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A Generalized Trust Region SQP Algorithm for Equality Constrained Optimization

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Abstract

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We introduce and analyze a class of generalized trust region sequential quadratic programming (GTRSQP) algorithms for equality constrained optimization. Unlike in standard trust region SQP (TRSQP) algorithms, the optimization subproblems arising in our GTRSQP algorithm can be generated from models of the objective and constraint functions that are not necessarily based on Taylor approximations. The need for such generalizations is motivated by optimal control problems for which model problems can be generated using, e.g., different discretizations.

Several existing TRSQP algorithms are special cases of our GTRSQP algorithm. Our first order global convergence result for the GTRSQP algorithm applied to TRSQP allows one to relax the condition that the so-called tangential step lies in the null-space of the linearized constraints.

The application of the GTRSQP algorithm to an optimal control problem governed by Burgers equation is discussed.

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Chapter 1

Introduction

We are interested in the solution of a class of large–scale nonlinear programming problems of the form

$$\begin{array}{ll} \min & f(y,u), \\ \text{s.t.} & c(y,u) = 0, \end{array} \tag{1.1}$$

where $y \in \mathbb{R}^M$, $u \in \mathbb{R}^{N-M}$, $f : \mathbb{R}^M \times \mathbb{R}^{N-M} \to \mathbb{R}$, $c : \mathbb{R}^M \times \mathbb{R}^{N-M} \to \mathbb{R}^M$, $M < N$. The problems we are interested in are obtained from discretizations of optimal control, optimal design or parameter identification problems. In these cases, y is called the (discretized) state and u represents the (discretized) control/design variables or unknown system parameters. The equality constraint $c(y,u) = 0$ is obtained from a discretization of a partial differential equation and is called the (discretized) state equation.

Most optimization methods for the solution of large-scale problems (1.1) improve a given approximation of the solution by optimizing a subproblem that is easier to solve than the original problem. For example, sequential quadratic programming (SQP) methods generate optimization subproblems from Taylor approximations of the objective and constraint functions and they require the solution of a large–scale quadratic problem at each iteration.

For our target problems, there are alternative ways to generate optimization subproblems. For example, optimization subproblems can be obtained by using different discretizations or by using model reduction techniques such as proper orthogonal decomposition. These subproblems are much smaller than the original problem and, hence, are easier to solve. They may be better representations of the original problem than the quadratic subproblems based on Taylor approximations of the original problem, which are being generated in SQP methods.

There exist algorithmic frameworks that allow one to use non-Taylor approximation based models. The ones most closely related to this work are reviewed in the next chapter. All of these are based on the trust-region framework. The subproblems are subject to a trust-region constraint, which estimates over which region of the variable space the model can be trusted to be a suitable approximation of the original problem.

All of the approaches for which convergence analyses are available do not consider (1.1), but instead tackle the so-called reduced problem

$$\min \tilde{f}(u), \tag{1.2}$$

possibly with additional equality or inequality constraints on u. In (1.2) $\tilde{f}(u) = f(y(u), u)$, where $y(u)$ is the solution of $c(y, u) = 0$ for a given u. It is assumed that $c(y, u) = 0$ has a unique solution for a given u. In all applications of interest, however, the solution $y(u)$ is not available analytically, but has to be computed numerically. Hence, only an approximation of $y(u)$ is available. This makes it impossible to verify the assumptions imposed in the convergence analyses for the existing approaches. We will provide more details in Chapter 2. Therefore, we tackle (1.1) directly. Since f and c as well as their derivatives can usually be evaluated easily at a given point (y, u), it is actually possible to compare function and derivative values of the models used in our approach against the 'truth'.

In its general form, our algorithm does not depend on the partitioning of the optimization variables into y and u, although the structure arising out of such a splitting can and is being used in concrete implementations. At this point it is sufficient to set $x = (y, u)$ and to consider

$$\begin{array}{ll} \min & f(x), \\ \text{s.t.} & c(x) = 0. \end{array} \tag{1.3}$$

Given a current approximation x_k of the solution of (1.3) and models m_k^l and m_k^c of the Lagrangian corresponding to (1.3) and the constraints in (1.3), we compute a trial iterate $x_k + s_k$ by solving

$$\begin{array}{ll} \min & m_k^l(\hat{x}_k + \hat{s}), \\ \text{s.t.} & m_k^c(\hat{x}_k + \hat{s}) = 0, \\ & \|P_k \hat{s}\| \leq \Delta_k. \end{array} \tag{1.4}$$

We do not assume that the domain of the models is equal to the domain of the original functions. In particular, we allow $\hat{x}_k, \hat{s} \in \mathbb{R}^{N_k}$. This is important, since many models are generated from coarse discretizations of the original problem. The matrix $P_k \in \mathbb{R}^{N \times N_k}$ is a prolongation that transfers the trial step $\hat{s}_k \in \mathbb{R}^{N_k}$ from the model domain into $\mathbb{R}^N$ and $R_k \in \mathbb{R}^{N_k \times N}$ is a restriction that maps the iterate $x_k \in \mathbb{R}^N$ into the model variable space $\mathbb{R}^{N_k}$. Thus, $\hat{x}_k = R_k x_k$ and $s_k = P_k \hat{s}_k$.

We use an extension of the trust-region SQP algorithm of [11] to manage the update of x_k, the update of Δ_k as well as the selection of models. The detailed formulation of the algorithm will be given in Section 3.2. As in trust-region SQP methods we have to deal with a possible incompatibility of the equality constraint and the trust-region constraint in (1.4). We use a composite step strategy. The trial step is the sum of the so-called quasi-normal step and the tangential step. The quasi-normal step is responsible to move toward feasibility and the tangential step is responsible to move toward optimality. Compared to trust-region SQP methods, which use model problems with linear constraints, it may be more computationally expensive to stay in a set $\{\hat{s} : m_k^c(\hat{x}_k+\hat{s}) = m_0^c\}$ where m_0^c is given. Therefore, the definition of the tangential step is more delicate in our context. Our approach detailed in Section 3.2.2 allows the tangential step to violate the level of model constraint satisfaction achieved by the quasi-normal step. Specialized to the context of trust-region SQP methods, this means that the tangential step may not be required to lie in the null-space of the linearized constraints. The amount by which the tangential step is allowed to violate the level of model constraint satisfaction achieved by the quasi-normal step depends on the norm of the model constraints and in the initial phase of the algorithm may be large. This can even be useful in the case of trust-region SQP methods when linearized constraints are solved using iterative methods. See also [22], where a different approach is discussed for trust-region SQP methods.

A first order convergence analysis for our algorithm is presented in Section 3.4. The convergence analysis of our algorithm is based on a careful adaptation of the convergence theory in [11]. Among the differences between our setting and that in [11] are our formulation of the predicted reduction and update of the penalty parameter, both of which are needed because we allow the tangential step to violate the level of model constraint satisfaction achieved by the quasi-normal step, and the bound for the difference between predicted and actual reduction, which is due to the fact that we do not use Taylor approximation based models.

The application of our algorithm to an optimal control problem governed by Burgers equation is discussed in Chapter 4. In this application, the models are constructed from a coarsening of the grid.

Chapter 2

Existing Work

In this section we review some of the optimization approaches that have motivated the work in this thesis. We consider optimization problems of the form

$$\begin{array}{ll} \min & f(y,u), \\ \text{s.t.} & c(y,u) = 0, \\ & e(y,u) = 0, \\ & h(y,u) \leq 0, \end{array} \tag{2.1}$$

where the optimization variables $x = (y,u) \in \mathbb{R}^N$ are partitioned into so-called states $y \in \mathbb{R}$ and controls/design parameters $u \in \mathbb{R}^{N-M}$. Here $f : \mathbb{R}^N \to \mathbb{R}$, $c : \mathbb{R}^N \to \mathbb{R}^M$, $e : \mathbb{R}^N \to \mathbb{R}^J$, $h : \mathbb{R}^N \to \mathbb{R}^I$, $J + M < N$, are sufficiently smooth functions. In several applications $c(y,u) = 0$ can be solved uniquely for y given u. Let $y(u)$ denote the function implicitly defined as the solution of $c(y,u) = 0$. In this case, we can rewrite (2.1) in the form

$$\begin{array}{ll} \min & \widetilde{f}(u), \\ \text{s.t.} & \widetilde{e}(u) = 0, \\ & \widetilde{h}(u) \leq 0, \end{array} \tag{2.2}$$

where

$$\widetilde{f}(u) = f(y(u), u), \quad \widetilde{e}(u) = e(y(u), u), \quad \widetilde{h}(u) = h(y(u), u). \tag{2.3}$$

We call (2.2) the reduced problem corresponding to (2.1).

Problems of the type (2.1) or (2.2) are typically being solved by iterative algorithms that use first or second order Taylor approximation models of the objective and constraint functions, such as quasi-Newton algorithms for unconstrained problems [13] or sequential quadratic programming (SQP) or interior-point algorithms [26] for constrained problems. In each iteration of these algorithms the nonlinear problem (2.1) or (2.2) is replaced by a quadratic problem. The motivation for this replacement is that the quadratic problem is easier to solve than the original problem, while (locally) retaining the structure of the original problem.

Recently nonlinear minimization algorithms have been developed that allow the use of more general, non–Taylor approximation based models. In all these algorithms trust–region techniques are used to manage the models.

The first such approach, applied to unconstrained problems

$$\min \widetilde{f}(u) \tag{2.4}$$

with twice continuously differentiable $\widetilde{f} : \mathbb{R}^{N-M} \to \mathbb{R}$ is due to [2] and is called the Approximation Management Framework (AMF). In each iteration k of the AMF a new trial iterate $u_k + s_k$ is obtained by approximately minimizing a model $\widetilde{m}_k^f : \mathbb{R}^{N-M} \to \mathbb{R}$

subject to a trust-region constraint,

$$\begin{aligned} \min \quad & \widetilde{m}_k^f(u_k + s), \\ \text{s.t.} \quad & \|s\| \le \Delta_k. \end{aligned} \tag{2.5}$$

The subproblem (2.5) does not need to be solved exactly. The trial step s_k only needs to satisfy a fraction of Cauchy decrease condition (FCD) for the model $\widetilde{m}_k^f$, which is given by

$$\widetilde{m}_k^f(u_k) - \widetilde{m}_k^f(u_k + s_k) \ge c_1 \|\nabla \widetilde{m}_k^f(u_k)\| \min\{c_2 \|\nabla \widetilde{m}_k^f(u_k)\| \ , \ \Delta_k\},$$

where c_1, c_2 are positive constants independent of k. The decision about the acceptance of the trial iterate $u_k + s_k$ and the update of the trust-region radius Δ_k is performed as in the well known trust-region algorithms for unconstrained optimization [10]. Global convergence of the AMF algorithm is proven under the assumption that the models $\widetilde{m}_k^f$ are twice continuously differentiable, satisfy

$$\widetilde{m}_k^f(u_k) = \widetilde{f}(u_k), \quad \nabla \widetilde{m}_k^f(u_k) = \nabla \widetilde{f}(u_k), \tag{2.6}$$

for all $k \in \mathbb{N}$ and $\|\nabla^2 \widetilde{m}_k^f(u_k + s)\| \le \kappa_H$ for all s with $\|s\| \le \Delta_k$ and all $k \in \mathbb{N}$. Under these assumptions it is shown in [2] that the sequence of iterates generated by the AMF satisfies

$$\liminf_{k \to \infty} \|\nabla \widetilde{f}(u_k)\| = 0.$$

The assumptions (2.6) on the model have been relaxed in [5]. In [5] the assumptions (2.6) on the models $\widetilde{m}_k^f$ are replaced by the following. Let s_k be the trust–region step. There exists a positive integer K, such that

$$\frac{|(\nabla \widetilde{m}_k^f(u_k) - \nabla \widetilde{f}(u_k))^T s_k|}{\|\nabla \widetilde{m}_k^f\| \|s_k\|} \le \xi \quad \forall k > K. \tag{2.7}$$

In (2.7), $\xi \in (0, 1)$ is a constant related to parameters in the update formula for the trust-region radius Δ_k. In [5] the trust–region convergence theories of [7] and [27] are used to prove that

$$\liminf_{k\to\infty} \|\nabla \widetilde{m}_k^f(u_k)\| = 0.$$

In the numerical examples shown in [5, 15], the condition (2.7) is monitored, but for the models used in [5, 15] no approach exists yet that allows one to enforce (2.7).

Extensions of the unconstrained AMF to constrained problems have been explored in [3], [4]. The AMF in [4] is based on the augmented Lagrangian method [9]. The problem to be solved is of the form

$$\begin{array}{ll} \min & \widetilde{f}(u), \\ \text{s.t.} & \widetilde{e}(u) = 0, \\ & u_l \leq u \leq u_u. \end{array}$$

The associated augmented Lagrangian is

$$\Phi(u, \lambda, \mu) = \widetilde{f}(u) + \lambda^T \widetilde{e}(u) + \frac{1}{2\mu}\|\widetilde{e}(u)\|^2.$$

In the kth iteration of the AMF in [4] a subproblem of the type

$$\begin{array}{ll} \min & \widetilde{m}_k^f(u_k + s) + \lambda_k^T \widetilde{m}_k^e(u_k + s) + \frac{1}{2\mu_k}\|\widetilde{m}_k^e(u_k + s)\|^2, \\ \text{s.t.} & u_l \leq u_k + s \leq u_u, \\ & \|s\| \leq \Delta_k \end{array}$$

is solved, where $\widetilde{m}_k^f$ and $\widetilde{m}_k^e$ are models of $\widetilde{f}$ and $\widetilde{e}$, respectively. The models are assumed to satisfy (2.6) and

$$\widetilde{m}_k^e(u_k) = \widetilde{e}(u_k), \nabla \widetilde{m}_k^e(u_k) = \nabla \widetilde{e}(u_k). \tag{2.8}$$

In [3], an AMF based on the Sl_1QP method of [16] has been discussed for the solution of (2.2). In the kth iteration of the AMF an approximation

$$m_k(u_k + s, \sigma_k) = \widetilde{m}_k^f(u_k + s) + \sigma_k \|\widetilde{m}_k^e(u_k + s)\|_1 + \sigma_k \| \max\{0, \widetilde{m}_k^h(u_k + s)\}\|_1$$

of the l_1 penalty function

$$\Phi(u_k + s, \sigma_k) = \widetilde{f}(u_k + s) + \sigma_k \|\widetilde{e}(u_k + s)\|_1 + \sigma_k \| \max\{0, \widetilde{h}(u_k + s)\}\|_1.$$

is minimized subject to a trust–region constraint. The models are assumed to satisfy (2.6), (2.8) and

$$\widetilde{m}_k^h(u_k) = \widetilde{h}(u_k), \nabla \widetilde{m}_k^h(u_k) = \widetilde{h}(u_k). \tag{2.9}$$

Another AMF is introduced in [1]. This approach is based on a class of multilevel methods for constrained optimization. For a problem of the form

$$\begin{array}{ll} \text{min} & \widetilde{f}(u), \\ \text{s.t.} & \widetilde{e}(u) = 0, \end{array}$$

the trial step s_k is computed as the sum of two substeps. The first substep s_k^1 is obtained by approximately minimizing a model of constraints $\widetilde{m}_k^e$ within a trust region,

$$\begin{array}{ll} \text{min} & \widetilde{m}_k^e(u_k + s), \\ \text{s.t.} & \|s\| \leq \alpha \Delta_k, \end{array}$$

where $\alpha \in (0, 1)$. The second substep s_k^2 is computed by approximately minimizing the model m_k^f of the objective function in the null–space of the linearized model constraints and subject to a trust–region. The subproblem for computing s_k^2 is given by

$$\begin{array}{ll} \text{min} & \widetilde{m}_k^f(u_k + s_k^1 + s), \\ \text{s.t.} & \nabla \widetilde{m}_k^e(u_k)^T s = 0, \\ & \|s\| \leq \sqrt{\Delta_k^2 - \|s_k^1\|^2}. \end{array} \tag{2.10}$$

It is assumed that the model constraints satisfy (2.8) and that the model of the objective function obeys

$$\widetilde{m}_k^f(u_k + s_1) = \widetilde{f}(u_k + s_1), \nabla \widetilde{m}_k^f(u_k + s_1) = \widetilde{f}(u_k + s_1). \tag{2.11}$$

The l_2 penalty function is used as a merit function. The scheme for the update of the penalty parameter is an extension of the updating scheme in [14]. Decision about acceptance of the trial step and update of trust region radius is performed as in [1].

In [1, 3, 4], it is noted that the conditions (2.6), (2.8) and (2.9) can be enforced using the so–called β-correction due to [8]. Given a true function $\widetilde{g} : \mathbb{R}^{N-M} \to \mathbb{R}$ and a model $\widetilde{m}_k^g$ of $\widetilde{g}$, which is not required to satisfy $\widetilde{m}_k^g(u_k) = \widetilde{g}(u_k)$ or $\nabla \widetilde{m}_k^g(u_k) = \nabla \widetilde{g}(u_k)$, the model is corrected as follows. One defines

$$\beta(u) = \frac{\widetilde{g}(u)}{\widetilde{m}_k^g(u)}$$

and

$$\beta_k(u) = \beta(u_k) + \nabla \beta(u_k)^T (u - u_k).$$

The corrected model of g is given by

$$(\widetilde{m}_k^g)^c(u) = \beta_k(u) \widetilde{m}_k^g(u).$$

Clearly, $(\widetilde{m}_k^g)^c(u_k) = \widetilde{g}(u_k)$ and $\nabla (\widetilde{m}_k^g)^c(u_k) = \nabla \widetilde{g}(u_k)$.

One difficulty that arises in the above mentioned approaches is that the implicitly defined function $y(u)$ often cannot be evaluated exactly. We discuss this for the unconstrained case (2.4). Since an analytic solution of $c(y(u), u) = 0$ is usually unavailable, one has to use iterative techniques to compute an approximation $\bar{y}$ of $y(u)$. Typically $\bar{y}$ only satisfy

$$\|c(\bar{y}, u)\| \leq \mathsf{tol}_c \tag{2.12}$$

where tol_c is a user specified constant. Applying Taylor series, we obtain

$$c(\bar{y}, u) - c(y, u) = c_y(y + \tau_c(y - \bar{y}), u)(y - \bar{y}),$$

where c_y is the partial Jacobian of c with respect to y. If $c_y(y + \tau_c(y - \bar{y}), u)$ is invertible, it follows that

$$\|y - \bar{y}\| \leq \kappa_c \mathsf{tol}_c,$$

where κ_c is an upper bound for $\|c_y^{-1}(y + \tau_c(y - \bar{y}), u)\|$. It is typically difficult to obtain a good estimate for κ_c. The bound for $\|y - \bar{y}\|$ imply

$$\begin{aligned} |f(y, u) - f(\bar{y}, u)| &\leq \|\nabla_y f(y + \tau_f(y - \bar{y}), u)\| \; \|y - \bar{y}\| \\ &\leq \kappa_f \mathsf{tol}_c, \end{aligned} \tag{2.13}$$

where $\kappa_f = \kappa_c \|\nabla_y f(y + \tau_f(y - \bar{y}), u)\|$, and

$$\begin{aligned} \|\nabla f(y(u), u) - \nabla f(\bar{y}, u)\| &\leq \|\nabla_y^2 f(y + \tau_g(y - \bar{y}), u)\| \; \|y(u) - \bar{y}\| \\ &\leq \kappa_g \mathsf{tol}_c \end{aligned} \tag{2.14}$$

where $\kappa_g = \kappa_c \|\nabla_y^2 f(y + \tau_g(y - \bar{y}), u)\|$. Again, it is typically difficult to obtain good estimates for the norms of the gradients and Hessians of f. From (2.13) and (2.14) we observe that although we can make the error between the computed gradient and true gradient smaller by enforcing a smaller tolerance tol_c, the actual errors $|f(y(u), u) - f(\bar{y}, u)|$

and $\|\nabla f(y(u), u) - \nabla f(\bar{y}, u)\|$ are difficult to determine. Hence, it is often impossible to enforce (2.7).

In the unconstrained case (2.4) it is possible to adapt an idea of [22, p. 291] and replace (2.7) by

$$\|\nabla \widetilde{m}_k^f(u_k) - \nabla \widetilde{f}(u_k)\| \leq \zeta \min\{\|\nabla \widetilde{m}_k^f\|, \Delta_k\} \quad \forall k > K. \tag{2.15}$$

In (2.15), $\zeta > 0$ is a constant independent of k. In contrast to ξ in (2.7), which is smaller than one, there is no restriction on the size of ζ. Hence, it may be possible to absorb all unknown bounds $\kappa_c, \kappa_f, \kappa_g$ into ζ.

The approach taken in this thesis includes y as an optimization variable and instead of (2.4) takes

$$\begin{array}{rl} \min & f(y, u), \\ \text{s.t.} & c(y, u) = 0 \end{array}$$

directly. The objective and the constraint functions are replaced by models. In many practical applications, the models are obtained from reduced bases approaches (see, e.g., [15, 23, 24]) or from coarse discretizations of the original problem (see e.g., [6]). In these cases, the dimension of the variables the models act on are different from the dimension N of the original variables. This is allowed in our approach. Our approach is a generalization of the trust–region SQP (TRSQP) method in [11]. In contrast to the algorithm in [11] the subproblems in our approach are nonlinear programming problems in which objective and the constraint functions are models of the original objective and the constraint functions. These models should be such that the subproblems are easier to solve than the original problem. As was stated before, the variable space for the model functions can be different than that for the original problem. We refer to our algorithms as generalized trust–region SQP (GTRSQP) algorithms, although the subproblems are not necessarily quadratic. Of the previously mentioned papers, our approach is most closely related to [1]. However, there are important differences between the algorithm in [1] and our algorithm. Unlike [1], we allow the variable space for the model functions to be different than those for the original problem. In addition, our tangential subproblem differs from that in [1]. The constraints in the tangential subproblem of [1] are linear and are obtained from the linearization of the constraint model. Our tangential subproblem is based on the full constraint model. Thus, if the constraint model is a good approximation of the actual constraints and inexpensive to work with, our approach has the potential to make better use of the model constraint. These two differences between the approach in [1] and our approach, also generate differences in other parts of the algorithms. We provide a first order convergence result, which generalizes that of [11]. Even if the models in our algorithm are chosen to be a quadratic approximation of the Lagrangian and a linearization of the constraints, respectively, i.e., our algorithm is a trust-region SQP algorithm, our first order global convergence result allows one to relax the condition in [11] that the so-called tangential step lies in the null-space of the linearized constraints. This is useful when the linearized constraints are solved by iterative methods.

Chapter 3

A Generalized Trust-Region SQP Algorithm

3.1 Introduction

Our generalized trust–region SQP (GTRSQP) method is a generalization of the trust–region SQP (TRSQP) method in [11]. In our generalization the subproblem may be nonlinear programming problems, obtained from 'simpler' models of the original problem. Among the AMF reviewed in the previous chapter, [1] seems most closely related to our approach. However, unlike [1], we allow the variable space for the model functions to be different than those for the original problem. In addition, our tangential subproblem differs from that in [1]. The constraints in the tangential subproblem of [1] are linear and are obtained from the linearization of the constraint model (see (2.10)). Our tangential subproblem is based on the full constraint model (see (3.7)). Thus, if the constraint model is a good approximation of the actual constraints and inexpensive to work with, our approach has the potential to make better use of the model constraint. These two differences between the approach in [1] and our approach, also generate difference in other parts of the algorithms.

Throughout this chapter, $||\cdot||$ denotes the 2-vector norm or the corresponding operator norm.

3.2 Formulation of the Algorithm

We consider problems of the form

$$\begin{array}{ll} \min & f(x), \\ \text{s.t.} & c(x) = 0, \end{array} \tag{3.1}$$

where $x \in \mathbb{R}^N, f : \mathbb{R}^N \longrightarrow \mathbb{R}, c : \mathbb{R}^N \longrightarrow \mathbb{R}^M, M < N$.

The Lagrangian associated with problem (3.1) is the function

$$l(x, \lambda) = f(x) + \lambda^T c(x),$$

where $\lambda \in \mathbb{R}^M$ is the Lagrange multiplier.

We compute a solution of (3.1) by solving a sequence of model problems. At iteration k, we are given an approximation x_k of a local solution x_* of (3.1) and models $m_k^l : \mathbb{R}^{N_k} \longrightarrow \mathbb{R}$ and $m_k^c : \mathbb{R}^{N_k} \longrightarrow \mathbb{R}^{M_k}$ of l and c, respectively. We allow $N_k \neq N$ and $M_k \neq M$. We use a restriction

$$R_k : \mathbb{R}^N \longrightarrow \mathbb{R}^{N_k}$$

to map the current iterate x_k into $\hat{x}_k = R_k x_k$ and we use a prolongation

$$P_k : \mathbb{R}^{N_k} \to \mathbb{R}^N$$

to map a trial step $\hat{s}_k$ in the model variable space into a trial step $s_k = P_k \hat{s}_k$ in the original variable space.

We assume that $m_k^l(\hat{x}_k + \hat{s})$ is a model of $l(x_k + P_k\hat{s}, \lambda_k)$ for small $\hat{s}$ and that $m_k^c(\hat{x}_k + \hat{s})$ is a model of $c(x_k + P_k\hat{s})$ for small $\hat{s}$. We compute a trial step by approximately solving

$$\begin{array}{ll} \min & m_k^l(\hat{x}_k + \hat{s}), \\ \text{s.t.} & m_k^c(\hat{x}_k + \hat{s}) = 0, \\ & \|P_k \hat{s}\| \le \Delta_k, \end{array} \tag{3.2}$$

where Δ_k is the trust-region radius in iteration k. The constraints in (3.2) can be incompatible. Therefore we will use a composite step algorithm. The trial step $\hat{s}_k$ is the sum of two substeps. The first substep is called the quasi-normal step $\hat{s}_k^{\mathrm{n}}$ and it is responsible to move towards feasibility. The second substep is called the tangential step $\hat{s}_k^{\mathrm{t}}$ and it is responsible to move towards optimality while approximately maintaining the model feasibility achieved by the quasi-normal step.

Progress of the algorithm is monitored using the augmented Lagrangian merit function

$$L(x, \lambda; \rho) = f(x) + \lambda^T c(x) + \frac{\rho}{2} c(x)^T c(x), \qquad \rho > 0.$$

We may use a restriction

$$R_k^\lambda : \mathbb{R}^M \to \mathbb{R}^{M_k}$$

to map the Lagrange multiplier estimate λ_k for the original problem (3.1) into a Lagrange multiplier estimate $\hat{\lambda}_k = R_k^\lambda \lambda_k$ for the model problem (3.2). Similarly, a prolongation

$$P_k^\lambda : \mathbb{R}^{M_k} \to \mathbb{R}^M$$

is used to map the Lagrange multiplier estimate $\hat{\lambda}_k$ for the model problem (3.2) into a Lagrange multiplier estimate $\lambda_k = P_k^\lambda \hat{\lambda}_k$ for the original problem (3.1).

3.2.1 The Quasi–Normal Step

The subproblem for the computation of the quasi-normal substep $\hat{s}_k^{\mathrm{n}}$ is given by

$$\begin{array}{ll} \min & \|m_k^c(\hat{x}_k + \hat{s}^{\mathrm{n}})\|, \\ \text{s.t.} & \|P_k \hat{s}^{\mathrm{n}}\| \leq \alpha^{\mathrm{n}} \Delta_k, \end{array} \tag{3.3}$$

where $0 < \alpha^{\mathrm{n}} < 1$ is some constant. The quasi-normal substep $\hat{s}_k^{\mathrm{n}}$ is not to required to solve (3.3) exactly. It is only required to satisfy the following three conditions. First, $\hat{s}_k^{\mathrm{n}}$ is required to satisfy

$$\|P_k \hat{s}_k^{\mathrm{n}}\| \leq \alpha^{\mathrm{n}} \Delta_k. \tag{3.4}$$

The second condition on the quasi-normal substep is a sufficient decrease condition which is given by

$$\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2 \;\geq c_1^{\mathrm{n}} \|m_k^c(\hat{x}_k)\| \min\{c_2^{\mathrm{n}} \|m_k^c(\hat{x}_k)\|, \alpha^{\mathrm{n}} \Delta_k\}, \tag{3.5}$$

where c_1^{n} and c_2^{n} are positive constants, which are independent of k. The third condition imposed on $\hat{s}_k^{\mathrm{n}}$ is

$$\|\hat{s}_k^{\mathrm{n}}\| \leq c_3^{\mathrm{n}} \|m_k^c(\hat{x}_k)\|, \tag{3.6}$$

where $c_3^{\mathrm{n}} > 0$ is independent of k.

Additional details on the actual computation of the quasi-normal step will be given in Section 3.6.

3.2.2 The Tangential Step

The tangential subproblem is related to

$$\begin{aligned} \min \quad & m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}}), \\ \text{s.t.} \quad & m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}}) = m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}}), \\ & \|P_k(\hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}})\| \le \Delta_k. \end{aligned} \tag{3.7}$$

The tangential step is responsible to move toward optimality. To measure progress in optimality, we typically use

$$\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n}}) = \|Z_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})^T \nabla m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|, \tag{3.8}$$

where $Z_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})$ is a matrix whose columns span the null-space of $\nabla m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})^T$, the Jacobian of m_k^c. If the Jacobian $\nabla m_k^c(\hat{x})^T$ of m_k^c at $\hat{x}$ has full rank, we can compute $I - \nabla m_k^c(\hat{x})[\nabla m_k^c(\hat{x})^T \nabla m_k^c(\hat{x})]^{-1} \nabla m_k^c(\hat{x})^T$. Since

$$\left(I - \nabla m_k^c(\hat{x})[\nabla m_k^c(\hat{x})^T \nabla m_k^c(\hat{x})]^{-1} \nabla m_k^c(\hat{x})^T\right) \nabla m_k^c(\hat{x}) = 0,$$

the columns of the matrix $\left(I - \nabla m_k^c(\hat{x})[\nabla m_k^c(\hat{x})^T \nabla m_k^c(\hat{x})]^{-1} \nabla m_k^c(\hat{x})^T\right)^T$ span the null space of $\nabla m_k^c(\hat{x})^T$. Thus, if the Jacobian of m_k^c at $\hat{x}$ has full rank, we can use

$$Z_k^c(\hat{x}_k) = \left(I - \nabla m_k^c(\hat{x}_k)[\nabla m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k)]^{-1} \nabla m_k^c(\hat{x}_k)^T\right)^T. \tag{3.9}$$

However, for specific applications, including optimal control problems other representations of the null-space representor $Z_k^c(\hat{x}_k)$ are favorable (see, e.g., [12, 25] and Section 4.3). Moreover, depending on the specific algorithm used to compute an approximate solution of (3.7) quantities other than (3.8) may have to be used to measure progress in optimality. For example, filter SQP methods use different quantities to measure progress in optimality, see, e.g., [10]. Our global convergence proof is largely independent of the specific quantity used to measure progress in optimality. Therefore, we simply denote such a quantity by $\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n}})$. Following [10] we call χ_k^M a first order criticality measure.

The tangential step $\hat{s}_k^{\mathrm{t}}$ is not to required to solve (3.7) exactly. We require that the tangential step $\hat{s}_k^{\mathrm{t}}$ satisfies three conditions. First, the tangential step is required to satisfy

$$\|P_k(\hat{s}_k^{\mathrm{n}} + \hat{s}_k^{\mathrm{t}})\| \le \Delta_k. \tag{3.10}$$

The second requirement is that

$$\|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}_k^{\mathrm{t}})\| \le \sqrt{\mu_k}\|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|, \tag{3.11}$$

where

$$\begin{aligned} \mu_k &= 1 \quad \text{if } \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\| = 0, \\ \mu_k &= \frac{\|m_k^c(\hat{x}_k)\|^2 + \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2}{2\|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2} \quad \text{else.} \end{aligned} \tag{3.12}$$

This means, if $\|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\| > 0$ we do not insist that the equality constraint in (3.7) is satisfied. We only require that the final tangential step iterate $\hat{s}_k^{\mathrm{t}}$ maintains an approximate level of feasibility for the model constraints $m_k^c(\hat{x}) = 0$ achieved by the quasi–normal step $\hat{s}_k^{\mathrm{n}}$. Since $\|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\| \le \|m_k^c(\hat{x}_k)\|$, we have $\mu_k \ge 1$.

The third requirement on $\hat{s}_k^{\mathrm{t}}$ is

$$\begin{aligned} & m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}}) - m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}_k^{\mathrm{t}}) \\ & \quad \ge c_1^{\mathrm{t}}\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n}}) \min\left\{c_2^{\mathrm{t}}\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n}}), (1-\alpha^{\mathrm{n}})\Delta_k\right\} \end{aligned} \tag{3.13}$$

where $c_1^{\mathrm{t}}, c_2^{\mathrm{t}} > 0$ are constants independent of k.

We have stated that the tangential subproblem is related to (3.7). The constraints in (3.7) are compatible ($\hat{s}^{\mathrm{t}} = 0$ is feasible), but the equality constraint is somewhat artificial. If it is possible to obtain a tangential step that meets the three requirements stated above and improves model feasibility, then it is feasible to take this step. Therefore a better statement of the tangential subproblem may be

$$\begin{array}{ll} \min & m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}}), \\ \text{s.t.} & m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}}) \leq \sqrt{\mu_k}|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})|, \\ & m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}}) \geq -\sqrt{\mu_k}|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})|, \\ & \|P_k(\hat{s}_k^{\mathrm{n}} + \hat{s}^{\mathrm{t}})\| \leq \Delta_k. \end{array} \tag{3.14}$$

where $|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})|$ denotes the vector whose ith component is $|(m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}}))_i|, i = 1, \cdots, M_k$.

Clearly, any point $\hat{s}^{\mathrm{t}}$ that is feasible for (3.7) is also feasible for (3.14). Moreover any point $\hat{s}^{\mathrm{t}}$ that is feasible for (3.14) satisfies (3.10) and (3.11). Since the tangential step only needs to satisfy (3.10), (3.11), (3.13) a variety of sub-optimization problems can be formulated whose approximate solution satisfies the tangential step conditions.

Additional details on the actual computation of the tangential step will be given in Section 3.7.

3.2.3 Evaluation of the Trial Step and Update of the Trust–Region Radius

The decision about acceptance of the step and update of trust region radius is based on the ratio of actual reduction given by

$$Ared(s_k, \rho_k) = L(x_k, \lambda_k; \rho_k) - L(x_k + s_k, \lambda_{k+1}; \rho_k) \tag{3.15}$$

and predicted reduction given by

$$
\begin{aligned}
Pred_k(\hat{s}_k, \rho_k) \;&=\; m_k^l(\hat{x}_k) + \frac{\rho_k}{2}\|m_k^c(\hat{x}_k)\|^2 - m_k^l(\hat{x}_k + \hat{s}_k) \\
&\quad -(\hat{\lambda}_{k+1} - \hat{\lambda}_k)^T m_k^c(\hat{x}_k + \hat{s}_k) - \frac{\rho_k}{2}\|m_k^c(\hat{x}_k + \hat{s}_k)\|^2. \qquad (3.16)
\end{aligned}
$$

Let $0 < \eta_1 < \eta_2 < 1, \alpha_1 < 1, \alpha_2 > 1$, and compute the ratio

$$
r_k = \frac{Ared(s_k, \rho_k)}{Pred_k(\hat{s}_k, \rho_k)}.
$$

The trust-region radius is updated as follows

$$
\Delta_{k+1} = \begin{cases} \alpha_1\|s_k\| & \text{if } r_k < \eta_1, \\ \max\{\Delta_k, \Delta_{min}\} & \text{if } \eta_1 \le r_k < \eta_2, \\ \min\{\Delta_{max}, \max\{\alpha_2\Delta_k, \Delta_{min}\}\} & \text{if } r_k \ge \eta_2, \end{cases} \qquad (3.17)
$$

where Δ_{min} is the lower bound on the trust region radius and Δ_{max} is an upper bound.

3.2.4 Updating the Lagrange Multipliers

The methods for updating the Lagrange multipliers are left unspecified at this point. We only require the Lagrange multipliers to be bounded, i.e., we require the existence of $\kappa_\lambda, \kappa_\lambda^m > 0$ such that

$$
\|\lambda_k\| \le \kappa_\lambda, \quad \|\hat{\lambda}_k\| \le \kappa_\lambda^m \quad \forall k \in \mathbb{N}.
$$

3.2.5 Updating the Penalty Parameters

The penalty parameter is updated to ensure that the predicted reduction in the merit function is positive at each iteration. We modify an update formula given in [14]. It ensures that the merit function is predicted to decrease at each iteration by at least a fraction of Cauchy decrease in the model of the constraint. In our algorithm we allow

$$m_k^l(\hat{x}_k + \hat{s}_k) \neq m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{t}})$$

and therefore the update of the penalty parameter is a little more delicate in our case.

At the kth iterate x_k, after s_k and $\hat{s}_k$ have been computed, evaluate

$$\begin{aligned} Pred_k(\hat{s}_k, \rho_{k-1}) &= m_k^l(\hat{x}_k) + \frac{\rho_{k-1}}{2}\|m_k^c(\hat{x}_k)\|^2 - m_k^l(\hat{x}_k + \hat{s}_k) \\ &\quad -(\hat{\lambda}_{k+1} - \hat{\lambda}_k)^T m_k^c(\hat{x}_k + \hat{s}_k) - \frac{\rho_{k-1}}{2}\|m_k^c(\hat{x}_k + \hat{s}_k)\|^2. \end{aligned}$$

If

$$Pred_k(\hat{s}_k, \rho_{k-1}) \geq \frac{\rho_{k-1}}{8}[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2],$$

then set $\rho_k = \rho_{k-1}$, else set

$$\rho_k = \beta + \frac{8[m_k^l(\hat{x}_k + \hat{s}_k) + (\hat{\lambda}_{k+1} - \hat{\lambda}_k)^T m_k^c(\hat{x}_k + \hat{s}_k) - m_k^l(\hat{x}_k)]}{\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2},$$

where $\beta > 0$.

In Lemmas 3.4.3 and 3.4.4 we will show that with this penalty parameter update the predicted reduction is 'sufficiently' positive.

3.2.6 Statement of the GTRSQP Algorithm

We present a complete description of our generalized trust–region SQP algorithm.

Algorithm 3.2.1 *1. Initialization. Given* $x_0, \lambda_0, \hat{\lambda}_0$. *Choose* $\Delta_{\min}, \Delta_{\max} > 0$, $\Delta_0 \in [\Delta_{\min}, \Delta_{\max}]$, $0 < \eta_1 < \eta_2 < 1$, $\alpha_1 < 1, \alpha_2 > 1$, $\beta > 0$ *and* $\epsilon_{tol} > 0$. *Set* $\rho_{-1} = 1$ *and* $k = 0$.

2. *Compute the first order criticality measure* $\chi_k(x_k)$ *and the constraint residual* $c(x_k)$.

   If $\chi_k(x_k) + \|c(x_k)\| < \epsilon_{tol}$ *then terminate.*

3. *Generate models* m_k^l, m_k^c, *a restriction* R_k *and a prolongation* P_k. *Set* $\hat{x}_k = R_k x_k$ *and compute* $\hat{\lambda}_k$.

4. *Compute a trial step*

   a. *Compute a quasi-normal step* $\hat{s}_k^{\mathrm{n}}$ *that satisfies* (3.4), (3.5) *and* (3.6). *Set* $s_k^{\mathrm{n}} = P_k \hat{s}_k^{\mathrm{n}}$.

   b. *Compute a tangential step* $\hat{s}_k^{\mathrm{t}}$ *that satisfies* (3.10), (3.11) *and* (3.13).

   c. *Set* $s_k = P_k(\hat{s}_k^{\mathrm{n}} + \hat{s}_k^{\mathrm{t}})$.

5. *Update the Lagrange multipliers* $\lambda_{k+1}, \hat{\lambda}_{k+1}$.

6. *Update the penalty parameter. Compute*

$$\begin{aligned} Pred_k(\hat{s}_k, \rho_{k-1}) &= m_k^l(\hat{x}_k) + \frac{\rho_{k-1}}{2}\|m_k^c(\hat{x}_k)\|^2 - m_k^l(\hat{x}_k + \hat{s}_k) \\ &\quad -(\hat{\lambda}_{k+1} - \hat{\lambda}_k)^T m_k^c(\hat{x}_k + \hat{s}_k) - \frac{\rho_{k-1}}{2}\|m_k^c(\hat{x}_k + \hat{s}_k)\|^2. \end{aligned}$$

   If

$$Pred_k(\hat{s}_k, \rho_{k-1}) \geq \frac{\rho_{k-1}}{8}[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2],$$

   then set $\rho_k = \rho_{k-1}$, *else set*

$$\rho_k = \beta + \frac{8[m_k^l(\hat{x}_k + \hat{s}_k) + (\hat{\lambda}_{k+1} - \hat{\lambda}_k)^T m_k^c(\hat{x}_k + \hat{s}_k) - m_k^l(\hat{x}_k)]}{\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2}.$$

7. *Evaluate the trial step s_k. Compute*

$$Ared(s_k, \rho_k) = L(x_k, \lambda_k; \rho_k) - L(x_k + s_k, \lambda_{k+1}; \rho_k)$$

and

$$r_k = \frac{Ared(s_k, \rho_k)}{Pred(s_k, \rho_k)}.$$

Set

$$\Delta_k = \begin{cases} \alpha_1 \|s_k\| & \textit{if } r_k < \eta_1, \\ \max\{\Delta_k, \Delta_{min}\} & \textit{if } \eta_1 \le r_k < \eta_2, \\ \min\{\Delta_{max}, \max\{\alpha_2 \Delta_k, \Delta_{min}\}\} & \textit{if } r_k \ge \eta_2. \end{cases}$$

If $r_k \ge \eta_1$, then set $x_{k+1} = x_k + s_k$, $\Delta_{k+1} = \Delta_k$, $k = k+1$ and goto 2, else goto 4.

We use the parameters

$$\eta_1 = 0.01, \eta_2 = 0.9, \alpha_1 = 0.5, \alpha_2 = 2, \Delta_{min} = 10^{-5}, \Delta_{max} = 10^5.$$

For convenience of the reader, we collect the notations most frequently used in the GTRSQP algorithm in Table 3.1.

Table 3.1: Notations used in the GTRSQP algorithm

$f(x)$	Objective function
$c(x)$	Constraint function
$l(x, \lambda)$	Lagrange function
$L(x, \lambda; \rho)$	Augmented Lagrange function
$\nabla c(x)^T$	Jacobian of the constraint
χ_k	Criticality measure
$m_k^l(\hat{x})$	Objective function for the model at the kth iteration
$m_k^c(\hat{x})$	Constraint function for the model in the kth iteration
$\nabla m_k^c(\hat{x})^T$	Jacobian of the model constraint in the kth iteration
$Z_k^c(\hat{x})$	Matrix whose columns span the null space of $\nabla m_k^c(\hat{x})^T$
χ_k^M	Criticality measure for the model
P_k	Prolongation matrix for the iterate in iteration k
R_k	Restriction matrix for the iterate in iteration k
P_k^λ	Prolongation matrix for the Lagrange multiplier in iteration k
R_k^λ	Restriction matrix for the Lagrange multiplier in iteration k
Δ_k	Trust region radius in iteration k
λ_k	Lagrange multiplier estimate in iteration k
$\hat{\lambda}_k$	Lagrange multiplier estimate for the model in iteration k
ρ_k	Penalty parameter in iteration k
x_k	kth iterate
s_k	Trial step in iteration k, $s_k = s_k^{\mathsf{n}} + s_k^{\mathsf{t}}$
s_k^{n}	Quasi-normal substep in iteration k
s_k^{t}	Tangential substep in iteration k
$\hat{x}_k$	$= R_k x_k$
$\hat{s}_k$	Trial step for the model in iteration k
$\hat{s}_k^{\mathsf{n}}$	Quasi-normal substep for the model in iteration k, $s_k^{\mathsf{n}} = P_k \hat{s}_k^{\mathsf{n}}$
$\hat{s}_k^{\mathsf{t}}$	Tangential substep for the model in iteration k, $s_k^{\mathsf{t}} = P_k \hat{s}_k^{\mathsf{t}}$

3.3 Comparison of GTRSQP and TRSQP

The trust–region SQP (TRSQP) method in [11] is a special case of our GTRSQP algorithm if we set $N_k = N$, $M_k = M$, $R_k = I$, $P_k = I$ and if we use the models

$$m_k^{\mathrm{c}}(x_k + s) = c(x_k) + \nabla c(x_k)^T s, \tag{3.18}$$

$$m_k^{l}(x_k + s) = l(x_k, \lambda_k) + \nabla_x l(x_k, \lambda_k)^T s + \frac{1}{2} s^T \nabla_x^2 l(x_k, \lambda_k) s. \tag{3.19}$$

The quasi-normal step subproblem (3.3) becomes the familiar subproblem

$$\begin{array}{ll} \min & \|c(x_k) + \nabla c(x_k)^T s^{\mathrm{n}}\|, \\ \text{s.t.} & \|s^{\mathrm{n}}\| \le \alpha^{\mathrm{n}} \Delta_k, \end{array} \tag{3.20}$$

Our conditions (3.4), (3.5), (3.6) become the conditions

$$\|s_k^{\mathrm{n}}\| \le \alpha^{\mathrm{n}} \Delta_k,$$

$$\|c(x_k)\|^2 - \|c(x_k) + \nabla c(x_k)^T s^{\mathrm{n}}\|^2 \ge c_1^{\mathrm{n}} \|c(x_k)\| \min \left\{ c_2^{\mathrm{n}} \|c(x_k)\|, \alpha^{\mathrm{n}} \Delta_k \right\},$$

and

$$\|s_k^{\mathrm{n}}\| \le c_3^{\mathrm{n}} \|c(x_k)\|,$$

which are precisely the condition imposed on the quasi-normal step in the TRSQP of [11].

With the choice (3.19), the tangential subproblem (3.7) becomes

$$\begin{array}{ll} \min & \nabla_x l(x_k, \lambda_k)^T (s_k^{\mathrm{n}} + s^{\mathrm{t}}) + \frac{1}{2} (s_k^{\mathrm{n}} + s^{\mathrm{t}})^T H_k (s_k^{\mathrm{n}} + s^{\mathrm{t}}) \\ \text{s.t.} & \nabla c_k(x_k)^T s^{\mathrm{t}} = 0, \\ & \|s_k^{\mathrm{n}} + s^{\mathrm{t}}\| \le \Delta_k. \end{array} \tag{3.21}$$

The problem (3.21) coincides with the tangential subproblem in [11].

An important difference between [11] and our algorithm is that we do not require the equality constraint in (3.7) to hold exactly. Instead, we only require that

$$\|c(x_k) + \nabla c(x_k)^T(s_k^{\mathrm{n}} + s_k^{\mathrm{t}})\| \leq \sqrt{\mu_k}\|c(x_k) + \nabla c(x_k)^T s_k^{\mathrm{n}}\|, \tag{3.22}$$

see (3.11), which implies

$$\|\nabla c(x_k)^T s_k^{\mathrm{t}}\| \quad \leq \quad \sqrt{1+\mu_k}\|c(x_k) + \nabla c(x_k)^T s_k^{\mathrm{n}}\|.$$

In many applications, see, e.g., [12, 22], the enforcement of the condition $\nabla c_k(x_k)^T s^{\mathrm{t}} = 0$ requires the solution of large scale linear systems. If these system solutions are performed iteratively, then $\nabla c_k(x_k)^T s^{\mathrm{t}} = 0$ cannot be maintained exactly.

For the models (3.18), (3.19) our requirements (3.10), (3.13) on the tangential step are identical to the requirements on the tangential step in [11], if we use $\chi_k^M(x_k + s_k^{\mathrm{n}})$ as defined in (3.8). In particular, (3.13) becomes

$$\begin{aligned}
& \nabla_s m_k^l(x_k + s_k^{\mathrm{n}} + s_k^{\mathrm{t}}) - \nabla_s m_k^l(x_k + s_k^{\mathrm{n}}) \\
\geq \quad & c_1^{\mathrm{t}}\chi_k^M(x_k + s_k^{\mathrm{n}}) \min\left\{c_2^{\mathrm{t}}\chi_k^M(x_k + s_k^{\mathrm{n}}), (1-\alpha^{\mathrm{n}})\Delta_k\right\}.
\end{aligned} \tag{3.23}$$

Our predicted reduction (3.16) becomes

$$\begin{aligned}
Pred_k(s_k, \rho_k) \quad = \quad & l(x_k, \lambda_k) + \frac{\rho_k}{2}\|c(x_k)\|^2 \\
& -l(x_k, \lambda_k) - \nabla_x l(x_k, \lambda_k)^T s - \tfrac{1}{2} s^T \nabla_x^2 l(x_k, \lambda_k) s \\
& -(\lambda_{k+1} - \lambda_k)^T(c(x_k) + \nabla c(x_k)^T s_k) - \frac{\rho_k}{2}\|c(x_k) + \nabla c(x_k)^T s_k\|^2,
\end{aligned}$$

which is identical to the predicted reduction in [11].

Our updating strategy for the penalty parameter is slightly different from that in [11]. We require

$$Pred_k(s_k, \rho_k) \geq \frac{\rho_k}{8}\left(\|c(x_k)\|^2 - \|c(x_k) + \nabla c(x_k)^T s_k^{\mathrm{n}}\|^2\right).$$

instead of

$$Pred_k(s_k, \rho_k) \geq \frac{\rho_k}{4}\left(\|c(x_k)\|^2 - \|c(x_k) + \nabla c(x_k)^T s_k\|^2\right)$$

(see [11], where also $\nabla c(x_k)^T s_k = \nabla c(x_k)^T s_k^{\mathrm{n}}$ holds). This technical difference is due to our relaxation (3.11) of the tangential subproblem equality constraints.

3.4 Global Convergence Theory

The convergence analysis of our algorithm is based on a careful extension of the convergence theory in [11].

To simplify the presentation of algorithm, we only increase the iteration count k when the iteration is successful. In the analysis that follows we need to show some properties of every trial step, not just the successful steps $\hat{s}_k$. Therefore, let Δ_k^j, $\hat{s}_k^j$ and ρ_k^j denote the quantities set by the main algorithm as it searches for an acceptable step. Thus, $\Delta_k^0 = \Delta_k$ at the first trial step of kth iteration, and $\hat{s}_k^0$ and ρ_k^0 are computed in steps 4 and 6 during the first trial iteration in step k. If the trial step $\hat{s}_k^j$ is acceptable, then $\hat{s}_k = \hat{s}_k^j$, $\rho_k = \rho_k^j$ and Δ_k^j is updated to become Δ_{k+1}.

3.4.1 Assumptions on the Problem and Models

Let $\Omega \subset \mathbb{R}^N$ be an open subset such that $x_k, x_k + s_k \in \Omega$ for all $k \in \mathbb{N}$. We make the following assumptions on the original problem.

A1. The functions f, c are twice continuously differentiable in Ω.

A2. The functions $f, \nabla f, \nabla^2 f, c, \nabla c^T, \nabla^2 c_i, i = 1, \cdots, m$ are bounded in Ω and the sequence $\{\lambda_k\}$ are bounded. i.e., there exist constants $k_g^c, k_H^c, k_H^l, k_\lambda > 0$ such that

$$
\begin{aligned}
\|\nabla c(x)^T\| &\leq k_g^c, \\
\|\nabla_x^2 l(x,\lambda)\| &\leq k_H^l, \\
\|\nabla^2 c_i(x)\| &\leq k_H^c, \; i = 1, \cdots, m
\end{aligned}
$$

for all $x \in \Omega$ and

$$\|\lambda_k\| \leq k_\lambda,$$

for all $k \in \mathbb{N}$.

Let $\hat{\Omega}_k \subset \mathbb{R}^{N_k}$ be open, convex subsets such that $\hat{x}_k, \hat{x}_k + \hat{s}_k \in \hat{\Omega}_k$ for all iterations $k \in \mathbb{N}$. We make the following assumptions on the model problems.

AM1. The functions m_k^l, m_k^c are twice continuously differentiable in $\hat{\Omega}_k$.

AM2. There exist constants $\kappa_m^l, \kappa_m^c, \kappa_g^l, \kappa_g^c, \kappa_H^l, \kappa_H^c, \kappa_\lambda^m > 0$ such that

$$
\begin{aligned}
\|m_k^l(\hat{x})\| &\leq \kappa_m^l, \\
\|m_k^c(\hat{x})\| &\leq \kappa_m^c, \\
\|\nabla m_k^l(\hat{x})\| &\leq \kappa_g^l, \\
\|\nabla m_k^c(\hat{x})\| &\leq \kappa_g^c, \\
\|\nabla^2 m_k^l(\hat{x})\| &\leq \kappa_H^l, \\
\|\nabla^2 (m_k^c)_i(\hat{x})\| &\leq \kappa_H^c, \; i = 1, \cdots, M_k,
\end{aligned}
$$

for all $\hat{x} \in \hat{\Omega}_k$ and all $k \in \mathbb{N}$, and

$$\|\hat{\lambda}_k\| \leq \kappa_\lambda^m,$$

for all $k \in \mathbb{N}$.

AM3. The matrices P_k, $k \in \mathbb{N}$, have full rank N_k and there exist a positive constant κ_{lp} such that $\|P_k \hat{x}\| \geq \kappa_{lp} \|\hat{x}\|$ for all $\hat{x} \in \mathbb{R}^{N_k}$ and for all $k \in \mathbb{N}$.

AM4. Let $s_k = P_k \hat{s}_k$, $\hat{x}_k = R_k x_k$. There exist $\kappa_{ubr1}, \kappa_{ubr2}, \kappa_{ubr3} > 0$ and $\gamma \in (0, 1]$ such that

$$
\begin{aligned}
&|Ared(s_k, \rho) - Pred_k(\hat{s}_k, \rho)| \\
&\leq \kappa_{ubr1} (\Delta_k)^\gamma \|s_k\| + \kappa_{ubr2} \rho (\Delta_k)^\gamma \|s_k\|^2 + \kappa_{ubr3} \rho (\Delta_k)^\gamma \|m_k^c(\hat{x}_k)\| \|s_k\|,
\end{aligned}
$$

for all $\rho > 0$, $\Delta_k > 0$ and all trial steps $\hat{s}_k$.

AM5. Let $\hat{x}_k = R_k x_k$. There exists a constant $\gamma_\chi > 0$ independent of k such that

$$\chi(x_k) \leq \gamma_\chi \chi_k^M(\hat{x}_k)$$

for all $k \in \mathbb{N}$.

AM6. Let $\hat{x}_k = R_k x_k$. There exists a constant $\gamma_c > 0$ independent of k such that

$$\|c(x_k)\| \leq \gamma_c \|m_k^c(\hat{x}_k)\|$$

for all $k \in \mathbb{N}$.

In the standard trust-region SQP method, Assumption AM4 with $\gamma = 1$ follows from A1, A2 and an application of the Taylor expansion, see, e.g., [11]. In our case, the models must be chosen properly to satisfy AM4 (as well as AM5, AM6) and the relaxation of AM4 to allow $\gamma \in (0, 1]$ is useful.

We will show later (see Lemma 3.4.1) that the following conditions imply AM4. The conditions AM7–AM10 will be useful in the context of the class of applications discussed in Chapter 4.

AM7. Let $\hat{x}_k = R_k x_k$. We assume $P_k^T c(x_k) = m_k^c(\hat{x}_k)$, $(P_k^\lambda)^T \nabla c(x_k)^T P_k = \nabla m_k^c(\hat{x}_k)^T$ for all $k \in \mathbb{N}$.

AM8. Let $\hat{x}_k = R_k x_k$. There exist constants $c_1 > 0$ and $\gamma \in (0, 1]$ independent of k such that

$$\left| \hat{s}_k^T \left(P_k^T \nabla c(x_k) \nabla c(x_k)^T P_k - \nabla m_k^c(\hat{x}_k) \nabla m_k^c(\hat{x}_k)^T \right) \hat{s}_k \right| \leq c_1 (\Delta_k)^\gamma \|\hat{s}_k\|^2$$

for all $k \in \mathbb{N}$ and all trial steps $\hat{s}_k$.

AM9. Let $\hat{x}_k = R_k x_k$. There exist constants $c_2 > 0$ and $\gamma \in (0, 1]$ independent of k such that

$$\left| c(x_k)^T \nabla c(x_k)^T P_k \hat{s}_k - m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k)^T \hat{s}_k \right| \leq c_2 (\Delta_k)^\gamma \| m_k^c(\hat{x}_k) \| \, \| \hat{s}_k \|$$

for all $k \in \mathbb{N}$ and all trial steps $\hat{s}_k$.

AM10. Let $\hat{x}_k = R_k x_k$. There exist constants $c_3 > 0$ and $\gamma \in (0, 1]$ independent of k such that

$$\| \nabla_x l(x_k, \lambda_k)^T P_k \hat{s}_k - \nabla m_k^l(\hat{x}_k) \hat{s}_k \| \leq c_3 (\Delta_k)^\gamma \| \hat{s}_k \|$$

for all $k \in \mathbb{N}$ and all trial steps $\hat{s}_k$.

With the SQP models specified in Section 3.3, the assumptions AM7-AM10 are satisfied with $\gamma = 1$.

Lemma 3.4.1 *If Assumptions A1, A2, AM1–AM3 and AM7–AM10 hold and if* $\lambda_{k+1} - \lambda_k = P_k^\lambda (\hat{\lambda}_{k+1} - \hat{\lambda}_k)$, *then AM4 is satisfied.*

Proof: Throughout the proof, we let $s_k = P_k \hat{s}_k$.

The definitions of $Ared(s_k, \rho)$ and $Pred_k(\hat{s}_k, \rho)$ imply

$$\begin{aligned}
& Ared(s_k, \rho) - Pred_k(\hat{s}_k, \rho) \\
= \; & l(x_k, \lambda_k) - l(x_k + s_k, \lambda_k) - (\lambda_{k+1} - \lambda_k)^T c(x_k + s_k) \\
& + \frac{\rho}{2} \left(\|c(x_k)\|^2 - \|c(x_k + s_k)\|^2 \right) \\
& - m_k^l(\hat{x}_k) + m_k^l(\hat{x}_k + \hat{s}_k) + (\hat{\lambda}_{k+1} - \hat{\lambda}_k)^T m_k^c(\hat{x}_k + \hat{s}_k) \\
& - \frac{\rho}{2} \left(\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k)\|^2 \right).
\end{aligned}$$

Applying Taylor expansion[1], it follows that

$$
\begin{aligned}
&|Ared(s_k,\rho) - Pred_k(\hat{s}_k,\rho)| \\
= \; & \Big| -\nabla_x l(x_k,\lambda_k)^T s_k - \tfrac{1}{2} s_k^T \nabla_x^2 l(x_k+\xi_1 s_k,\lambda_k) s_k \\
& -(\lambda_{k+1}-\lambda_k)^T \left(c(x_k) + \nabla c(x_k)^T s_k + \tfrac{1}{2} s_k^T \nabla^2 c(x_k+\xi_2 s_k) s_k\right) \\
& -\frac{\rho}{2}\Big(2c(x_k)^T \nabla c(x_k)^T s_k + c(x_k)^T s_k^T \nabla^2 c(x_k+\xi_2 s_k) s_k \\
& +\|\nabla c(x_k)^T s_k + \tfrac{1}{2} s_k^T \nabla^2 c(x_k+\xi_2 s_k) s_k\|^2\Big) \\
& +\nabla m_k^l(\hat{x}_k)^T \hat{s}_k + \tfrac{1}{2}(\hat{s}_k)^T \nabla^2 m_k^l(\hat{x}_k+\xi_3\hat{s}_k)\hat{s}_k \\
& +(\hat{\lambda}_{k+1}-\hat{\lambda}_k)^T \left(m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{s}_k + \tfrac{1}{2}(\hat{s}_k)^T \nabla^2 m_k^c(\hat{x}_k+\xi_4\hat{s}_k)\hat{s}_k\right) \\
& +\frac{\rho}{2}\Big(2m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k)^T \hat{s}_k + m_k^c(\hat{x}_k)^T (\hat{s}_k)^T \nabla^2 m_k^c(\hat{x}_k+\xi_4\hat{s}_k)\hat{s}_k \\
& +\|\nabla m_k^c(\hat{x}_k)^T \hat{s}_k + \tfrac{1}{2}(\hat{s}_k)^T \nabla^2 m_k^c(\hat{x}_k+\xi_4\hat{s}_k)\hat{s}_k\|^2\Big)\Big|.
\end{aligned}
$$

[1]Here and in the following we use $s_k^T \nabla^2 c(x_k+\xi s_k)s_k$ as a short hand for the vector function whose ith component function is $s_k^T \nabla^2 c_i(x_k+\xi_i s_k)s_k$ where ξ_i is the ith element of $\xi \in \mathbb{R}^m$ and $\hat{s}_k^T \nabla^2 m_k^c(\hat{x}_k+\xi\hat{s}_k)\hat{s}_k$ for the vector function whose ith component function is $\hat{s}_k^T \nabla^2 (m_k^c)_i(\hat{x}_k+\xi\hat{s}_k)\hat{s}_k$, where ξ_i is the ith element of $\xi \in \mathbb{R}^{M_k}$

Hence,

$$
\begin{aligned}
&|Ared(s_k,\rho) - Pred_k(\hat{s}_k,\rho)| \\
&\quad \leq \left|\nabla_x l(x_k,\lambda_k)^T s_k - \nabla m_k^l(\hat{x}_k)^T \hat{s}_k\right| \\
&\qquad + \tfrac{1}{2}\left|s_k^T \nabla_x^2 l(x_k+\xi_1 s_k,\lambda_k)s_k - (\hat{s}_k)^T \nabla^2 m_k^l(\hat{x}_k+\xi_3\hat{s}_k)\hat{s}_k\right| \\
&\qquad + \left|(\lambda_{k+1}-\lambda_k)^T\left(c(x_k)+\nabla c(x_k)^T s_k\right) - (\hat{\lambda}_{k+1}-\hat{\lambda}_k)^T\left(m_k^c(\hat{x}_k)+\nabla m_k^c(\hat{x}_k)^T\hat{s}_k\right)\right| \\
&\qquad + \tfrac{1}{2}\left|(\lambda_{k+1}-\lambda_k)^T s_k^T \nabla^2 c(x_k+\xi_2 s_k)s_k - (\hat{\lambda}_{k+1}-\hat{\lambda}_k)^T(\hat{s}_k)^T\nabla^2 m_k^c(\hat{x}_k+\xi_4\hat{s}_k)\hat{s}_k\right| \\
&\qquad + \rho\left|c(x_k)^T\nabla c(x_k)^T s_k - m_k^c(\hat{x}_k)^T\nabla m_k^c(\hat{x}_k)^T\hat{s}_k\right| \\
&\qquad + \frac{\rho}{2}\left|c(x_k)^T s_k^T\nabla^2 c(x_k+\xi_2 s_k)s_k - m_k^c(\hat{x}_k)^T(\hat{s}_k)^T\nabla^2 m_k^c(\hat{x}_k+\xi_4\hat{s}_k)\hat{s}_k\right| \\
&\qquad + \frac{\rho}{2}\Big|\|\nabla c(x_k)^T s_k + \tfrac{1}{2}s_k^T\nabla^2 c(x_k+\xi_2 s_k)s_k\|^2 \\
&\qquad - \|\nabla m_k^c(\hat{x}_k)^T\hat{s}_k + \tfrac{1}{2}(\hat{s}_k)^T\nabla^2 m_k^c(\hat{x}_k+\xi_4\hat{s}_k)\hat{s}_k\|^2\Big|.
\end{aligned}
$$

Using Assumptions A2, AM2, the above inequality yields

$$
\begin{aligned}
&|Ared(s_k,\rho) - Pred_k(\hat{s}_k,\rho)| \\
&\quad \leq \left|\nabla_x l(x_k,\lambda_k)^T s_k - \nabla m_k^l(\hat{x}_k)^T \hat{s}_k\right| + \tfrac{1}{2}\left(k_H^l\|s_k\|^2 + \kappa_H^l\|\hat{s}_k\|^2\right) \\
&\qquad + \left|(\lambda_{k+1}-\lambda_k)^T\left(c(x_k)+\nabla c(x_k)^T s_k\right) - (\hat{\lambda}_{k+1}-\hat{\lambda}_k)^T\left(m_k^c(\hat{x}_k)+\nabla m_k^c(\hat{x}_k)^T\hat{s}_k\right)\right| \\
&\qquad + \left(k_\lambda k_H^c\|s_k\|^2 + \kappa_\lambda^m\kappa_H^c\|\hat{s}_k\|^2\right) + \rho\left|c(x_k)^T\nabla c(x_k)^T s_k - m_k^c(\hat{x}_k)^T\nabla m_k^c(\hat{x}_k)^T\hat{s}_k\right| \\
&\qquad + \frac{\rho}{2}\Big(k_H^c\|c(x_k)\|\|s_k\|^2 + \kappa_H^c\|m_k^c(\hat{x}_k)\|\|\hat{s}_k\|^2 + \|\nabla c(x_k)^T s_k\|^2 - \|\nabla m_k^c(\hat{x}_k)^T\hat{s}_k\|^2 \\
&\qquad + k_g^c k_H^c\|s_k\|^3 + \kappa_g^c\kappa_H^c\|\hat{s}_k\|^3 + (k_H^c)^2\|s_k\|^4 + (\kappa_H^c)^2\|\hat{s}_k\|^4\Big).
\end{aligned}
$$

Assumption AM3 implies that $\|\hat{s}_k\| \leq \frac{1}{\kappa_{lp}}\|s_k\|$. Therefore

$$
\begin{aligned}
&|Ared(s_k,\rho) - Pred_k(\hat{s}_k,\rho)| \\
&\quad\leq \left|\nabla_x l(x_k,\lambda_k)^T s_k - \nabla m_k^l(\hat{x}_k)^T \hat{s}_k\right| + \rho\left|c(x_k)^T \nabla c(x_k)^T s_k - m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k)^T \hat{s}_k\right| \\
&\qquad + \left|(\lambda_{k+1}-\lambda_k)^T\left(c(x_k) + \nabla c(x_k)^T s_k\right) - (\hat{\lambda}_{k+1}-\hat{\lambda}_k)^T\left(m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{s}_k\right)\right| \\
&\qquad + \frac{\rho}{2}\left(k_H^c\|c(x_k)\|\|s_k\|^2 + \kappa_H^c\|m_k^c(\hat{x}_k)\|\|\hat{s}_k\|^2 + \|\nabla c(x_k)^T s_k\|^2 - \|\nabla m_k^c(\hat{x}_k)^T \hat{s}_k\|^2\right) \\
&\qquad + \tfrac{1}{2}\left(k_H^l + \kappa_H^l \kappa_{lp}^{-2} + 2k_\lambda k_H^c + 2\kappa_\lambda^m \kappa_H^c \kappa_{lp}^{-2}\right)\|s_k\|^2 \\
&\qquad + \frac{\rho}{2}\left[\left(k_g^c k_H^c + \kappa_g^c \kappa_H^c \kappa_{lp}^{-3}\right)\|s_k\|^3 + \left((k_H^c)^2 + (\kappa_H^c)^2 \kappa_{lp}^{-4}\right)\|s_k\|^4\right].
\end{aligned}
$$

Defining

$$
\tau_1 = \tfrac{1}{2}\left(k_H^l + \kappa_H^l \kappa_{lp}^{-2} + 2k_\lambda k_H^c + 2\kappa_\lambda^m \kappa_H^c \kappa_{lp}^{-2}\right)
$$

and

$$
\tau_2 = k_g^c k_H^c + \kappa_g^c \kappa_H^c \kappa_{lp}^{-3}, \quad \tau_3 = (k_H^c)^2 + (\kappa_H^c)^2 \kappa_{lp}^{-4}
$$

with $(\lambda_{k+1} - \lambda_k) = P_k^\lambda(\hat{\lambda}_{k+1} - \hat{\lambda}_k)$ and $s_k = P_k \hat{s}_k$ yield

$$
\begin{aligned}
&|Ared(s_k,\rho) - Pred_k(\hat{s}_k,\rho)| \\
&\quad\leq \left|\left(P_k^T \nabla_x l(x_k,\lambda_k) - \nabla m_k^l(\hat{x}_k)\right)^T \hat{s}_k\right| \\
&\qquad + \rho\left|\left(c(x_k)^T \nabla c(x_k)^T P_k - m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k)^T\right)\hat{s}_k\right| \\
&\qquad + \left|(\hat{\lambda}_{k+1}-\hat{\lambda}_k)^T\left[(P_k^\lambda)^T c(x_k) - m_k^c(\hat{x}_k) - \left((P_k^\lambda)^T \nabla c(x_k)^T P_k - \nabla m_k^c(\hat{x}_k)^T\right)\hat{s}_k\right]\right| \\
&\qquad + \frac{\rho}{2}\Big(k_H^c\|c(x_k)\|\|s_k\|^2 + \kappa_H^c\|m_k^c(\hat{x}_k)\|\|\hat{s}_k\|^2 \\
&\qquad + (\hat{s}_k)^T\left(P_k^T \nabla c(x_k)\nabla c(x_k)^T P_k - \nabla m_k^c(\hat{x}_k)\nabla m_k^c(\hat{x}_k)^T\right)\hat{s}_k\Big) \\
&\qquad + \tau_1\|s_k\|^2 + \frac{\tau_2\rho}{2}\|s_k\|^3 + \frac{\tau_3\rho}{2}\|s_k\|^4.
\end{aligned}
$$

Assumptions AM7–AM10 give

$$
\begin{aligned}
&|Ared(s_k;\rho) - Pred_k(\hat{s}_k,\rho)| \\
&\leq\ c_3(\Delta_k)^\gamma\|\hat{s}_k\| + \rho c_2\|m_k^c(\hat{x}_k)\|(\Delta_k)^\gamma\|\hat{s}_k\| \\
&\quad +\frac{\rho}{2}\left(\left(k_H^c\gamma_c + \frac{\kappa_H^c}{\kappa_{lp}^2}\right)\|m_k^c(\hat{x}_k)\|\|s_k\|^2 + c_1\kappa_{lp}^{-2}(\Delta_k)^\gamma\|s_k\|^2\right) \\
&\quad +\tau_1\|s_k\|^2 + \frac{\tau_2\rho}{2}\|s_k\|^3 + \frac{\tau_3\rho}{2}\|s_k\|^4.
\end{aligned}
$$

Assumption AM4 now follows with $\kappa_{ubr1} = \tau_1 + \kappa_{lp}^{-1}c_3$, $\kappa_{ubr2} = \frac{1}{2}(c_1\kappa_{lp}^{-2} + \tau_2\Delta_{max} + \tau_3\Delta_{max}^2)$, and $\kappa_{ubr3} = c_2/\kappa_{lp} + k_H^c\gamma_c/2 + \kappa_H^c/(2\kappa_{lp}^2)$. □

3.4.2 Properties of Predicted Reduction

In this section, the properties of the predicted decrease in the merit function obtained by the trial step is discussed.

The first lemma is an easy consequence of Assumptions AM2 and AM4 and the fact that $\|s_k\| \leq \Delta_k$.

Lemma 3.4.2 *If Assumptions AM2 and AM4 hold, then for any $\rho \geq 1$, we have*

$$|Ared(s_k;\rho) - Pred_k(\hat{s}_k,\rho)| \leq \kappa_{ubr}\rho(\Delta_k)^\gamma\|s_k\| \leq \kappa_{ubr}\rho\Delta_k\|s_k\|^\gamma, \tag{3.24}$$

for all trial steps $\hat{s}_k$, where $s_k = P_k\hat{s}_k$ and $\kappa_{ubr} = \kappa_{ubr1} + \kappa_{ubr2}\Delta_{max} + \kappa_{ubr3}\kappa_m^c$.

The following two lemmas are related to the updating strategy given by Algorithm 3.2.1. This strategy ensures that the merit function is predicted to decrease by a fraction of Cauchy decrease in the model constraint at each iteration.

Lemma 3.4.3 *For any $\rho > 0$, we have*

$$\begin{aligned} Pred_k(\hat{s}^j_k, \rho) &\geq \widetilde{Pred}_k(\hat{s}^j_k, \rho) \\ &\stackrel{\text{def}}{=} m^l_k(\hat{x}_k) - m^l_k(\hat{x}_k + \hat{s}^j_k) - (\hat{\lambda}^j_{k+1} - \hat{\lambda}^j_k)^T m^c_k(\hat{x}_k + \hat{s}^j_k) \\ &\quad + \frac{\rho^j_k}{4} \left(\|m^c_k(\hat{x}_k)\|^2 - \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 \right). \end{aligned} \tag{3.25}$$

Proof: The predicted reduction can be rewritten as

$$\begin{aligned} Pred_k(\hat{s}^j_k, \rho) &= m^l_k(\hat{x}_k) + \frac{\rho}{2} \left(\|m^c_k(\hat{x}_k)\|^2 - \mu_k \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 \right) \\ &\quad + \frac{\rho}{2} \left(\mu_k \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 - \|m^c_k(\hat{x}_k + \hat{s}^j_k)\|^2 \right) \\ &\quad - m^l_k(\hat{x}_k + \hat{s}^j_k) - (\hat{\lambda}^j_{k+1} - \hat{\lambda}^j_k)^T m^c_k(\hat{x}_k + \hat{s}^j_k). \end{aligned} \tag{3.26}$$

By (3.11) we have

$$\mu_k \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 - \|m^c_k(\hat{x}_k + \hat{s}^j_k)\|^2 \geq 0 \tag{3.27}$$

and the definition (3.12) of μ_k implies

$$\|m^c_k(\hat{x}_k)\|^2 - \mu_k \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 \geq \tfrac{1}{2} \left(\|m^c_k(\hat{x}_k)\|^2 - \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 \right). \tag{3.28}$$

The desired inequality (3.25) is obtained if we apply (3.27), (3.28) to (3.26). □

Lemma 3.4.4 *For the penalty parameter determined in step 6 of Algorithm* 3.2.1, *the predicted reduction satisfies*

$$Pred_k(\hat{s}^j_k, \rho^j_k) \geq \frac{\rho^j_k}{8} \left(\|m^c_k(\hat{x}_k)\|^2 - \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n},j}_k)\|^2 \right). \tag{3.29}$$

Proof: This inequality follows from the rule for the update of the penalty parameter, step 6 of Algorithm 3.2.1, and Lemma 3.4.3. □

The penalty parameter is increased only if the predicted reduction does not achieve a fraction of decrease in the model constraint. The following lemmas show that the predicted reduction is independent of penalty parameter ρ_k as long as the iterate $\hat{x}_k$ is close enough to the feasible region.

Lemma 3.4.5 *Under Assumptions AM1, AM2 there exist a constant*[2] $\kappa_{llq} > 0$ *independent of* k *such that*

$$m_k^l(\hat{x}_k) - m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) - (\hat{\lambda}_{k+1}^j - \hat{\lambda}_k^j)^T m_k^c(\hat{x}_k + \hat{s}_k^j) \;\geq -\kappa_{llq}\|m_k^c(\hat{x}_k)\|. \tag{3.30}$$

Proof: The boundedness of $\hat{\lambda}_{k+1}^j$ and (3.11) imply

$$\begin{aligned}
& m_k^l(\hat{x}_k) - m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) - (\hat{\lambda}_{k+1}^j - \hat{\lambda}_k^j)^T m_k^c(\hat{x}_k + \hat{s}_k^j) \\
\geq\; & -\|\nabla m_k^l(\hat{x}_k + \xi_1 \hat{s}_k^{\mathrm{n},j})\|\|\hat{s}_k^{\mathrm{n},j}\| - \sqrt{2}\kappa_\lambda^m(\|m_k^c(\hat{x}_k)\| + \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n},j})\|) \\
\geq\; & -\kappa_g^l\|\hat{s}_k^{\mathrm{n},j}\| - \kappa_\lambda^m\sqrt{2}(2\|m_k^c(\hat{x}_k)\| + \|\nabla m_k^c(\hat{x}_k + \xi_2\hat{s}_k^{\mathrm{n},j})\|\|\hat{s}_k^{\mathrm{n},j}\|) \\
\geq\; & -(\kappa_g^l + \kappa_\lambda^m\kappa_g^c\sqrt{2})\|\hat{s}_k^{\mathrm{n},j}\| - 2\sqrt{2}\kappa_\lambda^m\|m_k^c(\hat{x}_k)\|,
\end{aligned}$$

where $\xi_1, \xi_2 \in [0, 1]$. The last inequality and (3.6) imply the desired estimate if we set

$$\kappa_{llq} = (\kappa_g^l + \kappa_\lambda^m \kappa_g^c \sqrt{2})c_3^{\mathrm{n}} + 2\sqrt{2}\kappa_\lambda^m.$$

□

[2] κ_{llq} stands for Lower bound for difference between Lagrangian and Lagrangian on Quasi-normal step.

Lemma 3.4.6 *Let Assumptions AM1, AM2 be satisfied. For any $\rho > 0$, the predicted reduction obeys*

$$\begin{array}{rcl} Pred_k(\hat{s}_k^j, \rho) & \geq & \widetilde{Pred}_k(\hat{s}_k^j, \rho) \\ & \geq & c_1^{\mathrm{t}} \chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) \min\{c_2^{\mathrm{t}} \chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}), (1-\alpha^{\mathrm{n}})\Delta_k^j\} \\ & & -\kappa_{llq} \|m_k^c(x_k)\| + \frac{\rho}{4}\left(\|m_k^c(x_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n},j})\|^2\right) \end{array} \quad (3.31)$$

where κ_{llq} is given in Lemma 3.4.5.

Proof: From lemma 3.4.3, we have

$$Pred_k(\hat{s}_k^j, \rho) \;\geq\; \widetilde{Pred}_k(\hat{s}_k^j, \rho) \quad (3.32)$$

$$\begin{array}{rl} \geq & m_k^l(\hat{x}_k) - m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) - (\hat{\lambda}_{k+1}^j - \hat{\lambda}_k^j)^T m_k^c(\hat{x}_k + \hat{s}_k^j) \\ & +\frac{\rho}{4}\left(\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n},j})\|^2\right) \\ & +m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) - m_k^l(\hat{x}_k + \hat{s}_k^j). \end{array} \quad (3.33)$$

The inequality (3.31) is obtained if we apply (3.30) and (3.13) to (3.32). □

Lemma 3.4.7 *Let Assumptions AM1, AM2 hold and let the criticality measure satisfy*

$$\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) \geq \chi_k^M(\hat{x}_k) - \kappa_\chi \|\hat{s}_k^{\mathrm{n},j}\| \quad (3.34)$$

with some $\kappa_\chi > 0$, independent of k. If

$$\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$$

and if

$$\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^{\mathrm{n}})\Delta_k^j,$$

where σ satisfies

$$\begin{aligned} \sigma \quad \leq \quad \min\Big\{ & \frac{\epsilon}{3(1-\alpha^{\mathrm{n}})\Delta_{max}}, \frac{\epsilon}{3\kappa_\chi c_3^{\mathrm{n}}(1-\alpha^{\mathrm{n}})\Delta_{max}}, \\ & \frac{c_1^{\mathrm{t}}\epsilon}{6(1-\alpha^{\mathrm{n}})\kappa_{llq}} \min\{\frac{c_2^{\mathrm{t}}\epsilon}{3\Delta_{max}}, (1-\alpha^{\mathrm{n}})\} \Big\}, \end{aligned} \tag{3.35}$$

then for any $\rho > 0$,

$$\begin{aligned} Pred_k(\hat{s}_k^j, \rho) \quad & \geq \quad \widetilde{Pred}_k(\hat{s}_k^j, \rho) \\ & \geq \quad \tfrac{1}{2}c_1^{\mathrm{t}}\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) \min\left\{c_2^{\mathrm{t}}\chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}), (1-\alpha^{\mathrm{n}})\Delta_k^j\right\} \\ & \quad + \frac{\rho^j}{4}\left(\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n},j})\|^2\right). \end{aligned} \tag{3.36}$$

Proof: The inequalities $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^{\mathrm{n}})\Delta_k^j$ and $\sigma \leq \frac{\epsilon}{3(1-\alpha^{\mathrm{n}})\Delta_{max}}$ imply $\|m_k^c(\hat{x}_k)\| \leq \frac{\epsilon}{3}$. Thus, $\chi_k^M(\hat{x}_k) > \frac{2}{3}\epsilon$.

The conditions (3.34), (3.6) and (3.35) yield

$$\begin{aligned} \chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) \quad & \geq \quad \chi_k^M(\hat{x}_k) - \kappa_\chi\|\hat{s}_k^{\mathrm{n},j})\| \\ & \geq \quad \frac{2}{3}\epsilon - \kappa_\chi c_3^{\mathrm{n}}\|m_k^c(\hat{x}_k)\| \\ & \geq \quad \frac{2}{3}\epsilon - \kappa_\chi c_3^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}})\Delta_k^j \\ & \geq \quad \frac{1}{3}\epsilon. \end{aligned} \tag{3.37}$$

Combining the previous inequality and Lemma 3.4.6, we obtain

$$
\begin{aligned}
Pred_k(\hat{s}_k^j,\rho) &\geq \widetilde{Pred}_k(\hat{s}_k^j,\rho) \\
&\geq \tfrac{1}{2}c_1^{\mathrm{t}}\chi_k^M(\hat{x}_k+\hat{s}_k^{\mathrm{n},j})\min\{c_2^{\mathrm{t}}\chi_k^M(\hat{x}_k+\hat{s}_k^{\mathrm{n},j}),(1-\alpha^{\mathrm{n}})\Delta_k^j\} \\
&\quad +\frac{1}{6}c_1^{\mathrm{t}}\epsilon\min\{\frac{c_2^{\mathrm{t}}}{3}\epsilon,(1-\alpha^{\mathrm{n}})\Delta_k^j\} \\
&\quad -\kappa_{llq}\sigma(1-\alpha^{\mathrm{n}})\Delta_k^j+\frac{\rho}{4}\left(\|m_k^c(\hat{x}_k)\|^2-\|m_k^c(\hat{x}_k+\hat{s}_k^{\mathrm{n},j})\|^2\right) \\
&\geq \tfrac{1}{2}c_1^{\mathrm{t}}\chi_k^M(\hat{x}_k+\hat{s}_k^{\mathrm{n},j})\min\{c_2^{\mathrm{t}}\chi_k^M(\hat{x}_k+\hat{s}_k^{\mathrm{n},j}),(1-\alpha^{\mathrm{n}})\Delta_k^j\} \\
&\quad +\frac{1}{6}c_1^{\mathrm{t}}\epsilon\Delta_k^j\min\{\frac{c_2^{\mathrm{t}}}{3\Delta_{max}}\epsilon,1-\alpha^{\mathrm{n}}\} \\
&\quad -\kappa_{llq}\sigma(1-\alpha^{\mathrm{n}})\Delta_k^j+\frac{\rho}{4}\left(\|m_k^c(\hat{x}_k)\|^2-\|m_k^c(\hat{x}_k+\hat{s}_k^{\mathrm{n},j})\|^2\right)
\end{aligned}
$$

Since

$$
\sigma \leq \frac{c_1^{\mathrm{t}}\epsilon}{6\kappa_{llq}(1-\alpha^{\mathrm{n}})}\min\{\frac{c_2^{\mathrm{t}}\epsilon}{3\Delta_{max}},1-\alpha^{\mathrm{n}}\}
$$

it follows that

$$
\begin{aligned}
Pred_k(\hat{s}_k^j,\rho) &\geq \widetilde{Pred}_k(\hat{s}_k^j,\rho) \\
&\geq \tfrac{1}{2}c_1^{\mathrm{t}}\chi_k^M(\hat{x}_k+\hat{s}_k^{\mathrm{n},j})\min\{c_2^{\mathrm{t}}\chi_k^M(\hat{x}_k+\hat{s}_k^{\mathrm{n},j}),(1-\alpha^{\mathrm{n}})\Delta_k^j\} \\
&\quad +\frac{\rho^j}{4}\left(\|m_k^c(\hat{x}_k)\|^2-\|m_k^c(\hat{x}_k+\hat{s}_k^{\mathrm{n},j})\|^2\right).
\end{aligned}
$$

This completes the proof. $\square$

Lemma 3.4.7 asserts that if $\chi_k^M(\hat{x}_k)+\|m_k^c(\hat{x}_k)\|>\epsilon$. ρ_k is increased only when $\|m_k^c(\hat{x}_k)\|>\sigma(1-\alpha^{\mathrm{n}})\Delta_k^j$ with σ defined in (3.35).

The previous lemma imposed condition (3.34) on the criticality measure. This condition is not very restrictive and is often satisfied if the criticality measure is given by (3.8). This is the content of the following result.

Lemma 3.4.8 *Let Assumptions AM1, AM2 hold and let $Z_k^c(\hat{x})$ be a matrix whose columns span the null-space of $\nabla m_k^c(\hat{x})^T$. If there exist $\kappa_z^c > 0$ and $L_z > 0$ such that $\|Z_k^c(\hat{x})\| \le \kappa_z^c$ and*

$$\|Z_k^c(\hat{x}_1) - Z_k^c(\hat{x}_2)\| \le L_z \|\hat{x}_1 - \hat{x}_2\|$$

for all $\hat{x}, \hat{x}_1, \hat{x}_2 \in \hat{\Omega}_k$ and all $k \in \mathbb{N}$, then the criticality measure

$$\chi_k^M(\hat{x}_k) = \|Z_k^c(\hat{x}_k)^T \nabla m_k^l(\hat{x}_k)\|$$

satisfies (3.34) *with*

$$\kappa_\chi = L_z \kappa_g^l + \kappa_z^c \kappa_H^l.$$

Proof: Since

$$\begin{aligned}
& Z_k^c(\hat{x}_k + \hat{s}_k^{n,j})^T \nabla m_k^l(\hat{x}_k + \hat{s}_k^{n,j}) - Z_k^c(\hat{x}_k)^T \nabla m_k^l(\hat{x}_k) \\
= \;& (Z_k^c(\hat{x}_k + \hat{s}_k^{n,j}) - Z_k^c(\hat{x}_k))^T \nabla m_k^l(\hat{x}_k + \hat{s}_k^{n,j}) + Z_k^c(\hat{x}_k)^T (\nabla m_k^l(\hat{x}_k + \hat{s}_k^{n,j}) - \nabla m_k^l(\hat{x}_k)).
\end{aligned}$$

The assumptions on $Z_k^c(\hat{x})$ imply

$$\begin{aligned}
& \|Z_k^c(\hat{x}_k + \hat{s}_k^{n,j})^T \nabla m_k^l(\hat{x}_k + \hat{s}_k^{n,j})\| \\
\ge \;& \|Z_k^c(\hat{x}_k)^T \nabla m_k^l(\hat{x}_k)\| - L_z \|s_k^{n,j}\| \|\nabla m_k^l(\hat{x}_k + \hat{s}_k^{n,j})\| - \|Z_k^c(\hat{x}_k)^T \nabla^2 m_k^l(\hat{x}_k + \xi_z \hat{s}_k^{n,j}) \hat{s}_k^{n,j}\| \\
\ge \;& \|Z_k^c(\hat{x}_k)^T \nabla m_k^l(\hat{x}_k)\| - (L_z \kappa_g^l + \kappa_z^c \kappa_H^l) \|\hat{s}_k^{n,j}\|
\end{aligned}$$

where $\xi_z \in [0,1]$. □

Lemma 3.4.9 *Let the assumptions of Lemma* 3.4.7 *hold. If* $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$ *and* $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^n)\Delta_k^j$*, where* σ *is defined as in Lemma* 3.4.7 *then*

$$Pred_k(\hat{s}_k^j, \rho_k^j) \geq \kappa_{lpr}\Delta_k^j \tag{3.38}$$

where[3] $\kappa_{lpr} > 0$ *is a constant which depends on* ϵ*, but is independent of* k.

Proof: Using (3.37) in (3.36) and (3.5) we obtain

$$\begin{aligned} Pred_k(\hat{s}_k^j, \rho_k^j) &\geq \frac{1}{6}c_1^t \epsilon \min\{\frac{c_2^t}{3}\epsilon, (1-\alpha^n)\Delta_k^j\} \\ &\geq \frac{1}{6}c_1^t \epsilon \Delta_k^j \min\{\frac{c_2^t}{3\Delta_{max}}\epsilon, 1-\alpha^n\}. \end{aligned}$$

The last inequality implies the desired result if we set

$$\kappa_{lpr} = \frac{1}{6}c_1^t \epsilon \min\{\frac{c_2^t}{3\Delta_{max}}\epsilon, 1-\alpha^n\}.$$

□

3.4.3 Behavior of the Penalty Parameter

This section discusses the behavior of the penalty parameter. Algorithm 3.2.1 generates a nondecreasing sequence $\{\rho_k^i\}$ of the penalty parameters. The main result of this section is that this sequence is bounded.

[3] κ_{lpr} stands for Lower bound on Predicted Reduction

Lemma 3.4.10 *Let Assumptions AM1, AM2, AM3 hold and let the criticality measure satisfy* (3.34). *Let k, j be any pair of indices such that ρ_k^j is increased at jth trial step of the kth iteration. If $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$, then there exists*[4] $\kappa_{uptr} > 0$ *which depends on ϵ, but is independent of k and j such that*

$$\rho_k^j \Delta_k^j \quad \leq \kappa_{uptr}. \tag{3.39}$$

Proof: If ρ_k^j is increased at the jth trial step of kth iteration, then the rule for the update of the penalty parameter, step 6 of Algorithm 3.2.1, yields

$$\rho_k^j = \frac{8[m_k^l(\hat{x}_k + \hat{s}_k^j) + (\hat{\lambda}_{k+1}^j - \hat{\lambda}_k^j)^T m_k^c(\hat{x}_k + \hat{s}_k^j) - m_k^l(\hat{x}_k)]}{\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{n,j})\|^2} + \beta$$

or, equivalently,

$$\begin{aligned}
&\frac{\rho_k^j}{8}[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{n,j})\|^2] \\
= \;& m_k^l(\hat{x}_k + \hat{s}_k^j) + (\hat{\lambda}_{k+1}^j - \hat{\lambda}_k^j)^T m_k^c(\hat{x}_k + \hat{s}_k^j) - m_k^l(\hat{x}_k) \\
& + \frac{\beta}{8}[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{n,j})\|^2] \\
= \;& m_k^l(\hat{x}_k + \hat{s}_k^i) - m_k^l(\hat{x}_k + \hat{s}_k^{n,j}) \\
& - m_k^l(\hat{x}_k) + m_k^l(\hat{x}_k + \hat{s}_k^{n,j}) + (\hat{\lambda}_{k+1}^j - \hat{\lambda}_k^j)^T m_k^c(\hat{x}_k + \hat{s}_k^i) \\
& + \frac{\beta}{8}[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{n,i})\|^2]
\end{aligned} \tag{3.40}$$

[4] κ_{uptr} stands for Upper bound for Penalty parameter and Trust-region Radius

If we apply (3.5) to the left hand side of (3.40), (3.13) and Lemma 3.4.5 to the right hand side of (3.40), then we obtain

$$
\begin{aligned}
&\frac{\rho_k^j}{8} c_1^{\mathrm{n}} \|m_k^c(\hat{x}_k)\| \min\{c_2^{\mathrm{n}} \|m_k^c(\hat{x}_k)\|, \alpha^{\mathrm{n}}\Delta_k^j\} \\
\leq\;& -c_1^{\mathrm{t}} \chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}) \min\{c_2^{\mathrm{t}} \chi_k^M(\hat{x}_k + \hat{s}_k^{\mathrm{n},j}), (1-\alpha^{\mathrm{n}})\Delta_k^j\} \\
&+\kappa_{llq}\|m_k^c(\hat{x}_k)\| - \frac{\beta}{4}(\nabla m_k^c(\hat{x}_k + \xi \hat{s}_k^{\mathrm{n},j}) m_k^c(\hat{x}_k))^T \hat{s}_k^{\mathrm{n},j} - \frac{\beta}{8}\|\nabla m_k^c(\hat{x}_k + \xi \hat{s}_k^{\mathrm{n},j})^T \hat{s}_k^{\mathrm{n},j}\|^2 \\
\leq\;& \kappa_{llq}\|m_k^c(\hat{x}_k)\| - \frac{\beta}{4}(\nabla m_k^c(\hat{x}_k + \xi \hat{s}_k^{\mathrm{n},j}) m_k^c(\hat{x}_k))^T \hat{s}_k^{\mathrm{n},j} \\
\leq\;& \kappa_{llq}\|m_k^c(\hat{x}_k)\| + \frac{\beta}{4}\|\nabla m_k^c(\hat{x}_k + \xi \hat{s}_k^{\mathrm{n},j})\| \|m_k^c(\hat{x}_k)\| \|\hat{s}_k^{\mathrm{n},j}\| \\
\leq\;& \left(\kappa_{llq} + \frac{\beta}{4}\|\nabla m_k^c(\hat{x}_k + \xi \hat{s}_k^{\mathrm{n},j})\| \|\hat{s}_k^{\mathrm{n},j}\|\right) \|m_k^c(\hat{x}_k)\|.
\end{aligned}
$$

The last inequality implies

$$
\begin{aligned}
&\frac{\rho_k^j}{8} c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}} \|m_k^c(\hat{x}_k)\|, \alpha^{\mathrm{n}}\Delta_k^j\} \\
\leq\;& \kappa_{llq} + \frac{\beta}{4}\|\nabla m_k^c(\hat{x}_k + \xi \hat{s}_k^{\mathrm{n},j})\| \|\hat{s}_k^{\mathrm{n},j}\|. \qquad (3.41)
\end{aligned}
$$

Since the penalty parameter increases we have $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^j$ (otherwise, Lemma 3.4.7 implies that $Pred_k(\hat{s}_k^j, \rho_k^j) \geq (\rho_k^j/8)[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n},j})\|^2]$). Therefore (3.41) implies

$$
\rho_k^j \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}})\Delta_k^j, \alpha^{\mathrm{n}}\Delta_k^j\} \leq (8\kappa_{llq} + 2\beta\kappa_g^c\kappa_{lp}\Delta_{max})/c_1^{\mathrm{n}}.
$$

The assertion follows with

$$
\kappa_{uptr} = \frac{8\kappa_{llq} + 2\beta k_g^c \Delta_{max}}{c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}.
$$

$\square$

Lemma 3.4.11 *Let the Assumptions A1, A2, AM1–AM4 hold and let the criticality measure satisfy* (3.34). *If the penalty parameter is increased at the jth trial step of the kth iteration and if $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$, then there exists a $\Delta_* > 0$ which depends on ϵ but is independent of k such that*

$$\Delta_k^j \geq \Delta_*. \tag{3.42}$$

Proof: If we are at the first trial step of the iteration k, i.e., if $j = 0$, then the rule in (3.17) guarantees that

$$\Delta_k^0 \geq \Delta_{min}. \tag{3.43}$$

Thus we can restrict our attention to the case where $j \geq 1$. Since the penalty parameter is increased at the jth trial step, Lemma 3.4.7 implies $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^j$. We consider the following two cases:

i. $\|m_k^c(\hat{x}_k)\| > \sigma(1 - \alpha^{\mathrm{n}})\Delta_k^i$ for all $i = 0, \cdots, j$,

ii. $\|m_k^c(\hat{x}_k)\| > \sigma(1 - \alpha^{\mathrm{n}})\Delta_k^i$ does not hold for some $0 \leq i \leq j$.

i. Assume that $\|m_k^c(\hat{x}_k)\| > \sigma(1 - \alpha^{\mathrm{n}})\Delta_k^i$ for all $i = 0, \cdots, j$. From (3.24), we have

$$|Ared(s_k^i, \rho_k^i) - Pred_k(\hat{s}_k^i, \rho_k^i)| \leq \kappa_{ubr}\rho_k^i\Delta_k^i\|s_k^i\|^{\gamma}.$$

Since $\|m_k^c(\hat{x}_k)\| > \sigma(1 - \alpha^{\mathrm{n}})\Delta_k^i$, the rule for updating ρ_k^i and (3.5) imply

$$\begin{aligned} Pred_k(\hat{s}_k^i, \rho_k^i) &\geq \frac{\rho_k^i}{8}[\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n},i})\|^2] \\ &\geq \frac{\rho_k^i}{8}c_1^{\mathrm{n}}\|m_k^c(\hat{x}_k)\| \min\{c_2^{\mathrm{n}}\sigma(1 - \alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}\Delta_k^i. \end{aligned}$$

Thus

$$\frac{|Ared(s_k^i, \rho_k^i) - Pred_k(\hat{s}_k^i, \rho_k^i)|}{Pred_k(\hat{s}_k^i, \rho_k^i)} \leq \frac{8\kappa_{ubr}\|s_k^i\|^\gamma}{c_1^{\mathrm{n}}\|m_k^c(\hat{x}_k)\| \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}. \tag{3.44}$$

Since all the steps s_k^i for $i = 0, \cdots, j-1$ are rejected, we have

$$1 - \eta_1 < \left| \frac{Ared(s_k^i, \rho_k^i)}{Pred_k(\hat{s}_k^i, \rho_k^i)} - 1 \right|. \tag{3.45}$$

Combining (3.44) and (3.45), we obtain

$$\|s_k^i\|^\gamma \geq \frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}}\|m_k^c(\hat{x}_k)\| \;\forall i = 0, \cdots, j-1. \tag{3.46}$$

Since the step s_k^{j-1} is rejected, we have $\Delta_k^j = \alpha_1\|s_k^{j-1}\|$. Together with $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^0$ and $\Delta_k^0 \geq \Delta_{\min}$ it follows that

$$\begin{aligned} \Delta_k^j &= \alpha_1\|s_k^{j-1}\| \\ &\geq \alpha_1 \left[\frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}} \sigma(1-\alpha^{\mathrm{n}})\Delta_{\min} \right]^{1/\gamma} \\ &\stackrel{\text{def}}{=} \Delta_{1*}. \end{aligned} \tag{3.47}$$

ii. If $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^i$ does not hold for some $0 \leq i \leq j$, then there exists a largest index $l, 0 \leq l \leq j$ such that $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^{\mathrm{n}})\Delta_k^l$ holds.

If $j = l$, then $\Delta_k^j \geq \|s_k^l\|$.

If $j = l+1$, the rule of updating the trust-region radius implies $\Delta_k^j = \alpha_1\|s_k^l\|$.

If $j > l+1$, then $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^i$ for all $i = l+1, \cdots, j$, and analogously to the derivation of (3.46) we can show that

$$\|s_k^i\|^\gamma \geq \frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}}\|m_k^c(\hat{x}_k)\| \;\forall i = l+1, \cdots, j-1.$$

Since s_k^{j-1} and s_k^{l+1} are rejected trial steps and $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^{l+1}$, it follows that

$$\begin{aligned}
\Delta_k^j &= \alpha_1 \|s_k^{j-1}\| \\
&\geq \alpha_1 \left[\frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}} \|m_k^c(\hat{x}_k)\| \right]^{1/\gamma} \\
&\geq \alpha_1 \left[\sigma(1-\alpha^{\mathrm{n}}) \frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}} \Delta_k^{l+1} \right]^{1/\gamma} \\
&\geq \alpha_1 \left[\alpha_1\sigma(1-\alpha^{\mathrm{n}}) \frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}} \right]^{1/\gamma} \|s_k^l\|^{1/\gamma}. \qquad (3.48)
\end{aligned}$$

Let

$$\kappa_{2*} = \min\left\{ \alpha_1, \alpha_1 \left[\alpha_1\sigma(1-\alpha^{\mathrm{n}}) \frac{(1-\eta_1)c_1^{\mathrm{n}} \min\{c_2^{\mathrm{n}}\sigma(1-\alpha^{\mathrm{n}}), \alpha^{\mathrm{n}}\}}{8\kappa_{ubr}} \right]^{1/\gamma} \right\}.$$

In the subcases $j = l$ and $j = l+1$ we have (recall that $\gamma \in (0,1]$)

$$\Delta_k^j \geq \kappa_{2*}\|s_k^l\| \geq \kappa_{2*} \min\{1, \|s_k^l\|^{1/\gamma}\}.$$

In the subcase $j > l+1$ we have

$$\Delta_k^j \geq \kappa_{2*}\|s_k^l\|^{1/\gamma}.$$

Hence in all three subcases $j = l$, $j = l+1$, $j > l+1$ we have

$$\Delta_k^j \geq \kappa_{2*} \min\{1, \|s_k^l\|^{1/\gamma}\}. \qquad (3.49)$$

From Assumption AM4, Lemma 3.4.10, $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^{\mathrm{n}})\Delta_k^l$, and $\|s_k^l\| \leq \Delta_k^l$ we have

$$\begin{aligned}
&|Ared(s_k^l, \rho_k^l) - Pred_k(\hat{s}_k^l, \rho_k^l)| \\
&\leq \left(\kappa_{ubr1} + \kappa_{ubr2}\rho_k^l\Delta_k^l + \kappa_{ubr3}\sigma(1-\alpha^{\mathrm{n}})\rho_k^l\Delta_k^l\right) \|s_k^l\|^\gamma \Delta_k^l \\
&\leq \left(\kappa_{ubr1} + \kappa_{ubr2}\kappa_{uptr} + \kappa_{ubr3}\sigma(1-\alpha^{\mathrm{n}})\kappa_{uptr}\right) \|s_k^l\|^\gamma \Delta_k^l. \qquad (3.50)
\end{aligned}$$

Since $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^n)\Delta_k^l$, Lemma 3.4.9 implies

$$Pred_k(\hat{s}_k^l, \rho_k^l) \geq \kappa_{lpr}\Delta_k^l. \tag{3.51}$$

Combining (3.50) and (3.51) and the fact that s_k^l is rejected, we obtain

$$\begin{aligned} 1-\eta_1 &< \left| \frac{Ared(s_k^l, \rho_k^l)}{Pred_k(\hat{s}_k^l, \rho_k^l)} - 1 \right| \\ &\leq \frac{(\kappa_{ubr1} + \kappa_{ubr2}\kappa_{3*} + \kappa_{ubr3}\sigma(1-\alpha^n)\kappa_{uptr})\|s_k^l\|^\gamma}{\kappa_{lpr}}. \end{aligned}$$

Therefore

$$\|s_k^l\| \geq \left[\frac{(1-\eta_1)\kappa_{lpr}}{\kappa_{ubr1} + \kappa_{ubr2}\kappa_{3*} + \kappa_{ubr3}\sigma(1-\alpha^n)\kappa_{uptr}} \right]^{1/\gamma}. \tag{3.52}$$

Using (3.49) and (3.52), we obtain

$$\Delta_k^j \geq \kappa_{2*} \min\left\{1, \left[\frac{(1-\eta_1)\kappa_{lpr}}{\kappa_{ubr1} + \kappa_{ubr2}\kappa_{3*} + \kappa_{ubr3}\sigma(1-\alpha^n)\kappa_{uptr}} \right]^{1/\gamma^2} \right\} \stackrel{\text{def}}{=} \Delta_{2*}. \tag{3.53}$$

The bounds (3.43), (3.47) and (3.53) imply the desired result if we set

$$\Delta_* = \min\{\Delta_{\min}, \Delta_{1*}, \Delta_{2*}\}.$$

□

The following lemma guarantees the boundedness of the penalty parameters.

Lemma 3.4.12 *Let Assumptions A1, A2, AM1–AM4 hold and let the criticality measure satisfy* (3.34). *If* $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$, *then there exists a* $\rho_* > 0$ *which depends on* ϵ *but is independent of k such that*

$$\lim_{k\to\infty} \rho_k = \rho_* \tag{3.54}$$

Furthermore, there exist a $k_\rho \in \mathbb{N}$ such that $\rho_k = \rho_*$ for all $k \geq k_\rho$.

Proof: From Lemmas 3.4.10 and 3.4.11, we have that for any pair k, j of indices

$$\rho_k^j \leq \frac{\kappa_{uptr}}{\Delta_k^j} \leq \frac{\kappa_{uptr}}{\Delta_*}.$$

Thus the sequence $\{\rho_k\}$ is bounded by $\rho_* = \frac{\kappa_{uptr}}{\Delta_*}$. From the rule for the update of penalty parameter, every increase in ρ_k is by at least β. Since $\rho_* < \infty$, there can be at most finitely many increases, which implies the existence of k_ρ such that $\rho_k = \rho_*$ for all $k \geq k_\rho$. □

Lemma 3.4.13 *Let Assumptions A1, A2, AM1–AM4 hold and let the criticality measure satisfy* (3.34). *If $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$, the sequence $\{|L(x_k, \lambda_k; \rho_k)|\}$ is bounded.*

Proof: This result is a direct consequence of Lemma 3.4.12. □

Lemma 3.4.11 established the existence of a lower bound for the sub-sequence $\{\Delta_k^j\}$ of trust-region radii for iterations k, j in which the penalty parameter is increased. The following result established the existence of a lower bound for the entire sequence $\{\Delta_k^j\}$ of trust-region radii.

Lemma 3.4.14 *Let Assumptions A1, A2, AM1–AM4 hold and let the criticality measure satisfy* (3.34). *If $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$, then there exist a $\Delta_* > 0$ which depends on ϵ but is independent of k such that*

$$\Delta_k^j \geq \Delta_*. \tag{3.55}$$

Proof: The proof is similar to that of Lemma 3.4.11. The rule of updating trust-region radius in Algorithm 3.2.1 implies that Δ_k is bounded by $\Delta_{\min}$ if the first trial step is acceptable. Therefore we consider the case where at least one unsuccessful trial step occurs in search of an acceptable step. Assume that we have $j \geq 1$ unsuccessful steps. Similar to Lemma 3.4.11, we consider two cases:

i. $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^i$ for all $i = 0, \cdots, j$.

ii. $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^i$ does not hold for some $0 \leq i \leq j$.

i. In the first case, we proceed as in the proof of Lemma 3.4.11 to show the existence of $\Delta_{1*} > 0$, independent of k and i such that

$$\Delta_k^i \geq \Delta_{1*}.$$

ii. Suppose that $\|m_k^c(\hat{x}_k)\| > \sigma(1-\alpha^{\mathrm{n}})\Delta_k^i$ does not hold for all $i = 0, \ldots, j$. Let l be the largest index such that $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^{\mathrm{n}})\Delta_k^l$ holds. Since $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^{\mathrm{n}})\Delta_k^i$, $i = 0, \cdots, l$, Lemma 3.4.9 implies that $Pred_k(\hat{s}_k^i, \rho_k^i) \geq \kappa_{lpr}\Delta_k^i$ for all $i \leq l$. This inequality, the fact that s_k^i is rejected and (3.24) imply

$$1 - \eta_1 < \left| \frac{Ared(s_k^i, \rho_k^i)}{Pred_k(\hat{s}_k^i, \rho_k^i)} - 1 \right| \leq \frac{\kappa_{ubr}\rho_k^i \Delta_k^i \|s_k^i\|^\gamma}{\kappa_{lpr}\Delta_k^i} \leq \frac{\kappa_{ubr}\rho_* \|s_k^i\|^\gamma}{\kappa_{lpr}}.$$

Thus for all $i \leq l$,

$$\Delta_k^i \geq \|s_k^i\| \geq \left[\frac{(1-\eta_1)\kappa_{lpr}}{\kappa_{ubr}\rho_*}\right]^{1/\gamma}.$$

For all $i > l$ we have from (3.49) and the above inequality,

$$\Delta_k^i \geq \kappa_{2*} \min\{1, \|s_k^l\|^{1/\gamma}\} \geq \kappa_{2*} \min\left\{1, \left[\frac{(1-\eta_1)\kappa_{lpr}}{\kappa_{ubr}\rho_*}\right]^{1/\gamma^2}\right\} \stackrel{\text{def}}{=} \Delta_{2*}.$$

The desired result now follows with

$$\Delta_* = \min\{\Delta_{\min}, \Delta_{1*}, \Delta_{2*}\}.$$

□

3.4.4 First Order Global Convergence Result

The following Theorem shows that the criticality measure and the norm of the model constraint at iterates $\{\hat{x}_k\}$ generated by Algorithm 3.2.1 converge to zero.

Theorem 3.4.15 *Let Assumptions A1, A2, AM1–AM4 hold and let the criticality measure satisfy* (3.34). *If there exist* $\epsilon > 0$, *such that*

$$\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| \quad > \epsilon,$$

then

$$\lim_{k\to\infty} \|m_k^c(\hat{x}_k)\| = 0. \tag{3.56}$$

If in addition Assumption AM6 holds, then

$$\lim_{k\to\infty} \|c(x_k)\| = 0. \tag{3.57}$$

Proof: Assume that (3.56) is false. Then for every $\tau > 0$ there exists an infinite sequence of successful iterations $\{k_j\}$ such that $\|m_k^c(\hat{x}_{k_j})\| \geq \tau$ for all $j \in \mathbb{N}$.

Lemmas 3.4.12 and 3.4.14 and (3.5), (3.29) imply

$$
\begin{aligned}
Pred_{k_j} &\geq \frac{\rho_{k_j}}{8}\left[\|m_k^c(\hat{x}_{k_j})\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_{k_j}^n)\|^2\right] \\
&\geq \frac{c_1^n \rho_*}{8}\|m_k^c(\hat{x}_{k_j})\| \min\{c_2^n\|m_k^c(\hat{x}_{k_j})\|, \alpha^n \Delta_{k_j}\} \\
&\geq \frac{c_1^n \rho_* \tau}{8} \min\{c_2^n \tau, \alpha^n \Delta_*\} \\
&\stackrel{\text{def}}{=} \kappa_* > 0,
\end{aligned}
$$

for all $k_j \geq k_\rho$, where k_ρ is defined in Lemma 3.4.12. Hence, for $k_j \geq k_\rho$

$$L(x_{k_j}, \lambda_{k_j}; \rho_*) - L(x_{k_j+1}, \lambda_{k_j+1}; \rho_*) = Ared_{k_j} \geq \eta_1 Pred_{k_j} \geq \eta_1 \kappa_* > 0.$$

In particular

$$L(x_{k_J+1}, \lambda_{k_J+1}; \rho_*) \leq L(x_{k_\rho}, \lambda_{k_\rho}; \rho_*) - (k_J - k_\rho)\eta_1 \kappa_*.$$

for all $k_J - k_\rho$. This contradicts the fact that $\{|L(x_k, \lambda_k; \rho_k)|\}$. Hence (3.56) is proven.

The equation (3.57) is an immediate consequence of (3.56) and Assumption AM6. □

The following Theorem shows that the criticality measure and the norm of the constraint at iterates $\{x_k\}$ generated by Algorithm 3.2.1 converge to zero.

Theorem 3.4.16 *Let Assumptions A1, A2, AM1–AM4 hold and let the criticality measure satisfy* (3.34). *The sequences of iterates generated by Algorithm* 3.2.1 *satisfy*

$$\lim_{k\to\infty} \left(\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\|\right) = 0. \tag{3.58}$$

If, in addition, Assumptions AM5 and AM6 hold, then

$$\lim_{k\to\infty} \left(\chi_k(x_k) + \|c(x_k)\|\right) = 0. \tag{3.59}$$

Proof: Assume there exists $\epsilon > 0$ such that $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$. Theorem 3.4.15 implies $\|m_k^c(\hat{x}_k)\| \to 0$. Therefore, $\{\chi_k^M(\hat{x}_k)\}$ cannot converge to zero.

Suppose that $\chi_k^M(\hat{x}_k) \geq \tau_1$ for some $\tau_1 > 0$. Since $\|m_k^c(\hat{x}_k)\| \to 0$, there exists $K > k_\rho$ such that $\|m_k^c(\hat{x}_k)\| \leq \sigma(1-\alpha^n)\Delta_* \leq \sigma(1-\alpha^n)\Delta_k$ for all $k \geq K$. In the last inequality we have used Lemma 3.4.14. Thus Lemma 3.4.9 implies that $Pred_k \geq \kappa_{lpr}\Delta_k$ for all $k \geq K$. Using an argument analogous to the one in the proof of Theorem 3.4.15, we obtain

$$L(x_{k_j}, \lambda_{k_j}; \rho_*) - L(x_{k_j+1}, \lambda_{k_j+1}; \rho_*) = Ared_{k_j} \geq \eta_1 Pred_{k_j} \geq \eta_1 \kappa_{lpr} \Delta_* > 0,$$

which contradicts the boundedness of $\{|L(x_k, \lambda_k; \rho_k)|\}$. Hence the assumption $\chi_k^M(\hat{x}_k) + \|m_k^c(\hat{x}_k)\| > \epsilon$ is false.

Equation (3.59) is an immediate consequence of (3.58) from Assumptions AM5 and AM6. □

3.5 Specialization to Unconstrained Optimization Problems

Our GTRSQP algorithm and the corresponding convergence result can be specialized to unconstrained optimization problems of form

$$\min f(x) \tag{3.60}$$

where $x \in \mathbb{R}^N$, $f : \mathbb{R}^N \to \mathbb{R}$. In this section we sketch this specialization and we discuss how our algorithm and the corresponding convergence result relate to those in [2, 15]. Since the convergence results in [15] are stronger than those in [2], we concentrate on the former.

The specialization of our GTRSQP algorithm to unconstrained optimization problems of form (3.60) is given as follows.

Let $P_k \in \mathbb{R}^{N \times N_k}$. At iteration $x_k = P_k \hat{x}_k$ we have a model $m_k^f(\hat{x}_k + \hat{s})$ of $f(x_k + P_k \hat{s})$ and $\Delta_k > 0$ and we compute a trial step as the approximate solution of

$$\begin{array}{ll} \min & m_k^f(\hat{x}_k + \hat{s}), \\ \text{s.t.} & \|P_k \hat{s}\| \leq \Delta_k. \end{array} \tag{3.61}$$

We require that the trial step $\hat{s}_k$ satisfies

$$m_k^f(\hat{x}_k) - m_k^f(\hat{x}_k + \hat{s}_k) \geq c_1^{\mathrm{t}} \chi_k^M(\hat{x}_k) \min\{c_2^{\mathrm{t}} \chi_k^M(\hat{x}_k), \Delta_k\}, \tag{3.62}$$

where $c_1^{\mathrm{t}}, c_2^{\mathrm{t}} > 0$ are constants independent of k and

$$\chi_k^M(\hat{x}_k) = \|\nabla m_k^f(\hat{x}_k)\|.$$

Algorithm 3.5.1 *1. Initialization. Given* x_0. *Choose* $\Delta_0 > 0$, $0 < \eta_1 < \eta_2 < 1$, $\alpha_1 < 1 < \alpha_2$, *and* $\epsilon_{tol} > 0$. *Set* $k = 0$.

2. *If* $\|\nabla f(x_k)\| < \epsilon_{tol}$, *then terminate.*

3. *Generate a model* m_k^f, *a restriction* R_k *and a prolongation* P_k. *Set* $\hat{x}_k = R_k x_k$.

4. *Compute a trial step* $\hat{s}_k$ *that satisfies* (3.62).

5. *Evaluate the trial step* $s_k = P_k \hat{s}_k$. *Compute*

$$r_k = \frac{f(x_k) - f(x_k + s_k)}{m_k^f(\hat{x}_k) - m_k^f(\hat{x}_k + \hat{s}_k)}.$$

Set

$$\Delta_k = \begin{cases} \alpha_1 \|s_k\| & \textit{if } r_k < \eta_1, \\ \Delta_k & \textit{if } \eta_1 \le r_k < \eta_2, \\ \alpha_2 \Delta_k & \textit{if } r_k \ge \eta_2. \end{cases}$$

If $r_k \ge \eta_1$, *then set* $x_{k+1} = x_k + s_k$, $\Delta_{k+1} = \Delta_k$, $k = k+1$ *and goto 2, else goto 4.*

We impose the following assumptions on f and on the models m_k^f. Let $\Omega \subset \mathbb{R}^N$ be an open subset such that $x_k, x_k + s_k \in \Omega$ for all $k \in \mathbb{N}$. We assume that

A1. The objective function f is twice continuously differentiable in Ω.

A2. There exists a constant $k_H^f > 0$ such that $\|\nabla^2 f(x)\| \le k_H^f$ for all $x \in \Omega$.

A3. The objective function f is bounded from below on Ω.

Let $\hat{\Omega}_k \subset \mathbb{R}^{N_k}$ be open, convex subsets such that $\hat{x}_k, \hat{x}_k + \hat{s}_k \in \hat{\Omega}_k$ for all iteration $k \in \mathbb{N}$. We assume that

AM1. For all k, the model m_k^f is twice continuously differentiable in $\hat{\Omega}_k$.

AM2. There exists a constant $\kappa_H^f > 0$ such that $\|\nabla^2 m_k^f(\hat{x})\| \le \kappa_H^f$ for all $\hat{x} \in \hat{\Omega}_k$ and all $k \in \mathbb{N}$.

AM3. The matrices P_k, $k \in \mathbb{N}$, have full rank N_k and there exists $\kappa_{lp} > 0$ such that $\|P_k \hat{x}\| \ge \kappa_{lp} \|\hat{x}\|$ for all $\hat{x} \in \mathbb{R}^{N_k}$ and for all $k \in \mathbb{N}$, AM4. There exists a constant $c_1 > 0$ independent of k such that

$$\|P_k^T \nabla f(x_k) - \nabla m_k^f(\hat{x}_k)\| \leq c_1 \Delta_k.$$

AM5. There exists a constant $\gamma_\chi > 0$ independent of k such that $\|\nabla f(x_k)\| \leq \gamma_\chi \|\nabla m_k^f(\hat{x}_k)\|$ for all $k \in \mathbb{N}$.

Theorem 3.5.2 *Let Assumptions A1–A3, AM1–AM4 hold. The sequences of iterates generated by Algorithm* 3.5.1 *satisfy*

$$\lim_{k \to \infty} \|\nabla m_k^f(\hat{x}_k)\| = 0. \tag{3.63}$$

If, in addition, Assumption AM5 holds, then

$$\lim_{k \to \infty} \|\nabla f(x_k)\| = 0. \tag{3.64}$$

Proof: The somewhat lengthy proof can be obtained by rather straightforward extensions of the convergence proof for the basic trust-region algorithm found in [10, Sec. 6.4]. We omit the proof here. □

To compare our convergence result with existing ones, we assume $R_k = P_k = I$, i.e., $x_k = \hat{x}_k$ and $s_k = \hat{s}_k$.

The convergence result in [15] (which builds on [7, 27]) requires that

$$\|\nabla f(x_k) - \nabla m_k^f(x_k)\| \leq \xi \|\nabla m_k^f(x_k)\|, \tag{3.65}$$

where $0 < \xi < 1 - \eta_2$ and η_2 is the parameter in the trust-region algorithm 3.5.1. Using an idea in [22], it is possible to replace (3.65) by

$$\|\nabla f(x_k) - \nabla m_k^f(x_k)\| \leq \zeta \min\{\Delta_k, \|\nabla m_k^f(x_k)\|\}, \tag{3.66}$$

where $\zeta > 0$ is a constant independent of k. The constant ζ is not tied to parameters in the trust-region algorithm, in particular it is not required that $\zeta < 1$.

We note that (3.65) implies AM5 with $\gamma_X = 1 + \xi$. Condition (3.66) implies AM4 and AM5 with $c_1 = \zeta$ and $\gamma_X = 1 + \zeta$.

On the other hand, AM4 and AM5 imply (3.66) with $\zeta = \max\{c_1, 1 + \gamma_X\}$.

3.6 Computation of the Quasi-Normal Step

The quasi-normal step subproblem (3.3) is a norm-constrained nonlinear least squares problem. An approximate solution $\hat{s}_k^{\mathrm{n}}$ of subproblem (3.3) can be obtained by applying a trust–region method to (3.3). The ith iterate in this trust–region method applied to (3.3) is denoted by $\hat{s}_{i,k}^{\mathrm{n}}$, the ith step is denoted by $\hat{z}_i$, and the trust–region radius of the ith iteration is $\delta_{i,k}^{\mathrm{n}}$. The trust–region subproblem in the ith iteration is given by

$$\begin{array}{ll} \min & \|m_k^c(\hat{x}_k + \hat{s}_{i,k}^{\mathrm{n}}) + \nabla m_k^c(\hat{x}_k + \hat{s}_{i,k}^{\mathrm{n}})^T \hat{z}\|, \\ \text{s.t.} & \|P_k(\hat{s}_{i,k}^{\mathrm{n}} + \hat{z})\| \leq \alpha^{\mathrm{n}} \Delta_k, \\ & \|P_k \hat{z}\| \leq \delta_{i,k}^{\mathrm{n}}. \end{array} \tag{3.67}$$

To simplify this presentation, we only increase the iteration count i when the iteration is successful.

The initial iterate in the trust–region method applied to (3.3) is chosen to be $\hat{s}_{0,k}^{\mathrm{n}} = 0$

and the initial trust region radius $\delta^{\mathrm{n}}_{0,k} = \alpha^{\mathrm{n}}\Delta_k$. Thus the first subproblem (3.67) becomes

$$\begin{array}{ll} \min & \|m^c_k(\hat{x}_k) + \nabla m^c_k(\hat{x}_k)^T \hat{z}\|, \\ \text{s.t.} & \|P_k \hat{z}\| \le \delta^{\mathrm{n}}_{0,k}. \end{array} \tag{3.68}$$

We require that the approximate solution $\hat{z}_0$ of (3.68) satisfies the sufficient decrease condition

$$\begin{array}{rl} & \|m^c_k(\hat{x}_k)\|^2 - \|m^c_k(\hat{x}_k) + \nabla m^c_k(\hat{x}_k)^T \hat{z}_0\|^2 \\ \ge & c_1 \|m^c_k(\hat{x}_k)\| \min\{c_2 \|m^c_k(\hat{x}_k)\|, \delta^{\mathrm{n}}_{0,k}\}, \end{array} \tag{3.69}$$

where $c_1 \in (0,1)$, $c_2 > 0$ are independent of k.

Let $0 < \eta^{\mathrm{n}}, \gamma^{\mathrm{n}}_1, \gamma^{\mathrm{n}}_2 < 1$ be given parameters. The step $\hat{z}_0$ will be accepted if

$$\frac{\|m^c_k(\hat{x}_k)\|^2 - \|m^c_k(\hat{x}_k + \hat{z}_0)\|^2}{\|m^c_k(\hat{x}_k)\|^2 - \|m^c_k(\hat{x}_k) + \nabla m^c_k(\hat{x}_k)^T \hat{z}_0\|^2} \ge \eta^{\mathrm{n}}. \tag{3.70}$$

If $\hat{z}_0$ is accepted, the next trust–region iterate is $\hat{s}^{\mathrm{n}}_{1,k} = \hat{z}_0$ and the corresponding trust–region radius satisfies $\delta^{\mathrm{n}}_{1,k} \ge \gamma^{\mathrm{n}}_1 \delta^{\mathrm{n}}_{0,k}$. If (3.70) is not satisfied, the step $\hat{z}_0$ will be rejected, $\delta^{\mathrm{n}}_{0,k}$ will be decreased by a factor between γ^{n}_2 and γ^{n}_1, and (3.68) will be resolved.

We will show below that to satisfy (3.5) it is sufficient to perform only one successful step in the trust-region algorithm for (3.3). Additional trust–region iterates might be beneficial numerically. They are required to satisfy $\|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_{i,k})\|^2 \ge \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_{i+1,k})\|^2$. One could perform additional trust–region steps as long as $\|P_k \hat{z}\| \le \delta^{\mathrm{n}}_{i,k}$ implies $\|P_k(\hat{s}^{\mathrm{n}}_{i,k} + \hat{z}_i)\| \le \alpha^{\mathrm{n}}\Delta_k$, i.e., as long as (3.67) reduces to

$$\begin{array}{ll} \min & \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_{i,k}) + \nabla m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_{i,k})^T \hat{z}\|, \\ \text{s.t.} & \|P_k \hat{z}\| \le \delta^{\mathrm{n}}_{i,k}. \end{array}$$

Alternatively, one could also use a trust-region method for norm constrained problem [19].

A general algorithm is given as follows.

Algorithm 3.6.1 *1. Initialization. Let* $0 < \eta^{\mathsf{n}}, \gamma_1^{\mathsf{n}}, \gamma_2^{\mathsf{n}} < 1$ *be given and set* $\hat{s}_{0,k}^{\mathsf{n}} = 0, \delta_{0,k}^{\mathsf{n}} = \alpha^{\mathsf{n}}\Delta_k$.

2. *Compute a trial step* $\hat{z}_0$ *of subproblem* (3.68) *that satisfies* (3.69).

3. *Compute*

$$r_0^{\mathsf{n}} = \frac{\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{z}_0)\|^2}{\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{z}_0\|^2}.$$

If $r_0^{\mathsf{n}} < \eta^{\mathsf{n}}$, *reduce* $\delta_{0,k}^{\mathsf{n}}$ *by a factor of at least* γ_1^{n} *and at most* γ_2^{n}, *and goto step 2; else set* $\hat{s}_{1,k}^{\mathsf{n}} = \hat{z}_0$ *and choose* $\delta_{1,k}^{\mathsf{n}} \geq \gamma_1^{\mathsf{n}}\delta_{0,k}^{\mathsf{n}}$.

4. *For* $i = 1, \cdots, i_{\max} - 1$

   (a) *Compute a trial step* $\hat{z}_i$ *of subproblem* (3.67) *that satisfies*

$$\|m_k^c(\hat{x}_k + \hat{s}_{i,k}^{\mathsf{n}})\|^2 \geq \|m_k^c(\hat{x}_k + \hat{s}_{i,k}^{\mathsf{n}} + \hat{z}_i)\|^2 \tag{3.71}$$

   and $\|P_k(\hat{s}_{i,k}^{\mathsf{n}} + \hat{z}_i)\| \leq \alpha^{\mathsf{n}}\Delta_k$.

   (b) *Set* $\hat{s}_{i,k}^{\mathsf{n}} = \hat{s}_{i,k}^{\mathsf{n}} + \hat{z}_i$.

   End

5. *Set* $\hat{s}_k^{\mathsf{n}} = \hat{s}_{i_{\max},k}^{\mathsf{n}}$

Of course, for a practical algorithm one should specify another stopping criteria in addition to the maximum number of iterations. Also, the nondecreasing property (3.71) should be strengthened. Algorithm 3.6.1 was kept general on purpose so that it can cover different variants, but still ensures that the computed quasi-normal step meet the requirements (3.4), (3.5), and (3.6).

We call the 0th trial iterate successful if $r_0^{\mathrm{n}} \geq \eta^{\mathrm{n}}$.

Lemma 3.6.2 *Let assumptions AM1, AM2 and AM3 hold. For any $\hat{x}_k, \hat{x}_k + \hat{z}_0 \in \Omega_k$, we have*

$$\left|\|m_k^c(\hat{x}_k + \hat{z}_0)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{z}_0\|^2\right| \leq a\|\hat{z}_0\|^2 \tag{3.72}$$

where

$$a = \kappa_H^c \max\left\{\kappa_m^c, \frac{\kappa_g^c \Delta_{max}}{\kappa_{lp}}, \frac{\kappa_H^c \Delta_{max}^2}{2\kappa_{lp}^2}\right\}.$$

Proof: From assumption AM1 and AM2 on m_k^c and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \left|\|m_k^c(\hat{x}_k + \hat{z}_0)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{z}_0\|^2\right| \\
= & \left|\|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{z}_0 + \tfrac{1}{2}\hat{z}_0{}^T \nabla^2 m_k^c(\hat{x}_k + \xi\hat{z}_0)\hat{z}_0\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{z}_0\|^2\right| \\
\leq & \left|(m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{z}_0)^T (\hat{z}_0{}^T \nabla^2 m_k^c(\hat{x}_k + \xi\hat{z}_0)\hat{z}_0)\right| + \|\tfrac{1}{2}\hat{z}_0{}^T \nabla^2 m_k^c(\hat{x}_k + \xi\hat{z}_0)\hat{z}_0\|^2 \\
\leq & \kappa_m^c \kappa_H^c \|\hat{z}_0\|^2 + \kappa_g^c \kappa_H^c \|\hat{z}_0\|^3 + \frac{(\kappa_H^c)^2}{2}\|\hat{z}_0\|^4.
\end{aligned}$$

Assumption AM3, the choice of $\delta_{0,k}^{\mathrm{n}}$ and the boundedness of Δ_k imply

$$\|\hat{z}_0\| \leq \frac{1}{\kappa_{lp}}\|P_k \hat{z}_0\| \leq \frac{\delta_{0,k}^{\mathrm{n}}}{\kappa_{lp}} \leq \frac{\Delta_k}{\kappa_{lp}} \leq \frac{\Delta_{max}}{\kappa_{lp}}.$$

This gives the desired estimate. $\square$

Lemma 3.6.3 *Let assumption AM1,AM2 and AM3 hold, let c_1, c_2 be the constants in* (3.69) *and let a be the constant defined in Lemma* 3.6.2. *If $\|m_k^c(\hat{x}_k)\| \neq 0$ and if*

$$\delta_{0,k}^{n} \leq \frac{c_1(1-\eta^{n})}{\max\{a, 1/c_2\}}\|m_k^c(\hat{x}_k)\|, \tag{3.73}$$

then the trial step $\hat{z}_0$ will be accepted.

Proof: Notice that $0 < c_1 < 1$ and $0 < \eta^{n} < 1$ imply $c_1(1-\eta^{n}) < 1$. Hence, (3.73) implies

$$\delta_{0,k}^{n} \leq c_2\|m_k^c(\hat{x}_k)\|.$$

This inequality and (3.69) yield

$$\begin{aligned} \|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T\hat{z}_0\|^2 \;&\geq\; c_1\|m_k^c(\hat{x}_k)\|\min\left\{c_2\|m_k^c(\hat{x}_k)\|, \delta_{0,k}^{n}\right\} \\ &=\; c_1\|m_k^c(\hat{x}_k)\|\delta_{0,k}^{n}. \end{aligned} \tag{3.74}$$

Now we apply Lemma 3.6.2, (3.74), and (3.73) to obtain

$$\begin{aligned} |r_0^{n} - 1| \;&=\; \left|\frac{\|m_k^c(\hat{x}_k+\hat{z}_0)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T\hat{z}_0\|^2}{\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T\hat{z}_0\|^2}\right| \\ &\leq\; \frac{a\delta_{0,k}^{n}}{c_1\|m_k^c(\hat{x}_k)\|} \\ &\leq\; 1-\eta^{n}. \end{aligned}$$

Therefore $r_0^{n} > \eta^{n}$, which means $\hat{z}_0$ will be accepted. □

Theorem 3.6.4 *Let assumptions AM1,AM2 and AM3 be satisfied. There exist $c_1^{n}, c_2^{n} > 0$, independent of k, such that the quasi-normal step $\hat{s}_k^{n}$ computed in Algorithm* 3.6.1 *satisfies*

$$\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k+\hat{s}_k^{n})\|^2 \geq c_1^{n}\|m_k^c(\hat{x}_k)\|\min\{c_2^{n}\|m_k^c(\hat{x}_k)\|, \alpha^{n}\Delta_k\}. \tag{3.75}$$

Proof: Let c_1, c_2 be the constants in (3.69) and let a be the constant defined in Lemma 3.6.2.

We distinguish two cases.

Case 1: Let

$$\alpha^{\mathrm{n}} \Delta_k^{\mathrm{n}} \leq \frac{c_1(1-\eta^{\mathrm{n}})}{\max\{a, 1/c_2\}} \|m_k^c(\hat{x}_k)\|.$$

Since $\delta_{0,k}^{\mathrm{n}} = \alpha^{\mathrm{n}} \Delta_k^{\mathrm{n}}$ satisfies (3.73), Lemma 3.6.3 guarantees that the first trial step is successful,

$$\hat{s}_{1,k}^{\mathrm{n}} = \hat{z}_0.$$

The definition of a successful iterate and (3.69) yield

$$\begin{aligned} \|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_{1,k}^{\mathrm{n}})\|^2 &\geq \eta^{\mathrm{n}} \left(\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k) + \nabla m_k^c(\hat{x}_k)^T \hat{s}_{1,k}^{\mathrm{n}}\|^2\right) \\ &\geq \eta^{\mathrm{n}} c_1 \|m_k^c(\hat{x}_k)\| \min\{c_2\|m_k^c(\hat{x}_k)\|, \alpha^{\mathrm{n}} \Delta_k^{\mathrm{n}}\}. \end{aligned}$$

Since all subsequent iterates satisfy (3.71), we obtain

$$\|m_k^c(\hat{x}_k)\|^2 - \|m_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})\|^2 \geq \eta^{\mathrm{n}} c_1 \|m_k^c(\hat{x}_k)\| \min\{c_2\|m_k^c(\hat{x}_k)\|, \alpha^{\mathrm{n}} \Delta_k^{\mathrm{n}}\}.$$

Case 2: Let

$$\alpha^{\mathrm{n}} \Delta_k^{\mathrm{n}} > \frac{c_1(1-\eta^{\mathrm{n}})}{\max\{a, 1/c_2\}} \|m_k^c(\hat{x}_k)\|.$$

Let $\delta_{0,k}^{\mathrm{n}}$ be the trust-region radius that corresponds to the first successful trial iterate. Lemma 3.6.3 and the fact that at the end of an unsuccessful trial iterate the trust-region radius is reduced by at most γ_2^{n} imply the inequality

$$\delta_{0,k}^{\mathrm{n}} > \gamma_2^{\mathrm{n}} \frac{c_1(1-\eta^{\mathrm{n}})}{\max\{a, 1/c_2\}} \|m_k^c(\hat{x}_k)\|. \tag{3.76}$$

The definition of a successful iterate, (3.69), and (3.76) yield

$$\begin{aligned}\|m_k^{\rm c}(\hat{x}_k)\|^2 - \|m_k^{\rm c}(\hat{x}_k + \hat{s}^{\rm n}_{1,k})\|^2 &\geq \eta^{\rm n}\left(\|m_k^{\rm c}(\hat{x}_k)\|^2 - \|m_k^{\rm c}(\hat{x}_k) + \nabla m_k^{\rm c}(\hat{x}_k)^T \hat{s}^{\rm n}_{1,k}\|^2\right)\\ &\geq \eta^{\rm n} c_1 \|m_k^{\rm c}(\hat{x}_k)\| \min\{c_2\|m_k^{\rm c}(\hat{x}_k)\|, \delta^{\rm n}_{0,k}\}\\ &\geq \eta^{\rm n} c_1 \min\left\{c_2, \gamma_2^{\rm n}\frac{c_1(1-\eta^{\rm n})}{\max\{a, 1/c_2\}}\right\} \|m_k^{\rm c}(\hat{x}_k)\|^2.\end{aligned}$$

Since all subsequent iterates satisfy (3.71), this implies

$$\|m_k^{\rm c}(\hat{x}_k)\|^2 - \|m_k^{\rm c}(\hat{x}_k + \hat{s}^{\rm n}_{1,k})\|^2 \geq \eta^{\rm n} c_1 \min\left\{c_2, \gamma_2^{\rm n}\frac{c_1(1-\eta^{\rm n})}{\max\{a, 1/c_2\}}\right\} \|m_k^{\rm c}(\hat{x}_k)\|^2.$$

The desired result now follows with

$$c_1^{\rm n} = \eta^{\rm n} c_1, \quad c_2^{\rm n} = \min\left\{c_2, \gamma_2^{\rm n}\frac{c_1(1-\eta^{\rm n})}{\max\{a, 1/c_2\}}\right\}.$$

□

Theorem 3.6.5 *Let the standard assumptions AM1, AM2, and AM3 on* $m_k^{\rm c}$ *be satisfied. If the trust–region steps* $\hat{z}_i$ *satisfy*

$$\|\hat{z}_i\| \leq c_3 \|m_k^{\rm c}(\hat{x}_k + \hat{s}^{\rm n}_{i,k})\| \tag{3.77}$$

for some $c_3 > 0$, *independent of* i *and* k, *then the quasi-normal step* $\hat{s}_k^{\rm n}$ *computed in Algorithm* 3.6.1 *satisfies*

$$\|\hat{s}_k^{\rm n}\| \leq c_3^{\rm n} \|m_k^{\rm c}(\hat{x}_k)\| \tag{3.78}$$

with $c_3^{\rm n} = i_{\max} c_3$.

Proof: The conditions (3.77) and (3.71) imply

$$\|\hat{z}_i\| \leq c_3 \|m_k^c(\hat{x}_k)\|, \quad i = 0, \ldots, i_{\max} - 1$$

Since $\hat{s}^{\mathrm{n}}_{j+1,k} = \sum_{i=0}^{j} \hat{z}_i$, $j = 0, \ldots, i_{\max} - 1$, we obtain

$$\|\hat{s}^{\mathrm{n}}_k\| \leq i_{\max} c_3 \|m_k^c(\hat{x}_k)\|.$$

$\square$

3.7 Computation of the Tangential Step

3.7.1 Well-Posedness of the Tangential Step Conditions

We require that a tangential step $\hat{s}^{\mathrm{t}}$ satisfies (3.10), (3.11) and (3.13). However, it is not obvious that a point exists that satisfies these three conditions simultaneously. The existence of such a point is the goal of this section. We will show that under certain conditions a solution $\hat{s}^{\mathrm{t}}_*$ of (3.7) satisfies (3.10), (3.11) and (3.13). Obviously, since any local solution $\hat{s}^{\mathrm{t}}_*$ of (3.7) satisfies the constraints, it obeys (3.10) and (3.11). Hence we only need to concentrate on the condition (3.13).

3.7.1.1 A Modified Tangential Subproblem

We begin by looking at the modified tangential subproblem

$$\begin{array}{ll} \min & m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}), \\ \text{s.t.} & m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}) = m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k), \\ & \|P_k \hat{s}\| \leq (1 - \alpha^{\mathrm{n}})\Delta_k. \end{array} \tag{3.79}$$

Note that since $\|P_k \hat{s}^{\mathrm{n}}_k\| \leq \alpha^{\mathrm{n}}\Delta_k$, any point $\hat{s}$ that is feasible for (3.79) is also feasible for (3.7) and (3.14).

To alleviate the notation we introduce the functions

$$\phi_k(\hat{s}) \;=\; m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}), \tag{3.80}$$

$$h_k(\hat{s}) \;=\; m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}) - m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k). \tag{3.81}$$

With this notation, the modified tangential subproblem (3.79) reads

$$\begin{array}{ll} \min & \phi_k(\hat{s}), \\ \text{s.t.} & h_k(\hat{s}) = 0 \\ & \|P_k \hat{s}\| \leq (1 - \alpha^{\mathrm{n}})\Delta_k. \end{array} \tag{3.82}$$

Notice that the Jacobian $\nabla h_k(\hat{s})^T$ of h_k at $\hat{s}$ is $\nabla h_k(\hat{s})^T = \nabla m_k^{\mathrm{c}}(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s})^T$. Therefore the matrix $Z_k^{\mathrm{c}}(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s})$ that defined in (3.9) spans the null space of $\nabla h_k(\hat{s})^T$.

3.7.1.2 Reformulation of the Tangential Subproblem

Theorem 3.7.1 *Let AM1 and AM3 hold and define* $D_k = P_k^T P_k \in \mathbb{R}^{N_k}$. *Assume that* $(\nabla h(\hat{s}_{k,*}), 2D_k\hat{s}_{k,*})^T$ *has full rank if* $\|P_k\hat{s}_{k,*}\| = (1-\alpha^{\mathrm{n}})\Delta_k$ *and* $\nabla h(\hat{s}_{k,*})^T$ *has full rank if* $\|P_k\hat{s}_{k,*}\| < (1-\alpha^{\mathrm{n}})\Delta_k$. *The following statements are true.*

i. If $\hat{s}_{k,*}$ *is a solution of tangential subproblem* (3.82), *then there exist Lagrange multipliers* $\hat{\lambda}_* = \hat{\lambda}(\hat{s}_{k,*})$ *and* $\mu_* = \mu(\hat{s}_{k,*})$ *such that*

$$\begin{aligned} \nabla\phi_k(\hat{s}_{k,*}) + \nabla h_k(\hat{s}_{k,*})\hat{\lambda}_* + 2\mu_* D_k\hat{s}_{k,*} &= 0, \\ h_k(\hat{s}_{k,*}) &= 0, \\ \mu_*((1-\alpha^{\mathrm{n}})^2\Delta_k^2 - \|P_k\hat{s}_{k,*}\|^2) &= 0, \\ \mu_* &\geq 0. \end{aligned} \tag{3.83}$$

ii. If $\hat{s}_{k,*}$, $\hat{\lambda}_*$ *and* μ_* *satisfy* (3.83) *and if there exists* $\gamma(\hat{s}_{k,*}) > 0$ *such that*

$$\hat{v}^T\left(\nabla^2(\phi_k(\hat{s}_{k,*}) + h_k(\hat{s}_{k,*})^T\hat{\lambda}_*) + 2\mu_* D_k\right)\hat{v} \geq \gamma(\hat{s}_{k,*})\|\hat{v}\|^2 \tag{3.84}$$

for all $\hat{v} \neq 0$ *with*

$$\begin{aligned} \nabla h_k(\hat{s}_{k,*})^T\hat{v} = 0 \quad &\text{if } \|P_k\hat{s}_{k,*}\| < (1-\alpha^{\mathrm{n}})\Delta_k, \\ \nabla h_k(\hat{s}_{k,*})^T\hat{v} = 0 \text{ and } \hat{v}^T D_k\hat{s}_{k,*} = 0 \quad &\text{if } \|P_k\hat{s}_{k,*}\| = (1-\alpha^{\mathrm{n}})\Delta_k, \end{aligned} \tag{3.85}$$

then $\hat{s}_{k,*}$ *is a local minimum of* (3.82).

Our goal is to relate (3.82) to a problem without equality constraints. For this purpose we introduce

$$\Psi_k(\hat{s}, \rho, \mu) = \phi_k(\hat{s}) + h_k(\hat{s})^T \hat{\lambda}(\hat{s}, \mu) + \frac{\rho}{2}\|h_k(\hat{s})\|_2^2, \tag{3.86}$$

where $\hat{\lambda}(\hat{s}, \mu)$ is the unique solution of

$$\nabla h_k(\hat{s})^T \left(\nabla \phi_k(\hat{s}) + \nabla h_k(\hat{s})\hat{\lambda} + 2\mu D_k \hat{s}\right) = 0$$

i.e.,

$$\hat{\lambda}(\hat{s}, \mu) = -[\nabla h_k(\hat{s})^T \nabla h_k(\hat{s})]^{-1} \nabla h_k(\hat{s})^T (\nabla \phi_k(\hat{s}) + 2\mu D_k \hat{s}). \tag{3.87}$$

Theorem 3.7.2 *Assume that AM1 hold and the Jacobian $\nabla h_k(\hat{s})$ has full rank. Let $\Psi_k(\hat{s}, \rho, \mu)$ be defined as in* (3.86) *with $\hat{\lambda}(\hat{s}, \mu)$ as in* (3.87). *Then*

i. Ψ_k is differentiable with respect to $\hat{s}$ and

$$\nabla \Psi_k(\hat{s}, \rho, \mu) = \nabla \phi_k(\hat{s}) + \nabla h_k(\hat{s})\hat{\lambda}(\hat{s}, \mu) + \rho \nabla h_k(\hat{s}) h_k(\hat{s}) + \nabla \hat{\lambda}(\hat{s}, \mu) h_k(\hat{s}), \tag{3.88}$$

where the Jacobian of $\hat{\lambda}(\hat{s}, \mu)$ with respect to $\hat{s}$ is given by[5]

$$\begin{aligned} \nabla \hat{\lambda}(\hat{s}, \mu)^T \;=\; & -[\nabla h_k(\hat{s})^T \nabla h_k(\hat{s})]^{-1} \\ & \times \Big[\nabla h_k(\hat{s})^T \Big(\nabla^2[\phi_k(\hat{s}) + h_k(\hat{s})^T \hat{\lambda}]|_{\hat{\lambda}=\hat{\lambda}(\hat{s},\mu)} + 2\mu D_k\Big) \\ & + \nabla^2 h_k(\hat{s})^T \Big(\nabla \phi_k(\hat{s}) + \nabla h_k(\hat{s})\hat{\lambda}(\hat{s}, \mu) + 2\mu D_k \hat{s}\Big)\Big]. \end{aligned} \tag{3.89}$$

[5]Here and in the following we use $\nabla^2 h_k(\hat{s})^T v$ as a shorthand for $\sum_{i=1}^{m_k} \nabla^2 (h_k(\hat{s}))_i v_i$, where $(h_k(\hat{s}))_i$ is the ith component function of h_k.

ii. Ψ_k *is twice differentiable with respect to* $\hat{s}$ *and*

$$\begin{aligned}\nabla^2\Psi_k(\hat{s},\rho,\mu) &= \nabla^2[\phi_k(\hat{s})+h_k(\hat{s})^T\hat{\lambda}]|_{\hat{\lambda}=\hat{\lambda}(\hat{s},\mu)}+\rho\nabla h_k(\hat{s})\nabla h_k(\hat{s})^T \\ &\quad +\nabla\hat{\lambda}(\hat{s},\mu)\nabla h_k(\hat{s})^T+\nabla h_k(\hat{s})\nabla\hat{\lambda}(\hat{s},\mu)^T \\ &\quad +\nabla^2\hat{\lambda}(\hat{s},\mu)^T h_k(\hat{s})+\rho\nabla^2 h_k(\hat{s})^T h_k(\hat{s}). \qquad (3.90)\end{aligned}$$

Proof: i. By definition (3.87), $\hat{\lambda}(\hat{s},\mu)$ solves

$$[\nabla h_k(\hat{s})^T\nabla h_k(\hat{s})]\hat{\lambda}(\hat{s},\mu)+\nabla h_k(\hat{s})^T(\nabla\phi_k(\hat{s})+2\mu D_k\hat{s})=0.$$

By the implicit function theorem

$$\begin{aligned}&[\nabla h_k(\hat{s})^T\nabla h_k(\hat{s})]\nabla\hat{\lambda}(\hat{s},\mu)^T \\ &+\nabla h_k(\hat{s})^T\Big(\nabla^2[\phi_k(\hat{s})+h_k(\hat{s})^T\hat{\lambda}]\Big|_{\hat{\lambda}=\hat{\lambda}(\hat{s},\mu)}+2\mu D_k\Big) \\ &+\nabla^2 h_k(\hat{s})^T\Big(\nabla\phi_k(\hat{s})+\nabla h_k(\hat{s})\hat{\lambda}(\hat{s},\mu)+2\mu D_k\hat{s}\Big) \qquad = 0.\end{aligned}$$

ii. Differentiation of (3.88) gives (3.90). □

Consider the subproblem

$$\begin{aligned}\min\quad & \Psi_k(\hat{s},\rho,\mu), \\ \text{s.t.}\quad & \|P_k\hat{s}\|\le(1-\alpha^{\text{n}})\Delta_k.\end{aligned} \qquad (3.91)$$

Theorem 3.7.3 *Let AM1 and AM3 hold and define* $D_k=P_k^TP_k\in\mathbb{R}^{N_k}$.

i. *If* $\hat{s}_{k,*}$ *is a solution of* (3.91), *then there exists* $\sigma_* = \sigma(\hat{s}_{k,*})$ *such that*

$$\begin{aligned} \nabla \Psi_k(\hat{s}_{k,*}, \rho, \mu) + 2\sigma_* D_k \hat{s}_{k,*} &= 0, \\ \sigma_*((1-\alpha^{\mathrm{n}})^2 \Delta_k^2 - \|P_k \hat{s}_{k,*}\|^2) &= 0, \\ \sigma_* &\geq 0. \end{aligned} \tag{3.92}$$

ii. *If* $\hat{s}_{k,*}$ *and* σ_* *satisfy* (3.92) *and if there exists* $\gamma(\hat{s}_{k,*}) > 0$

$$\hat{v}^T \left(\nabla^2 \Psi_k(\hat{s}_{k,*}, \rho, \mu) + 2\sigma_* D_k \right) \hat{v} \geq \gamma(\hat{s}_{k,*}) \|\hat{v}\|^2 \tag{3.93}$$

for all

$$\begin{aligned} \hat{v} \neq 0 \quad & \textit{if } \|P_k \hat{s}_{k,*}\| < (1-\alpha^{\mathrm{n}})\Delta_k; \\ \hat{v} \neq 0 \textit{ with } \hat{v}^T D_k \hat{s}_{k,*} = 0 \quad & \textit{if } \|P_k \hat{s}_{k,*}\| = (1-\alpha^{\mathrm{n}})\Delta_k; \end{aligned} \tag{3.94}$$

then $\hat{s}_{k,*}$ *is a local minimum of* (3.91).

Theorem 3.7.4 *Assume that AM1 and AM3 hold and define* $D_k = P_k^T P_k \in \mathbb{R}^{N_k}$. *Assume that* $(\nabla h(\hat{s}_{k,*}), 2D_k \hat{s}_{k,*})^T$ *has full rank if* $\|P_k \hat{s}_{k,*}\| = (1-\alpha^{\mathrm{n}})\Delta_k$ *and* $\nabla h(\hat{s}_{k,*})^T$ *has full rank if* $\|P_k \hat{s}_{k,*}\| < (1-\alpha^{\mathrm{n}})\Delta_k$,.

i. *Consider* (3.91) *with* $\mu = \mu_*$. *Let* $\rho \geq 0$ *be arbitrary. If* $\hat{s}_{k,*}, \hat{\lambda}_*, \mu_*$ *solve* (3.83), *then* $\hat{s}_{k,*}$, $\sigma_* = \mu_*$ *solve* (3.92).

ii. *Consider* (3.91) *with* $\mu = \mu_*$. *Let* $\hat{s}_{k,*}, \hat{\lambda}_*, \mu_*$ *satisfy* (3.83), (3.84), (3.85). *There exists* $\rho_* > 0$, *dependent on* $\hat{s}_{k,*}$, *such that for all* $\rho \geq \rho_*$ *the point* $\hat{s}_{k,*}$ *and Lagrange multiplier* $\sigma_* = \mu_*$ *satisfy* (3.92), (3.93), (3.94).

Proof: i. Let $\hat{s}_{k,*}, \hat{\lambda}_*, \mu_*$ solve (3.83).

The first equation in (3.83) and the definition (3.87) imply that $\hat{\lambda}_* = \hat{\lambda}(\hat{s}_{k,*}, \mu_*)$.

Since $h_k(\hat{s}_{k,*}) = 0$, (3.87) and (3.88) imply

$$\begin{aligned}\nabla\Psi_k(\hat{s}_{k,*}, \rho, \mu_*) + \mu_*\hat{s}_{k,*} &= \left(I - \nabla h_k(\hat{s}_{k,*})[\nabla h_k(\hat{s}_{k,*})^T \nabla h_k(\hat{s}_{k,*})]^{-1} \nabla h_k(\hat{s}_{k,*})^T\right) \\ &\quad \times \left(\nabla \phi_k(\hat{s}_{k,*}) + \mu_* \hat{s}_{k,*}\right).\end{aligned}$$

Since

$$\left(I - \nabla h_k(\hat{s}_{k,*})[\nabla h_k(\hat{s}_{k,*})^T \nabla h_k(\hat{s}_{k,*})]^{-1} \nabla h_k(\hat{s}_{k,*})^T\right) \nabla h_k(\hat{s}_{k,*}) = 0,$$

we can multiply the first equation in (3.83) by $I - \nabla h_k(\hat{s}_{k,*})[\nabla h_k(\hat{s}_{k,*})^T \nabla h_k(\hat{s}_{k,*})]^{-1} \nabla h_k(\hat{s}_{k,*})^T$ to show that $\hat{s}_{k,*}$, $\sigma_* = \mu_*$ solve (3.92).

ii. Let $\hat{s}_{k,*}, \hat{\lambda}_*, \mu_*$ solve (3.83). The first equation in (3.83) implies

$$\begin{aligned}\nabla\hat{\lambda}(\hat{s}_{k,*}, \mu_*)^T &= -[\nabla h_k(\hat{s}_{k,*})^T \nabla h_k(\hat{s}_{k,*})]^{-1} \nabla h_k(\hat{s}_{k,*})^T \\ &\quad \times \left(\nabla[\phi_k(\hat{s}_{k,*}) + h_k(\hat{s}_{k,*})^T \hat{\lambda}]|_{\hat{\lambda}=\hat{\lambda}(\hat{s}_{k,*},\mu_*)} + 2\mu_* D_k\right). \qquad (3.95)\end{aligned}$$

Since $h_k(\hat{s}_{k,*}) = 0$,

$$\begin{aligned}\nabla^2\Psi_k(\hat{s}_{k,*}, \rho, \mu_*) &= \nabla^2[\phi_k(\hat{s}_{k,*}) + h_k(\hat{s}_{k,*})^T \hat{\lambda}]|_{\hat{\lambda}=\hat{\lambda}(\hat{s}_{k,*},\mu_*)} + \rho \nabla h_k(\hat{s}_{k,*}) \nabla h_k(\hat{s}_{k,*})^T \\ &\quad + \nabla\hat{\lambda}(\hat{s}_{k,*}, \mu_*) \nabla h_k(\hat{s}_{k,*})^T + \nabla h_k(\hat{s}_{k,*}) \nabla\hat{\lambda}(\hat{s}_{k,*}, \mu_*)^T.\end{aligned}$$

With

$$\begin{aligned}Q(\hat{s}_{k,*}) &= \nabla^2[\phi_k(\hat{s}_{k,*}) + h_k(\hat{s}_{k,*})^T \hat{\lambda}]|_{\hat{\lambda}=\hat{\lambda}(\hat{s}_{k,*},\mu_*)} + 2\mu_* D_k, \\ H(\hat{s}_{k,*}) &= \nabla h_k(\hat{s}_{k,*})[\nabla h_k(\hat{s}_{k,*})^T \nabla h_k(\hat{s}_{k,*})]^{-1} \nabla h_k(\hat{s}_{k,*})^T,\end{aligned}$$

we have

$$
\begin{aligned}
&\nabla^2\Psi_k(\hat{s}_{k,*}, \rho, \mu_*) + 2\mu_* D_k \\
= \quad & Q(\hat{s}_{k,*}) + \rho\nabla h_k(\hat{s}_{k,*})\nabla h_k(\hat{s}_{k,*})^T - Q(\hat{s}_{k,*})H(\hat{s}_{k,*}) - H(\hat{s}_{k,*})Q(\hat{s}_{k,*}).
\end{aligned}
$$

Case 1: Let $\|P_k\hat{s}_{k,*}\| < (1-\alpha^{\mathrm{n}})\Delta_k$.

Let v be arbitrary and decompose $v = u + w$ with $u \in N(\nabla h_k(\hat{s}_{k,*})^T)$ and $w \in N(\nabla h_k(\hat{s}_{k,*})^T)^{\perp}$.

If $\tau(\hat{s}_{k,*})$ is the smallest positive singular value of $\nabla h_k(\hat{s}_{k,*})^T$, then

$$
\|\nabla h_k(\hat{s}_{k,*})^T w\|^2 \geq \tau(\hat{s}_{k,*})^2\|w\|^2 \quad \forall w \in \mathbb{R}^{N_k}.
$$

Using the definition of $Q(\hat{s}_{k,*})$ and $H(\hat{s}_{k,*})$ and (3.84) we obtain

$$
\begin{aligned}
&(u+w)^T(\nabla^2\Psi_k(\hat{s}_{k,*},\rho,\mu_*)+2\mu_* D_k)(u+w)\\
&= u^TQ(\hat{s}_{k,*})u + 2u^TQ(\hat{s}_{k,*})w + w^TQ(\hat{s}_{k,*})w + \rho\|\nabla h_k(\hat{s}_{k,*})^T w\|^2\\
&\quad -2u^TQ(\hat{s}_{k,*})H(\hat{s}_{k,*})w - 2w^TQ(\hat{s}_{k,*})H(\hat{s}_{k,*})w,\\
&\geq \gamma(\hat{s}_{k,*})\|u\|^2 - 2\|Q(\hat{s}_{k,*})\|\|u\|\|w\| - \|Q(\hat{s}_{k,*})\|\|w\|^2 + \rho\tau(\hat{s}_{k,*})^2\|w\|^2\\
&\quad -2\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\|\|u\|\|w\| - 2\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\|\|w\|^2\\
&= \frac{\gamma(\hat{s}_{k,*})}{2}\|u\|^2 + \left(\frac{\sqrt{\gamma(\hat{s}_{k,*})}}{2}\|u\| - \frac{2\|Q(\hat{s}_{k,*})\|}{\sqrt{\gamma(\hat{s}_{k,*})}}\|w\|\right)^2\\
&\quad -\left(\|Q(\hat{s}_{k,*})\| + \frac{4\|Q(\hat{s}_{k,*})\|^2}{\gamma(\hat{s}_{k,*})}\right)\|w\|^2 + \rho\tau(\hat{s}_{k,*})^2\|w\|^2\\
&\quad +\left(\frac{\sqrt{\gamma(\hat{s}_{k,*})}}{2}\|u\| - \frac{2\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\|}{\sqrt{\gamma(\hat{s}_{k,*})}}\|w\|\right)^2\\
&\quad -\left(2\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\| + \frac{4\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\|^2}{\gamma(\hat{s}_{k,*})}\right)\|w\|^2,\\
&\geq \frac{\gamma(\hat{s}_{k,*})}{2}\|u\|^2 + \Big(\rho\tau(\hat{s}_{k,*})^2 - \|Q(\hat{s}_{k,*})\| - \frac{4\|Q(\hat{s}_{k,*})\|^2}{\gamma(\hat{s}_{k,*})}\\
&\quad -2\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\| - \frac{4\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\|^2}{\gamma(\hat{s}_{k,*})}\Big)\|w\|^2.
\end{aligned}
$$

The assertion now follows if we choose

$$
\rho_* > \frac{1}{\tau(\hat{s}_{k,*})^2}\left(\|Q(\hat{s}_{k,*})\| + \frac{4\|Q(\hat{s}_{k,*})\|^2}{\gamma(\hat{s}_{k,*})} + 2\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\| + \frac{4\|Q(\hat{s}_{k,*})H(\hat{s}_{k,*})\|^2}{\gamma(\hat{s}_{k,*})}\right).
$$

Case 2: Let $\|P_k\hat{s}_{k,*}\| = (1-\alpha^n)\Delta_k$.

Let v be arbitrary and decompose $v = u + w$ with $u \in N((\nabla h_k(\hat{s}_{k,*})|D_k\hat{s}_{k,*})^T)$ and $w \in N((\nabla h_k(\hat{s}_{k,*})|D_k\hat{s}_{k,*})^T)^\perp$. Since $N((\nabla h_k(\hat{s}_{k,*})^T) \subset N((\nabla h_k(\hat{s}_{k,*})|D_k\hat{s}_{k,*})^T)$, $w \in$

$N((\nabla h_k(\hat{s}_{k,*})|D_k\hat{s}_{k,*})^T)^\perp \subset N((\nabla h_k(\hat{s}_{k,*})^T)^\perp.$

If $\tau(\hat{s}_{k,*})$ is the smallest positive singular value of $\nabla h_k(\hat{s}_{k,*})^T$, then

$$\|\nabla h_k(\hat{s}_{k,*})^T w\|^2 \geq \tau(\hat{s}_{k,*})^2 \|w\|^2 \quad \forall w \in \mathbb{R}^{N_k}.$$

The remainder of the proof is identical to the proof of case 1. □

3.7.1.3 Conceptual Algorithm for the Reformulated Tangential Subproblem

Let $\hat{s}_{k,*}$ be a global solution of (3.82), let μ_* be the corresponding Lagrange multiplier and let ρ_* be the penalty parameter specified in Theorem 3.7.4 ii.

The subproblem (3.91) with $\rho = \rho_*$ and $\mu = \mu_*$ is solved using a trust–region method. The ith iterate in this trust–region method applied to (3.91) is denoted by $\hat{s}_{i,k}$, the ith step is denoted by $\hat{z}_i$, and the trust–region radius of the ith iteration is $\delta^{\mathrm{t}}_{i,k}$.

$$\begin{array}{ll} \min & \nabla\Psi_k(\hat{s}_{i,k}, \rho_*, \mu_*)^T\hat{z} + \frac{1}{2}\hat{z}^T\nabla^2\Psi_k(\hat{s}_{i,k}, \rho_*, \mu_*)\hat{z}, \\ \text{s.t.} & \|P_k(\hat{s}_{i,k} + \hat{z})\| \leq (1 - \alpha^{\mathrm{n}})\Delta_k, \\ & \|P_k\hat{z}\| \leq \delta^{\mathrm{t}}_{i,k}. \end{array} \tag{3.96}$$

To simplify this presentation, we only increase the iteration count i when the iteration is successful.

The initial iterate is chosen to be $\hat{s}_{0,k} = 0$ and the initial trust-region radius is $\delta^{\mathrm{t}}_{0,k} = (1 - \alpha^{\mathrm{n}})\Delta_k$. With this choice of initial data the trust-region subproblem (3.96) with $i = 0$ reduces to

$$\begin{array}{ll} \min & \nabla\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T\hat{z} + \frac{1}{2}\hat{z}^T\nabla^2\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\hat{z}, \\ \text{s.t.} & \|P_k\hat{z}\| \leq \delta^{\mathrm{t}}_{0,k}. \end{array} \tag{3.97}$$

For reasons that will become apparent later, we replace the Hessian in (3.97) by zero. Thus, our initial trust-region subproblem is

$$\begin{array}{ll} \min & \nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z} \\ \text{s.t.} & \|P_k \hat{z}\| \leq \delta^{\mathrm{t}}_{0,k}. \end{array} \tag{3.98}$$

A standard result in trust-region methods states that the solution $\hat{z}_0$ of (3.98) satisfies the sufficient decrease condition

$$\begin{aligned} & -\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0 \\ \geq \;& c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \min\{c_2 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, \delta^{\mathrm{t}}_{0,k}\}, \end{aligned} \tag{3.99}$$

where $c_1 \in (0, 1)$, $c_2 > 0$ are independent of k.

Remark 3.7.5 *Note that c_2 depends on the Hessian of the model. For the model in* (3.98) *this Hessian is zero and, in particular, c_2 is independent of k. If we had used the exact Hessian $\nabla^2 \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)$ as in* (3.97)*, then it would have been unrealistic to assume that c_2 is independent of k, since the norm of the Hessian depends on ρ_*, μ_*, which in turn depend on $\hat{s}_{k,*}$*

Let $0 < \eta^{\mathrm{t}}, \gamma^{\mathrm{t}}_1, \gamma^{\mathrm{t}}_2 < 1$ be given parameters. The step $\hat{z}_0$ will be accepted if

$$\frac{\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) - \Psi_k(\hat{s}_{0,k} + \hat{z}_0, \rho_*, \mu_*)}{-\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0} \geq \eta^{\mathrm{t}}. \tag{3.100}$$

If $\hat{z}_0$ is accepted, the next trust–region iterate is $\hat{s}^{\mathrm{n}}_{1,k} = \hat{z}_0$ and the corresponding trust–region radius satisfies $\delta^{\mathrm{t}}_{1,k} \geq \gamma^{\mathrm{t}}_1 \delta^{\mathrm{t}}_{0,k}$. If (3.100) is not satisfied, the step $\hat{z}_0$ will be rejected, $\delta^{\mathrm{t}}_{0,k}$ will be decreased by a factor between γ^{t}_2 and γ^{t}_1, and (3.98) will be resolved.

It is sufficient to perform only one successful step in the trust-region algorithm for (3.91). If additional trust–region iterates are computed, they are required to satisfy

$$\Psi_k(\hat{s}_{i,k}, \rho_*, \mu_*) \geq \Psi_k(\hat{s}_{i,k} + \hat{z}_i, \rho_*, \mu_*)$$

and $\|P_k(\hat{s}_{i,k} + \hat{z}_i)\| \leq (1 - \alpha^{\mathrm{n}})\Delta_k$.

The conceptual algorithm is given as follows.

Algorithm 3.7.6 *1. Initialization. Let* $0 < \eta^{\mathrm{t}}, \gamma_1^{\mathrm{t}}, \gamma_2^{\mathrm{t}} < 1$ *be given and set* $\hat{s}_{0,k} = 0$, $\delta_{0,k}^{\mathrm{t}} = (1 - \alpha^{\mathrm{n}})\Delta_k$.

2. *Compute a trial step* $\hat{z}_0$ *of subproblem* (3.98) *that satisfies* (3.99).

3. *Compute*

$$r_0^{\mathrm{t}} = \frac{\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) - \Psi_k(\hat{s}_{0,k} + \hat{z}_0, \rho_*, \mu_*)}{-\nabla\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0}.$$

If $r_0^{\mathrm{t}} < \eta^{\mathrm{t}}$, *reduce* $\delta_{0,k}^{\mathrm{t}}$ *by a factor of at least* γ_1^{t} *and at most* γ_2^{t}, *and goto step 2; else set* $\hat{s}_{1,k} = \hat{s}_{0,k} + \hat{z}_0$ *and choose* $\delta_{1,k}^{\mathrm{t}} \geq \gamma_1^{\mathrm{t}}\delta_{0,k}^{\mathrm{t}}$.

4. *For* $i = 1, \cdots$

   (a) *Compute a trial step* $\hat{z}_i$ *of subproblem* (3.96) *that satisfies*

   $$\Psi_k(\hat{s}_{i,k}, \rho_*, \mu_*) \geq \Psi_k(\hat{s}_{i,k} + \hat{z}_i, \rho_*, \mu_*) \tag{3.101}$$

   and $\|P_k(\hat{s}_{i,k} + \hat{z}_i)\| \leq (1 - \alpha^{\mathrm{n}})\Delta_k$.

   (b) *Set* $\hat{s}_{i+1,k} = \hat{s}_{i,k} + \hat{z}_i$.

End

We call the 0th trial iterate successful if $r_0^{\rm t} \geq \eta^{\rm t}$.

Lemma 3.7.7 *Let assumption AM1, AM2 and AM3 hold. Let $L_k > 0$ be a Lipschitz constant such that*

$$[\nabla \Psi_k(\hat{s}_{0,k} + \xi \hat{z}_0, \rho_*, \mu_*) - \nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)]^T \hat{z}_0 \leq L_k \xi^2 \|\hat{z}_0\|^2 \quad \forall \xi \in [0, 1].$$

If $\|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \neq 0$ there exists constant ξ independent of k such that if

$$\delta_{0,k}^{\rm t} \leq \frac{c_1(1-\eta^{\rm t})}{\max\{L_k, 1/c_2\}} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, \tag{3.102}$$

then the trial step $\hat{z}_0$ will be accepted.

Proof: Notice that $0 < c_1 < 1$ and $0 < \eta^{\rm t} < 1$ imply $c_1(1-\eta^{\rm t}) < 1$. Hence, (3.102) implies

$$\delta_{0,k}^{\rm t} \leq c_2 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|.$$

This inequality and (3.99) yield

$$\begin{aligned} -\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0 &\geq c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \min\{c_2 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, \delta_{0,k}^{\rm t}\} \\ &= c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \delta_{0,k}^{\rm t}. \end{aligned} \tag{3.103}$$

The definition of $r_0^{\rm t}$, (3.103), and (3.102) imply that, with some $\xi \in [0, 1]$,

$$\begin{aligned} |r_0^{\rm t} - 1| &= \left| \frac{\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) + \nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0 - \Psi_k(\hat{s}_{0,k} + \hat{z}_0, \rho_*, \mu_*)}{-\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0} \right| \\ &= \left| \frac{[\nabla \Psi_k(\hat{s}_{0,k} + \xi \hat{z}_0, \rho_*, \mu_*) - \nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)]^T \hat{z}_0}{-\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0} \right| \\ &\leq \frac{L_k \delta_{0,k}^{\rm t}}{c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|} \\ &\leq 1 - \eta^{\rm t}. \end{aligned}$$

Therefore $r_0^{\mathrm{t}} > \eta^{\mathrm{t}}$, which means $\hat{z}_0$ will be accepted. $\square$

Remark 3.7.8 *Since Ψ_k is twice continuously differentiable with respect to $\hat{x}$, we have that*

$$L_k \leq \max_{\xi \in (0,1)} \|\nabla^2 \Psi_k(\xi \hat{z}_0, \rho_*, \mu_*)\|. \tag{3.104}$$

The definition (3.90) *of $\nabla^2 \Psi_k$ shows that the right hand side in* (3.104) *depends on $\rho_* = \rho(\hat{s}_{k,*})$ and $\mu_* = \mu(\hat{s}_{k,*})$ which in turn depends on the iteration k. Hence it is not clear whether L_k can be uniformly bounded from above.*

Lemma 3.7.9 *Let assumptions AM1, AM2 and AM3 be satisfied. There exist $c_1^{\mathrm{t}} > 0$, independent of k and $c_2^{\mathrm{t}} > 0$ dependent on L_k such that the steps $\hat{s}_{i,k}$, $i = 1, \ldots$, computed in Algorithm* 3.7.6 *satisfy*

$$\begin{aligned} & \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) - \Psi_k(\hat{s}_{i,k}, \rho_*, \mu_*) \\ \geq \;& c_1^{\mathrm{t}} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \min\{c_2^{\mathrm{t}} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, (1-\alpha^{\mathrm{n}})\Delta_k\}. \end{aligned} \tag{3.105}$$

Proof: The proof is similar to the proof of Theorem 3.6.4. We distinguish two cases.

Case 1: Let

$$(1-\alpha^{\mathrm{n}})\Delta_k \leq \frac{c_1(1-\eta^{\mathrm{t}})}{\max\{L_k, 1/c_2\}} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|.$$

Since $\delta_{0,k}^{\mathrm{t}} = (1-\alpha^{\mathrm{n}})\Delta_k$ satisfies (3.102), Lemma 3.7.9 guarantees that the first trial step is successful,

$$\hat{s}_{1,k}^{\mathrm{t}} = \hat{z}_0.$$

The definition of a successful iterate and (3.69) yield

$$\begin{aligned}
&\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) - \Psi_k(\hat{s}_{1,k}, \rho_*, \mu_*) \\
&\quad \geq \; -\eta^{\mathrm{t}} \nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0 \\
&\quad \geq \; \eta^{\mathrm{t}} c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \min\{c_2 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, (1-\alpha^{\mathrm{n}})\Delta_k\}.
\end{aligned}$$

Since all subsequent iterates satisfy (3.101), we obtain

$$\begin{aligned}
&\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) - \Psi_k(\hat{s}_{i,k}, \rho_*, \mu_*) \\
&\quad \geq \; \eta^{\mathrm{t}} c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \min\{c_2 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, (1-\alpha^{\mathrm{n}})\Delta_k\}.
\end{aligned}$$

Case 2: Let

$$(1-\alpha^{\mathrm{n}})\Delta_k > \frac{c_1(1-\eta^{\mathrm{t}})}{\max\{L_k, 1/c_2\}} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|.$$

Let $\delta^{\mathrm{t}}_{0,k}$ be the trust-region radius that corresponds to the first successful trial iterate. Lemma 3.6.3 and the fact that at the end of an unsuccessful trial iterate the trust-region radius is reduced by at most γ^{t}_2 imply the inequality

$$\delta^{\mathrm{t}}_{0,k} > \gamma^{\mathrm{t}}_2 \frac{c_1(1-\eta^{\mathrm{n}})}{\max\{L_k, 1/c_2\}} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|. \tag{3.106}$$

The definition of a successful iterate, (3.99), and (3.106) yield

$$\begin{aligned}
&\Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*) - \Psi_k(\hat{s}_{1,k}, \rho_*, \mu_*) \\
&\quad \geq \; -\eta^{\mathrm{t}} \nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)^T \hat{z}_0 \\
&\quad \geq \; \eta^{\mathrm{t}} c_1 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\| \min\{c_2 \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|, \delta^{\mathrm{t}}_{0,k}\} \\
&\quad \geq \; \eta^{\mathrm{t}} c_1 \min\left\{c_2, \gamma^{\mathrm{t}}_2 \frac{c_1(1-\eta^{\mathrm{t}})}{\max\{L_k, 1/c_2\}}\right\} \|\nabla \Psi_k(\hat{s}_{0,k}, \rho_*, \mu_*)\|^2.
\end{aligned}$$

Since all subsequent iterates satisfy (3.101), this implies

$$\begin{aligned}&\Psi_k(\hat{s}_{0,k},\rho_*,\mu_*)-\Psi_k(\hat{s}_{i,k},\rho_*,\mu_*)\\ \geq\ &\eta^{\mathrm{t}}c_1\min\left\{c_2,\gamma_2^{\mathrm{t}}\frac{c_1(1-\eta^{\mathrm{t}})}{\max\{L_k,1/c_2\}}\right\}\|\nabla\Psi_k(\hat{s}_{0,k},\rho_*,\mu_*)\|^2\end{aligned}$$

The desired result now follows with

$$c_1^{\mathrm{t}}=\eta^{\mathrm{t}}c_1,\quad c_2^{\mathrm{t}}=\min\left\{c_2,\gamma_2^{\mathrm{t}}\frac{c_1(1-\eta^{\mathrm{t}})}{\max\{L_k,1/c_2\}}\right\}.$$

□

Remark 3.7.10 *In algorithm* 3.7.6, *the first iterate* $\hat{s}_{i,k}$ *is required to satisfy the sufficient decrease condition*

$$\begin{aligned}&\Psi_k(\hat{s}_{0,k},\rho_*,\mu_*)-\Psi_k(\hat{s}_{1,k},\rho_*,\mu_*)\\ \geq\ &\eta^{\mathrm{t}}c_1\|\nabla\Psi_k(\hat{s}_{0,k},\rho_*,\mu_*)\|\min\left\{c_2\|\nabla\Psi_k(\hat{s}_{0,k},\rho_*,\mu_*)\|,\delta_{0,k}^{\mathrm{t}}\right\}.\end{aligned}$$

The following sequent iterates are required to satisfy

$$\Psi_k(\hat{s}_{i,k},\rho_*,\mu_*)\geq\Psi_k(\hat{s}_{i+1,k},\rho_*,\mu_*).$$

If we require the stronger condition

$$\begin{aligned}&\Psi_k(\hat{s}_{i,k},\rho_*,\mu_*)-\Psi_k(\hat{s}_{i+1,k},\rho_*,\mu_*)\\ \geq\ &\eta^{\mathrm{t}}c_1\|\nabla\Psi_k(\hat{s}_{i,k},\rho_*,\mu_*)\|\min\left\{c_2\|\nabla\Psi_k(\hat{s}_{i,k},\rho_*,\mu_*)\|,\delta_{i,k}^{\mathrm{t}}\right\},\end{aligned}$$

$i=1,2,\ldots$, *then the convergence theory for trust region methods guarantees that* $\liminf_{i\to\infty}\nabla\Psi_k(\hat{s}_{i,k},\rho_*,\mu_*)=0$, *and under stronger assumptions even that* $\lim_{i\to\infty}\hat{s}_{i,k}=\widetilde{\hat{s}_{k,*}}$, *where* $\widetilde{\hat{s}_{k,*}}$ *is a local solution of* (3.91) *(although not necessarily* $\widetilde{\hat{s}_{k,*}}=\hat{s}_{k,*}$.

Theorem 3.7.11 *Let assumptions AM1, AM2 and AM3 be satisfied. Assume that the solution $\hat{s}_{k,*}$ satisfies*

$$\Psi_k(\hat{s}_{k,*}, \rho_*, \mu_*) \leq \Psi_k(\hat{s}_{1,k}, \rho_*, \mu_*).$$

There exist c_1^{t}, independent of k and $c_2^{\mathrm{t}} > 0$ dependent on L_k such that the steps $\hat{s}_{i,k}$, $i = 1, \ldots$, computed in Algorithm 3.7.6 *satisfy*

$$\begin{aligned} & m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}}) - m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}_{k,*}) \\ \geq \; & c_1^{\mathrm{t}} \| Z_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})^T \nabla m_k^f(\hat{x}_k + \hat{s}_k^{\mathrm{n}}) \| \\ & \times \min\{ c_2^{\mathrm{t}} \| Z_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})^T \nabla m_k^f(\hat{x}_k + \hat{s}_k^{\mathrm{n}}) \|, (1 - \alpha^{\mathrm{n}})\Delta_k \}, \end{aligned} \tag{3.107}$$

where $Z_k^c(\hat{x}_k + \hat{s}_k^{\mathrm{n}})$ is defined in (3.9).

Proof: By definition of (3.81) of h_k, we have $h_k(0) = 0$. Since $\hat{s}_{0,k} = 0$, (3.87) and (3.88) imply

$$\nabla \Psi_k(0, \rho_*, \mu_*) = \left(I - \nabla h_k(0) [\nabla h_k(0)^T \nabla h_k(0)]^{-1} \nabla h_k(0) \right) \nabla \phi_k(0).$$

Since $\Psi_k(\hat{s}_{k,*}, \rho_*, \mu_*) \leq \Psi_k(\hat{s}_{1,k}, \rho_*, \mu_*)$, Lemma 3.7.9 implies

$$\begin{aligned} & \Psi_k(0, \rho_*, \mu_*) - \Psi_k(\hat{s}_{k,*}, \rho_*, \mu_*) \\ \geq \; & c_1^{\mathrm{t}} \| \nabla \Psi_k(0, \rho_*, \mu_*) \| \min\{ c_2^{\mathrm{t}} \| \nabla \Psi_k(0, \rho_*, \mu_*) \|, (1 - \alpha^{\mathrm{n}})\Delta_k \}. \end{aligned}$$

Since $h_k(0) = h_k(\hat{s}_{k,*}) = 0$, $\Psi_k(0, \rho_*, \mu_*) = \phi_k(0) = m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}})$ and $\Psi_k(\hat{s}_{k,*}, \rho_*, \mu_*) = \phi_k(\hat{s}_{k,*}) = m_k^l(\hat{x}_k + \hat{s}_k^{\mathrm{n}} + \hat{s}_{k,*})$, this implies the assertion. □

3.7.2 Algorithm for the Computation of the Tangential Step

The Algorithm 3.7.6 for the computation of the tangential step, was used in a constructive proof to show that, under certain conditions the requirements (3.10), (3.11) and (3.13) can be satisfied. For practical purposes, however, the algorithm is less useful. First, the feasible set in (3.79) is only a small subset of the feasible set in (3.7). Therefore, using (3.79) might slow the progress of the GTRSQP algorithm down unnecessarily. Secondly, Algorithm 3.7.6 operates with the penalty function. The penalty parameter ρ_* is available theoretically, but not practically. In our example, we therefore use a different algorithm for the computations of the tangential step.

Our algorithm used is a slight modification of the trust-region algorithm in [11] applied to (3.7) with initial iterate $\hat{s}^{\mathrm{t}}_{0,k} = 0$. The constraint $\|P_k(\hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i,k} + \hat{z})\| \leq \Delta_k$ is incorporated using a simple barrier approach [26]. That is, we consider the tangential subproblem

$$\begin{array}{ll} \min & m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i,k} + \hat{z}) - \varsigma_k \ln(\Delta^2_k - \|P_k(\hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i,k} + \hat{z})\|^2), \\ \text{s.t.} & m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i,k} + \hat{z}) = m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k), \\ & \|P_k \hat{z}\| \leq \delta^{\mathrm{t}}_{i,k}. \end{array} \tag{3.108}$$

where ς_k is a positive scalar, the barrier parameter.

At the first successful iteration the augmented Lagrangian merit function

$$\begin{aligned} M(\hat{s}, \hat{\sigma}, \rho^{\mathrm{t}}) &= m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}) - \varsigma_k \ln(\Delta^2_k - \|P_k(\hat{s}^{\mathrm{n}}_k + \hat{s})\|^2) \\ & \quad + \hat{\sigma}^T \left(m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}) - m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k)\right) \\ & \quad + \frac{\rho^{\mathrm{t}}}{2} \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}) - m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k)\|^2 \end{aligned} \tag{3.109}$$

satisfies

$$\begin{aligned} & M(0, \hat{\sigma}_{0,k}, \rho^{\mathrm{t}}_{0,k}) - M(\hat{s}^{\mathrm{t}}_{1,k}, \hat{\sigma}_{1,k}, \rho^{\mathrm{t}}_{0,k}) \\ \geq \; & c^{\mathrm{t}}_1 \chi^M_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k) \min\left\{ c^{\mathrm{t}}_2 \chi^M_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k), \delta^{\mathrm{t}}_{0,k} \right\} \end{aligned} \tag{3.110}$$

where $c^{\mathrm{t}}_1, c^{\mathrm{t}}_2 > 0$ are constants independent of k (cf. (3.13)). Note that $M(0, \hat{\sigma}_{0,k}, \rho^{\mathrm{t}}_{0,k}) = m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k) - \varsigma_k \ln(\Delta^2_k - \|P_k \hat{s}^{\mathrm{n}}_k\|^2)$. We continue the trust-region SQP iteration in [11] until

$$\begin{aligned} & m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k) - \varsigma_k \ln(\Delta^2_k - \|P_k \hat{s}^{\mathrm{n}}_k\|^2) \\ & - m^l_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i+1,k}) + \varsigma_k \ln(\Delta^2_k - \|P_k(\hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i+1,k})\|^2) \\ \geq \; & M(0, \hat{\sigma}_{0,k}, \rho^{\mathrm{t}}_{0,k}) - M(\hat{s}^{\mathrm{t}}_{1,k}, \hat{\sigma}_{1,k}, \rho^{\mathrm{t}}_{0,k}) \end{aligned}$$

(cf. (3.13), (3.110)) and

$$\|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k + \hat{s}^{\mathrm{t}}_{i+1,k})\| \leq \sqrt{\mu_k} \|m^c_k(\hat{x}_k + \hat{s}^{\mathrm{n}}_k)\|$$

(cf. (3.11)).

Chapter 4

Optimal Control of Burgers Equation

4.1 Problem Formulation

We consider an optimal control problem governed by Burgers equation. The infinite dimensional optimization problem is given by

$$\min \quad \frac{1}{2}\int_0^1 (y(x) - z(x))^2 dx + \frac{\alpha}{2}\int_0^1 u(x)^2 dx \tag{4.1a}$$

$$\text{s.t.} \quad -\nu y_{xx}(x) + y(x)y_x(x) = r(x) + u(x) \quad x \in (0, 1), \tag{4.1b}$$

$$y(0) = y(1) = 0,$$

where $\alpha, \nu \in \mathbb{R}$, $\nu > 0$, $r \in L^2(0, 1)$ and $z \in L^2(0, 1)$ are given. Results about existence and characterization of solutions of (4.1) and related problems can be found, e.g., in [28, 29, 30]. We consider a finite element discrimination of (4.1).

Let $\phi_1, \cdots, \phi_{n_y}$ denote the basis functions for the state and let $\psi_1, \cdots, \psi_{n_u}$ denote the basis functions for the control. We assume that $\phi_i(0) = \phi_i(1) = 0$, $i = 1, \ldots, n_y$. The state and the control variables are approximated by

$$y_h(x) = \sum_{i=1}^{n_y} y_i \phi_i(x), \qquad u_h(x) = \sum_{i=1}^{n_u} u_i \psi_i(x). \tag{4.2}$$

We define

$$\vec{y} = (y_1, \cdots, y_{n_y})^T, \qquad \vec{u} = (u_1, \cdots, u_{n_u})^T,$$

We require that the weak form of (4.1b),

$$\int_0^1 \nu y_x(x) v_x(x) + y(x) y_x(x) v(x) - r(x) v(x) - u(x) v(x) dx \quad \forall v \in H_0^1(0,1),$$

is satisfied for $y = y_h$, $u = u_h$ and $v = \phi_1, \cdots, \phi_{n_y}$. This leads to the discredited state equations

$$A\vec{y} + N(\vec{y}) - \vec{r} - B\vec{u} = 0, \tag{4.3}$$

where

$$A \in \mathbb{R}^{n_y \times n_y},\ B \in \mathbb{R}^{n_y \times n_u},\ N(\vec{y}) \in \mathbb{R}^{n_y},\ \vec{r} \in \mathbb{R}^{n_y}$$

are matrices and vectors with entries

$$\begin{aligned}
A_{ij} &= \nu \int_0^1 (\phi_i)_x(x) (\phi_j)_x(x) dx, \\
B_{ij} &= \int_0^1 \phi_i(x) \psi_j(x) dx, \\
(N(\vec{y}))_i &= \int_0^1 \left(\sum_{j=1}^{n_y} y_j \phi_j(x) \right) \left(\sum_{j=1}^{n_y} y_j (\phi_j)_x(x) \right) \phi_i(x) dx, \\
\vec{r}_i &= \int_0^1 r(x) \phi_i(x) dx.
\end{aligned}$$

The objective function (4.1a) is discredited as follows

$$\frac{1}{2}(\vec{y} - \vec{z})^T M_y (\vec{y} - \vec{z}) + \frac{\alpha}{2} \vec{u}^T M_u \vec{u}. \tag{4.4}$$

Here $M_y \in \mathbb{R}^{n_y \times n_y}$, $M_u \in \mathbb{R}^{n_u \times n_u}$, $\vec{z} \in \mathbb{R}^{n_y}$, are matrices and a vector with entries

$$\begin{aligned}
(M_y)_{ij} &= \int_0^1 \phi_i(x)\phi_j(x)dx, \\
(M_u)_{ij} &= \int_0^1 \psi_i(x)\psi_j(x)dx, \\
\vec{z}_i &= \int_0^1 z(x)\phi_i(x)dx.
\end{aligned}$$

Thus, our discredited optimal control problem is of the form

$$\begin{array}{ll}
\min & \frac{1}{2}(\vec{y}-\vec{z})^T M_y(\vec{y}-\vec{z}) + \frac{\alpha}{2}\vec{u}^T M_u \vec{u}, \\
\text{s.t.} & A\vec{y} + N(\vec{y}) - \vec{r} - B\vec{u} = 0.
\end{array} \tag{4.5}$$

If we subdivide the interval $[0,1]$ into n_x subinterval of length $\Delta x = 1/n_x$ and if we use the standard piecewise linear basis functions

$$\phi_i(x) = \begin{cases} (\Delta x)^{-1}(x-(i-1)\Delta x) & x \in [(i-1)\Delta x, i\Delta x], \\ (\Delta x)^{-1}(-x+(i+1)\Delta x) & x \in [i\Delta x, (i+1)\Delta x], \quad i = 1, \cdots n_y, \\ 0 & \text{else} \end{cases}$$

$$\psi_i(x) = \begin{cases} (\Delta x)^{-1}(x-(i-2)\Delta x) & x \in [(i-2)\Delta x, (i-1)\Delta x], \\ (\Delta x)^{-1}(-x+i\Delta x) & x \in [(i-1)\Delta x, i\Delta x], \qquad i = 1, \cdots n_u, \\ 0 & \text{else} \end{cases}$$

where $n_y \stackrel{\text{def}}{=} n_x - 1$ and $n_u \stackrel{\text{def}}{=} n_x + 1$, then

$$M_y = \frac{\Delta x}{6} \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ \ddots & \ddots & \ddots & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 \end{pmatrix} \in \mathbb{R}^{n_y \times n_y},$$

$$A = \frac{\nu}{\Delta x} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ \ddots & \ddots & \ddots & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n_y \times n_y}, \tag{4.6}$$

$$N(y) = \frac{1}{6}\begin{pmatrix} y_1y_2 + y_2^2 \\ -y_1^2 - y_1y_2 - y_2y_3 + y_3^2 \\ \vdots \\ -y_{i-1}^2 - y_{i-1}y_i + y_iy_{i+1} + y_{i+1}^2 \\ \vdots \\ -y_{n_y-2}^2 - y_{n_y-2}y_{n_y-1} + y_{n_y-1}y_{n_y} + y_{n_y}^2 \\ -y_{n_y-1}^2 - y_{n_y-1}y_{n_y} \end{pmatrix} \in \mathbb{R}^{n_y}$$

and

$$B = \frac{\Delta x}{6}\begin{pmatrix} 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 & \\ & & & & 1 & 4 & 1 \end{pmatrix} \in \mathbb{R}^{n_y \times n_u},$$

$$M_u = \frac{\Delta x}{6}\begin{pmatrix} 2 & 1 & & \\ 1 & 4 & 1 & \\ \ddots & \ddots & \ddots & \\ & 1 & 4 & 1 \\ & & 1 & 2 \end{pmatrix} \in \mathbb{R}^{n_u \times n_u}. \tag{4.7}$$

4.2 Hierarchical Basis

We use the piecewise linear hierarchical basis [31, 32, 33] for the discrimination of the state and of the state equation. Let $l \in \mathbb{N}$, $n_x = 2^l$, $\Delta x = 1/n_x$,

$$\mathcal{V}_l = \operatorname{span}\{\phi_{0,l}, \ldots, \phi_{n_x,l}\},$$

where

$$\phi_{i,l}(x) = \begin{cases} (\Delta x)^{-1}(x - (i-1)\Delta x) & x \in [(i-1)\Delta x, i\Delta x], \\ (\Delta x)^{-1}(-x + (i+1)\Delta x) & x \in [i\Delta x, (i+1)\Delta x], \quad i = 0, \cdots n_x, \\ 0 & \text{else} \end{cases}$$

and

$$\mathcal{N}_l = \{0, \Delta x, 2\Delta x, \ldots, 1\}.$$

The hierarchical basis of $\mathcal{V}_{l+1}$ is obtained by augmenting the basis of $\mathcal{V}_l$. In particular,

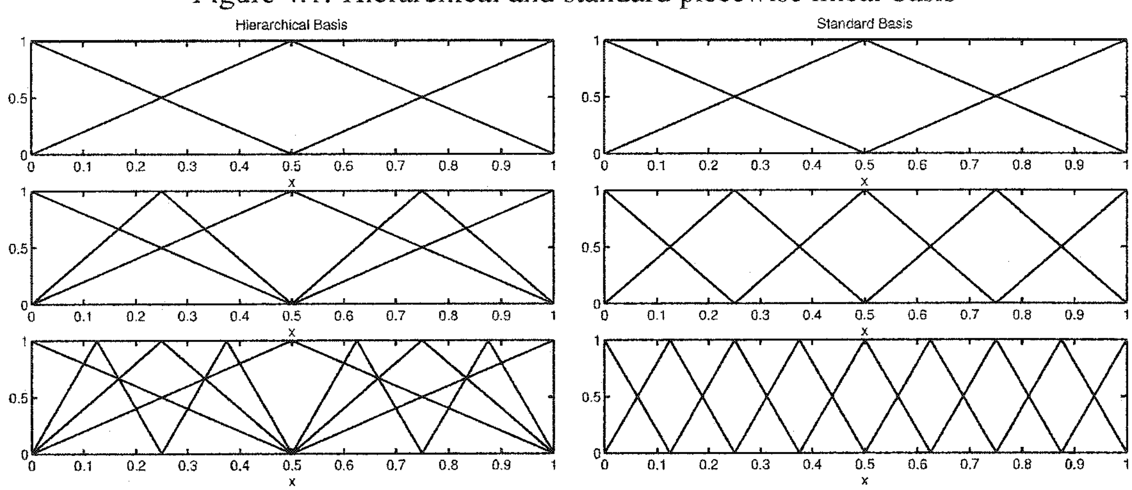
$$\mathcal{V}_{l+1} = \mathcal{V}_l \oplus \mathcal{W}_l$$

where

$$\mathcal{W}_l = \text{span}\{\phi_{i,l+1} \,:\, x_i \in \mathcal{N}_{l+1} \setminus \mathcal{N}_l\}.$$

The hierarchical bases for $l = 1, 2, 3$ are illustrated in Figure 4.1. For comparison, Figure 4.1 also shows the standard bases consisting of the hat functions $\phi_{0,l}, \ldots, \phi_{n_x,l}$.

Figure 4.1: Hierarchical and standard piecewise linear basis



If $y = \sum_{i=0}^{2^l} y_i \phi_{i,l} \in \mathcal{V}_l$, then the coefficients $\vec{y} = (y_0, \ldots, y_{2^l})^T$ of its representation $y = \sum_{i=0}^{2^l} y_i \phi_{i,l}$ in the standard basis and the coefficients $\vec{y}^H = (y_0^H, \ldots, y_{2^l}^H)^T$ of its representation $y = \sum_{i=0}^{2^l} y_i \phi_{i,l}^H$ in the hierarchical basis are related by

$$\vec{y}^H = T_H \vec{y}.$$

For example for $l = 2$,

$$T_H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular,

$$\begin{aligned} & A\vec{y} + N(\vec{y}) - \vec{r} - B\vec{u} \\ = \;& T_H^{-T} A T_H^{-1} \vec{y}^H + T_H^{-T} N(T_H^{-1}\vec{y}^H) - T_H^{-T}\vec{r} - T_H^{-T} B\vec{u}, \qquad (4.8) \\ & \frac{1}{2}(\vec{y} - \vec{z})^T M_y (\vec{y} - \vec{z}) + \frac{\alpha}{2}\vec{u}^T M_u \vec{u} \\ = \;& \frac{1}{2}(\vec{y}^H - T_H\vec{z})^T T_H^{-T} M_y T_H^{-1} (\vec{y}^H - T_H\vec{z}) + \frac{\alpha}{2}\vec{u}^T M_u \vec{u}. \qquad (4.9) \end{aligned}$$

(cf. (4.3), (4.4)). In practice one never forms $T_H^{-T} A T_H^{-1}$, $T_H^{-T} N(T_H^{-1}\vec{y}^H)$, etc., but instead works with A, $N(\vec{y})$, etc. and applies the mappings $\vec{y} \mapsto \vec{y}^H = T_H\vec{y}$ and $\vec{y}^H \mapsto \vec{y} = T_H^{-1}\vec{y}^H$. These can be computed efficiently using the algorithms discussed, e.g., in [31, 32].

We set

$$c(\vec{y}^H, \vec{u}) = T_H^{-T} A T_H^{-1} \vec{y}^H + T_H^{-T} N(T_H^{-1}\vec{y}^H) - T_H^{-T}\vec{r} - T_H^{-T} B\vec{u}, \qquad (4.10)$$

$$f(\vec{y}^H, \vec{u}) = \frac{1}{2}(\vec{y}^H - T_H\vec{z})^T T_H^{-T} M_y T_H^{-1} (\vec{y}^H - T_H\vec{z}) + \frac{\alpha}{2}\vec{u}^T M_u \vec{u}, \qquad (4.11)$$

and we consider the discredited optimal control problem

$$\begin{aligned} \min \quad & f(\vec{y}^H, \vec{u}), \\ \text{s.t.} \quad & c(\vec{y}^H, \vec{u}) = 0. \end{aligned} \qquad (4.12)$$

In the following we will consider the discredited problem (4.12) for a fixed fine grid of level L as our given problem and we will omit the $\vec{\cdot}^H$, $\vec{\cdot}$'s for simplicity.

4.3 Problem Structure

For problems like (4.5) the optimization variables group into states y and controls u. This grouping of variables as well as additional structure can be used in optimization algorithms (see, e.g., [20, 21]). We briefly review the problem structure here to indicate how it is used in our implementation of the GTRSQP algorithm.

The Jacobian $\nabla c(y,u)^T$ of $c(y,u)$ is given by

$$\nabla c(y,u)^T = (c_y(y,u) \quad c_u(y,u)),$$

where $c_y(y,u)$ and $c_u(y,u)$ are the partial Jacobians of $c(y,u)$ with respect to y and u, respectively. If $c_y(y,u)$ is invertible, then the columns of

$$Z(y,u) = \begin{pmatrix} -c_y(y,u)^{-1}c_u(y,u) \\ I \end{pmatrix} \tag{4.13}$$

span the null-space of $\nabla c(y,u)^T$. This null-space representation is used in the context of our application (4.5).

With the partitioning of the optimization variables into y and u, the Lagrangian reads

$$l(y,u,\lambda) = f(y,u) + \lambda^T c(y,u)$$

and its partial gradients are given by

$$\nabla_y l(y,u,\lambda) = f_y(y,u) + c_y(y,u)^T\lambda$$

and

$$\nabla_u l(y,u,\lambda) = f_u(y,u) + c_u(y,u)^T\lambda.$$

If we assume that $c_y(y,u)$ is invertible, a Lagrange multiplier estimate can be computed by solving the so-called adjoint equation

$$f_y(y,u) + c_y(y,u)^T\lambda = 0. \tag{4.14}$$

With this Lagrange multiplier estimate and (4.13) we have

$$Z_k(y,u)^T \begin{pmatrix} \nabla_y l(y,u,\lambda) \\ \nabla_u l(y,u,\lambda) \end{pmatrix} = \nabla_u l(y,u,\lambda).$$

In our particular application where the objective and constraint functions are given by (4.10), (4.11), we obtain

$$Z_k(y,u)^T \begin{pmatrix} \nabla_y l(y,u,\lambda) \\ \nabla_u l(y,u,\lambda) \end{pmatrix} = \nabla_u l(y,u,\lambda) = \alpha M_u u - B^T T_H^{-1}\lambda. \tag{4.15}$$

In our application the vectors $y \in \mathbb{R}^{n_y}$, $u \in \mathbb{R}^{n_u}$ represent functions $\sum_{i=0}^{2^l} y_i \phi_{i,l}^H \in H_0^1(0,1)$ and $\sum_{i=0}^{n_u} u_i \psi_i \in L^2(0,1)$, respectively. Therefore, we do not use the Euclidean norm to measure their lengths, but the following weighted norms:

$$\|u\|_{L^2}^2 = u^T M_u u, \tag{4.16}$$

where M_u is given by (4.7), and

$$\|y\|_{H_0^1}^2 = y^T T_H^{-T} A_1 T_H^{-1} y, \tag{4.17}$$

where A_1 is given by (4.6) with $\nu = 1$.

4.4 Model Construction

The problem (4.12) to solve is a problem with $n_y = 2^L - 1$ state variables and $n_u = 2^L + 1$ control variables. At iteration k a model is generated by selecting a coarse grid for the states. It would also be possible to reduce the control space, but in this example we have chosen to keep the original control space. Hence the projection and reduction operators are given by

$$P_k = \begin{pmatrix} P_k^y & \\ & I \end{pmatrix}, \quad R_k = \begin{pmatrix} R_k^y & \\ & I \end{pmatrix}.$$

Furthermore, $P_k^\lambda = P_k^y$ and $R_k^\lambda = R_k^y$.

Suppose the coarse grid consists of grid points $x_{i_1}^k, \ldots, x_{i_{n_y^k}}^k$. If a hierarchical basis is used, then the restriction and prolongation operators R_k^y, P_k^y satisfy $R_k^y = (P_k^y)^T$ and $P_k^y \in \mathbb{R}^{n_y \times n_y^k}$ is the matrix with

$$(P_k^y)_{ij} = \begin{cases} 1 & \text{if } i = i_j, \\ 0 & \text{else.} \end{cases} \tag{4.18}$$

The models of the objective and constraint functions are given by

$$m_k^f(\hat{y}, u) = f(P_k^y \hat{y}, u)$$

and

$$m_k^c(\hat{y}, u) = (P_k^y)^T c(P_k^y \hat{y}, u), \tag{4.19}$$

respectively. We set

$$\begin{aligned} m_k^l(\hat{y}_k, u_k) &= m_k^f(\hat{y}_k, u_k) + \hat{\lambda}_k^T m_k^c(\hat{y}_k, u_k) \\ &= f(P_k^y \hat{y}_k, u_k) + \hat{\lambda}_k^T (P_k^y)^T c(P_k^y \hat{y}_k, u_k). \end{aligned} \tag{4.20}$$

Clearly, our model inherits all differentiability properties from the original problem. If there exists a bounded subset Ω such that $(y_k, u_k) \in \Omega$ for all k, then assumptions A1, A2, AM1, AM2 are satisfied. The prolongation P_k^y given by (4.18) satisfies $\|P_k^y\hat{y}\| \geq \|\hat{y}\|$ for all $\hat{y}$. Hence AM3 is satisfied with $\kappa_{lp} = 1$.

If $y_k = P_k^y \hat{y}_k$, then (4.19) and (4.20) imply

$$\begin{aligned} (P_k^y)^T c(y_k, u_k) &= (P_k^y)^T c(P_k^y \hat{y}_k, u_k) = m_k^c(\hat{y}_k, u_k), \\ (P_k^y)^T \nabla_y c(y_k, u_k) P_k^y &= \nabla_{\hat{y}} m_k^c(\hat{y}_k, u_k). \end{aligned}$$

Similarly, if $y_k = P_k^y \hat{y}_k$, $\lambda_k = P_k^y \hat{\lambda}_k$, then

$$\begin{aligned} (P_k^y)^T \nabla_x l(x_k, \lambda_k) &= (P_k^y)^T \left(f_y(y_k, u_k) + \nabla_y c(y_k, u_k)^T \lambda_k \right) \\ &= (P_k^y)^T \left(f_y(P_k^y \hat{y}_k, u_k) + \nabla_y c(V_k \hat{y}_k, u_k)^T P_k^y \hat{\lambda}_k \right) \\ &= \nabla_{\hat{y}} m_k^l(\hat{y}_k, u_k). \end{aligned}$$

Thus, if $y_k = P_k^y \hat{y}_k$, Assumptions AM7 and AM10 are automatically satisfied independent of the coarse grid chosen. Note that the special property (4.18) of P_k^y is not used, i.e., Assumptions AM7 and AM10 are also automatically satisfied if we do not use the hierarchical basis.

In the previous paragraph we have assumed that $y_k = P_k^y \hat{y}_k$. Since $\hat{y}_k = R_k^y y_k$, the identity $y_k = P_k^y \hat{y}_k$ implies that

$$y_k = P_k^y R_k^y y_k. \tag{4.21}$$

Thus, in general, if we assume $y_k = P_k^y \hat{y}_k$, then the grid constructed in iteration k must contain the grid constructed in iteration $k-1$.

The model, i.e., the coarse grid needs to be chosen such that Assumptions AM5, AM6 and AM8, AM9 are satisfied. We select our coarse grid so that AM6 is enforced and we apply a heuristic that attempts to satisfy AM5. Currently, the conditions AM8 and AM9 are not explicitly included in our construction of the coarse grid, but are monitored in our implementation of the GTRSQP algorithm.

We first discuss how we construct our coarse grid to ensure AM6. Assumption AM6 requires that

$$\|c(y_k, u_k)\| \leq \gamma_c \|m_k^c(\hat{y}_k, u_k)\|.$$

Let γ_c be given (in our computations we use $\gamma_c = 2$). To satisfy Assumption AM6, we select the smallest subset of nodes $G^c_{n_y^k}$ such that

$$\sum_{x_i \in G^c_{n_y^k}} (c(y_k, u_k))_i^2 \geq \frac{\|c(y_k, u_k)\|^2}{\gamma_c^2}.$$

The coarse grid $G_{n_y^k}$ used in iteration k satisfies $G^c_{n_y^k} \subset G_{n_y^k}$. (Here $G_{n_y^k}$ is the set of indices of nodes used in the coarse grid of iteration k.) Hence,

$$\sum_{x_i \in G_{n_y^k}} (c(y_k, u_k))_i^2 \geq \frac{\|c(y_k, u_k)\|^2}{\gamma_c^2}.$$

This condition is equivalent to $\|c(y_k, u_k)\| \leq \gamma_c \|P_k^y c(y_k, u_k)\|$. Thus, since $y_k = P_k^y \hat{y}_k$ and $m_k^c(\hat{y}_k, u_k) = P_k^y c(y_k, u_k)$, AM6 is satisfied.

To enforce Assumption AM5, we proceed as follows. Recall (cf. (4.15)) that our criticality measure for the original problem is given by

$$\chi_k(y_k, u_k) = \|\nabla_u l(y_k, u_k, \lambda_k^{\text{fine}})\|_{L^2},$$

where λ_k^{fine} is the solution of the adjoint equation (4.14) (on the fine grid), i.e.,

$$c_y(y_k, u_k)^T \lambda_k^{\text{fine}} = -\nabla_y f(y_k, u_k).$$

Let $\hat{\lambda}_k$ be the Lagrange multiplier estimate obtained by solving the adjoint equation corresponding to the coarse grid, i.e, let $\hat{\lambda}_k$ solve

$$(P_k^y)^T c_y(P_k^y \hat{y}_k, u_k)^T P_k^y \hat{\lambda}_k = -(P_k^y)^T f(P_k^y \hat{y}_k, u_k).$$

The criticality measure for the model is given by

$$\chi_k^M(\hat{y}_k, u_k) = \|\nabla_u l(\hat{y}_k, u_k, \hat{\lambda}_k)\|_{L^2}.$$

Since $y_k = P_k^y \hat{y}_k$,

$$\begin{aligned}
&\chi_k(y_k, u_k) - \chi_k^M(\hat{y}_k, u_k) \\
&= \|\nabla_u f(y_k, u_k) + c_u(y_k, u_k)^T \lambda_k^{\text{fine}}\|_{L^2} - \|\nabla_u f(y_k, u_k) + c_u(y_k, u_k)^T P_k^y \hat{\lambda}_k\|_{L^2}, \\
&\le \|c_u(y_k, u_k)^T\| \, \|\lambda_k^{\text{fine}} - P_k^y \hat{\lambda}_k\|_{L^2}.
\end{aligned}$$

Therefore, we want our coarse grid to be such that the adjoint solution on the coarse grid can capture most of the features of the adjoint solution on the fine grid. We apply the following heuristic. Given λ_k^{fine} and a parameter $\tilde{\gamma}_\chi > 1$ (in our computations we use $\tilde{\gamma}_\chi = 2$), we select smallest subset of nodes $G_{n_y^k}^\lambda$ such that

$$\sum_{i \in G_{n_y^k}^\lambda} (\lambda_k^{\text{fine}})_i^2 \ge \frac{\|\lambda_k^{\text{fine}}\|^2}{\tilde{\gamma}_\chi^2}.$$

The coarse grid $G_{n_y^k}$ used in iteration k satisfies $G_{n_y^k}^\lambda \subset G_{n_y^k}$. Hence,

$$\sum_{i \in G_{n_y^k}} (\lambda_k^{\text{fine}})_i^2 \ge \frac{\|\lambda_k^{\text{fine}}\|^2}{\tilde{\gamma}_\chi^2},$$

which is equivalent to $\|\lambda_k^{\text{fine}}\| \le \tilde{\gamma}_\chi \|R_k^y \lambda_k^{\text{fine}}\|$.

Let $G_{n_y^{k-1}}$ be the set of indices of nodes used in the coarse grid of iteration $k-1$. To enforce (4.21), the indices of nodes used in the coarse grid of iteration k have to satisfy $G_{n_y^{k-1}} \subset G_{n_y^k}$. To compute the coarse grid for iteration k, we successively subdivide the intervals, if necessary, until all grid points with indices in $G^c_{n_y^k} \cup G^\lambda_{n_y^k} \cup G_{n_y^{k-1}}$ are contained in the coarse grid.

As we have stated before, Assumptions AM8, AM9 are difficult to enforce, since they depend on the computed trial step $\hat{s}_k$. These assumptions currently do not enter the construction of the coarse grid, but are monitored during the iteration.

4.5 Numerical Test

We consider the optimal control problem (4.1) with the desired state

$$z(x) = \begin{cases} 1 & \text{in } (0, \frac{1}{2}], \\ 0 & \text{otherwise,} \end{cases}$$

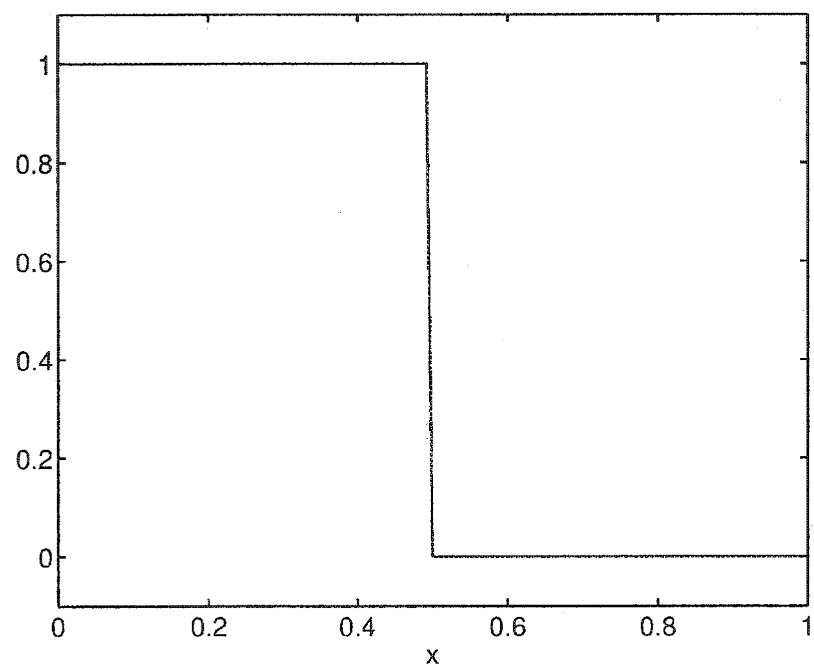
right hand side $r(x) = 1$ and

$$\alpha = 10^{-5}, \quad \nu = 10^{-2}.$$

For the discrimination we use $n_x = 128$ subintervals. The desired state and solution of Burgers equation with $u = 0$ are shown in Figure 4.2 and Figure 4.3 respectively.

We first apply the standard trust region SQP (TRSQP) algorithm to the problem. The TRSQP algorithm converges within 17 iterations and the convergence behavior is displayed in Table 4.1. In Table 4.1, k denotes the iteration number, $f_k = f(y_k, u_k)$, $c_k = c(y_k, u_k)$,

Figure 4.2: Desired state



$\nabla l_k = (\nabla_y l(y_k, u_k, \lambda_k)^T, \nabla_u l(y_k, u_k, \lambda_k)^T)$, s_k^{n} denotes the quasi-normal step, s_k^{t} the tangential step, Δ_k denotes the trust region radius, ρ_k denotes the penalty parameter, and i_c denotes the number of conjugate gradient iterations executed for the computation of tangential step. Moreover,

$$\|s_k\|^2_{H_0^1 \times L^2} = \|(s_k)_y\|^2_{H_0^1} + \|(s_k)_u\|^2_{L^2}$$

(cf. (4.16), (4.17)). In Tables 4.1 and 4.2, we use $\|s_k^{\mathrm{n}}\|$ as a short hand for $\|s_k^{\mathrm{n}}\|_{H_0^1 \times L^2}$ and $\|s_k^{\mathrm{t}}\|$ for $\|s_k^{\mathrm{t}}\|_{H_0^1 \times L^2}$, $\|\nabla l_k\|$ as a short hand for $\|\nabla l_k\|_{H_0^1 \times L^2}$.

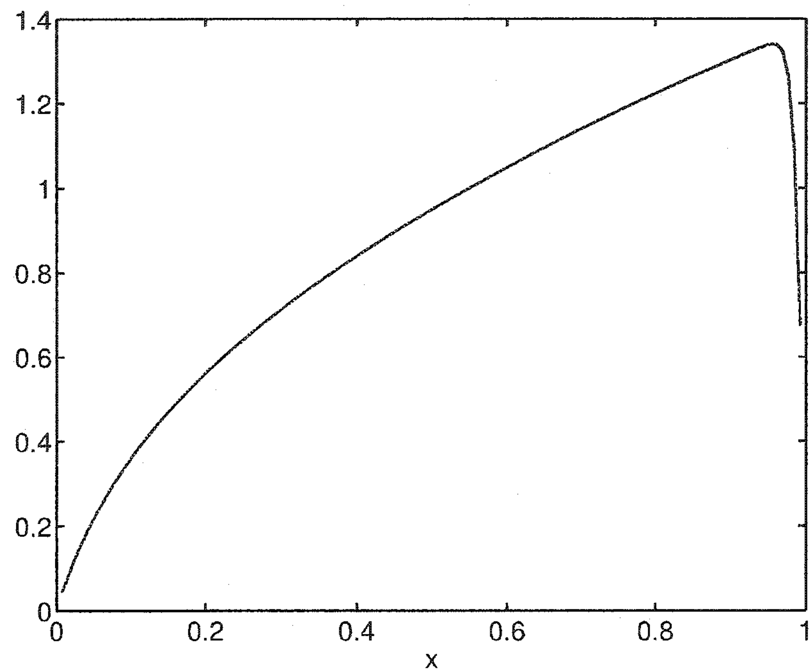
The computed state and control are shown in Figure 4.4 and Figure 4.5, respectively.

The convergence behavior for our generalized trust region SQP (GTRSQP) algorithm is

Table 4.1: Convergence history of the SQP method

k	f_k	$\|c_k\|_{L^2}$	$\|\nabla l_k\|$	Δ_k	$\|s_k^{\mathtt{n}}\|$	$\|s_k^{\mathtt{t}}\|$	ρ_k	i_c
0	2.447e-1	5.233e-1	3.632e-2	1.000e+5			1	
0	2.447e-1	5.233e-1	3.632e-2	1.000e+5	1.122e+1	9.303e+0	65	1
1	1.130e-1	1.673e-1	1.727e-3	1.000e+5	1.863e+0	4.761e+0	65	4
2	1.499e-2	1.132e-1	1.898e-3	1.000e+5	4.351e+0	2.798e+0	65	1
3	3.974e-2	6.357e-2	4.589e-3	1.000e+5	1.865e+0	3.868e+0	65	3
4	1.587e-2	4.526e-2	3.944e-4	2.000e+5	2.167e+0	8.708e-1	65	2
5	1.474e-2	6.915e-3	7.755e-4	1.195e+1	3.399e-1	3.670e+0	65	6
5	1.474e-2	6.915e-3	7.755e-4	1.195e+1	3.399e-1	2.236e+0	65	5
6	9.617e-3	6.356e-3	8.356e-4	1.195e+1	2.738e-1	1.935e+0	65	6
7	6.949e-3	2.761e-3	1.941e-4	2.391e+1	1.127e-1	4.405e-1	65	4
8	6.613e-3	5.048e-4	6.127e-5	1.195e+1	1.402e-2	3.071e+0	65	6
8	6.613e-3	5.048e-4	6.127e-5	5.976e+0	1.402e-2	1.599e+0	65	5
8	6.613e-3	5.048e-4	6.127e-5	5.976e+0	1.402e-2	8.387e-1	65	5
9	5.837e-3	4.942e-4	2.895e-5	5.976e+0	1.488e-2	8.513e-1	65	6
10	5.151e-3	5.778e-4	2.228e-5	5.976e+0	2.576e-2	8.401e-1	65	6
11	4.618e-3	5.216e-4	6.701e-5	5.976e+0	2.781e-2	9.437e-1	65	6
12	4.179e-3	7.152e-4	2.019e-5	5.976e+0	3.752e-2	9.580e-1	65	6
13	3.848e-3	5.118e-4	4.871e-5	5.976e+0	2.341e-2	9.310e-1	65	8
14	3.479e-3	3.345e-4	1.252e-4	5.976e+0	1.978e-2	8.460e-1	65	8
15	3.210e-3	3.013e-4	1.674e-4	5.976e+0	1.352e-2	8.207e-1	65	8
16	2.947e-3	2.229e-4	5.184e-4	1.195e+1	1.360e-2	4.834e-2	65	1
17	2.941e-3	1.329e-5	8.095e-6	1.195e+1	8.479e-4	1.339e+0	65	9

Figure 4.3: Solution of Burgers equation with $u = 0$



shown in Table 4.2. The GTRSQP algorithm converges within 25 iterations. The notation used in the header of Table 4.2 is identical to that used in Table 4.1. In addition, n_y^k denotes the number of grid points used in the model of the GTRSQP iteration k, i_T denotes the number of SQP iterations required for the computation of the tangential step s_k^{t}, and i_c is the cumulative number of CG iterations executed in the SQP algorithm for the computation of the tangential step s_k^{t}.

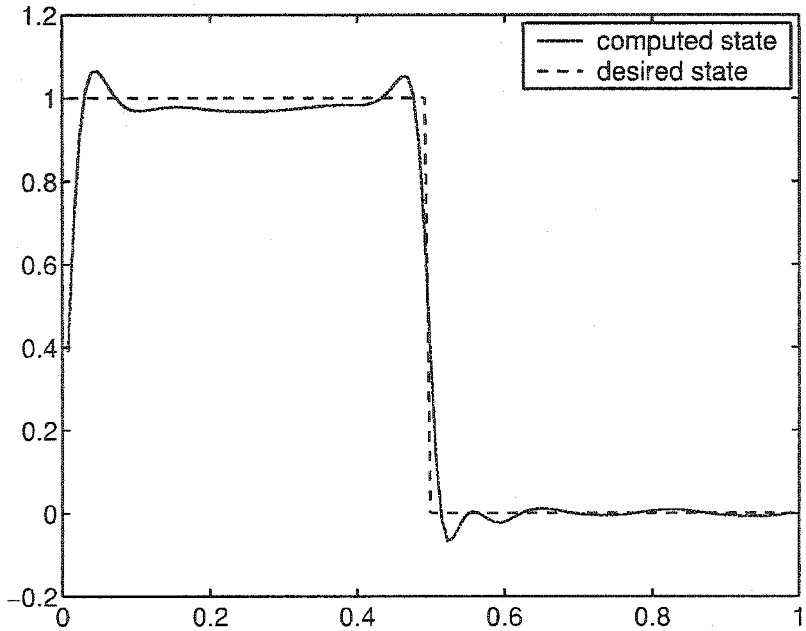
We monitor the ratio

$$\frac{\left| \hat{s}_k^T \left(P_k^T \nabla c(x_k)^T \nabla c(x_k) P_k - \nabla m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k) \right) \hat{s}_k \right|}{\Delta_k \|\hat{s}_k\|^2} \tag{4.22}$$

Table 4.2: Convergence history of the GSQP method

k	n_y^k	f_k	$\|c_k\|_{L^2}$	$\|\nabla l_k\|$	Δ_k	$\|s_k^n\|$	$\|s_k^t\|$	ρ_k	i_T	i_c
0	1	2.44e-1	5.23e-1	1.72e+0	1.00e+5			1	1	1
0	1	2.44e-1	5.23e-1	1.72e+0	2.00e+5	9.53e+0	8.04e+0	96	1	1
1	2	1.53e-1	1.71e-1	8.00e-1	1.98e+0	1.17e+0	3.41e+0	96	1	2
1	2	1.53e-1	1.71e-1	8.00e-1	3.97e+0	1.17e+0	1.23e+0	96	1	1
2	4	1.46e-1	9.50e-2	9.31e-1	7.94e+0	2.29e+0	3.16e+0	96	11	19
3	8	4.81e-2	3.80e-2	1.18e+0	2.02e+0	8.15e-1	2.32e+0	96	9	25
3	8	4.81e-2	3.80e-2	1.18e+0	1.15e+0	8.15e-1	2.12e+0	96	2	4
3	8	4.81e-2	3.80e-2	1.18e+0	1.15e+0	8.15e-1	4.17e-1	96	2	4
4	12	5.07e-2	3.91e-2	7.15e-1	2.31e+0	1.13e+0	1.14e+0	96	2	8
5	14	4.61e-2	2.54e-2	1.50e+0	2.31e+0	9.03e-1	1.13e+0	96	4	5
6	16	4.22e-2	1.80e-2	1.58e+0	2.31e+0	6.78e-1	8.46e-1	96	2	4
7	18	3.71e-2	1.98e-2	4.02e-1	4.62e+0	5.07e-1	5.20e-1	96	2	6
8	21	3.37e-2	1.30e-2	4.60e-1	1.13e+0	4.44e-1	1.28e+0	96	2	8
8	21	3.37e-2	1.30e-2	4.60e-1	2.26e+0	4.44e-1	2.12e-1	96	2	6
9	26	3.08e-2	7.60e-3	4.92e-1	2.26e+0	2.63e-1	4.26e-1	96	1	4
10	29	2.81e-2	6.67e-3	4.79e-1	2.26e+0	2.38e-1	3.03e-1	96	1	4
11	33	2.61e-2	6.21e-3	2.27e-1	2.26e+0	2.34e-1	2.69e-1	96	1	4
12	37	2.40e-2	4.69e-3	2.15e-1	4.53e+0	1.65e-1	2.12e-1	96	1	4
13	41	2.27e-2	4.16e-3	1.92e-1	4.53e+0	1.26e-1	6.24e-1	96	1	5
14	45	1.94e-2	4.34e-3	1.31e-1	4.53e+0	1.13e-1	4.63e-1	96	1	4
15	48	1.74e-2	3.90e-3	3.48e-2	9.06e+0	1.36e-1	3.63e-1	96	1	5
16	51	1.59e-2	2.94e-3	2.76e-2	1.81e+1	8.82e-2	9.45e-1	96	1	6
17	54	1.24e-2	2.87e-3	1.69e-2	3.62e+1	8.71e-2	1.51e+0	96	2	11
18	58	8.69e-3	1.68e-3	6.56e-3	7.25e+1	6.18e-2	2.66e+0	96	2	15
19	61	5.18e-3	5.70e-4	2.79e-3	1.45e+2	1.62e-2	2.96e+0	96	2	20
20	63	3.60e-3	3.22e-4	1.33e-3	2.90e+2	1.44e-2	2.41e+0	96	2	23
21	64	3.18e-3	1.90e-4	6.35e-4	6.21e+0	6.20e-3	1.13e+0	96	1	27
21	64	3.18e-3	1.90e-4	6.35e-4	1.24e+1	6.20e-3	9.05e-3	96	1	6
22	66	3.18e-3	1.69e-4	4.29e-4	2.48e+1	6.67e-3	2.22e-2	96	1	8
23	68	3.18e-3	1.32e-4	3.59e-4	4.96e+1	4.50e-3	7.18e-2	96	2	11
24	71	3.18e-3	9.61e-5	1.86e-4	9.93e+1	3.20e-3	2.08e-1	96	1	14
25	74	3.16e-3	5.27e-5	9.67e-5	9.93e+1	2.18e-3	4.36e-1	96	1	22

Figure 4.4: Optimal state y



and

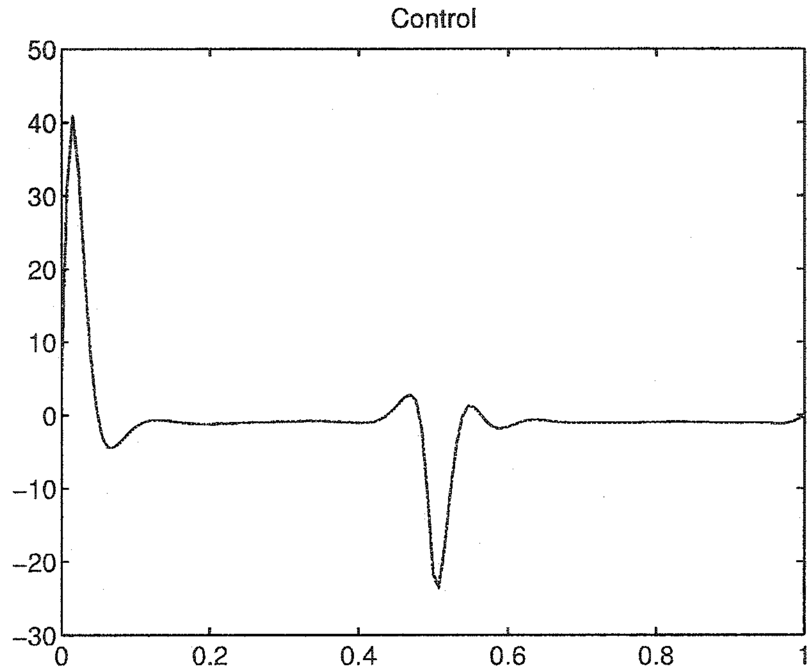
$$\frac{\left| c(x_k)^T \nabla c(x_k)^T P_k \hat{s}_k - m_k^c(\hat{x}_k)^T \nabla m_k^c(\hat{x}_k)^T \hat{s}_k \right|}{\Delta_k \| m_k^c(\hat{x}_k) \|_{L^2} \| \hat{s}_k \|} \tag{4.23}$$

at each iteration and they are given in Figure 4.6 and 4.7.

The grid points used to construct the models are shown in Figure 4.8. Finally, Figure 4.9 and Figure 4.10 display the state and control at each iteration respectively.

Our numerical results indicate the feasibility of our approach. Our GTRSQP algorithm reaches the required tolerances and the model in the final GTRSQP iteration is based on only 74 grid points compared to 128 grid points in the fine grid. However, in the SQP sub–iterations of the GTRSQP algorithm more CG iterations are executed (see column i_c in Table 4.2) than in the 'pure' SQP method applied to the fine grid problem (see column

Figure 4.5: Optimal control u



i_c in Table 4.1). These CG iterations are an indicator of the overall performance of the algorithms. Note that in the GTRSQP algorithm each CG sub-iteration is cheaper, since it requires coarse grid operations, than each CG sub-iteration in Table 4.1. Given the relative sizes of the problems and the relative number of CG iterations, Tables 4.2 and 4.1 indicate that the performance of both algorithms measured in flops is similar. Further research in the implementation of the tangential step algorithm inside the GTRSQP algorithm (which is where the CG iterations are executed) is needed to determine if this step can be executed more efficiently. Moreover, our model construction described in Section 4.4 is somewhat ad-hoc. Improved models clearly will boost the performance of the GTRSQP methods. Both issues are part of future research.

Figure 4.6: Ratio in (4.22)

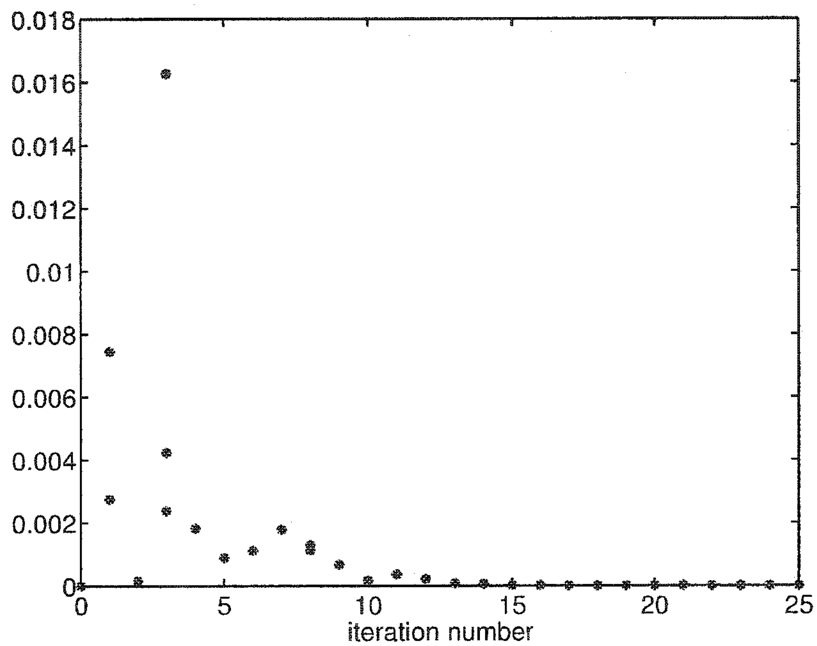


Figure 4.7: Ratio in (4.23)

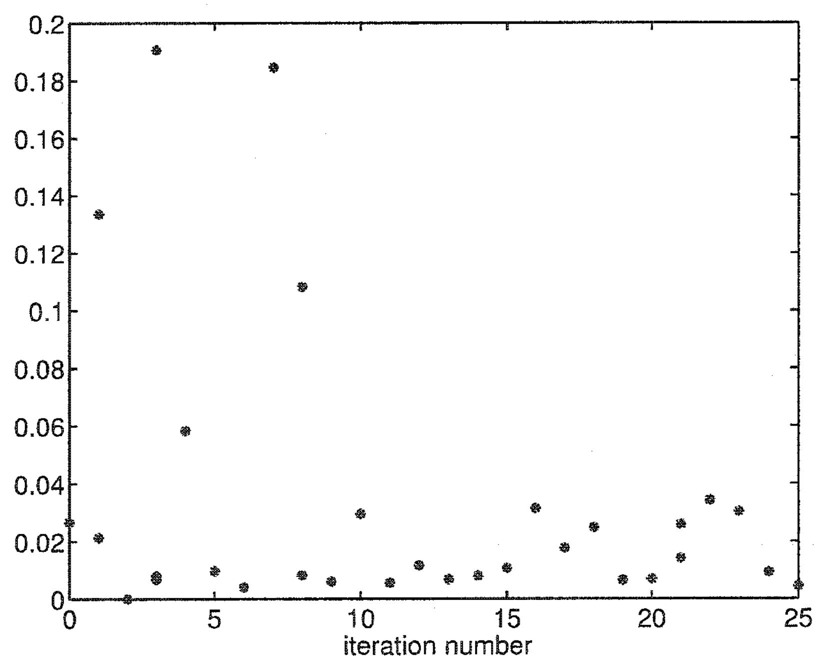


Figure 4.8: Grid points used to construct the model in GTRSQP iteration k

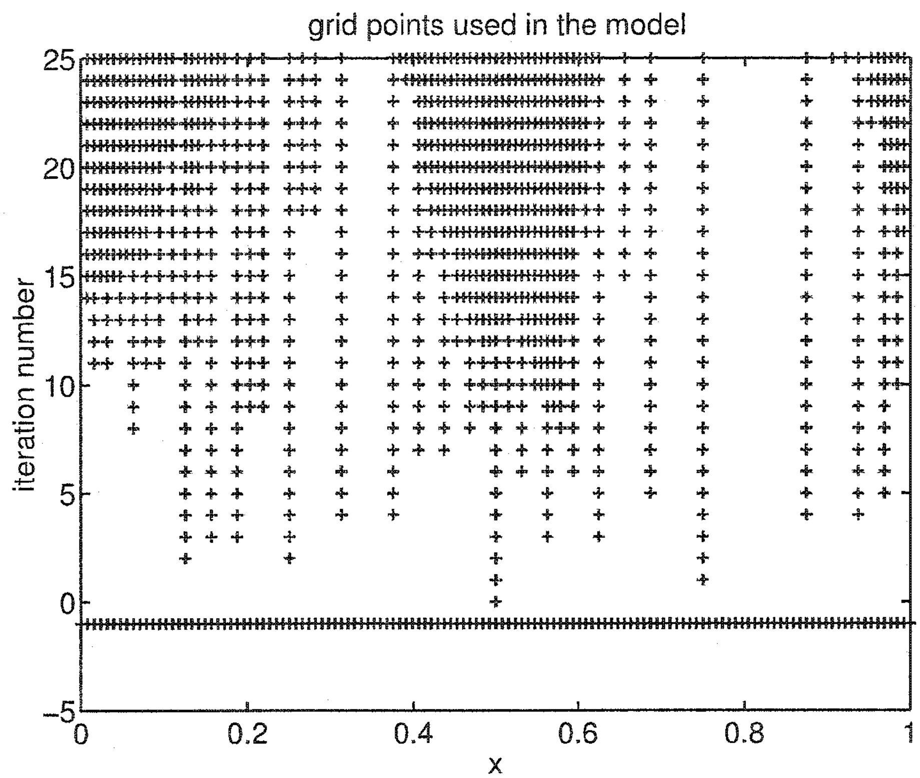


Figure 4.9: Computed state in GTRSQP iteration k

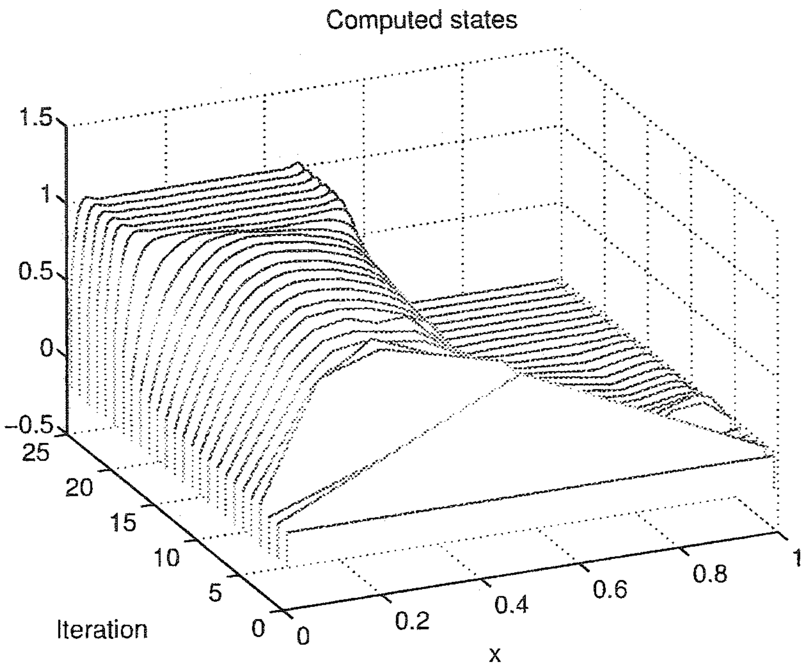
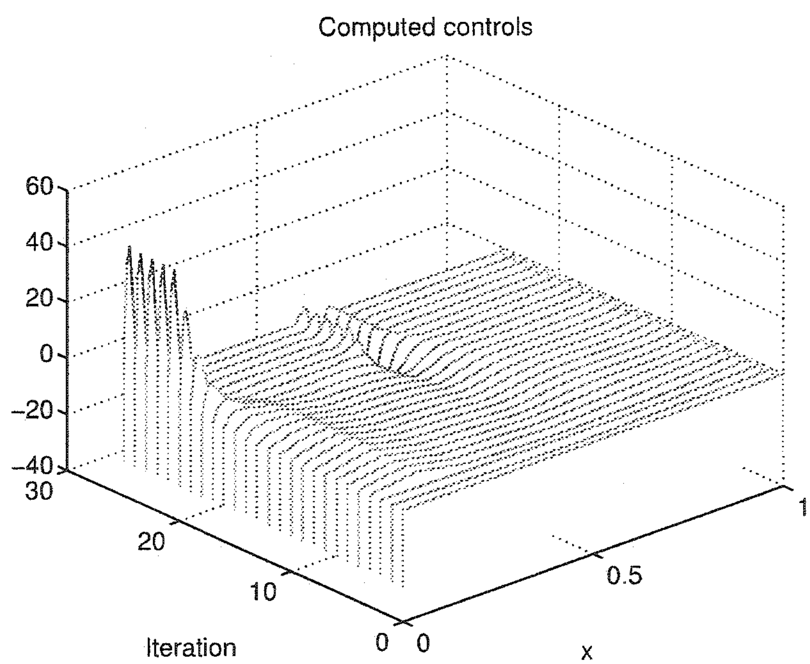


Figure 4.10: Computed control in GTRSQP iteration k



Chapter 5

Summary and Future Work

We have presented and analyzed a class of generalized trust region sequential quadratic programming (GTRSQP) algorithms for equality constrained optimization. The GTRSQP algorithms solve the original equality constrained optimization problem by solving a sequence of equality constrained optimization model problems. The optimization variable space for these model problems can be different from that of the original problem. Typically, the dimension of the optimization variable space for the model problems is much smaller than that of the original problem. In this case, the small model problems are easier to solve.

The design of our GTRSQP algorithm follows that of composite step trust region SQP (TRSQP) algorithms. Several algorithmic modifications were necessary, however, because the model constraints in our subproblems may be nonlinear. These modifications primarily concern the formulation of the tangential step and of the predicted reduction. In addition, since we allow non-Taylor approximation based models, we have to ensure a suitable bound on the difference between actual and predicted reduction.

We have proven a first order global convergence result for the GTRSQP algorithm, extending the analysis of [11]. Our first order global convergence result for the GTRSQP algorithm applied to TRSQP allows one to relax the condition that the so-called tangential step lies in the null-space of the linearized constraints. This is useful in TRSQP methods when the linearized constraints are solved by iterative methods.

Our GTRSQP algorithm has been applied to an optimal control problem governed by Burgers equation in which the models are generated by coarsening of the original grid.

Our convergence analysis requires several assumptions on the model. Possibly the most critical one is an assumption on the bound on the difference between actual and predicted reduction (see AM4). Further investigations into ways to weaken this assumption or how to enforce this assumption in concrete applications would be useful. In this context, the exploration of an extension of the SQP filter algorithm [17, 18] could be interesting.

Our GTRSQP algorithm allows great flexibility in the computation of the quasi-normal and the tangential step. The algorithms used for these tasks, especially the algorithm for the computation of the tangential step, can greatly influence the overall performance of the GTRSQP algorithm. More experiments with different algorithms for quasi-normal and the tangential step computation would be interesting.

Finally, the application of our algorithms to different problems and the generation of different models, e.g, through the use of reduced basis approaches, should be investigated.

Bibliography

[1] N. Alexandrov. A trust–region framework for managing approximations in constrained optimization problems. In *Proceedings of the First ISSMO/NASA Internet Conference on Approximation and Fast Reanalysis Techniques in Engineering Optimization, June 14–27, 1998*, 1998.

[2] N. Alexandrov, J. E. Dennis Jr., R. M. Lewis, and V. Torczon. A trust region framework for managing the use of approximation models in optimization. *Structural Optimization*, 15:16–23, 1998. Appeared also as ICASE report 97–50.

[3] N. Alexandrov and R. M. Lewis. First-order approximation and model management in optimization. In L. T. Biegler, O. Ghattas, M. Heinkenschloss, and B. van Bloemen Waanders, editors, *Large-Scale PDE-Constrained Optimization*, Lecture Notes in Computational Science and Engineering, Vol. 30, pages 65–81, Heidelberg, 2003. Springer-Verlag.

[4] N. Alexandrov, R. M. Lewis, C. R Gumbert, L. L. Green, and P. A. Newman. Optimization with variable fidelity models applied to wing design. *AIAA paper 2000–*

0841, 2000. Appeared also as ICASE Report 99–45, http://www.icase.edu/.

[5] E. Arian, M. Fahl, and E. W. Sachs. Trust–region proper orthogonal decomposition for flow control. Technical Report 2000–25, ICASE, NASA Langley Research Center, Hampton VA 23681–2299, 2000.

[6] R. Becker. Mesh adaptation for stationary flow control. *J. Math. Fluid Mech.*, 3:317–341, 2001. http://gaia.iwr.uni-heidelberg.de/~roland/publications.html.

[7] R. G. Carter. On the global convergence of trust region algorithms using inexact gradient information. *SIAM J. Numer. Anal.*, 28:251–265, 1991.

[8] K. J. Chang, R. T.Haftka, G. L. Gils, and P. J. Kao. Sensitivity-based scaling for approximation structural response. *Journal of Aircraft*, 30:283–288, 1993.

[9] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. A globally convergent augmented Lagrangian algorithm for optimization with general constraints and simple bounds. *SIAM J. Numer. Anal.*, 28:545–572, 1991.

[10] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust–Region Methods*. SIAM, Philadelphia, 2000.

[11] J. E. Dennis, M. El-Alem, and M. C. Maciel. A global convergence theory for general trust–region–based algorithms for equality constrained optimization. *SIAM J. Optimization*, 7:177–207, 1997.

[12] J. E. Dennis, M. Heinkenschloss, and L. N. Vicente. Trust–region interior–point algorithms for a class of nonlinear programming problems. *SIAM J. Control and Optimization*, 36:1750–1794, 1998.

[13] J. E. Dennis, Jr. and R. B. Schnabel. *Numerical Methods for Nonlinear Equations and Unconstrained Optimization.* Prentice-Hall, Englewood Cliffs, N. J, 1983. Republished by SIAM, Philadelphia, 1996.

[14] M. El-Alem. A global convergence theory for the Celis–Dennis–Tapia trust region algorithm for constrained optimization. *SIAM J. Numer. Anal.*, 28:266–290, 1991.

[15] M. Fahl and E.W. Sachs. Reduced order modelling approaches to PDE–constrained optimization based on proper orthogonal decompostion. In L. T. Biegler, O. Ghattas, M. Heinkenschloss, and B. van Bloemen Waanders, editors, *Large-Scale PDE-Constrained Optimization*, Lecture Notes in Computational Science and Engineering, Vol. 30, Heidelberg, 2003. Springer-Verlag.

[16] R. Fletcher. *Practical Methods of Optimization.* John Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore, second edition, 1987.

[17] R. Fletcher and S. Leyffer. Nonlinear programming without a penalty function. *Math. Program.*, 91(2, Ser. A):239–269, 2002.

[18] R. Fletcher, S. Leyffer, and P. L. Toint. On the global convergence of a filter-SQP algorithm. *SIAM J. Optim.*, 13(1):44–59 (electronic), 2002.

[19] M. Heinkenschloss. A trust–region method for norm constrained problems. *SIAM J. Numerical Analysis*, 35:1594–1620, 1998.

[20] M. Heinkenschloss and L. N. Vicente. TRICE: A package of trust–region interior–point algorithms for the solution of optimal control and engineering design problems. User's guide. Technical report, Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005–1892, 1996. `http://www.caam.rice.edu/~trice`.

[21] M. Heinkenschloss and L. N. Vicente. An interface between optimization and application for the numerical solution of optimal control problems. *ACM Transactions on Mathematical Software*, 25:157–190, 1999.

[22] M. Heinkenschloss and L. N. Vicente. Analysis of inexact trust–region SQP algorithms. *SIAM J. Optimization*, 12:283–302, 2001.

[23] K. Ito and S. S. Ravindran. A reduced order method for simulation and control of fluid flows. *J. Computational Physics*, 143:403–425, 1998.

[24] K. Kunisch and S. Volkwein. Control of Burger's equation by a reduced order approach using proper orthogonal decomposition. *Journal of Optimization Theory and Applications*, 102:345–371, 1999.

[25] F.-S. Kupfer and E. W. Sachs. A prospective look at SQP methods for semilinear parabolic control problems. In K.-H. Hoffmann and W. Krabs, editors, *Optimal Con-*

trol of Partial Differential Equations, Irsee 1990, Lect. Notes in Control and Information Sciences, Vol. 149, pages 143–157. Springer Verlag, 1991.

[26] J. Nocedal and S. J. Wright. *Numerical Optimization.* Springer Verlag, Berlin, Heidelberg, New York, 1999.

[27] Ph. L. Toint. Global convergence of a class of trust-region methods for nonconvex minimization in Hilbert space. *IMA Journal of Numerical Analysis*, 8:231–252, 1988.

[28] S. Volkwein. *Mesh-Independence of an Augmented Lagrangian-SQP Method in Hilbert Spaces and Control Problems for the Burgers Equation.* PhD thesis, Fachbereich Mathematik, Technische Universität Berlin, Berlin, Germany, 1997.

[29] S. Volkwein. Application of the augmented Lagrangian-SQP method to optimal control problems for the stationary Burgers equation. *Comput. Optim. Appl.*, 16(1):57–81, 2000.

[30] S. Volkwein. Distributed control problems for the Burgers equation. *Comput. Optim. Appl.*, 18(2):115–140, 2001.

[31] H. Yserentant. On the multi–level splitting of finite element spaces. *Numer. Math.*, 49:379–412, 1986.

[32] H. Yserentant. Hierarchical bases. In R. E. O'Malley, editor, *ICIAM 91*, pages 256–276, Philadelphia, 1992. SIAM.

[33] O. C. Zienkiewicz, D. W. Kelley, J. Gago, and I. Babuska. Hierarchical finite element approaches, error estimates and adaptive refinement. In J. R. Whiteman, editor, *The Mathematics of Finite Elements and Applications IV, MAFELAP 1981*, pages 313–346. Academic Press, London, New York, 1982.