A THEOREM ON PERIODS OF INTEGRALS OF ALGEBRAIC MANIFOLDS

by Phillip A. Griffiths

1. I want to state and prove a theorem about a certain transcendental invariant, the *period matrix*,[1] which is attached to every non singular algebraic manifold $V \subset \boldsymbol{P}_N$. In order to illustrate the basic ideas and avoid a lot of discussion about general algebraic manifolds, let me do things in case V is an *algebraic surface*, i.e., $\dim V = 2$.

What I am interested in is *algebraic families of algebraic surfaces.*

Example. Let $\xi^0, \xi^1, \xi^2, \xi^3$ be homogeneous coordinates in $\boldsymbol{P}_3$. Then an algebraic surface $V \subset \boldsymbol{P}_3$ of degree n is given by

$$\sum_{i_0+i_1+i_2+i_3=n} \lambda_{i_0 i_1 i_2 i_3}(\xi^0)^{i_0}(\xi^1)^{i_1}(\xi^2)^{i_2}(\xi^3)^{i_3} = 0.$$

The $\lambda_{i_0 i_1 i_2 i_3}$ are determined up to a non zero constant, and so correspond uniquely to a point $\lambda = [\cdots, \lambda_{i_0 \cdots i_3}, \cdots]$ in some big $\boldsymbol{P}_M$. We write V_λ for the surface with the above equation and may think of $\{V_\lambda\}_{\lambda \in \boldsymbol{P}_M}$ as an *algebraic family of algebraic surfaces.*

Now, still looking at this example, a "general" $V_\lambda \subset \boldsymbol{P}_3$ will be non singular. More precisely, there exists a hypersurface $H \subset \boldsymbol{P}_M$ such that V_λ is non singular for $\lambda \in \boldsymbol{P}_M - H$.

In general, an algebraic family of algebraic surfaces is given by polynomial equations

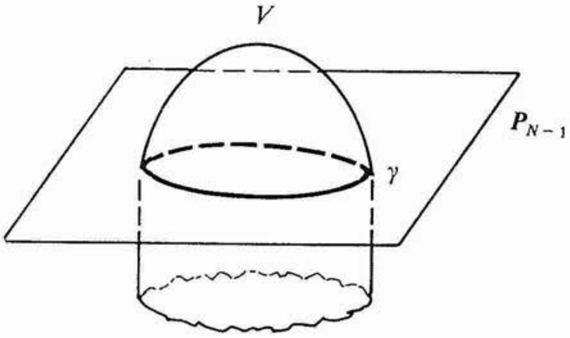
$$f_\alpha(\xi^0, \cdots, \xi^N;\ \lambda^0, \cdots, \lambda^M) = 0$$

$$g_\rho(\lambda^0, \cdots, \lambda^M) = 0$$

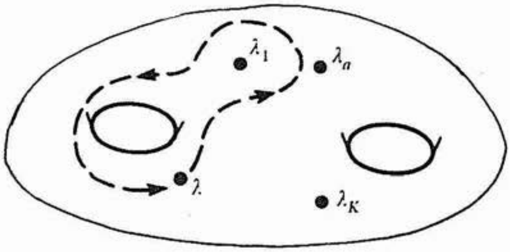
with parameter space a *non singular* projective algebraic manifold B, and then our family is $\{V_\lambda\}_{\lambda \in B}$ with V_λ non singular for $\lambda \in B - H = B^*$.

2. In our family $\{V_\lambda\}_{\lambda \in B}$ we let V be a typical non singular surface. We want to define the *period matrix* $\Omega(V)$ of V. For this we first choose a basis $\gamma_1, \cdots, \gamma_b$ for $H_2(V, \boldsymbol{Z})$ (mod torsion). There is a distinguished class

γ given by the homology class of $\boldsymbol{P}_{N-1} \cdot V$ where $\boldsymbol{P}_{N-1} \subset \boldsymbol{P}_N$ is a generic hyperplane:



To see how unique this choice of basis is, suppose for example that B is a curve:



Here H is the finite set $\lambda_1, \cdots, \lambda_K$ of *critical points* where V_{λ_α} is singular. Then the basis of $H_2(V, \boldsymbol{Z})$ is determined up to the action of the fundamental group $\pi_1(B^*)$ on $H_2(V, \boldsymbol{Z})$ where $B^* = B - \{\lambda_1, \cdots, \lambda_K\}$ and $\pi_1(B^*)$ acts by displacing cycles around a closed path on B^*. Such an action is given by an integral matrix $T\gamma_\rho = \sum_{\sigma=1}^{b} T_\rho^\sigma \gamma_\sigma$ where i) $T\gamma = \gamma$ and ii) $TQ^tT = Q$ where $Q = (\gamma_\rho \cdot \gamma_\sigma)$ is the *intersection matrix* on $H_2(V, \boldsymbol{Z})$. We let Γ be the group of all such matrices.

Now choose a basis $\omega^1, \cdots, \omega^m$ for the holomorphic 2-forms on V. This basis is determined up to $\tilde{\omega}^\alpha = \sum_{\beta=1}^{m} A_\beta^\alpha \omega^\beta$, det $A \neq 0$. We then form the *period matrix*

$$\Omega = \begin{pmatrix} \int_{\gamma_1} \omega^1 & \cdots & \int_{\gamma_b} \omega^1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega^m & \cdots & \int_{\gamma_b} \omega^m \end{pmatrix}.$$

This matrix satisfies[2]

$$\left.\begin{aligned} &\Omega Q^t \Omega = 0 \qquad &(1)\\ &\Omega Q^t \overline{\Omega} > 0 \qquad &(2) \end{aligned}\right\} \quad \textit{Riemann bilinear relations},$$

$$(3) \qquad \begin{cases} \Omega(\lambda) \text{ depends holomorphically on } \lambda \text{ and} \\ d\Omega Q^t \Omega = 0 \qquad (\textit{infinitesimal period relation}). \end{cases}$$

It is determined up to the equivalences[3]

$$(4) \qquad \Omega \sim A\Omega, \qquad \det A \neq 0;$$

$$(5) \qquad \Omega \sim \Omega T, \qquad T \in \Gamma.$$

3. With these preliminaries we can state our theorem. Let D be the complex manifold of all matrices Ω which satisfy (1), (2), and with the equivalence relation (4). We call D the *period matrix domain*. It can be proved that:[4]

> The group Γ acts *properly discontinuously* on D so that D/Γ is an analytic space (in fact, D/Γ is locally the quotient of a polycylinder factored by a finite group).

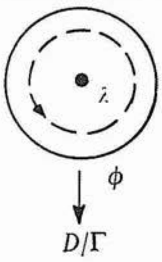
We should think of D/Γ as the totality of all possible (inequivalent) period matrices which might turn up as an $\Omega(V_\lambda)$ for $\lambda \in B^*$.

Main Theorem. *Let $\phi: B^* \to D/\Gamma$ be the period mapping, and let $\phi(B^*) \subset D/\Gamma$ be the image of B^* ($=$ all period matrices $\Omega(V_\lambda)$ for $\lambda \in B^*$). Then the closure $\overline{\phi(B^*)} \subset D/\Gamma$ is an analytic set in which $\phi(B^*)$ is a Zariski open set.*

In more concrete terms, given $\Omega_0 \in D$ there exists a neighborhood U of Ω_0 in D and local analytic functions $F_\alpha(\Omega)$, $G_\rho(\Omega)$ defined for $\Omega \in U$ such that a given potential period matrix Ω is an $\Omega(V_\lambda)$ if, and only if, $F_\alpha(\Omega) = 0,\ G_\rho(\Omega) \neq 0$.

4. The proof of the main theorem is based on *hyperbolic complex analysis* (to be explained) with the essential ingredient being the *infinitesimal period relation* (3).

Let us suppose first that B is a curve and let us take a look at the period mapping around a critical point:



$\Delta^* =$ punctured disc $0 < |\lambda| < \varepsilon$.

Displacing cycles around the origin gives the *Picard-Lefschetz transformation* $T: H_2(V,\mathbf{Z}) \to H_2(V,\mathbf{Z})$. The first step in the proof is to show that:

(6) $\phi: \Delta^* \to D/\Gamma$ extends across the origin if T is of finite order.

This says intuitively that, even though V_0 may have nasty singularities, the homological assumption $T^N = I$ implies that we can still define its period matrix as $\lim_{\lambda\to 0} \Omega(\lambda) = \Omega(0) \in D/\Gamma$.

We will not prove (6) now but will come back to it later. Using (6) we may then assume that our period mapping $\phi: B^* \to D/\Gamma$ has the property that *all Picard-Lefschetz transformations around critical points are of infinite order*. With this assumption we will prove

(7) $\phi: B^* \to D/\Gamma$ is a *proper* holomorphic mapping.

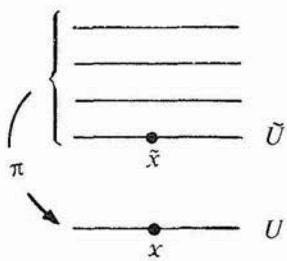
If this is done, a standard result (*proper mapping theorem*) in several complex variables says that $\phi(B^*)$ is an analytic set, and then the Main Theorem is clear.

We shall prove (7) by contradiction. If it is false, then there is a compact set $K \subset D/\Gamma$ such that $\phi^{-1}(K)$ is *non compact* in B^*. From this we find:

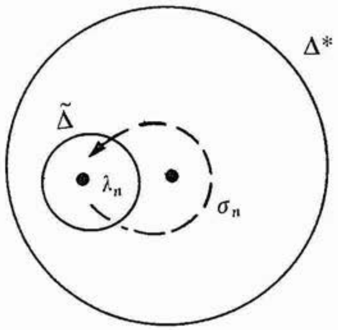
(8) There exists a punctured disc Δ^* around a critical point and a sequence $\{\lambda_n\} \in \Delta^*$ with $\lambda_n \to 0$ such that $\phi(\lambda_n) = x_n$ tends to a point $x \in D/\Gamma$.

We must show that (8) leads to trouble.

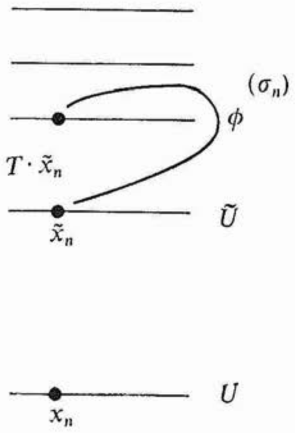
Let $\pi: D \to D/\Gamma$ be the projection and let $\tilde{x} \in D$ lie over x; i.e., $\pi(\tilde{x}) = x$. For simplicity suppose also that $\tilde{x}$ is *not* a fixed point of any $S \in \Gamma$—at worst the stabilizer $\Gamma_{\tilde{x}}$ of $\tilde{x}$ is a *finite* subgroup of Γ. Then $\pi^{-1}(x) = \bigcup_{S\in\Gamma} S\tilde{x}$ and we can choose a small neighborhood U of x such that $\pi^{-1}(U)$ is the *disjoint* union $\bigcup_{B\in\Gamma} S\cdot\tilde{U}$ where $\tilde{U}$ is a neighborhood of $\tilde{x}$ lying over U. The picture is



Around λ_n we choose a small disc $\tilde{\Delta}$ and lift ϕ to a holomorphic mapping $\tilde{\phi}: \tilde{\Delta} \to \tilde{U}$



In fact we may think of $\tilde{\phi}(\lambda)$ as the period matrix $\Omega(\lambda) = (\int_{\gamma_\rho(\gamma)} \omega^\alpha(\lambda))$ of the surface V_λ for λ in $\tilde{\Delta}$. Now analytic continuation of the local mapping $\tilde{\phi}: \tilde{\Delta} \to D$ around the circle σ_n passing through λ_n brings $\tilde{\phi}(\lambda)$ back to $T \cdot \tilde{\phi}(\lambda)$ where $T: H_2(V_{\lambda_n}, \mathbf{Z}) \to H_2(V_{\lambda_n}, \mathbf{Z})$ is the Picard-Lefschetz transformation around the origin. A priori we might have a picture



However, suppose we can prove:

(9) There exists a Γ-invariant hermitian metric ds_D^2 on D such that $\phi: \Delta^* \to D/\Gamma$ satisfies $\phi^* ds_D^2 \leqq ds_{\Delta^*}^2$ where $ds_{\Delta^*}^2$ is the complete metric of constant negative curvature on Δ^*.

Then we will be done, because i) the circle σ_n passing through λ_n has (non euclidean) length

$$l_{\Delta^*}(\sigma_n) = \frac{1}{\log \frac{1}{|\lambda_n|}}$$

and so $l_{\Delta^*}(\sigma_n) \to 0$ as $n \to \infty$; ii) using this together with (9), for the length $l_D(\tilde{\phi}(\sigma_n))$ of the image curve of σ_n under $\tilde{\phi}$ we have $l_D(\tilde{\phi}(\sigma_n)) \to 0$ as $n \to \infty$; iii) it follows that $\tilde{\phi}(\sigma_n)$ lies entirely in $\tilde{U}$ for large enough n since the distance $d_D(\tilde{U}, S\tilde{U}) \geqq \delta > 0$ is bounded below for all $S \in \Gamma$, $S \neq I$; and iv) if $\tilde{\phi}(\sigma_n)$ lies entirely in $\tilde{U}$, then $\tilde{x}_n$ is a fixed point of T, which is impossible since T has infinite order.

Thus the whole point really is to prove (9), which we may designate as the *distance decreasing property* of ϕ. The idea is to use a suitable version of the *Schwarz-Ahlfors-Pick Theorem*, which says that if we have a holomorphic mapping $\phi: \Delta^* \to X$ where Δ^* has the canonical metric $ds^2_{\Delta^*}$ of constant negative curvature -1 and X is a complex hermitian manifold whose ds^2_X has holomorphic sectional curvatures all $\leqq -1$, then $\phi^* ds^2_X \leqq ds^2_D$.

In the problem at hand it can be seen that there is no such Γ-invariant, negatively curved metric on D. However, the Γ-invariant metric $\mathrm{Trace}(\bar{\partial}(H^{-1}\partial H))$, where $H(\Omega) = \Omega Q^t \bar{\Omega}$, turns out to have all holomorphic sectional curvatures $\leqq -1$ *in the subspace* $d\Omega Q^t \Omega = 0$ of the *tangent space* T_Ω to D at the point Ω. Using the infinitesimal bilinear relation (3) we see then that ϕ is in fact *negatively curved* in the sense that $K_D(\phi_* \partial/\partial\lambda) \leqq -1$ where K_D is the sectional curvature in the 2-plane $\phi_* \partial/\partial\lambda \wedge \phi_* \overline{\partial/\partial\lambda}$ in D. From this we can mildly generalize the proof of the Schwarz-Ahlfors-Pick Theorem to our situation so that we can prove (9) and are done.

5. The points left open in the above argument are: i) the possibility that $\tilde{x}$ might be a fixed point of some $S \in \Gamma$, $S \neq I$ (in that case, the stability group $\Gamma_{\tilde{x}}$ is a finite group, $\pi^{-1}(x) = \bigcup_{S \in \Gamma/\Gamma_{\tilde{x}}} S \cdot \tilde{X}$, $\pi^{-1}(U) = \bigcup_{S \in \Gamma/\Gamma_{\tilde{x}}} S \cdot \tilde{U}$ and the argument proceeds essentially as before); ii) the assumption that the parameter space B is a curve; and iii) the assertion (6) that the period map extends across critical points where the local Picard-Lefschetz transformation is of finite order.

To prove (6) we may go to a finite covering of Δ^*, ramified at the origin, and assume that we have then a single-valued holomorphic mapping $\phi: \Delta^* \to D$ which satisfies the infinitesimal bilinear relation (3) and is consequently *negatively curved* in the sense described above. We want to extend ϕ across the origin. This is more or less the problem of trying to prove the usual *Riemann extension theorem* (where $D = \Delta$ is a disc) *without* using Laurent series, but instead utilizing only the generalized Schwarz lemma. The only proof I know is based on the following two facts:

(10) There exists a properly discontinuous group $\hat{\Gamma}$ of automorphisms of D such that $\hat{\Gamma}$ acts without fixed points and the quotient space $D/\hat{\Gamma}$ is *compact*.

In fact, for the period matrix domain D we can write down $\hat{\Gamma}$ explicitly. Or we could quote a general theorem of Borel and Harish-Chandra on the existence of such $\hat{\Gamma}$.

(11) The composed mapping $\phi: \Delta^* \to D/\hat{\Gamma}$ is still negatively curved and maps into a *compact* manifold. Under these conditions, Mrs. M. Kwack (Berkeley thesis) has proved that ϕ extends to the whole disc.

Once we have an extension of $\phi: \Delta^* \to D/\hat{\Gamma}$ we can lift this around the origin to get an extension of $\phi: \Delta^* \to D$.

There remains finally the question of what to do when the parameter space B has arbitrary dimension. If H is the subvariety of points $\lambda \in B$ for which V_λ is singular, it is clear that we can blow up B along H without changing the problem. Thus, applying Hironaka we may assume that $H = H_1 + \cdots + H_l$ where the H_j are non singular divisors on B which cross transversely. If T_j is the Picard-Lefschetz transformation on $H_2(V, \boldsymbol{Z})$ obtained by displacing cycles around a loop surrounding H_j, then as before we can locally extend the period mapping across H_j whenever T_j is of finite order. Thus we may assume that we have

$$\phi: B - (H_1 + \cdots + H_l) \to D/\Gamma$$

where each Picard-Lefschetz transformation T_j is of infinite order. As before, we want to prove now that ϕ is proper.

In the above argument when $\dim B = 1$, a neighborhood in $B - (H_1 + \cdots + H_l)$ of a critical point H_j was a punctured disc. Now such a neighborhood is of the form $(\Delta^*)^r \times \Delta^{n-r} = \underbrace{\Delta^* \times \cdots \times \Delta^*}_{r} \times \underbrace{\Delta \times \cdots \times \Delta}_{n-r}$ where the Δ^* are punctured discs and the Δ are ordinary discs. However $(\Delta^*)^r \times \Delta^{n-r}$ still has an obvious complete metric of negative holomorphic sectional curvatures, and so we can proceed as before.

6. I want to give some concluding remarks and applications.

The first is that the above theorem and proof illustrate the following general principle: *The global study of periods in an algebraic family of algebraic varieties fits into the general subject of hyperbolic complex analysis; which is by definition the study of negatively curved holomorphic mappings between complete hermitian complex manifolds.*

Such mappings are *distance decreasing* (Schwarz-Ahlfors-Pick), *volume decreasing* (Chern), and in general enjoy very strong *rigidity properties.* For example we mention the following

(12) *Rigidity Theorem.* Let $\{V_\lambda\}_{\lambda \in B}$ and $\{V'_\lambda\}_{\lambda \in B}$ be two families of algebraic varieties with the same parameter space B. Suppose that at one point $\lambda_0 \in B$ we have $V_{\lambda_0} = V'_{\lambda_0} = V$ and that, using this identification, the fundamental group $\pi_1(B^*, \lambda_0)$ acts the same on $H^*(V_{\lambda_0}, \mathbf{Z})$ and $H^*(V'_{\lambda_0}, \mathbf{Z})$. Then the two period mappings $\phi: B^* \to D/\Gamma$ and $\phi': B^* \to D/\Gamma$ are the same.

(In case V is an abelian variety this theorem is due to Grothendieck and Borel-R. Narasimhan.)

Another example of the very strong global properties of the period mapping is the following result of Borel, which is proved using hyperbolic complex analysis and reduction theory:

(13) *Extension Theorem (Borel).* Let $\{V_\lambda\}_{\lambda \in B}$ be an algebraic family as above and assume that i) the divisor $H \subset B$ of points corresponding to singular V_λ has normal crossings, and ii) the period matrix domain D is *hermitian symmetric.* Then there exists a compactification $\widehat{D/\Gamma}$ of D/Γ (Baily-Borel) and the period mapping $\phi: B - H \to D/\Gamma$ extends uniquely to a holomorphic mapping $\hat{\phi}: B \to \widehat{D/\Gamma}$.

In case V is a curve this result is due to Mayer-Mumford. The period matrix domain D is hermitian symmetric in case $q = 1$ (periods of holomorphic one forms) but is generally *not* hermitian symmetric if $q > 1$.

As an application of our main theorem we mention the following:

(14) *Application.* Let $V \subset \mathbf{P}_N$ be a polarized algebraic manifold and suppose that V has a *variety of moduli* $\{V_\lambda\}_{\lambda \in B^*}$ ($V = V_{\lambda_0}$ for some $\lambda_0 \in B^*$). Then the set of period matrices $\Omega(V') \in D/\Gamma$ corresponding to all algebraic manifolds V' which are deformations of V forms an analytic set.

For example, V has a variety of moduli if i) V is a curve (Mumford), or ii) V is a surface with positive canonical bundle (Matsusaka-Mumford).

7. This is an addendum to the results given above. There I have used the fact that the period map was negatively curved to conclude that it was distance decreasing (outside a compact set; cf. note 6). However, in our case the period mapping $\phi: B^* \to D/\Gamma$ is also *volume decreasing*—this fact seems to be somewhat more subtle than the distance decreasing property. Assuming as always that our divisor $H \subset B$ has only normal crossings and using the fact that the hyperbolic volume of $(\Delta^*)^r \times \Delta^{n-r}$ is *finite*, we have the following:

(15) *Complement to Main Theorem.* Let $\phi: B^* \to D/\Gamma$ be the period mapping as in the statement of the main theorem. Then the closure $\overline{\phi(B^*)}$ of $\phi(B^*)$ in D/Γ is an analytic set with finite volume $v_D(\overline{\phi(B^*)})$, where v_D is computed from the complete hermitian structure on D.

We remark that the volume $v_D(D/\Gamma)$ of D/Γ is finite (Borel and Harish-Chandra), but since D/Γ is non compact this does not necessarily mean that the volume of a closed subset of D/Γ is finite.

This volume business is also probably relevant as regards Borel's extension theorem (13). Namely, as mentioned, his proof uses the whole of reduction theory and we might like to find an argument which uses only hyperbolic function theory. A little reflection will show that we can at most hope to prove that ϕ extends to a *rational* map $\hat{\phi}: B \to \widehat{D/\Gamma}$, since there may be trouble about whether or not $\widehat{D/\Gamma}$ is a minimal model and it seems that only explicit information on the compactification can help out on this point.

At any rate here is an *incomplete* function-theoretic argument to prove the

(16) *Weak Extension Theorem.* With the notations and assumptions of (13), the period map extends to a *meromorphic map* $\hat{\phi}: B \to \widehat{D/\Gamma}$.

Proof. Let M be D/Γ, $\hat{M}$ be $\widehat{D/\Gamma}$, and $A = \hat{M} - M$. Then $\hat{M}$ and A are analytic sets and we claim the following

(17) **Lemma.** The *graph* $G(\phi)$ is a closed analytic subset of

$$(B \times \hat{M}) - (H \times A).$$

Proof. We will only prove that $G(\phi)$ is closed. If not there is a sequence $(\lambda_n, \phi(\lambda_n)) \in G(\phi)$ which converges to a point (λ, x) in $B \times \hat{M}$ which is not in $G(\phi)$. We let $x_n = \phi(\lambda_n)$. We may assume that $\lambda_n \to \lambda$ in B and $x_n \to x$ in $\hat{M}$. If $\lambda \in B - H$, then $x = \phi(\lambda)$. Thus $\lambda \in H$ and we have a sequence $\{\lambda_n\}$ in $B - H$ such that $\lambda_n \to \lambda \in H$ and $\phi(\lambda_n) = x_n \to x \in M$ (since $(\lambda, x) \in B \times \hat{M} - H \times A$). This is the same situation (8) that led to a contradiction before.

Now the graph $G(\phi) \subset B \times M$ and the volume $v_{B \times M}(G(\phi)) \leqq v_B + v_M(\phi(B^*)) < \infty$ since ϕ is volume decreasing. Thus we have the situation:

(18) $G \subset (B \times \hat{M}) - (H \times A)$ is an analytic set with $v_{B \times M}(G) < \infty$.

If instead of (18) we had:

(18′) $G \subset (B \times \hat{M}) - (H \times A)$ is an analytic set with $\mathscr{H}(G) < \infty$ where $\mathscr{H}$ is the appropriate dimensional *Hausdorff measure* on $B \times \hat{M}$,

then we could use Bishop's theorem (cf. the Springer notes by Stoltzenberg) to conclude that $\bar{G} \subset B \times \hat{M}$ is an analytic set, which is what we want.

Now it seems quite likely that (18) $\Rightarrow$ (18′) since v_{B+M} is computed with respect to a $ds_B^2 \times ds_M^2$ where ds_M^2 is *complete*. We do not as yet have a proof of (18) $\Rightarrow$ (18′) ((18′) is true a fortiori by Borel's theorem) and so will stop here.

NOTES

1. Our general reference is the author's paper, "On the Periods of Integrals on Algebraic Manifolds," *Rice University Studies*, Vol. 54, No. 4 (1968), 21–38.
2. Ibid., p. 28.
3. Ibid., p. 22.
4. Ibid., p. 23.
5. Ibid., p. 34.
6. This is somewhat of a simplification. Actually $\phi: B^* — D/\Gamma$ is only negatively curved outside a suitable compact set $K \subset B^*$. It is clear that this is all that really matters.

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