.

Lemma 2. If a feasible allocation $\boldsymbol{x}$ is optimal, then a set of cones $\{G_i(x_i)\}$ is positively semi-independent. If a set of cones $\{F_i(x_i)\}$ is positively semi-independent, then the allocation $\boldsymbol{x}$ is optimal. Furthermore, a sum $G(\boldsymbol{x}) = \sum_{i=1}^{m} G_i(x_i)$ is a pointed convex cone provided $\boldsymbol{x}$ is an optimum.

Proof) Let $\boldsymbol{x}$ be an optimum. Suppose $\{G_i(x_i)\}$ is not positively semi-independent. Then there exist a vector v_i in $G_i(x_i) \cup \{0\}$ for all i, not all zero, such that $\sum_{i=1}^{m} v_i = 0$. For each i with a nonzero v_i, there exists a number $\alpha_i > 0$ such that $u_i(x_i + \alpha_i v_i) > u_i(x)$. Let $\alpha = \min\{\alpha_i\}$. By Lemma 1 (i), we have $u_i(x_i + \alpha v_i) > u_i(x)$ for all i. However, we obtain $\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} \alpha v_i = \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} w_i$., contradicting the optimality of $\boldsymbol{x}$. Let us prove the second argument. Suppose that x is not optimal. Pick an allocation y that Pareto dominates x. Then for some i, $u_i(y_i) > u_i(x_i)$. The consumption vector y_i can be decomposed as $y_i = x_i + (y_i - x_i)$, implying $y_i - x_i \in F_i(x_i)$ for such an i. However the relation $\sum_{i=1}^{m}(y_i - x_i) = 0$ contradicts the positive semi-independence of $\{F_i(x_i)\}$. Since each $G_i(x_i)$ is pointed and $\{G_i(x_i)\}$ is positive semi-independent, the same argument as in proving Proposition 1 implies that $G(x)$ is pointed. *Q.E.D.*

Let $D(x)$ be a set $\{p \in R^l : py \geq 0$ for all y in the union of all $G_i(x_i)\}$. We remark that $L^{\perp}(x) \supset D(x)$. Indeed, we will locate an equilibrium price system in $D(x)$ in proving the existence of an equilibrium. Assume that x is an optimum. By the supporting hyperplane theorem, we have the nonempty $D(x)$ because $G(x)$ is a pointed convex cone. Hence any vector p in $D(x)$ supports $G_i(x_i) + x_i$ at x_i, implying it supports $P_i(x_i)$ since $P_i(x)$ is in a set $x + \mathrm{cl}\ G_i(x_i)$. We have the following proposition as a corollary to Lemma 2.

Proposition 3. For an optimum x, $D(x) \cap S^{l-1}$ is not empty.

We recall that a pair (x, p) with $p \neq 0$ and x in $\Omega(w)$ is a quasi-equilibrium of the economy w if for each i, $pw_i = px_i$ and $py \geq px_i$ whenever $y \geq_i x_i$. Proposition 3 says that every optimum x is a quasi-equilibrium of the economy x with respect to some nonzero prices.

In a monotonic economy, it is easy to check the equivalence between optimality and weak optimality. Mas-Colell (1985) uses the following price-dependent hypothesis to restore the equivalence under $\mathbf{a}_1$ - $\mathbf{a}_3$.

The allocation x and the price system p are such that

$$px_i > \inf\ \{pz : z \in X_i\} \text{ for all } i.$$

Instead, we make the following assumption to show that an individually rational weak optimum is an optimum.

$\mathbf{a}_4$: Let x be an allocation in $\Omega(w)$ with $u_i(x_i) > u_i(w_i)$ for all i. Then for each i, there exists an allocation y in $A(w)$ such that $u_i(y_i) > u_i(x_i)$.

Under the condition $\mathbf{a}_4$, it is always possible to find an income transfer to consumer i from the rest of an economy that makes him better off, when an economy is in a state of individual rationality. If an allocation x in $\Omega(w)$ is not weakly optimal, such an income transfer is always possible. What is required by $\mathbf{a}_4$ is that in a weakly optimal state of an economy, a reallocation can be implemented in such a way that consumer i gets better off at a cost of deteriorating the rest of an economy even to a state in which some consumer j is worse off than the initial welfare $u_j(w_j)$. The condition $\mathbf{a}_4$ can be considered a reflection of an economic trade-off between the conflicting goals to be achieved by scarce means, which is common to the process of making economic decisions. We take some cases for which $\mathbf{a}_4$ holds. Suppose all consumption sets X_i are affine subspaces of R^l. The assumption $\mathbf{a}_4$ is satisfied provided there exists a set of strongly desirable directions (v_i) such that each v_i can be positively spanned by the rest of directions, i.e. $v_i = \sum_{j \neq i}^{m} \alpha_i v_i$ for some $\alpha_i \geq 0$. In cases that X_i is not an affine space, the same condition applies when a weakly optimal consumption occurs in the boundary of a consumption set for all consumers. The positive spanning condition is fulfilled if consumption patterns are not too much different from each other. We remark that the assumption $\mathbf{a}_4$ always holds if preferences are monotonic with $X_i = R^l_+$ for all i or if all consumers have the same consumption set which is an affine subspace of R^l. We have the following lemma.

Lemma 3. Under the assumptions $\mathbf{a}_1$ - $\mathbf{a}_4$, every weak optimum in $\Omega(w)$ is optimal.

Proof) Let z be a weakly optimal allocation in $\Omega(w)$. Suppose it is not optimal. We can pick an allocation x which Pareto dominates z. First we consider the case as simple as possible in which $u_1(z_1) = u_1(x_1)$ and $u_i(z_i) < u_i(x_i)$ for all $i > 1$. Assumption $\mathbf{a}_4$ enables us to pick an allocation y in $A(w)$ such that $u_1(y_1) > u_1(x_1)$. Without loss of generality, we may assume that $u_i(y_i) < u_i(x_i)$ for all $i > 1$. Let $z(\alpha) = \alpha y + (1 - \alpha)x$ for $0 \leq \alpha \leq 1$. By the convexity of preferences, we have $u_1(z_1(\alpha)) > u_1(x_1)$ and for $i > 1$, $u_i(z_i(\alpha)) > u_i(y_i)$ for all $0 < \alpha < 1$. By the continuity of utility functions, we can choose a positive number α^* sufficiently close to 0 such that $u_1(z_1) = u_1(x_1) < u_1(z_1(\alpha^*))$ and $u_i(z_i) < u_i(z_i(\alpha^*))$ for all $i \geq 2$. Indeed, we have $\sum_{i=1}^{m} z_i(\alpha^*) = \alpha^* \sum_{i=1}^{m} y_i + (1 - \alpha^*) \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} w_i$. We can construct such an allocation in more complex cases in the same way. The existence of an allocation $z(\alpha^*)$ contradicts the weak optimality of z. *Q.E.D.*

If $\mathbf{a}_4$ is assumed and allocations are restricted to be individually rational in the context, we will use optimality and weak optimality interchangeably.

CHAPTER 4

The Existence of an Equilibrium

We normalize a utility function for each i such that $u_i(w_i) = 0$. Define a mapping $U : \Omega(w) \rightarrow R^m$ by $U(x) = (u_1(x_1),..., u_m(x_m))$ and introduce the augmented utility possibility set $W = \{ \upsilon \in R^m_+ : \upsilon \leq U(x)$ for some x in $\Omega(w)\}$ following Cheng (1991). In chapter 2, we have showed the rationally feasible set for the projection economy is compact in the grand model. It implies the set W for the projection economy is compact. Specifically, Figure 3 shows that W is compact for the economy in the example with $r_2 = \frac{4r_1}{1 + 3r_2}$ in chapter 1, which is not covered by the grand model. These results can be incorporated into the Closedness Hypothesis introduced by Mas-Colell (1986).

> **Closedness Hypothesis.** The set W is bounded and for a sequence (x_n) in $\Omega(w)$ with $x_{in} \leq_i x_{i(n+1)}$ for all i and n, there exists x in $\Omega(w)$ such that $x_{in} \leq_i x$ for all i and n.

It is obvious under the Closedness Hypothesis that the set W is compact. We note that $U(x)$ is nonnegative for all x in $\Omega(w)$. If w is optimal, then it is already a quasi-equilibrium allocation with a supporting price system by Proposition 3. In order to avoid triviality, we will assume w is not weakly optimal. Let $\Delta = \{s \in R^m_+ : \sum_{i=1}^{m} s_i = 1\}$. For each $s \in \Delta$, define a mapping $f(s) = \max \{\alpha \in R : \alpha s \in W\}$. Since W has the nonempty interior, we obtain $f(s) > 0$ on Δ. Let $\Psi(w)$ denote the set of optimal allocations in $\Omega(w)$.

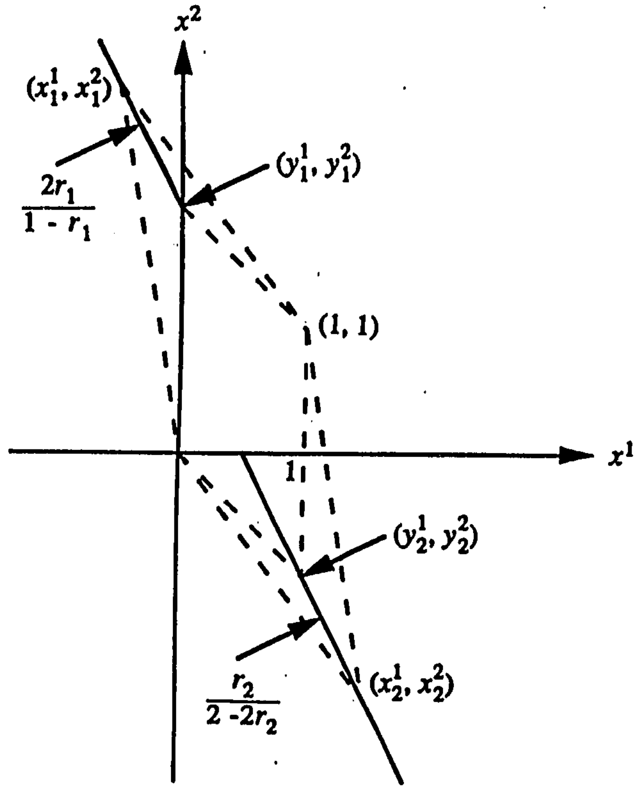


Figure 3

For an allocation $x = \{(x_1^1, x_1^2), (x_2^1, x_2^2)\}$ indicated on the dotted parallelogram, it is always possible to find a bounded allocation $y = \{(y_1^1, y_1^2), (y_2^1, y_2^2)\}$ located on the inner parallelogram which is at least as good as x to each agent.

The following lemma shows that to each point of the upper part of the boundary of W corresponds an optimal allocation in $\Psi(w)$.

Lemma 4. To all s in Δ, there corresponds a weakly optimal allocation $x \in \Psi(w)$ such that $u_i(x_i) = f(s)s_i$ for all i. The mapping $f : \Delta \to R_+$ is continuous.

Proof) Let υ be a point in W such that $f(s)s = \upsilon$ for some $s \in \Delta$. By definition, there exists an allocation $z \in \Omega(w)$ such that $\upsilon \leq U(z)$. If $\upsilon = U(z)$, it is easy to check that z is weakly optimal. However, the case with $\upsilon_i < u_i(z_i)$ for some i is impossible by the same argument in Lemma 3.

Let $s_n \to s$ in Δ. We must show that $f(s_n) \to f(s)$. There exist sequences (υ_n) in Δ and (x_n) in $\Psi(w)$ such that $f(s_n)s_n = \upsilon_n = U(x_n)$. Since $f(s_n)$ is bounded, it has a subsequence $f(s_{n'}) \to \alpha$, implying $f(s_{n'})s_{n'} \to \alpha s$. Then there exists an optimal allocation x such that $U(x) = \alpha s$. Since $\{f(s_{n'})\}$ is a subsequence of $\{f(s_n)\}$, it is true that liminf $f(s_n) \leq \alpha \leq$ limsup $f(s_n)$. Suppose liminf $f(s_n) < \alpha$. Since $s_{n'} \to s$, there exists a sufficiently large number N such that $U(x_N) = f(s_N)s_N < \alpha s = \lim f(s_{n'})s_{n'} = U(x)$, contradicting the weak optimality of both x_N and x. Analogously, we can prove that limsup $f(s_n) = \alpha$. In conclusion, we have $\lim f(s_n) = f(s)$. *Q.E.D.*

We are ready to prove the existence of a quasi-equilibrium of an economy w by the Negishi approach.

Theorem 1. Under assumptions $\mathbf{a}_1$-$\mathbf{a}_4$, an economy w has a quasi-equilibrium if the Closedness Hypothesis is satisfied.

Proof) For a point $s \in \Delta$, we can choose an optimal allocation $x(s)$ from $\Psi(w)$ such that $U(x(s)) = f(s)s$ by Lemma 4. We denote by $p(s)$ the set $D(x) \cap S^{l-1}$. Obviously, $p(s)$ is convex and by Proposition 3, non-empty. Define a correspondence $\Phi : \Delta \to R^m$ by

$$\Phi(s) = \{e \in R^m : \text{there exists } p \in p(s) \text{ such that } e_i = p(w_i - x_i(s)) \text{ for each } i\}.$$

A point in $\Phi(s)$ defines a vector of values of excess demands for each supporting price system. The set $\Phi(s)$ is non-empty, compact, and convex. We note that $\sum_{i=1}^{m} e_i = 0$ for each $e \in \Phi(s)$ because of Walras' law, and that $0 \in \Phi(s)$ if and only if $x(s)$ is a quasi-equilibrium.

Now show that the correspondence $\Phi : \Delta \to R^m$ is upper hemicontinuous. Let $s_n \to s$ and $e_n \in \Phi(s)$ with $e_n \to e$. It is enough to show that $e \in \Phi(s)$. For each n, we can pick x_n from $\Psi(w)$ satisfying $f(s_n)s_n = U(x_n)$ for all n and p_n from $p(s_n)$ such that $e_{in} = p_n(w_i - x_{in})$ for all i and n. By the continuity of f and U, we have $U(x(s)) = f(s)s$. On the other hand, (p_n) has a subsequence, denoted by the same notation, converging to a point p, because (p_n) is bounded. Suppose p does not support $x(s)$. Then for some i, there exists $u_i(z) > u_i(x_i(s))$ such that $pz < px_i(s)$. By the continuity of preferences, we obtain $u_i(z) > u_i(x_{in})$ and $p_n z < p_n x_{in}$ for sufficiently large n. It contradicts the fact that p_n supports x_n, implying $p \in p(s)$. Obviously, $p_n(w_i - x_{in}) \to e$. We conclude that $e \in \Phi(s)$ and therefore, the mapping Φ is upper hemicontinuous.

Since $\Phi(s)$ is compact, we can pick a positive number β satisfying $\sum_{i=1}^{m} |e_i| \leq \beta$ for all $e \in \Phi(s)$. Let T be a set $\{t \in R^m : \sum_{i=1}^{m} t_i = 0$ and $\sum_{i=1}^{m} |t_i| \leq \beta\}$. It contains all $\Phi(s)$.

Define a mapping $r : \Delta \mathrm{x} T \to \Delta$ by

$$r(s, t) = (1/\sum_{i=1}^{m} (s_i + t_i)^+)((s_1 + t_1)^+, \ldots, (s_m + t_m)^+),$$

where $(s_i + t_i)^+ = \max(0, s_i + t_i)$ for all i. The mapping r is well-defined because $\sum_{i=1}^{m} (s_i + t_i)^+ \geq \sum_{i=1}^{m} (s_i + t_i) = 1 + \sum_{i=1}^{m} t_i = 1$. Define a correspondence $\Phi_r : \Delta \mathrm{x} T \to \Delta \mathrm{x} T$ by $\Phi_r(s, t) = (r(s, t), \Phi(s))$. Then Φ_r is upper hemicontinuous and $\Phi_r(s, t)$ is non-empty, compact, and convex for all (s, t) in $\Delta \mathrm{x} T$. By Kakutani's fixed point theorem, Φ_r has a fixed point (s^*, t^*), i. e. $\Phi_r(s^*, t^*) = (s^*, t^*)$. Thus for all i,

$$s_i^* = (s_i^* + t_i^*)^+ / \sum_{i=1}^{m} (s_i^* + t_i^*)^+$$

and there exists p^* in $p(s^*)$ and x^* with $U(x^*) = f(s^*)s^*$ such that

$t_i^* = p^* (w_i - x_i^*)$ for all i.

To show that p^* is a quasi-equilibrium is equivalent to proving that $t^* = 0$. If $s_i^* = 0$ for some i, it implies $u_i(x_i^*) = 0$ and $t_i^* \leq 0$. Since p^* supports x_i^*, we have $t_i^* = p^* (w_i - x_i^*) \geq 0$, yielding $t_i^* = 0$. If $s_j^* > 0$ for some j, we have $(s_j^* + t_j^*)^+ = (s_j^* + t_j^*) > 0$, yielding $\sum_{h=1}^{m} (s_h^* + t_h^*)^+ = \sum_j (s_j^* + t_j^*) = 1$. Then

$$s_j^* = (s_j^* + t_j^*)^+ / \sum_{h=1}^{m} (s_h^* + t_h^*)^+ = s_j^* + t_j^*, \text{ i.e. } t_j^* = 0. \qquad Q.E.D.$$

Let S be the union of $D(\boldsymbol{x})$'s for all $\boldsymbol{x}$ in $\Psi(\boldsymbol{w})$. Then a quasi-equilibrium $(\boldsymbol{x}, p)$ is an equilibrium under the following minimum wealth constraint,

MWC $\quad pw_i > \inf pX_i$ for all p in cl $(S \cap S^{l-1})$.

Otherwise, $z >_i x_i$ implies $px_i = pz$. Pick y in X_i such that $pw_i > py$. For some small $\alpha > 0$, we have $\alpha y + (1-\alpha)z >_i x_i$ by the continuity of preferences. Since $p(\alpha y + (1-\alpha)z) < px_i$, it contradicts the hypothesis. As a corollary of Proposition 2 and Theorem 1, we prove the existence of an equilibrium for the grand model in 2.4.

Corollary. Under the assumptions $\mathbf{a}_1$-$\mathbf{a}_4$ and MWC, a symbiosis-free economy $\boldsymbol{w}$ has an equilibrium.

CHAPTER 5

Regular Economies

We demonstrate in this chapter that the properties of regular economies investigated in Debreu (1972) for the bounded consumption sets are preserved in the case of the unbounded consumption sets under some natural conditions. We need to adapt the assumptions stated in the previous chapter to regular economies such that $X_i = R^l$ and u_i satisfies the following regularity conditions for all $i=1,\ldots,m$.

$\mathbf{a}_1'$. (i) u_i is C^2 (partial derivatives up to the second order exist and are continuous),

(ii) $\mathrm{D}u_i(x) \neq 0$ for all i and x in R^l (local non-satiation).

(iii) $v^{\mathrm{T}}\mathrm{D}^2u_i(x)v < 0$ for all nonzero vectors v in R^l with $\mathrm{D}u_i(x)v = 0$ (strict quasi-concavity).

The condition $\mathbf{a}_1'$ implies a cone $0^+P_i(x)$ has the nonempty interior and the trivial linearity space for all x in R^l. We add the following assumption.

$\mathbf{a}_2'$. Every economy w in R^{ml} is free from an economic symbiosis.

By Proposition 1, the condition $\mathbf{a}_2'$ implies every economy w in R^{ml} has the compact rationally feasible set. A sufficient condition for $\mathbf{a}_2'$ to be fulfilled is the unions $\bigcup_{x \in R^l} K_i(x)$ for all i are positively semi-independent.

For the endowment w_i in R^l and given prices $p \in S^{l-1}$, consumer i is faced with the problem

$$\max u_i(z) \qquad \text{s.t.} \quad p(z - w_i) \leq 0.$$

We define mappings f_1^i, f_2^i for each i, and f_3 and f_4 as follows

$$f_1^i(x_i, \mu_i, p) = \mathrm{D}u_i(x_i) - \mu_i p,$$

$$f_2^i(w_i, x_i, p) = - p(x_i - w_i),$$

$$f_3(p) = 2p \text{ and } f_4 = \sum_{i=1}^{m} (x_i^1 - w_i^1).$$

where we denotes by x^1 a subvector of a vector x with the first coordinate deleted. If x_i is determined by $f_1^i(x_i, \mu_i, p) = 0$ and $f_2^i(w_i, x_i, p) = 0$ for some positive μ_i, then x_i represents a demand of consumer i for the endowment w_i and prices p. We put $f_j = (f_j^i)_{i=1}^m$ for $j = 1, 2$, $Y = R^{2ml} \times R_{++}^m \times S^{l-1}$ and $T = R^{ml+m+l}$. For a mapping $f : Y \rightarrow T$ defined by $f = (f_1, \ldots, f_4)$, we introduce a set

$$E = \{(w, x, \mu, p) \in Y : f(w, x, \mu, p) = 0\},$$

where μ denotes the m-tuple $(\mu_i)_{i=1}^m$. The set E turns out to be an equilibrium manifold.

We presents more notations;

$$\delta f_1 = (\delta f_1^1, \ldots, \delta f_1^m) \in R^{lm} ; \delta f_2 = (\delta f_2^1, \ldots, \delta f_2^m) \in R^l ;$$

$$\delta f_3 \in R ; \delta f_4 \in R^{l-1}; \delta f = (\delta f_1, \ldots, \delta f_4) \in T.$$

The assumption $\mathbf{a}_2'$ will lead to the compactness of a set of equilibria for a compact set of economies, which is equivalent to the properness of the projection mapping $\pi : E \rightarrow R^{lm}$.

Proposition 4. Under $\mathbf{a}_1'$ and $\mathbf{a}_2'$, the set E is a submanifold of Y of dimension ml and the projection mapping $\pi : E \rightarrow R^{lm}$ is proper.

Proof) The set E is an inverse image $f^{-1}(0)$. First we will show the mapping f has 0 as a regular value, which is equivalent to showing the surjectivity of the linear mappings $\mathrm{D}f(y)$

for all y in $f^{-1}(0)$. Then the regular value theorem says that $f^{-1}(0)$ is a submanifold of dimension ml. Suppose a vector δf in T solves the homogeneous linear system of equations

(5) $\delta f \mathrm{D} f(w, x, \mu, p) = 0$ for a pair $(w, x, \mu, p) \in f^{-1}(0)$.

In particular, we have the following subsystems; for each i

(5-1) $\delta f_2^i p - \delta f_4[0\ \mathrm{I}_{l-1}] = 0,$

(5-2) $\delta f_1^i \mathrm{D}^2 u_i(x_i) - \delta f_2^i p + \delta f_4[0\ \mathrm{I}_{l-1}] = 0,$

and

(5-3) $\delta f_1^i p = 0,$

where $[0\ \mathrm{I}_{l-1}]$ is a $(l-1) \mathrm{x} l$ matrix with the identity matrix I_{l-1} of order $l - 1$.

It is immediately obtained from (5-1) that $\delta f_4 = 0$ and $\delta f_2^i = 0$ for each i. Then (5-2) and (5-3) combined with a_1' yield $\delta f_1^i = 0$ for each i. Substituting those results into the system (5) results in $\delta f_3 = 0$. Thus $\delta f = 0$ is a unique solution to (5) for all $(w, x, \mu, p) \in f^{-1}(0)$, which leads to the surjectivity of $\mathrm{D}f$ on $f^{-1}(0)$.

Let F be a compact set in R^{lm} and (w_n, x_n, μ_n, p_n) a sequence in $\pi^{-1}(F)$. We have to show that $\pi^{-1}(F)$ is compact, which is equivalent to showing that (w_n, x_n, μ_n, p_n) has a subsequence convergent to a point in $\pi^{-1}(F)$. Since F is compact, (w_n) has a subsequence converging to a point w in R^{lm}. Without any loss, we may assume that the sequence (w_n) itself is convergent. We put $\alpha_n = \sum_{i=1}^{m} |x_{in}|$. Suppose that (x_n) does not have a convergent subsequence. Since $\lim \alpha_n = \infty$, without loss of generality we can assume that $\alpha_n > 0$ for all n. Put $\xi_{in} = x_{in}/\alpha_n$ for each i. Since (ξ_{in}) is bounded, it has a subsequence, denoted by the same notation, converging to a vector ξ_i in R^l. Let w_i^* be a point in R^l such that $u_i(w_i^*) \le \min \{u_i(w_{in})\}$. In fact, for each i we have $\xi_i \in 0^+P_i(w_i^*)$ by Lemma A1 because $P_i(w_i^*) \supset$ the union of $P_i(w_{in})$ over $n \supset (x_{in})$. It is clear that $\lim (\sum_{i=1}^{m} |x_{in}|)/\alpha_n = 1$ and for some i, $\xi_i \ne 0$. But we obtain

$$\lim \sum_{i=1}^{m} (x_{in} - w_{in})/\alpha_n = \sum_{i=1}^{m} \xi_i = 0.$$

It contradicts the condition $\mathbf{a}_2'$. Thus (x_n) has a subsequence converging to a vector x in R^{lm}. Again we may assume that (x_n) itself is convergent. Clearly $(\mathrm{D}u_i(x_{in}))$ converges to a vector $\mathrm{D}u_i(x_i)$ which is nonzero by $\mathbf{a}_1'$ (ii). Since $(\mathrm{D}u_i(x_{in}))$ and (p_n) is bounded, (μ_{in}) is bounded, too. Thus (p_n) and (μ_{in}) have subsequences converging to points p and $\mu_i >$ 0, respectively. Since f is continuous, the limit point (w, x, μ, p) is in $\pi^{-1}(F)$. *Q.E.D.*

The existence of an equilibrium for every economy w in R^{lm} under $\mathbf{a}_1'$ and $\mathbf{a}_2'$ is immediate from Theorem 1 and the remarks following the assumption $\mathbf{a}_4$, because every consumer has a consumption set R^l. By applying Sard's theorem and the inverse function theorem, we have the genericity of local uniqueness of equilibria.

Theorem 2. Under the conditions $\mathbf{a}_1'$ and $\mathbf{a}_2'$, there exists an equilibrium for every economy and there exists an open dense subset Θ of R^{lm} with the null complement such that every economy in Θ has a finite number of equilibria.

Balasko (1988) investigates the existence of an equilibrium in the same framework except for that he hypothesizes that the preferences are bounded from below, which has the same effect as imposing a lower bound on consumption sets. In the book, the hypothesis is taken to be 'not essential but merely convenient' in proving the properness of the projection mapping on an equilibrium manifold. Rather, we have shown that the nonexistence of economic symbiosis is essential to verifying the properness of the projection mapping when preferences are not bounded-below.

CHAPTER 6

Infinite Dimensional Markets

6.1. The Price Support of Weak Optima

A need for an infinite dimensional commodity space arises when we are interested in an economy operating in a continuous time, an infinite time horizon or facing infinite states of the world in the future. In this chapter, we consider an economy with infinite dimensional consumption sets which may not be bounded from below. An example is an economy with infinite contingent claims markets in which unlimited short sales are allowed. We assume all consumption sets X_i are subsets of a normed linear space E. In a real world, every agent does not have the same opportunity to participate in securities markets. In order to encompass cases with asymmetric participation, we allow consumption sets to vary with agents. Let $X = \sum_{i=1}^{m} X_i$. We make the following assumptions on consumption sets.

b$_1$. All X_i are closed, convex sets with the non-empty relative interior in a closed subspace T_i of E such that $E = \sum_{i=1}^{m} T_i$, and T_i has a closed subspace Z_i with $X = Z_1 \oplus Z_2 \oplus \ldots \oplus Z_m$, i.e. X is their topological direct sum.

In case that all X_i are closed subspaces, the above conditions are interpreted as follows. The space E can be considered a set of portfolios that can be spanned by potentially feasible portfolios for all consumers. Thus the condition $E = \sum_{i=1}^{m} X_i$ is natural. Every vector space can be written as an algebraic sum of its subspaces. The real restriction is the closedness of subspaces Z_i spanning X. The closedness assumption trivially holds if there exists a set X_i which is so big that it has a finite dimensional complemented subspace because every finite dimensional subspace is closed in E. It is easy to find examples satisfying the closedness condition in Hilbert spaces because Hilbert spaces allow complemented subspaces. For example, let X be a Hilbert space with a direct sum $X = Z_1 \oplus Z_2 \oplus \ldots \oplus Z_m$. Then a set of closed subspaces X_i containing Z_i satisfies the condition. We provide the price supportability of weakly optimal allocations in an infinite dimensional commodity space.

Proposition 5. Under the assumptions $\mathbf{a_2}$-$\mathbf{a_3}$ and $\mathbf{b_1}$, a weak optimum x can be supported by some continuous linear functional p in E^*, i.e. $py \geq px_i$ whenever $y \geq_i x_i$ for all i.

Proof) It is well known that there exists a subspace Y_i of T_i such that $T_i = Z_i + Y_i$ and $Z_i \cap Y_i = \{0\}$. Define the projection mapping $\varphi_i : T_i \to Z_i$ by $\varphi_i(x) = z$ where $x = z + y$ with $z \in Z_i$ and $y \in Y_i$. The mapping φ_i is open. In fact, $\varphi_i(V)$ is open in Z_i for an open set V in T_i since $\varphi_i(V) = (V + Y_i) \cap Z_i$ with $V + Y_i$ open in T_i. Let x be a weakly optimal allocation for an economy w. Put $P = \sum_{i=1}^{m} \text{int } P_i(x_{in})$ and $V_i = \varphi_i(\text{int } P_i(x_{in}))$. First, we will show that P has the non-empty interior in E. Since int $P_i(x_{in})$ is open in T_i, V_i is open in Z_i. It is sufficient to show that $\sum_{i=1}^{m} V_i$ has the non-empty interior. Let $\pi_i : E \to Z_i$ be the canonical projection mapping for each i. Under the assumption $\mathbf{b_1}$, π_i is continuous by Lemma A6 in Appendix. Thus $\pi_i^{-1}(V_i)$ is open in E. Put $Z_{-i} = \sum_{j \neq i} Z_j$. Since $\pi_i^{-1}(V_i) = V_i + Z_{-i}$ for each i and $\sum_{i=1}^{m} V_i = \bigcap_{i=1}^{m} (V_i + Z_{-i})$, $\sum_{i=1}^{m} V_i$ is open. Therefore we conclude that the interior of P is not empty.

We claim $\sum_{i=1}^{m} w_i \notin P$. Otherwise, there exists an allocation y such that $x_i <_i y_i$ for all i and $\sum_{i=1}^{m} w_i = \sum_{i=1}^{m} y_i$. It contradicts the weak optimality of x. Since $\sum_{i=1}^{m} w_i \notin$ int P, the separating hyperplane theorem guarantees the existence of a nonzero linear functional p in E^* such that

$$p(\sum_{i=1}^{m} w_i) \leq pz \quad \text{for all } z \in P.$$

Thus we obtain $p(\sum_{i=1}^{m} w_i) \leq pz$ for all $z \in$ cl P. Since cl $P \supset \sum_{i=1}^{m} P_i(x_i)$ under $\mathbf{a}_3$, in particular we have $p(\sum_{i=1}^{m} x_i) \leq p(y + \sum_{i \neq j}^{m} x_i)$ whenever $y \in P_j(x_j)$ for all i, or $px_j \leq pz$ for all $z \in P_j(x_j)$. *Q.E.D.*

The compactness requirement for the feasible set of an economy is problematic in an infinite dimensional economy in general. Before proceeding with it, we introduce some concepts and notations of a vector space with the order structure. By an ordered normed linear space E, we mean a normed linear space together with a reflexive, transitive, anti-symmetric relation $\leq$ on E, satisfying the following additional properties: (*a*) if $x \leq y$ and $\alpha \in R_+$ then $\alpha x \leq \alpha y$, (*b*) if $x \leq y$ and $0 \leq z$ then $x + z \leq y + z$. We define the positive cone $E^+ = \{x \in E : 0 \leq y\}$. For example, let (Ω, F, μ) be a Lebesgue measure space and $L_p(\Omega, F, \mu)$ the space of p-th power integrable functions on (Ω, F, μ) for $1 \leq p \leq \infty$. If the underlying measure space (Ω, F, μ) is clear from the context, we write L_p for $L_p(\Omega, F, \mu)$. The ordering on $L_p(\Omega, F, \mu)$ is defined pointwise, i.e. $f \leq g$ if $f(\omega) \leq g(\omega)$ almost everywhere for f and g in $L_p(\Omega, F, \mu)$. Then $L_p^+ = \{f \in L_p(\Omega, F, \mu) : 0 \leq f(\omega)$ almost everywhere $\}$.

To contrast with an extended economy an economy with consumption sets in a positive cone, we call it a positive economy. For an economy w on an infinite dimensional space E, the boundedness problem of $\Omega(w)$ is very delicate because without extra conditions on E^+, $\Omega(w)$ may not be bounded even if consumption sets are restricted to E^+. The following example which is due to Krasnosel'skii (1964) and Mas-Colell and Zame

(1991) is conflicting to the finite dimensional perspective because every economy in the nonnegative orthant of a finite dimensional Euclidean space has the bounded feasible set.

Example 1. Let $E = C^1([0, 1])$, the space of continuously differentiable functions on [0, 1], with the norm

$$\|x\| = \max|x(t)| + \max|x'(t)|$$

and pointwise ordering. In an economy with two consumers having consumption sets $X_1 = X_2 = E^+$ and endowments w_1 and w_2 in E^+, the feasible set

$$\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_1 \leq w_1 + w_2\}$$

is not norm-bounded. For there exist arbitrarily small functions with arbitrarily large derivatives.

This example can arise because there exist two positive vectors in the positive cone of $C^1([0, 1])$ with certain length which may be added up to a vector arbitrarily close to zero. This phenomenon does not seem to be natural in terms of economic intuition because every aggregation of sizable positive quantities in a real economy is bounded away from zero. Fortunately, the boundedness of the feasible set for a positive economy is restored if that counter-intuitive addup is assumed away. We need to put some restrictions on a positive cone. A cone C in a normed linear space E is *normal* if there exists a number $\delta > 0$ such that $\|x_1 + x_2\| \geq \delta$ for all x_1 and x_2 in C with $\|x_1\| = \|x_2\| = 1$. A norm $\| \ \|$ in the space E is *semi-monotonic* if $x \leq y$ implies $\|x\| \leq N\|y\|$ for all x, y in E^+ and some constant N. When the normality is imposed on a positive cone, it simply requires that a sum of two positive vectors with a unit length be bounded away from zero. So the normality of a positive cone is a natural condition in light of economic aggregation. Moreover, we have the following consequences, which are quoted from Krasnosel'skii (1964) or Peressini (1967).

Lemma 5. Let E be an ordered linear space with a norm $\| \ \|$. Then the following statements are equivalent.

(i) The positive cone E^+ is normal.

(ii) The norm $\| \ \|$ is semi-monotonic.

(iii) The set $\{x \in E : 0 \le x \le y\}$ for a vector y in E^+ is norm-bounded.

Thus the normality of a positive cone of normed linear spaces with order structures is necessary and sufficient for the feasible sets for a positive economy to be bounded. If E is a normed lattice, the norm in E is semi-monotonic by definition and therefore, the positive cone is normal. In particular, the positive cone in the Lebesgue spaces L_p is normal. Since the normality is sufficient and necessary conditions for the boundedness of $\Omega(w)$ in a positive economy, the lattice structure in a Banach space seems to be a little stronger condition than being required as far as the boundedness of $\Omega(w)$ is concerned.

We provide an existence theorem for an extended economy. The proof is very similar to the one for a finite dimensional case except for applying some theorems specific to infinite dimensional spaces like Alaouglou's theorem, which is well demonstrated in Magill (1981), Mas-Colell (1986) or Cheng (1991).

Theorem 3. Under $\mathbf{a}_2$-$\mathbf{a}_4$, and $\mathbf{b}_1$, an economy w has an equilibrium provided the Closedness Hypothesis is satisfied.

There are two cases for which the Closedness Hypothesis holds; $\Omega(w)$ is bounded or unbounded. If $\Omega(w)$ is not bounded, it is very difficult to verify the hypothesis in general. But Cheng (1991) shows that if preferences can be represented by the expected utility, the Closedness Hypothesis is fulfilled for an economy in $L_p(\Omega, F, \mu)$ with a single commodity in each state of the world. For expository convenience, we will analyze the case with the bounded $\Omega(w)$ in 6.2 and the case with the unbounded $\Omega(w)$ modelled on the expected utility assumption in 6.3.

6.2. The Bounded $\Omega(w)$

We consider a normed linear space E with a normal positive cone E^+. For a positive economy of infinite dimension, the Closedness Hypothesis comes true crucially depending on the topology of the space E. A sufficient condition for the Closedness Hypothesis to hold is the compactness of $\Omega(w)$ with respect to the topology on E. But the topology on E should be sensible from the economic point of view. Under the normality of E^+, $\Omega(w)$ is always compact in the weak* topology because $\Omega(w)$ is closed and bounded. For reflexive spaces, $\Omega(w)$ is compact in the weak topology because the weak* topology coincides with the weak topology. In particular, $\Omega(w)$ for a positive economy on L_p spaces with $1 < p < \infty$ is weakly compact because these are reflexive with the normal positive cone. For more details about the compactness of $\Omega(w)$ for a positive economy, we refer to Mas-Colell (1986).

It is easy to guess the situation will worsen for an extended economy. First of all, we may ask which conditions guarantee the boundedness of $\Omega(w)$ in an extended economy. There is nothing special about an extended economy compared to a positive one as far as the Closedness Hypothesis is concerned if the boundedness of $\Omega(w)$ is restored in an extended economy. In the following we will examine the conditions under which $\Omega(w)$ is bounded in an extended economy. We provide some conditions on cones in E, which is similar to the normality.

Definition 2. A cone C in a normed linear space with a norm $\| \ \|$ is strictly pointed if there exists a number $\delta > 0$ such that $\|z_1 + z_2\| \geq \delta \max\{\|z_1\|, \|z_2\|\}$ for all $z_1, z_2 \in C$.

It is easy to check that the positive cone E^+ is normal if and only if it is strictly pointed. But the strict pointedness condition is more restrictive than the normality condition in general because the former requires every convex combination of two vectors with unit length and with smaller length, respectively, be bounded away from zero. The following example borrowed from Krasnosel'skii et al (1989) shows that there exists a strictly pointed cone with the nonempty interior even if the positive cone has empty interior.

Example 2. Consider a closed convex cone in l_2

$$C = \{ x = (\alpha_1, \alpha_2, \ldots) : \alpha_1 \geq 0 \text{ and } \alpha_1^2 \geq \alpha_2^2 + \alpha_3^2 + \ldots \}.$$

This cone has the nonempty interior in l_2. Let $x = (\alpha_1, \alpha_2, \ldots)$ and $y = (\beta_1, \beta_2, \ldots)$ be two vectors in C. Then we have $\|x\| \leq \sqrt{2}\alpha_1$ and $\|y\| \leq \sqrt{2}\beta_1$, which leads to

$$\|x + y\| \geq \alpha_1 + \beta_1 \geq \frac{1}{\sqrt{2}} \{\|x\| + \|y\|\} \geq \frac{1}{\sqrt{2}} \max\{\|x\|, \|y\|\}.$$

It implies that C is strictly pointed.

We claim that if a cone C is strictly pointed, then for a positive integer k there exists $\delta' > 0$ such that $\|\sum_{i=1}^{k} z_i\| \geq \delta' \max\{\|z_1\|, \ldots, \|z_k\|\}$ for all $z_i \in C$. It will be sufficient to show the claim is true only for $k = 3$ because it is possible to induce the general case. For $k = 3$, we have

$$\|z_1 + z_2 + z_3\| \geq \delta \max\{\|z_1\|, \|z_2 + z_3\|\} \geq \delta \max\{\|z_1\|, \delta \max\{\|z_2\|, \|z_3\|\}\}$$

for some $\delta > 0$, by the strict pointedness of C. If $\delta \geq 1$, it leads to

$$\delta\max\{\|z_1\|, \delta\max\{\|z_2\|, \|z_3\|\}\} \geq \delta\max\{\|z_1\|, \max\{\|z_2\|, \|z_3\|\}\} = \delta\max\{\|z_1\|, \|z_2\|, \|z_3\|\}.$$

If $0 < \delta < 1$, we obtains

$$\delta\max\{\|z_1\|, \delta\max\{\|z_2\|, \|z_3\|\}\} = \delta^2\max\{\|z_1\|/\delta, \|z_2\|, \|z_3\|\} \geq \delta^2\max\{\|z_1\|, \|z_2\|, \|z_3\|\}.$$

For all $z_i \in C$ with $\sum_{i=1}^{m} \|x_{in}\| \neq 0$, put $y_i = z_i/\max\{\|z_1\|, \ldots, \|z_k\|\}$. Then C is strictly pointed if and only if there exists a number $\delta' > 0$, such that $\|\sum_{i=1}^{k} y_i\| \geq \delta'$ for all $y_i \in C$ with $\|y_i\| = 1$ for some i and $\|y_j\| \leq 1$ for $j \neq i$.

For a convex cone C in E, let $R(z, \varepsilon; C)$ denote a set $\{x \in E^m : \sum_{i=1}^{m} x_i \in B(z, \varepsilon) \cap C$ and $x_i \in C$ for all $i\}$ and $R(z; C)$ a set $\{x \in E^m : \sum_{i=1}^{m} x_i = z$ and $x_i \in C$ for all $i\}$ for some z in E and $\varepsilon \geq 0$. The strict pointedness condition leads to the following lemma.

Lemma 6. Let C be a convex cone in E. Then the cone C is strictly pointed if and only if $R(z, \varepsilon; C)$ is norm-bounded for any $z \in C$ and $\varepsilon > 0$. Suppose that C has the nonempty relative interior with respect to some affine subspace. Then it is strictly pointed if and only if $R(z; C)$ is norm-bounded for any z in the relative interior of C.

Proof) On the contrary, suppose that $R(z, \varepsilon; C)$ is not bounded. Then there exists a sequence (x_n) in $R(z, \varepsilon; C)$ with $\sum_{i=1}^{m} \|x_{in}\| \to \infty$. Put $\delta_n = \sum_{i=1}^{m} \|x_{in}\|$. Since $\sum_{i=1}^{m} x_{in} \in B(z, \varepsilon)$, we have $\| \sum_{i=1}^{m} x_{in}/\delta_n \| \to 0$. Since $\max \{\|x_{in}/\delta_n\|\} \geq 1/m$ for all n, it contradicts the strict pointedness of C. Conversely, suppose C is not strictly pointed. Then there exist sequences (x_n) in C with $\|x_{in}\| = 1$ for some i and (y_n) with $\|y_n\| \to 0$ such that $\sum_{i=1}^{m} x_{in} = y_n$. For notational convenience, we assume $\|x_{1n}\| = 1$ for all n. Since C is a convex cone, y_n is in C for all n. Pick a sequence (λ_n) of positive numbers with $\lambda_n \to \infty$ such that $\|\lambda_n y_n\| \to 0$. For some nonzero z in C, we have $\lambda_n x_{1n} + z + \sum_{i=2}^{m} \lambda_n x_{in} = \lambda_n y_n + z$. Since C is a convex cone, $\lambda_n x_{1n} + z$, $\lambda_n x_{in}$ for all $i \geq 2$ and $\lambda_n y_n + z$ are in C for all n. But $\|\lambda_n y_n\| \to 0$ implies $\lambda_n y_n + z \in B(z, \varepsilon) \cap C$ for sufficiently large n. It contradicts the boundedness of $R(z, \varepsilon; C)$ because in particular, $\lambda_n x_{1n} + z \to \infty$ with $\lambda_n x_{1n} + z$ in C.

Let F be the affine subspace of E with respect to which C has the nonempty relative interior. Pick a nonzero vector v in the relative interior of C such that $B(v, \varepsilon) \cap F$ is in C for some $\varepsilon > 0$. Then for sufficiently large n, $v - \lambda_n y_n \in B(v, \varepsilon) \cap F$ where $B(v, \varepsilon)$ is an open ball in E with the radius ε centered at v. The conclusion comes from putting $z = v - \lambda_n y_n$ in the previous argument. *Q.E.D.*

One immediate consequence of the above lemma is that $\Omega(w)$ is bounded if consumption sets are subsets of a set $y + C$ where $y \in E$ and C is a strictly pointed convex cone which has the nonempty relative interior with respect to the affine subspace F. This is a generalization of the boundedness of $\Omega(w)$ for a positive economy to an economy with consumption sets bounded below by some strictly pointed cone.

We consider imposing some conditions on preferences. We recall the properness condition of preferences which is due to Mas-Colell (1986). A preference $\leq_i$ is proper at $x \in X_i$ if there exists an open convex cone V such that $(x - V) \cap P_i(x)$ is empty. We propose some conditions on preferences which is similar to the normality of a cone. For some nonnegative number k, we denote a set $\{x \in P_i(x_i) : \|x\| \geq k\}$ by $P_i^k(x_i)$.

Definition 3. A preference $\leq_i$ is normal at x in X_i if there exists numbers $\delta > 0$ and $k > 0$ such that $\|z_1 + z_2\| \geq \delta \max\{\|z_1\|, \|z_2\|\}$ for all $z_1, z_2 \in P_i^k(x_i)$.

The normality of preferences is well defined in the sense that if $\leq_i$ is normal at x, it is normal at any consumption in $P_i(x)$. It is a proper generalization of two well-known cases to an extended economy on an infinite dimensional space. First, it is innocuous on finite dimensional consumption sets because it holds whenever indifference curves does not contain non-trivial affine subspaces. Also by definition, preferences in a normal positive cone of normed linear spaces are normal. We introduce the following condition

b$_2$. For each i, every vector v_i in $0^+P_i(w_i)$ satisfies the relation

$$\inf_{z \in P_i(w_i)} \| \lambda v_i + z \| \to \infty \text{ as } \lambda \to \infty.$$

This condition turns out to encompass the normality as well as the properness property of preferences.

Lemma 7. The assumption **b$_2$** holds true under one of the following conditions; (i) $\leq_i$ is normal at w_i or (ii) under the MWC condition, $\leq_i$ is proper at w_i with the half-line free indifference curve through w_i.

Proof) Let $\leq_i$ be normal at w_i and $v_i \in 0^+P_i(w_i)$. Suppose $\inf_{z \in P_i(w_i)} \| \lambda_n v_i + z \|$ is bounded as $\lambda_n \to \infty$. It implies there exists (z_n) in $P_i(w_i)$ with $\|z_n\| \to \infty$ such that $\| \lambda_n v_i$

$+ z_n \|$ is bounded. But we have $\| \lambda_n v_i + z_n \| + \|w_i\| \geq \| (\lambda_n v_i + w_i) + z_n\|$ which contradicts the normality of $\leq_i$ because

$\|\lambda_n v_i + w_i\| \to \infty$ and $\|z_n\| \to \infty$ with $\lambda_n v_i + w_i$ and z_n in $P_i(w_i)$.

By the properness of $\leq_i$ at w_i, there exists a continuous linear functional p supporting $P_i(w_i)$ at w_i, i.e. $pw_i \leq pz$ for all $z \in P_i(w_i)$. Since the indifference curve through w_i has no half-lines, we have $w_i <_i w_i + \lambda v_i$ for all $v_i \in 0^+P_i(w_i)$ and $\lambda > 0$. Suppose $pw_i = p(w_i + \lambda v_i)$ for some $\lambda > 0$. Under the MWC condition, there exists a point y such that $py < pw_i$. For all $0 \leq \alpha < 1$, we obtain $\alpha p(w_i + \lambda v_i) + (1 - \alpha)py < pw_i$. But all $\alpha < 1$ sufficiently close to 1 implies $w_i <_i \alpha(w_i + \lambda v_i) + (1 - \alpha)y$, which leads to contradiction. We conclude that $pv_i > 0$ for all $v_i \in 0^+P_i(w_i)$. Without loss of generality, we may assume $\|p\| \leq 1$. For any $\lambda > 0$ and z in $P_i(w_i)$, we have $p(w_i + \lambda v_i) \leq p(z + \lambda v_i)$. As $p(w_i + \lambda v_i) \to \infty$ with $\lambda \to \infty$, so does $p(y + \lambda v_i) \to \infty$. On the other hand, we have

$$p(y + \lambda v_i) \leq \|p\| \, \|y + \lambda v_i\| \leq \|y + \lambda v_i\|,$$

implying $\|y + \lambda v_i\| \to \infty$ with $\lambda \to \infty$. *Q.E.D.*

For some $v_i \in 0^+P_i(w_i)$ and $\lambda > 0$, let $\Gamma_i(v_i; \lambda)$ be the smallest closed cone containing the set $\lambda v_i + P_i(w_i)$. If $\leq_i$ is either normal or proper at w_i, then there exists $\lambda > 0$ by Lemma 7 such that $\lambda v_i + P_i(w_i)$ is away from the origin. Thus for such an $\lambda > 0$, $\Gamma_i(v_i; \lambda)$ is a pointed closed cone. Fix v_i and put $\Gamma_i(v_i) = \bigcap_{\lambda \geq 0} \Gamma_i(v_i; \lambda)$.

Lemma 8. Under the condition $\mathbf{b_2}$, the cone $\Gamma_i(v_i)$ coincides with the recession cone $0^+P_i(w_i)$.

Proof) We recall the definition of asymptotic cones of a set in a finite dimensional space in Debreu (1959). Let $C_i^k(w_i)$ be the closure of the cone generated by $P_i^k(w_i)$, implying $C_i^k(w_i)$ is a cone, too. The asymptotic cone of $P_i(w_i)$ is defined in Debreu (1959) by a cone $\cap_{k\geq 0} C_i^k(w_i)$. It is easy to check that this cone is in fact the recession cone because $P_i(w_i)$ is a closed convex set. Thus $0^+P_i(w_i) = \cap_{k\geq 0} C_i^k(w_i)$. By assumption, for each k

there exists a positive number λ_k such that $\inf_{z\in P_i(w_i)} \|\lambda_k v_i + z\| \geq k$. Thus we have $C_i^k(w_i) \supset \lambda_k v_i + P_i(w_i)$ since $v_i \in 0^+P_i(w_i)$ implies $P_i(w_i) \supset \lambda_k v_i + P_i(w_i)$. It implies $C_i^k(w_i) \supset \Gamma_i(v_i;\, \lambda_k)$, which yields $0^+P_i(w_i) \supset \Gamma_i(v_i)$. Conversely, let $v \in \Gamma_i(v_i)$. Then for a sequence $\lambda_k \to \infty$, there exist sequences (x_k) in $P_i(w_i)$, (α_k) with $\alpha_k \geq 0$ and (r_k) in a closed ball $B(\varepsilon_k) = \{x \in E\colon \|x\| \leq \varepsilon_k\}$ with $\varepsilon_k \to 0$ such that $v = \alpha_k(\lambda_k v_i + x_k) + r_k$. By assumption, we have $\|\lambda_k v_i + x_k\| \to \infty$ with $\lambda_k \to \infty$, which implies $\alpha_k \to \infty$. By Lemma A1 in Appendix, we obtain $v \in 0^+P_i(w_i)$ since $\alpha_k(\lambda_k v_i + x_k) \to v$ with $\lambda_k v_i + x_k$ in $P_i(w_i)$. *Q.E.D.*

We impose the last condition on preferences at an aggregate level.

> $\mathbf{b}_3$. There exists a number $\delta > 0$ and an m-tuple (λ^i) of nonnegative numbers such that $\|\sum_{i=1}^{k} z_i\| \geq \delta \max\{\|z_1\|,\ldots, \|z_k\|\}$ for all $z_i \in \Gamma_i(v_i;\, \lambda^i)$.

The condition $\mathbf{b}_3$ can be considered the aggregate normality condition of preferences because the sum $\sum_{i=1}^{m} \Gamma_i(v_i;\, \lambda^i)$ is strictly pointed. On the other hand, due to Lemma 8 a cone $\Gamma_i(v_i;\, \lambda^i)$ can be considered an approximation of the recession cone $0^+P_i(w_i)$ so that $\mathbf{b}_3$ may be interpreted as an asymptotic version of (3). Theses consequences lead to the boundedness of the feasible set for an extended economy in an infinite dimensional setting.

Proposition 6. For each i, assume that X_i is a subset of a normed linear space E. Then under $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{b}_2$, the set $\Omega(w)$ is convex, closed and bounded if $\mathbf{b}_3$ is satisfied.

Proof) That $\Omega(w)$ is convex and closed is immediate. Under $\mathbf{b}_3$, there exists $\delta > 0$ and an m-tuple (λ^i) such that $\|\sum_{i=1}^{k} z_i\| \geq \delta \max\{\|z_1\|,\ldots, \|z_k\|\}$ for all $z_i \in \Gamma_i(v_i;\, \lambda^i)$. Put $C = \sum_{i=1}^{m} \Gamma_i(v_i;\, \lambda^i)$ and $z = \sum_{i=1}^{m}(\lambda^i v_i + w_i)$. Since C is a strictly pointed convex cone, $R(z;\, C)$ is bounded by Lemma 6, which implies $\Omega(w)$ is bounded. *Q.E.D.*

As remarked before, in an extended economy $\Omega(w)$ is always compact in the weak* topology under $\mathbf{a}_1$-$\mathbf{a}_2$ and $\mathbf{b}_2$-$\mathbf{b}_3$ since $\Omega(w)$ is bounded and closed. Moreover for reflexive spaces, the set $\Omega(w)$ is compact in the weak topology because the weak* topology coincides with the weak topology. In particular, $\Omega(w)$ for an extended economy on L_p spaces with $1<p<\infty$ is weakly compact.

6.3. The Unbounded $\Omega(w)$

All possible states of the world are represented by a measure space (Ω, F). For each state, there are l commodities in each state in (Ω, F). We assume each agent has the same probability assessments over the state of the world which are represented by a probability measure μ. Let $L_p(\Omega, F, \mu)^l$ be the commodity space for $1 \leq p \leq \infty$. The individual trading opportunities for contingent commodities may differ with the exogenously imposed constraint on each agent, which can be generated by some institutional arrangements or individual trading capability. Such an asymmetry of trading opportunities is represented by the consumption set X_i which is a subspace of $L_p(\Omega, F, \mu)^l$. A realization $x(s)$ in a state s in (Ω, F) of a random vector x represents a commodity vector contingent on s. We assume preferences are represented by the expected utility function v_i which has a von Neumann-Morgenstern utility function u_i on R^l. Thus the utility function $v_i : L_p(\Omega, F, \mu)^l \to R$ is defined by $v_i(x) = \int u_i(x(s))d\mu(s)$. We impose the following assumptions on u_i.

$\mathbf{c}_1$. $u_i : R^l \to R$ is strictly concave and have no satiation point.

$\mathbf{c}_2$. Let K_i be the recession cone of u_i. A set of cones $\{K_i\}$ is positively semi-independent.

For $l = 1$ and $1 < p \leq \infty$, under the above assumptions Cheng (1991) shows that $\Omega(w)$ may not be bounded if the endowment w is not weakly optimal and the marginal utility of income is bounded from above for some agent i. But he verifies the Closedness Hypothesis under the expected utility assumption for $l = 1$, which seems to need complicated arguments. In fact, the set W is bounded. For a finite number l, we show through simple geometric intuition that the Closedness Hypothesis is fulfilled in a special case where the aggregate endowment is constant.

Lemma 9. Assume that each X_i contains a non-zero constant. Let w_i be a vector in X_i for all i. Under the condition $\mathbf{c}_2$, the augmented utility possibility set W is bounded. Furthermore, if $\sum_{i=1}^{m} w_i(s) = a$ for all s in (Ω, F) with a vector a in R^l, the Closedness Hypothesis holds under $\mathbf{c}_1$ and $\mathbf{c}_2$.

Proof) Let (x_n) be a sequence in $\Omega(w)$ with $x_{in} \leq_i x_{i(n+1)}$ for all i and n. We define a hyperplane H by $H = \{y \in L_p(\Omega, F, \mu) : \int y(s)d\mu(s) = 0\}$. Since a unit random variable 1 is not in H, $L_p(\Omega, F, \mu)$ can be written as a direct sum of H and <b>, i.e. $L_p(\Omega, F, \mu) = H \oplus \langle 1 \rangle$, where $\langle 1 \rangle$ denotes a one-dimensional subspace of $L_p(\Omega, F, \mu)$ spanned by a vector 1. Thus for all n and i, a random vector x_{in} is uniquely decomposed into $x_{in} = z_{in} + \lambda_{in}$ with z_{in} in H^l and λ_{in} in R^l. Similarly there exists a pair of y_i in H^l and κ_i in R^l such that $w_i = y_i + \kappa_i$. Thus we have $\sum_{i=1}^{m} (z_{in} + \lambda_{in}) = \sum_{i=1}^{m} (y_i + \kappa_i)$. By the uniqueness of the decomposition, we obtain

$$\sum_{i=1}^{m} z_{in} = \sum_{i=1}^{m} y_i \text{ and } \sum_{i=1}^{m} \lambda_{in} = \sum_{i=1}^{m} \kappa_i \text{ for all } n.$$

On the other hand, individual rationality and Jensen's inequality lead to

$$\int u_i(w_i(s))d\mu(s) \leq \int u_i(x_{in}(s))d\mu(s) \leq u_i\left(\int z_{in}(s)d\mu(s) + \lambda_{in}\right) = u_i(\lambda_{in}) \text{ for all } n.$$

Since $u_i(\lambda_{in})$ is bounded below, for each i there exists a vector c_i in R^l such that $u_i(c_i) \leq u_i(\int z_{in}(s)d\mu(s) + \lambda_{in}) = u_i(\lambda_{in})$ for all n. Thus by the assumption $\mathbf{c}_2$, (λ_{in}) is bounded for all i since $\sum_{i=1}^{m} \lambda_{in} = \sum_{i=1}^{m} \kappa_i$ for all n, implying W is bounded.

Let (λ_{in}) itself denote a subsequence converging to a vector λ_i in R^l. If $\sum_{i=1}^{m} w_i(s) = a$ for all s in (Ω, F), we have $\sum_{i=1}^{m} \lambda_{in} = \mathrm{a}$. Since $U_i(x_{in}(s))$ is increasing, we obtain $U_i(x_{in}) \leq U_i(\lambda_i)$ for all n and furthermore, $\sum_{i=1}^{m} \lambda_i = \mathrm{a}$ Thus the Closedness Hypothesis is satisfied.

Q.E.D.

We provide two remarks about the above lemma. In the proof of the Closedness Hypothesis in the lemma, we chose an allocation located in a finite dimensional subspace which Pareto dominates the original allocation by projecting it onto the subspace along the hyperplane H. In that special case, the closedness result is robust to the choice of the topologies in the sense that every Hausdorff vector space topology is identical in a finite dimensional subspace. Second, the recession cones of U_i's are positive semi-independent since U_i is strictly concave and all agents has the same belief μ. But as shown in Cheng (1991), the feasible set $\Omega(w)$ may not be bounded. Thus the positive semi-independence of the recession cones of preferences may not lead to the boundedness of $\Omega(w)$ in an infinite dimensional economy, which cannot happen in a finite dimensional economy.

We conjecture that the Closedness Hypothesis holds for a finite $l > 1$ in every economy under $\mathbf{c}_1$ and $\mathbf{c}_2$. The following example is a variant of the security market model in Kreps (1979).

Example 3. Consider an economy that endures in two dates, $t = 0, 1$. A probability space (Ω, F, μ) represent states of the world at date 1 as well as a unanimously held subjective probability assessment concerning the state of the world. There is a single consumption good, the numeraire, and agents are interested in certain consumption at date zero and state contingent consumption at date 1 which is a random variable on (Ω, F). We take $L_2(\Omega, F, \mu)$ as the space of contingent commodities. Let a subspace X_i of $L_2(\Omega, F, \mu)$ denote the consumption set for agent i. Thus we consider consumption bundles of the form (λ, x) in RxX_i. Here (λ, x) represents λ units of consumption at date zero and $x(s)$ units of consumption at date 1 if the state is s. Let a vector (γ_i, w_i) in RxX_i denote the endowment of agent i. A price system p for contingent commodities is a continuous linear functional on $L_2(\Omega, F, \mu)$. We assume agent i chooses an optimal pair (λ_i, x_i) which maximizes the expected utility

$$v_i(\lambda, x) = \int u_i(\lambda, x(s))d\mu(s) \qquad \text{subject to } \lambda + px \leq \gamma_i + pw_i, x \in X_i,$$

where $u_i : R\text{x}R \to R$ is a von Neumann-Morgenstern utility function. The expected utility presupposes that u_i and μ are sufficiently well behaved so that all integrals of the indicated form exists and are finite. Under the assumptions $\mathbf{c_1}$-$\mathbf{c_2}$ and $\mathbf{b_1}$, this economy has an equilibrium if our conjecture about the Closedness Hypothesis is realized.

APPENDIX

Review of Convex Sets and Cones

Let Γ be a convex subset of a topological vector space E. It is immediate from the definition of the recession cone that $0^{+}\Gamma$ contains the origin of E and that for convex subsets Γ_1 and Γ_2 of R^l, $\Gamma_2 \supset \Gamma_1$ implies $0^{+}\Gamma_2 \supset 0^{+}\Gamma_1$. Extensive discussion about the recession cone in R^l is found in Rockafellar (1970). Lemma A1 and A2 are available in Rockafellar (1970) in case for a finite dimensional vector space, which presents their proofs as a consequence of applying preliminary results. We provide alternative proofs which can be directly induced from definitions and furthermore, hold in a topological vector space. Those lemmas address the useful properties of the recession cone which find repeated application in this paper.

Lemma A1. Suppose Γ is a non-empty closed convex set in E. Then $0^{+}\Gamma$ is closed, and it consists of all possible limits of sequences (x_n/λ_n) with $x_n \in \Gamma$, $\lambda_n > 0$, and $\lambda_n \to \infty$.

Proof) Let x be a point in Γ and (y_n) a sequence in $0^{+}\Gamma$ that converges to a vector v in R^l. By definition, $x + \lambda y_n \in \Gamma$ for any $\lambda \geq 0$ and n. The sequence $(x + \lambda y_n)$ converges to $x + \lambda v$ in Γ because Γ is closed, which implies $v \in 0^{+}\Gamma$.

Let $y \in 0^{+}\Gamma$, $x \in \Gamma$ and (λ_n) be a sequence of positive numbers with $\lambda_n \to \infty$. For each n, there is x_n in Γ such that $x_n = x + \lambda_n y$. Or we have, $y = (x_n - x) / \lambda_n$, which implies that y is the limit of a sequence (x_n/λ_n). Conversely, let (x_n/λ_n) be a sequence converging to a vector v in E with $x_n \in Y$, $\lambda_n > 0$, and $\lambda_n \to \infty$. It is clear that if Γ is non-empty, Γ and the translation $b + \Gamma$ with b in R^l have the same recession cone. Thus without loss of generality, we may assume that $0 \in \Gamma$. Let x be a point in Γ and λ any nonnegative number. By convexity of Γ, we have $(\lambda x_n/\lambda_n + x)/2 = (\lambda/\lambda_n)(x_n + x)/2 + (1 - \lambda/\lambda_n)(x/2) \in \Gamma$ for sufficiently large n with $\lambda/\lambda_n < 1$, which implies that $\lambda x_n/\lambda_n + x \in \Gamma$. Since Γ is closed, $\lambda v + x \in \Gamma$ for all $\lambda \geq 0$ and $x \in \Gamma$. *Q.E.D.*

We have the following corollary as an immediate application of Lemma A1.

Corollary. Let Γ be a non-empty closed convex set in R^l. (i) Γ is bounded if and only if $0^+\Gamma = \{0\}$. (ii) Let v be a vector in E. If $z + \lambda v \in \Gamma$ for some $z \in \Gamma$ and all $\lambda \geq 0$, then $v \in 0^+\Gamma$.

The following lemma shows that a concave function has a simple property in terms of the recession cones of the level sets.

Lemma A2. Suppose that $f(x)$ is a concave function which is continuous on a convex subset Γ of E. Then the level set $P(x) = \{y \in \Gamma: f(y) \geq f(x)\}$ has the same recession cone for all $x \in \Gamma$.

Proof) Since $f(x)$ is concave, $P(x)$ is convex for all $x \in \Gamma$. If $f(x)$ is constant for all $x \in \Gamma$, the lemma is trivially true. Thus we may assume that there are points x_1 and x_2 in Γ such that $f(x_1) < f(x_2)$. Clearly, $P(x_1) \supset P(x_2)$, which implies $0^+P(x_1) \supset 0^+P(x_2)$. Let $v \in 0^+P(x_1)$. We have only to show that $f(y+\lambda v) \geq f(y)$ for all $y \in P(x_2)$ and $\lambda \geq 0$. Let $y \in P(x_2)$. Then by the concavity of f,

(5) $$f\{\alpha x_1 + (1-\alpha)(y+\lambda v)\} \geq \alpha f(x_1 + \{(1-\alpha)/\alpha\}\lambda v) + (1-\alpha)f(y) \text{ for all } \alpha \in (0,1],$$

Since $x_1 + \{(1-\alpha)/\alpha\}\lambda v \in P(x_1)$,

(6) $$\alpha f(x_1 + \{(1-\alpha)/\alpha\}\lambda v) + (1-\alpha)f(y) \geq \alpha f(x_1) + (1-\alpha)f(y) \text{ for all } \alpha \in (0,1]$$

As $\alpha \to 0$, the left-hand side of (5) converges to $f(y + \lambda v)$ by the continuity of f while the right-hand side of (6) converges to $f(y)$, which leads to $f(y + \lambda v) \geq f(y)$. *Q.E.D.*

It is known in Kannai (1977) that every quasi-concave function with the identical recession cone for all level sets is not concavifiable. We provide conic relationships between a finite set of closed convex sets and their sum.

Lemma A3. Let $\{C_j\}$ be a set of d closed convex sets in R^l and C the sum of these sets C_j. Then 0^+C is pointed if and only if every 0^+C_j is pointed and $\{0^+C_j\}$ is positively semi-independent.

Proof). We claim C is a closed convex set provided that every 0^+C_j is pointed and $\{0^+C_j\}$ is positively semi-independent. Pick a sequence (y_n) in C converging to a point y in R^l. Then there exists a sequence $(\boldsymbol{x}_n)$ in $\boldsymbol{C}$ such that $y_n = \sum_{j=1}^{d} x_{jn}$ for all n. By the same reasoning as in proving the second equivalence of Proposition 1, $(\boldsymbol{x}_n)$ is bounded, implying it has a convergent subsequence. Let $\boldsymbol{x}$ be the limit of the convergent subsequence. Obviously, we have $y = \sum_{j=1}^{d} x_j$ and therefore $y \in C$. Since C is closed and convex, the recession cone 0^+C is closed and $0^+C = d0^+C$. Let 0^+C be pointed. Then a set of d cones 0^+C is positively semi-independent. Since each 0^+C_j is a subset of 0^+C, the conclusion is immediate.

Suppose 0^+C is not pointed. It means a set of d cones 0^+C is not positively semi-independent. Then by Proposition 1, a set $\{\boldsymbol{x} \in R^{dl} : \sum_{j=1}^{d} x_j = w \text{ and } x_j \in C \text{ for all } j\}$ is not bounded for a vector w in R^l. Pick a sequence $(\boldsymbol{x}_n)$ with $\sum_{j=1}^{d} \|x_{jn}\| \to \infty$ as $n \to \infty$ and let $(\boldsymbol{z}_{jn})$ be a sequence in $\boldsymbol{C}$ such that $z_{hjn} \in C_h$ for each h and $\sum_{h=1}^{d} z_{hjn} = x_{jn}$. We have $\sum_{j=1}^{d} x_{jn}/d = \sum_{h=1}^{d} (\sum_{j=1}^{d} z_{hjn}/d) = w/d$ and $\sum_{j=1}^{d} z_{hjn}/d \in C_h$ for each h, n because C_h is convex. It implies a set $\{\boldsymbol{x} \in R^{dl} : \sum_{j=1}^{d} x_j = w/d \text{ and } x_j \in C_j \text{ for all } j\}$ is not bounded. It contradicts the results of Proposition 1. *Q.E.D.*

The following lemma is a consequence of applying the supporting hyperplane theorem to the pointed cone. We refers to Werner (1987) for detailed proof.

Lemma A4. If C is a pointed convex cone in R^l, there exists $p \in R^l$ such that $px > 0$ for all $x \in K$.

A vector x in a convex set Γ in R^l is an extreme point if for y and z in Γ and $\lambda \in [0, 1]$, $x = \lambda y + (1-\lambda)z$ implies either $x = y$ or $x = z$. A closed convex set Γ is strongly convex if every boundary point of Γ is an extreme point. Let Y be a subspace of R^l of positive dimension. For any nonzero $p \in Y$, the subspace $T_p = \{v \in Y : p\, v = 0\}$ is called the hyperplane perpendicular to p. The set $H_p = \{v \in Y : pv \geq 0\}$ is the halfspace above T_p. The vector p is said to supports a convex set Γ at x if $H_p + x \supset \Gamma$, or $p\Gamma \geq px$. If $py > px$ for all $y \in \Gamma$, $y \neq x$, we say that p supports strictly.

Let Γ be a closed convex set in Y with the non-empty $\text{int}_Y(\Gamma)$ and Y' a subspace of R^l which is a translate of Y. For a point x in the boundary $\partial\Gamma$ of Γ, we define a set

$$N(x) = \{p \in Y' : \ p \text{ supports } \Gamma \text{ at } x\}.$$

Let $C(N(x))$ be the cone generated by $N(x)$ and we call it the normal cone to Γ at x. Clearly, it is non-empty due to the supporting hyperplane theorem, and closed and convex. We denote by $C(\Gamma)$ the union of $C(N(x))$ over x in $\partial\Gamma$.

Lemma A5. Let Γ be a closed convex set in an affine space Y with the non-empty $\text{int}_Y(\Gamma)$. Then $(0^+\Gamma)^0 \cap Y'$ is a subset of $C(\Gamma)$.

Proof) Let $p \in (0^+\Gamma)^0 \cap Y'$. If $px \geq 0$ for all $x \in \Gamma$, p supports Γ at some point $z \in \partial\Gamma$ by the supporting hyperplane theorem, which implies $p \in C(\Gamma)$. Suppose that $px < 0$ for some $x \in \Gamma$. We claim that a set $\Gamma(p) = \Gamma \cap \{x \in X: \ px \leq 0\}$ is a compact convex set. Convexity and closedness of $\Gamma(p)$ is immediate. Suppose that it is not bounded. By the corollary of Lemma A1, there exists a nonzero direction of recession v in $0^+\Gamma(p)$. Since $\Gamma(p)$ is a subset of Γ, we have $v \in 0^+\Gamma$, which yields

$p(x + \lambda v) \leq 0$ for a point $x \in \Gamma(p)$ and all $\lambda \geq 0$.

It implies $pv \leq 0$, which is a contradiction. By applying the supporting hyperplane theorem to the compact convex set $\Gamma(p)$, we show that p supports $\Gamma(p)$, and therefore Γ, at some point in $\partial\Gamma$. *Q.E.D.*

Corollary. Suppose that Γ is a closed and strongly convex set in an affine space Y with the non-empty $\mathrm{int}_Y(\Gamma)$. Then $C(\Gamma) = (0^+\Gamma)^0 \cap Y^t$.

Proof) According to Lemma A5, we have only to show that $C(\Gamma)$ is a subset of $(0^+\Gamma)^0 \cap Y^t$. By the supporting hyperplane theorem, there exists p in Y^t which supports Γ at some $x \in \partial\Gamma$. Since Γ is strongly convex with the non-empty relative interior, $x + (0^+\Gamma \cap \Gamma) = x$. Thus p strictly supports $x + 0^+\Gamma$ at x, which implies $p \in (0^+\Gamma)^0 \cap Y^t$. *Q.E.D.*

We introduce basic materials about topological vector spaces that fit our needs. Let M be a closed subspace of a topological vector space E. If there exists a closed subspace N of E such that $E = N + M$ and $N \cap M = \{0\}$, then M is said to be complemented in X. In this case, we write $E = N \oplus M$ and call it the direct sum of M and N. A linear mapping $\pi : E \to E$ is called a projection mapping in X if $\pi^2 = \pi$. A metric d on a vector space E is invariant if $d(x + z, y + z) = d(x, y)$ for all x, y, z in E. A topological vector space E is an F-space if its topology is induced by a complete invariant metric d. If E is a locally convex F-space, it is an Frechet space. Banach spaces and Hilbert spaces are examples of Frechet spaces. The following lemma reveals important properties of an F-space.

Lemma A6. If E is an F-space with $E = N \oplus M$, the projection mapping $\pi : E \to E$ with range M and null space N is continuous.

Proof) See p 126 in Rudin (1973).

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