RICE UNIVERSITY

A LUMPED PARAMETER APPROACH TO LONGITUDINAL AND TORSIONAL VIBRATIONS OF OIL WELL DRILL STRINGS

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Science

Houston, Texas
May 1962

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ABSTRACT

The effect of tool joints on the passage of plane longitudinal and torsional waves along a drill pipe was studied. An approximate solution to the governing equations of motion found, and an idealized tool joint constructed. Calculations were made for the effect of the idealized joint on an example drill string. The results showed that tool joints had negligible effect for exciting frequencies of the same order as common rotary speeds and the drill pipe could be taken as a uniform pipe with negligible error.

Equivalent systems of rigid inertias and massless springs were developed for a drill string with a uniform drill pipe and one drill collar section, for fixed-fixed or fixed-free end boundary conditions. Undamped responses of the equivalent systems were found for arbitrary periodic forces applied to any point of the drill string. Approximate responses were derived for small damping. Specific formulas were derived for responses at the top of the drill string for given periodic displacement or force at the bottom, for fixed-fixed or fixed-free boundary conditions.

ACKNOWLEDGMENT

The author gratefully acknowledges the advice and encouragement given by Dr. J. C. Wilhoit during this study, as well as for suggestions of the problem studied here. Also the author wishes to thank Mr. Shinya Ochiai for his generous help on the computer problems involved.

Special thanks are due the American Petroleum Institute and the Rice University Mechanical Engineering Department, whose generosity allowed the author to devote full time to the study of this problem.

INTRODUCTION

Vibrations of oil well drilling strings have long been of interest to the petroleum production industry because of damage to surface drilling equipment and to the drill string itself.

The purpose of this study is to place the governing mathematical equations on as sound a basis as possible and to develop a method of approximating drill string dynamics with an equivalent system of rigid inertias and massless springs. The study is limited to torsional and longitudinal vibrations only, since the lateral vibrations of a drill string involves possible buckling and indefinable contact with the sides of the bore hole.

Several authors[1-5*] have studied longitudinal and torsional drill string vibrations using the classical wave equation for a uniform bar. All have assumed the drill pipe to be one or more bars of uniform section rigidly joined together. However, the drill pipe which accounts for most of the string length, is not uniform, but often has very heavy couplings which may account for 20% of the drill pipe weight. Also, at the junction of drill pipe and coupling (or tool joint), area and area moment of inertia may change by factors as large as 4 and 6 respectively. The change may be either abrupt or gradual. This makes suspect the common practice of ignoring the tool joints. Hence, a major portion of this study is devoted to the effects of the tool joints. It will be shown that tool joints have negligible effect on drill string vibrations in the usual range of rotary drilling speeds.

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* Numbers refer to the bibliography at the end of the paper.

Not knowing the effect of the heavy drill pipe couplings on drill string dynamics makes the idea of a "lumped parameter" equivalent dynamic system attractive. However, the study of the coupling problem indicates they can be ignored and hence an equivalent system of rigid masses and massless springs is developed for two bars rigidly joined together. The technique allows finding the drill string natural frequencies with approximately the same accuracy as the wave equation approach and reduces the degree of freedom of the system from infinity to a more managable size. The lumped system leads to relatively simple equations for the response of a damped system; lack of experimental data for comparison eliminates a numerical comparison of results.

Of special importance to this study is the frequency range involved, i.e., less than 200 revolutions per minute rotary drilling speed, which limits longitudinal and torsional driving frequencies to 20 radians per second. For roller cone bits, harmonics may be present with frequencies up to 30 times the drilling speed, but probably of small amplitude. Evaluation of their importance must await actual motion measurements at hole bottoms during drilling. Even at this much higher excitation frequency of 600 radians per second, methods of this paper should give reasonably accurate results.

Sections I and II of this paper are devoted to deriving the equations of motion and their solutions. Section III is devoted to application of the results of sections I and II to the drill string. The remaining sections deal with equivalent lumped systems for the drill string and their responses. For reference, a typical drill string configuration is shown in Figure 1.

CONTENTS

SYMBOLS

ρ	mass density, 0.283 #/in^3 for steel
g	gravitational constant
E	modulus of elasticity
	11.5×10^6 #/in^2 for steel in torsion
	30×10^6 #/in^2 for steel in tension or compression
c	velocity of propagation of a plane wave
	1.25×10^5 in/sec for torsional wave
	2.04×10^5 in/sec for longitudinal wave
I	polar area moment in^4 for torsional case; area in^2 for longitudinal case
θ	displacement
	angular twist for torsional case
	local axial displacement for longitudinal case
x, y, w, z	space coordinates along the bar axis
ω	vibration frequency, radians per second
F	general force
	#-in for torsional case
	# for longitudinal case
ϵ, γ	phase shift angle
i, j, k, l	summation indices

I. EQUATIONS OF MOTION

Torsional and longitudinal vibrations of bars are governed by the well known wave equation[7]:

$$\frac{\partial}{\partial x}\left[E(x)\, I(x) \frac{\partial \theta}{\partial x} \right] = I(x) \frac{\rho}{g} \frac{\partial^2 \theta}{\partial t^2}$$

Since in the problem at hand, material properties E , ρ , and g , are constant, the equation becomes:

$$\frac{\partial}{\partial x}\left[I(x) \frac{\partial \theta}{\partial x} \right] = \frac{I(x)}{c^2} \frac{\partial^2 \theta}{\partial t^2} \qquad c^2 = \frac{Eg}{\rho} \qquad \text{I-1)}$$

For a harmonic solution, the form below is assumed:

$$\theta(x,t) = \bar{\theta}(x)\, e^{i\omega t}$$

$$\frac{d}{dx}\left[I(x) \frac{d\bar{\theta}}{dx} \right] + \left(\frac{\omega}{c}\right)^2 I(x)\, \bar{\theta} = 0 \qquad \text{I-2)}$$

For a uniform bar $I(x)$ is constant and the equation becomes

$$\frac{d^2 \bar{\theta}}{dx^2} + \left(\frac{\omega}{c}\right)^2 \bar{\theta} = 0 \qquad \text{I-3)}$$

$$\text{I-4)}$$

$$\bar{\theta}(x) = A \sin \frac{\omega x}{c} + B \cos \frac{\omega x}{c}$$

$$= C \sin\left(\frac{\omega x}{c} + \epsilon\right) \qquad C = \sqrt{A^2 + B^2}$$

$$\sin \epsilon = \frac{B}{C} \qquad \cos \epsilon = \frac{A}{C}$$

For bars with axial holes the forms of $I(x)$ in equation I-2) are listed in Figure 2 for internal and external tapers for both the torsional and longitudinal case. To the writer's knowledge, no closed form solutions exist for any of these cases.

The problem may be attacked by the method of successive approximations, based on an existence theorem for ordinary linear differential equations[8,9]. A solution of form I-5) is assumed, where the ϕ_n are functions of the independent variable x, as yet undetermined. Sufficient conditions for the existence of such a solution are that $I(x)$ be continuous and nowhere zero in the interval $x_1 \leq x \leq x_2$, which is the case.

Substitution of equation I-5) into I-2) yields:

$$\frac{d}{dx}\left(I(x) \frac{d\Theta}{dx} \right) + \left(\frac{\omega}{c}\right)^2 I(x)\, \Theta = 0 \qquad \text{I-2)}$$

$$\Theta(x) = \phi_0 + \left(\frac{\omega}{c}\right)^2 \phi_1 + \left(\frac{\omega}{c}\right)^4 \phi_2 + \cdots + \left(\frac{\omega}{c}\right)^{2n} \phi_n + \cdots\cdots \qquad \text{I-5)}$$

$$\frac{d}{dx}\left(I(x) \frac{d\phi_0}{dx} \right) + \left(\frac{\omega}{c}\right)^2 \left\{ \frac{d}{dx}\left(I(x) \frac{d\phi_1}{dx} \right) + \phi_0 \right\}$$
$$+ \left(\frac{\omega}{c}\right)^4 \left\{ \frac{d}{dx}\left(I(x) \frac{d\phi_2}{dx} \right) + \phi_1 \right\} + \cdots + \left(\frac{\omega}{c}\right)^{2n} \left\{ \frac{d}{dx}\left(I(x) \frac{d\phi_n}{dx} \right) + \phi_{n-1} \right\} + \cdots\cdots$$

Equating coefficients of powers of $\left(\frac{\omega}{c}\right)^{2n}$, $n = 1, 2, 3, 4 \ldots\ldots$, to zero yields the values of $\phi_0, \phi_1, \ldots\ldots$ etc. below.

$$\phi_0 = C_1 \int \frac{dx}{I(x)} + C_2$$

$$\phi_1 = -\int \frac{\int \phi_0\, ds}{I(x)}\, dx$$

$$\phi_1 = -C_1 \int \frac{\int \left\{ \int \frac{du}{I(u)} \right\} ds}{I(x)}\, dx \quad - C_2 \int \frac{x}{I(x)}\, dx$$

$$\vdots$$

$$\phi_n = -\int \frac{\int \phi_{n-1}\, ds}{I(x)}\, dx$$

Obviously, succeeding coefficients become very difficult to evaluate. For the frequency range of interest $\left(\frac{\omega}{c}\right)^2$ is very small and succeeding powers of $\left(\frac{\omega}{c}\right)^{2n}$ much smaller. For example, the maximum values of

ϕ_0 and ϕ_1 are compared below using dimensions of a typical tool-joint external taper.

$$\omega = 600 \text{ RAD/SEC} \qquad mx_1 = 2.25'' \qquad mx_2 = 3.00''$$

Longitudinal: $\phi_0(x_1) = -0.172\,C_1 + C_2$ $\qquad \phi_0(x_2) = -0.113\,C_1 + C_2$

$\frac{\omega}{c} = 0.003$ $\qquad \phi_1(x_1) \cong -5.07\,C_1 + 0.14\,C_2$ $\qquad \phi_1(x_2) \cong -6.48\,C_1 - 0.28\,C_2$

Torsional: $\phi_0(x_1) = 0.028\,C_1 + C_2$ $\qquad \phi_0(x_2) = 0.47\,C_1 + C_2$

$\frac{\omega}{c} = 0.0048$ $\qquad \phi_1(x_1) \cong -0.31\,C_1 + 1.63\,C_2$ $\qquad \phi_1(x_2) \cong -0.001\,C_1 + 0.006\,C_2$

Based on these results, ϕ_0 alone should be enough for quite accurate numerical calculations. For all following work ϕ_0 will be assumed to be the solution to equation I-2).

Formulas are given below for the form of $\bar{\Theta}(x) = \phi_0$, which will be used in succeeding work*.

Internal Taper:

Longitudinal $\qquad \bar{\Theta}(x) = C_1 \left\{ \log \sqrt{\frac{r_0 + mx}{r_0 - mx}} \right\} + C_2$

Torsional $\qquad \bar{\Theta}(x) = C_1 \left\{ \log \sqrt{\frac{r_0 + mx}{r_0 - mx}} + \mathrm{TAN}^{-1} \frac{mx}{r_0} \right\} + C_2$

External Taper:

Longitudinal $\qquad \bar{\Theta}(z) = C_1 \left\{ \log \sqrt{\frac{mz - r_2}{mz + r_2}} \right\} + C_2$

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* See Figure 6 for tool joint configuration.

Torsional $\bar{\Theta}(z) = C_1 \left\{ \log \sqrt{\dfrac{mz - r_2}{mz + r_2}} + \tan^{-1} \dfrac{mz}{r_2} \right\} + C_2$

Adjusting the constants in the above equations so that the variable term drops out at start of taper in all cases, and noting r = mx and r = mz gives the final form of the equations which will be used in the succeeding section.

Internal Tapered Increase in Section:

Longitudinal $\bar{\Theta}(x) = B_2 \left\{ \log \sqrt{\dfrac{r_0 + r}{r_0 - r} \; \dfrac{r_0 - r_1}{r_0 + r_1}} \right\} + C_2$

$$= B_2 f_L(r) + C_2$$

I-6)

Torsional $\bar{\Theta}(x) = B_2 \left\{ \log \sqrt{\dfrac{r_0 + r}{r_0 - r} \; \dfrac{r_0 - r_1}{r_0 + r_1}} + \tan^{-1} \dfrac{r}{r_0} - \tan^{-1} \dfrac{r_1}{r_0} \right\} + C_2$

$$= B_2 f_T(r) + C_2$$

External Tapered Section Increase:

Longitudinal $\bar{\Theta}(z) = B_3 \left\{ \log \sqrt{\dfrac{r - r_2}{r + r_2} \; \dfrac{r_0 + r_2}{r_0 - r_2}} \right\} + C_3$

$$= B_3 \, g_L(r) + C_3$$

I-7)

Torsional $\bar{\Theta}(z) = B_3 \left\{ \log \sqrt{\dfrac{r - r_2}{r + r_2} \; \dfrac{r_0 + r_2}{r_0 - r_2}} - \tan^{-1} \dfrac{r}{r_2} + \tan^{-1} \dfrac{r_0}{r_2} \right\} + C_3$

$$= B_3 \, g_L(r) + C_3$$

In a similar manner the following equations for the two types of tapered decrease in section can be derived.

External Tapered Section Decrease:

Longitudinal $$\bar{\Theta}(Y) = D_2\left\{\log\sqrt{\frac{r-r_2}{r+r_2}\,\frac{r_3+r_2}{r_3-r_2}}\right\} + E_2$$

$$= D_2\,h_L(r) + E_2$$

I-8)

Torsional $$\bar{\Theta}(Y) = D_2\left\{\log\sqrt{\frac{r-r_2}{r+r_2}\,\frac{r_3+r_2}{r_3-r_2}} - \mathrm{TAN}^{-1}\frac{r}{r_2} + \mathrm{TAN}^{-1}\frac{r_3}{r_2}\right\} + E_2$$

$$= D_2\,h_T(r) + E_2$$

Internal Tapered Section Decrease:

Longitudinal $$\bar{\Theta}(w) = D_3\left\{\log\sqrt{\frac{r_0+r}{r_0-r}\,\frac{r_0-r_2}{r_0+r_2}}\right\} + E_3$$

$$= D_3\,k_L(r) + E_3$$

I-9)

Torsional $$\bar{\Theta}(w) = D_3\left\{\log\sqrt{\frac{r_0+r}{r_0-r}\,\frac{r_0-r_2}{r_0+r_2}} + \mathrm{TAN}^{-1}\frac{r}{r_0} - \mathrm{TAN}^{-1}\frac{r_2}{r_0}\right\} + E_3$$

$$= D_3\,k_T(r) + E_3$$

II. EQUATIONS OF MOTION ACROSS DISCONTINUITIES IN A BAR

Abrupt Change in Section

For an abrupt increase in section as in Figure 3-a solutions of the wave equation of the form of equation I-4) may be patched together by matching force or torque and longitudinal or angular displacement of the discontinuity. An arbitrary harmonic input will be assumed and the resulting motion developed for the other side of the discontinuity.

$$x \leq x_1 \qquad \theta_1(x,t) = A\sin\left(\frac{\omega x}{c}+\epsilon_1\right)e^{i\omega t}$$

$$x \geq x_1 \qquad \theta_2(x,t) = B\sin\left(\frac{\omega x}{c}+\epsilon_2\right)e^{i\omega t}$$

Boundary Contition 1: $\theta_1(x_1,t) = \theta_2(x_1,t)$

$$2: \quad EI_P \left.\frac{\partial\theta_1}{\partial x}\right|_{x_1} = EI_B \left.\frac{\partial\theta_2}{\partial x}\right|_{x_1}$$

or

$$A\sin\left(\frac{\omega x_1}{c}+\epsilon_1\right) = B\sin\left(\frac{\omega x_1}{c}+\epsilon_2\right)$$

$$I_P A\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right) = I_B B\cos\left(\frac{\omega x_1}{c}+\epsilon_2\right)$$

The equation determining ϵ_2 results from dividing boundary condition equation 2 into 1.

$$\tan\left(\frac{\omega x_1}{c}+\epsilon_2\right) = \frac{I_B}{I_P}\tan\left(\frac{\omega x_1}{c}+\epsilon_1\right) \qquad \text{II-1)}$$

by B.C. 2: $$\frac{B}{A} = \frac{I_P}{I_B}\,\frac{\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)}{\cos\left(\frac{\omega x_1}{c}+\epsilon_2\right)} \qquad \text{II-2)}$$

Determination of ϵ_2 from equation II-1) allows solving equation II-2) for the amplitude ratio, completely determining $\theta_2(x,t)$

For an abrupt decrease in section the connecting equations are determined in an exactly similar way.

$$x \leq x_1' \qquad \theta_1'(x,t) = A' \sin\left(\frac{\omega x}{c} + \gamma_1\right)$$

$$x \geq x_1' \qquad \theta_2'(x,t) = B' \sin\left(\frac{\omega x}{c} + \gamma_2\right)$$

Boundary Condition 1: $\theta_1'(x_1',t) = \theta_2'(x_1',t)$

$$2: \quad EI_B \left.\frac{\partial \theta_1'}{\partial x}\right|_{x_1'} = EI_P \left.\frac{\partial \theta_2'}{\partial x}\right|_{x_1'}$$

$$\tan\left(\frac{\omega x_1'}{c} + \gamma_2\right) = \frac{I_P}{I_B} \tan\left(\frac{\omega x_1'}{c} + \gamma_1\right) \qquad \text{II-3)}$$

$$\frac{B'}{A'} = \frac{I_B}{I_P}\,\frac{\cos\left(\frac{\omega x_1'}{c}+\gamma_1\right)}{\cos\left(\frac{\omega x_1'}{c}+\gamma_2\right)} \qquad \text{II-4)}$$

Dual Taper Change in Section

The assumed transition from small to large cross section will be that of Figure 3-b. The approximate equations of motion derived in section I will be used to represent the motion of the tapered sections. The equations with the appropriate boundary conditions are listed below.

$$x \leq x_1 \qquad \Theta_1(x,t) = A\sin\left(\frac{\omega x}{c} + \epsilon_1\right) e^{i\omega t}$$

$$x_1 \leq x \leq x_2 \qquad \Theta_2(x,t) = [B_2 f(r) + C_2]\, e^{i\omega t} \qquad f(r) = \begin{cases} f(r_1) = 0 & @x_1 \\ f(r_2) & @x_2 \end{cases}$$

by equation I-6)

$$x_2 \leq x \leq x_3 \qquad \Theta_3(x,t) = [B_3 g(r) + C_3]\, e^{i\omega t} \qquad g(r) = \begin{cases} g(r_0) = 0 & @x_2 \\ g(r_3) & @x_3 \end{cases}$$

by equation I-7)

$$x \geq x_3 \qquad \Theta_4(x,t) = \bar{B}\sin\left(\frac{\omega x}{c} + \epsilon_2\right) e^{i\omega t}$$

Boundary Condition 1: $\Theta_1(x_1,t) = \Theta_2(x_1,t)$

2: $EI_p \left.\frac{\partial\Theta_1}{\partial x}\right|_{x_1} = EI_p \left.\frac{\partial\Theta_2}{\partial x}\right|_{x_1}$

3: $\Theta_2(x_2,t) = \Theta_3(x_2,t)$

4: $EI_N \left.\frac{\partial\Theta_2}{\partial x}\right|_{x_2} = EI_N \left.\frac{\partial\Theta_3}{\partial x}\right|_{x_2}$

5: $\Theta_3(x_3,t) = \Theta_4(x_3,t)$

6: $EI_B \left.\frac{\partial\Theta_3}{\partial x}\right|_{x_3} = EI_B \left.\frac{\partial\Theta_4}{\partial x}\right|_{x_3}$

or B.C. 1: $A\sin\left(\frac{\omega x_1}{c} + \epsilon_1\right) = C_2$

2: $\frac{\omega}{c} A\cos\left(\frac{\omega x}{c} + \epsilon_1\right) = B_2 f'(r_1)$*

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* Prime denotes differentiation with respect to x.

$$\therefore \quad \theta_2(x,t) = A\left[\frac{\omega}{c}\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)\frac{f(r)}{f'(r_1)} + \sin\left(\frac{\omega x_1}{c}+\epsilon_1\right)\right] e^{i\omega t}$$

by B. C. 3: $$A\left[\frac{\omega}{c}\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)\frac{f(r_2)}{f'(r_1)} + \sin\left(\frac{\omega x_1}{c}+\epsilon_1\right)\right] = C_3$$

by B. C. 4: $$A\frac{\omega}{c}\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)\frac{f'(r_2)}{f'(r_1)} = B_3\, g'(r_0)$$

but $$\frac{f'(r_2)}{f'(r_1)} = \frac{I_P}{I_N}$$

$$\theta_3(x,t) = A\left[\frac{\omega}{c}\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)\left(\frac{I_P}{I_N}\,\frac{g(r_0)}{g'(r_0)} + \frac{f(r_2)}{f'(r_1)}\right) + \sin\left(\frac{\omega x_1}{c}+\epsilon_1\right)\right] e^{i\omega t}$$

by B. C. 5: $$A\left[\frac{\omega}{c}\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)\left(\frac{I_P}{I_N}\,\frac{g(r_3)}{g'(r_0)} + \frac{f(r_2)}{f'(r_1)}\right) + \sin\left(\frac{\omega x_1}{c}+\epsilon_1\right)\right]$$

$$= \bar{B}\sin\left(\frac{\omega x_1}{c}+\epsilon_1\right)$$

B. C. 6: $$A\left[\frac{\omega}{c}\cos\left(\frac{\omega x_1}{c}+\epsilon\right)\frac{I_P}{I_N}\,\frac{g'(r_3)}{g'(r_0)}\right] = \bar{B}\,\frac{\omega}{c}\cos\left(\frac{\omega x_3}{c}+\bar{\epsilon}_2\right)$$

but $\dfrac{g'(r_3)}{g'(r_0)} = \dfrac{I_N}{I_B}$ $\quad \therefore \quad \dfrac{\bar{B}}{A} = \dfrac{I_P}{I_B}\,\dfrac{\cos\left(\frac{\omega x_1}{c}+\epsilon_1\right)}{\cos\left(\frac{\omega x_1}{c}+\bar{\epsilon}_2\right)}$ II-5)

B.C. 5 $\div$ B.C. 6 $$\tan\left(\frac{\omega x_3}{c}+\bar{\epsilon}_2\right) = \frac{\omega}{c}\left(\frac{g(r_3)}{g'(r_3)} + \frac{I_B}{I_P}\,\frac{f(r_2)}{f'(r_1)}\right) + \frac{I_B}{I_P}\tan\left(\frac{\omega x_1}{c}+\epsilon_1\right)$$

II-6)

Hence for $\frac{\omega}{c}$ very small the value of $\bar{\epsilon}_2$ approaches ϵ_2 of equation II-1). for the same total change in section if $x_1 = x_1'$.

Taking the inverse tangent of both equations II-1) and II-6) yields:

$$\frac{\omega x_1}{c} + \epsilon_2 = \tan^{-1}\left\{ \frac{I_B}{I_P}\tan\left(\frac{\omega x_1}{c}+\epsilon_1\right)\right\}$$

$$\frac{\omega x_3}{c} + \bar{\epsilon}_2 = \tan^{-1}\left\{ \frac{\omega}{c}\left(\frac{g(r_3)}{g'(r_3)} + \frac{I_B}{I_P}\frac{f(r_1)}{f'(r_1)} \right) + \frac{I_B}{I_P}\tan\left(\frac{\omega x_1}{c}+\epsilon_1\right) \right\}$$

$$\epsilon_2 - \bar{\epsilon}_2 = \frac{\omega}{c}(x_3 - x_1) + \tan^{-1}\left\{ \frac{I_B}{I_P}\tan\left(\frac{\omega x_1}{c}+\epsilon_1\right)\right\} - \tan^{-1}\left\{ \frac{\omega}{c}\left[\frac{g(r_3)}{g'(r_3)} + \frac{I_B}{I_P}\frac{f(r_1)}{f'(r_1)} \right] + \frac{I_B}{I_P}\tan\left(\frac{\omega x_1}{c}+\epsilon_1\right) \right\} \qquad \text{II-7)}$$

By equations II-2)

$$\frac{\bar{B}}{B} - 1 = \frac{\cos\left(\frac{\omega x_3}{c} + \bar{\epsilon}_2\right)}{\cos\left(\frac{\omega x_1}{c} + \epsilon_2\right)} - 1 \qquad \text{II-8)}$$

Equations II-7) and II-8) indicate the error involved in replacing a conventional tapered section change by an abrupt change. The results of equation II-7 at various input angles $\frac{\omega x_1}{c} + \epsilon$ are plotted versus ω in Figures 4 and 5 for a typical* tool joint.

Similarly, use of equations I-8) and I-9) as equations of motion through the dual taper decrease in section of figure yield:

$$x \le x_1' \qquad \theta_1'(x,t) = A' \sin\left(\frac{\omega x}{c} + \gamma\right) e^{i\omega t}$$

$$x_1' \le x \le x_2' \qquad \theta_2'(x,t) = [D_2 h(r) + E_2] e^{i\omega t}$$

by equation I-8) $\qquad h(r) = \begin{cases} h(r_3) = 0 & @ x_1' \\ h(r_0) & @ x_2' \end{cases}$

$$x_2' \le x \le x_3' \qquad \theta_3'(x,t) = [D_3 k(r) + E_3]\, e^{i\omega t}$$

$$k(r) = \begin{cases} k(r_3) = 0 & @ x_2' \\ k(r_1) & @ x_3' \end{cases}$$

by equation I-9)

$$x_3' \le x \qquad \theta_4'(x,t) = \bar{B}' \sin\left(\frac{\omega x}{c} + \bar{\gamma}_2\right) e^{i\omega t}$$

$$\frac{\bar{B}'}{A'} = \frac{I_B}{I_P}\,\frac{\cos\left(\frac{\omega x_1'}{c} + \gamma_1\right)}{\cos\left(\frac{\omega x_3'}{c} + \bar{\gamma}_2\right)} \qquad \text{II-9)}$$

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* See Figure 6.

$$\tan\left(\frac{\omega x_3'}{c}+\bar{\gamma}_2\right) = \frac{\omega}{c}\left(\frac{h(r_0)}{h'(r_3)}\frac{I_P}{I_B}+\frac{k(r_1)}{k'(r_1)}\right)+\frac{I_P}{I_B}\tan\left(\frac{\omega x_1'}{c}+\gamma_1\right) \qquad \text{II-10)}$$

Combination of equations II-3), II-4), II-10), and II-11)result in "error" equations II-12) and II-13) below:

$$\begin{aligned}\gamma_2-\bar{\gamma}_2 &= \frac{\omega}{c}(x_3'-x_1') + \tan^{-1}\left\{\frac{I_P}{I_B}\tan\left(\frac{\omega x_1'}{c}+\gamma\right)\right\} \\ &\quad - \tan^{-1}\left\{\frac{\omega}{c}\left[\frac{h(r_0)}{h'(r_3)}\frac{I_P}{I_B}+\frac{k(r_1)}{k'(r_1)}\right]+\frac{I_P}{I_B}\tan\left(\frac{\omega x_1'}{c}+\gamma_1\right)\right. \qquad \text{II-11)}\end{aligned}$$

$$\frac{B'}{\bar{B}'}-1 = \frac{\cos\left(\frac{\omega x_3'}{c}+\bar{\gamma}_2\right)}{\cos\left(\frac{\omega x_1'}{c}+\gamma_1\right)}-1 \qquad \text{II-12)}$$

The results of equation II-11) at various values of input angle $\frac{\omega x_1}{c}+\epsilon_1$ are plotted versus ω in Figures 4 and 5.

III. DRILL STRING DYNAMICS

As shown in Figure 1 the drill string consists of a kelly joint, square or hexagonal in section, a very long section of drill pipe, and a much shorter section of drill collar, and finally the cutting bit. For this analysis, the effect of the kelly joint and the bit will be ignored.

Drill Pipe Dynamics

The large section couplings or tool joints, normally at 30'-0" to 32'-0" intervals along the drill pipe, may account for as much as 20% of the total mass of the drill pipe string. In addition area and polar area moment of inertia may change by factors of 4 and 6 respectively. Their effect on the passage of longitudinal or torsional waves along the drill pipe will be analyzed using the results of Section II.

Typical tool joint configurations are shown in Figure 6. Although only those for 4.5" diameter drill pipe will be examined here, the work is more general than might appear. The most commonly used drill pipe sizes are 4.5" and 5.0" diameter. Each has a more or less standard tool joint configuration. Total change in area and area polar moment of inertia from pipe to tool joint barrel are by factors of 3.8 and 4.7 for typical 5.0" tool joints. The importance of these factors is seen from equations II-5), II-6), II-9), and II-10). Since these factors are 4.17, and 6.12 for the 4.5" drill pipe tool joint combination, it alone will be considered.

The results of equations II-7) and II-11), plotted in Figures 4 and 5, show that replacing a tapered increase or decrease in section by an abrupt change results in a large phase shift than the actual tapered change. It follows that phase shift calculations using the idealized tool joint of Figure 6 instead of the type 1 or 2 will result in larger values than is actually the case. If idealized joint calculations show small variation from the results for a uniform straight pipe, the actual tool joint effect is even less.

For calculation purposes Test Well "A" or reference 3 was selected.* The drill pipe string configurations used are shown in Figure 8. For phase shift calculations of string 1, equation I-4) was assumed to hold in each constant area section, with solution of the form $A_n \sin(\frac{\omega x}{c} + \epsilon_n) e^{i\omega t}$; that II-1,3) apply at discontinuities. For string 2 no tool joints were used except the last, since a half tool joint is commonly used to connect the pipe string to the drill collar. The resulting phase angles are plotted in Figure 9. Natural frequencies of the drill pipe string alone are tabulated in Figure 10; a phase angle at the end of the string of 90° indicating a free end, a phase angle of 0° indicating a fixed end.

Combination of the equations for the phase shift at the beginning and end of an idealized tool joint gives:

$$\tan\left(\frac{\omega x_n}{c} + \epsilon_n\right) = \frac{I_B}{I_P} \tan\left(\frac{\omega x_n}{c} + \epsilon_{n-1}\right)$$

$$\tan\left(\frac{\omega x_{n+1}}{c} + \epsilon_{n+1}\right) = \frac{I_P}{I_B} \tan\left(\frac{\omega x_{n+1}}{c} + \epsilon_n\right) \qquad x_{n+1} = x_n + \ell \qquad \ell = 18''$$

$$\epsilon_{n+1} = \tan^{-1}\left\{ \frac{I_P}{I_B} \tan\left\{ \tan^{-1}\left\{ \frac{I_B}{I_P} \tan\left(\frac{\omega x_n}{c} + \epsilon_{n-1}\right) + \frac{\omega \ell}{c} \right\}\right\}\right\} - \frac{\omega}{c}(x_n + \ell)$$

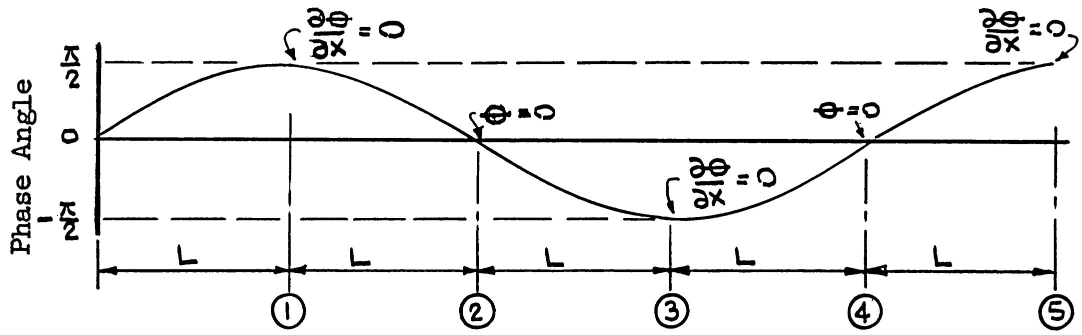
---

* Test Well "A" is shown in Figure 7.

For small values of ω the total phase shift is very small, and would become large only when the angle $\frac{\omega \ell}{c}$ becomes large. Figures 9 and 10 indicate this is the case.

It is concluded that the effect of the idealized tool joints on a traveling wave is negligible except for very high frequencies, and the effect of actual tool joints is even less. Since displacement was assumed continuous in the derivation of equations II-1) and II-3), if the phase angle change through a tool joint is approximately zero, the amplitude of the drill pipe motion must be very nearly that of a uniform bar.

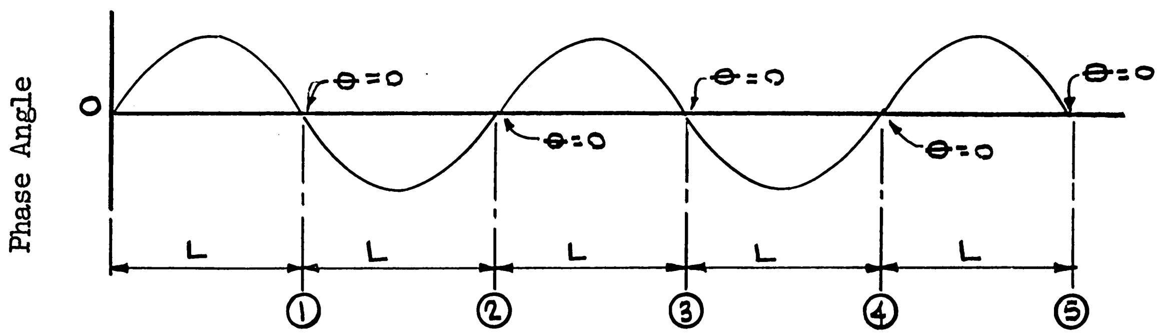
Although calculations were made for only 3445'-8" of drill pipe, lower modes of longer lengths can be found from the curves of Figure 9 because the direction of calculation is unimportant. For example, the lowest fix-free frequency of 3445'-8" of drill pipe corresponds to the first fix-fix frequency of a pipe twice as long; the second fix-free mode of a pipe three times as long; and the second fix-fix mode of a pipe four times as long. Similar results may be found for any multiple of 3445'-8" as illustrated in sketches below. As a result, it is concluded that tool joints have negligible effect on pipe phase, regardless of pipe length, for low frequencies.



Point	Mode	Pipe Length
1	1st fix-free	L
2	1st fix-fix	2L
3	2nd fix-free	3L
4	2nd fix-fix	4L
5	3rd fix-free	5L

ω = 4.65 Radians/Second

$L = L_p$ = 3445'-8"



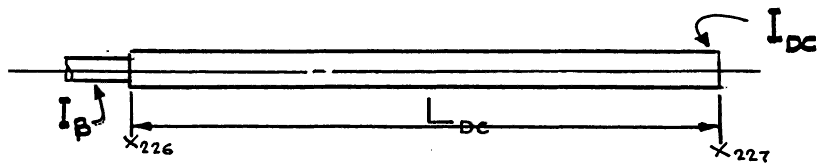
Point	Mode	Pipe Length
1	1st fix-fix	L
2	2nd fix-fix	2L
3	3rd fix-fix	3L
4	4th fix-fix	4L
5	5th fix-fix	5L

ω = 9.40 Radians/Second

$L = L_p$ 3445'8"

<u>Drill Collar Dynamics</u>

Since the drill collar is a uniform bar, wave equation I-3) and the corresponding solution I-4) apply. Calculations can be made for the phase angle at the tool joint drill pipe junction which will satisfy either a fixed or free boundary condition at the bottom of the drill collar. The method of calculation is presented below; results are plotted in Figure 11.



$$x_{225} \le x \le x_{226} \qquad \theta_{225}(x,t) = A_{225}\,\mathrm{SIN}\left(\frac{\omega x}{c} + \epsilon_{225}\right) e^{i\omega t}$$

$$x_{226} \le x \le x_{227} \qquad \theta_{226}(x,t) = A_{226}\,\mathrm{SIN}\left(\frac{\omega x}{c} + \epsilon_{226}\right) e^{i\omega t}$$

Fix-Fix
$$\left.\frac{\partial \theta_{226}}{\partial x}\right|_{x_{227}} = A_{226}\frac{\omega}{c}\,\mathrm{COS}\left(\frac{\omega x_{227}}{c} + \epsilon_{226}\right) = 0$$

$$\frac{\omega x_{227}}{c} + \epsilon_{226} = \frac{\pi}{2}$$

Fix-Free
$$\left.\theta_{226}\right|_{x_{227}} = A_{226}\,\mathrm{SIN}\left(\frac{\omega x_{227}}{c} + \epsilon_{226}\right)$$

$$\frac{\omega x_{227}}{c} + \epsilon_{226} = \pi$$

Either case
$$\mathrm{TAN}\left(\frac{\omega x_{226}}{c} + \epsilon_{225}\right) = \frac{I_B}{I_{DC}}\,\mathrm{TAN}\left(\frac{\omega x_{226}}{c} + \epsilon_{226}\right)$$

$$\frac{\omega x_{226}}{c} + \epsilon_{225} = \mathrm{TAN}^{-1}\left\{\frac{I_B}{I_{DC}}\,\mathrm{TAN}\left(\frac{\omega x_{226}}{c} + \epsilon_{226}\right)\right\}$$

$$\frac{\omega x_{227}}{c} = \frac{\omega x_{226}}{c} + \frac{\omega L_{DC}}{c}$$

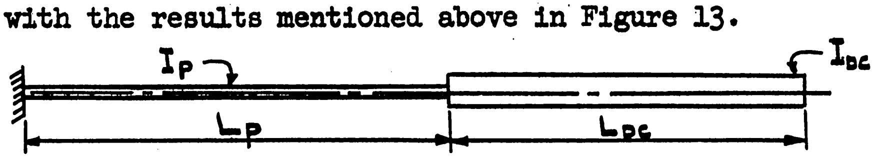
Fix-Free
$$\frac{\omega x_{226}}{c} + \epsilon_{225} = \mathrm{TAN}^{-1}\left\{\frac{I_B}{I_{DC}}\,\mathrm{TAN}\left(\frac{\pi}{2} - \frac{\omega L_{DC}}{c}\right)\right\}$$

Fix-Fix
$$\frac{\omega x_{226}}{c} + \epsilon_{225} = \mathrm{TAN}^{-1}\left\{\frac{I_B}{I_{DC}}\,\mathrm{TAN}\left(\pi - \frac{\omega L_{DC}}{c}\right)\right\}$$

Drill String Dynamics

The natural frequencies of Test Well "A" of reference 3 may be found by a combination of the results of the preceding section, which are shown in Figure 11. Intersections of the "Drill Pipe Curves" with the "Drill Collar Curves" are natural frequencies of the system. The two families of "Drill Pipe Curves" for the uniform drill pipe and the drill pipe with 113 tool joints give only slightly different natural frequencies; again indicating tool joint effects may be ignored.

Since the drill pipe with tool joints behaves so nearly like a uniform bar, the case of a uniform pipe directly connected to a uniform drill collar is considered below. Resulting frequency curves are shown in Figure 12 and resulting natural frequencies are tabulated for comparison with the results mentioned above in Figure 13.



$$0 \le x \le L_p \qquad \theta_p(x,t) = A_p \sin\left(\frac{\omega x}{c}\right) e^{i\omega t}$$

$$L_p \le x \le L_p + L_{DC} \qquad \theta_{DC}(x,t) = A_{DC} \sin\left(\frac{\omega x}{c} + \epsilon\right) e^{i\omega t}$$

Matching force or torque and displacement at $x = L_p$ gives:

Free end:
$$\left.\frac{\partial \theta_{DC}}{\partial x}\right|_{L_p+L_{DC}} = A_{DC}\frac{\omega}{c}\cos\left(\frac{\omega}{c}L_p + \frac{\omega}{c}L_{DC} + \epsilon\right) = 0$$

$$\frac{\omega L_p}{c} + \epsilon = \frac{\pi}{2} - \frac{\omega L_{DC}}{c}$$

Fixed end: $\Theta_{DC}\Big|_{L_P+L_{DC}} = A_{DC}\,\mathrm{SIN}\left(\frac{\omega}{c}L_P + \frac{\omega}{c}L_{DC} + \epsilon\right) = 0$

$$\frac{\omega L_P}{c} + \epsilon = \pi - \frac{\omega L_{DC}}{c}$$

Free end: $\mathrm{TAN}\left(\frac{\omega L_P}{c} + \epsilon\right) = \mathrm{TAN}\left(\frac{\pi}{2} - \frac{\omega L_{DC}}{c}\right) = \mathrm{COT}\left(\frac{\omega L_{DC}}{c}\right)$

Fixed end: $\mathrm{TAN}\left(\frac{\omega L_P}{c} + \epsilon\right) = \mathrm{TAN}\left(\pi - \frac{\omega L_{DC}}{c}\right) = -\mathrm{TAN}\left(\frac{\omega L_{DC}}{c}\right)$

The resulting frequency equation for a free drill collar end is:

$$\mathrm{TAN}\left(\frac{\omega L_P}{c}\right) = \frac{I_P}{I_{DC}}\,\mathrm{COT}\left(\frac{\omega L_{DC}}{c}\right) \qquad \text{III-1)}$$

$$\frac{\omega L_P}{c} - n\pi = \mathrm{TAN}^{-1}\left\{\frac{I_P}{I_{DC}}\,\mathrm{COT}\left(\frac{\omega L_{DC}}{c}\right)\right\}$$

$$n = 1, 2, 3, \ldots$$

The resulting frequency equation for a fixed drill collar end is:

$$\mathrm{TAN}\left(\frac{\omega L_P}{c}\right) = -\frac{I_P}{I_{DC}}\,\mathrm{TAN}\left(\frac{\omega L_{DC}}{c}\right) \qquad \text{III-2)}$$

$$\frac{\omega L_P}{c} - n\pi = -\mathrm{TAN}^{-1}\left\{\frac{I_P}{I_{DC}}\,\mathrm{TAN}\left(\frac{\omega L_{DC}}{c}\right)\right\}$$

For later reference the modes shapes are given below for the nth mode where ω_n is the nth root of the corresponding frequency equation.

Fix-free: $0 \leq x \leq L_P \qquad \Theta_n(x) = \mathrm{SIN}\,\frac{\omega_n x}{c}$ II-3)

$$L_P \leq x \leq L_P + L_{DC} \qquad \Theta_n(x) = \frac{\mathrm{SIN}\left(\frac{\omega_n x}{c}\right)}{\cos\left(\frac{\omega_n L_{DC}}{c}\right)}\cos\left(\frac{\omega_n}{c}(L_P + L_{DC} - x)\right)$$

Fix-fix: $0 \leq x \leq L_p \quad \theta_n(x) = \sin\left(\frac{\omega_n}{c}x\right)$

$$L_p \leq x \leq L_p + L_{DC} \quad \theta_n(x) = \frac{\sin\left(\frac{\omega_n}{c}L_p\right)}{\sin\left(\frac{\omega_n}{c}L_{DC}\right)} \sin\left(\frac{\omega_n}{c}(L_p + L_{DC} - x)\right) \tag{III-4}$$

IV. EQUIVALENT LUMPED PARAMETER SYSTEMS

Equivalent System for Single Uniform Bar

In this section equivalent lumped systems will be derived for uniform single bars with ends fixed-fixed and fixed-free as shown in Figures 14-a and 14-b. A standard length l_0 of a uniform bar will be defined and all its elasticity put in one spring of the equivalent system, Figure 14-e, and all its inertia placed in one mass of the lumped system. Equations will be derived for the fixed end case and the fixed-free end case for slightly different length bars and conditions specified so that results may be applied to bars of the same length.

Fixed-fixed Case: $l_0 = \frac{L}{n+1}$

$$K = \frac{EI}{l_0} \qquad J = \rho \frac{l_0}{g} I \qquad \text{IV-1)}$$

$$\frac{K}{J} = \frac{E}{\rho} g \frac{1}{l_0^2} = \frac{c^2}{l_0^2}$$

Equating elastic and inertial torques for each mass of the lumped system gives:

$$k = 1 \qquad 2K\theta_1 - J\ddot{\theta}_1 - K\theta_2 = 0$$

$$1 < k < n \qquad -K\theta_{k-1} + 2K\theta_k - J\ddot{\theta}_k - K\theta_{k+1} = 0$$

$$k = n \qquad -K\theta_{n-1} + 2K\theta_n - J\ddot{\theta}_n = 0$$

For a periodic solution: $\theta_k = \bar{\theta}_k e^{i\omega t}$

$$k = 1 \qquad (2K - J\omega^2)\bar{\theta}_1 - K\bar{\theta}_2 = 0$$

$$1 < k < n \qquad -K\bar{\theta}_{k-1} + (2K - J\omega^2)\bar{\theta}_k - K\bar{\theta}_{k+1} = 0$$

$$k=n \qquad -K\bar{\Theta}_{n-1} + (2K - J\omega^2)\bar{\Theta}_n = 0$$

Division of all n equations by K yields:

$$k=1 \qquad (2-R)\bar{\Theta}_1 - \bar{\Theta}_2 = 0 \qquad R = \frac{J\omega^2}{K}$$

$$1 < k < n \qquad -\bar{\Theta}_{k-1} + (2-R)\bar{\Theta}_k - K\bar{\Theta}_{k+1} = 0$$

$$k=n \qquad -\bar{\Theta}_{n-1} + (2-R)\bar{\Theta}_n = 0$$

The frequency determinant is:

$$\begin{vmatrix} 2-R & -1 & & & & \\ -1 & 2-R & -1 & & & \\ & -1 & 2-R & -1 & & \\ & & & & & \\ & & & -1 & 2-R & -1 \\ & & & & -1 & 2-R \end{vmatrix} = 0$$

This can be factored by identity 1 page 53:

$$\frac{\sin(n+1)\phi}{\sin\phi} = 0 \qquad 2-R = 2\cos\phi$$

$$R = 4\sin^2\frac{\phi_p}{2} \qquad \text{IV-3)}$$

$$\phi_p = \frac{\pi}{n+1}p \qquad p = 1,2,3,\ldots$$

By equation IV-2):

$$\omega_p^2 = \frac{K}{J}R = 4\frac{c^2}{l_0^2}\sin^2\left(\frac{\pi}{2(n+1)}p\right)$$

For ϕ_p small, i.e., n large $\sin\left(\frac{\phi_p}{2}\right)$ is approximately equal to $\frac{\phi_p}{2}$.

$$\omega_p \simeq \frac{c}{l_0}\frac{\pi}{n+1}p \simeq \frac{\pi c}{L_1}p$$

which is the same as the wave equation for a fixed-fixed uniform bar.

i.e. $$\omega_p = \frac{\pi c}{L_1} p \qquad \theta_p(x,t) = A \sin\left(\frac{\pi x}{L} p\right) e^{i\omega t} \qquad \text{IV-4)}$$

Since the mode shapes are unique only within a constant factor, a legitimate choice constant for the pth mode is:

$$\bar{\theta}_1 = \sin\left(\frac{\pi}{n+1} p\right)$$

Application of the recurrance formula for the sine allows evaluation of all succeeding $\bar{\theta}$'s.

$$\bar{\theta}_k = (2-R)\,\bar{\theta}_{k-1} - \bar{\theta}_{k-1} \qquad 2-R = 2\cos\phi_p$$

$$\sin l\phi_p = 2\cos\phi_p \sin(l-1)\phi_p - \sin(l-2)\phi_p$$

$$\bar{\theta}_k = \sin\frac{k\pi}{n+1} p \simeq \sin\frac{\pi x p}{L_1} \qquad x = k l_0 \qquad \text{IV-5)}$$

Fixed-free Case:

$$l_0' = \frac{L_1'}{n+\frac{1}{2}}$$

$$\frac{K}{J} = \frac{c^2}{l_0'^2}$$

The equations of motion are the same as equations IV-2 with the exception of the last equation since $K_{n+1} = 0$.

$$k = 1 \qquad (2-R)\,\bar{\theta}_1 - \bar{\theta}_2 = 0$$

$$1 < k < n \qquad -\bar{\theta}_{k-1} + (2-R)\,\bar{\theta}_k - \bar{\theta}_{k+1} = 0$$

$$k = n \qquad -\bar{\theta}_{n-1} + (1-R)\,\bar{\theta}_n = 0$$

The frequency determinant is:

$$\begin{vmatrix} 2-R & -1 & & & \\ -1 & 2-R & -1 & & \\ & & \ddots & & \\ & & -1 & 2-R & -1 \\ & & & -1 & 1-R \end{vmatrix} = 0$$

This can be factored by identify 2, page 53:

$$\frac{\cos(n+\frac{1}{2})\phi}{\sin\frac{\phi}{2}} = 0 \qquad 2-R = 2\cos\phi$$

$$\phi_p = \frac{\pi}{2n+1}p \quad p = 1,3,5,7,\ldots \qquad \omega^2 = 4\frac{K}{J}\sin^2\frac{\phi_p}{2} \qquad \text{IV-6)}$$

As before for ϕ_p small, i.e., n large $\sin\frac{\phi_p}{2}$ is approximately equal to $\frac{\phi_p}{2}$.

$$\omega_p \simeq \frac{c}{l_0'}\frac{\pi}{2n+1}p \cong \frac{c\pi}{2L_1'}p$$

This is the same as the wave equation solution for a fixed-free bar:

$$\omega_p = \frac{\pi c}{2L'}p \qquad p = 1,3,5,\cdots \qquad \theta_p(x,t) = A_p'\sin\left(\frac{\pi x}{2L_1'}p\right)e^{i\omega t}$$

Determining the modes shapes as in equation IV-5) yields:

$$\theta_k = \sin\left(\frac{k\pi}{2n+1}p\right) \simeq \sin\left(\frac{\pi x}{2L_1'}p\right)$$

If the two uniform bars of Figures 14-a and 14-b are the same, $L_1 = L_1'$. To give correct results for lower frequencies for both the fix-fix case and fix-free case, the value of K/J must be the same.

$$\frac{K}{J} = \frac{c^2}{\ell_0^2} = \frac{c^2}{\ell_0'^2} \qquad \ell_0 = \frac{L_1}{n+1} = \frac{L_1'}{n+\frac{1}{2}}$$

$$n+1 = n+\tfrac{1}{2}$$

which is obviously not an identity. However, if n is large enough the two values become approximately equal. For the case n = 50, the difference is less than 1%. Figure 15 shows the error resulting from the lumped approximation in natural frequency for n = 50.)

Equivalent Lumped System for a Dual Section Bar

The results of section III show that tool joints may be ignored on drill pipe section of a drill string, and that it can be replaced by a uniform bar of the same length. Hence, the drill string will be taken to be two bars of uniform cross section, rigidly joined together, as in Figure 14-c and 14-d. The equivalent lumped system is shown in Figure 14-f. Equivalent systems will be derived for the cases; top and bottom fixed, top fixed and bottom free, and top free and bottom fixed. In all cases the systems reduce to that of a single uniform bar if the bars are of the same section.

The fixed-fixed case will be considered first and as before, all elasticity and inertia of a standard length ℓ_o will be placed in the springs and inertias, respectively, of the lumped system.

$$\ell_0 = \frac{L_P}{m+\frac{1}{2}} = \frac{L_{DC}}{n+\frac{1}{2}}$$

$$K_1 = \frac{EI_P}{\ell_0} \qquad K_2 = \frac{EI_{DC}}{\ell_0} \qquad \text{IV-7)}$$

$$J_1 = \rho\frac{\ell_0}{g}I_P \qquad J_2 = \rho\frac{\ell_0}{g}I_{DC}$$

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* A similar treatment for a uniform string is given in reference 11 pp. 122-126.

$$\frac{K_1}{J_1} = \frac{c^2}{l_0^2} \qquad\qquad \frac{K_2}{J_2} = \frac{c^2}{l_0^2} \qquad \text{IV-7) Cont'd.}$$

$$\alpha = \frac{I_{oc}}{I_p} \qquad \frac{K_2}{K_1} = \frac{EI_{oc}}{l_0}\,\frac{l_0}{EI_p} = \alpha$$

$$\frac{J_2}{K_1} = \frac{J_2}{K_2}\,\frac{K_2}{K_1} = \alpha\,\frac{J_1}{K_1}$$

$$K_3 = \frac{2K_1K_2}{K_1+K_2} = \frac{2\alpha K_1}{1+\alpha} = \beta K_1 \qquad \beta = \frac{2\alpha}{1+\alpha}$$

Equating elastic and inertial torques for each inertia yield the equations of motion below:

$$k=1 \qquad 2K_1\theta_1 - J_1\ddot{\theta}_1 - K_1\theta_2 = 0 \qquad \text{IV-8)}$$

$$1 < k < m \qquad -K_1\theta_{k-1} + 2K_1\theta_k - J_1\ddot{\theta}_k - K_1\theta_{k+1} = 0$$

$$k=m \qquad -K_1\theta_{m-1} + (K_1+K_3)\theta_m - J_1\ddot{\theta}_m - K_3\theta_{m+1} = 0$$

$$k=m+1 \qquad -K_3\theta_{m+1} + (K_2+K_3)\theta_{m+1} - J_2\ddot{\theta}_{m+1} - K_2\theta_{m+1} = 0$$

$$m+1 < k < n \qquad -K_2\theta_{k-1} + 2K_2\theta_k - J_2\ddot{\theta}_k - K_2\theta_{k+1} = 0$$

$$k=n \qquad -K_2\theta_{n-1} + 2K_2\theta_n - J_n\ddot{\theta}_n = 0$$

Assuming a periodic solution and dividing all m + n equations by K_1, and using the constants defined in equation IV-7) yields:

$$\theta_k = \bar{\theta}_k e^{i\omega t} \qquad R = \frac{J_1\omega^2}{K_1} = \frac{J_2\omega^2}{K_2} = \left(\frac{c\,\omega}{l_0}\right)^2$$

$$k=1 \qquad (2-R)\bar{\theta}_1 -$$

$$1 < k < m \qquad -\bar{\theta}_{k-1} + (2-R)\bar{\theta}_k - \bar{\theta}_{k+1} = 0$$

$$k=m \qquad -\bar{\theta}_{m-1} + (\beta+1-R)\bar{\theta}_m - \beta\bar{\theta}_{m+1} = 0$$

$$k = m+1 \qquad -\beta\,\bar{\theta}_m + \left(\beta+\alpha(1-R)\right)\bar{\theta}_{m+1} - \alpha\,\bar{\theta}_{m+2} = 0$$

$$m+1 < k < m+n \qquad -\alpha\,\bar{\theta}_{k-1} + \alpha(2-R)\,\bar{\theta}_k - \alpha\,\bar{\theta}_{k+1} = 0$$

$$k = m+n \qquad -\alpha\,\bar{\theta}_{m+n-1} + \alpha(2-R)\,\bar{\theta}_{m+n} = 0$$

The frequency determinant is then:

m + n rows and columns; row "for m^{th} inertia" marked at the $(1+\beta-R)$ row:

$$\begin{vmatrix} 2-R & -1 \\ -1 & 2-R & -1 \\ & & \ddots \\ & & -1 & 2-R & -1 \\ & & & -1 & (1+\beta-R) & -\beta \\ & & & & -\beta & (\beta+\alpha(1-R)) & -\alpha \\ & & & & & -\alpha & \alpha(2-R) & -\alpha \\ & & & & & & \ddots \\ & & & & & & -\alpha & \alpha(2-R) & -\alpha \\ & & & & & & & -\alpha & \alpha(2-R) \end{vmatrix} = 0 \qquad \text{IV-10)}$$

Expanded on the mth row and column by Sylvester's identity:*

(m rows and columns) (n rows and columns) $-\beta^2$ (m-1 rows and columns) (n-1 rows and columns)

$$\begin{vmatrix} 2-R & -1 \\ -1 & 2-R & -1 \\ & & \ddots \\ & -1 & 2-R & -1 \\ & & -1 & 1+\beta-R \end{vmatrix} \begin{vmatrix} \beta+\alpha(1-R) & -\alpha \\ -\alpha & \alpha(2-R) & -\alpha \\ & & \ddots \\ & -\alpha & \alpha(2-R) & -\alpha \\ & & -\alpha & \alpha(2-R) \end{vmatrix} - \beta^2 \begin{vmatrix} 2-R & -1 \\ -1 & 2-R & -1 \\ & & \ddots \\ & -1 & 2-R & -1 \\ & & -1 & 2-R \end{vmatrix} \begin{vmatrix} \alpha(2-R) & -\alpha \\ -\alpha & \alpha(2-R) & -\alpha \\ & & \ddots \\ & -\alpha & \alpha(2-R) & -\alpha \\ & & -\alpha & \alpha(2-R) \end{vmatrix} = 0$$

Expanding the first determinant on its last row and the second on its first row yields:

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* Reference 10, v. 2, page 422.

$$\left\{(1+\beta-R)\overset{(m-1)\times(m-1)}{\begin{vmatrix} 2-R & -1 & & & \\ -1 & 2-R & -1 & & \\ & & \ddots & & \\ & & -1 & 2-R & -1 \\ & & & -1 & 2-R \end{vmatrix}} - \overset{(m-2)\times(m-2)}{\begin{vmatrix} 2-R & -1 & & & \\ -1 & 2-R & -1 & & \\ & & \ddots & & \\ & & -1 & 2-R & -1 \\ & & & -1 & 2-R \end{vmatrix}}\right\}\left\{[\beta+\alpha(1-R)]\overset{(n-1)\times(n-1)}{\begin{vmatrix} \alpha(2-R) & -\alpha & & \\ -\alpha & & & \\ & & \ddots & -\alpha \\ & & -\alpha & \alpha(2-R) \end{vmatrix}} - \alpha^2 \overset{(n-2)\times(n-2)}{\begin{vmatrix} \alpha(2-R) & -\alpha & & \\ -\alpha & & & \\ & & \ddots & -\alpha \\ & & -\alpha & \alpha(R-2) \end{vmatrix}}\right\}$$

$$-\beta^2\alpha^{n-1}\underset{(m-1)\times(m-1)}{\begin{vmatrix} 2-R & -1 & & & \\ -1 & 2-R & -1 & & \\ & & \ddots & & \\ & & -1 & 2-R & -1 \\ & & & -1 & 2-R \end{vmatrix}}\;\underset{(n-1)\times(n-1)}{\begin{vmatrix} 2-R & -1 & & & \\ -1 & 2-R & -1 & & \\ & & \ddots & & \\ & & -1 & 2-R & -1 \\ & & & -1 & 2-R \end{vmatrix}} = 0 \qquad \text{IV-11)}$$

Factoring out powers of α and applying identity 1 page 53 yields:

$$\left\{(1+\beta-R)\frac{\sin m\phi}{\sin\phi} - \frac{\sin(m-1)\phi}{\sin\phi}\right\}\left\{[\beta+\alpha(1-R)]\,\alpha^{n-1}\frac{\sin n\phi}{\sin\phi} - \frac{\sin(n-1)\phi}{\sin\phi}\right\}$$

$$-\beta^2\alpha^{n-1}\frac{\sin m\phi}{\sin\phi}\,\frac{\sin n\phi}{\sin\phi}$$

$$2-R = 2\cos\phi$$

$$1-R = 2\cos\phi - 1$$

$$R = 4\sin^2\frac{\phi}{2}$$

Using the definition of R and the recurrance formula for the sine gives:

$$(1-R)\sin m\phi - \sin(m-1)\phi = 2\cos\phi\sin m\phi - \sin(m-1)\phi - \sin m\phi$$

$$= \sin(m+1)\phi - \sin m\phi$$

Restricting $\sin\phi \neq 0$ and canceling out the $\sin\phi$ terms:

$$\left\{(\beta-1)\sin m\phi+\sin(m+1)\phi\right\}\left\{\left(\frac{\beta}{\alpha}-1\right)\sin n\phi+\sin(n+1)\phi\right\}$$

$$-\frac{\beta^2}{\alpha}\sin m\phi\cdot\sin n\phi=0$$

$$\beta\sin m\phi\left[\sin(n+1)\phi-\sin n\phi\right]+\frac{\beta}{\alpha}\sin n\phi\left[\sin(m+1)\phi-\sin m\phi\right]$$

$$-\sin m\phi\left[\sin(n+1)\phi-\sin n\phi\right]+\sin(m+1)\phi\left[\sin(n+1)\phi-\sin n\phi\right]=0$$

Dividing the expression by $\frac{\beta}{2\alpha}$ from equation IV-7) the expression simplifies to:

$$\alpha\left[\sin(n+1)\phi-\sin n\phi\right]\left[\sin m\phi+\sin(m+1)\phi\right]$$

$$+\left[\sin(m+1)\phi-\sin m\phi\right]\left[\sin(n+1)\phi+\sin n\phi\right]=0$$

By the identifies:

$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A+\sin B=2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

The final form of the frequency equation is:

$$\left\{4\alpha\cos(n+\tfrac{1}{2})\phi\sin(m+\tfrac{1}{2})\phi+4\cos(m+\tfrac{1}{2})\phi\sin(n+\tfrac{1}{2})\phi\right\}\sin\frac{\phi}{2}\cos\frac{\phi}{2}=0$$

$$\alpha=\frac{I_{pc}}{I_p}=-\tan(n+\tfrac{1}{2})\phi\cot(m+\tfrac{1}{2})\phi \qquad \text{IV-12)}$$

By definition of R:

$$\omega_p^2=R_p\frac{K_1}{J_1}=4\frac{c^2}{l_0^2}\sin^2\frac{\phi_p}{2}$$

For ϕ_p small:

$$\omega_p\simeq\frac{c}{l_0}\phi_p$$

$$\phi_P \simeq \frac{\omega}{c} \ell_0$$

$$(n+\tfrac{1}{2})\phi_P \simeq \frac{\omega}{c} L_{DC} \qquad (m+\tfrac{1}{2})\phi_P \simeq \frac{\omega}{c} L_P$$

$$\alpha = \frac{I_{DC}}{I_P} \simeq -\tan\left(\frac{\omega L_{DC}}{c}\right)\cot\left(\frac{\omega}{c}L_P\right)$$

Which is the same as frequency equation III-2).

Equation IV-12 can be rewritten as:

$$1-\alpha = \frac{\sin(m+n+1)\phi}{\cos(n+\tfrac{1}{2})\phi \sin(m+\tfrac{1}{2})\phi}$$

which has the same roots for $\alpha = 1$ as equation IV-3) for a uniform bar of length $(m+n+1)\ell_0$. Both frequency equation III-2 and IV-12) are indeterminate when $L_P = L_{DC}$. In this case the two bars have the same fixed-fixed frequencies and can vibrate independently. In an actual drill string, the drill pipe is usually many times longer than the drill collar.

If γ_P is a root of the equation

$$\alpha = \frac{I_{DC}}{I_P} = -\tan\gamma \cot\delta\gamma \qquad \delta = \frac{m+\tfrac{1}{2}}{n+\tfrac{1}{2}} = \frac{L_P}{L_{DC}}$$

The natural frequency by the wave equation is

$$\omega_P = \frac{c\gamma_P}{L_{DC}} = \frac{c\gamma_F}{(n+\tfrac{1}{2})\ell_0} = \frac{c\phi}{\ell_0} \qquad (n+\tfrac{1}{2})\phi = \gamma$$

For the lumped equivalent system the natural frequency is:

$$\omega_P' = 2\frac{c}{\ell_0}\sin\frac{\phi}{2}$$

Error of the lumped system is:

$$\frac{\omega_p - \omega'_p}{\omega_p} = 1 - \frac{2}{\phi} \sin \frac{\phi}{2}$$

Since the approximate roots of IV-12) occur when $\alpha = 1$, at $\phi_p = \frac{\pi}{m+n+1} p$ the maximum error can be computed as a function of n and $\frac{L_p}{L_{oc}}$. This is shown in Figure 16.

$$\text{i.e.} \quad \frac{\omega_p - \omega'_p}{\omega_p} \simeq 1 - \frac{2}{p\pi}(m+n+1) \sin\left\{\frac{\pi}{2(m+n+1)} p\right\} \qquad \text{IV-13)}$$

Similar to the equivalent system of a single uniform bar, the mode shapes can be evaluated by assuming that $\bar{\theta}_1 = \sin \phi_p$ for the pth mode. The recurrance formula for the sine allows the first m coefficients to be evaluated:

$$\bar{\theta}_1 = \sin \phi_p$$

$$\begin{aligned}
\bar{\theta}_k &= (2-R)\,\bar{\theta}_{k-1} - \bar{\theta}_{k-2} \\
&= 2\cos\phi \sin(k-1)\phi_p - \sin(k-2)\phi_p \\
&= \sin(k\phi_p)
\end{aligned}$$

By the mth equation of motion IV-9):

$$\begin{aligned}
\beta\,\bar{\theta}_{m+1} &= (\beta+1-R)\,\bar{\theta}_m - \bar{\theta}_{m-1} \\
&= \beta \sin m\phi_p + (2\cos\phi_p - 1)\sin m\phi_p - \sin(m-1)\phi_p \\
&= (\beta - 1)\sin m\phi_p + \sin(m+1)\phi_p
\end{aligned}$$

$$\beta = \frac{2\alpha}{1+\alpha} \qquad \beta - 1 = \frac{\alpha - 1}{\alpha + 1} \qquad \text{by IV-7)}$$

Divided by $\frac{\beta}{\alpha}$ the equation becomes:

$$2\alpha\,\bar{\Theta}_{m+1} = \alpha\left\{\sin m\phi_p + \sin(m+1)\phi_p\right\} + \sin(m+1)\phi_p - \sin m\phi_p$$

$$2\bar{\Theta}_{m+1} = 2\sin(m+\tfrac{1}{2})\phi_p\cos\frac{\phi_p}{2} + \frac{2}{\alpha}\cos(m+\tfrac{1}{2})\phi_p\sin\frac{\phi_p}{2}$$

$$\alpha = -\tan(n+\tfrac{1}{2})\phi_p\cot(m+\tfrac{1}{2})\phi_p$$

$$\begin{aligned}\bar{\Theta}_{m+1} &= \sin(m+\tfrac{1}{2})\phi_p\left\{\cos\frac{\phi_p}{2} - \frac{\cos(n+\frac{1}{2})\phi_p\sin\frac{\phi_p}{2}}{\sin(n+\frac{1}{2})\phi_p}\right\}\\ &= \frac{\sin(m+\frac{1}{2})\phi_p}{\sin(n+\frac{1}{2})\phi_p}\left\{\sin(n+\tfrac{1}{2})\phi_p\cos\frac{\phi_p}{2} - \cos(n+\tfrac{1}{2})\phi_p\sin\frac{\phi_p}{2}\right\}\\ &= \frac{\sin(m+\frac{1}{2})\phi_p}{\sin(n+\frac{1}{2})\phi_p}\sin n\phi_p\end{aligned}$$

By the m + 1 st equation of motion IV-9:

$$\bar{\Theta}_{m+2} = \left(\frac{\beta}{\alpha} + 1 - R\right)\bar{\Theta}_{m+1} - \bar{\Theta}_m$$

$$\frac{\beta}{\alpha} = \frac{2}{1+\alpha} = \frac{2\sin(m+\frac{1}{2})\phi_p\cos(n+\frac{1}{2})\phi_p}{\sin(m-n)\phi_p}$$

$$1 - R = 2\cos\phi - 1$$ by IV-7 and IV-12)

$$\left(\frac{\beta}{\alpha} + 1 - R\right) = \frac{2\cos\phi_p\sin(m-n)\phi_p + \sin(m+n+1)\phi_p}{\sin(m-n)\phi_p}$$

$$\begin{aligned}\bar{\Theta}_{m+2} = \frac{\sin(m+\frac{1}{2})\phi_p}{\sin(m-n)\phi_p\sin(n+\frac{1}{2})\phi_p}\Big\{&2\cos(n+\tfrac{1}{2})\phi_p\sin(m+\tfrac{1}{2})\phi_p\sin n\phi_p\\ &-2\sin(n+\tfrac{1}{2})\phi_p\cos(n+\tfrac{1}{2})\phi_p\sin m\phi_p\\ &+(2\cos\phi_p - 1)\sin(m-n)\phi_p\sin n\phi_p\Big\}\end{aligned}$$

$$\bar{\Theta}_{m+2} = \frac{\sin(m+\frac{1}{2})\phi_p}{\sin(n+\frac{1}{2})\phi_p}\sin(n-1)\phi_p$$

The remaining can be found by rewriting the recurrance formula for the sine and solving the remaining equations of motion IV-9).

$$\sin(\ell-1)\phi = 2\cos\phi\sin\ell\phi - \sin(\ell+1)\phi$$

$$r = 1, 2, 3\ldots \qquad \bar{\Theta}_{m+r} = \frac{\sin(m+\frac{1}{2})\phi_p}{\sin(n+\frac{1}{2})\phi_p}\sin(n+1-r)\phi_p$$

The mode shapes for the pth mode are then:

$$k \leq m \qquad \bar{\theta}_{kp} = \sin k\phi_p$$

$$k \geq m \qquad \bar{\theta}_{kp} = \frac{\sin(m+\frac{1}{2})\phi_p}{\sin(n+\frac{1}{2})\phi_p} \sin(m+n+1-k)\phi_p$$

which are of the same form as equation III-v).

Defining the normalization factor below*:

$$N_p^2 = \sum_{k=1}^{m+n} J_{kk}\bar{\theta}_{kp}^2$$

Application of identity 4, page 54, yields the result below:

$$\sum_{k=1}^{m} J_{kk}\bar{\theta}_{kp}^2 = J_1 \sum_{k=1}^{m} \sin^2 k\theta_p = \frac{J_1}{4}\left\{2m+1-\frac{\sin(2m+1)\phi_p}{\sin\phi_p}\right\}$$

$$\sum_{k=m+1}^{m+n} J_{kk}\bar{\theta}_{kp}^2 = \alpha J_1 \frac{\sin^2(m+\frac{1}{2})\phi_p}{\sin^2(n+\frac{1}{2})\phi_p} \sum_{s=1}^{n} \sin^2 s\phi_p$$

$$= \frac{\alpha J_1}{4} \frac{\sin^2(m+\frac{1}{2})\phi_p}{\sin^2(n+\frac{1}{2})\phi_p}\left\{2n+1-\frac{\sin(2n+1)\phi_p}{\sin\phi_p}\right\}$$

$$N_p^2 = J_1\left\{\sum_{k=1}^{m}\bar{\theta}_k^2 + \alpha\sum_{k=m+1}^{m+n}\bar{\theta}_k^2\right\}$$

$$= \frac{J_1}{4}\left\{2m+1-\frac{\sin(2m+1)\phi_p}{\sin\phi_p} - \frac{\cos(m+\frac{1}{2})\phi_p\sin(m+\frac{1}{2})\phi_p}{\cos(n+\frac{1}{2})\phi_p\sin(n+\frac{1}{2})\phi_p}\left(2n+1-\frac{\sin(2n+1)\phi_p}{\sin\phi_p}\right)\right\}$$

$$N_p^2 = \frac{J_1}{4}\left\{2m+1-\frac{\sin(2m+1)\phi_p}{\sin(2n+1)\phi_p}(2n+1)\right\}$$

---

* See reference 7, pp 40-42, for explanation of definition.

Elements of the solution vector to the fixed-fixed system of equations may be written:

$$k \leq m \qquad e_{kp} = \frac{\sin k\phi_p}{N_p} \qquad \text{IV-15)}$$

$$k \geq m \qquad e_{kp} = \frac{1}{N_p}\frac{\sin(m+\frac{1}{2})\phi_p}{\sin(n+\frac{1}{2})\phi_p}\sin(m+n+1-k)\phi_p$$

Where e_{kp} is the amplitude of the motion of the k^{th} mass when the system is vibrating at the p^{th} natural frequency.

Consider the product:

$$N_p N_q \sum_{k=1}^{m+n} e_{kp} J_{kk} e_{kq} = J_1 \sum_{k=1}^{n} \sin k\phi_p \sin k\phi_q$$

$$+\alpha J_1 \frac{\sin(m+\frac{1}{2})\phi_p \sin(m+\frac{1}{2})\phi_q}{\sin(n+\frac{1}{2})\phi_p \sin(n+\frac{1}{2})\phi_q} \sum \sin k\phi_p \sin\phi_q$$

Substitution of identify 6, page 54, and substitution of the frequency IV-12) for α, yield a fraction with the numerator below:

$$= \sin(m+\tfrac{1}{2})\phi_p \cos(m+\tfrac{1}{2})\phi_q \cos\frac{\phi_p}{2}\sin\frac{\phi_p}{2}$$
$$- \cos(m+\tfrac{1}{2})\phi_p \sin(m+\tfrac{1}{2})\phi_q \sin\frac{\phi_p}{2}\cos\frac{\phi_q}{2}$$

$$- \frac{\sin(m+\frac{1}{2})\phi_p \cos(m+\frac{1}{2})\phi_q}{\sin(n+\frac{1}{2})\phi_p \cos(n+\frac{1}{2})\phi_q}\Big\{\sin(n+\tfrac{1}{2})\phi_p \cos(n+\tfrac{1}{2})\phi_q \cos\frac{\phi_p}{2}\sin\frac{\phi_q}{2}$$
$$- \cos(n+\tfrac{1}{2})\phi_p \sin(n+\tfrac{1}{2})\phi_q \sin\frac{\phi_p}{2}\cos\frac{\phi_q}{2}\Big\}$$

$$= \frac{2\sin\frac{\phi_p}{2}\cos\frac{\phi_q}{2}}{\sin(n+\frac{1}{2})\phi_p\cos(n+\frac{1}{2})\phi_q}\left\{\sin(m-n)\phi_p\sin(m+n+1)\phi_q - \sin(m-n)\phi_q\sin(m+n+1)\phi_p\right\}$$

$$= \frac{2\sin\frac{\phi_p}{2}\cos\frac{\phi_q}{2}\left[(1-\alpha)(1+\alpha)-(1-\alpha)(1+\alpha)\right]}{\sin(n+\frac{1}{2})\phi_p\cos^2(n+\frac{1}{2})\phi_q\sin(m+\frac{1}{2})\phi_q\sin(m+\frac{1}{2})\phi_p\cos(n+\frac{1}{2})\phi_p} = 0$$

$$\therefore \sum_{k=1}^{m+n} e_{kp} J_{kk} e_{kq} = 0 \quad p \neq q \qquad \text{IV-15-a)}$$

For the fixed top, free bottom case the equations of motion are the same as IV-9) except for the last equation. $K_{m+n+1} = 0$, as shown in Figure 14.f. The same constants defined in equations IV-7) will also apply here.

$$\theta_k(t) = \bar{\theta}_k e^{i\omega t}$$

$$R = \frac{J_1\omega^2}{K_1} = \left(\frac{c\omega}{l_0}\right)^2$$

$$k=1 \qquad (2-R)\bar{\theta}_1 - \bar{\theta}_2 = 0$$

$$1 < k < m \qquad -\bar{\theta}_{k-1} + (2-R)\bar{\theta}_k - \bar{\theta}_{k+1} = 0$$

$$k=m \qquad -\bar{\theta}_{m-1} + (\beta+1-R)\bar{\theta}_m - \beta\bar{\theta}_{m+1} = 0$$

$$k=m+1 \qquad -\beta\bar{\theta}_m + (\beta+\alpha(1-R))\bar{\theta}_{m+1} - \alpha\bar{\theta}_{m+2} = 0$$

$$m+n < k < n \qquad -\alpha\bar{\theta}_{k-1} + \alpha(2-R)\bar{\theta}_k - \alpha\bar{\theta}_{k+1} = 0$$

$$k=m+n \qquad -\alpha\bar{\theta}_{m+n-1} + \alpha(1-R)\bar{\theta}_{m+n} = 0$$

The frequency determinant is then:

$$
\begin{array}{l}
m + n \text{ rows} \\
\& \text{ columns} \\
\\
\\
\text{row } m \text{ ———} \\
\\
\\
\\
\\
\end{array}
\begin{vmatrix}
2-R & -1 & & & & & & \\
-1 & 2-R & -1 & & & & & \\
 & & & & & & & \\
 & -1 & 2-R & -1 & & & & \\
 & & -1 & (1+\beta-R) & -\beta & & & \\
 & & & -\beta & (\beta+\alpha(1-R)) & -\alpha & & \\
 & & & & -\alpha & \alpha(2-R) & -\alpha & \\
 & & & & & -\alpha & \alpha(2-R) & -\alpha \\
 & & & & & & -\alpha & \alpha(1-R)
\end{vmatrix} = 0
$$

Expansion on the mth row and column by Sylvester's identity; and reduction of the two highest order determinants in the fix-fix case yields results similar to equation IV-11).

$$
\left\{ (1+\beta-R) \underset{(m-1)x(m-1)}{\begin{vmatrix} 2-R & -1 & & \\ -1 & 2-R & -1 & \\ & & & \\ & -1 & 2-R & -1 \\ & & -1 & 2-R \end{vmatrix}} - \underset{(m-2)x(m-2)}{\begin{vmatrix} 2-R & -1 & & \\ -1 & 2-R & -1 & \\ & & & \\ & -1 & 2-R & -1 \\ & & -1 & 2-R \end{vmatrix}} \right\}
\left\{ [\beta+\alpha(1-R)] \underset{(n-1)x(n-1)}{\begin{vmatrix} \alpha(2-R) & -\alpha & & \\ -\alpha & \alpha(2-R) & -\alpha & \\ & & & \\ & -\alpha & \alpha(2-R) & -\alpha \\ & & -\alpha & \alpha(1-R) \end{vmatrix}} - \alpha^2 \underset{(n-2)x(n-2)}{\begin{vmatrix} \alpha(2-R) & -\alpha & & \\ -\alpha & \alpha(2-R) & -\alpha & \\ & & & \\ & -\alpha & \alpha(2-R) & -\alpha \\ & & -\alpha & \alpha(1-R) \end{vmatrix}} \right\}
$$

$$
-\beta^2 \alpha^{n-1} \underset{(m-1)x(m-1)}{\begin{vmatrix} 2-R & -1 & & \\ -1 & 2-R & -1 & \\ & & & \\ & -1 & 2-R & -1 \\ & & -1 & 2-R \end{vmatrix}} \underset{(n-1)x(n-1)}{\begin{vmatrix} 2-R & -1 & & \\ -1 & 2-R & -1 & \\ & & & \\ & -1 & 2-R & -1 \\ & & -1 & 1-R \end{vmatrix}} = 0
$$

Applying identify 1, page 53, to the first, second, and fifth determinants; identify 2, page 43, to the third, fourth, and sixth determinants yields:

$$\left\{(1+\beta-R)\frac{\sin m\phi - \sin(m-1)\phi}{\sin\phi}\right\}\left\{\left(\frac{\beta}{\alpha}+1-R\right)\frac{\sin n\phi - \sin(n-1)\phi}{\sin\phi} - \frac{\sin(n-1)\phi - \sin(n-2)\phi}{\sin\phi}\right\}\alpha^n$$

$$-\alpha^{n-1}\beta^2\frac{\sin m\phi}{\sin^2\phi}[\sin n\phi - \sin(n-1)\phi] = 0$$

$$(1-R)\sin m\phi - \sin(m-1)\phi = 2\cos\phi\sin m\phi - \sin(m-1)\phi - \sin m\phi$$

$$= \sin(m+1)\phi - \sin m\phi$$

Restricting ϕ so $\sin\phi \neq 0$, and canceling $\sin\phi$ out:

$$\left\{(\beta-1)\sin m\phi + \sin(m+1)\phi\right\}\left\{\frac{\beta}{\alpha}[\sin n\phi - \sin(n-1)\phi] + [\sin(n+1)\phi - 2\sin n\phi + \sin(n-1)\phi]\right\}$$

$$-\frac{\beta^2}{\alpha}\sin m\phi[\sin n\phi - \sin(n-1)\phi] = 0$$

$$\frac{\beta}{\alpha}[\sin n\phi - \sin(n-1)\phi][\sin(m+1)\phi - \sin m\phi]$$

$$+\beta\sin m\phi[\sin(n+1)\phi - 2\sin n\phi + \sin(n-1)\phi]$$

$$+[\sin(m+1)\phi - \sin m\phi][\sin(n+1)\phi - 2\sin n\phi + \sin(n-1)\phi] = 0$$

Dividing the equation by $\frac{\beta}{2\alpha}$ yields:

$$\beta = \frac{2\alpha}{1+\alpha} \qquad \frac{\beta}{2\alpha} = \frac{1}{1+\alpha}$$

$$\alpha[\sin(n+1)\phi-2\sin n\phi+\sin(n-1)\phi][2\sin m\phi+\sin(m+1)\phi-\sin m\phi]$$

$$+[\sin(m+1)\phi-\sin m\phi][\sin(n+1)\phi-\sin n\phi]=0$$

$$\sin A \pm \sin B = 2\sin\left(\frac{A\pm B}{2}\right)\cos\left(\frac{A\mp B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin(n+1)\phi-2\sin n\phi+\sin(n-1)\phi = 2\sin\frac{\phi}{2}[\cos(n+\tfrac{1}{2})\phi-\cos(n-\tfrac{1}{2})\phi]$$

$$= -4\sin^2\frac{\phi}{2}\sin n\phi$$

$$\sin(m+1)\phi+\sin m\phi = 2\sin(m+\tfrac{1}{2})\phi\cos\frac{\phi}{2}$$

$$\sin(m+1)\phi-\sin m\phi = 2\cos(m+\tfrac{1}{2})\phi\sin\frac{\phi}{2}$$

$$\sin(n+1)\phi-\sin(n-1)\phi = 2\cos n\phi\sin\phi$$

Applying the above identities gives:

$$-8\alpha\sin^2\frac{\phi}{2}\sin n\phi\sin(m+\tfrac{1}{2})\phi\cos\phi = -8\cos(m+\tfrac{1}{2})\phi\cos n\phi\sin^2\frac{\phi}{2}\cos\frac{\phi}{2}$$

The final form of the natural frequency equation is:

$$\alpha = \frac{I_{DC}}{I_p} = \cot(m+\tfrac{1}{2})\phi\cot n\phi \qquad \text{IV-16)}$$

For ϕ_p the pth root of the frequency equation, by the definition of R:

$$\omega_p^2 = R_p\frac{k}{J}$$

$$= 4\frac{c^2}{l_0^2}\sin^2\frac{\phi_p}{2}$$

For ϕ_p small: $\omega_p \simeq \frac{c}{l_0}\phi_p$

$$\phi_p \simeq \omega_p\frac{l_0}{c}$$

$$(m+\tfrac{1}{2})\ell_o = L_{DC} \qquad n\ell_o \simeq L_{DC}$$

$$\alpha = \frac{I_{DC}}{I_P} \simeq \cot\frac{\omega L_P}{c}\cot\frac{\omega L_{DC}}{c}$$

Which is the same as equation III-1).

Equation IV-16) may also be written:

$$\alpha - 1 = \frac{\cos\left(\frac{2(m+n)+1}{2}\phi\right)}{\sin(m+\frac{1}{2})\phi\ \sin n\phi}$$

which has the same roots for $\alpha = 1$ as equation IV-6) for a uniform bar of length $(m + n)\ell_o$.

If γ_P is a root of the equation:

$$\alpha = \cot\gamma\cot\delta\gamma \qquad \delta = \frac{m+\frac{1}{2}}{n-\frac{1}{2}} = \frac{L_P}{L_{DC}}$$

The natural frequency by the wave equation solution is:

$$\omega_P = \frac{c\gamma_P}{L_{DC}} = \frac{c\gamma_P}{(n-\frac{1}{2})\ell_o} = \frac{cn\phi_P}{(n-\frac{1}{2})\ell_o} \qquad \phi = \frac{\gamma}{n}$$

For the lumped equivalent system, the natural frequency is:

$$\omega_P' = 2\ \frac{c}{\ell_o}\sin\frac{\phi}{2}$$

Error of the lumped system natural frequencies is then:

$$\frac{\omega_P - \omega_P'}{\omega_P} = 1 - \frac{2n-1}{n\phi}\sin\frac{\phi}{2}$$

Since the maximum roots of IV-16) occur when $\alpha = 1$, at $\phi_P = \frac{(2P-1)\pi}{2(m+n)+1}$, the maximum error can be computed as a function of n and $\underline{L}$P.

$$\text{i.e.}\quad \frac{\omega_P-\omega_P'}{\omega_P} \leq 1-(2-\frac{1}{n})\,\frac{2(m+n)+1}{(2P-1)\pi}\sin\frac{\phi}{2} \qquad \text{IV-17)}$$

As in the fixed-fixed case, the first coefficient $\bar{\Theta}_1$ will be assumed as $\sin\phi_p$ for the pth natural mode. As before, the recurrance formula for the sine can be used to evaluate successive coefficients up to $\bar{\Theta}_m$.

$$k \leq m \qquad \bar{\Theta}_k = (2-R)\bar{\Theta}_{k-1} - \bar{\Theta}_{k-2} \qquad 2-R = 2\cos\phi_p$$
$$= \sin k\phi_p$$

By the mth equation of motion IV-9)

$$\beta\bar{\Theta}_{m+1} = (\beta + 1 - R)\,\bar{\Theta}_m - \bar{\Theta}_{m-1}$$

$$= \beta\sin m\phi_p + (2\cos\phi_p - 1)\sin m\phi_p - \sin(m-1)\phi_p$$

$$= (\beta-1)\sin m\phi_p + \sin(m+1)\phi_p$$

Divided by $\frac{\beta}{\alpha}$ the equation becomes:

$$\frac{\beta}{\alpha} = \frac{2}{1+\alpha}$$

$$2\alpha\,\bar{\Theta}_{m+1} = \alpha(\sin m\phi_p + \sin(m+1)\phi_p$$

$$+ \sin(m+1)\phi_p - \sin m\phi_p$$

$$\bar{\Theta}_{m+1} = \sin(m+\tfrac{1}{2})\phi_p\cos\frac{\phi_p}{2} + \frac{1}{\alpha}\cos(m+\tfrac{1}{2})\phi_p\sin\frac{\phi_p}{2}$$

$$\frac{1}{\alpha} = \tan(m+\tfrac{1}{2})\phi_p \tan n\phi_p \qquad \text{by IV-16}$$

$$\bar{\Theta}_{m+1} = \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\left\{\cos n\phi_p\cos\frac{\phi_p}{2} + \sin n\phi_p\sin\frac{\phi_p}{2}\right\}$$

$$\bar{\Theta}_{m+1} = \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\cos(n-\tfrac{1}{2})\phi_p$$

By the m+1 at equation of motion IV-9):

$$\bar{\Theta}_{m+2} = \left(\frac{\beta}{\alpha} + 1 - R\right)\bar{\Theta}_{m+1} - \bar{\Theta}_m$$

$$= \frac{\beta}{\alpha}\left\{\frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\cos(n-\tfrac{1}{2})\phi_p - \sin m\phi_p\right\}$$

$$+ (2\cos\phi_p - 1)\frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\cos(n-\tfrac{1}{2})\phi_p$$

$$= \frac{\beta}{\alpha}\left\{\frac{\cos(m-n+\frac{1}{2})\phi_p}{\cos n\phi_p}\sin\frac{\phi_p}{2}\right\}$$

$$+ (2\cos\phi_p - 1)\frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\cos(n-\tfrac{1}{2})\phi_p$$

$$\frac{\beta}{\alpha} = \frac{2}{1+\alpha} = \frac{\sin(m+\frac{1}{2})\phi_p \sin n\phi_p}{\cos(m-n+\frac{1}{2})\phi_p} \qquad \text{by IV-16)}$$

$$= \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\left\{2\sin\frac{\phi_p}{2}\sin n\phi_p + (2\cos\phi_p - 1)\cos(n-\tfrac{1}{2})\phi_p\right\}$$

$$= \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\left\{\cos(n-\tfrac{1}{2})\phi_p - \cos(n+\tfrac{1}{2})\phi_p\right.$$

$$\left. - (2\cos\phi_p - 1)\cos(n-\tfrac{1}{2})\phi_p\right\}$$

$$= \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\left\{2\cos\phi_p\cos(n-\tfrac{1}{2})\phi_p - \cos(n+\tfrac{1}{2})\phi_p\right\}$$

$$\bar{\Theta}_{m+2} = \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\cos(n-\tfrac{3}{2})\phi_p$$

Writing the recurrance formula for the cosine and solving the remaining equations of motion IV-9), the remaining $\bar{\Theta}_k$ are found to be:

$$\cos(l-1)\phi_p = 2\cos\phi_p\cos l\phi_p - \cos(l+1)\phi_p$$

$$r=1,2,\ldots \qquad \bar{\Theta}_{m+r} = \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p}\cos(n+\tfrac{1}{2}-r)\phi_p$$

The amplitude coefficients for the pth mode are then:

$$k \leq m \quad \bar{\Theta}_{kp} = \sin k\phi_p$$

$$k > m \quad \bar{\Theta}_{kp} = \frac{\sin(m+\frac{1}{2})\phi_p}{\cos n\phi_p} \cos(m+n+\tfrac{1}{2}-k)\phi_p$$

which are of the form of equation III-3).

Defining the normalizing factor for the pth mode as:

$$N_p^2 = \sum_{k=1}^{m+n} J_{kk} \bar{\Theta}_{kp}^2$$

Application of identities 4 and 5, page 54, give the results below:

$$\sum_{k=1}^{m} \bar{\Theta}_k^2 = \sum_{k=1}^{m} \sin^2 k\phi_p$$

$$= \frac{2m+1}{4} + \frac{1}{4}\frac{\sin(2m+1)\phi_p}{\sin\phi_p}$$

$$\sum_{k=m+1}^{m+n} \bar{\Theta}_k^2 = \frac{\sin^2(m+\frac{1}{2})\phi_p}{\cos^2 n\phi_p} \sum_{k=m+1}^{m+n} \cos^2(m+n+\tfrac{1}{2}-k)\phi_p$$

$$\sum_{k=m+1}^{m+n} \cos^2(m+n+\tfrac{1}{2}-k)\phi_p = \sum_{s=1}^{n} \cos^2\frac{(2s-1)\phi_p}{2}$$

$$= \sum_{s=1}^{2n-1} \cos^2 s\frac{\phi_p}{2} - \sum_{s=1}^{n-1} \cos^2 s\phi_p$$

$$= \frac{n}{2} - \frac{1}{8}\frac{\cos(2n+\frac{1}{2})\phi_p - \cos(2n-\frac{1}{2})\phi_p}{\sin\frac{\phi_p}{2}\sin\phi_p}$$

$$= \frac{n}{2} + \frac{1}{4}\frac{\sin 2n\phi_p}{\sin\phi_p}$$

$$N_p^2 = J_1 \sum_{k=1}^{m} \bar{\Theta}_{kp}^2 + \alpha J_1 \sum_{k=m+1}^{m+n} \bar{\Theta}_{kp}^2$$

The final form of the normalization factor for the fixed-free case is:

$$N_p^2 = \frac{2m+1}{4} + \frac{n}{2}\,\frac{\sin(2m+1)\phi_p}{\sin 2n\phi_p} \qquad \text{IV-18)}$$

Elements of the solution vector to the fixed-free system of equations may be written

$$k \leq m \qquad e_{kp} = \frac{\sin k\phi_p}{N_p}$$

$$\text{IV-19)}$$

$$k > m \qquad e_{kp} = \frac{\sin(m+\frac{1}{2})\phi_p}{N_p \cos n\phi_p}\cos(m+n+\tfrac{1}{2}-k)\phi_p$$

Where e_{kp} is the amplitude of the motion of the k^{th} mass when the system is vibrating at the p^{th} natural frequency.

Consider the product:

$$N_p N_q \sum_{k=1}^{m+n} e_{kp} J_{kk} e_{kq}$$

$$= J_1 \sum_{k=1}^{m} \sin k\phi_p \sin k\phi_q$$

$$+ \alpha\,\frac{\sin(m+\frac{1}{2})\phi_p \sin(m+\frac{1}{2})\phi_q}{\cos n\phi_q \cos n\phi_p}\,.$$

$$\cdot \sum_{k=m+1}^{m+n} \cos \tfrac{k}{2}\phi_p \cos \tfrac{k}{2}\phi_q$$

Substitution of identities 6 and 7, page 43, for the summation terms and use of the frequency equation IV-16) will reduce the term to zero just as in the fixed-fixed case.

$$\text{i.e.} \quad \sum_{k=1}^{m+n} e_{kp} J_{kk} e_{kq} = 0 \quad p \neq q \qquad \text{IV-19-a)}$$

For the fixed bottom, free top case mode coefficients will be counted from the bottom of the string.* Results are then the same as the fixed top-free bottom case with the subscripts interchanged and α^{-1} replacing α.

$$\cot(n+\tfrac{1}{2})\phi \cot m\phi = \frac{I_P}{I_{DC}} = \frac{1}{\alpha} \qquad \text{IV-20)}$$

$$k \leq n \qquad \bar{\theta}_{kp} = \sin k\phi_p \qquad \text{IV-21)}$$

$$k > n \qquad \bar{\theta}_{kp} = \frac{\sin(n+\frac{1}{2})\phi_p}{\cos m\phi_p} \cos(m+n+\tfrac{1}{2}-k)\phi_p$$

$$N_p^2 = \left\{ \frac{2n+1}{4} + \frac{m}{2} \frac{\sin(2n+1)\phi_p}{\sin 2m\phi_p} \right\} J_2 \qquad \text{IV-22)}$$

$$k > n \qquad e_{kp} = \frac{\sin k\phi_p}{N_p} \qquad \text{IV-23)}$$

$$k \leq n \qquad e_{kp} = \frac{\sin(n+\frac{1}{2})\phi_p}{N_p \cos m\phi_p} \cos(m+n+\tfrac{1}{2}-k)\phi_p$$

---

* $L_{DC} = (n + \frac{1}{2})\, \ell_o \qquad J_2 = \dfrac{I_{DC}\, \ell_o}{g}$

$L_P = m\, \ell_o$

V. RESPONSE OF THE EQUIVALENT LUMPED SYSTEMS

The equations of motion of the lumped system of section IV can be put into the matrix form given below. By matrix algebra the response of the system may be found to an arbitrary periodic forcing function acting on any or all of the masses of the system. The general equation of motion for the r^{th} inertia may be written:

$$\theta_r(K_{r+1}+K_{r-1})+\ddot{\theta}_r J_r+\dot{\theta}_r C_r-K_{r-1}\theta_{r-1}-K_{r+1}\theta_{r+1}=F_r e^{i\omega t}$$

$$\theta_r S_{rr}+\ddot{\theta}_r J_{rr}+\dot{\theta} C_{rr}+S_{r\,r-1}\theta_{r-1}+S_{r\,r+1}\theta_{r+1}=F_r e^{i\omega t}$$

$$S_{rr}=K_{r+1}+K_{r-1} \qquad S_{r\,r+1}=-K_{r+1}$$

$$S_{r\,r-1}=-K_{r-1}$$

$$S_{rs}=0 \quad |r-s|\geq 2$$

$$J_{rs}=0 \quad r\neq s$$

$$[J][\ddot{\theta}]+[C][\dot{\theta}]+[S][\theta]=[F]e^{i\omega t}$$

$$[J]=J_1\begin{array}{cl}\begin{bmatrix}1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & \alpha & & \\ & & & & & & \alpha & \\ & & & & & & & \ddots \\ & & & & & & & & \alpha \\ & & & & & & & & & \alpha\end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \text{row } m \\ \\ \\ \\ \\ \\ \end{array}\end{array}$$

The damping matrix is diagonal since only surface forces on a length of drill string ℓ_o are considered:

$$[C] = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & \ddots \end{bmatrix}$$

$$[S] = K_1 \begin{bmatrix} \delta & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -1 & 1+\beta & -\beta & & \\ & & & -\beta & \alpha+\beta & -\alpha & \\ & & & & \ddots & \ddots & \\ & & & & & -\alpha & \gamma\alpha \end{bmatrix}$$

Top Fixed-Fixed Bottom $\delta = 2$ $\gamma = 2$

row m

Top Fixed-Free Bottom $\delta = 2$ $\gamma = 1$

Top Free-Fixed Bottom $\delta = 1$ $\gamma = 2$

$$[F] = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{m+n} \end{bmatrix} \qquad [\theta] = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{m+n} \end{bmatrix}$$

Assuming a linear periodic response in terms of the natural modes:

$$[\theta] = [E][X]e^{i\omega t} \quad [X] = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ \vdots \end{bmatrix} \qquad \text{V-3)}$$

Where [E] is made up of the elements defined by equation IV-15) or IV-19).

$$[E] = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1\,m+n} \\ e_{21} & & & \vdots \\ e_{31} & & & \vdots \\ \vdots & & & \vdots \\ e_{m+n\,1} & \cdots & \cdots & \end{bmatrix}$$

Substitution of the above into equation V-I and premultiplication of the entire equation by ${}^{T}[E]$ yields:

$$-\omega^2\,{}^{T}[E][J][E][X] + i\omega\,{}^{T}[E][C][E][X] + {}^{T}[E][S][E][X] = {}^{T}[E][F]$$

Calling the first matrix product $\bar{J}$, the second $\bar{C}$, the third $\bar{S}$, the resulting elements of the k^{th} row and p^{th} column of each is:

$$\bar{J}_{kp} = \sum_{r}^{m+n} {}^{T}e_{rk} \sum_{s}^{m+n} J_{rs}\, e_{sp}$$

$$J_{rs} = 0 \quad r \neq s$$

$${}^{T}e_{rk} = e_{kr}$$

$$\bar{J}_{kp} = \sum_{r}^{m+n} J_{rr}\, e_{rk}\, e_{rp}$$

$= 1 \quad p = k$ by definition of the normalization factor

$= 0 \quad p \neq k$ by equation IV-15-a) or IV-19-a)

$$\bar{S}_{kp} = \sum_{r}^{m+n} {}^{T}e_{kr} \sum_{s}^{m+n} S_{rs}\, e_{sp}$$

$$= \sum_{r}^{m+n} e_{rk}\left(S_{r,r-1}\, e_{r-1,p} + S_{rr}\, e_{rp} + S_{r,r+1}\, e_{r+1,p}\right)$$

$$S_{rs} = 0 \quad |r-s| \geq 2$$

But $e_{r-1,p}$ etc. are the normalized amplitudes of the system when it is vibrating in a natural mode at frequency ω_p. Hence they satisfy the equation of motion for the r^{th} inertia.

$$S_{r,r-1}\, e_{r-1,p} + S_{r,r+1}\, e_{r+1,p} + e_{rp}(S_{rr} - \omega_p^2 J_{rr}) = 0$$

$$\therefore \quad \bar{S}_{kp} = \omega_p^2 \sum_{r}^{m+n} e_{rk}\, e_{rp}\, J_{rr}$$

$$= \omega_p^2 \qquad k = p$$

$$= 0 \qquad k \neq p$$

Therefore the inertia matrix and the spring matrix are diagonalized by the transformation V-3). In general the damping matrix is not.

$$\bar{C}_{kp} = \sum_{r}^{m+n} {}^{T}e_{kr} \sum_{s}^{m+n} c_{rs}\, e_{sp}$$

$$c_{rs} = 0 \quad r \neq s$$

$$= \sum_{r}^{m+n} e_{rk}\, e_{rp}\, c_{rr}$$

Since the E matrix is orthogonal to the inertia matrix and hence to the spring matrix, the damping matrix will be diagonalized exactly if it is proportional to the inertia matrix.

i.e. $C_{rr} = C_0 \qquad r \leq m$

$C_{rr} = \alpha C_0 \qquad r > m$

Since the above is physically unlikely, an approximate approach must be used.

For small damping the damping matrix may be assumed diagonal and the diagonal terms assumed proportional to the natural frequencies[12].

i.e. $\bar{C}_{kp} \simeq 0 \quad k \neq p$

$$\bar{c}_{kk} = \frac{\omega_k}{A_k} \qquad A_k \doteq 20 \qquad A_k = \omega_k \div \sum_r^{m+n} e_{rk}^2 c_{rr}$$

The equations separate and the X's may be solved for.

$$-\omega^2 X_k + i\omega \frac{\omega_k}{A_k} X_k + \omega_k^2 X_k = \sum_r^{m+n} {}^T e_{kr} F_r$$
$$= \sum_r^{m+n} e_{kr} F_r$$

Transforming back to the original coordinates, the final result is:

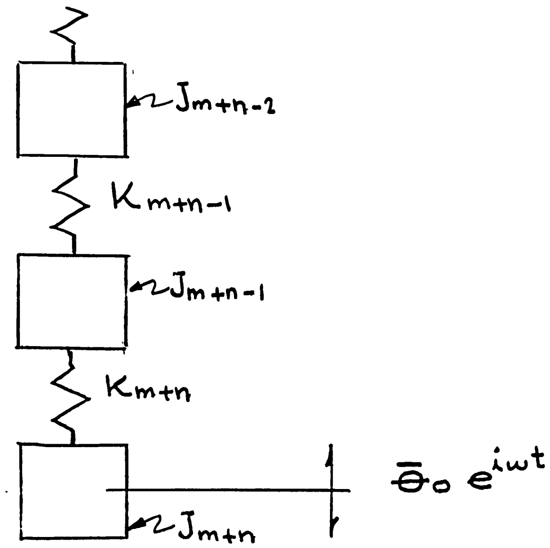
$$[\theta] = [E][X]\, e^{i\omega t}$$
$$\theta_l = \sum_k^{m+n} e_{lk} X_k$$
$$= \sum_k^{m+n} \sum_r^{m+n} \frac{e_{lk} e_{kr} F_r}{\omega_k^2 - \omega^2 + i\omega \frac{\omega_k}{A_k}}$$

If the only driving force is at the bottom of the hole:

$$\theta_l = \sum_k^{m+n} \frac{e_{lk} e_{m+n,k} F_{m+n}}{\omega_k^2 - \omega^2 + i\omega \frac{\omega_k}{A_k}}$$

The formula is written out on page 41 for the three types of lumped systems of section IV, for the response at the top of the hole for a periodic force at the bottom of the hole. The real part of the expression should be used.

For a given displacement at the bottom of the hole, the force on the next to last inertis becomes merely the disvlacement of the last inertia times the spring constant. The situation is illustrated below:



Given: $\Theta_{m+n} = \bar{\Theta}_0 e^{i\omega t}$ $\qquad\qquad$ $\Theta_k = \bar{\Theta}_k e^{i\omega t}$

$$-\omega^2 J_{m+n-1}\bar{\Theta}_{m+n-1} + K_{m+n-1}(\bar{\Theta}_{m+n-1} - \bar{\Theta}_{m+n-2}) = K_{m+n}(\bar{\Theta}_0 - \bar{\Theta}_{m+n-1})$$

$$\bar{\Theta}(K_{m+n-1} - K_{m+n} - \omega^2 J_{m+n-1}) - K_{m+n-1}\bar{\Theta}_{m+n-2}$$

$$= K_{m+n}\bar{\Theta}_0 = F_{m+n-1}$$

which is the same as a system with the bottom end fixed of one less degree of freedom. Responses are given below for the top of the string.

<u>Top</u> <u>Fixed</u>

$$\Theta_1 = \alpha \frac{c^2}{\ell_0^2}\sum_{k=1}^{m+n-1} \frac{\sin(m+\frac{1}{2})\phi_k \sin^2\phi_k \,\Theta_{BOTTOM}}{\sin(n-\frac{1}{2})\phi_k\left\{\frac{2m+1}{4} - \frac{2n-1}{4}\frac{\sin(2m+1)\phi_k}{\sin(2n-1)\phi_k}\right\}} \cdot \left\{\frac{e^{i\omega t}}{\omega_k^2 - \omega^2 + i\omega\frac{\omega_k}{A_k}}\right\}$$

Where ϕ_k is the k^{th} root of: $\dfrac{I_{oc}}{I_p} = -\cot(m+\frac{1}{2})\phi \tan(n-\frac{1}{2})\phi$

Top Free

$$\Theta_1 = \frac{c^2}{l_o^2} \sum_{k=1}^{m+n-1} \frac{\sin(n-\frac{1}{2})\phi_k \sin\phi_k \cos\frac{\phi_k}{2}\,\Theta_{BOTTOM}}{\cos m\phi_k \left\{ \frac{2n-1}{4} + \frac{m}{2} \frac{\sin(2n-1)\phi_k}{\sin 2m\phi_k} \right\}} \cdot \left\{ \frac{e^{i\omega t}}{\omega_k^2 - \omega^2 + i\omega \frac{\omega_k}{A_k}} \right\}$$

Where ϕ_k is the k^{th} root of: $\frac{I_{DC}}{I_P} = \cot(m+\frac{1}{2})\phi \cot(n-1)\phi$

Both cases: $m+\frac{1}{2} = (n-\frac{1}{2}) \frac{L_P}{L_{DC}}$ $\quad l_o = \frac{L_{DC}}{n+\frac{1}{2}}$

$$\omega_k^2 = \frac{4c^2}{l_o^2} \sin^2 \frac{\phi_k}{2}$$

RESPONSE AT TOP HOLE FOR PERIODIC FORCE AT BOTTOM

Fixed-Fixed

$$\Theta_1 = \frac{(m+\frac{1}{2})g}{\rho I_P L_P} \sum_{k=1}^{m+n} \frac{\sin(m+\frac{1}{2})\phi_k \sin^2\phi_k \, F_{BOTTOM}}{\sin(n+\frac{1}{2})\phi_k \left\{ \frac{2m+1}{4} - \frac{2n+1}{4} \frac{\sin(2m+1)\phi_k}{\sin(2n+1)\phi_k} \right\}} \cdot \left\{ \frac{e^{i\omega t}}{\omega_k^2 - \omega^2 + i\omega \frac{\omega_k}{A_k}} \right\}$$

Where ϕ_k is the k^{th} root of equation IV-12).

Top Fixed-Free Bottom

$$\Theta_1 = \frac{(m+\frac{1}{2})g}{\rho I_P L_P} \sum_{k=1}^{m+n} \frac{\sin(m+\frac{1}{2})\phi_k \sin\phi_k \cos\frac{\phi_k}{2} \, F_{BOTTOM}}{\cos n\phi_k \left\{ \frac{2m+1}{4} + \frac{n}{2} \frac{\sin(2m+1)\phi_k}{\sin 2n\phi_k} \right\}} \cdot \left\{ \frac{e^{i\omega t}}{\omega_k^2 - \omega^2 + i\omega \frac{\omega_k}{A_k}} \right\}$$

Where ϕ_k is the k^{th} root of equation IV-16).

Top Free Bottom Fixed

$$\Theta_1 = \frac{(n+\frac{1}{2})}{\rho\, I_{DC} L_{DC}}\, g \sum_{k=1}^{m+n} \frac{\sin(n+\frac{1}{2})\phi_k \sin\phi_k \cos\frac{\phi_k}{2} F_{BOTTOM}}{\cos m\phi_k \left\{\frac{2n+1}{4} + \frac{m}{2}\frac{\sin(2n+1)\phi_k}{\sin 2m\phi_k}\right\}} \cdot \left\{\frac{e^{i\omega t}}{\omega_k^2 - \omega^2 + i\omega\frac{\omega_k}{A_k}}\right\}$$

Where ϕ_k is the k^{th} root of equation IV-20).

For all cases

$$(m+\tfrac{1}{2}) = (n+\tfrac{1}{2})\frac{L_P}{L_{DC}} \qquad \ell_0 = \frac{L_{DC}}{n+\frac{1}{2}}$$

$$\omega_k^2 = 4\frac{c^2}{\ell_0^2}\sin^2\frac{\phi_k}{2}$$

VI. SUMMARY AND CONCLUSIONS

The results of sections I, II, and III, show that the heavy tool joints have negligible effect on the vibratory motion of the drill string, within the usual range of rotary drilling speeds. Therefore the drill pipe string may be considered a uniform bar with the same properties as the pipe itself. This allows a simple wave equation treatment of the drill string, as on page 17.

Based on the simple wave equation treatment above, an equivalent lumped parameter system may be constructed for various system boundary conditions. Placing the equations in matrix form allows the derivation of the response of any point of the drill string to a periodic disturbing force or periodic displacement applied to any or all points of the drill string. For small damping the approximate response can be found. The lack of experimental data prevents actual comparison of numerical results calculated for the lumped system. Also, actual system damping by the drilling fluid in and around the string may be proportional to a power of velocity and not viscous as was assumed here[14]. Consideration of velocity power terms makes the system equations of motion non-linear, and very difficult to treat. Approximate viscous coefficients can be calculated from experimental results, however, and the work of this paper in that direction must await these calculations for any proof of its usefulness. In any case, the undamped response of the lumped system may be easily calculated, with the minimum of computing power, from the expressions of pages 50 and 51. The minimum number of lumps to give responses of desired accuracy may be found from figures 16 and 17 for given boundary conditions.

VII. APPENDIX

I. The identify for the special determinant below, called a continuent, is based on the fact that its expansion follows the recurrence formula for the sine*.

$$\text{m rows and columns}\quad \begin{vmatrix} c & -1 & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -1 & c \end{vmatrix} = \frac{\sin(m+1)\phi}{\sin\phi} \qquad \text{for} \qquad c = 2\cos\phi$$

II. The identify below is based on identify I following expansion on the bottom row:

$$\text{m rows and columns}\quad \begin{vmatrix} c & -1 & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -1 & (c-1) \end{vmatrix} = (c-1) \underbrace{\begin{vmatrix} c & -1 & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -1 & c \end{vmatrix}}_{\text{m-1 rows and columns}} - \underbrace{\begin{vmatrix} c & -1 & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -1 & c \end{vmatrix}}_{\text{m-2 rows and columns}}$$

$$c = 2\cos\phi$$

$$= \frac{(2\cos\phi - 1)\sin m\phi - \sin(m-1)\phi}{\sin\phi}$$

$$= \frac{\sin(m+1)\phi - \sin m\phi}{\sin\phi} = \frac{\cos(m+\frac{1}{2})\phi}{\cos\frac{\phi}{2}}$$

III. The identify below is stated without proof[13]:

$$\frac{1}{2} + \cos\phi + \cos 2\phi + \cos 3\phi + \dots \quad \dots + \cos n\phi = \frac{\sin(n+\frac{1}{2})\phi}{2\sin\phi}$$

- -

* Reference 10, v 3, pg. 413

IV.
$$\sum_{k=1}^{n} \sin^2 k\phi = \frac{1}{2}\sum_{k=1}^{n}(1-\cos 2k\phi)$$
$$= \frac{1}{2}\left(n - \frac{\sin(n+\frac{1}{2})(2\phi)}{2\sin\phi} - \frac{1}{2}\right)$$
$$= \frac{2n-1}{4} - \frac{\sin(2n+1)\phi}{4\sin\phi}$$

V.
$$\sum_{k=1}^{n} \cos^2 k\phi = n - \sum_{k=1}^{n} \sin^2 k\phi =$$
$$= \frac{2n+1}{4} + \frac{\sin(2n+1)\phi}{4\sin\phi}$$

VI.
$$\sum_{k=1}^{m} \sin kA \sin kB = \frac{1}{2}\sum_{k=1}^{m} \cos k(A-B) - \frac{1}{2}\sum_{k=1}^{m} \cos k(A+B)$$
$$= \frac{1}{2}\left\{\frac{\sin(m+\frac{1}{2})(A-B)}{2\sin(\frac{A-B}{2})} - \frac{\sin(m+\frac{1}{2})(A+B)}{2\sin(\frac{A+B}{2})}\right\}$$
$$= \frac{1}{4}\left\{\frac{\sin(m+\frac{1}{2})A\cos(m+\frac{1}{2})B}{\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})}\left(\sin\left(\frac{A+B}{2}\right) - \sin\left(\frac{A-B}{2}\right)\right)\right.$$
$$\left. - \frac{\cos(m+\frac{1}{2})A\sin(m+\frac{1}{2})B}{\sin(\frac{A+B}{2})\sin(\frac{A+B}{2})}\left(\sin\left(\frac{A+B}{2}\right) - \sin\left(\frac{A-B}{2}\right)\right)\right\}$$
$$= \frac{1}{2}\left\{\frac{\sin(m+\frac{1}{2})A\cos(m+\frac{1}{2})B\cos\frac{A}{2}\sin\frac{B}{2}}{\sin(\frac{A-B}{2})\sin(\frac{A+B}{2})}\right.$$
$$\left. - \frac{\cos(m+\frac{1}{2})A\sin(m+\frac{1}{2})B\sin\frac{A}{2}\cos\frac{B}{2}}{\sin(\frac{A-B}{2})\sin(\frac{A+B}{2})}\right\}$$

VII.

$$\sum_{k=1}^{m} \cos kA \cos kB = \frac{1}{2} \sum_{k=1}^{m} \cos k(A-B) + \frac{1}{2} \sum_{k=1}^{m} \cos k(A+B)$$

$$= \frac{1}{4} \left\{ \frac{\sin(m+\frac{1}{2})(A-B)}{\sin(\frac{A-B}{2})} + \frac{\sin(m+\frac{1}{2})(A+B)}{\sin(\frac{A+B}{2})} - 2 \right\}$$

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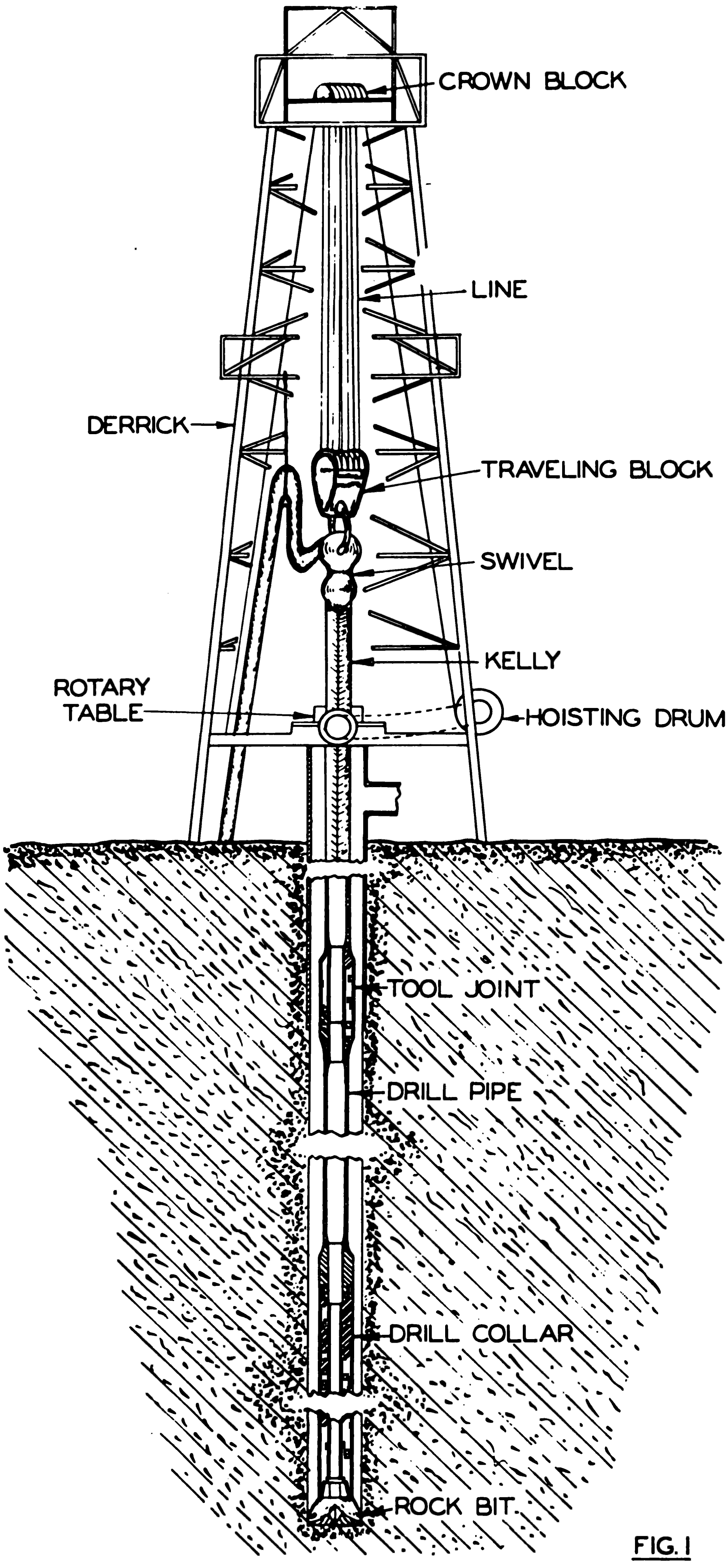
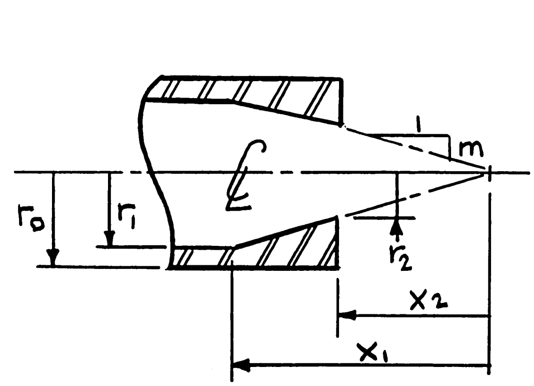
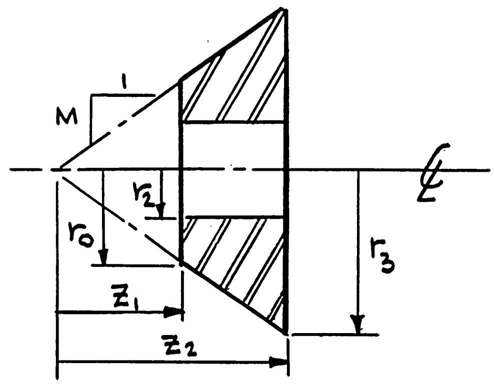

CROWN BLOCK
LINE
DERRICK
TRAVELING BLOCK
SWIVEL
KELLY
ROTARY
TABLE
HOISTING DRUM
TOOL JOINT
DRILL PIPE
DRILL COLLAR
ROCK BIT

FIG. 1

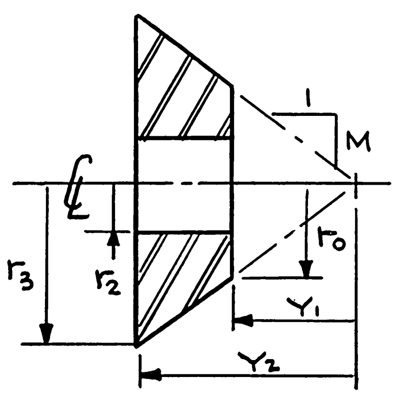


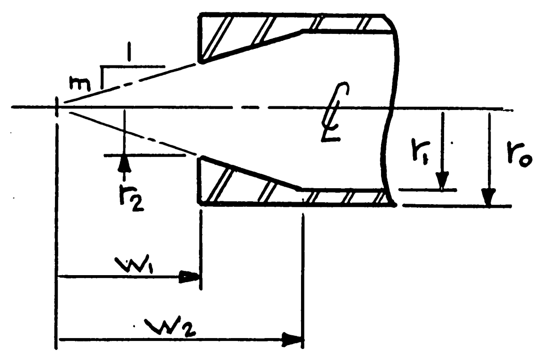


Longitudinal: $I(x) = \pi(r_0^2 - mx^2)\ \mathrm{IN}^2$ $\quad I(z) = \pi(M^2z^2 - r_2^2)\ \mathrm{IN}^2$

Torsional: $I(x) = \frac{\pi}{2}(r_0^4 - m^4x^4)\ \mathrm{IN}^4$ $\quad I(z) = \frac{\pi}{2}(M^4z^4 - r_2^4)\ \mathrm{IN}^4$

Increase in Section



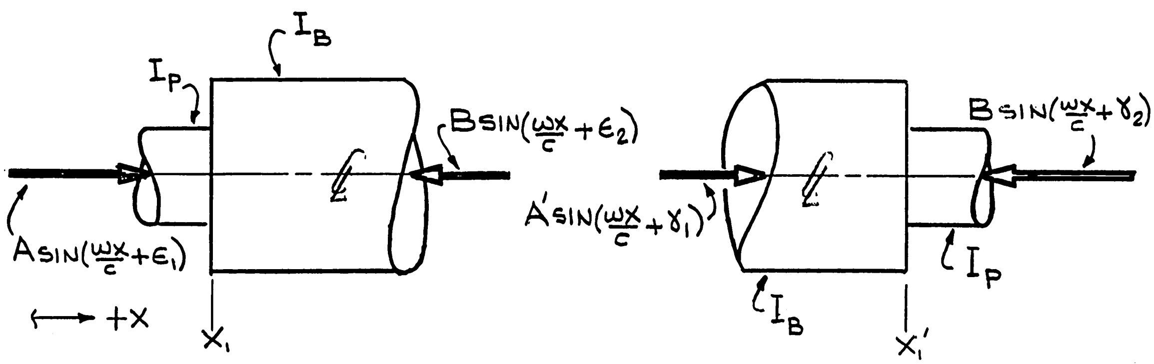


Longitudinal: $I(Y) = \pi(M^2Y^2 - r_2^2)\ \mathrm{IN}^2$ $\quad I(W) = \pi(r_0^2 - m^2W^2)\ \mathrm{IN}^2$

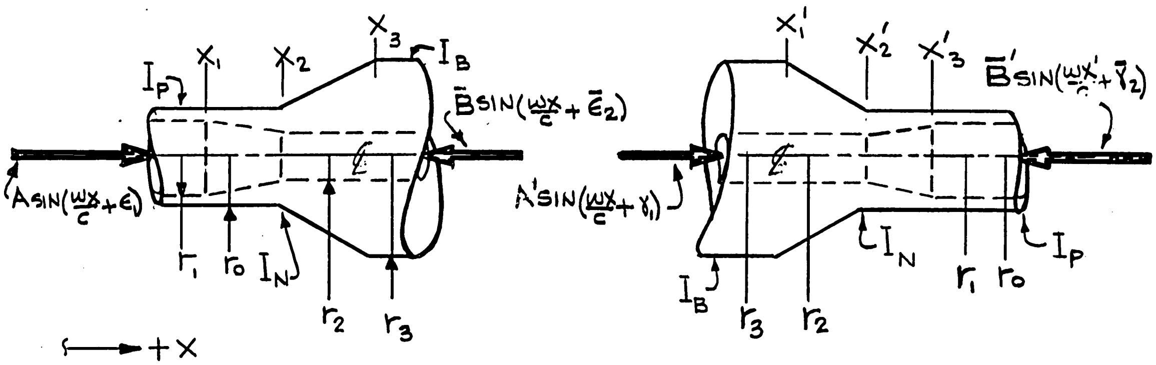
Torsional $I(Y) = \frac{\pi}{2}(M^4Y^4 - r_2^4)\ \mathrm{IN}^4$ $\quad I(W) = \frac{\pi}{2}(r_0^4 - m^4W^4)\ \mathrm{IN}^4$

Decrease in Section

Dual Taper Change
in Section
Fig. 2



a Abrupt Change



b Dual Taper Change

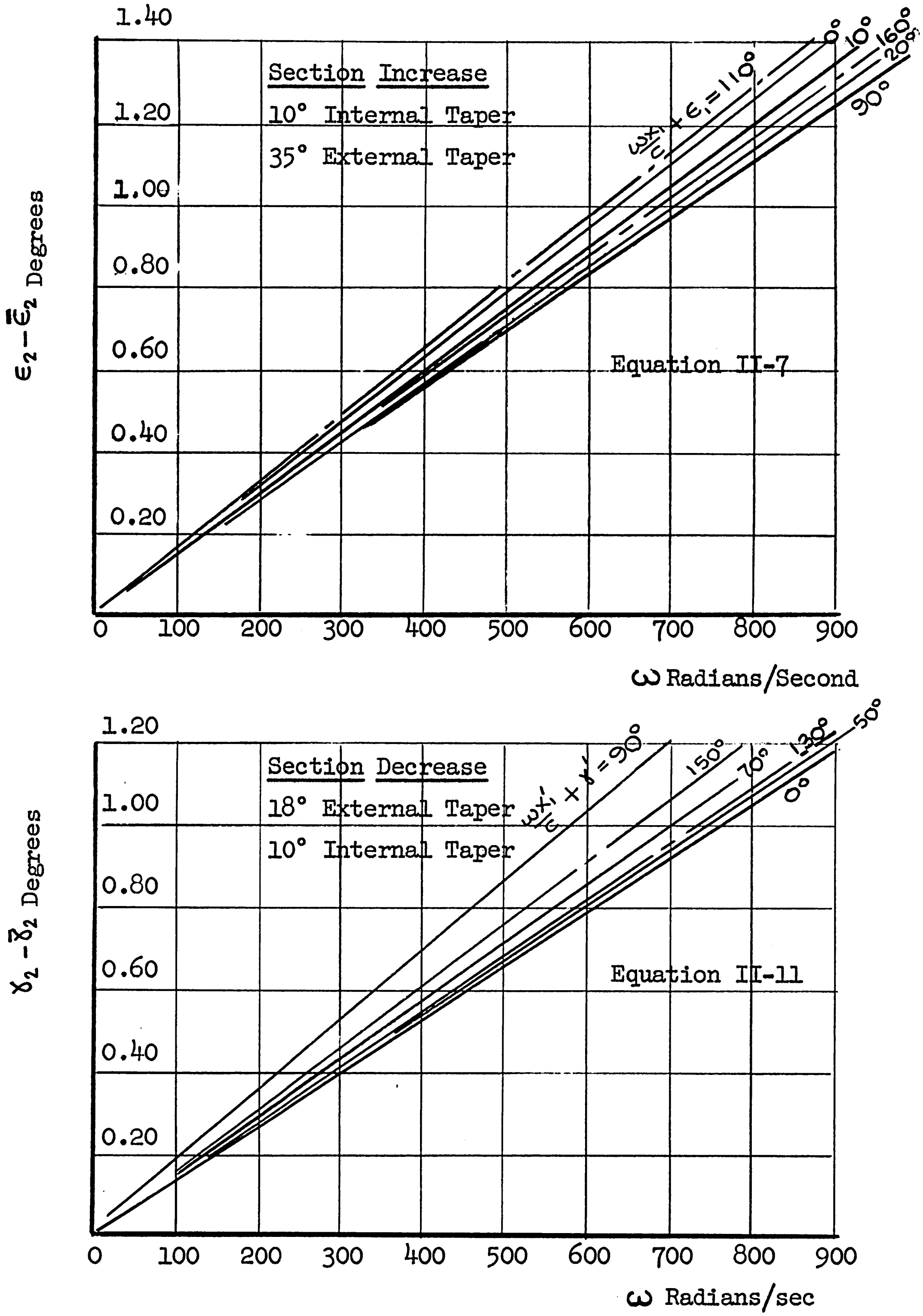
Subscripts: P - Pipe

N - Neck

B - Barrel

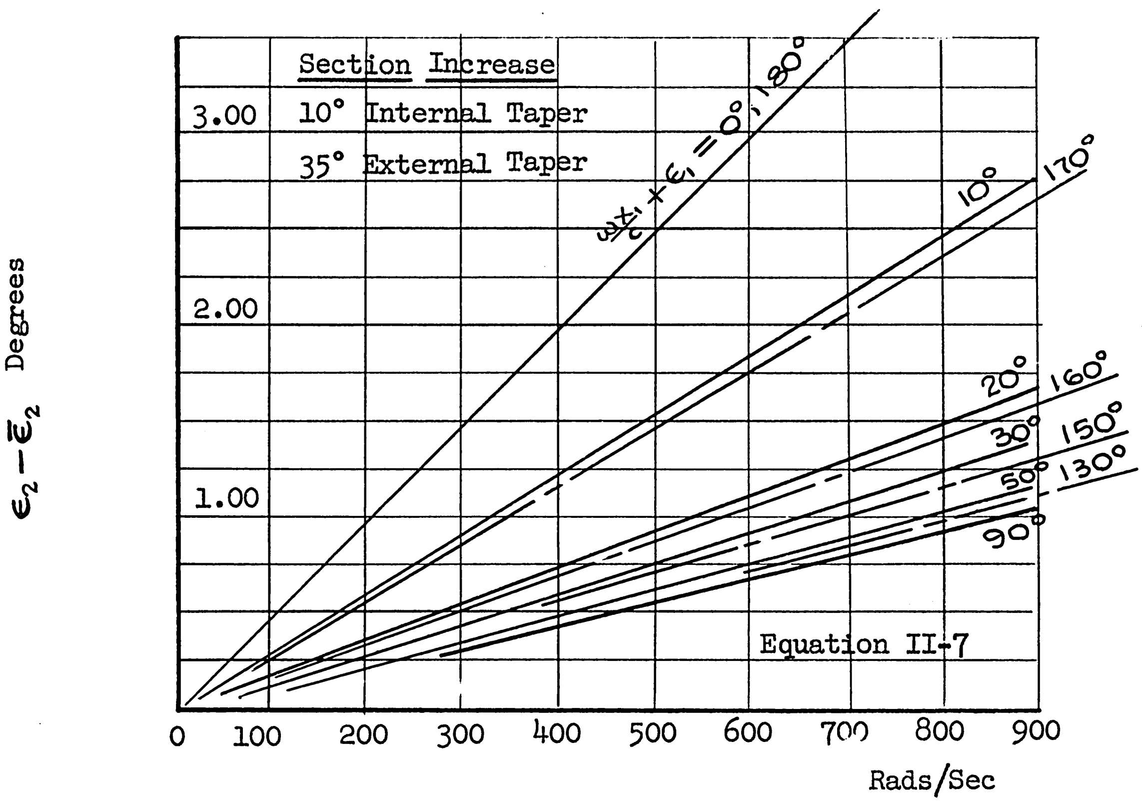
Changes in Drill-Pipe
Section at Tool Joint

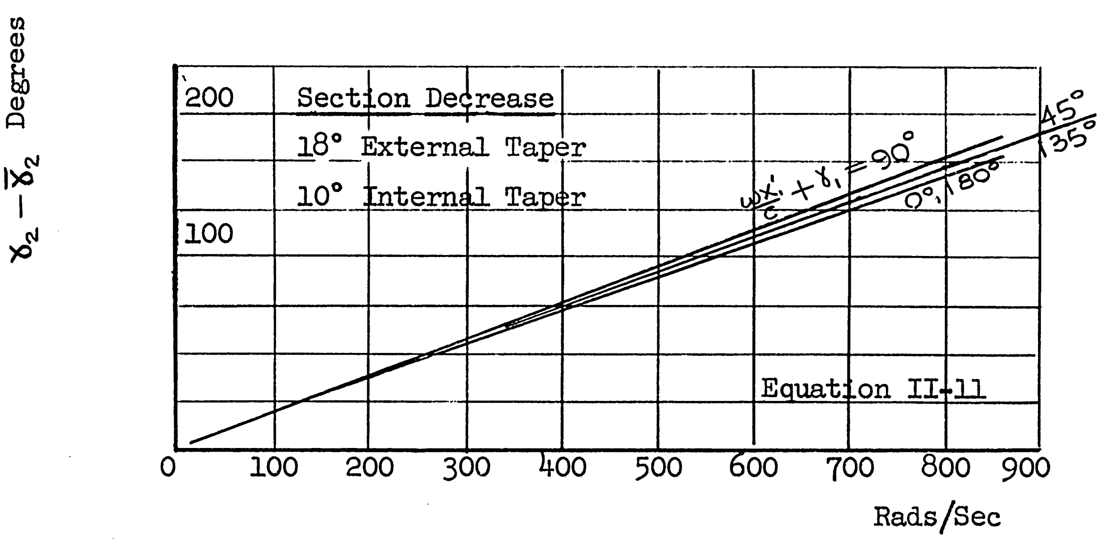
Figure 3



Phase Shift Error for Abrupt Change
Compared to Dual Taper Change
Longitudinal - Tool Joint of Figure 6

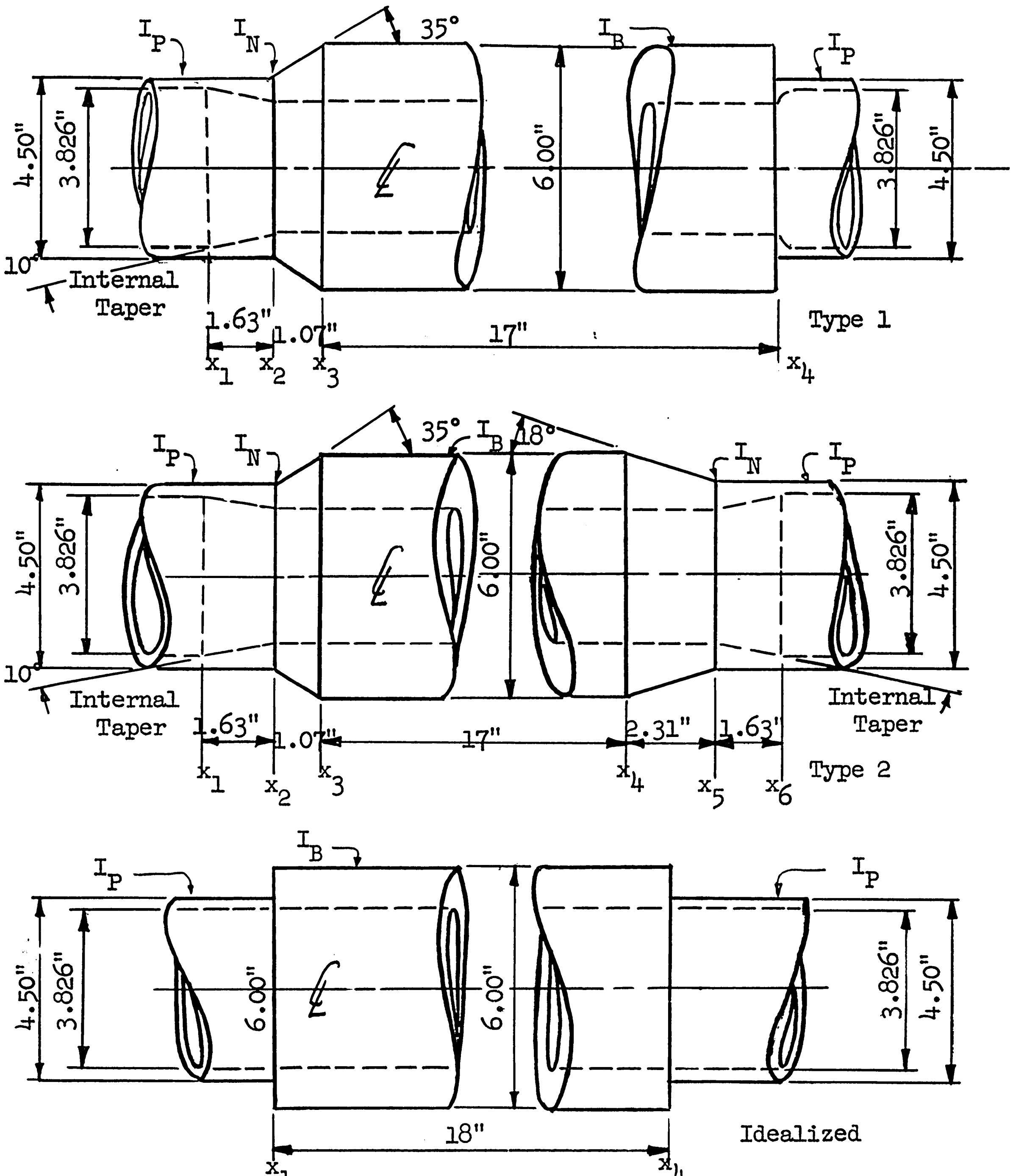
Figure 4





Phase Shift Error for Abrupt Change
Compared to Dual Taper Change
Torsion - Tool Joint of Figure 6

Figure 5

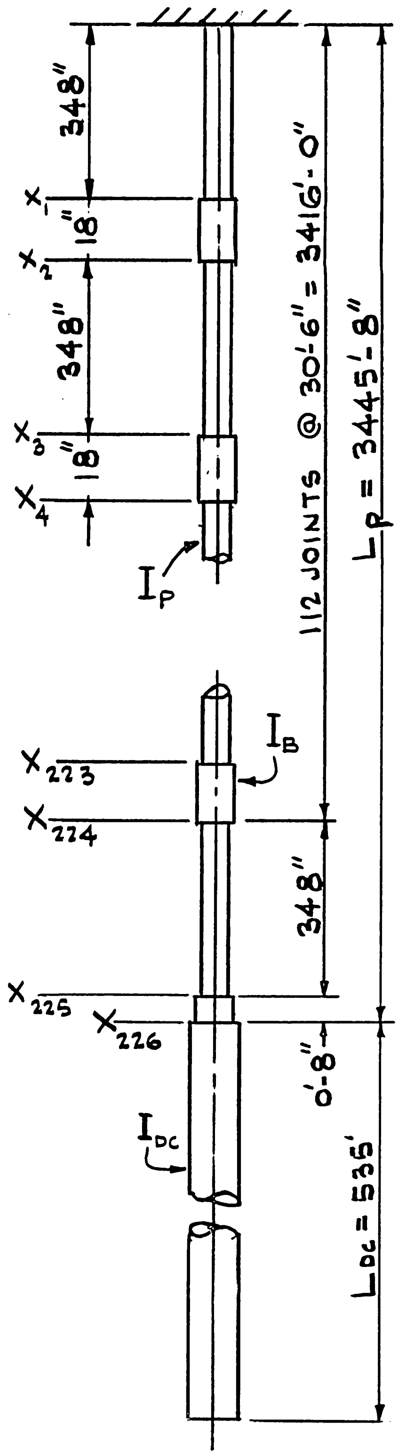


Tors: $I_P = 19.25 \text{ in}^4$ Long: $I_P = 4.42$ $r_o = 2.25"$ $r_2 = 1.625"$

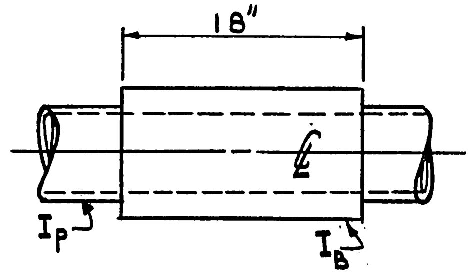
$I_N = 29.6 \text{ in}^4$ $I_N = 7.61$ $r_1 = 1.913"$ $r_3 = 3.00"$

$I_B = 116.9 \text{ in}^4$ $I_B = 21.3$

Figure 6



Fixed or Free



Tool Joint Used For
Computation Purposes

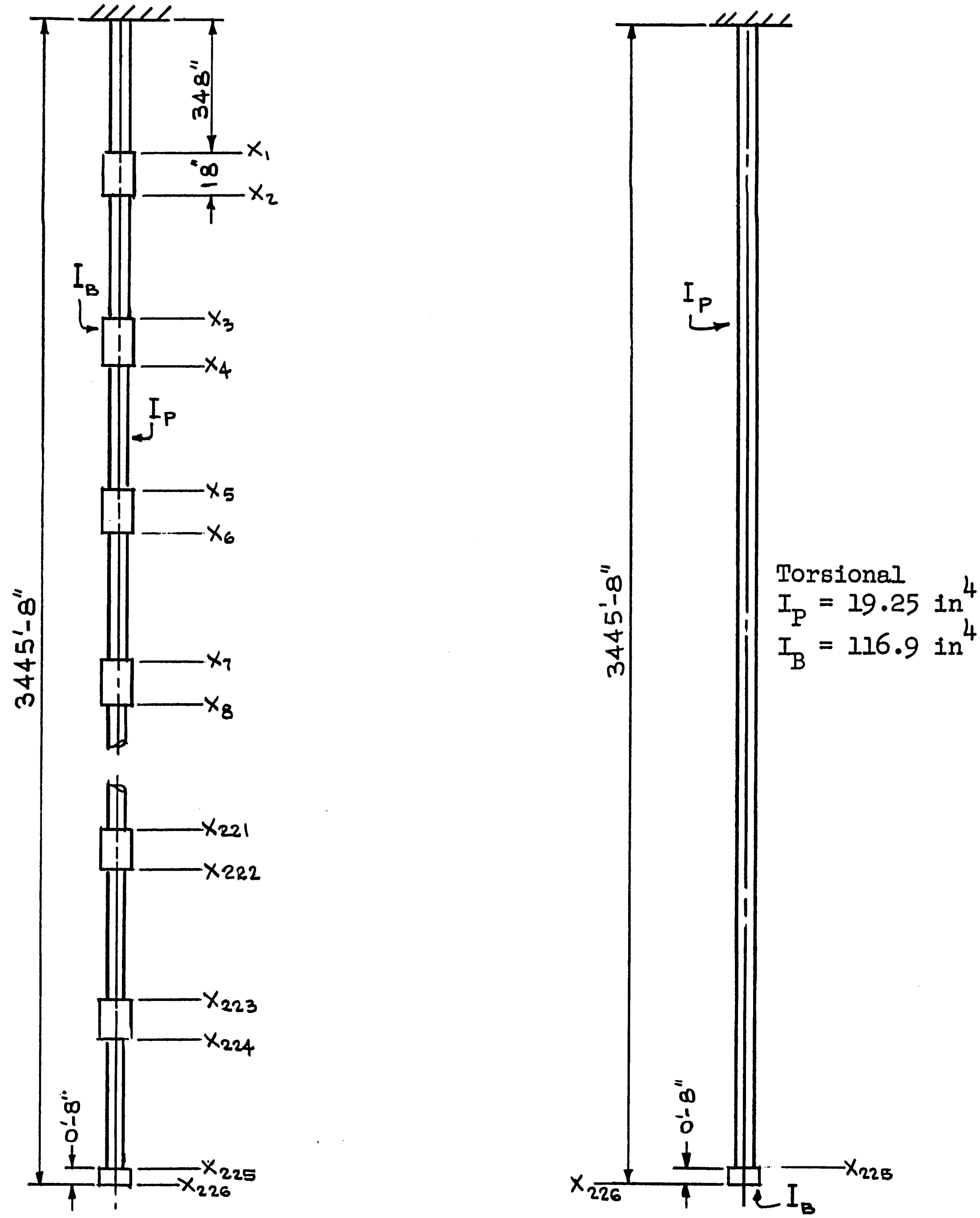
$I_P = 19.2 \text{ in}^4$

$I_B = 116.9 \text{ in}^4$

$I_{DC} = 198 \text{ in}^4$

Finnie and Bailey
Text Well "A"

Figure 7



$x_n \le x \le x_{n+1}$ $\quad \theta_n(x,t) = A_n \sin\left(\frac{\omega x}{c} + \epsilon_n\right) e^{i\omega t}$

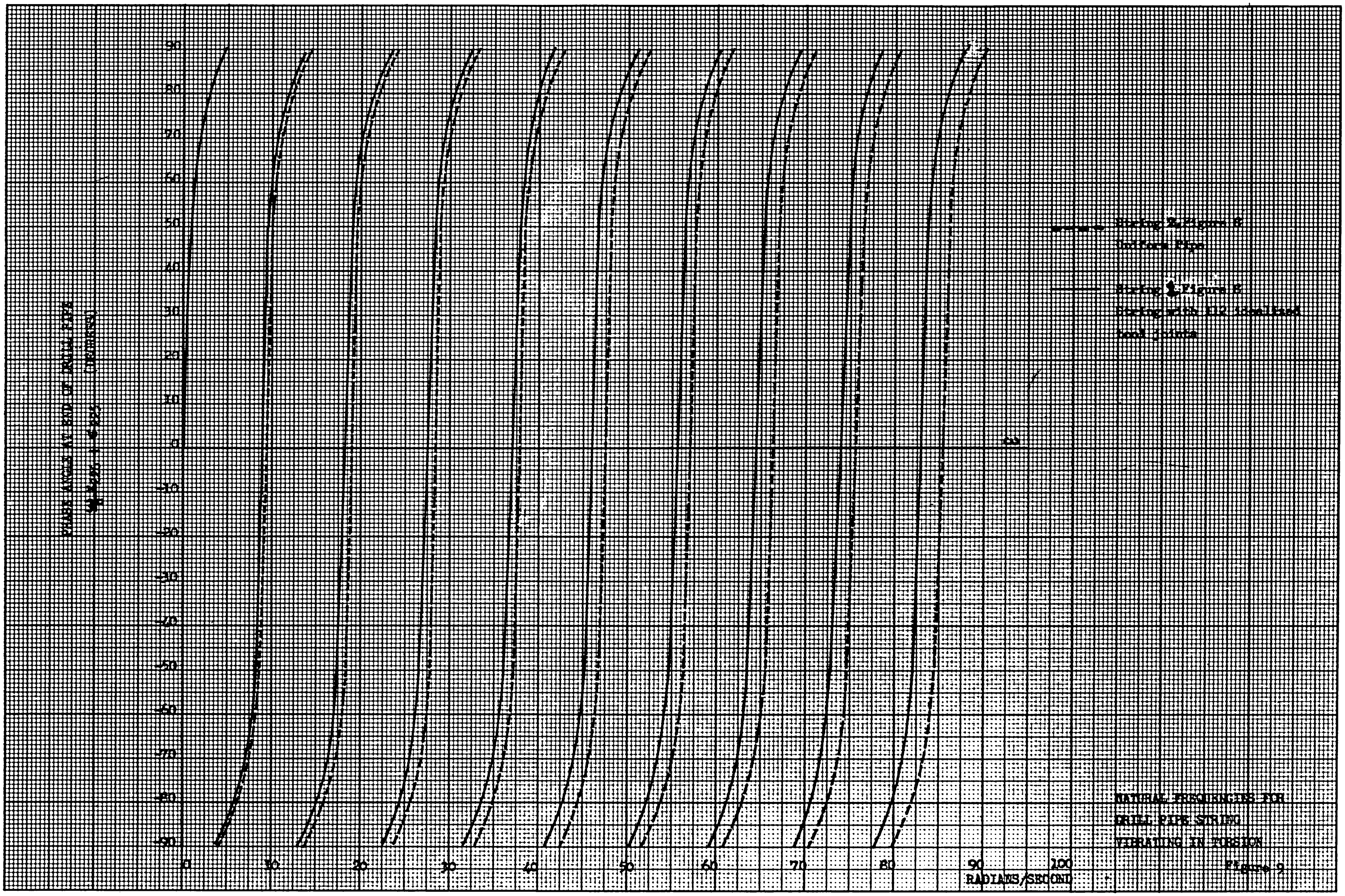
String 1

$x \le x_{225}$ $\quad \theta(x,t) = A \sin\left(\frac{\omega x}{c}\right) e^{i\omega t}$

$x \ge x_{225}$ $\quad \theta(x,t) = A \sin\left(\frac{\omega x}{c} + \epsilon\right) e^{i\omega t}$

String 2

Example Drill Pipe String
Figure 8



NATURAL FREQUENCIES FOR
DRILL PIPE STRING
VIBRATING IN TORSION

Figure 9

NATURAL FREQUENCIES OF DRILL PIPE

ONLY-- Finnie & Bailey Test Well "A"*

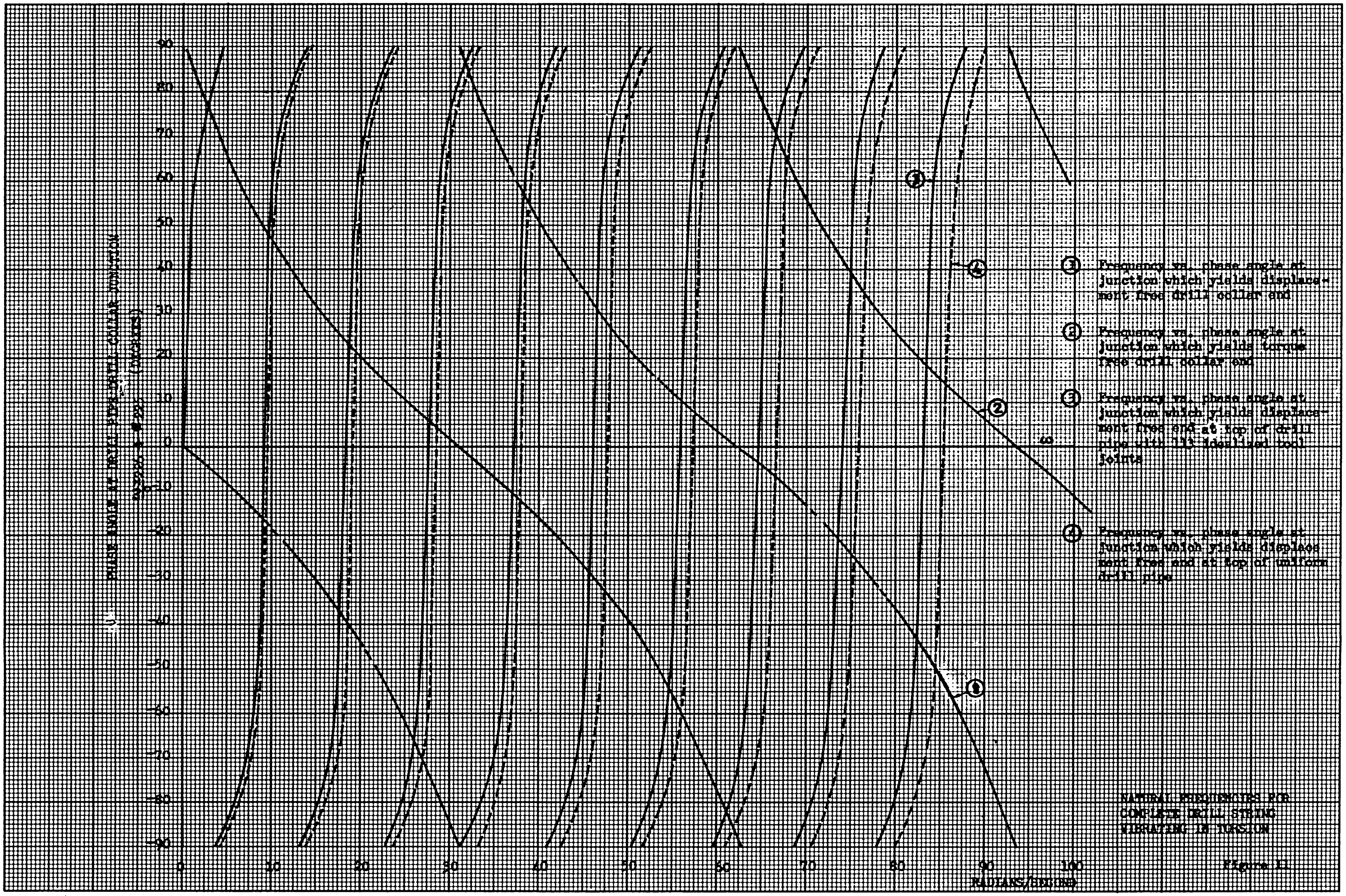
Mode Number	Top Fixed-Free Bottom ω_1	ω_2	Top Fixed-Fixed Bottom ω'_1	ω'_2	radians/sec
1	4.65	4.73	9.4	9.4	
2	14.25	14.2	18.5	18.9	
3	23.50	23.7	27.7	28.4	
4	32.25	33.1	37.5	37.9	
5	41.70	42.7	46.0	47.3	
6	51.00	52.1	55.7	56.8	
7	60.25	61.5	64.7	66.3	
8	69.50	71.0	74.0	75.8	
9	78.00	80.5	83.2	85.2	
10	88.00	89.8	92.4	94.7	
11	97.00	99.4			

ω_1, ω'_1 Results of Graphical Solution of Figure 9 for Drill Pipe String with 113 Tool Joints (Figure 8, String 1)

ω_2, ω'_2 Results of Graphical Solution of Figure 9 for Drill Pipe String with one Tool Joint (Figure 8, String 2)

* Figure 7,8

Figure 10



NATURAL FREQUENCIES FOR COMPLETE DRILL STRING VIBRATING IN TORSION

Figure 11

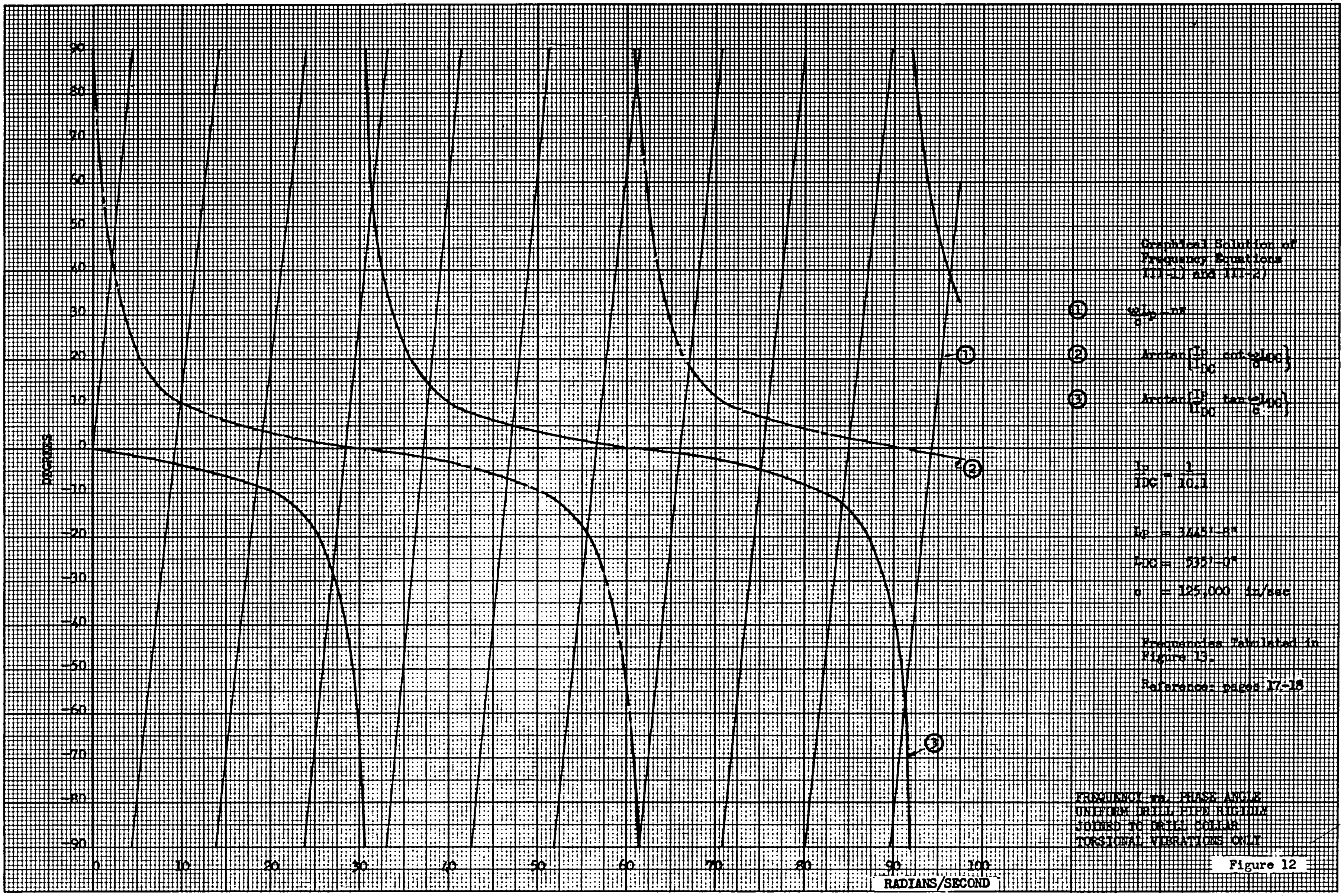

DEGREES
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
0
10
20
30
40
50
60
70
80
90
100
RADIANS/SECOND
1
2
3
Graphical Solution of Frequency Equations (11-1) and (11-2)
① $\frac{\omega L_P}{c} - n\pi$
② $\mathrm{Arctan}\left\{\frac{I_P}{I_{DC}} \cot \frac{\omega L_{DC}}{c}\right\}$
③ $\mathrm{Arctan}\left\{\frac{I_P}{I_{DC}} \tan \frac{\omega L_{DC}}{c}\right\}$
$\frac{I_P}{I_{DC}} = \frac{1}{10.1}$
$L_P = 3445'-8''$
$L_{DC} = 535'-0''$
$c = 125,000$ in/sec
Frequencies Tabulated in Figure 13.
Reference: pages 17-18
FREQUENCY vs. PHASE ANGLE
UNIFORM DRILL PIPE RIGIDLY
JOINED TO DRILL COLLAR
TORSIONAL VIBRATIONS ONLY

Figure 12

NATURAL FREQUENCIES OF COMPLETE DRILL

STRING--Finnie & Bailey Test Well "A"*

Mode Number	Top Fixed-Free Bottom ω_1	ω_2	ω_3	Top Fixed-Fixed Bottom ω'_1	ω'_2	ω'_3	radians/sec
1	2.6	2.6	2.2	9.00	9.2	9.0	
2	10.0	10.2	10.0	17.1	18.5	18.3	
3	18.9	19.2	19.0	26.1	26.7	27.0	
4	28.0	28.5	28.0	31.7	32.0	31.0	
5	37.3	37.3	37.0	38.7	37.8	38.0	
6	45.5	47.3	46.5	46.6	48.1	47.0	
7	54.0	55.8	55.5	56.0	57.3	56.5	
8	61.2	61.8	61.0	64.6	66.7	66.0	
9	66.6	68.0	67.0	73.4	75.5	75.0	
10	74.8	76.5	75.5	82.7	84.6	84.0	
11	83.3	85.8	85.0	90.3	91.3	91.5	
12	92.3	95.0	94.0	95.2	96.7	96.0	

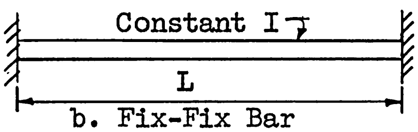
ω_1, ω'_2 Results of Graphical Solution Figure 11 for Drill Pipe String 1 of Figure 6 (With 113 Tool Joints)

ω_2, ω'_2 Results of Graphical Solution Figure 11 for Drill Pipe String 2, Figure 8 (Tool Joint at Drill Collar Junction only)

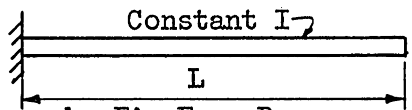
ω_3, ω'_2 Results of Graphical Solution, Figure 12 for Uniform Drill Pipe directly connected to Uniform Drill Dollar (no tool joints) Reference - pages 17-18

* Figure 7

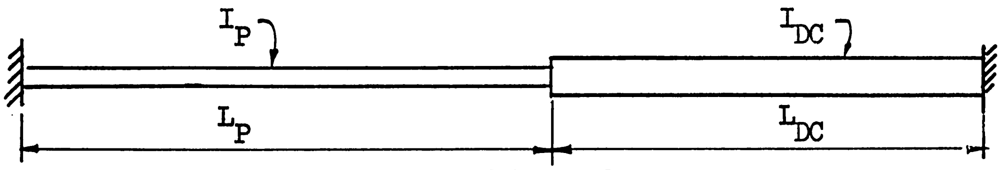
Figure 13



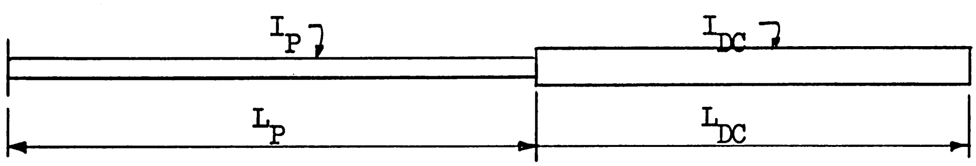
b. Fix-Fix Bar



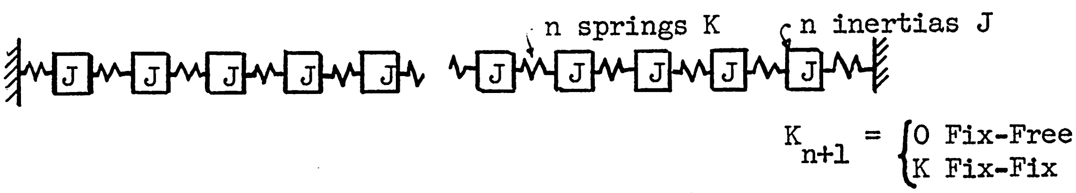
b. Fix-Free Bar



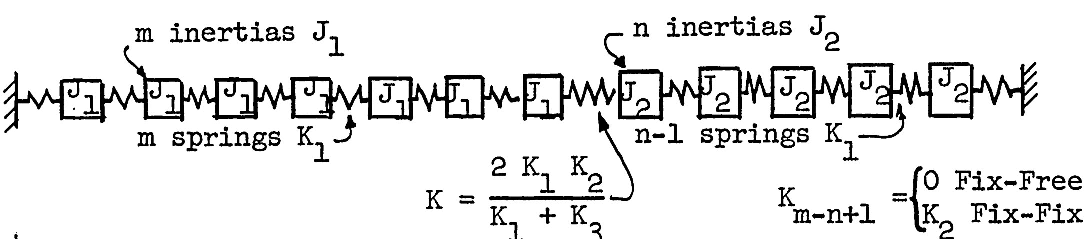
c. Fix-Fix Dual Bar



d. Fix-Free Dual Bar

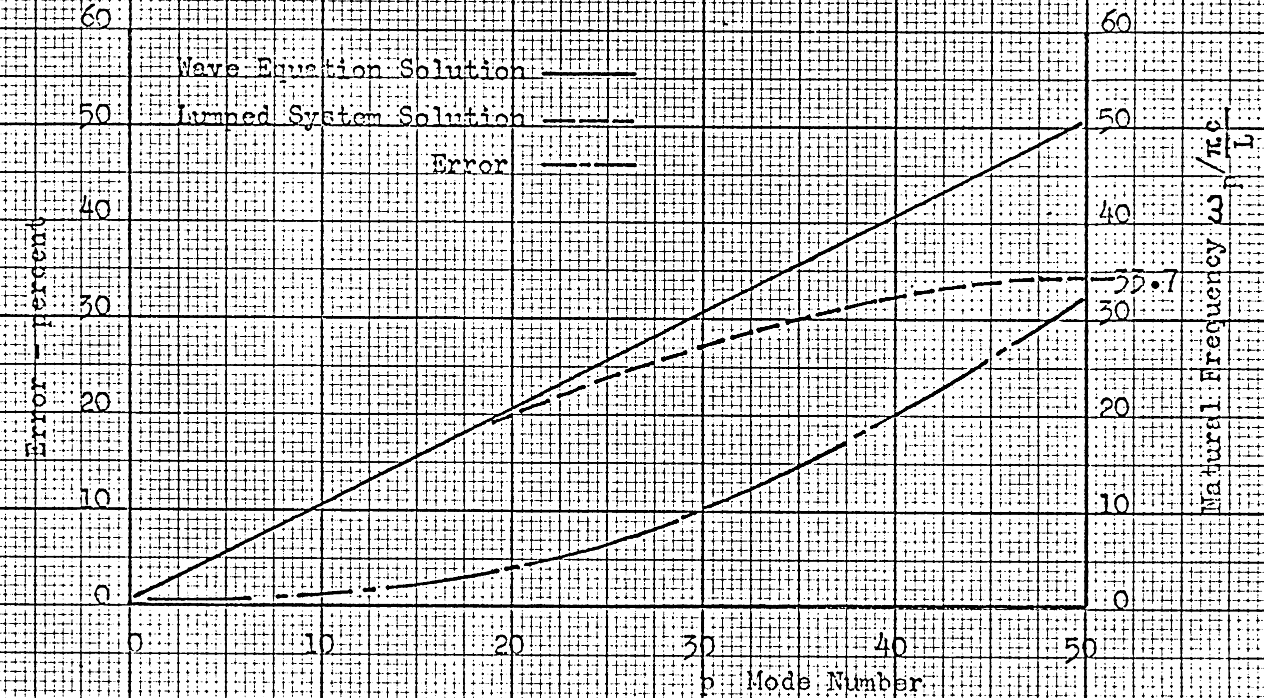


e. Equivalent Lumped System For
Uniform Bar



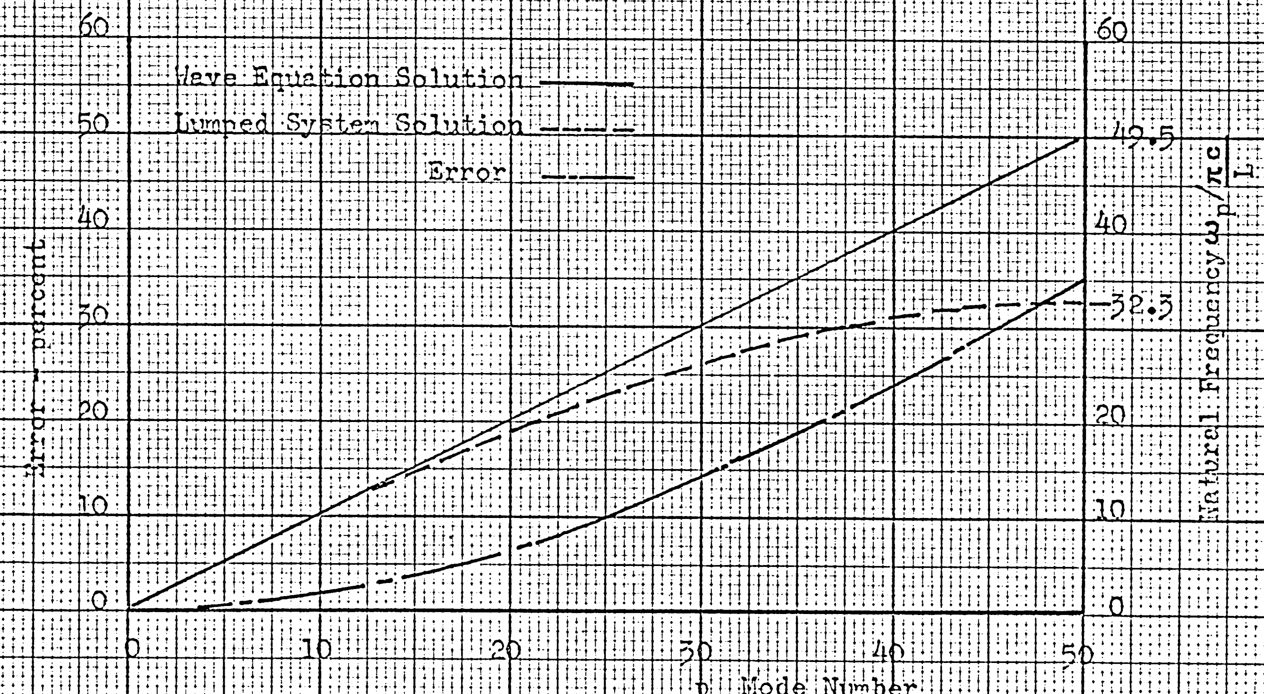
f. Equivalent Lumped System For
Dual Section Bar

Figure 14



Frequency Spectrum

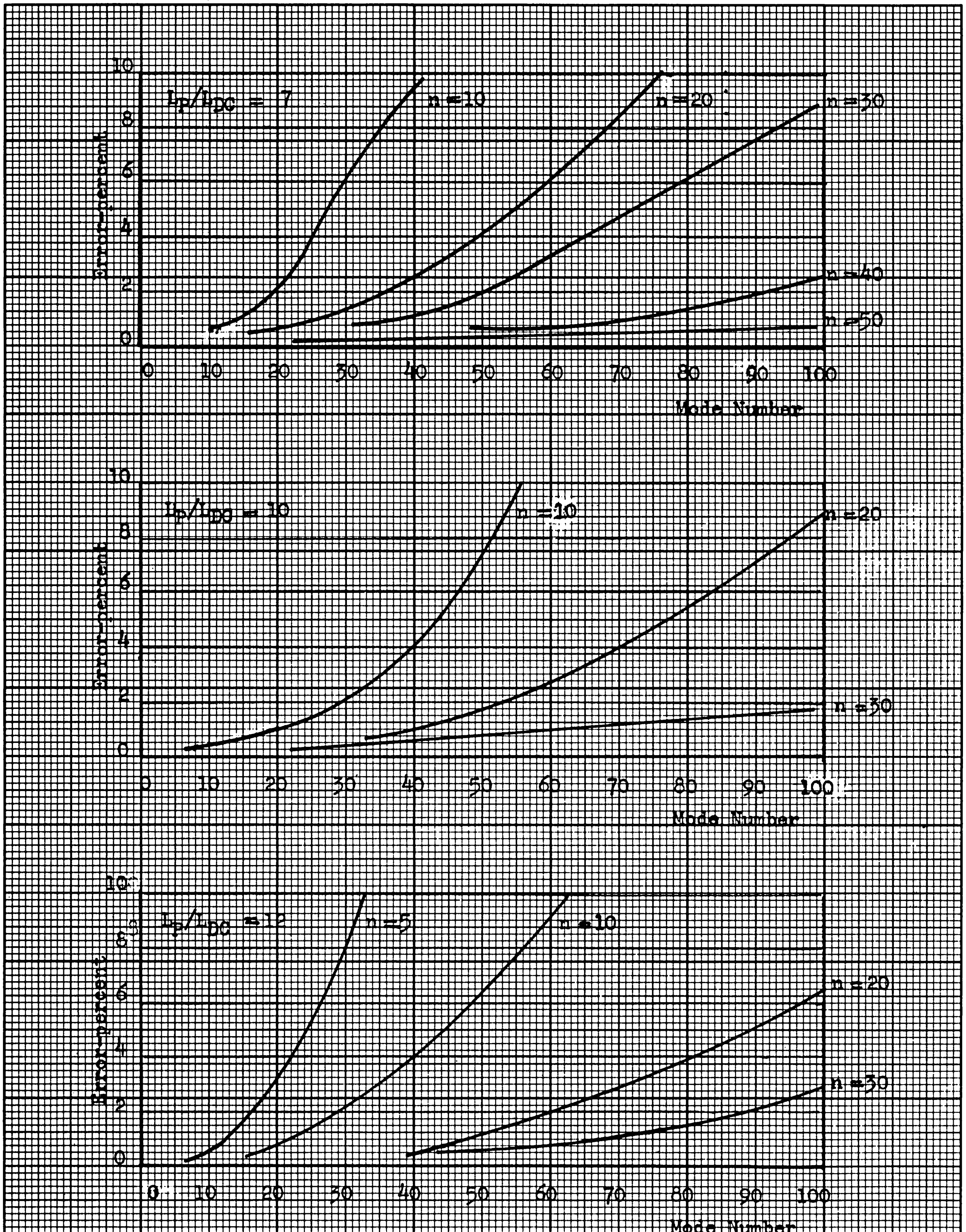
Uniform Fixed-Fixed Bar
Equivalent 50 inertia
Lumped Parameter System



Frequency Spectrum

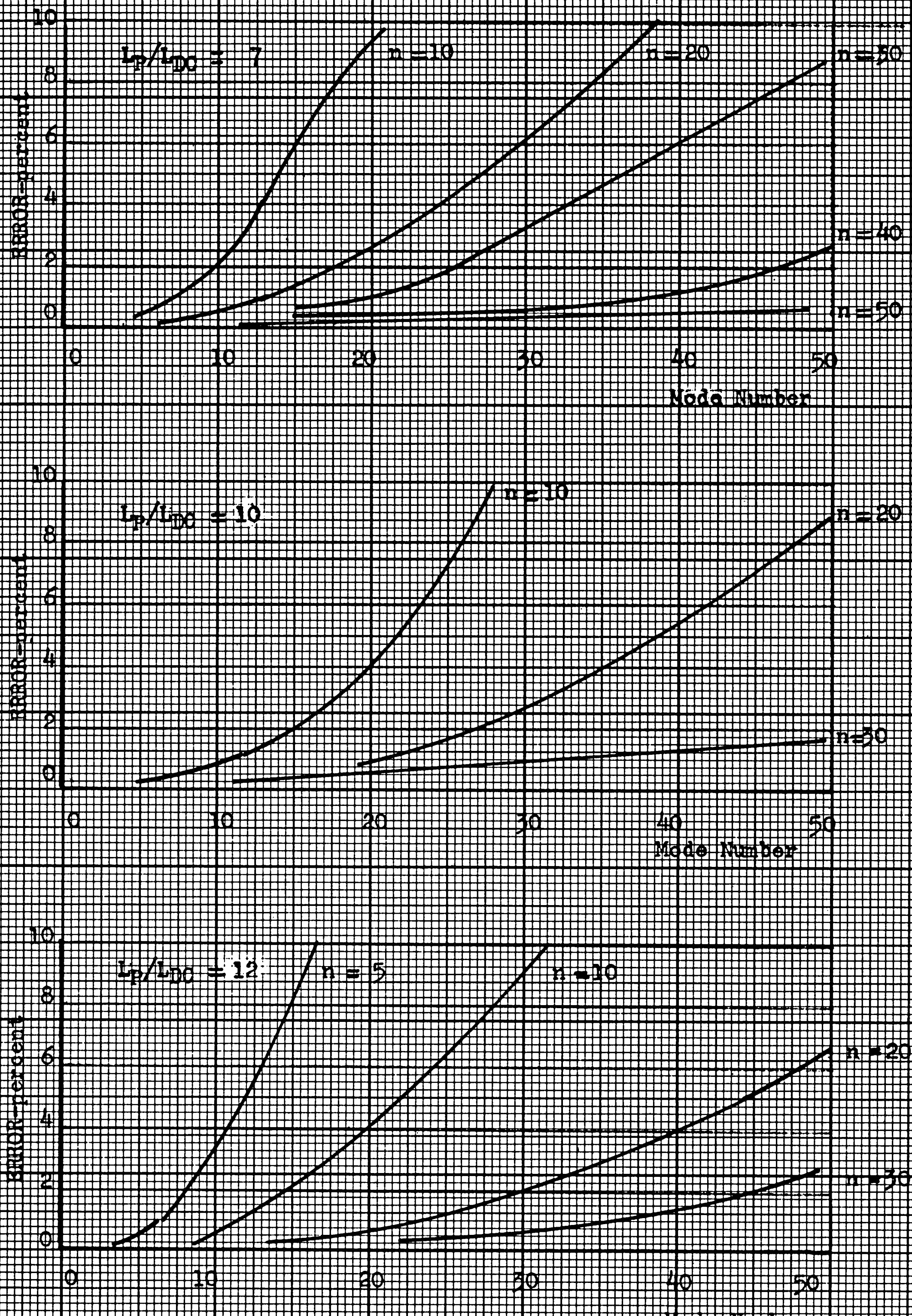
Uniform Fixed-Free Bar
Equivalent 50 inertia
Lumped Parameter System

Figure 15



MAXIMUM FREQUENCY ERROR vs MODE NUMBER
FOR FIXED-FIXED EQUIVALENT SYSTEM
(Re: Equation IV-13)

Figure 16



MAXIMUM FREQUENCY ERROR vs MODE NUMBER
FOR FIXED-FREE EQUIVALENT SYSTEMS
(Re: Equation IV-17)

Figure 17