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EQUIVALENT LINEARIZATION OF A RANDOMLY EXCITED YIELDING OSCILLATOR

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ABSTRACT

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Equivalent linearization of bilinear hysteretic systems subjected to a white noise excitation is attempted by using 2nd and 3rd order linear systems. The bilinear hysteretic systems considered have the slope ratio between the initial and the reduced stiffness of $\alpha = 1/2$ (moderately nonlinear case) and $\alpha = 1/21$ (nearly elasto-plastic case). The technique is to match both energy dissipation per unit of time and average frequency between the original system and its equivalent linear system in stationary motion. In the 2nd order linearization these criteria are essentially the same as the requirements from the Krylov-Bogoluibov method. However, special attention is given to the estimation of the hysteretic energy dissipation per unit of time, resulting in improved predictions of stationary levels of root-mean-square displacement and velocity response.

Satisfying the above matching criteria does not require explicit specification of the parameters in the equivalent linear system. In this investigation several linearization matching the above criteria are considered. These include: the usual 2nd

order linear system, a model with two uncorrelated 2nd order modes whose undamped natural frequencies correspond to the initial stiffness and the reduced stiffness of the bilinear hysteretic system, a 3rd order linear system which has the same stiffness arrangement as the bilinear hysteretic system model but replaces the Coulomb friction slider in the original system by a viscous damper, and a model with two uncorrelated 3rd order modes which have the same root-mean-square displacement.
A severe test of the equivalence to the original system is executed by comparing the response power spectral densities.

After getting the specified equivalent linear systems, their transient root-mean-square responses are compared with the experimental results. For this response analyses, the Rice method is applied for the 2nd order linear systems and the Markov process approach is taken for the 3rd order linear systems. As a result, a correlation between the stationary response power spectral density matching and the transient root-mean-square response matching is found. As a whole, the two-mode 3rd order linear system proves to be the best linearization among those considered herein.

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SUMMARY OF NOMENCLATURE

Symbol	Explanation or Definition
A	amplitude of vibration
$\{A_i\}$	see Eq.(5.8)
$[B_{ij}]$	see Eq.(5.8)
$E[\cdot]$	expectation
$(E.D.)_b$	energy dissipation per unit of time in the bilinear hysteretic system (BHS)
$(E.D.)_{2nd}$	energy dissipation per unit of time in the 2nd order linear system (2LS)
$(E.D.)_{3rd}$	energy dissipation per unit of time in the 3rd order linear system (3LS)
F_o	amplitude of the periodic excitation
$H_{2,2}(\omega)$	frequency response function of the two-mode 2nd order linear system (2M2LS)
$H_{2,eq}(\omega)$	frequency response function of the equivalent 2nd order linear system (E2LS)
$H_{3,1}(\omega)$	frequency response function of the mass in the 3LS
$H_{3,2}(\omega)$	frequency response function of point 2 in the 3LS(see Fig.10)
$H_{3,1,2}(\omega)$	transfer function of the two-mode 3rd order linear system (2M3LS)
$H_{3,1,L}(\omega)$	transfer function of the low frequency 3rd order linear system (LF3LS)
$H_{3,1,H}(\omega)$	transfer function of the high frequency 3rd order linear system (HF3LS)
H.E.D.	hysteresis energy dissipation per cycle
N	measure of the excitation intensity, see Eq.(2.5)
$R(\tau)$	auto-correlation function

Symbol	Explanation or Definition
S_o	level of white noise
$\underline{S}$	covariance matrix of y_i
$\|\underline{S}\|$	determinant of $\underline{S}$
$[\underline{S}]_{ij}$	adjoint matrix of the (i,j) element of $\underline{S}$
Y	yield level
$\lceil a \rfloor$	see Eq.(5.7)
c	viscous damping coefficient in the BHS
c_1	equivalent viscous damping coefficient in the 3LS
c_2	equivalent viscous damping coefficient representing the hysteresis energy dissipation in the 3LS
c_{eq}	viscous damping coefficient of the E2LS
erfc(·)	complementary error function
$\{f_i(t)\}$	see Eq.(5.7)
h(t)	impulse response function of the linear system
$\underline{i}$	imaginary unit
k	small amplitude stiffness of BHS
k_1	reduced stiffness after BHS yielding
k_{eq}	equivalent stiffness of E2LS
m	mass
ms	mean-square response
p	transition probability density function
$p_A(u)$	probability density function for random amplitude
$\{q\}$	operation matrix defined in Eq.(5.12)
t	time
x	displacement

Symbol	Explanation or Definition
$\{y_i\}$	state vector, see Eq.(5.7)
$\{z_i\}$	state vector, see Eq.(5.11)
$\{z_{io}\}$	initial state vector
α	slope ratio in the BHS
β_o	fraction of the critical viscous damping in BHS
β_{eq}	fraction of the critical viscous damping in E2LS
β_{2L}, β_{2H}	damping factors in the 2M2LS
β_{31}, β_{32}	damping factors in the 3LS
γ	fraction of the ms response contributed from the LF3LS in the 2M3LS
$\delta(\cdot)$	Dirac delta function
ζ	fraction of the critical viscous damping in the 2LS
η	$2\sigma_x^2/Y^2$
λ_i	eigen values of characteristic equation (5.13)
μ_{ij}	see Eq.(5.18)
ν	$\sqrt{2/\eta}$
σ_x	root-mean-square (rms) displacement
$\sigma_{\dot{x}}$	rms velocity
$\sigma_{x_{3,1,L}}$	rms displacement of the LF3LS
$\sigma_{\dot{x}_{3,1,L}}$	rms velocity of the LF3LS
$\sigma_{x_{3,1,H}}$	rms displacement of the HF3LS
$\sigma_{\dot{x}_{3,1,H}}$	rms velocity of the HF3LS
$\varphi(x)$	normalized restoring force-deformation
ω	circular frequency

Symbol	Explanation or Definition
ω_o	undamped natural frequency of small amplitude
ω_{eq}	undamped natural frequency of the E2LS
ω_{KB}	average frequency prediction from Krylov-Bogoliubov method with small nonlinearity
ω_R	average frequency prediction from the harmonic resonance frequency
$\omega_{3,1}$	average frequency of the 3LS
$\omega_{b,1}$	average frequency of BHS
ω_d	$\omega_{eq}\sqrt{1 - \beta_{eq}^2}$
ω_{od}	$\omega_{eq}\sqrt{\beta_{eq}^2 - 1}$
$\frac{d^i}{dt^i}$	ith order time derivative

Dots over variables also denote derivatives with respect to time.

I. INTRODUCTION

1.1. Background

When the earthquake-resistant design of structures is discussed, the basic questions concern how well the structures will survive earthquakes, and which design parameters are the most preferable ones. Furthermore, it may be that these questions are best answered probabilistically rather than deterministically because of the uncertainties involved in the excitation, even assuming the structural properties are fixed. This means that the response randomness comes soley from the excitation through a completely prescribed structure. Hence, the finding of a design solution must involve the statistical excitation characteristics.

When the excitation intensity is small, the corresponding structural response remains within the yield level of strength of the composing material, then the linear response analysis will be enough for designing. However, when a structure is exposed to a strong motion, its response will certainly exceed its yield level, and the force-deformation characteristics are no longer linear but result in a deformed shape. It must be noted here that such a situation does not lead instantly to the structural collapse. We can observe many structures which have survived earthquakes although responding beyond their yield level. This fact tells us that the existence of another branch beyond the yield level in the force-deformation characteristics and the energy dissipation from the resulting hysteresis loop can contribute much to earthquake-resistance. Several types of such curves have been proposed depending on the type of structure, and the corresponding response analyses have been carried out by many people using particular earthquake records[1)] or some simulations[2)] as input motion.

However, the stochastic treatment seems to be more appropriate from the aforementioned reason. Rice's method[3), 4), 5)] and the Markov process approach[6), 7)] are available in this direction. But the former method is not applicable in the nonlinear system response analysis. In this case the governing equation is easily constructed by the latter method, but the solution is found only in limited cases.[6)] A hysteretic system is yet to be solved. One possible approach to such a system involves seeking an equivalent linear system whose response characteristics can approximate those of the original system. This procedure is very attractive for two reasons: first, the well established random linear theory[4), 5)] can then be applied for the response analysis and secondly, the efficient use of modern digital computers are then available.

Many people have been concerned so far with finding an equivalence between a hysteretic system and the 2nd order viscously damped linear system in steady state harmonic motion. Among them Jennings[8)] listed six conceivable linearization techniques for the elasto-plastic hysteretic system, where the hysteretic energy dissipation per cycle is equated between the original system and its linearization. Rea[9] presented a geometrical interpretation for the Jennings results by using the Ramberg-Osgood type force-deformation curve and added one more approach.

If a bilinear hysteretic system is exposed to a random excitation, its response becomes a function of the ratio of the excitation intensity level to the system yield level, as pointed out by Iwan and Lutes.[10] Liu[11] extended one of the above harmonic results assuming a random amplitude distribution, but his approach turned out to give poor results when applied to the bilinear hysteretic system. Essentially the linearization techniques for random motion may be classified into the following three types;

(1) the matching of response characteristics

(2) the Krylov-Bogoliubov method

(3) the dissipative energy equivalence per unit of time

The first method was taken by Hudson.[12] This author from response spectra curves evaluated the equivalent viscous damping factor in the 2nd order linear system with the same mass and the initial tangent stiffness of the hysteretic system. Lutes[13] showed several conceivable linearizations coming from different response matching criteria, using analog computer results. These methods, however, are not practical for general response prediction because of their dependence on preceeding experiments. The application of the second approach was first discussed by Caughey[14] and was applied to the bilinear hysteretic system in small nonlinearity situations by the author.[15] According to the

Iwan and Lutes investigation,[10] this method can predict the yielding system response satisfactorily for a moderately nonlinear system but not for a severly nonlinear case. The third method was proposed by Karnopp[16] in the analysis of an elasto-plastic system and was called the power balance method. Karnopp and Brown[17] assumed that the random hysteretic energy dissipation cycle is equal to the average frequency of the hysteretic system. That is, the authors assumed that the average energy dissipated per unit of time could be found by multiplying the average energy dissipated per cycle times an average frequency of the system. It is shown in this work that that assumption is unsatisfactory for a severely nonlinear system.

Another possible method predicting the hysteretic system response is to find an equivalent nonlinear nonhysteretic system and to use Caughey's solution[18] of the corresponding Fokker-Plank equation, as Lutes[19] attempted. However, this approach is hard to extend to other than white Gaussian excitation cases. At the same time, the transient response is yet to be found for such a system.

As for an equivalent linearization in transient motion, which is more important in earthquake engineering than is the linearization in stationry motion, few works have been done so far. Hudson's work[12] belongs to this field but his investigation was limited only to giving a crude estimation of the equivalent viscous damping representing hysteresis energy dissipation. Only Shah's work[20] presents the response comparison in a transient situation between experimental results for a bilinear hysteretic system and its equivalent linearizations obtained in reference (13).

1.2. Object and Scope

The objective of this investigation is two-fold: first, to find a linear system with response characteristics nearly equivalent to those of a bilinear hysteretic system in stationary random motion when subjected to a stationary Gaussian white noise excitation; secondly, to check this "equivalent" linear system to see whether it is also able to predict the transient response of the original system.

As for the bilinear hysteretic system, two typical ones whose experimental results are available[10] are considered, one of which has the slope ratio between the initial sfiffness and the reduced stiffness after the system yielding as $\alpha = 1/2$; and the other of which has $\alpha = 1/21$. In this thesis, the former system will be referred to as the moderately bilinear hysteretic system and the latter as the nearly elasto-plastic hysteretic system.

Herein, the term "equivalent linear system" is used for a linear system which can approximate both the root-mean-square (rms) displacement and rms velocity of the bilinear hysteretic system. Both of these response characteristics are needed in predicting the maximum response or in dealing with the first passage problem of the original system. These quantities are functions of the ratio of the excitation intensity level to the system yielding level, as is reviewed in Chapter II.

The linearization technique adopted is based on matching energy dissipation per unit of time between the bilinear hysteretic system and the linear system, as well as matching average frequency between these systems. First, these criteria are applied to the 2nd order linear system in Chapter III. In this case, these requirements are the same, in essense, as the Krylov-Bogoliubov linearization. The energy matching criterion, requiring no specific linear system due to the fact that any 2nd order system dissipates the same amount of energy per unit of time,[16) is concerned with the prediction of the rms displacement response. On the other hand, the average frequency matching criterion is related to the prediction of the rms velocity response.

It is possible to specify the 2nd order system linearizations satisfying the above matching criteria. Then, as a severe test of the equivalence to the original system, response power spectral density matching is attempted for those systems. Two systems are considered in this part of the investigation. The first is the usual 2nd order linear system with the same mass as the bilinear hysteretic system but with the stiffness and the damping coefficient depending on the nonlinearity situation. The second system has two uncorrelated 2nd order modes, where the one modalfrequency corresponds to the initial stiffness and the other to the reduced stiffness of the bilinear hysteretic system.

As a further step to linearization of bilinear hysteretic systems, the 3rd order linear model of Fig.10, which has the same stiffness arrangement as the bilinear hysteretic system model of Fig.2 but replaces the Coulomb slider in the original system by a viscous damper, is considered in Chapter IV. As the linearization technique, the energy dissipation matching per unit of time is applied. Since the 3rd order linear system is found to dissipate the same amount of energy per unit of time as the 2nd order system, it results in the same prediction of the rms displacement. But in this case the average frequency is confined in the vicinity of the natural frequency of the initial stiffness or of the reduced stiffness of the bilinear hysteretic system, such that velocity matching cannot be achieved, in general. Therefore, a model having two uncorrelated 3rd order modes is devised and then average frequency matching is executed. This two-mode model of the 3rd order linear system is checked in the frequency domain (response power spectral density) for representative nonlinearity situations.

After getting the above equivalent linear systems, all of which yield the same rms displacement and velocity response in stationary motion, they are compared in the transient response region with the experimental results in Chapter V, where Rice's method[3], [4], [5] is taken for the response analysis of the 2nd order linear systems and the Markov process approach[7] for the 3rd order linear systems.

The last Chapter VI gives the conclusions drawn from the investigation reported herein.

II. BILINEAR HYSTERETIC SYSTEM

2.1. Description of the system

A bilinear hysteretic system is the nonlinear system having the normalized restoring force-deformation characteristics of Fig.1. This figure represents steady state response with amplitude A and yield level of Y. Fig.2 is an illustration of such a conceptual mechanical model, and the corresponding governing equation can be written as :

$$m\ddot{x} + c\dot{x} + K\varphi(x) = mf(t) \qquad (2.1)$$

or

$$\ddot{x} + 2\beta_0\omega_0\dot{x} + \omega_0^2\varphi(x) = f(t) \qquad (2.1)'$$

where m , the mass

$\omega_0 = \sqrt{\frac{K}{m}}$, undamped natural circular frequency of small amplitude response

$K = k_1 + k_2$, small amplitude stiffness

k_1 , reduced stiffness after the system yielding

$\beta_0 = \frac{c}{2\omega_0 m}$, fraction of critical viscous damping for small amplitudes

$\varphi(x)$, bilinear hysteretic restoring force characteristics chosen to have an initial slope of unity and the second slope $\alpha = k_1/K$

$mf(t)$, excitation force

and the dots over variable x denote derivatives with respect to time.

No exact response characteristics of such a hysteretic system due to a random excitation of f(t) have yet been obtained analytically. Experimental investigations on such systems, on the other hand, have been made so far by many people, and extensively by Iwan and Lutes.[10)]

2.2. Root-mean-square Response

This section gives a brief description of the experimental results for the rms response of the bilinear hysteretic systems, as obtained by Iwan and Lutes[10)] from an analog computer investigation. These results will be used to check the accuracy of the approximate methods presented here. The authors considered the excitation as a Gaussian white noise. Using a two-sided power spectral density, which is convenient for analytical response investigation, gives the auto-correlation function as

$$R(\tau) = 2\pi S_0 \delta(\tau) \tag{2.2}$$

The S_o here is one-half of that used by the authors since they used a one-sided definition of power spectral density.

In the hysteretic system, the response level is a function of the excitation level and the yield level. This is the essential difference of the hysteretic system from the linear system for which the response is direct proportional to the excitation level. For a linear system represented by the governing equation of

$$\ddot{x} + 2\beta\omega_0\dot{x} + \omega_0^2 x = f(t) \qquad (2.3)$$

the rms displacement σ_x and rms velocity $\sigma_{\dot{x}}$ 4), 5) can be expressed in a nondimensional form as

$$\frac{\sigma_x \omega_0^2}{\sqrt{2S_0\omega_0}} = \frac{\sigma_{\dot{x}}\omega_0}{\sqrt{2S_0\omega_0}} = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \qquad (2.4)$$

The above authors presented the experimental rms displacement and velocity normalized by

$$N = \sqrt{\frac{2S_0}{\omega_0^3}} \qquad (2.5)$$

and $N\omega_0$, respectively. The dashed lines in Figs.5,6,7,8,13 and 14 are such experimental results reproduced from the reference (10), which clearly indicate the hysteretic system response deviation from the linear system response of the initial stiffness. Note that a change in the yield level sometimes results in significant decrease and sometimes significant increase in rms displacement response and that there exists a yield level which produces the least response for a given excitation level in the slightly viscously damped bilinear hysteretic system and the rms displacement increases monotonically with the decrease of the yield level in the 5 % fraction of critical damped case of the nearly elasto-plastic hysteretic system. The rms velocity is always decreased and has the least response at a certain yield level for a given excitation level.

The limiting cases of the extreme nonlinearity situations of $Y/N \to \infty$ or $Y/N \to 0$ are intuitively taken as linear systems governed by the equation of

$$m\ddot{x} + c\dot{x} + kx = mf(t) \tag{2.6}$$

or

$$\ddot{x} + 2\zeta\Omega\dot{x} + \Omega^2 x = f(t) \tag{2.6$'$}$$

where $\Omega = \omega_0$ or $\Omega = \sqrt{\alpha}\,\omega_0$, depending on $Y/N \to \infty$ or $Y/N \to 0$, respectively, and $\zeta\Omega = \beta_0\omega_0$. Then the rms response characteristics are found as:

for the rms displacement

$$\left.\begin{aligned} \frac{\sigma_x}{N} &= \frac{1}{2}\sqrt{\frac{\pi}{\beta_0}} && \text{as } Y/N \to \infty \\ \frac{\sigma_x}{N} &= \frac{1}{2}\sqrt{\frac{\pi}{\beta_0\alpha}} && \text{as } Y/N \to 0 \end{aligned}\right\} \tag{2.7}$$

for the rms velocity

$$\frac{\sigma_{\dot{x}}}{\omega_0 N} = \frac{1}{2}\sqrt{\frac{\pi}{\beta_0}} \qquad \text{as } Y/N \to \infty \text{ or as } Y/N \to 0 \tag{2.8}$$

These values make the response asymptotes for the corresponding response, which are drawn in chain lines in Figs.7,8,13 and 14.

2.3. Average frequency

One measure of the average frequency of the mass of the oscillator of Fig.2 is given by the ratio of the rms velocity response $\sigma_{\dot{x}}$ to the rms displacement response σ_x, i.e.

$$\omega_{av} = \frac{\sigma_{\dot{x}}}{\sigma_x} \tag{2.9}$$

It will be shown that knowledge of this quantity as well as the energy dissipation per unit of time give enough information for predicting the bilinear hysteretic system displacement and velocity response.

Lutes' experimental results defined by Eq.(2.9) is reproduced from reference (13) for the cases of α = 1/2 with β_0 = 0 % and of α = 1/21 with β_0 = 0 % in Fig.3 by the dashed lines. One can note from this figure that the average frequency of the bilinear hysteretic system varys continuously from the natural frequency ω_0 of small amplitude response to the reduced frequency $\sqrt{\alpha}\,\omega_0$ of the second branch stiffness as the value σ_x/Y grows.

III. EQUIVALENT LINEARIZATION BY THE 2ND ORDER SYSTEM

3.1. Equivalent 2nd order linear system

Consider the 2nd order linear system of Fig.4 as a linearization of the bilinear hysteretic system of Eq.(2.1) when subjected to the random excitation of Eq.(2.2). The governing equation of motion of this system is expressed as

$$m\ddot{x} + c_{eq}\dot{x} + k_{eq}x = mf(t) \qquad (3.1)$$

or

$$\ddot{x} + 2\beta_{eq}\omega_{eq}\dot{x} + \omega_{eq}^2 x = f(t) \qquad (3.1)'$$

where

$$\omega_{eq} = \sqrt{\frac{K_{eq}}{m}} \qquad \beta_{eq} = \frac{C_{eq}}{2\sqrt{mK_{eq}}}$$

and these parameters are to be determined as demonstrated below.

The restoring forces in Eqs.(2.1) and (3.1) can then written as $C\dot{x} + K\varphi(x)$ and $C_{eq}\dot{x} + K_{eq}x$, respectively. Multiplying these restoring forces by $\dot{x}$ gives the rates of energy dissipation in the two systems. Equating the expectations of these values of energy dissipation per unit of time yields

$$CE[\dot{X}^2] + KE[\varphi(X)\dot{X}] = C_{eq}E[\dot{X}^2] + K_{eq}E[X\dot{X}] \qquad (3.2)$$

Noting that $E[x\dot{x}] = 0$ in the stationary state motion, one can get

$$CE[\dot{X}^2] + KE[\varphi(X)\dot{X}] = C_{eq}E[\dot{X}^2] \qquad (3.3)$$

or the equivalence per unit mass per unit of time

$$\beta_{eq}\omega_{eq} = \beta_0\omega_0 + \frac{1}{2}\omega_0^2 \frac{E[\varphi(X)\dot{X}]}{E[\dot{X}^2]} \tag{3.3$'$}$$

Similarly, if both restoring forces are multiplied by x and then their expectations are equated, one can get

$$K\,E[\varphi(X)X] = k_{eq}\,E[X^2] \tag{3.4}$$

or the equivalence per unit mass

$$\left(\frac{\omega_{eq}}{\omega_0}\right)^2 = \frac{E[\varphi(X)X]}{E[X^2]} \tag{3.4$'$}$$

Eqs.(3.3)$'$ and (3.4)$'$ are exactly the same as are obtained from the Krylov-Bogoliubov method.[14] However, in calculating $E[\varphi(x)x]$ and $E[\varphi(x)\dot{x}]$ the procedure used in this work differs from that previously used[15] with the Krylov-Bogoliubov method.

3.2. Average frequency

The quantity ω_{eq} in Eq.(3.1)$'$ is the same as the average frequency as defined in Eq.(2.9). When the narrow-band process is assumed for the bilinear hysteresis response,

$$E[X^2] = \frac{1}{2}E[A^2] = \frac{1}{2}\int_0^\infty u^2 p_A(u)\,du \tag{3.5}$$

$$E[\varphi(X)X] = \frac{1}{2}E[AC(A)] = \frac{1}{2}\int_0^\infty u\,c(u)\,p_A(u)\,du \tag{3.6}$$

where $p_A(u)$ is the probability density function of random amplitude A, and

$$C(A) = \frac{1}{\pi}\int_0^{2\pi} \cos\psi \cdot \varphi(A\cos\psi)\,d\psi \tag{3.7}$$

Under the above assumption, Caughey[15] evaluated Eq.(3.4)' by taking the Rayleigh distribution for A, which has probability density of the form

$$p_A(u) = \frac{u}{\sigma_x^2} e^{-\frac{u^2}{2\sigma_x^2}} \tag{3.8}$$

where σ_x is the rms displacement. The result is

$$\left(\frac{\omega_{KB}}{\omega_0}\right)^2 = 1 - \frac{8(1-\alpha)}{\pi}\int_1^{\infty}(\xi^{-3} + \eta^{-1}\xi^{-1})(\xi-1)^{\frac{1}{2}} e^{-\frac{\xi^2}{\eta}} d\xi \tag{3.9}$$

where $\quad \eta = \dfrac{2\sigma_x^2}{Y^2}$

This average frequency prediction was compared with the analog computer result by Lutes[13] and was comfirmed to be able to match very closely the latter for the moderately nonlinear case but deviates appreciably for the nearly elasto-plastic case of σ_x/Y between 0.7 and 10.

Another possible prediction of the average frequency is an extension from the harmonic resonance frequency of a bilinear hysteretic system.[22] Using the Rayleigh amplitude assumption of Eq.(3.8), one can evaluate the expectation of the square of the resonance frequency to obtain an average frequency of

$$\left(\frac{\omega_R}{\omega_0}\right)^2 = \alpha + \frac{1-\alpha}{\pi}\int_{\nu}^{\infty}\left[\cos^{-1}\left(1-\frac{2\nu}{\xi}\right) - 2\left(1-\frac{2\nu}{\xi}\right)\left\{\frac{\nu}{2}\left(1-\frac{\nu}{\xi}\right)\right\}^{\frac{1}{2}}\xi\, e^{-\frac{\xi^2}{2}}\right]d\xi$$
$$+ (1-\alpha)\left(1-e^{-\frac{\nu^2}{2}}\right) \qquad (3.10)$$

where

$$\nu = \sqrt{\frac{2}{\eta}}$$

This derivation is given in Appendix A.

The average frequencies found by Eqs.(3.9) and (3.10) are compared with that from the experimental investigation in Fig.3. Note that Eq.(3.9) is able to predict the average frequency defined by Eq.(2.9) for the bilinear hysteretic system closer than Eq.(3.10). Furthermore, it was shown in Ref.(13) that there is no appreciable variation of the average frequency due to the viscous damping effect except for $\sigma_x/Y < 0.7$. Hence, Eq.(3.9) will be used to predict the average frequency of the original system in the later investigation.

3.3. Energy dissipation per unit of time

As already noted, Eq.(3.2) represents the energy dissipation per unit of time. In this investigation the hysteresis energy dissipation per unit of time $E[\varphi(x)\dot{x}]$ is estimated from the hysteresis energy dissipation per cycle which is the integrated quantity over the hysteresis cycle. Furthermore, this quantity is a convenient measure to use since its magnitude is the area enclosed in the hysteresis loop, and can be found, referring to the restoring force-deformation diagram of Fig.1, as

$$(\text{H.E.D.}) = \begin{cases} 0 & , \text{ for } A \leq Y \\ 4Y(A-Y)K(1-\alpha) & , \text{ for } A > Y \end{cases} \qquad (3.11)$$

In converting from the energy dissipation per cycle to the energy dissipation per unit of time it must be noted that the energy dissipation cycle due to system yielding may be of different duration than the cycle of the mass response of the bilinear hysteretic system which affects the viscous energy dissipation. One can easily recognize from Fig.2 that the former frequency ω_{b2} is to be determined from the response across the Coulomb damper (point 2 in the same figure), while the latter frequency ω_{b1} is to be determined from the mass response. Therefore, the total expected energy dissipation per unit of time in the bilinear hysteretic system can be expressed as

$$E\,(\text{E.D./unit time})_b = c\,E[\dot{x}^2] + E\left[\frac{\omega_{b2}}{2\pi}(\text{H.E.D.})\right] \qquad (3.12)$$

It is anticipated here that ω_{b2} varies such that it is almost equal to ω_o when the response exceeds slightly the yield level Y and as the response greatly exceeds Y it approaches $\sqrt{\alpha}\,\omega_o$ because the stiffness component αK (spring 1 in Fig.2) then dominates the bilinear hysteretic system response which is related to the motion of the point 2 in Fig.2. Thus one can reasonably expect that

$$\sqrt{\alpha}\,\omega_0 \leq \omega_{b2} \leq \omega_0 \tag{3.13}$$

This relation confines the expected hysteresis energy dissipation per unit of time as

$$\frac{\sqrt{\alpha}\,\omega_0}{2\pi} E[H.E.D.] \leq E\left[\frac{\omega_{b2}}{2\pi}(H.E.D.)\right] \leq \frac{\omega_0}{2\pi} E[H.E.D.] \tag{3.14}$$

In the extrme cases such as $A/Y \to 1$ or $A/Y \to \infty$, the bilinear system reduces into the linear system with the initial stiffness K,or the one with the reduced stiffness αK, respectively. Furthermore, the corresponding response appears as a narrow-band process if the viscous damping effect is small. In these cases the distribution of the amplitude A is certainly represented by the Rayleigh distribution of Eq.(3.8). This distribution is also assumed to apply for all other cases in this investigation, although this may turn out to be a crude assumption when A/Y is not either very large or very small.Hence,the expectation of hysteresis energy dissipation per cycle is evaluated from Eq.(3.11) as

$$\begin{aligned} E[H.E.D.] &= \int_Y^\infty 4Y(u-Y)K(1-\alpha)\,p_A(u)\,du \\ &= 2\sqrt{2\pi}K(1-\alpha)Y^2\,\frac{\sigma_x}{Y}\,\mathrm{erfc}\left(\frac{Y}{\sqrt{2}\sigma_x}\right) \end{aligned} \tag{3.15}$$

where erfc(·) denotes the complementary error function defined by

$$erfc(u) = 1 - \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt \tag{3.16}$$

3.4. Prediction of the rms response characteristics

The linearization requirements of Eqs.(3.3) and (3.4) are to match the energy dissipation per unit of time and the average frequency between the bilinear system and the linear system, as is stated in section 3.1. In this section, matching energy dissipation per unit of time is first considered in stationary state motion. This method was used before by Karnopp and Brown[17] who called it the power balance method, but the authors took the average frequency of Eq.(3.9) as the energy dissipation cycle. Therefore, their rms response predictions come out the same as those obtained by Iwan and Lutes[10] using the Krylov-Bogoliubov method.

In what follows, an improved prediction of the rms displacement is achieved by considering the bounds for the expected hysteresis energy dissipation of Eq.(3.14). This technique, however, does not necessarily result in velocity response matching with the original system even when displacement response matching is achieved. To overcome this shortcoming, the average frequency matching between the hysteretic system and its linearization is also considered, taking the prediction by Eq.(3.9). Since the average frequency is related to the viscous damping effect in the bilinear hysteretic system, this matching criterion is of great significance in the linearization of the viscously damped bilinear hysteretic system.

For convenience sake and at the same time for a check of the accuracy in predicting the bilinear hysteretic system response by the following method, a Gaussian white noise excitation f(t) of Eq.(2.2) is used. Then the corresponding response characteristics of the linearized system are easy to obtain from the input-output relationship in linear random vibration theory,[4), 5)] when the frequency response function of the concerned response $H(\omega)$ is known, as

$$E\left[\left(\frac{d^{\ell}}{dt^{\ell}}x\right)^{2}\right] = \sigma^{2}_{\frac{d^{\ell}x}{dt^{\ell}}} = S_0 \int_{-\infty}^{\infty} \left| \omega^{\ell} H(\omega) \right|^{2} d\omega \qquad (3.17)$$

The frequency response function of Eq.(3.1)' is given as

$$\omega_{eq}^{2} H_{2,eq}(\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_{eq}}\right)^{2} + \underline{i}\, 2\beta_{eq}\left(\frac{\omega}{\omega_{eq}}\right)} \qquad (3.18)$$

where $\underline{i}$ is the imaginary unit. Substitution of Eq.(3.18) into Eq.(3.17) for $H(\omega)$ yields the rms displacement response σ_x and the rms velocity response $\sigma_{\dot{x}}$ of

$$\frac{\sigma_x \,\omega_{eq}^{2}}{\sqrt{S_0\, \omega_{eq}}} = \frac{\sigma_{\dot{x}}\, \omega_{eq}}{\sqrt{S_0\, \omega_{eq}}} = \sqrt{\frac{\pi}{2\beta_{eq}}} \qquad (3.19)$$

The 2nd order system of Eq.(3.1) can dissipate the expected energy per unit of time through its viscous damper

$$E[(\text{E.D./ unit time})_{2nd}] = C_{eq}E[\dot{x}^2] = 2m\omega_{eq}\beta_{eq}\sigma_{\dot{x}}^2 \tag{3.20}$$

Substitution of the rms velocity response from Eq.(3.19) into Eq.(3.20) yields

$$E[(\text{E.D./ unit time})_{2nd}] = m\pi S_o \tag{3.21}$$

Note that the energy dissipation per unit of time is independent of K_{eq} and C_{eq}. Then the energy dissipation equivalence criterion between Eq.(2.1) and Eq.(3.1) gives, from Eq.(3.3)

$$2m\beta_0\omega_0\sigma_{\dot{x}_{b1}}^2 + E\left[\frac{\omega_{b2}}{2\pi}(\text{H.E.D})\right] = m\pi S_0 \tag{3.22}$$

Introducing the average frequency ω_{b1}, one can get

$$2m\beta_0\omega_0\omega_{b1}^2\sigma_{x_{b1}}^2 + E\left[\frac{\omega_{b2}}{2\pi}(\text{H.E.D})\right] = m\pi S_0 \tag{3.23}$$

where the Krylov-Bogoliubov prediction ω_{KB} in Eq.(3.9) will be substituted for ω_{b1} from the average frequency matching criterion.

3.4.1. No viscous damping case (C= 0)

In Eq.(3.23), substitute the bounds of Eq.(3.14) for the expected hysteresis energy dissipation per unit of time, for the case where there is no viscous damping effect in the hysteretic system. Hence,

$$\frac{\sqrt{\alpha}\,\omega_0}{2\pi} E[H.E.D.] \leq m\pi S_0 \leq \frac{\omega_0}{2\pi} E[H.E.D] \tag{3.24}$$

where E[H.E.D.] is given in Eq.(3.15). Since the hysteretic system response varies with the ratio of excitation level to yield level, as stated in section 2.2., it is convenient to normalize Eq.(3.24) by the measure of excitation given in Eq.(2.5). Thus,

$$\frac{\pi\sqrt{\pi}\left(\frac{Y}{\sqrt{2}\sigma_x}\right)}{2(1-\alpha)\,\mathrm{erfc}\left(\frac{Y}{\sqrt{2}\sigma_x}\right)} \leq \left(\frac{Y}{N}\right)^2 \leq \frac{\pi\sqrt{\pi}\left(\frac{Y}{\sqrt{2}\sigma_x}\right)}{2\sqrt{\alpha}(1-\alpha)\,\mathrm{erfc}\left(\frac{Y}{\sqrt{2}\sigma_x}\right)} \tag{3.25}$$

This relationship should give upper and lower bounds for the response σ_x/Y (or σ_x/N) for a given Y/N. It is anticipated that in the limiting case $Y/N \to 0$, the upper bound will give a good prediction for σ_x/Y in connection with the remarks in section2.2. On the other hand, if $Y/N \to \infty$, the lower bound is expected to be a good approximation for σ_x/Y.

Another possible estimation of the hysteresis energy dissipation per unit of time is to postulate that its cycle is equal to the average frequency of response. This gives

$$\left(\frac{Y}{N}\right)^2 = \frac{\pi\sqrt{\pi}\,\frac{Y}{\sqrt{2}\sigma_x}}{2\left(\frac{\omega_{bl}}{\omega_0}\right)(1-\alpha)\,\mathrm{erfc}\left(\frac{Y}{\sqrt{2}\sigma_x}\right)} \tag{3.26}$$

where ω_{bl} may be substituted from ω_{KB} in Eq.(3.9) or from ω_{R} in Eq.(3.10). The solution using ω_{KB} for ω_{bl} has been proposed by Karnopp and Brown.[17)]

The above derived "upper" and "lower" bounds for σ_x/Y as well as the predictions from Eq.(3.26) are compared with the experimental results from reference (10) in Fig.5. From the results, it is recognized that the "upper bound", which is derived from the lower energy equivalence in Eq.(3.25), is always the best approximation of σ_x/N. In the region of Y/N of most concern, say, from 2 to 8, this "upper bound' gives almost the same value as the experimental results for the moderately nonlinear case (see Fig.5.1) and estimates the experimental results within 20 % underestimate for the nearly elasto-plastic case. (see Fig.5.2) The "lower bound" on the other hand, turns out to be a poor prediction in the whole region of Y/N. In the small nonlinearity situation ($\sigma_x/Y < 1$), the hysteresis energy dissipation cycle might be expected to be the natural frequency ω_0 for small amplitude response as discussed before. Nevertheless this assumption fails to match the rms displacement response.
A conceivable reason for this may be attributed to the Rayleigh distribution assumption for the peak response process. It is reported in reference (19) that the distribution of the bilinear hysteretic system response is not Gaussian, especially around $\sigma_x/Y = 0.5$ its deviation is strong. The relationship between this strongly non-Gaussian probability and the failure of equivalent linearization is further investigated in this reference.

The solution from Eq.(3.26) with $\omega_{bl} = \omega_{KB}$ is seen to give underestimation within 15 % error in the moderately nonlinear case while failing in its prediction in the region of most concern for the nearly elasto-plastic case. Based on the above observation, the "low energy equivalence" of Eq.(3.25) will be used henceforth as the linearization technique. The reason why the "upper " bound of Eq.(3.25) is not really higher than the experimental results in the nearly elaso-plastic case, as is seen in Fig.5.2, may also be due to the Rayleigh distribution assumption for the random amplitude of bilinear hysteretic system response. In this case, the response comes out as a rather broad-band process in a fairly large range of σ_x/Y, as is investigated in section 3.5., so that the corresponding peak distribution will be between the Rayleigh and the Gaussian distribution.

The rms velocity response is easily found, using the average frequency predicted by Eq.(3.9), as

$$\sigma_{\dot{x}}^2 = \omega_{KB}^2 \sigma_x^2 \qquad (3.27)$$

The bold solid line in Fig.6 shows the computation results for the nearly elasto-plastic hysteretic system while the thin solid line represents the prediction from the Krylov-Bogoliubov method with small nonlinearity,[10)] and both are compared with the experimental results of the dashed curve. Although the rms velocity is directly affected by the prediction of the average frequency of the original system, Eq.(3.27) approximates well the experimental results, giving an underestimate for the latter within 20 % error in the region of Y/N less than 0.5, an estimate almost the same in the vicinity of Y/N = 1 and an overestimate within 17 % error in Y/N between 2 and 15. On the other hand, the Krylov-Bogoliubov method with small nonlinearity gives always an underestimate within 20 % error for the experimental results. In the moderately nonlinear case, the rms velocity prediction by Eq.(3.27) is almost the same as the experimental results in the most concerned region of Y/N between 1 and 10.

3.4.2. Viscously damped case (C $\neq$ 0)

For the viscously damped bilinear hysteretic system, the energy dissipation matching criterion Eq.(3.3) is put into the following form.

$$C\,\omega_{KB}^2 \sigma_x^2 + \sqrt{\frac{\alpha}{\pi}}\,\omega_0\,K(1-\alpha)\,Y^2\left(\frac{\sqrt{2}\sigma_x}{Y}\right)\mathrm{erfc}\left(\frac{Y}{\sqrt{2}\sigma_x}\right) = m\pi S_0 \qquad (3.28)$$

The use of the normalization measure of Eq.(2.5) leads to

$$\beta_0\left(\frac{\omega_{KB}}{\omega_0}\right)^2\left(\frac{\sqrt{2}\sigma_x}{Y}\right)^2 + \sqrt{\frac{\alpha}{\pi}}\,(1-\alpha)\left(\frac{\sqrt{2}\sigma_x}{Y}\right)\mathrm{erfc}\left(\frac{Y}{\sqrt{2}\sigma_x}\right) = \frac{\pi}{2}\left(\frac{N}{Y}\right)^2 \qquad (3.29)$$

This gives the ratio of rms displacement to yield level for a given value of Y/N. The dimensionless rms displacements obtained in this way are plotted in Fig.7 together with the previous results of $\beta_0 = 0$ for the moderately nonlinear case for $\beta_0 = 1$ % and for the nearly elasto-plastic cases for $\beta_0 = 1$ and 5 %. The dashed lines in the same figure are the analog computer results reproduced from reference (10). Note that the above prediction gives almost the same rms displacement response with the experimental results in the whole region of Y/N for the moderately nonlinear hysteretic system with 1 % viscous damping effect, and an underestimate within 15 % error for the nearly elasto-plastic case with the same viscousdamping but it loses the accuracy in the most concerned region of Y/N as the viscous damping effect grows and results in only 75 % prediction of the experimental result in the worst case when 5 % viscous damping exists. This is due to the descrepancy of the prediction of average frequency by Eq.(3.9) from ω_{b1} as was seen in Fig.3. However, it must be emphasized that the method presented here indicates a remarkable improvement of the response prediction when compared with other methods.[10), 19)]

When the nonlinearity situation becomes great such that $\sigma_x/Y \gg 1$, the average frequency ω_{KB} approaches $\sqrt{\alpha}\,\omega_0$ and the complementary error function can be expanded in a series as

$$erfc\left(\frac{Y}{\sqrt{2}\sigma_x}\right) = 1-\sqrt{\frac{2}{\pi}}\left[\frac{Y}{\sigma_x} - \frac{1}{3\cdot 1}\left(\frac{Y}{\sigma_x}\right)^3 + \frac{1}{5\cdot 2^2\cdot 2!}\left(\frac{Y}{\sigma_x}\right)^5 - \cdots\right]$$

Using only the first term of this series in Eq.(3.29) gives

$$\alpha\beta_0\left(\frac{\sqrt{2}\sigma_x}{Y}\right)^2 + \sqrt{\frac{\alpha}{\pi}}(1-\alpha)\left(\frac{\sqrt{2}\sigma_x}{Y}\right)\left(1-\sqrt{\frac{2}{\pi}}\frac{Y}{\sigma_x}\right) = \frac{\pi}{2}/\left(\frac{Y}{N}\right)^2$$

The first term being great compared with the second term, then in the limit one can get

$$\frac{\sigma_x}{N} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha\beta_0}} \tag{3.30}$$

On the other hand, when the nonlinearity situation becomes small such that $\sigma_x/Y \ll 1$, the average frequency ω_{KB} approaches to ω_0 and the complementary error function has the expanded form of

$$erfc\left(\frac{Y}{\sqrt{2}\sigma_x}\right) = \sqrt{\frac{2}{\pi}}\frac{\sigma_x}{Y}\exp\left(-\frac{Y^2}{2\sigma_x^2}\right)\left[1-\frac{\sigma_x^2}{Y^2}+\frac{1\cdot 3\,\sigma_x^4}{Y^4}-\cdots\right]$$

so that taking only the first term leads Eq.(3.29) into

$$\beta_0\left(\frac{\sqrt{2}\sigma_x}{Y}\right)^2 + \frac{\sqrt{\alpha}}{\pi}(1-\alpha)\left(\frac{\sqrt{2}\sigma_x}{Y}\right)^2\exp\left(-\frac{Y^2}{2\sigma_x^2}\right) = \frac{\pi}{2}/\left(\frac{Y}{N}\right)^2$$

The second term being again negligible compared with the first term, so that in the limit one can get

$$\frac{\sigma_x}{N} = \frac{1}{2}\sqrt{\frac{\pi}{\beta_0}} \tag{3.31}$$

Eqs.(3.30) and (3.31) indicate that in the above extreme nonlinearity situations the hysteretic system turns out as the 2nd order system with initial stiffness or with reduced stiffness as was discussed intuitively in Chapter II. These phenomena are clearly recognized in the corresponding regions of Fig.7, where the analytical predictions coincide with the experimental results and both approach to the asymptotes of Eqs.(3.30) or (3.31).

The rms velocity response of the viscously damped bilinear hysteretic system is again found using the average frequency prediction of Eq.(3.9), i.e.,

$$\sigma_{\dot{x}}^2 = \omega_{KB}^2 \sigma_x^2 \tag{3.32}$$

Fig.8 shows the computation results, where one can see the almost coincidence between the prediction and the experimental result in the whole region of Y/N for the moderately nonlinear hysteretic system with 1 % viscous damping factor. However, the prediction gives an overestimate within 20 % error of the experimental result in the 1 % viscously damped nearly elasto-plastic hysteretic system and an overestimate within 15 % error in the 5 % viscously damped case.

3.5. Power spectral density matching

In the previous section, reasonably good predictions both for the rms displacement and rms velocity response of the bilinear hysteretic system in stationary state motion are obtained based on the criteria of matching energy dissipation and average frequency. However, different linearizations satisfying both these matchings with the original system are now conceivable. Another severe check on the accuracy of the equivalence between the bilinear hysteretic system and its linearization can be made in the frequency domain by comparing the response power spectral density. This equivalence was first attempted by Lutes[13] for the equivalent 2nd order linear system and its two-mode model. Note that Eq.(3.17) guarantees the rms response characteristics matching up to the higher order derivative of response if it exists when the response power spectral density is matched between two systems. However, a more complicated linear system will generally be required for this matching rather than the system matching only the rms displacement or the rms velocity, or both.

3.5.1. Bilinear hysteretic system

As is stated before, the bilinear hysteretic system response varys with the ratio of excitation to yield level. At the same time it must be emphasized that its response power spectral density shows a remarkable variation of shape depending on this ratio. Hence, it may be worthwhile to review this relationship from Iwan and Lutes' analog computer investigation.[10] These authors presented the response power spectral density of the bilinear hysteretic system in the form of $\omega_0^2\sqrt{\frac{S_x(\omega)}{S_0}}$.

In Fig.9, the dashed lines show the experimental results for representative cases of σ_x/Y with no viscous damping.

It is noted in Fig.9.1 that the shape of the power spectral density of the moderately nonlinear case is close to that for a lightly damped 2nd order linear system, indicating a noticeable peak. However, the location of this peak is affected by the yield level Y and shifts continuously from the undamped natural frequency ω_0 of the small amplitude stiffness to the reduced frequency $\sqrt{\alpha}\,\omega_0$ of the second branch stiffness as the value of σ_x/Y becomes large. The peak is always bounded and it has its lowest value at approximately $\sigma_x/Y = 1.3$. On the other hand, the nearly elasto-plastic hysteretic system is much more affected by the yielding. Fig.9.2 shows a predominant peak at ω_0 for small σ_x/Y and at $\sqrt{\alpha}\,\omega_0$ for large σ_x/Y; however, the peak frequency shifting does not occur in the same way for this case as it does for the moderately nonlinear case for intermediate values of σ_x/Y. Rather, it appears that for this system the peak at ω_0 decreases while the lower frequency components increase at the same time. This gives a monotonically decreasing curve with no predominant peak at all for σ_x/Y between 4 and 9. This is indicative of a broad-band response. At the same time, it is found that the low frequency power spectral density always appear like that for a linear system with the reduced stiffness.

3.5.2. Equivalent linear systems

For linear systems the response power spectral density in dimensionless form as above used is nothing but the normalized frequency transfer function, $\omega_0^2|H(\omega)|$.

3.5.2.a. Equivalent 2nd order linear system

Consider the system governed by Eq.(3.1). Its frequency transfer function can be written as

$$\omega_0^2|H_{2,eq}(\omega)| = \left[\frac{1}{\left(\frac{\omega_{eq}}{\omega_0}\right)^2\left[\left\{1-\left(\frac{\omega}{\omega_0}\right)^2\Big/\left(\frac{\omega_{eq}}{\omega_0}\right)^2\right\}^2 + 4\beta_{eq}\left(\frac{\omega}{\omega_0}\right)^2\Big/\left(\frac{\omega_{eq}}{\omega_0}\right)^2\right]}\right]^{\frac{1}{2}} \tag{3.33}$$

The equivalent linearization criteria requiring both matching energy dissipation per unit of time and average frequency with the bilinear hysteretic system yield

$$\omega_{eq} = \omega_{KB} \tag{3.34}$$

$$\beta_{eq} = \frac{\pi}{4}\Big/\left\{\left(\frac{\omega_{eq}}{\omega_0}\right)^3\left(\frac{\sigma_x}{N}\right)^2\right\} \tag{3.35}$$

where ω_{KB} and σ_x/N are already obtained by Eqs.(3.9) and (3.28), respectively. This system, as is seen from Eq.(3.33), can have only one predominant peak at frequency ω_{eq} when β_{eq} is small. Therefore, it may be of use in approximating the response power spectral density of the moderately nonlinear hysteretic system. But for the nearly elasto-plastic hysteretic system case its application will be limited only to the nonlinearity situation $\sigma_x/Y < 1$, considering the remarks in section 3.5.1. Fig.9.1 shows the comparison of the analog computer results and its a approximation by using Eq.(3.33) for the moderately nonlinear case. Note that this system with stiffness and damping factor chosen as functions of σ_x/Y can match satisfactorily the response power spectral of the hysteretic system only in great nonlinearity situations ($\sigma_x/Y \geqslant 1$). In this case the transfer function of Eq.(3.33) has

$$\omega_0^2 |H_{2,eq}(\omega)| \longrightarrow \frac{1}{\alpha} \qquad \text{as } \omega \longrightarrow 0 \qquad (3.36)$$

This is the same tendency as for the bilinear hysteretic system described in section 3.5.1. But the approximation by Eq.(3.33) still has somewhat of a shortcoming for the small nonlinearity situation case, for then

$$\omega_0^2 |H_{2,eq}(\omega)| \longrightarrow 1 \qquad \text{as } \omega \longrightarrow 0 \qquad (3.37)$$

This is contrary to the remarks in section 3.5.1.

3.5.2.b. Uncorrelated two-mode 2nd order linear system

Lutes[13] has suggested that an improved linear model can be devised for the nearly elasto-plastic case by taking a two-mode model of usual 2nd order linear systems, where the modal responses are assumed uncorrelated and one modal frequency corresponds to the initial stiffness and the other to the reduced stiffness of the bilinear hysteretic system. This gives the linear transfer function of

$$\omega_0^2 |H_{2,2}(\omega)| = \left[\frac{1}{\{1-(\frac{\omega}{\omega_0})^2\}^2 + 4\beta_{2H}^2(\frac{\omega}{\omega_0})^2} + \frac{1}{\{\alpha-(\frac{\omega}{\omega_0})^2\}^2 + 4\beta_{2L}^2\alpha(\frac{\omega}{\omega_0})^2} \right]^{\frac{1}{2}} \tag{3.38}$$

where

$$\beta_{2H} = \frac{\pi(\alpha-1)}{4\{\alpha(\frac{\sigma_x}{N})^2 - (\frac{\sigma_{\dot{x}}}{\omega_0 N})^2\}} \qquad \beta_{2L} = \frac{\pi(1-\alpha)}{4\alpha\sqrt{\alpha}\{(\frac{\sigma_x}{N})^2 - (\frac{\sigma_{\dot{x}}}{\omega_0 N})^2\}} \tag{3.39}$$

The response characteristics included above are already obtained by Eqs.(3.29) and (3.32), respectively. However, this method may be limited to the case where the above two modes are well separated, since the response of two modes whose natrual frequencies are not well separated can not be expected to be uncorrelated. The correlation effect in this latter case usually reduces the two peaks of of each mode into one.[22)]

Fig.9.2 shows the approximation obtained in this way for the nearly elasto-plastic hysteretic system. Note that it has the same general tendency as the experimental response power spectral density; however, a slight descrepancy can be observed, which gives a lower value in the region $\frac{\omega}{\omega_0}<1$ but a higher value in the vicinity of $\frac{\omega}{\omega_0} = 1$. This discrepancy is partly due to the inaccurate prediction of the rms displacement and rms velocity response. Fig.9.3 shows the approximation by Eq.(3.38) for the moderately nonlinear cases. It is noticed here that such approximation is only good in the strong nonlinearity situation ($\sigma_x/Y \gg 1$).

The worst case is $\sigma x/Y = 1.3$ as shown in Fig.9.4, where the least response is observed. In this figure the response power spectral density approximation by Eq.(3.33) is also presented.

IV. EQUIVALENT LINEARIZATION BY THE 3RD ORDER SYSTEM

4.1. Description of the system

In this chapter, the linear model of Fig.10 is considered, with all elements taken the same as in the bilinear hysteretic system except that the Coulomb friction slider is replaced by an additional viscous damper. This kind of model is frequently used in the visco-elastic theory. The corresponding governing equations of motion can be expressed as:

$$\left.\begin{aligned} m\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) &= mf(t) \\ c_2\dot{x}_2 - k_2(x_1 - x_2) &= 0 \end{aligned}\right\} \quad (4.1)$$

where x_1 and x_2 are the displacements of the mass and across the second damper as indicated in Fig.10, and C_1 and C_2 represent the equivalent viscous damping coefficients for the viscous and hystersis energy loss in the bilinear hysteretic system, respectively. Then the energy dissipation in the original system due to yielding is replaced by that due to viscous damping in the proposed linear system. Intuitively one can recognize that this system, when $C_2 \to \infty$, corresponds to the original system with $Y \to \infty$ and both give a 2nd order linear system with a stiffness $K = k_1 + k_2$. Similarly the original system, when $Y \to 0$, corresponds to the new system with $C_2 \to 0$ and both give the 2nd order linear system with reduced stiffness αK.

To obtain the rms response characteristics of the proposed linear system from Eq.(3.17), its frequency response function first must be found. It is easily calculated from the steady state motion as is seen in Appendix B. The result is

$$\omega_0^2 H_{3,1}(\omega) = \frac{i\frac{1-\alpha}{2\beta_{32}}\left(\frac{\omega}{\omega_0}\right) + 1}{-i\frac{1-\alpha}{2\beta_{32}}\left(\frac{\omega}{\omega_0}\right)^3 - \left\{\frac{\beta_{31}}{\beta_{32}}(1-\alpha) + 1\right\}\left(\frac{\omega}{\omega_0}\right)^2 + i\left(2\beta_{31} + \frac{1-\alpha}{2\beta_{32}}\right)\left(\frac{\omega}{\omega_0}\right) + \alpha} \tag{4.2}$$

where β_{31} is the fraction of the critical viscous damping effect defined by

$$\beta_{31} = \frac{c_1}{2\omega_0 m} \tag{4.3}$$

and β_{32} is the additional viscous damping factor representing the hysteresis energy dissipation defined by

$$\beta_{32} = \frac{1-\alpha}{2\omega_0} \cdot \frac{k_2}{c_2} \tag{4.4}$$

Note that Eq.(4.2) has the 3rd order power of frequency since the system is a 3rd order linear system. Similarly, the frequency response function of the point 2 in Fig.10 is found as

$$\omega_0^2 H_{3,2}(\omega) = \frac{1}{-i\frac{1-\alpha}{2\beta_{32}}\left(\frac{\omega}{\omega_0}\right)^3 - \left\{\frac{\beta_{31}}{\beta_{32}}(1-\alpha) + 1\right\}\left(\frac{\omega}{\omega_0}\right)^2 + i\left(2\beta_{31} + \frac{1-\alpha}{2\beta_{32}}\right)\left(\frac{\omega}{\omega_0}\right) + \alpha} \tag{4.5}$$

The limiting cases of the bilinear hysteretic system with an infinite or zero yield level, being 2nd order linear systems, are governed by Eq.(2.6) and then their frequency response functions are expressed as

$$\Omega^2 H(\omega) = \frac{1}{\{1-(\frac{\omega}{\Omega})^2\} + i2\zeta(\frac{\omega}{\Omega})} \tag{4.6}$$

where $\Omega = \omega_0$ or $\Omega = \alpha\omega_0$, depending on the yield level $Y \to \infty$ or $Y \to 0$, respectively, and $\zeta\Omega = \beta_0\omega_0$. These extreme situations are the same in the 3rd order linear system. That is

$$\omega_0^2 H_{3,1}(\omega) = \frac{1}{\{1-(\frac{\omega}{\omega_0})^2\} - i2\beta_{31}(\frac{\omega}{\omega_0})} \quad , \text{ as } \beta_{32} \to 0 \tag{4.7}$$

and

$$\omega_0^2 H_{3,1}(\omega) = \frac{1}{\alpha[\{1-(\frac{\omega}{\sqrt{\alpha}\,\omega_0})^2\} - i2\beta_{31}'(\frac{\omega}{\sqrt{\alpha}\,\omega_0})]} \quad , \text{ as } \beta_{32} \to \infty \tag{4.8}$$

where

$$\beta_{31}' = \frac{C_1}{2\sqrt{\alpha}\,\omega_0 m}$$

These frequency response functions imply that the natural frequency is ω_0 when $\beta_{32} \to 0$ and $\sqrt{\alpha}\,\omega_0$ when $\beta_{32} \to \infty$, and that the value of the viscous damping factor β_{31} or β_{31}' is the same as β_0 in the above extreme cases.

Substitution of Eqs.(4.2) or (4.5) for H(ω) in Eq.(3.17) yields for the mass response the mean-square (ms) displacement

$$\sigma_{x_{3,1}}^2 = \frac{\pi S_0}{2\beta_{32}\omega_0^3} \cdot \frac{1 + \frac{4\beta_{32}^2}{\alpha(1-\alpha)^2} + \frac{4\beta_{31}\beta_{32}}{\alpha(1-\alpha)}}{\{1+\frac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\}\{1+\frac{\beta_{31}}{\beta_{32}}(1-\alpha)\} + \alpha\frac{\beta_{31}}{\beta_{32}}} \tag{4.9}$$

and the ms velocity

$$\sigma_{\dot{x}_{3,1}}^{2} = \frac{\pi S_0}{2\beta_{32}\omega_0} \cdot \frac{1 + \dfrac{4\beta_{32}^{2}}{(1-\alpha)^2} + \dfrac{4\beta_{31}\beta_{32}}{1-\alpha}}{\left\{1 + \dfrac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left\{1 + \dfrac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\} + \alpha\dfrac{\beta_{31}}{\beta_{32}}} \tag{4.10}$$

and for the point 2 response the ms displacement

$$\sigma_{x_{3,2}}^{2} = \frac{\pi S_0}{2\beta_{32}\omega_0^3} \cdot \frac{\dfrac{4\beta_{32}}{\alpha(1-\alpha)}\left\{\dfrac{\beta_{32}}{1-\alpha} + \beta_{31}\right\}}{\left\{1 + \dfrac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left\{1 + \dfrac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\} + \alpha\dfrac{\beta_{31}}{\beta_{32}}} \tag{4.11}$$

and the ms velocity

$$\sigma_{\dot{x}_{3,2}}^{2} = \frac{\pi S_0}{2\beta_{32}\omega_0} \cdot \frac{\dfrac{4\beta_{32}^{2}}{(1-\alpha)^2}}{\left\{1 + \dfrac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left\{1 + \dfrac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\} + \alpha\dfrac{\beta_{31}}{\beta_{32}}} \tag{4.12}$$

where the integration operation has been carried out by the residue integral formula.[23] An alternative method for obtaining the above response characteristics appears in Appendix C. When the normalization is made for Eqs.(4.9) and (4.10) by the measure of Eq.(2.5), one obtains

$$\left(\frac{\sigma_{x_{3,1}}}{N}\right)^{2} = \frac{\pi}{4} \cdot \frac{1 + \dfrac{4\beta_{32}^{2}}{\alpha(1-\alpha)^2} + \dfrac{4\beta_{31}\beta_{32}}{\alpha(1-\alpha)}}{(1-\alpha)\left\{1 + \dfrac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left(\dfrac{\beta_{32}}{1-\alpha} + \beta_{31}\right) + \alpha\beta_{31}} \tag{4.9'}$$

and

$$\left(\frac{\sigma_{\dot{x}_{31}}}{\omega_0 N}\right)^2 = \frac{\pi}{4} \cdot \frac{1 + \frac{4\beta_{32}^2}{(1-\alpha)^2} + \frac{4\beta_{31}\beta_{32}}{1-\alpha}}{(1-\alpha)\left\{1 + \frac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left(\frac{\beta_{32}}{1-\alpha} + \beta_{31}\right) + \alpha\beta_{31}} \qquad (4.10)'$$

4.2. Prediction of the rms response

The 3rd order linear system defined by Eq.(4.1), when subjected to the random input motion of Eq.(2.2), can be expected to dissipate the enrgy per unit of time

$$E\left[(\text{E.D./unit time})_{3rd}\right] = C_1 E[\dot{x}_1^2] + C_2 E[\dot{x}_2^2] \qquad (4.13)$$

and when $E[\dot{x}_1^2]$ and $E[\dot{x}_2^2]$ are substituted from Eqs.(4.9) and (4.10), the right hand side of Eq.(4.13) gives $m\pi S_o$, that is

$$E\left[(\text{E.D./unit time})_{3rd}\right] = m\pi S_o \qquad (4.14)$$

This energy dissipation per unit of time is exactly the same as that for the 2nd order linear system, as is seen in Eq.(3.21). This fact might be expected from the fact that the energy dissipation per unit of time is related only to the mass as far as a white noise excitation is assumed.[16] Hence, the same rms displacement response values as shown in Fig.7 can also be predicted from the above 3rd order linear system by matching energy dissipation per unit of time, taking the same procedure as for the 2nd order linearization. However, the rms velocity which is directly related to the average frequency may not be matched with the bilinear hysteretic system. But the previous consideration on the frequency response function suggests that the 3rd order linear system may have more flexibility than the specified 2nd order linear system.

4.2.1. No viscous damping case (C = 0)

When no viscous damping is assumed, as a special case, for the bilinear hysteretic system, the corresponding damping factor β_{31} in the 3rd order linear system vanishes. The average frequency of this system is then found from Eqs.(4.9) and (4.10) by setting $\beta_{31} = 0$ as

$$\frac{\omega_{3,1}}{\omega_0} = \sqrt{\alpha}\left\{ 1 + \frac{(1-\alpha)^3}{\alpha(1-\alpha)^2 + 4\beta_{32}^2} \right\}^{\frac{1}{2}} \qquad (4.15)$$

which implies that $\omega_{3,1}/\omega_0 = 1$ when $\beta_{32} = 0$ and $\omega_{3,1}/\omega_0 = \sqrt{\alpha}$ when $\beta_{32} \to \infty$, as are seen in Eqs.(4.7) and (4.8), and monotonically decreases as β_{32} grows. However, the value of β_{32} is constrained if energy dissipation matching is achieved. In particular, from Eq.(4.9) β_{32} is found to be

$$\beta_{32} = \frac{1-\alpha}{2\pi}\left\{\alpha(1-\alpha)\left(\frac{\sigma_{x_{3,1}}}{N}\right)^2 \pm \sqrt{\alpha}\sqrt{\alpha(1-\alpha)^2\left(\frac{\sigma_{x_{3,1}}}{N}\right)^4 - \pi^2}\right\} \quad (4.16)$$

Note that there exist two different 3rd order linear systems which give the same rms displacement response but have different average frequencies. The variation β_{32} is shown versus Y/N in Fig.11, where β_{32} always falls in the region less than 0.053 or more than 0.60 for the moderately nonlinear hysteretic system. The corresponding values for the nearly elasto-plastic case are 0.031 and 0.36. Fig.12 illustrates how the rms displacement $\sigma_{x_{3,1}}/N$ and the average frequency ω_{31} vary with the value of β_{32}. The small value of β_{32} in the region I in this figure gives the average frequency as near the natural frequency for the initial stiffness, ω_0 and the large value of β_{32} in the region II gives it fairly near the reduced frequency for the second branch stiffness $\sqrt{\alpha}\,\omega_0$. Henceforth, the 3rd order system of the smaller β_{32} is referred to as the high frequency 3rd order linear system and the one of the large β_{32} as the low frequency 3rd order linear system. These values of average frequency are compared with the analog computer results in Fig.3. It is noted here that one may reasonably choose the low frequency 3rd order linear system as a linearization of the bilinear hysteretic system in the large nonlinearity situation and the high frequency 3rd order linear system in the small nonlinearity situation both for the moderately and nearly elasto-plastic cases. But in the most concerned nonlinearity situation, say in the region of σ_x/Y from 0.6 to 10, neither of the above 3rd order linear systems can approximate the experimental results for average frequency. This results in failure in matching the velocity response in that situation. The one-dotted and the two-dotted chain lines in Fig.6 are the case for the nearly elasto-plastic hysteretic system.

To overcome this shortcoming of individual 3rd order systems, their combination is considered so as to produce average frequency matching with the original system. The simplest such model takes the total mean-square response as the weighted sum of individual responses, neglecting any coupling effects. Thus, the average frequency is expressed as

$$\omega_{b1}^2 = \gamma \frac{\sigma_{\dot{x}_{3,1,L}}^2}{\sigma_{x_{3,1,L}}^2} + (1-\gamma)\frac{\sigma_{\dot{x}_{3,1,H}}^2}{\sigma_{x_{3,1,H}}^2} \tag{4.17}$$

where $\sigma_{x_{3,1,L}} = \sigma_{x_{3,1,H}} = \sigma_{x_{3,1}}$. The notations $\sigma_{x_{3,1,L}}$ and $\sigma_{x_{3,1,H}}$ are the rms displacements of the low frequency 3rd order system and the high frequency 3rd order system, respectively, and $\sigma_{\dot{x}_{3,1,L}}$ and $\sigma_{\dot{x}_{3,1,H}}$ are the corresponding rms velocities. The weighting factor γ is the fraction of the mean-square response contributed by the low frequency system. This procedure is to adopt a two-mode linear system whose modes are 3rd order and uncoupled as a linearization of the bilinear hysteretic system. Thus this model will be referred to as the uncorrelated two-mode 3rd order linear system. The left hand side of Eq.(4.17) will be replaced by Eq.(3.9) following the remarks in section 3.2. Then, γ is solved as

$$\gamma = \frac{\omega_{KB}^2 \sigma_{x_{3,1}}^2 - \sigma_{\dot{x}_{3,1,H}}^2}{\sigma_{\dot{x}_{3,1,L}}^2 - \sigma_{\dot{x}_{3,1,H}}^2} \tag{4.18}$$

Thus obtained rms velocity response is indicated in Fig.6 by the bold solid line, which is of course identical with the prediction by the 2nd order linearization, since both are based on matching the average frequency to ω_{KB}.

4.2.2. Viscously damped case ($C \neq 0$)

In principle it should be possible to choose the two damping parameters C_1 and C_2 (or β_{31} and β_{32}) for the modes such that the response of the uncorrelated two-mode 3rd order system matches both the rms displacement and velocity response coming from Eqs.(3.29) and (3.32). When the hysteretic system also contains some viscous damping ($C \neq 0$), however, the evaluation of the equivalence parameters becomes very difficult. The explicit evaluation of these parameters is desired since they are needed for consideration of response power spectral density and transient response. The following section presents an alternative approximate method for evaluating the parameters in the uncorrelated two-mode 3rd order linearization of a hysteretic system containing viscous damping.

4.3. Alternative response prediction

In this method the viscous damping coefficient C_2 is taken to be the same as if C_1 were zero. Thus Eq.(4.16) is used to obtain β_{32}, giving both a high frequency and a low frequency 3rd order systems, as before. In order to match the energy dissipation between the other viscous damper and the viscous damper in the bilinear hysteretic system, the damping coefficient C_1 is determined from

$$C_1 E[\dot{X}_1^2] = C\, E[\dot{X}_{b1}^2] \tag{4.19}$$

Therefore, the assumptions are that the effect of the viscous damping on the hysteresis energy dissipation is considered negligible and also that the effect of the damping C_1 on the motion of the point 2 in the 3rd order linear system is considered negligible. The 3rd order linear system with such obtained damping coefficients will certainly yield a slightly different response than that from Eq.(4.14) when the viscous damping effect grows in the bilinear hysteretic system.

Introducing the average frequencies ω_{b1} and ω_{31} gives

$$C_1 = \left(\frac{\omega_{b1}}{\omega_{31}}\right)^2 C \tag{4.20}$$

or

$$\beta_{31} = \left(\frac{\omega_{b1}}{\omega_{31}}\right)^2 \beta_0 \tag{4.20$'$}$$

Furthermore, this equation is replaced by the following from the average frequency matching criterion:

$$\beta_{31} = \left(\frac{\omega_{KB}}{\omega_{31}}\right)^2 \beta_0 \qquad (4.21)$$

When the forementioned two-mode 3rd order linear system is adopted as a linearization of the bilinear hysteretic system, this equation produces two different values for the viscous damping factor β_{31}, depending on the average frequency of the 3rd order linear system. Substituion of β_{31} and β_{32} thus obtained both for the low and high frequency 3rd order systems into Eqs.(4.9) and (4.10) gives the corresponding system response characteristics. In applying the two-mode concept the rms displacements from the two modes are assumed to be the same, which is not exactly true but is is found to be acceptable when the viscous damping is small.

The normalized rms displacement and velocity response obtained in this way are plotted in Figs.13 and 14 for the moderately nonlinear systems with $\beta_0 = 0$, and 1 % and for the nearly elasto-plastic ones with $\beta_0 = 0$, 1, and 5 %, and they are compared with the analog computer results. For the moderately nonlinear case the uncorrelated two-mode 3rd order system predicts almost the same values of rms displacement and velocity as the experimental results in the most concerned region when $\beta_0 = 1$ %, but it loses accuracy in the small nonlinearity situation of $\sigma_x/Y < 0.7$, slightly underestimating the experimental results, and in the large nonlinearity situation of $\sigma_x/Y > 5$, slightly overestimating them. For the nearly elasto-plastic case a remarkable improvement within 10 % error underestimation is attained in predicting both the rms displacement and velocity response in the most concerned nonlinearity situation. Furthermore, for the 5 % viscously damped case the rms displacement is almost the same as the experimental results even in the small nonlinearity situation and the rms velocity is underpredicted within 13 % error. In the large nonlinearity situation both response characteristics are slightly overpredicted.

4.4. Response power spectral density

In what follows the response power spectral densities of the 3rd order system and of the uncorrelated two-mode 3rd order system whose parameter are determined in section 4.3 are investigated. Since the frequency response function of the 3rd order system is found by Eq.(4.2) for the mass response, its normalized transfer function is expressed as

$$\omega_0^2 |H_{3,1}(\omega)| = \left[\frac{1 + \left\{ \frac{1-\alpha}{2\beta_{32}} \left(\frac{\omega}{\omega_0}\right) \right\}^2}{\left[\alpha - \left\{ \frac{\beta_{31}}{\beta_{32}}(1-\alpha) + 1 \right\} \left(\frac{\omega}{\omega_0}\right)^2 \right]^2 + \left[\frac{1-\alpha}{2\beta_{32}} \left\{ 1 - \left(\frac{\omega}{\omega_0}\right)^2 \right\} + 2\beta_{31} \right]^2 \left(\frac{\omega}{\omega_0}\right)^2} \right]^{\frac{1}{2}} \tag{4.22}$$

Neglecting correlation between the modal responses, the previously adopted two-mode system has a transfer function of

$$\omega_0^2|H_{3,1,2}(\omega)| = \omega_0^2\{\gamma|H_{3,1,L}(\omega)|^2 + (1-\gamma)|H_{3,1,H}(\omega)|^2\}^{\frac{1}{2}} \tag{4.23}$$

where the low frequency 3rd order linear system transfer function, $H_{3,1,L}(\omega)$ and the high frequency one, $H_{3,1,H}(\omega)$ are found from Eq.(4.22) by substituting the appropriate values of β_{31}, β_{32} and γ. When $\beta_{31} = 0$, the above 3rd order system or the two-mode system coincides with the corresponding system obtained from the energy dissipation matching and average frequency matching criteria in section 4.2.

The plotting of Eq.(4.22) and Eq.(4.23) are made for the representative cases of nonlinearity situation used in section 3.5 for the 2nd order linearization. Fig.15.1 for a nearly elasto-plastic case shows that for the nonlinearity situation of $\sigma_x/Y = 1.0$ the two-mode 3rd order system can approximate the experimental results quite well except in the less significant very low frequency region, having a small peak at the natural frequency ω_0 of initial stiffness of the bilinear hysteretic system and relatively flat frequency contents below ω_0. But as the nonlinearity situation grows the two-mode 3rd order system fails in matching the response power spectral density in the vicinity of ω_0. The experimental results lose the predominant peak at ω_0 but the two-mode system does still have a small spike there. In this nonlinearity situation, it appears that the low frequency 3rd order system by itself can better approximate the experimental results as is seen in Fig.15.2. For a moderately nonlinear hysteretic case, the response power spectral density approximation by the 3rd order system is limited to large or small nonlinearity situationsdue to its rather fixed average frequency at either the natural frequency of initial stiffness or of the reduced stiffness of the original system. Compared to the 2nd order system approximation, the 3rd order system can improve the matching with the experimental results in the low frequency region, although the peak location matching is not so good as the former system (see Fig.15.3). The uncorrelated two-mode approach by the 3rd order system in the intermediate nonlinearity situation results in a failure, showing two appreciable peaks at the above locations. The computation result at $\sigma_x/Y = 1.3$ is given in Fig.9.4 for comparison from other linearizations.

V. TRANSIENT RESPONSE

The preceeding two chapters presented several equivalent linear systems which give reasonable predictions for both the rms displacement and velocity response for bilinear hysteretic systems in stationary state motion. When one is concerned with the structural response due to earthquake motion, however, it should be noted that structures are usually forced to vibrate from an initial state of rest. An earthquake excitation is generally of relatively short duration, say around 30 seconds at most, and its main part might be considered as a stationary Gaussian process. Structures designed to resist earthquakes generally have a fundamental vibration period of 0.5 to 5 seconds. Hence, from the structural design point of view the most significant portion of the response may be in the transient region before reaching the stationary state motion. Note the fact now that different equivalent linear systems may have different response features in this region, even though they have the same stationary response levels. For these reasons, transient response comparisons are made between these linear systems and the bilinear hysteretic system, assuming the Gaussian white noise of Eq.(2.2) as input motion.

5.1. The 2nd order linear systems

The transient response characteristics of the linear system expressed by Eq.(3.1) are found from the Rice method [3), 4), 5)] as

$$E\left[\frac{d^i x(t_1)}{dt^i}\cdot\frac{d^j x(t_2)}{dt^j}\right] = \int_0^{t_1}\int_0^{t_2}\frac{d^i}{dt^i}h(t_1-\tau_1)\,\frac{d^j}{dt^j}h(t_2-\tau_2)\,E[f(\tau_1)f(\tau_2)]\,d\tau_1\,d\tau_2 \tag{5.1}$$

where h(t) is the impulse response function of the concerned system, which is the inverse Fourier transform of the frequency response function of Eq.(3.18). Substitution of h(t) and Eq.(2.2) into Eq.(5.1) for $E[f(\tau_1)f(\tau_2)]$ [24)] gives the mean-square (ms) displacement as

$$\left(\frac{\Delta x}{N}\right)^2 = \frac{\pi}{4\beta_{eq}\left(\frac{\omega_{eq}}{\omega_0}\right)^3}\left[1 - e^{-2\beta_{eq}\omega_{eq}t}\left\{1 + \frac{\beta_{eq}\omega_{eq}}{\omega_d}\sin 2\omega_d t + \frac{2\beta_{eq}^2\omega_{eq}^2}{\omega_d^2}\sin^2\omega_d t\right\}\right] \tag{5.2}$$

and the ms velocity as

$$\left(\frac{\Delta \dot{x}}{\omega_0 N}\right)^2 = \frac{\pi}{4\beta_{eq}\left(\frac{\omega_{eq}}{\omega_0}\right)}\left[1 - e^{-2\beta_{eq}\omega_{eq}t}\left\{1 - \frac{\beta_{eq}\omega_{eq}}{\omega_d}\sin 2\omega_d t + \frac{2\beta_{eq}^2\omega_{eq}^2}{\omega_d^2}\sin^2\omega_d t\right\}\right] \tag{5.3}$$

for $\beta_{eq} \leq 1$, where $\omega_d = \omega_{eq}\sqrt{1-\beta_{eq}^2}$ and the measure of excitation N is given in Eq.(2.5). The parameters included above, ω_{eq} and β_{eq} can be obtained by Eqs.(3.34) and (3.35), respectively.

When the uncorrelated two-mode model indicated by Eq.(3.38) is considered as a linearization, the transient ms response is the sum of the ms responses from each mode. In this case, the damping factors which are determined by Eq.(3.39) are not always less than 1.0. For the representative cases investigated in section 3.5, the high frequency mode always has the damping factor less than 1.0 but the low frequency mode has values greater than 1.0 in small nonlinearity situations (small σ_x/Y). Thus, when both the high and the low frequency systems have damping factors less than 1.0, the transient response can be calculated from Eqs.(5.2) and (5.3) by replacing ω_{eq} by ω_0 or $\sqrt{\alpha}\,\omega_0$, and β_{eq} by β_{2H} or β_{2L} in Eq.(3.39), respectively. For the system with damping factor greater than 1.0, the transient response can be calculated by the following:

for the ms displacement

$$\left(\frac{\sigma_x}{N}\right)^2 = \frac{\pi}{4\beta_{eq}\left(\frac{\omega_{eq}}{\omega_0}\right)^3}\left[1 - e^{-2\beta_{eq}\omega_{eq}t}\left\{1 + \frac{\beta_{eq}\omega_{eq}}{\omega_{od}}\sinh 2\omega_{od}t + \frac{2\beta_{eq}^2\omega_{eq}^2}{\omega_{od}^2}\sinh^2\omega_{od}t\right\}\right] \tag{5.4}$$

and for the ms velocity

$$\left(\frac{\sigma_{\dot{x}}}{\omega_0 N}\right)^2 = \frac{\pi}{4\beta_{eq}\frac{\omega_{eq}}{\omega_0}}\left[1 - e^{-2\beta_{eq}\omega_{eq}t}\left\{1 - \frac{\beta_{eq}\omega_{eq}}{\omega_{od}}\sinh 2\omega_{od}t + \frac{2\beta_{eq}^2\omega_{eq}^2}{\omega_{od}^2}\sinh^2\omega_{od}t\right\}\right] \tag{5.5}$$

where $\omega_{od} = \omega_{eq}\sqrt{\beta_{eq}^2 - 1}$ and for the low frequency system $\omega_{eq} = \sqrt{\alpha}\,\omega_0$ and $\beta_{eq} = \beta_{2L}$.

5.2. The 3rd order linear systems

When a linear system is subjected to the Gaussian white noise excitation of Eq.(2.2), the response process is a Markov process. Then the system response characteristics are completely determined by its initial probability density function and its transition probability density. The latter function is the solution of the associated Chapman-Kolomogorov-Smoluchowski integral equation or, equivalently, of the associated Fokker-Plank partial differential equation.

For the 3rd order linear system it is easier to find the rms response characteristics by the above method than to use the previously described Rice method, which involves finding the impulse response and the following double integration.

The governing equation of the 3rd order system of Eq.(4.1) is converted into simultaneous partial differential equations of the first order with the change of variables as

$$y_1 = x_1, \; y_2 = x_2, \; y_3 = \dot{x}_1 \tag{5.6}$$

Then Eq.(4.1) is expressed in a matrix-vector form as

$$\frac{d}{dt}\{y_i\} = [a]\{y_i\} + \{f_i(t)\} \qquad (i=1,2,3) \tag{5.7}$$

where
$$\{y_i\} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} \quad [a] = \begin{bmatrix} 0 & 0 & 1 \\ \frac{k_2}{C_2} & -\frac{k_2}{C_2} & 0 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & -\frac{C_1}{m} \end{bmatrix} \qquad \{f_i(t)\} = \begin{Bmatrix} 0 \\ 0 \\ f(t) \end{Bmatrix}$$

When the excitation of Eq.(2.2) is input into the 3rd order system, the corresponding Fokker-Plank equation becomes[7)]

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{3}\frac{\partial}{\partial y_i}(A_i P) - \frac{1}{2}\sum_{i,j}^{3}\frac{\partial^2}{\partial y_i \partial y_j}(B_{ij}P) \tag{5.8}$$

where $P = P(y_1, y_2, y_3;\ t\ \ y_{10}, y_{20}, y_{30};\ t_0)$ is the transition probability density function, representing the state probability density of y_1, y_2, and y_3 at time t, given state y_{10}, y_{20}, and y_{30} at t_0, and

$$\{A_i\} = [a]\{y_i\} \qquad [B_{ij}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\pi S_0 \end{bmatrix}$$

or, explicitly

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial y_1}(y_3 P) - \frac{\partial}{\partial y_2}\left[\frac{k_2}{c_2}(y_1 - y_2)P\right] - \frac{\partial}{\partial y_3}\left[-\left(\frac{K}{m}y_1 - \frac{k_2}{m}y_2 - \frac{c_1}{m}y_3\right)P\right]$$
$$+ \pi S_0 \frac{\partial^2 P}{\partial y_3^2} \tag{5.9}$$

A convenient method to solve this Fokker-Plank equation is to transform it into the form of

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{3}\lambda_i \frac{\partial}{\partial z_i}(z_i P) - \frac{1}{2}\sum_{i,j=1}^{3}\mu_{ij}\frac{\partial^2 P}{\partial z_i \partial z_j} \tag{5.10}$$

For this purpose, the following transformation of co-ordinates is made as

$$\{z_i\} = [q]\{y_i\} \tag{5.11}$$

where the operation matrix $[q]$ is the one which can diagonalize $[a]$ such that

$$[q][a] = [\lambda_i][q] \qquad (5.12)$$

where $[\lambda_i]$ is the diagonal matrix of the eigen values λ_i of $[a]$ which is obtained from

$$\left| [a] - \lambda_i I \right| = 0 \qquad (5.13)$$

For the 3rd order system the λ_i are the complex roots of

$$\lambda_i^3 + \left(\frac{k_2}{C_2} + \frac{C_1}{m}\right)\lambda_i^2 + \frac{k_2 C_1 + K C_2}{m C_2}\lambda_i + \frac{k_1 k_2}{m C_2} = 0 \qquad (5.13)'$$

Using this solution, one can get the inverse of $[q]$, $[q]^{-1}$ from Eq.(5.12) as

$$[q]^{-1} = \begin{bmatrix} q_{11}\begin{Bmatrix} 1 \\ \dfrac{k_2}{K_2 + C_2\lambda_1} \\ \lambda_1 \end{Bmatrix} & q_{12}\begin{Bmatrix} 1 \\ \dfrac{k_2}{K_2 + C_2\lambda_2} \\ \lambda_2 \end{Bmatrix} & q_{13}\begin{Bmatrix} 1 \\ \dfrac{k_2}{K_2 + C_2\lambda_3} \\ \lambda_3 \end{Bmatrix} \end{bmatrix} \qquad (5.14)$$

and furthermore one can set $q_{11} = q_{12} = q_{13} = 1$ without losing generality. Using the solutions of Eq.(5.13)', the $[q]$ matrix is found as

$$
[q] = \begin{bmatrix}
\dfrac{\{k_2^2 + (\lambda_2+\lambda_3)k_2C_2\}(k_2+\lambda_1C_2)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)\,C_2^2k_2} & -\dfrac{(k_2+\lambda_1C_2)(k_2+\lambda_2C_2)(k_2+\lambda_3C_2)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)\,C_2^2k_2} & -\dfrac{k_2C_2(k_2+\lambda_1C_2)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)\,C_2^2k_2} \\[2ex]
\dfrac{\{k_2^2 + (\lambda_3+\lambda_1)k_2C_2\}(k_2+\lambda_2C_2)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)\,C_2^2k_2} & -\dfrac{(k_2+\lambda_1C_2)(k_2+\lambda_2C_2)(k_2+\lambda_3C_2)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)\,C_2^2k_2} & -\dfrac{k_2C_2(k_2+\lambda_2C_2)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)\,C_2^2k_2} \\[2ex]
\dfrac{\{k_2^2 + (\lambda_1+\lambda_2)k_2C_2\}(k_2+\lambda_3C_2)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)\,C_2^2k_2} & -\dfrac{(k_2+\lambda_1C_2)(k_2+\lambda_2C_2)(k_2+\lambda_3C_2)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)\,C_2^2k_2} & -\dfrac{k_2C_2(k_2+\lambda_3C_2)}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)\,C_2^2k_2}
\end{bmatrix} \tag{5.15}
$$

The statistical characteristics of the transformed response $\{z_i\}$ have already obtained been obtained by Wang and Uhlenbeck[7] for the initial conditions of $\{z_{io}\}$. The mean value is

$$\{E[z_i]\} = \{z_{io}\} \lceil e^{\lambda_i t} \rfloor \tag{5.16}$$

The variances and the covariances are given by

$$E\{(z_i - E(z_i))(z_j - E(z_j))\} = -\frac{\mu_{ij}}{\lambda_i + \lambda_j}\left[1 - e^{(\lambda_i + \lambda_j)t}\right] \tag{5.17}$$

where

$$[\mu_{ij}] = [q][B][q]^T \tag{5.18}$$

and the superscript T denotes the transpose of a matrix.
The initial condition in the original co-ordinates, i.e.

$$P(y_1, y_2, y_3; 0 \mid y_{10}, y_{20}, y_{30}; 0) = \delta(y_1 - y_{10})\delta(y_2 - y_{20})\delta(y_3 - y_{30}) \tag{5.19}$$

makes $\{z_{io}\} = \{0\}$. Thus the above response characteristics become

$$[\sigma^2_{z_i z_j}] = E[z_i z_j] = -\frac{\mu_{ij}}{\lambda_i + \lambda_j}\left[1 - e^{(\lambda_i + \lambda_j)t}\right] \tag{5.20}$$

The back transformation of $E[z_i z_j]$ into the original co-ordinates can be carried out by using the matrices $[q]^{-1}$ and $([q]^{-1})^T$.

$$[\sigma^2_{y_i y_j}] = \begin{bmatrix} \sigma^2_{y_1} & \sigma^2_{y_1 y_2} & \sigma^2_{y_1 y_3} \\ & \sigma^2_{y_2} & \sigma^2_{y_2 y_3} \\ \text{symmetric} & & \sigma^2_{y_3} \end{bmatrix} = [q]^{-1}[\sigma^2_{z_i z_j}]([q]^{-1})^T \tag{5.21}$$

This gives the mean-square response characteristics of the 3rd order system.

5.3. Analysis

The transient mean-square displacement and velocity response of the previously obtained equivalent linear systems are first investigated in the case of no viscous damping in the bilinear hysteretic system, i.e., $\beta_0 = 0$. As the basis of comparison, the experimental results by Shah,[20] which were obtained from ensemble averages, after integrating the bilinear hysteretic system step by step with a digitally generated white noise excitation, are used. Figs.16 and 17 show some representative cases for both moderately nonlinear and nearly elasto-plastic hysteretic systems, where the transient response of the equivalent 2nd order linear system of Eq.(3.1) is presented by the dotted line, that of the two-mode model of the 2nd order system of Eq.(3.38) by the thin solid line, that of the low frequency 3rd order linear system by the one-dotted chain line, that of the high frequency 3rd order system by the two-dotted line, that of the two-mode model of the 3rd order system by the bold solid line, and that of the above experimental results by the dashed line. The selection of an equivalent linear system is made with the help of the response power spectral density matching in sections 3.5 and 4.4, assuming the existence of some correlation between this and transient response matching.

First note that the transient response duration of the bilinear hysteretic system differs strongly depending on the value of Y/N. Figs.16 and 17 show that this duration is minimum when the mean-square velocity response in stationary state is minimum. For the moderately nonlinear hysteretic system this occurs between Y/N = 2 and 3 and for the nearly elasto-plastic hysteretic system between Y/N = 1 and 2, both giving the transient duration of about 3 to 4 times the undamped natural period of small amplitude response. On the other hand, the strong-motion portion of earthquakes usually have an excitation duration of at least 5 to 10 times the natural period of the concerned structure. Hence, for the bilinear hysteretic system in the situation of Y/N giving the minimum mean-square velocity response the stationary state response is more significant than the transient one from the structural design point of view. However, for the bilinear hysteretic system in other situations the transient response becomes significant due to the long duration time to reach the stationary state motion. Note, though, that this duration is shortened by the presence of viscous damping in the bilinear hysteretic system.

Secondly, compare the transient response of the bilinear hysteretic system with that of the equivalent linear systems. In the moderately nonlinear hysteretic case the equivalent 2nd order linear system and the two-mode model of the 3rd order linear system give almost the same transient response and both can approximate well the experimental mean-square displacement and velocity response, except near the linear system response region, say Y/N = 15. The failure in this last situation is apparently due to the poor prediction of the stationary state response.

On the other hand, in the nearly elasto-plastic hysteretic case, the transient response differs appreciably among the investigated equivalent linear systems, depending on the value of Y/N. When Y/N = 1.0, the equivalent 2nd order linear system is the best equivalent linearization of the original system. At this situation, σ_x/Y is nearly 10 so that the response power spectral density approximation is also best by this system, referring to section 3.5. When Y/N = 5, the two-mode model of the 3rd order system is the best linearization. At this situation, σ_x/Y is nearly 1.0 so that this system is also the best to approximate the response power spectral density of the original system. Note, however, that in this situation, near the linear system response, all the linearizations considered give almost the same transient response and can approximate reasonably the experimental results.

From the above observation it is concluded that if the response power spectral density is matched between the bilinear hysteretic system and its linearization in the stationary state motion, then the transient response of the corresponding linear system can well approximate that of the original system. As a whole the two-mode model of the 3rd order linear system is the best linearization among those considered herein.

Hence, the transient response comparison between the viscously damped bilinear hysteretic system and its linearization by the two-mode 3rd order system is made in Figs.18 and 19. In this case, the system parameters of the linear model as determined in section 4.3 are used. In these figures the stationary state response from section 4.1 and from the analog computer are also presented. Note that the two-mode 3rd order linear system succeeds quite well in matching both the mean-square displacement and velocity response in viscously damped bilinear hysteretic cases, except in the particular case of Y/N = 15 for the moderately nonlinear hysteretic system.

VI. SUMMARY AND CONCLUSIONS

Two analytical linearization techniques for hysteretic systems in stationary random motion which have been separately proposed in the past are combined here. One of the techniques is Krylov-Bogoliubov method with certain small nonlinearity assumptions as proposed by Caughey,[15] and the other is the power balance method proposed by Karnopp.[17] The former method is derived from the mean square minimization of the difference between the equations of motion for the bilinear hysteretic system and the 2nd order linear system with the same mass. This minimization gives two equivalence conditions concerning the equivalent frequency and the equivalent damping factor. The latter method is based on establishing an energy loss equivalence per unit of time between the two systems. However, it was demonstrated in section 3.1 that this equivalence gives a requirement which is closely related to one of those in the Krylov-Bogoliubov method. This requirement is referred to as the energy dissipation matching criterion. In this study this requirement is used in conjunction with the other requirement from the Krylov-Bogoliubov method, which is called the average frequency matching criterion.

When predicting the root-mean-square (rms) displacement from the criterion of matching energy dissipation per unit of time, it was found that much attention must be paid to the average duration of hysteresis energy dissipation cycle. This cycle was found to have different duration than the cycle of the mass response (or the average frequency of the mass response). Reference (10) reports that for moderate values of response level over yield level the bilinear hysteretic system response exhibits a typical wandering motion of the central axis in its time history, and that the response is like small oscillations at the small amplitude natural frequency superposed on this low frequency wandering. It appears that the low frequency component primarly represents motion across the Coulomb friction slider in the model. The average frequency across this slider is strongly related to the slope of restoring force curve beyond yielding, denoted by αK, which justifies taking the reduced frequency after system yielding as the average frequency of the hysteresis energy dissipation cycle. The rms displacement comparison between the above prediction and the experimental results[10)] confirmed this hysteresis energy dissipation cycle. For the prediction of the rms velocity response, the average frequency matching criterion was used, together with the above prediction of the rms displacement. As a result, significantly improved predictions both for the rms displacement and velocity response were attained, without specifying the parameters of the linearized system.

When one is concerned with a bilinear hysteretic system response analysis due to earthquake excitation, response matchings in non-stationary motion are desired for the equivalent linear system. If the earthquake motion is simply approximated by a stationary motion of a limited duration, as is frequently assumed, then the more general nonstationary problem is reduced to the transient problem of the system response building up to its stationary state. Various equivalent linear systems are conceivable, all of which can satisfy the energy dissipation matching per unit time and the average frequency matching criteria mentioned above. In this investigation, the usual 2nd order linear system and a model with two uncorrelated 2nd order modes are first considered. These linear systems, although they approach to the same stationary response levels, show different transient response behavior. It was shown that there exists a correlation between matching this transient response behavior and matching the stationary response power spectral density. For the comparison the experimental results from Iwan and Lutes[10)] were used for power spectral density and Shah's experimental results[20)] were used for the transient response.

As a further step of equivalent linearization, a 3rd order linear system and its uncorrelated two-mode system are also considered. The 3rd order linear system considered herein has the same mass, springs and dashpot as the hysteretic model, but it also has an additional viscous damper replacing the Coulomb slider and representing the hysteresis energy dissipation due to system yielding. This system was shown to have an average frequency of $\sqrt{\alpha}\,\omega_0$ across the additional viscous damper when no viscous damping effect existed in the original system, which is consistent with the hysteresis energy dissipation cycle used in the above energy dissipation matching criterion. Since the energy dissipation per unit of time is the same for any conceivable linear system in stationary motion as long as they have identical mass and are subjected to identical Gaussian white noise excitation, the energy dissipation matching criterion gives the same rms displacement for the 3rd order system as for the 2nd order system. However, the 3rd order system considered in this investigation, has a rather fixed average frequency either near the natural frequency of the initial stiffness --- high frequency 3rd order system, or fairly near the reduced frequency after the system yielding --- low frequency 3rd order system. Thus this system can not necessarily result in velocity matching with the original system. Of course, one can get a good prediction for the rms velocity response by using the average frequency matching criterion. In this case, however, one must take the two-mode system as the equivalent linear system.

For the response power spectral density matching and the transient response matching investigations of the 3rd order systems, the system parameters must be explicitly determined. In the viscously damped hysteretic case, it is very difficult to evaluate these parameters for the two-mode 3rd order system in order to satisfy the energy dissipation and average frequency matching criteria. Fortunately, an alternative determination adopted in section 4.3 resulted in better prediction for the rms displacement than that from the energy dissipation matching criterion in the most concerned nonlinearity situation for the nearly elasto-plastic hysteretic system.

The transient response matching was also investigated for the two-mode 3rd order system and was compared with the results of the 2nd order systems. It was observed that there was a greater variation of velocity response than displacement response among the different linear models, particularly near the nonlinearity situation where the minimum value of stationary velocity was attained. As a whole, the two-mode 3rd order system proved to be the best linearization, and the transient response matching by this system was improved for the nearly elasto-plastic case as the viscous damping effect was increased in the bilinear hysteretic system up to 5 % of critical damping.

In structural design practice, the transient time before reaching stationary response is one of the significant factors. This time depends strongly on the ratio of yield level to excitation level in the bilinear hysteretic system. For the yield level which gave the smallest stationary velocity response for a particular excitation level, this duration of transient response was about 3 to 4 times the initial natural period. In other situations the transient response continued for a much longer time.

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Appendix A

A prediction of the average frequency

When the bilinear hysteretic system is subjected to random excitations, its average frequency defined by the ratio of the rms velocity response to the rms displacement response is expected to be a function of σ_x/Y. This theoretical prediction is made here based on the resonance frequency in the harmonic mtion.

This frequency is deduced by Caughey,[21] assuming the slowly varying response amplitude and phase, as

$$\left[\frac{\omega_R(A)}{\omega_0}\right]^2 = \begin{cases} 1 & , \text{ for } A \le Y \\ \alpha + \frac{1-\alpha}{\pi}\left[\cos^{-1}\left(1-\frac{2Y}{A}\right) - 2\left(1-\frac{2Y}{A}\right)\sqrt{\frac{Y}{A}\left(1-\frac{Y}{A}\right)}\ \right] & \text{ for } A > Y \end{cases} \tag{A.1}$$

where A is the amplitude of the bilinear hysteretic system in resonant motion.

For random excitations, if the system is assumed to behave with the above resonance frequency corresponding to its amplitude at any cycle, then the mean square expectation of the resonance frequency is found from

$$E\left[\left(\frac{\omega_R(A)}{\omega_0}\right)^2\right] = \int_0^\infty \left(\frac{\omega_R(A)}{\omega_0}\right)^2 p(A)\,dA \tag{A.2}$$

The assumption of the Rayleigh distribution of Eq.(3.8) for p(A) yields

$$E\left[\left(\frac{\omega_R(A)}{\omega_0}\right)^2\right] = \alpha + \frac{1-\alpha}{\pi}\int_{\nu}^{\infty}\left[\cos^{1}\left(1-\frac{2\nu}{\xi}\right) - 2\left(1-\frac{2\nu}{\xi}\right)\left\{\frac{\nu}{2}\left(1-\frac{\nu}{\xi}\right)\right\}^{\frac{1}{2}}\xi\, e^{-\frac{\xi^2}{2}}\right]d\xi$$
$$+ (1-\alpha)\left(1-e^{-\frac{\nu^2}{2}}\right) \qquad \text{(A.3)}$$

where $\nu = \frac{Y}{\sigma_x}$

This expression is plotted as the dotted line in Fig.3 for comparison with the average frequency obtained experimentally.

Appendix B

Response characteristics of the 3rd order linear system

The governing equation of motion of Eq.(4.1) is rewritten in a vector-matrix form as

$$\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} mf(t) \\ 0 \end{Bmatrix} \tag{B.1}$$

This steady state harmonic solution can be found by substituting

$$\left.\begin{aligned} x_1 &= X_1 e^{i\omega t} \\ x_2 &= X_2 e^{i\omega t} \\ f(t) &= F_0\, e^{i\omega t} \end{aligned}\right\} \tag{B.2}$$

then

$$X_1 = \frac{\frac{k_2}{m} + i\frac{c_2}{m}\omega}{\{(-\omega^2 + \frac{k_1}{m} + \frac{k_2}{m}) + i\frac{c_1}{m}\omega\}(\frac{k_2}{m} + i\frac{c_2}{m}\omega) - (\frac{k_2}{m})^2} F_0 \tag{B.3}$$

$$X_2 \quad \frac{\frac{k_2}{m}}{\{(-\omega^2 + \frac{k_1}{m} + \frac{k_2}{m}) + i\frac{c_1}{m}\omega\}(\frac{k_2}{m} + i\frac{c_2}{m}\omega) - (\frac{k_2}{m})^2} F_0 \tag{B.4}$$

If the following notations are used

$$\alpha = \frac{k_1}{k_1+k_2} \qquad \omega_0 = \sqrt{\frac{k_1+k_2}{m}} \qquad \beta_{31} = \frac{c_1}{2\omega_0 m} \qquad \beta_{32} = \frac{1-\alpha}{2\omega_0} \cdot \frac{k_2}{c_2} \tag{B.5}$$

the frequency response functions are obtained for the mass as

$$H_{3,1}(\omega) = \frac{X_1}{F_0} = \frac{i\frac{1-\alpha}{2\beta_{32}}\omega + \omega_0}{-i\frac{1\ \alpha}{2\beta_{32}}\omega^3 - \{\frac{\beta_{31}}{\beta_{32}}(1-\alpha) + 1\}\omega_0\omega^2 + i(2\beta_{31} + \frac{1\ \alpha}{2\beta_{32}})\omega_0^2\omega + \alpha\omega_0^3} \tag{B.6}$$

and for the point 2 indicated in Fig.10 as

$$H_{3,2}(\omega) = \frac{X_2}{F_0}$$

$$= \frac{\omega_0}{-i\frac{1\ \alpha}{2\beta_{32}}\omega^3 - \left\{\frac{\beta_{31}}{\beta_{32}}(1-\alpha)-1\right\}\omega_0\omega^2 + i\left(2\beta_{31}+\frac{1-\alpha}{2\beta_{32}}\right)\omega_0^2\omega + \alpha\omega_0^3} \tag{B.7}$$

The rms response characteristics are calculated [4), 5)]

for the mass

$$\sigma^2_{\frac{dx_1}{dt^l}} = \int_{-\infty}^{\infty} \left|\omega^l H_{3,1}(\omega)\right|^2 S(\omega)\, d\omega \tag{B.8}$$

for the point 2

$$\sigma^2_{\frac{dx_2}{dt^l}} = \int_{-\infty}^{\infty} \left|\omega^l H_{3,2}(\omega)\right|^2 S(\omega)\, d\omega \tag{B.9}$$

When the excitation is a stationary noise with intensity level S_0, these integrals can be readly evaluated in an analytical way.[23)] The results are

$$\sigma^2_{x_1} = \frac{\pi S_0}{2\beta_{32}\omega_0^3} \cdot \frac{1+\frac{4\beta_{32}^2}{\alpha(1-\alpha)^2}+\frac{4\beta_{31}\beta_{32}}{\alpha(1-\alpha)}}{\left\{1+\frac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left\{1+\frac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\}+\alpha\frac{\beta_{31}}{\beta_{32}}} \tag{B.10}$$

$$\sigma^2_{\dot{x}_1} = \frac{\pi S_0}{2\beta_{32}\omega_0} \cdot \frac{1+\frac{4\beta_{32}^2}{(1-\alpha)^2}+\frac{4\beta_{31}\beta_{32}}{1-\alpha}}{\left\{1+\frac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left\{1+\frac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\}+\alpha\frac{\beta_{31}}{\beta_{32}}} \tag{B.11}$$

Appendix C

Stationary state response (Alternative method 1)

Since the real part of the λ_i of Eq.(5.13) is negative when $t \to \infty$, Eq.(5.20) becomes

$$[\sigma^2_{z_i z_j}]_{st.} = -\frac{\mu_{ij}}{\lambda_i + \lambda_j} \qquad \text{(C.1)}$$

or, in the matrix form

$$[\lambda_i][\sigma^2_{z_i z_j}]_{st.} + [\sigma^2_{z_i z_j}]_{st.}[\lambda_j] = -[\mu_{ij}] \qquad \text{(C.2)}$$

where $[\lambda_i]$ is the diagonal matrix and $[\sigma^2_{z_i z_j}]_{st}$ denotes the covariance matrix in the stationary state response. Substitution of Eq.(5.12) and Eq.(5.18) into Eq.(C.2) and using Eq.(5.21) yields the requirement relationship for $[\sigma^2_{y_i y_j}]_{st}$.

$$[a][\sigma^2_{y_i y_j}]_{st.} + [\sigma^2_{y_i y_j}]_{st.}[a]^T = -[B] \qquad \text{(C.3)}$$

After solving this equation componentwise, the covariance matrix for y_i is found as

$$[\sigma^2_{y_i y_j}] = \begin{bmatrix} \sigma^2_{y_1} & \sigma^2_{y_1 y_2} & \sigma^2_{y_1 y_3} \\ & \sigma^2_{y_2} & \sigma^2_{y_2 y_3} \\ \text{symmetric} & & \sigma^2_{y_3} \end{bmatrix}$$

$$
= \frac{\pi S_0}{\dfrac{c_2}{k_2}\dfrac{k_1}{m}\left\{ \dfrac{k_2}{m}\left(1+\dfrac{c_1}{c_2}\right)\left(\dfrac{k_2}{c_2}+\dfrac{c_1}{m}\right)+\dfrac{k_1 c_1}{m^2}\right\}} \times
$$

$$
\begin{bmatrix}
\dfrac{k_2}{c_2}+\dfrac{c_1}{m}+\dfrac{k_1 c_2}{m k_2} & \dfrac{k_2}{c_2}+\dfrac{c_1}{m} & 0 \\[2ex]
 & \dfrac{k_2}{c_2}+\dfrac{c_1}{m} & -\dfrac{k_1}{m} \\[2ex]
\text{symmetric} & & \left(\dfrac{k_2}{c_2}+\dfrac{c_1}{m}+\dfrac{c_2}{m}+\dfrac{k_1 c_2}{m k_2}\right)\dfrac{k_1}{m}
\end{bmatrix}
\tag{C.4}
$$

Using the notations of Eq.(B.5), the above is

$$
\underline{\delta} = \frac{\pi S_0}{2\beta_{32}\omega_0^3} \Big/ \left\{ \left[1+\frac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right]\left\{1+\frac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\}+\alpha\frac{\beta_{31}}{\beta_{32}} \right\}
$$

$$
\times \begin{bmatrix}
1+\dfrac{4\beta_{32}^2}{\alpha(1-\alpha)^2}+\dfrac{4\beta_{31}\beta_{32}}{\alpha(1-\alpha)} & \dfrac{2\beta_{32}}{\alpha(1-\alpha)}\left\{\dfrac{\beta_{32}}{1-\alpha}+\beta_{31}\right\} & 0 \\[2ex]
 & \dfrac{2\beta_{32}}{\alpha(1-\alpha)}\left\{\dfrac{\beta_{32}}{1-\alpha}+\beta_{31}\right\} & -\dfrac{\beta_{32}}{1-\alpha}\omega_0 \\[2ex]
\text{symmetric} & & 1+\dfrac{4\beta_{32}}{(1-\alpha)^2}+\dfrac{4\beta_{31}\beta_{32}}{1-\alpha}
\end{bmatrix}
\tag{C.4$'$}
$$

As for the mean-square velocity at point 2 in the 3rd order system $\sigma_{\dot{x}_2}^2$, one can deduce it from

$$\sigma_{\dot{x}_2}^2 = \left(\frac{k_2}{c_2}\right)^2 E[(x_1 - x_2)^2]$$

$$= \left(\frac{k_2}{c_2}\right)^2 (\sigma_{x_1}^2 - 2\sigma_{x_1}\sigma_{x_2} + \sigma_{x_2}^2) \qquad (C.5)$$

Substituting $\sigma_{y_1}^2$, $\sigma_{y_1 y_2}^2$, and $\sigma_{y_2}^2$ into this from Eq.(C.4) gives

$$\sigma_{\dot{x}_2}^2 = \frac{\frac{k_1}{m}\frac{k_2}{c_2}\pi S_0}{\frac{c_2 k_1}{k_2 m}\left\{\frac{k_2}{m}\left(1 + \frac{c_1}{c_2}\right)\left(\frac{k_2}{c_2} + \frac{c_1}{m}\right) + \frac{k_1 c_1}{m^2}\right\}} \qquad (C.6)$$

or

$$\sigma_{\dot{x}_2}^2 = \frac{\pi S_0}{2\beta_{32}\omega_0} \cdot \frac{\frac{4\beta_{32}^2}{(1-\alpha)^2}}{\left\{1 + \frac{4\beta_{31}\beta_{32}}{(1-\alpha)^2}\right\}\left\{1 + \frac{\beta_{31}}{\beta_{32}}(1-\alpha)\right\} + \alpha\frac{\beta_{31}}{\beta_{32}}} \qquad (C.6)'$$

Stationary response (Alternative method 2)

From the linearity property of the adopted 3rd order system, one can take the 3-dimensional Gaussian distribution for the transition probability P in the Fokker-Plank equation in a stationary state. That is

$$P(y_1, y_2, y_3; \infty | y_{1o}, y_{2o}, y_{3o}; t_o)$$

$$= P(y_1, y_2, y_3) = \frac{1}{(2\pi)^{3/2} \, |\underline{S}|^{1/2}} \exp\left[-\frac{1}{2|\underline{S}|} \sum_{i}^{3} \sum_{j}^{3} |\underline{S}|_{ij} \, y_i y_j \right] \quad \text{(C.7)}$$

where $|\underline{S}|$ is the determinant of the covariance matrix for y_i, i.e.

$$\underline{S} = \begin{bmatrix} \sigma^2_{y_1} & \sigma^2_{y_1 y_2} & \sigma^2_{y_1 y_3} \\ & \sigma^2_{y_2} & \sigma^2_{y_2 y_3} \\ \text{symmetric} & & \sigma^2_{y_3} \end{bmatrix}$$

and $|\underline{S}|_{ij}$ is the adjoint matrix of the (i,j) element of $\underline{S}$.
After substituting Eq.(C.6) into Eq.(5.9), one can get the following six simultaneous algebraic equations in the above six unknown mean-square responses but one of which, $\sigma^2_{y_1 y_3}$ should be identically zero due to the stationarity of the process.

$$\begin{cases} -\dfrac{k_2}{C_2}|\underline{S}|_{12} + \dfrac{k_1+k_2}{m}|\underline{S}|_{31} - \dfrac{\pi S_0}{2}\dfrac{|\underline{S}|^2_{31}}{|\underline{S}|} = 0 \\[2ex] \dfrac{k_2}{C_2}|\underline{S}|_{22} - \dfrac{K_2}{m}|\underline{S}|_{23} - \dfrac{\pi S_0}{2}\dfrac{|\underline{S}|^2_{23}}{|\underline{S}|} = 0 \\[2ex] |\underline{S}|_{31} + \dfrac{c_1}{m}|\underline{S}|_{33} - \dfrac{\pi S_0}{2}\cdot\dfrac{|\underline{S}|^2_{33}}{|\underline{S}|} = 0 \\[2ex] -|\underline{S}|_{12} + \dfrac{k_2}{C_2}|\underline{S}|_{23} - \dfrac{K_2}{m}|\underline{S}|_{33} + \dfrac{C_1}{m}|\underline{S}|_{23} - \pi S_0 \dfrac{|\underline{S}|_{33}|\underline{S}|_{23}}{|\underline{S}|} = 0 \end{cases} \quad \text{(C.8)}$$

$$\begin{cases} -|\underline{S}|_{11} - \frac{k_2}{c_2}|\underline{S}|_{23} + \frac{k_1 + k_2}{m}|\underline{S}|_{33} + \frac{c_1}{m}|\underline{S}|_{31} - \pi S_0 \frac{|\underline{S}|_{31}|\underline{S}|_{33}}{|\underline{S}|} = 0 \\ \frac{\pi S_0}{2}|\underline{S}|_{33} - \left(\frac{k_2}{c_2} + \frac{c_1}{m}\right)|\underline{S}| = 0 \end{cases}$$

These equations give the same solutions as Eq.(C.4).

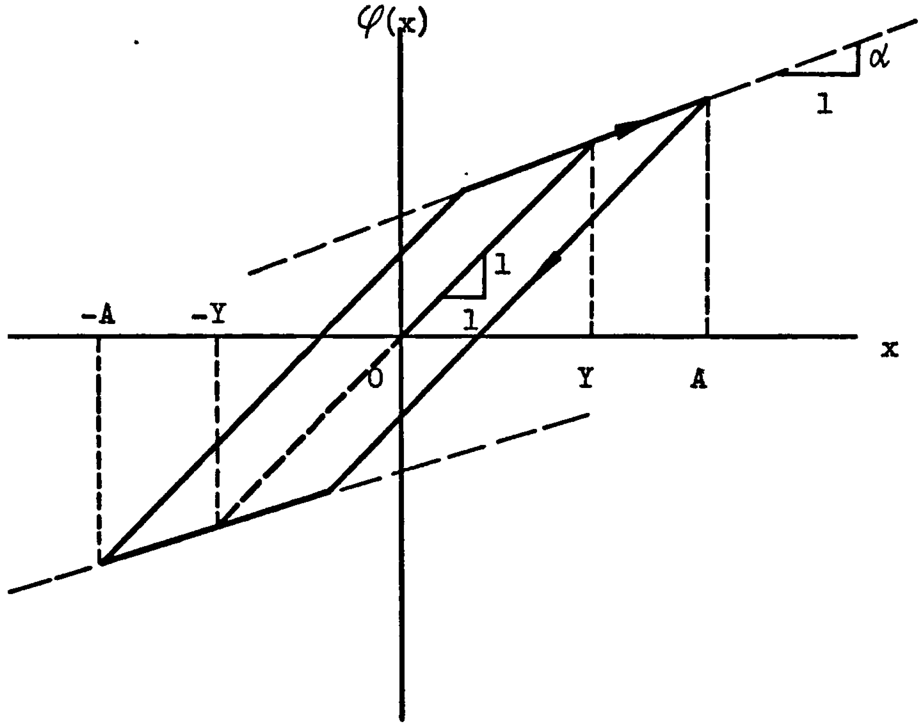
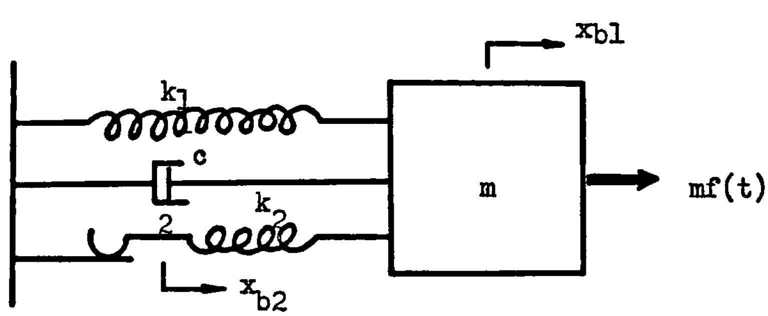


Fig.1 Bilinear Hysteretic Restoring Force



Coulomb Friction Slider with
Maximum Force = k_2Y

Fig.2 Bilinear Hysteretic Mechanical Oscillator

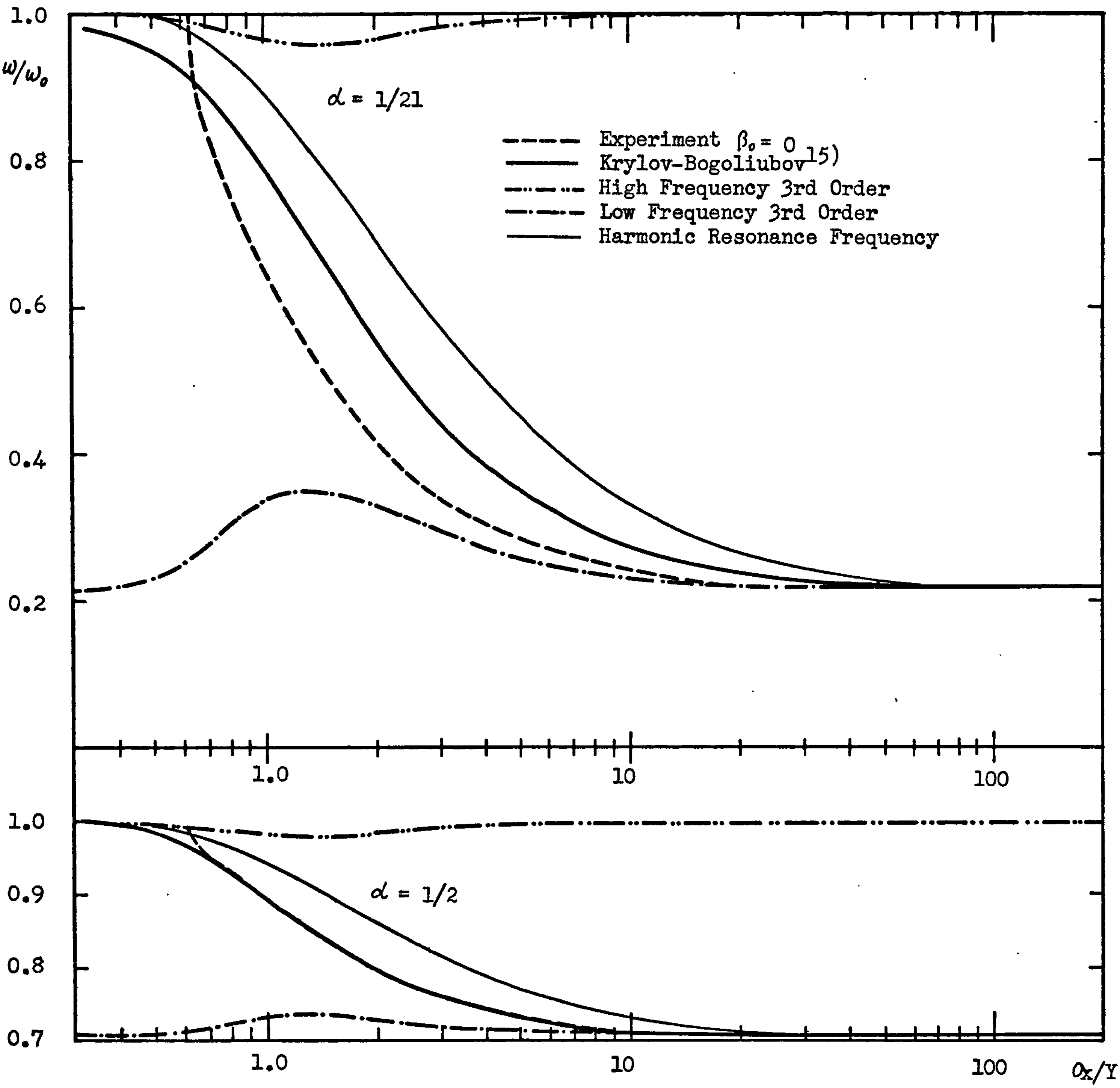


Fig.3 Average frequency

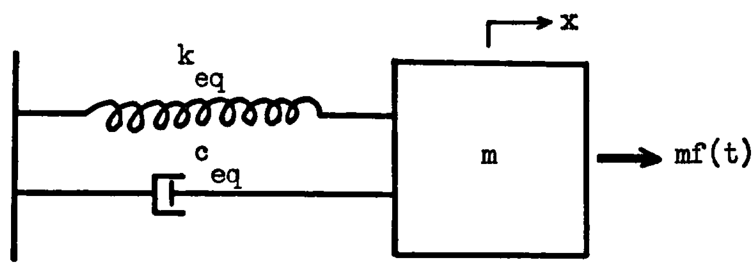


Fig.4 Equivalent 2nd order linear system

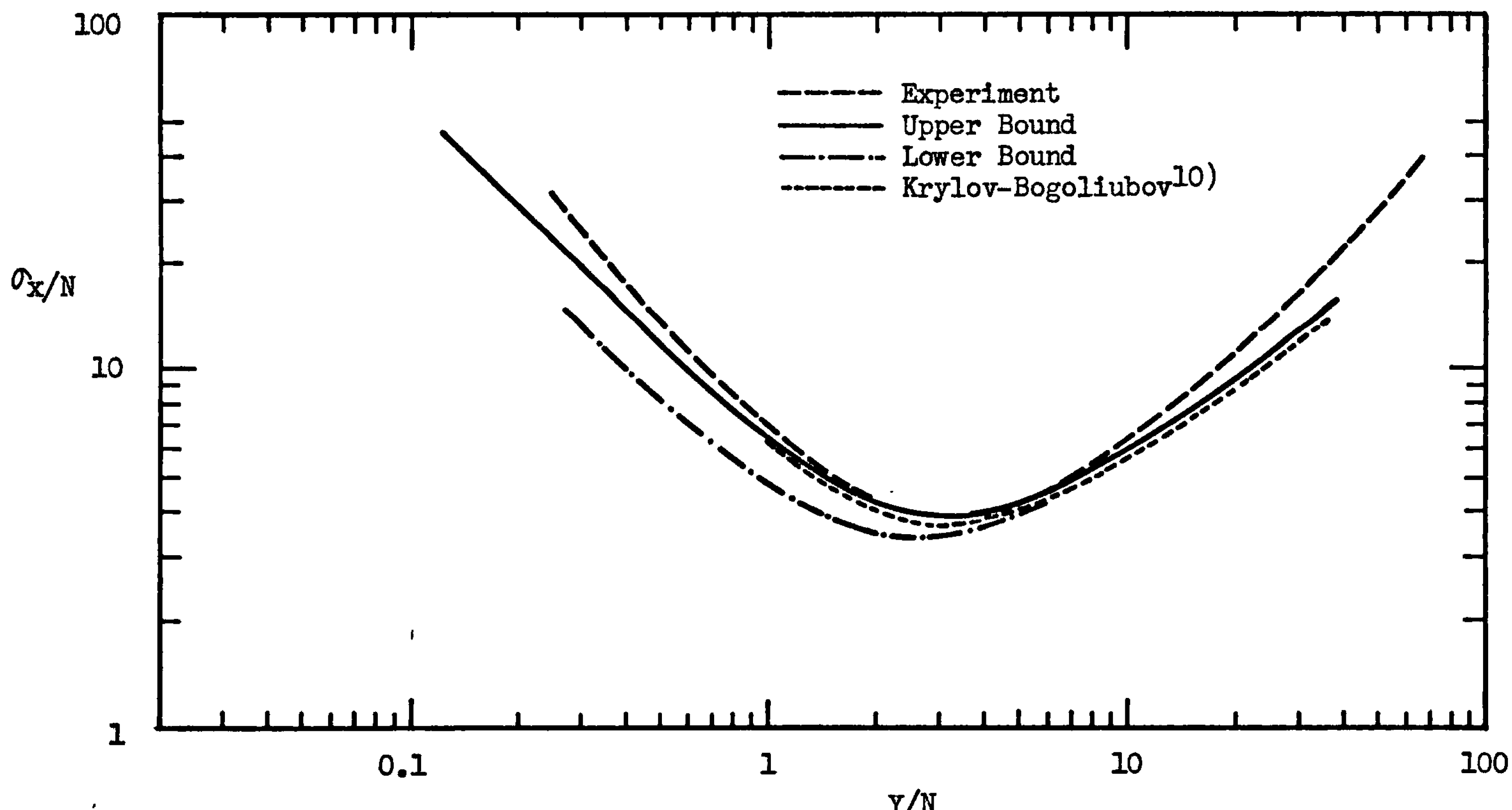


Fig.5.1 rms Displacement

$\alpha = 1/2, \quad \beta_o = 0$

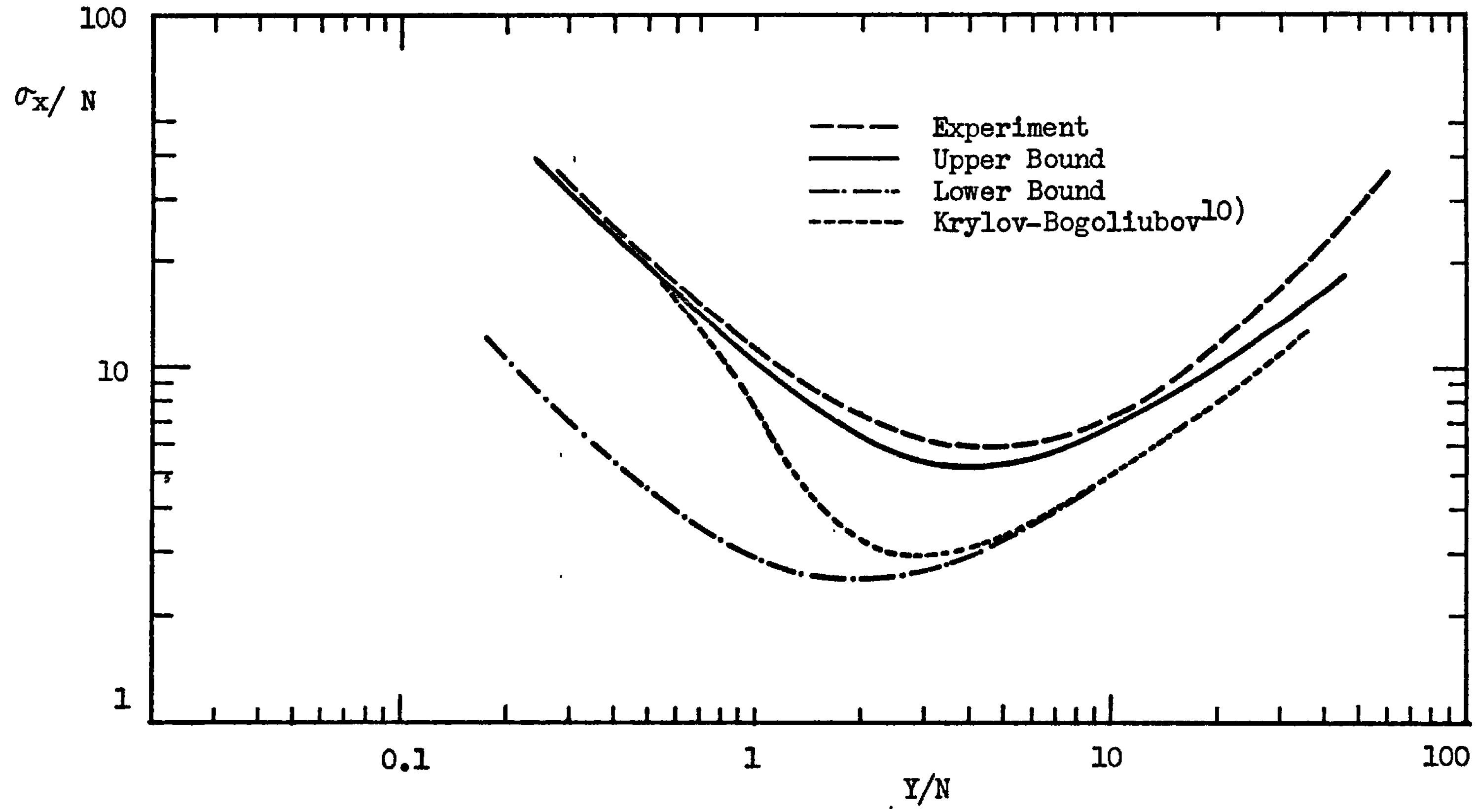


Fig.5.2 rms Displacement

$\alpha = 1/21, \quad \beta_0 = 0$

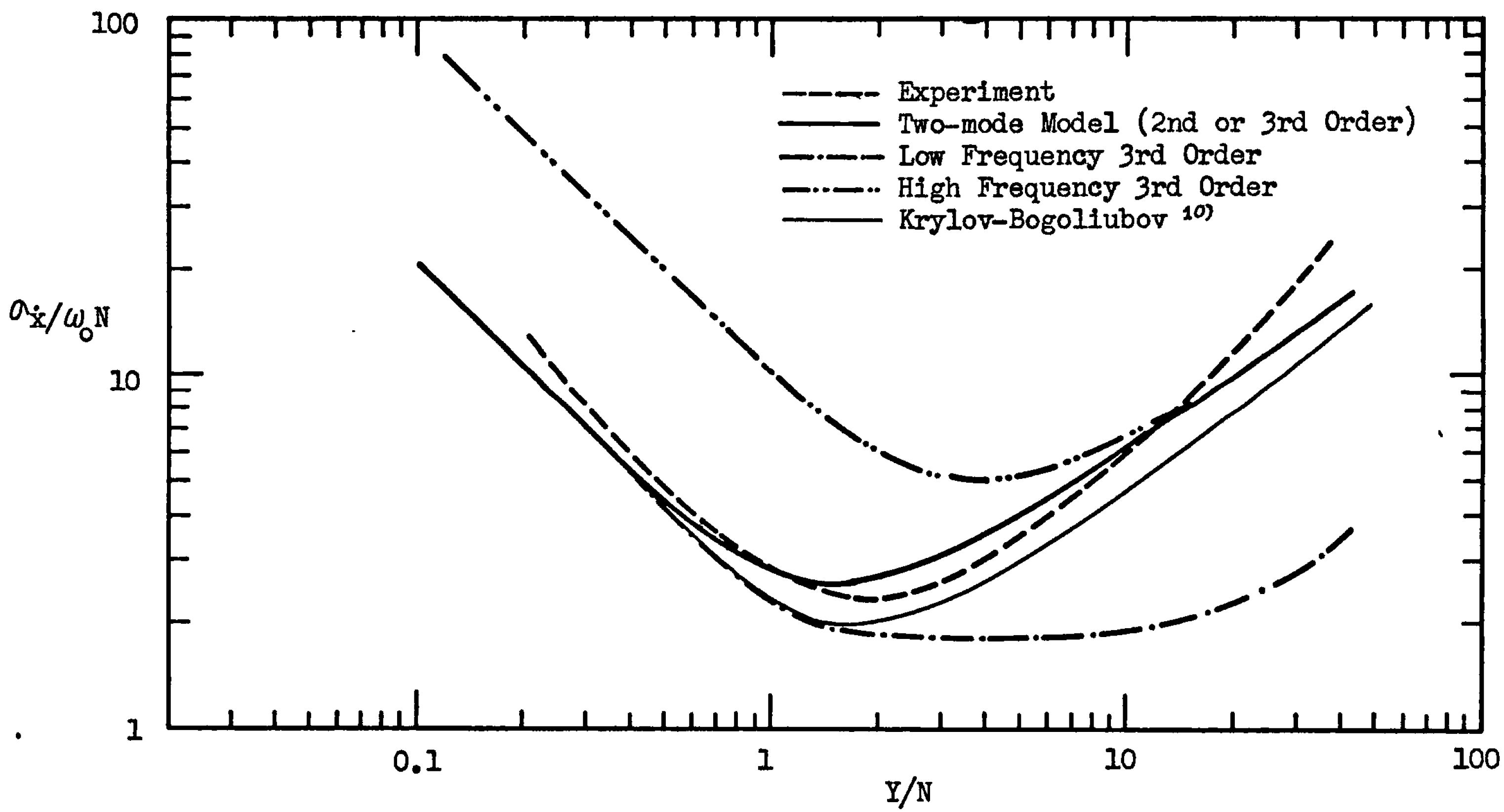


Fig.6 rms Velocity Response

$\alpha = 1/21, \quad \beta_o = 0$

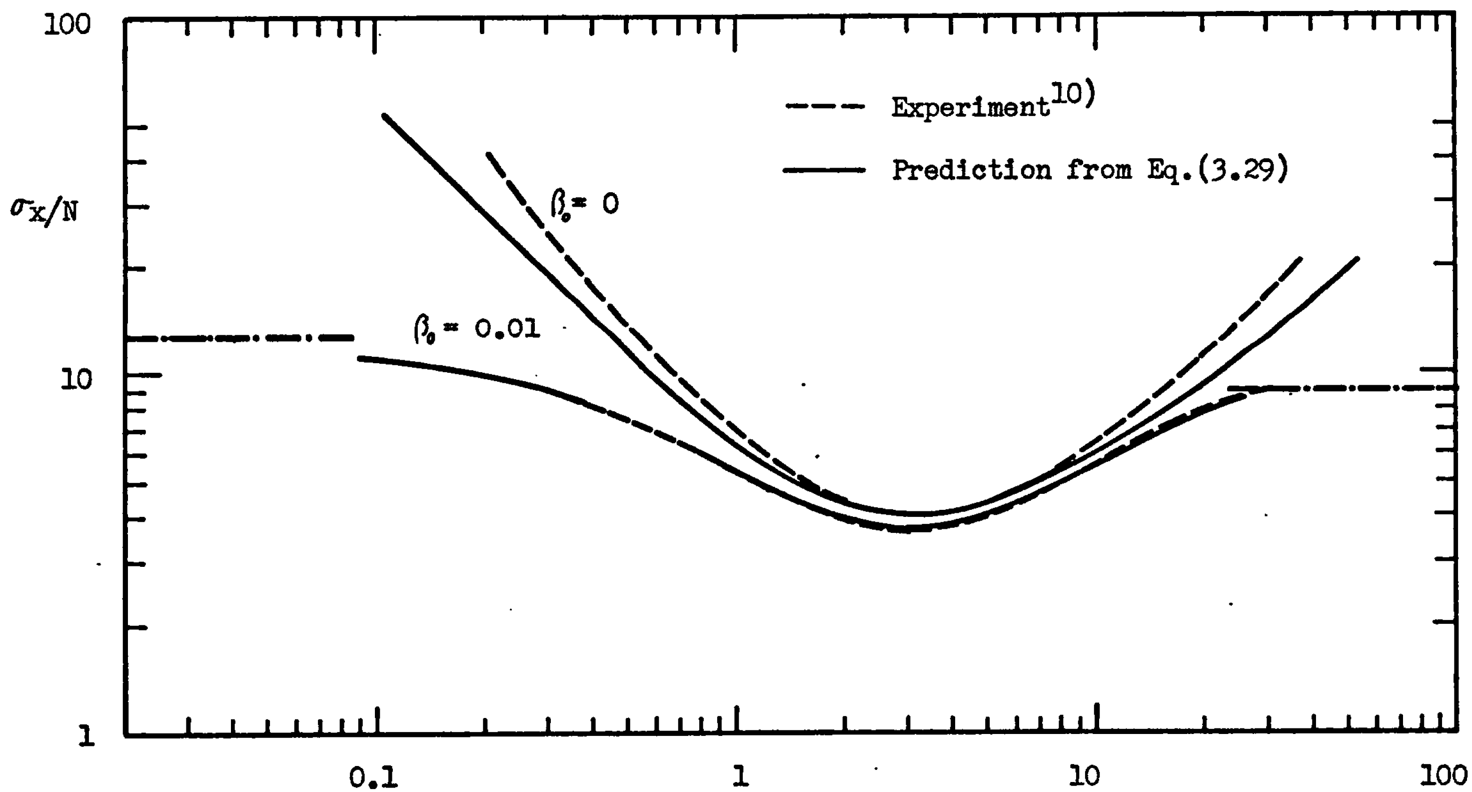


Fig.7.1 rms Displacement

$\alpha = 1/2, \quad \beta_0 = 0,\ 1\ \%$

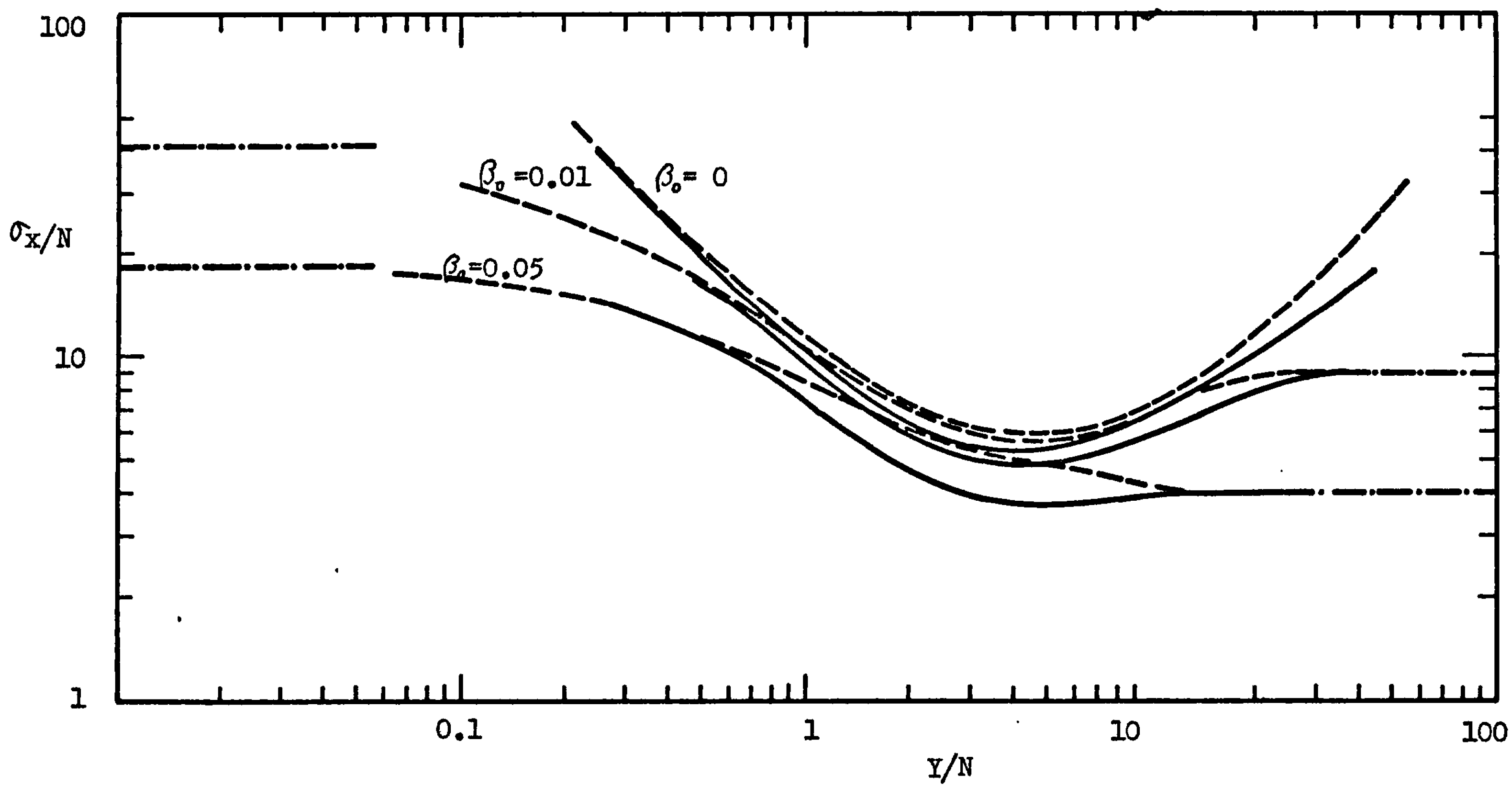


Fig.7.2 rms Displacement

$\alpha = 1/21, \quad \beta_0 = 0,\ 1,\ 5\ \%$

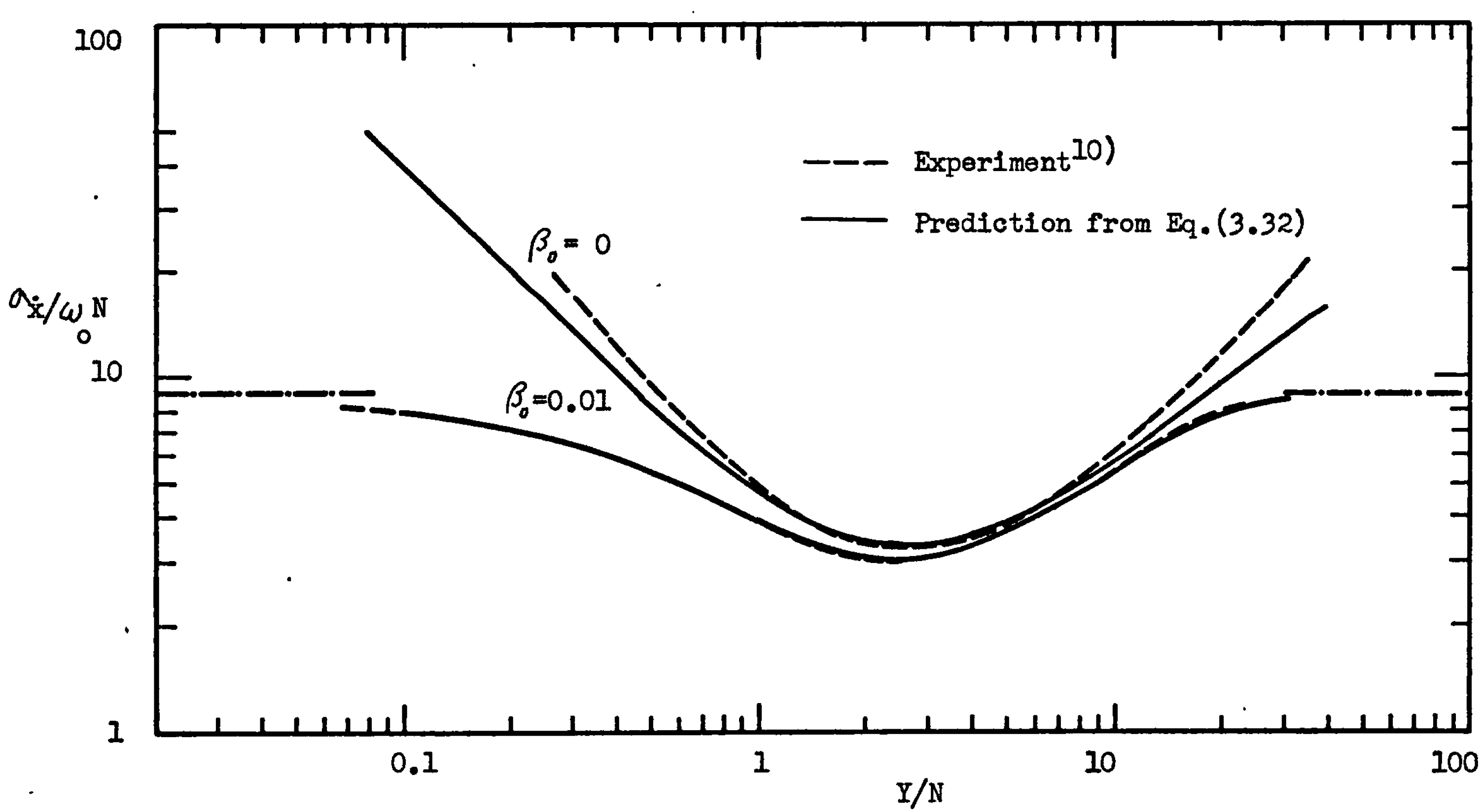


Fig.8.1 rms Velocity Response

$\alpha = 1/2, \quad \beta_o = 0,\ 1\ \%$

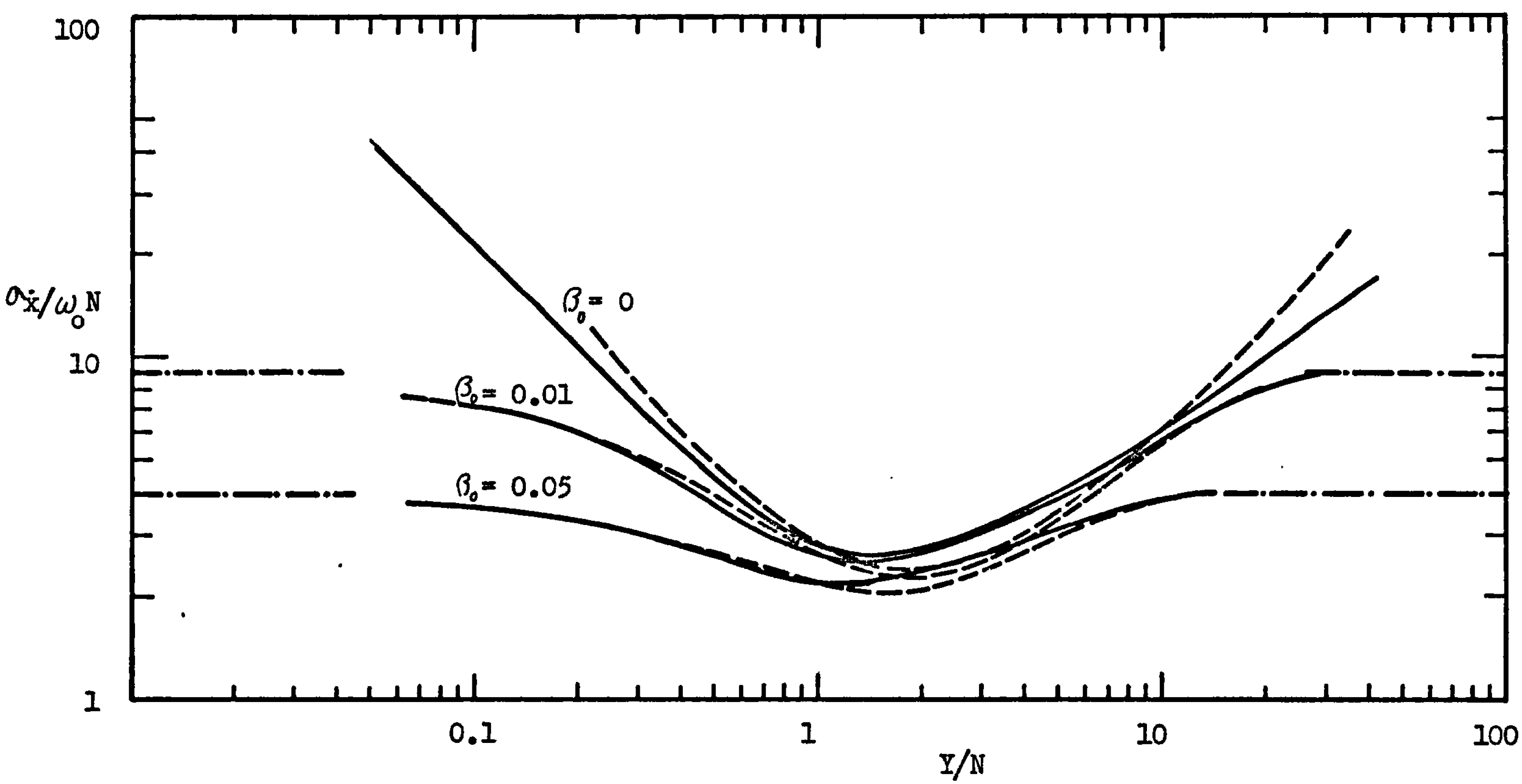


Fig.8.2 rms Velocity Response

$\alpha = 1/21, \quad \beta_o = 0,\ 1,\ 5\ \%$

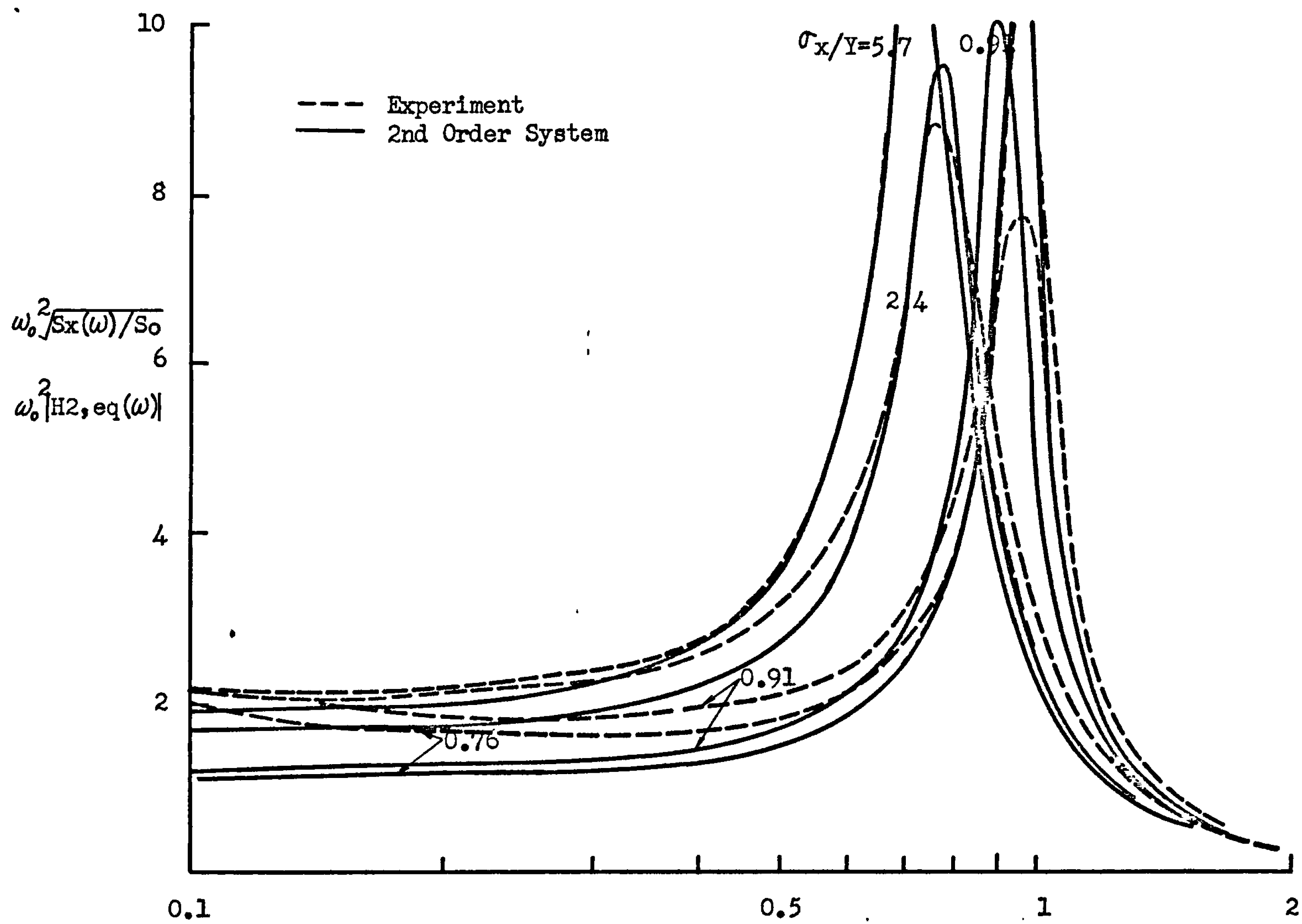


Fig.9.1 Response Power Spectral Density

$\alpha = 1/2, \quad \beta_o = 0$

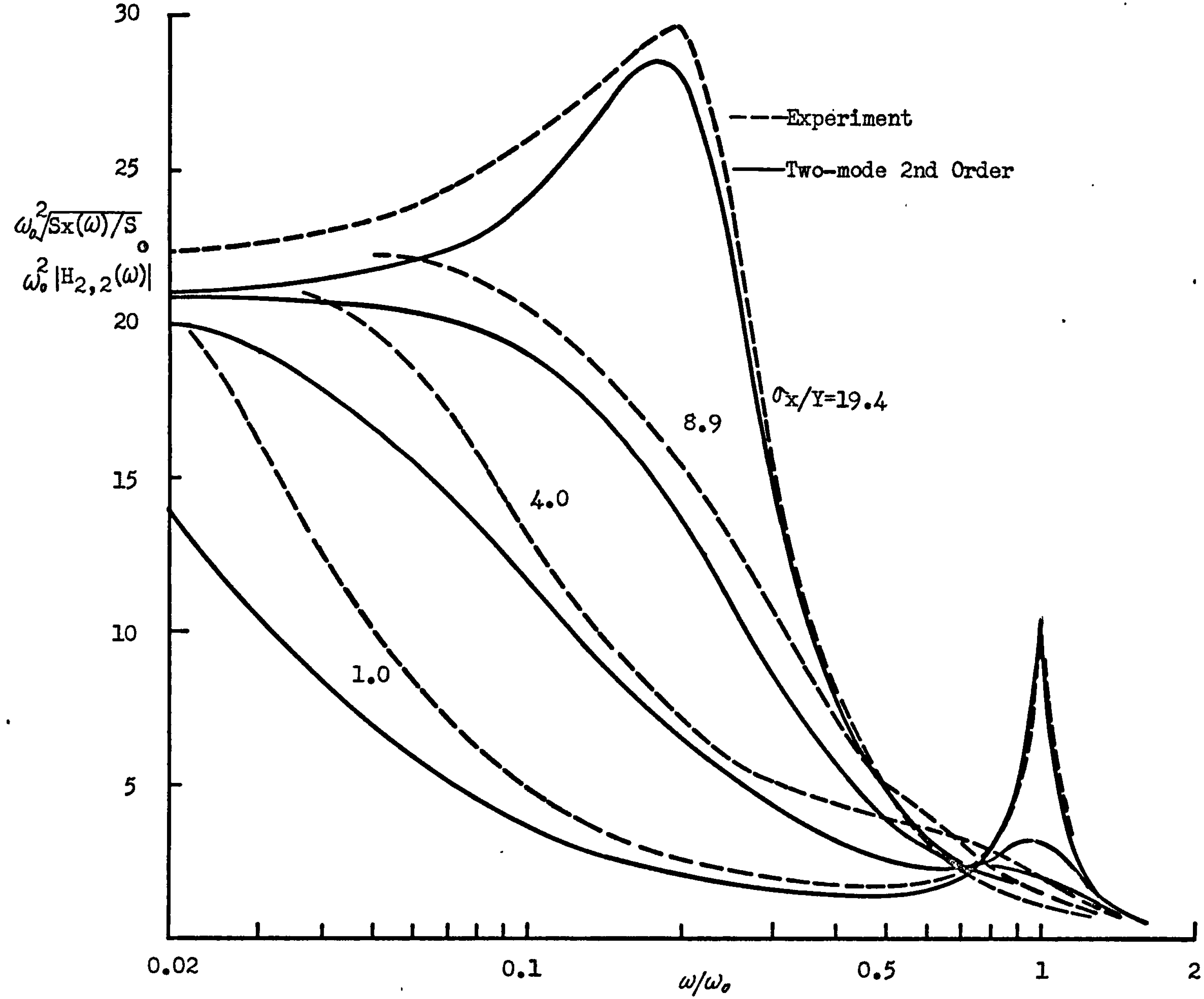


Fig.9.2 Response Power Spectral Density

$\alpha = 1/21$ $\beta = 0$

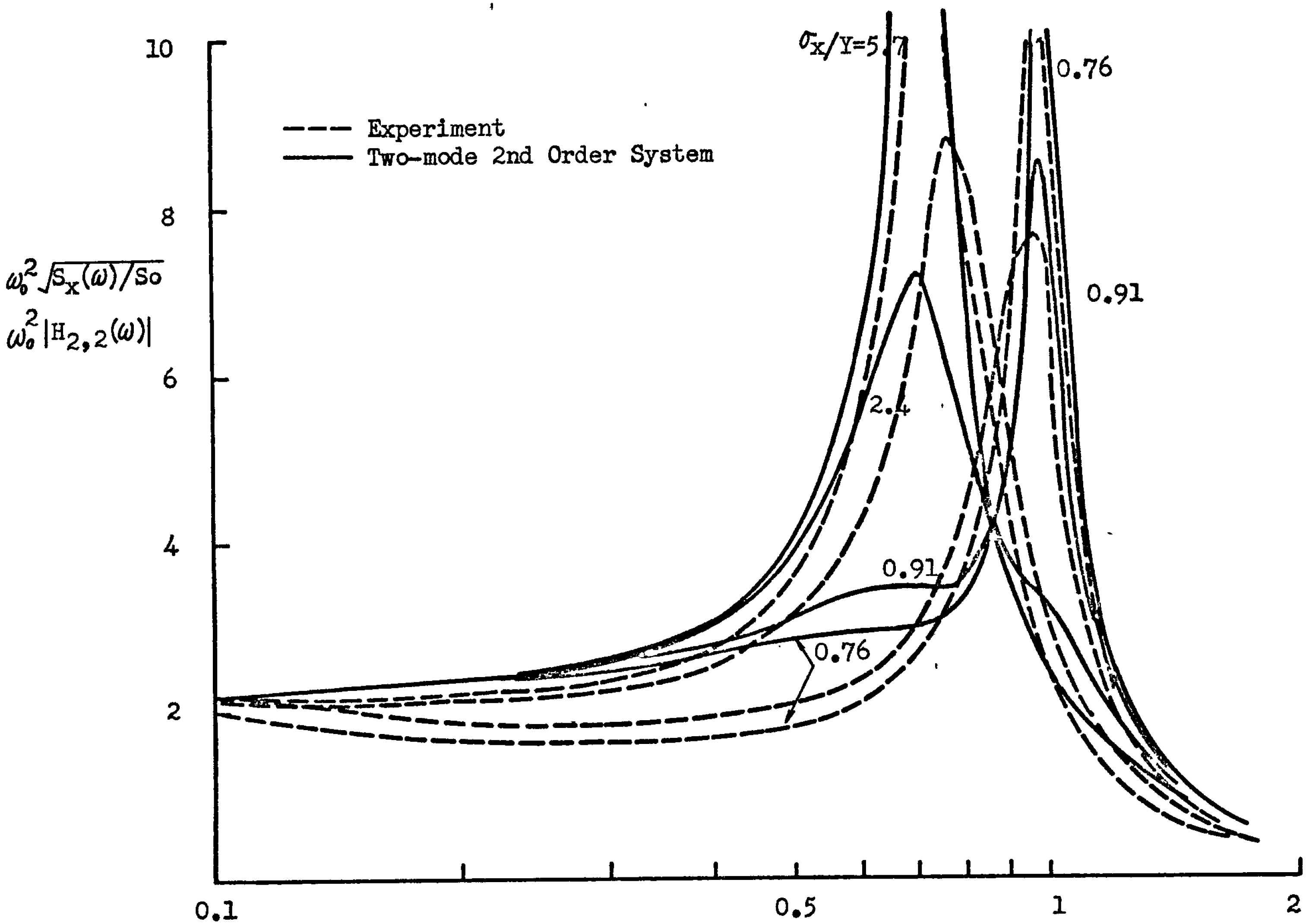


Fig.9.3 Response Power Spectral Density

$\alpha = 1/2,\ \beta_0 = 0$

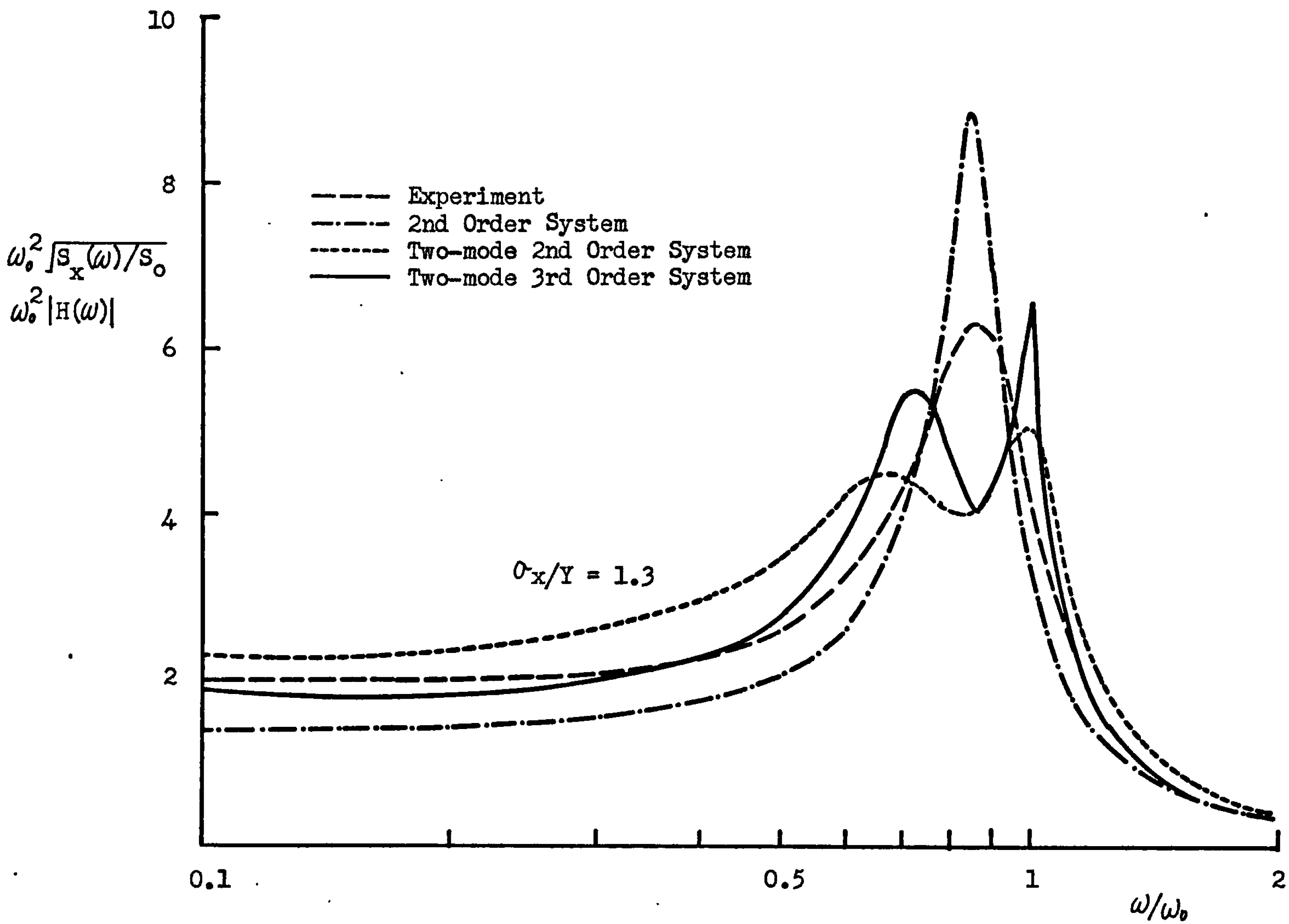


Fig. 9.4 Response Power Spectral Density

$\alpha = 1/2, \ \beta_o = 0$

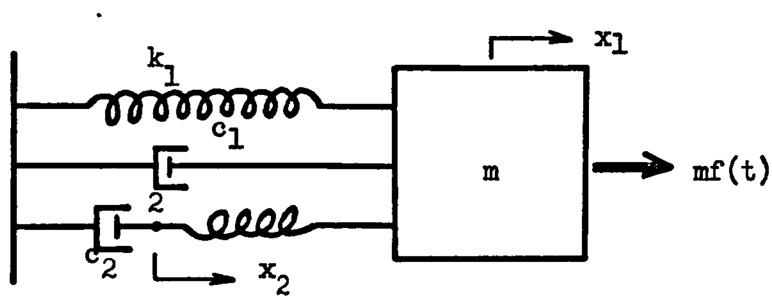


Fig. 10 3rd Order Linear System

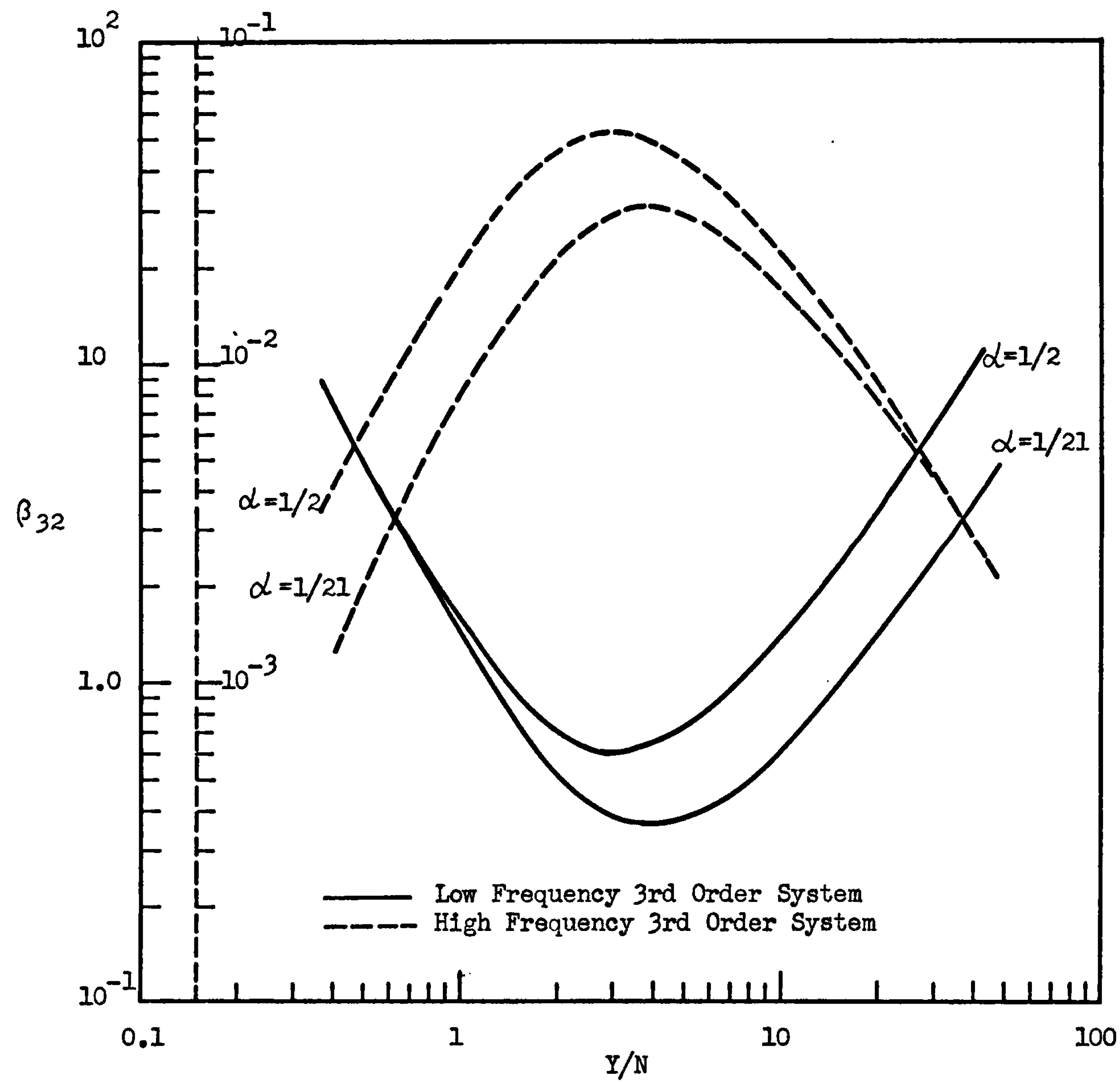


Fig.11 Equivalent Viscous Damping Factor β_{32}

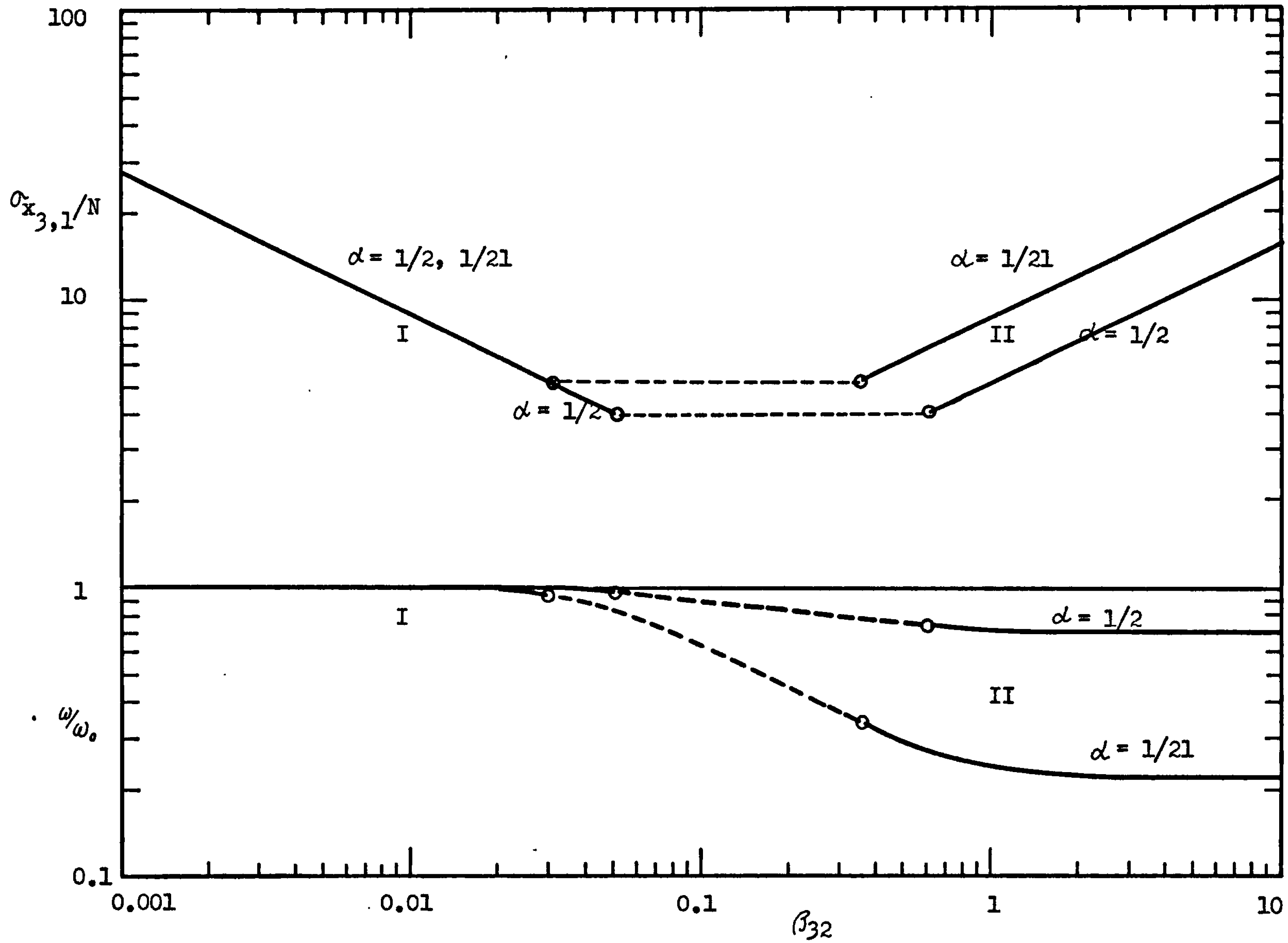


Fig.12 rms Displacement- β_{32}

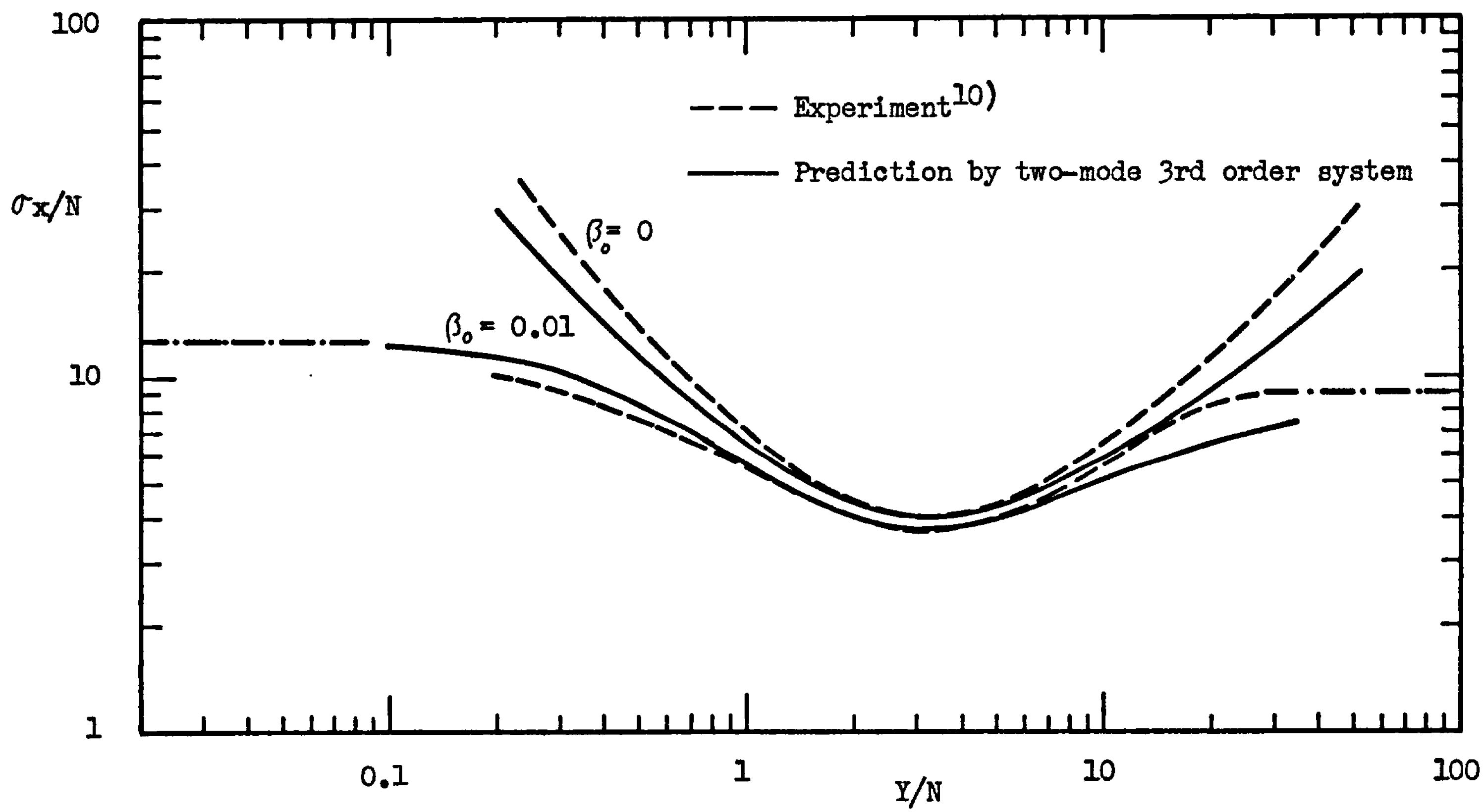


Fig.13.1 rms Displacement of Two-mode 3rd Order System

$\alpha = 1/2, \beta_o = 0, 1$ %

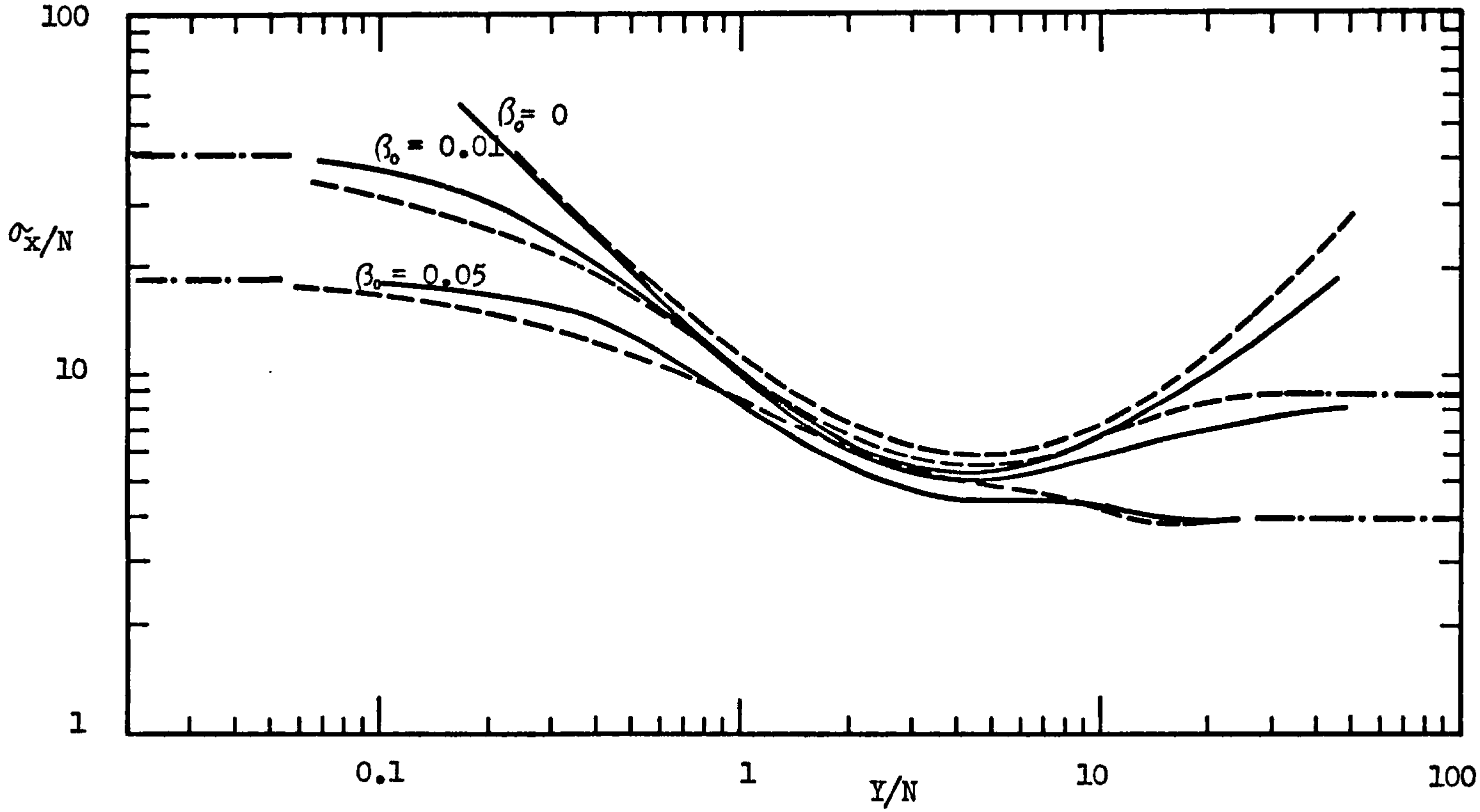


Fig.13.2 rms Displacement of Two-mode 3rd Order System

$\alpha = 1/21$, $\beta_o = 0, 1, 5\ \%$

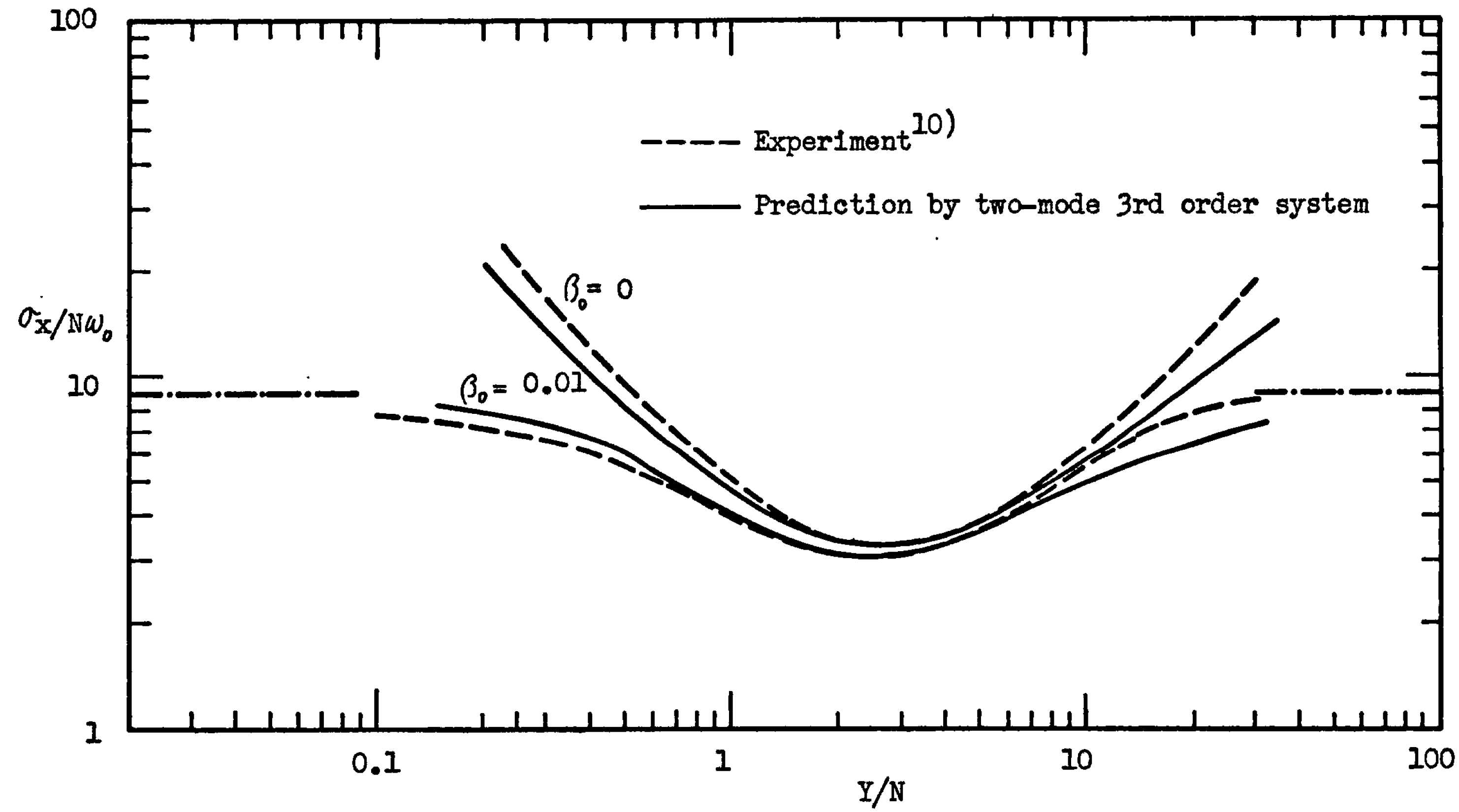


Fig.14.1 rms Velocity of Two-mode 3rd Order System

$\alpha = 1/2$, $\beta_0 = 0, 1$ %

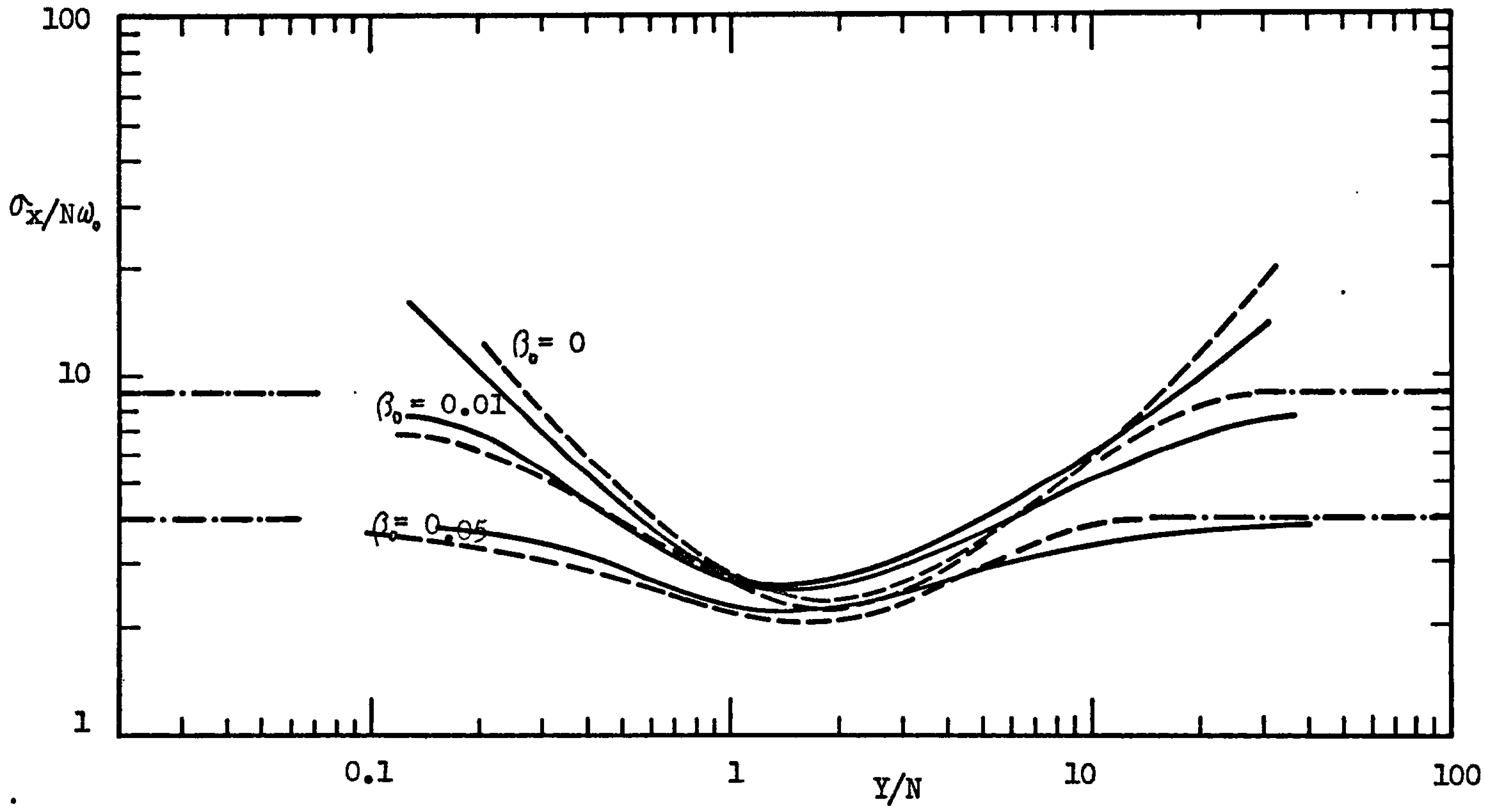


Fig.14.2 rms Velocity of Two-mode 3rd Order System

$\alpha = 1/21, \quad \beta_0 = 0,\ 1,\ 5\ \%$

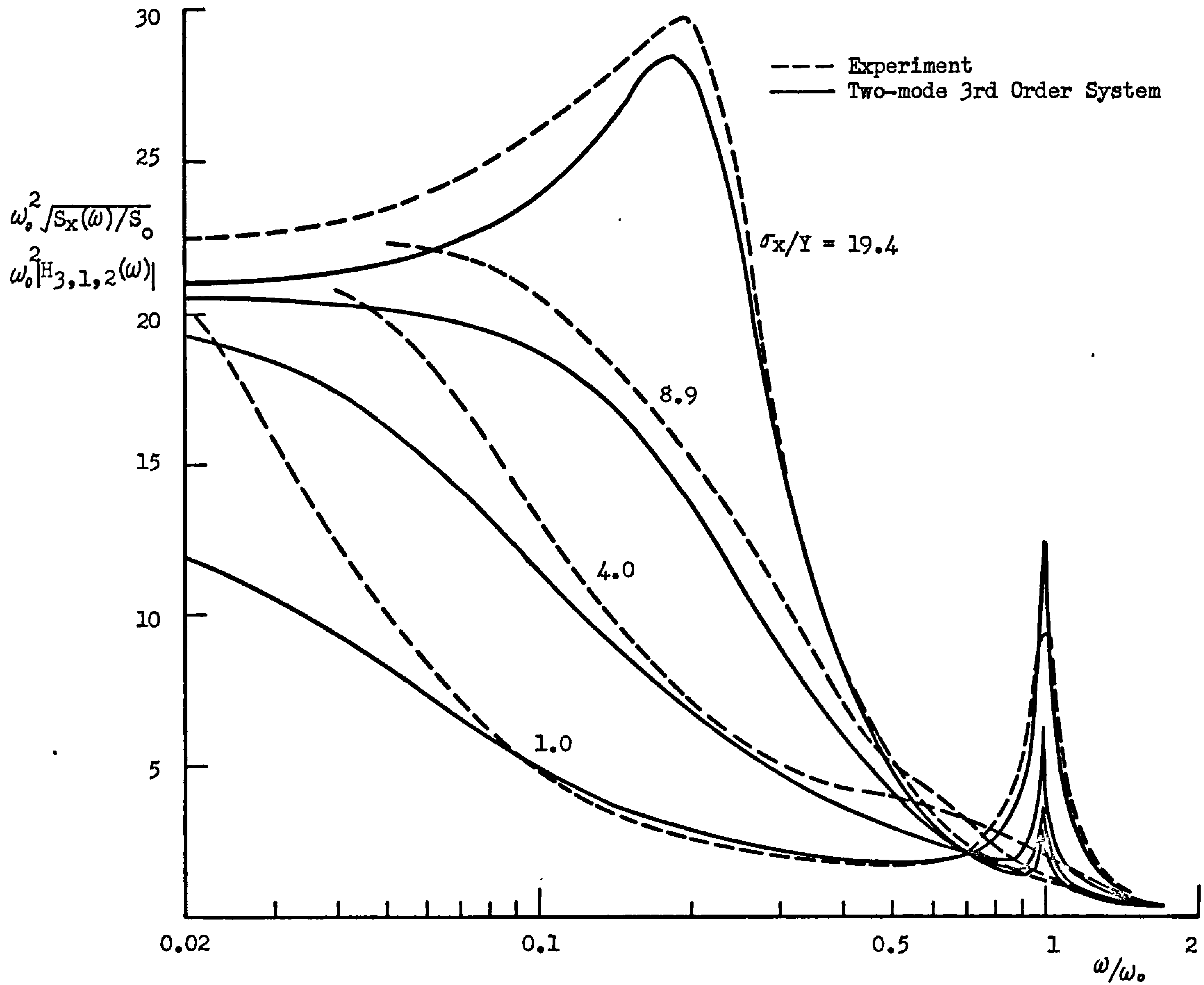


Fig.15.1 Response Power Spectral Density

$\alpha = 1/21, \quad \beta_0 = 0$

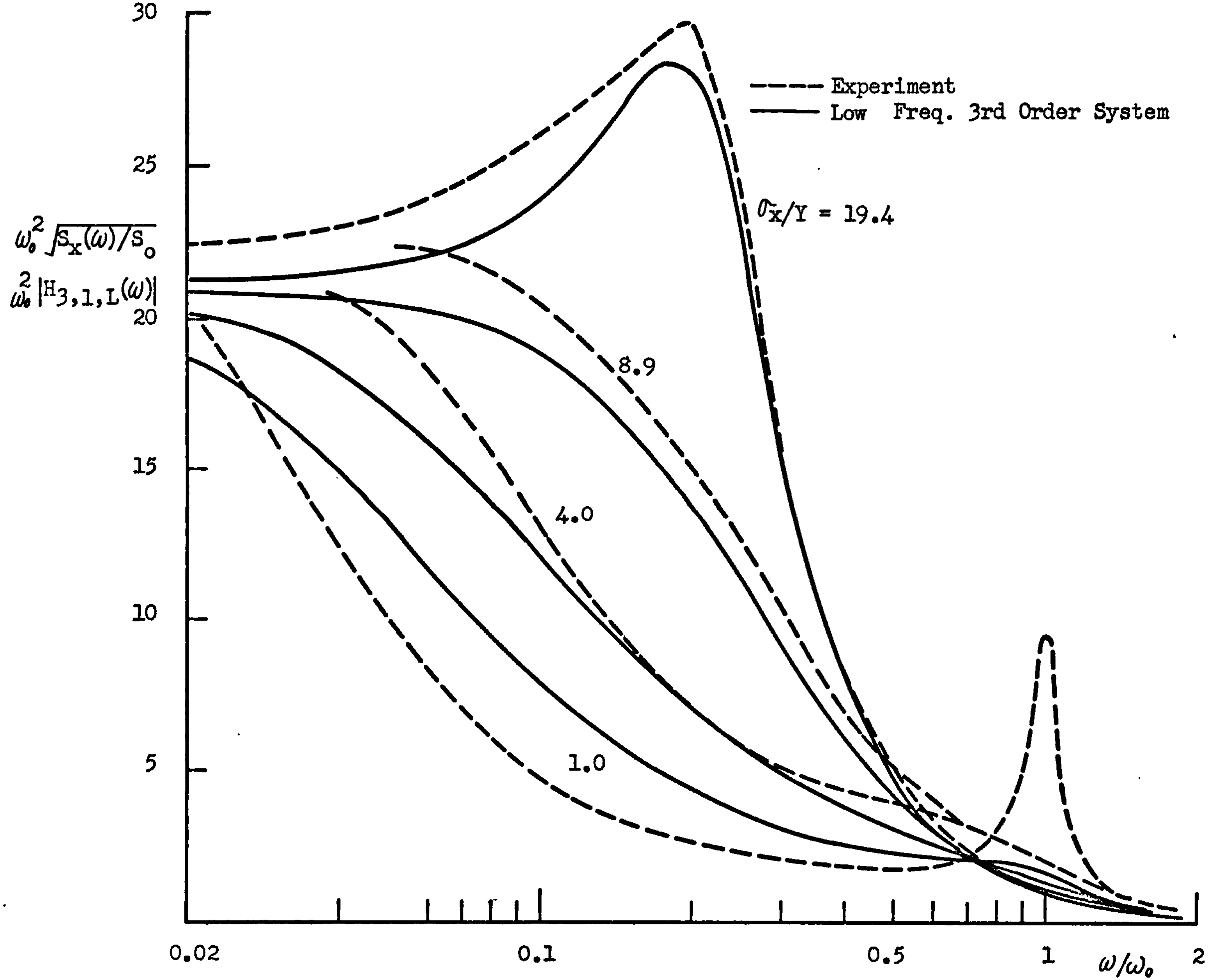


Fig.15.2 Response Power Spectral Density

$\alpha = 1/21, \quad \beta_0 = 0$

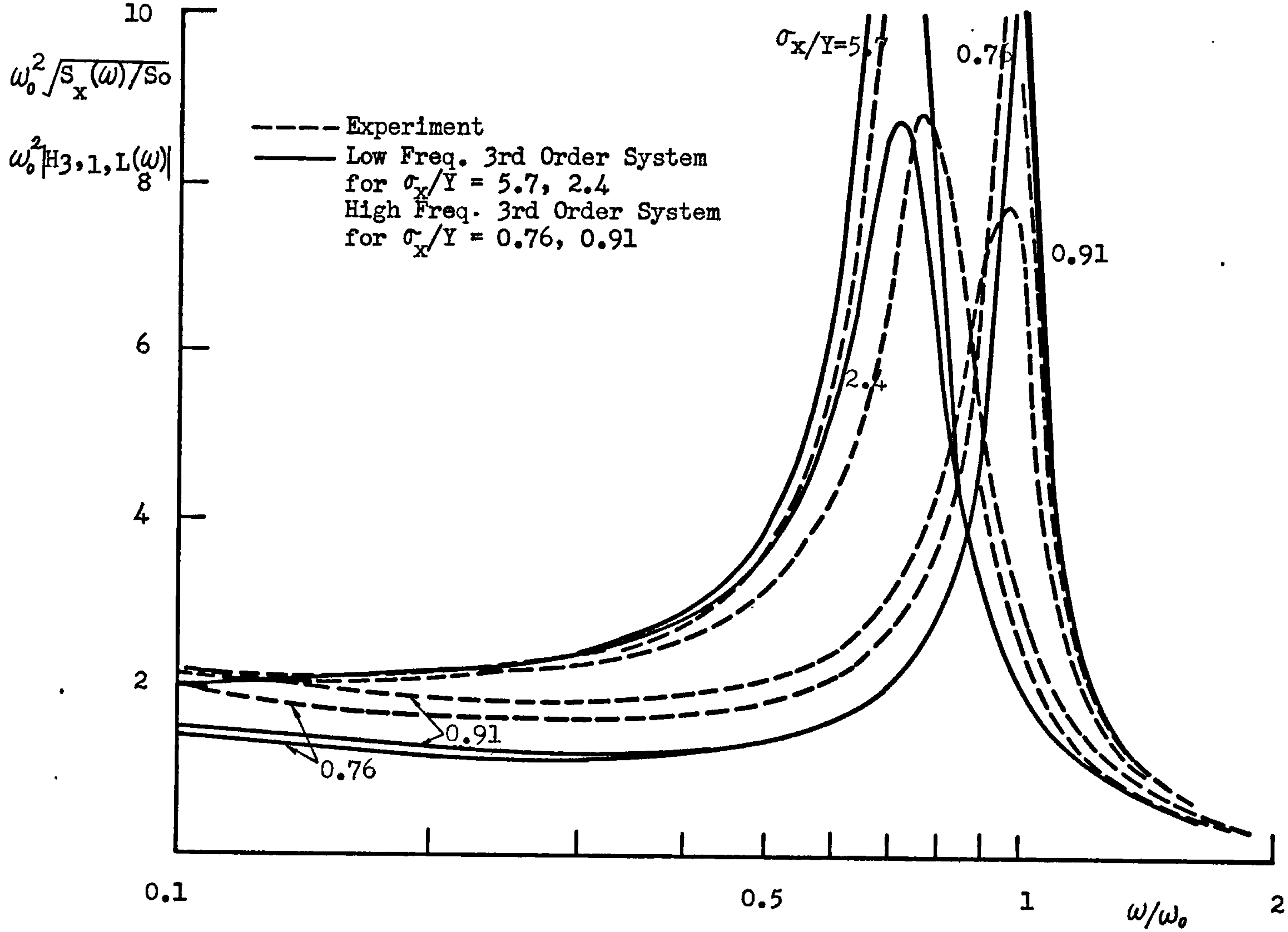


Fig.15.3 Response Power Spectral Density

$\alpha = 1/2,\ \beta_0 = 0$

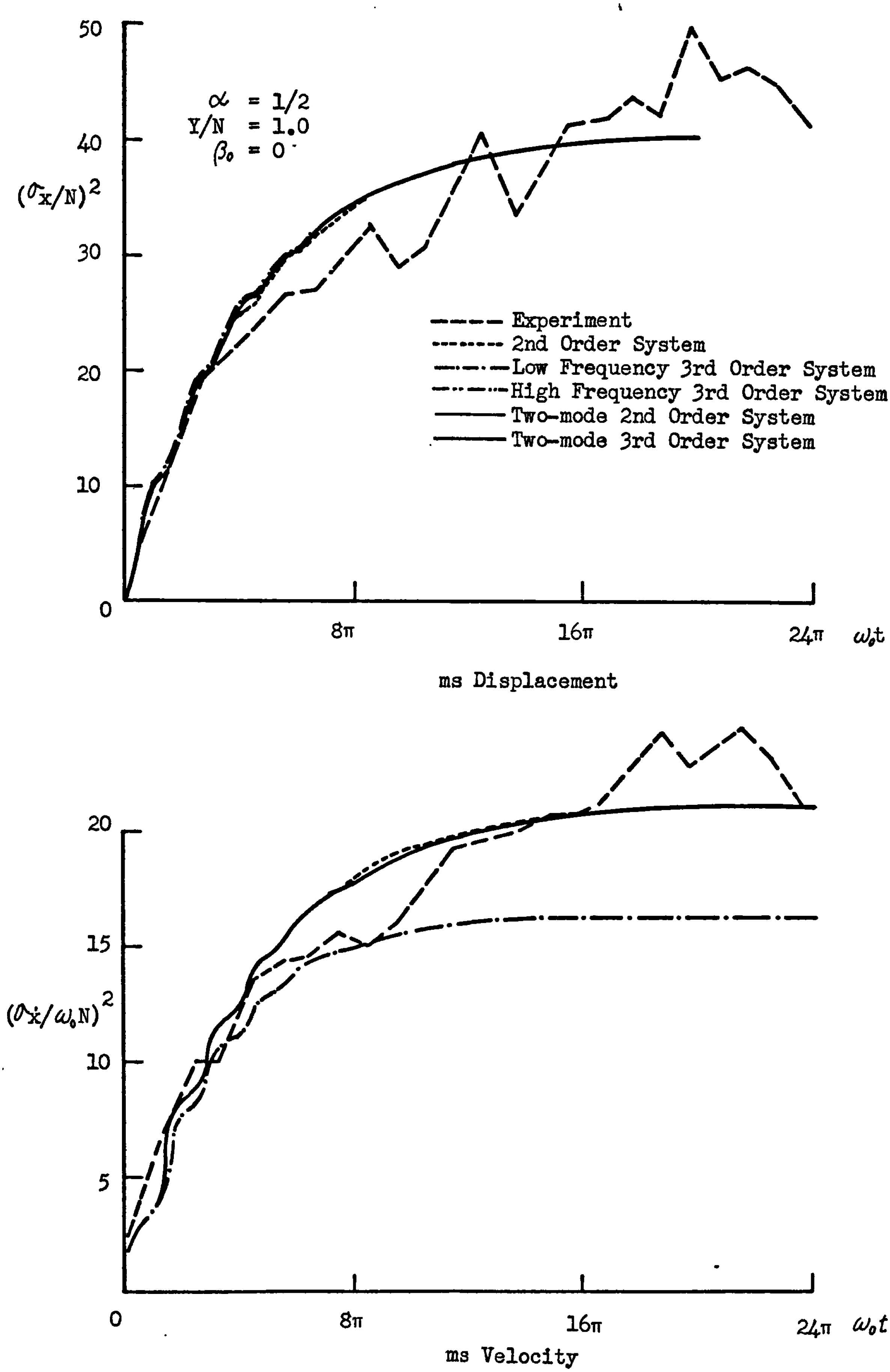
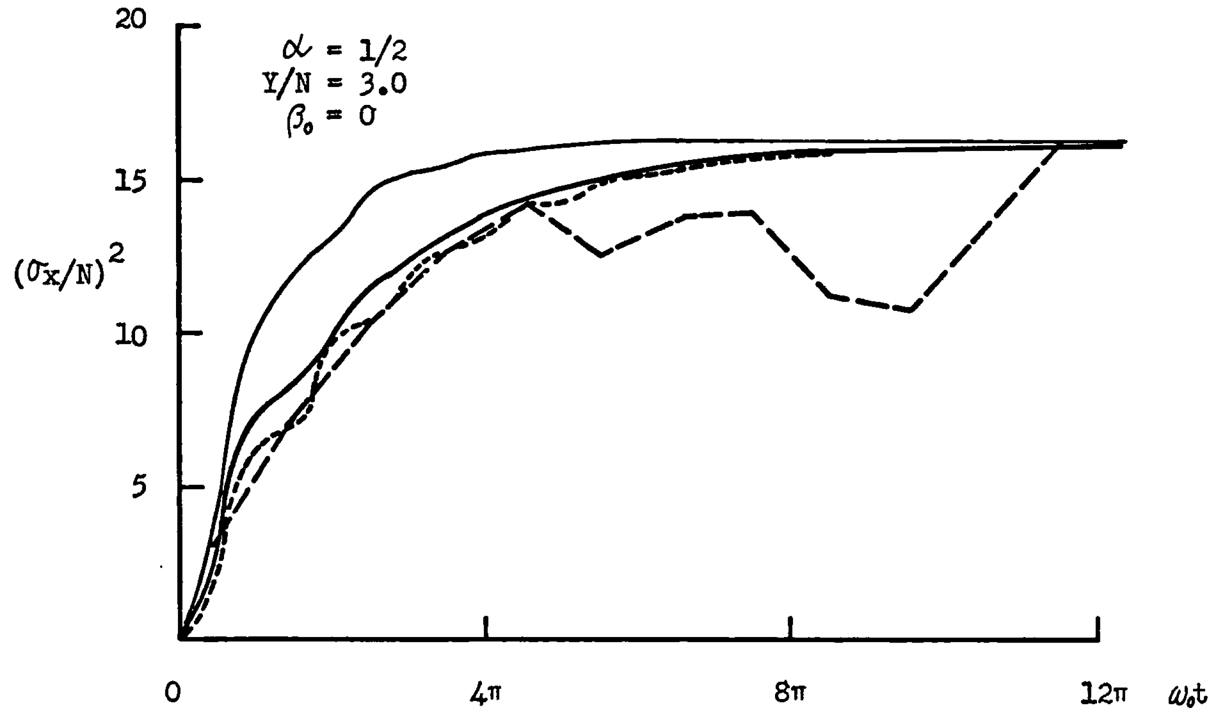
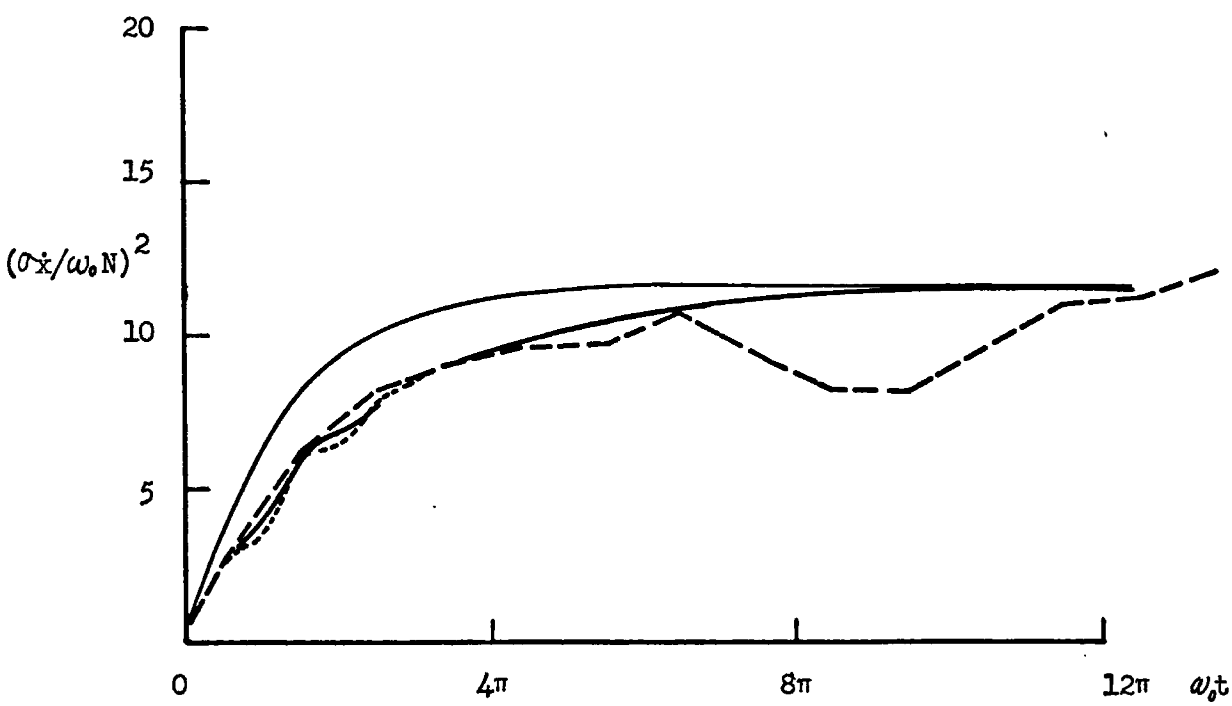


Fig. 16.1 Transient Mean-square Response

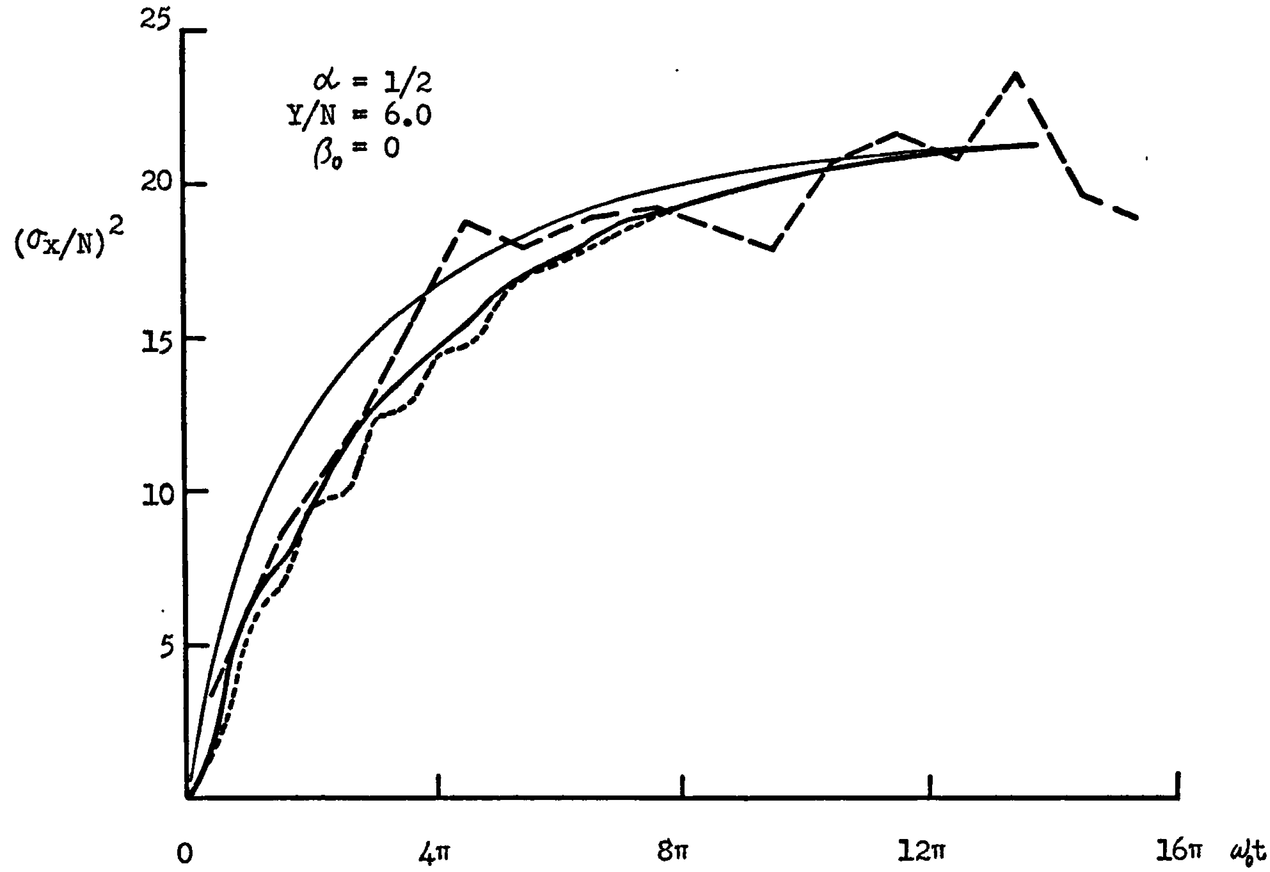


ms Displacement

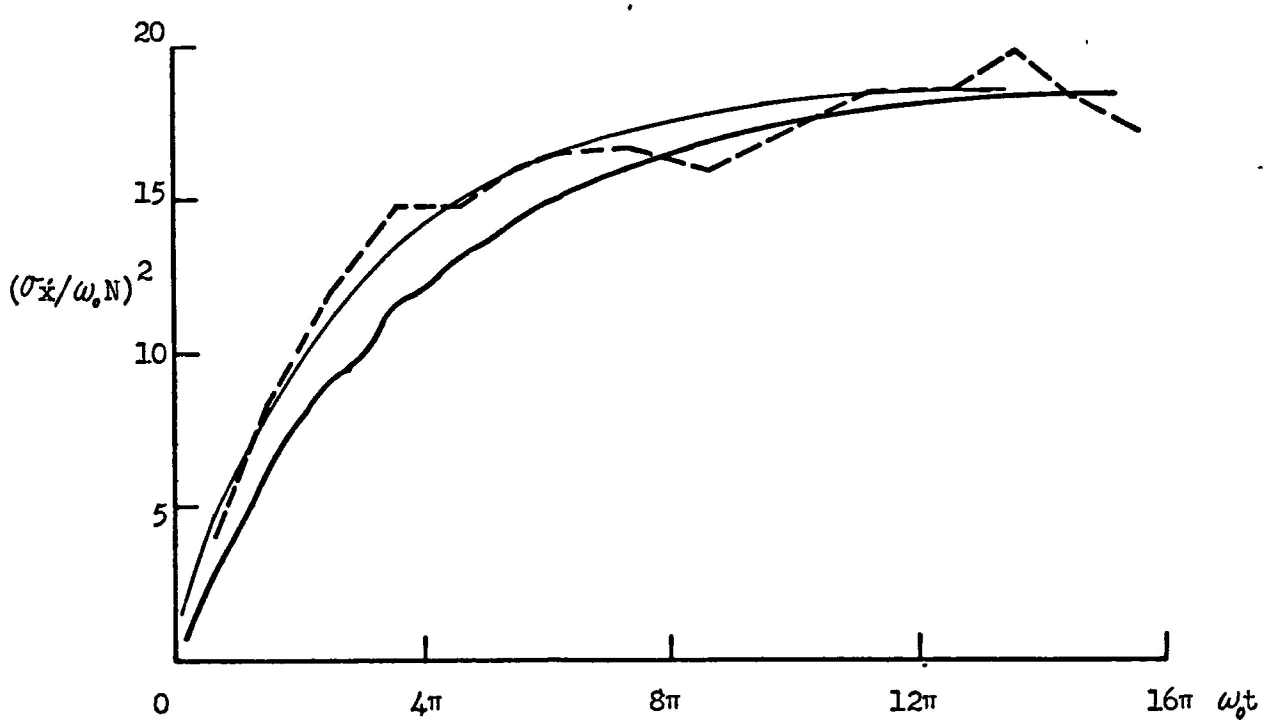


ms Velocity

Fig. 16.2 Transient Mean-square Response

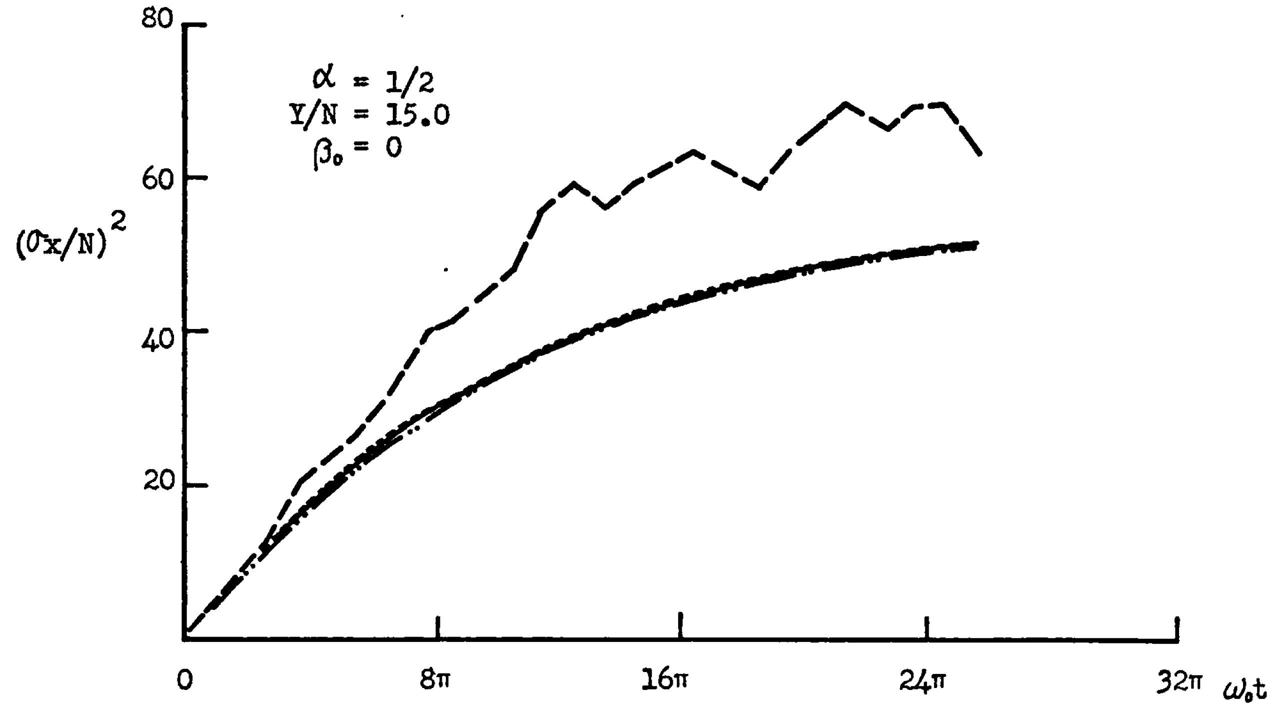


ms Displacement

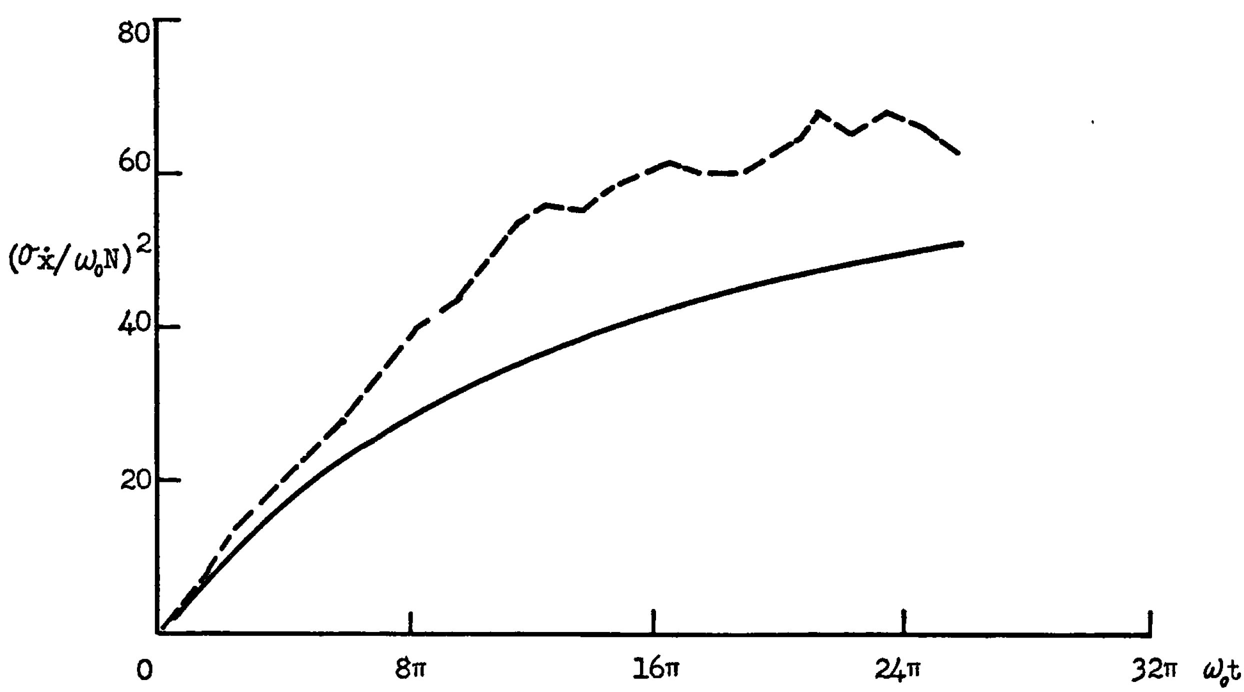


ms Velocity

Fig. 16.3 Transient Mean-square Response



ms Displacement



ms Velocity

Fig. 16.4 Transient Mean-square Response

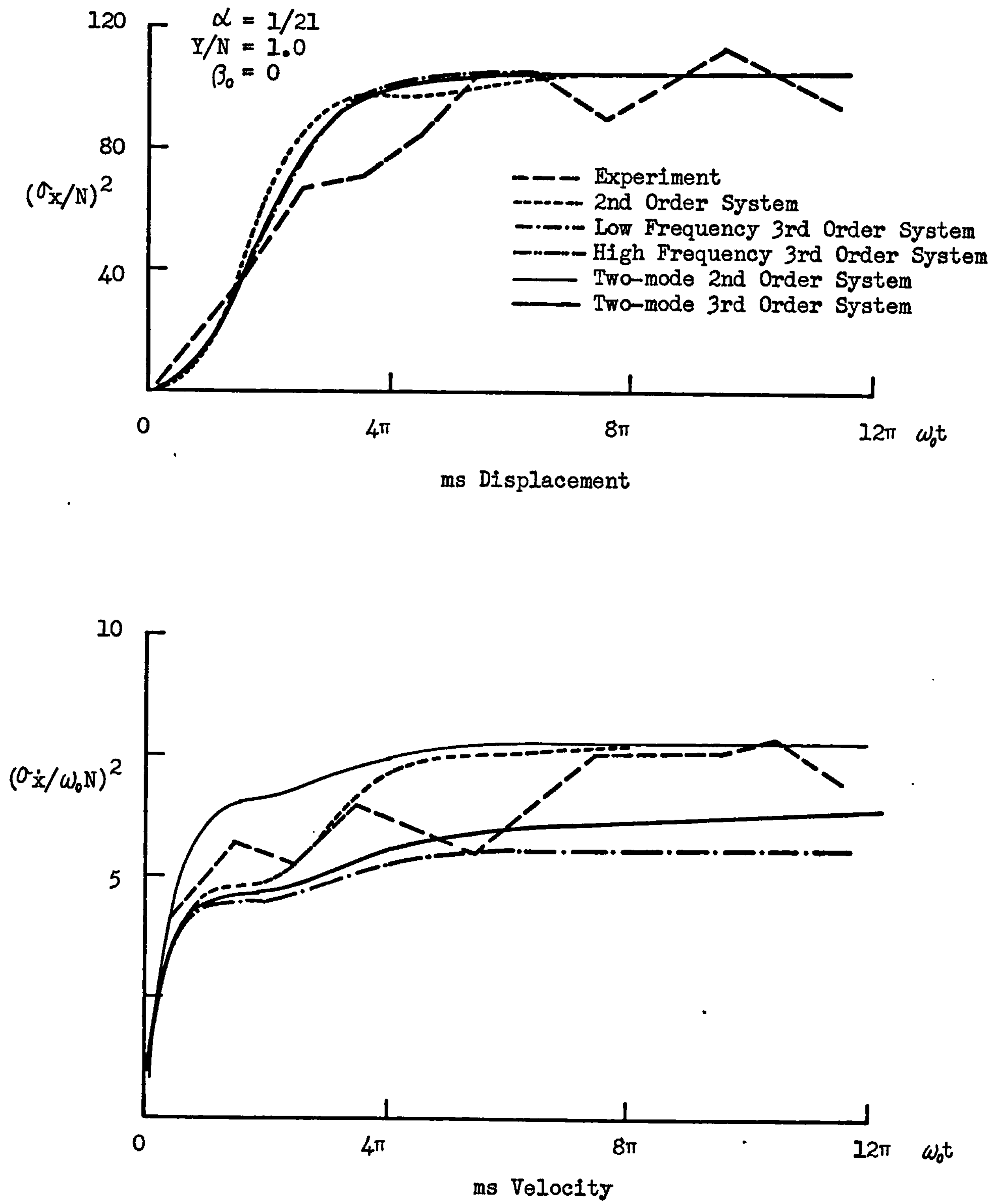
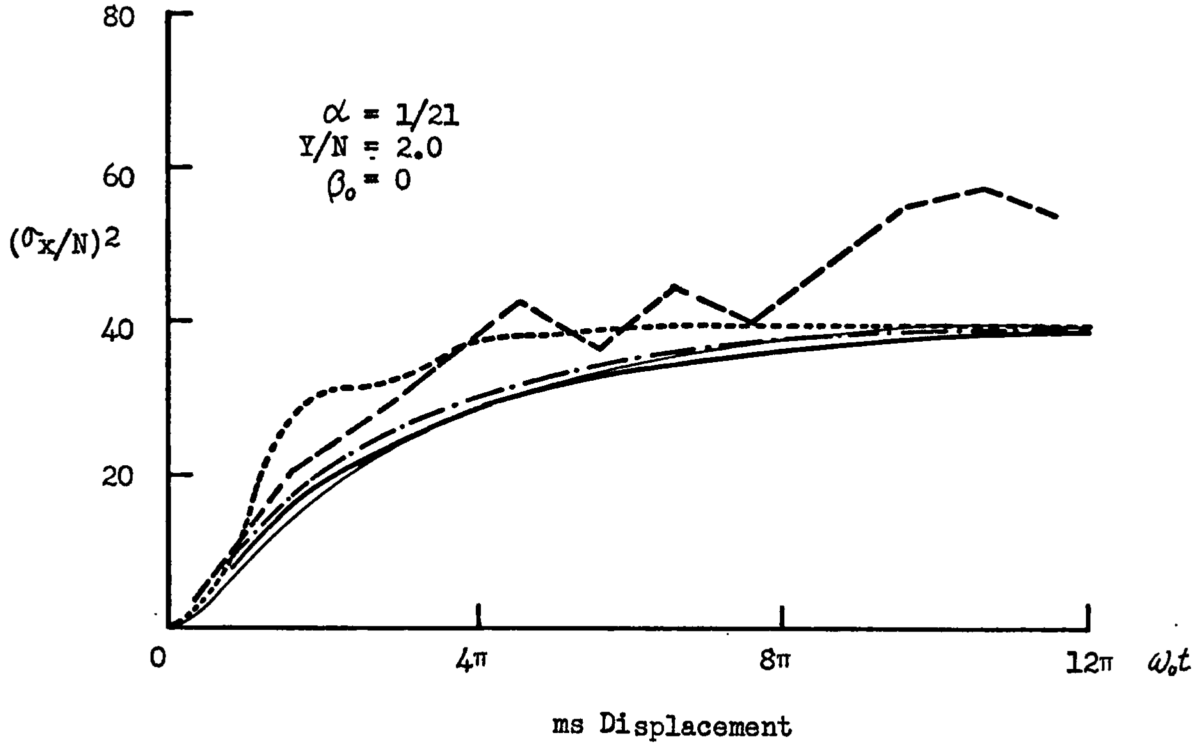


Fig. 17.1 Transient Mean-square Response



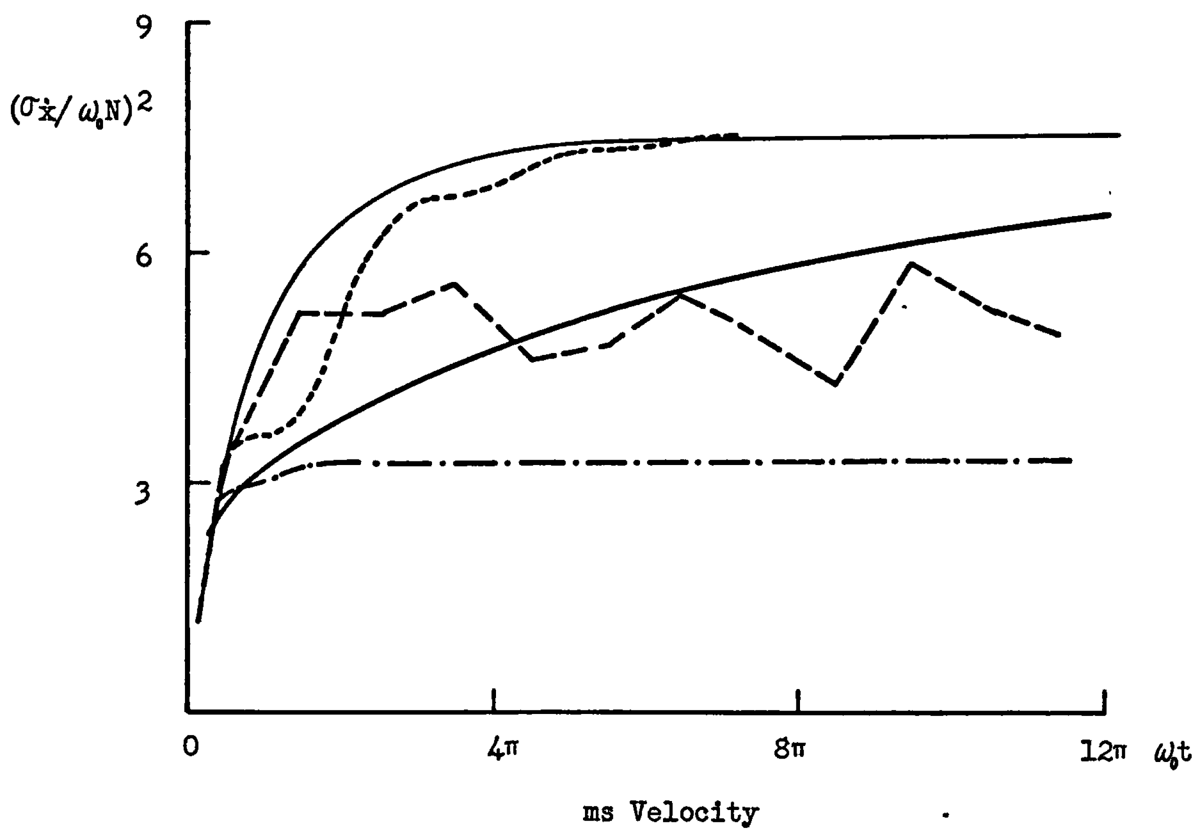
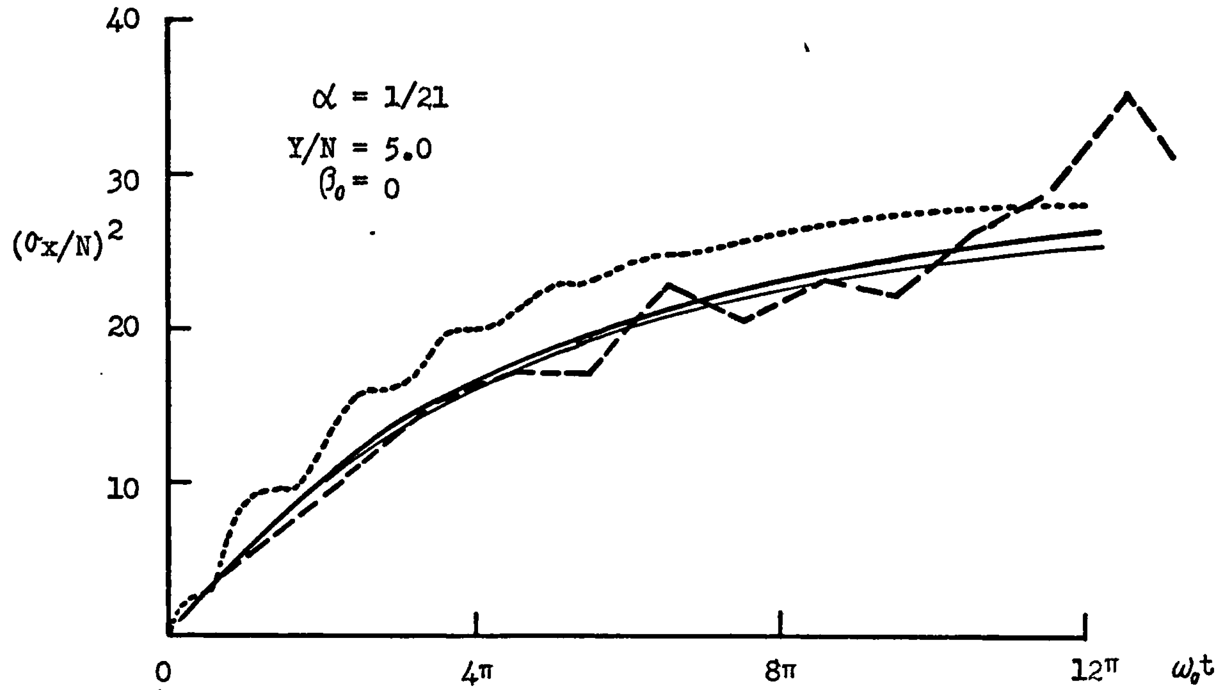
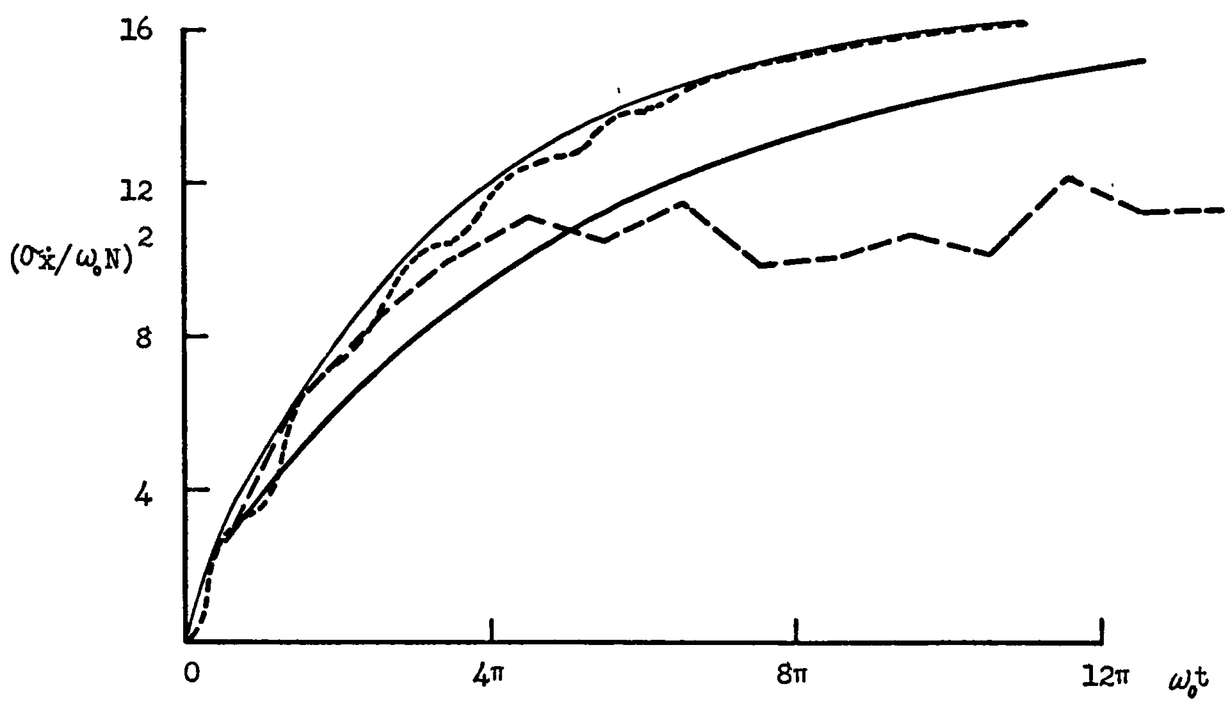


Fig. 17.2 Transient Mean-square Response

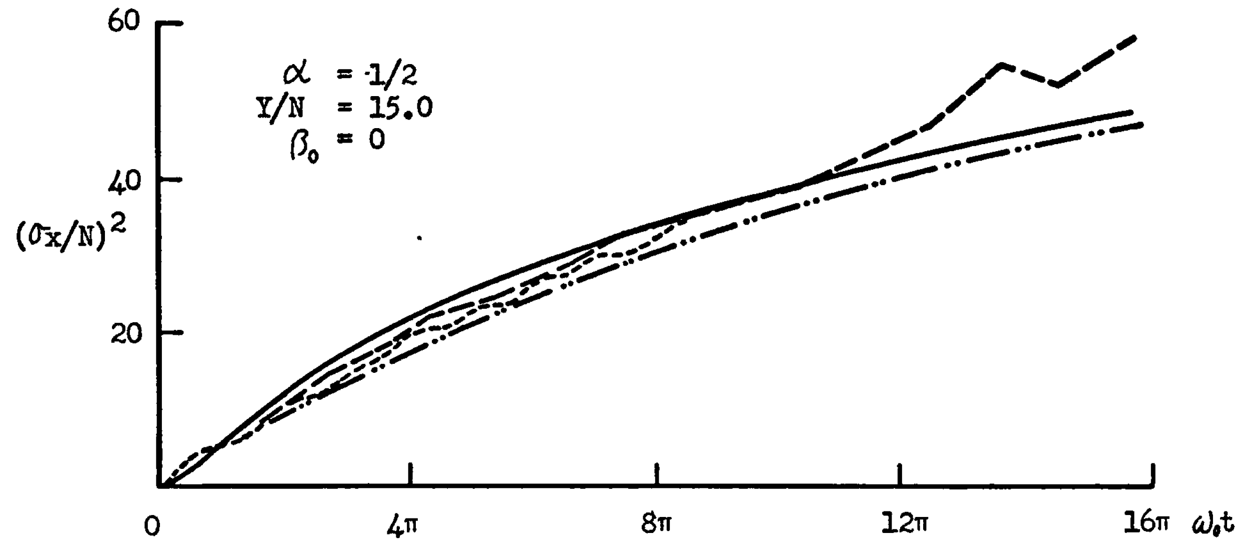


ms Displacement

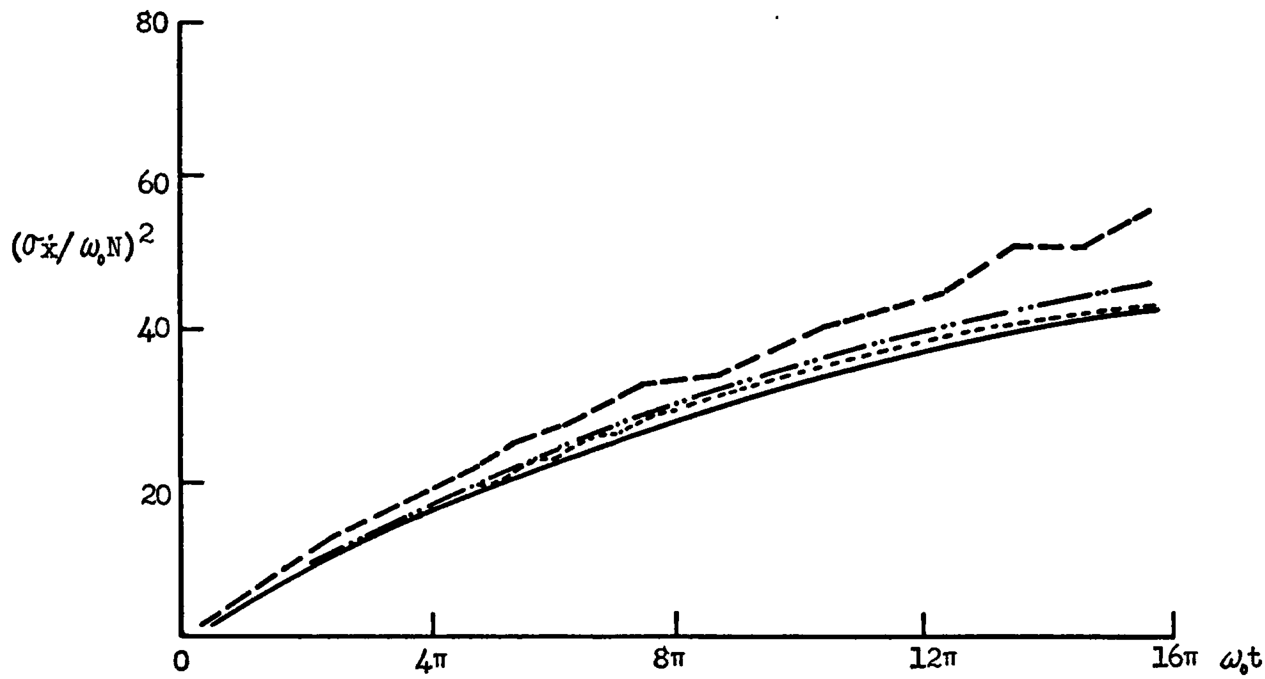


ms Velocity

Fig. 17.3 Transient Mean-square Response



ms Displacement



ms Velocity

Fig. 17.4 Transient Mean-square Response

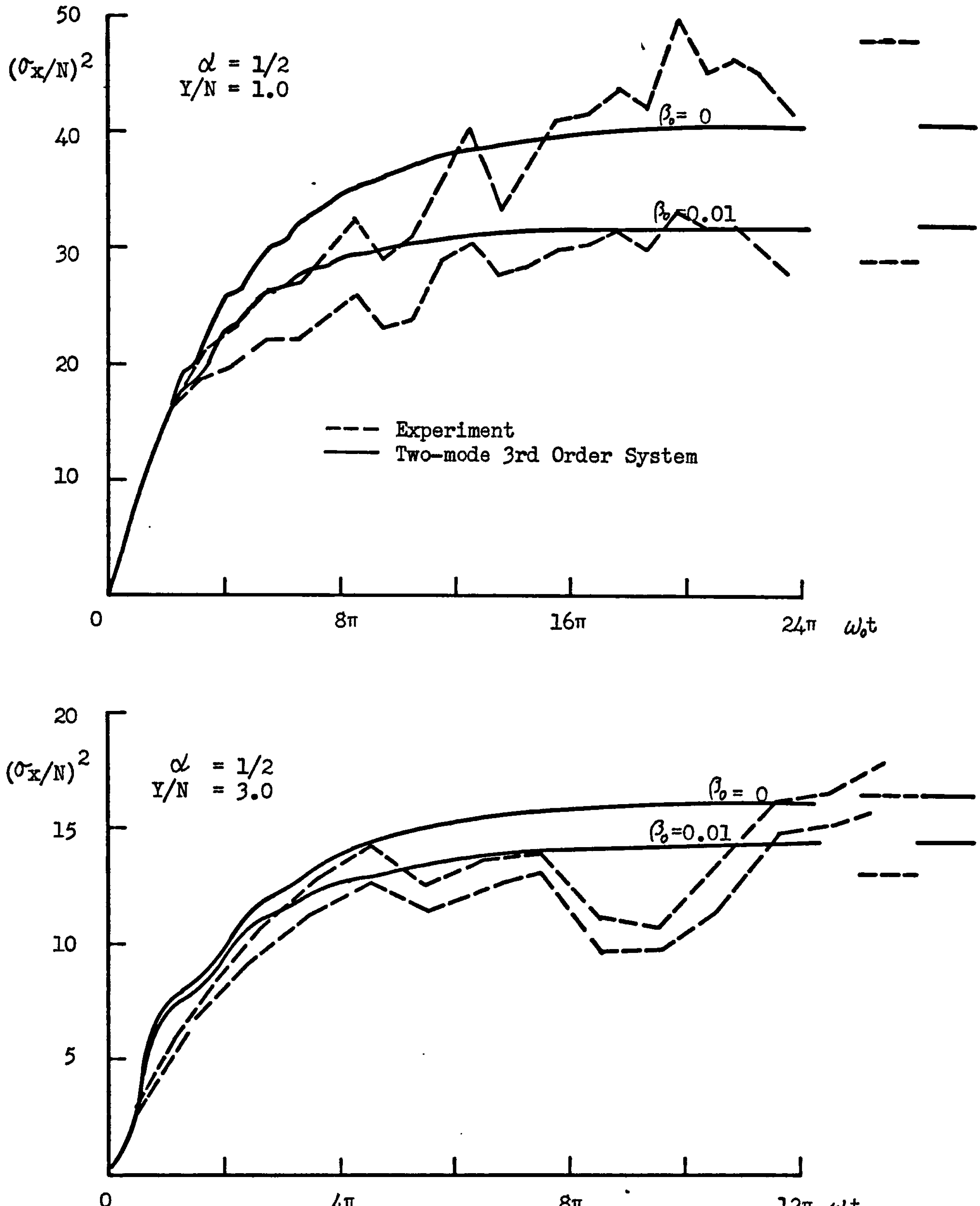


Fig. 18.1 Mean-square Displacement of the Two-mode 3rd Order System

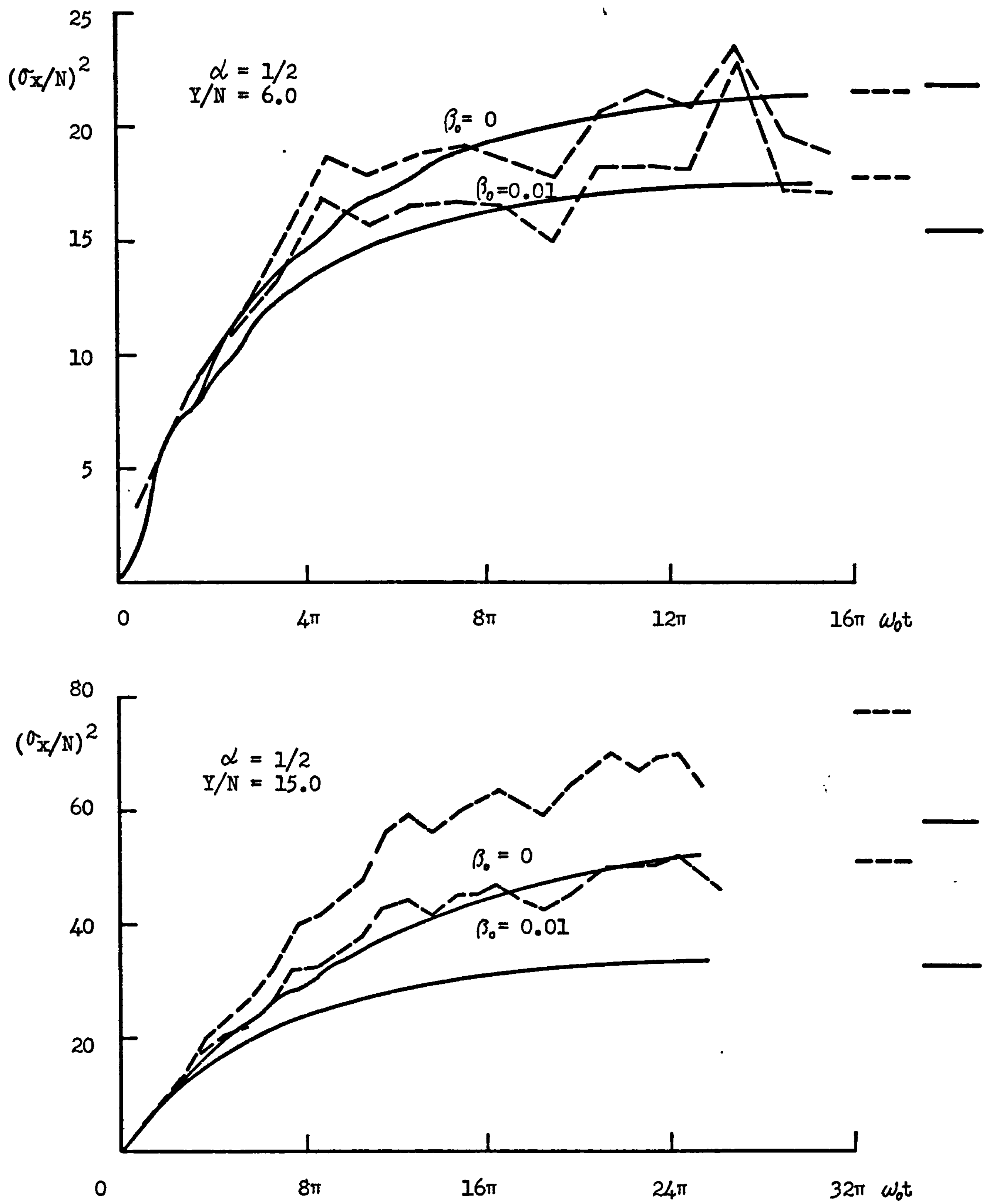


Fig.18.2 Mean-square Displacement of the Two-mode 3rd Order System

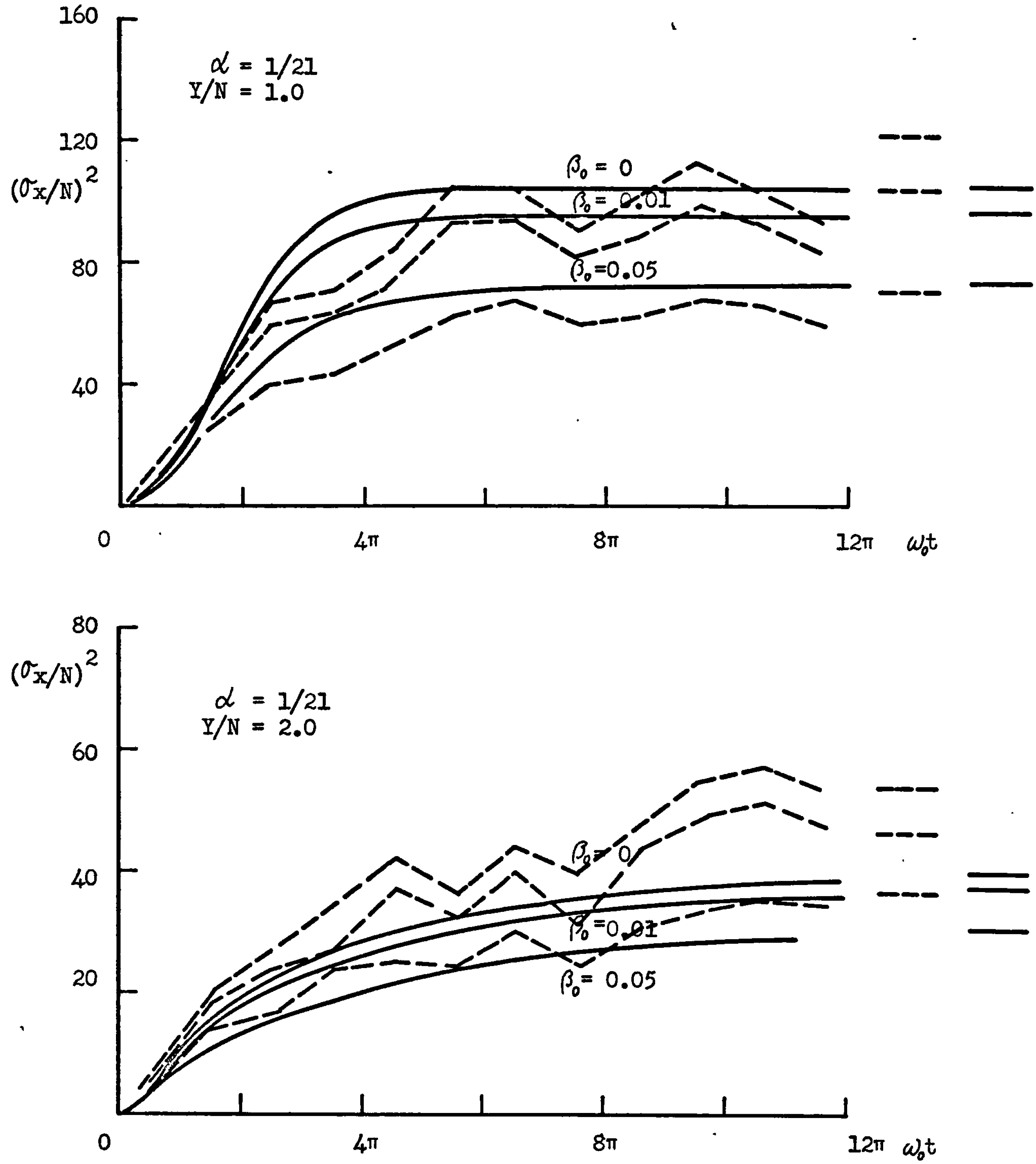


Fig.18.3 Mean-square Displacement of the Two-mode 3rd Order System

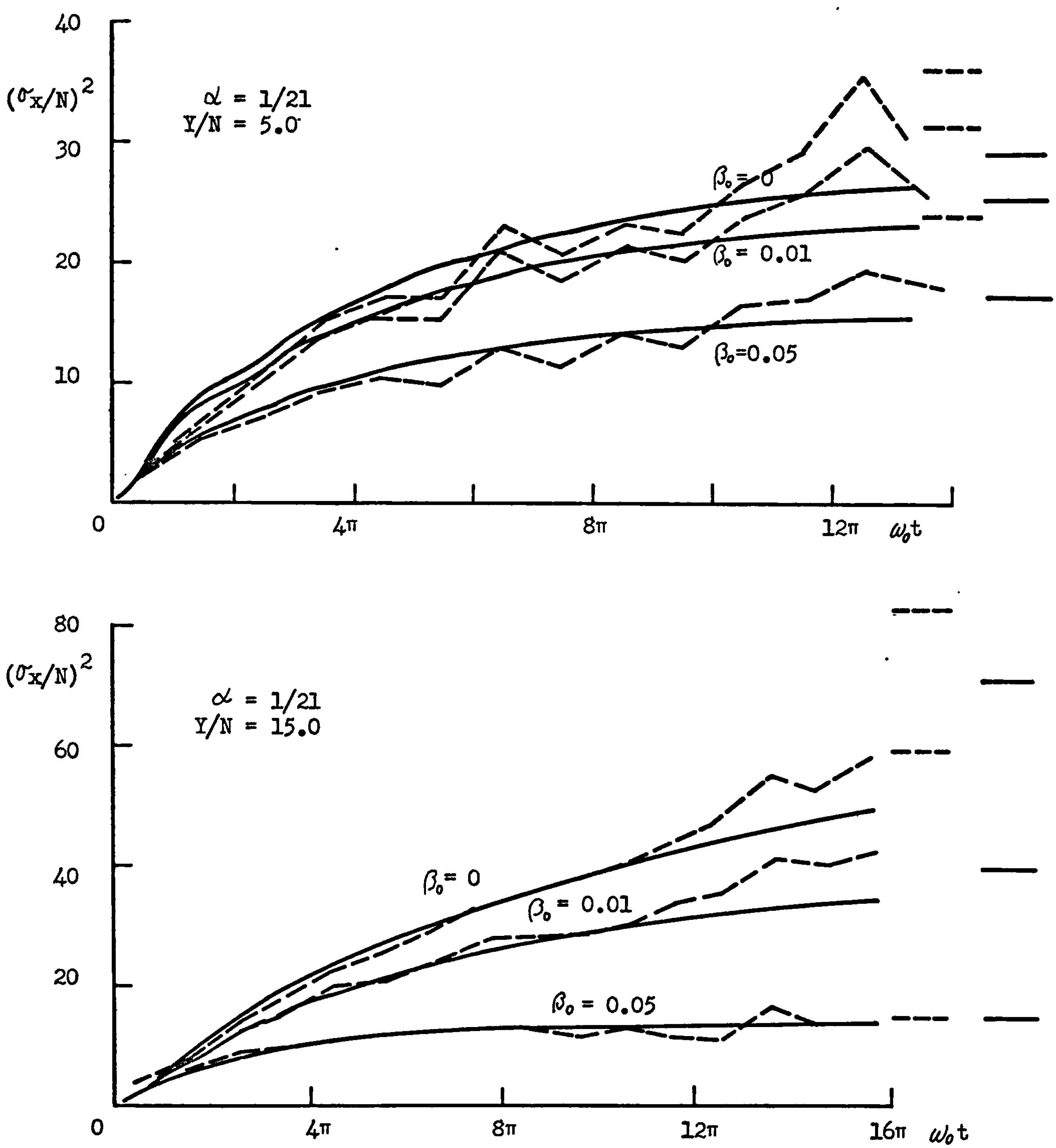


Fig.18.4 Mean-square Displacement of the Two-mode 3rd Order System

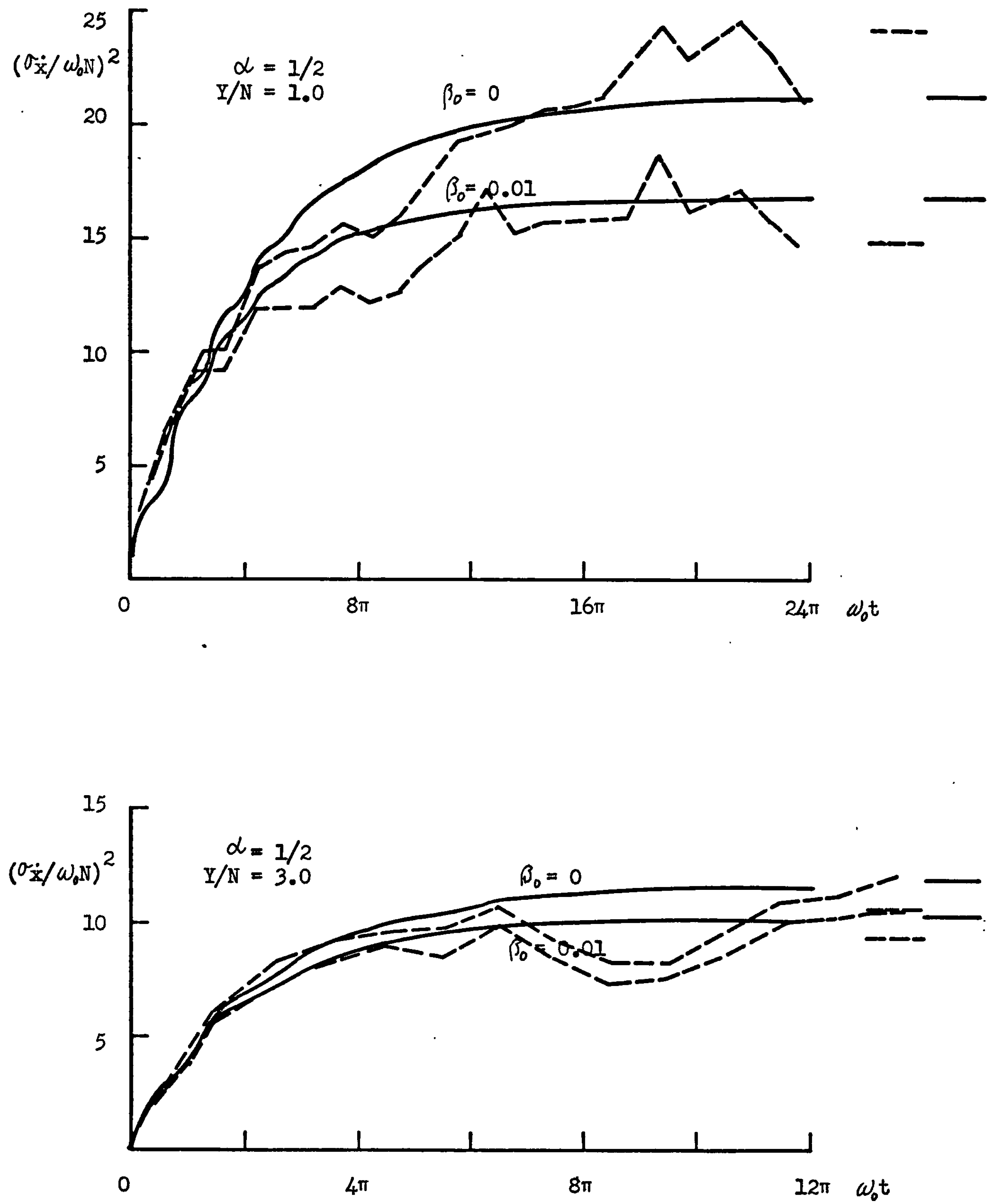


Fig.19.1 Mean-square Velocity of the Two-mode 3rd Order System

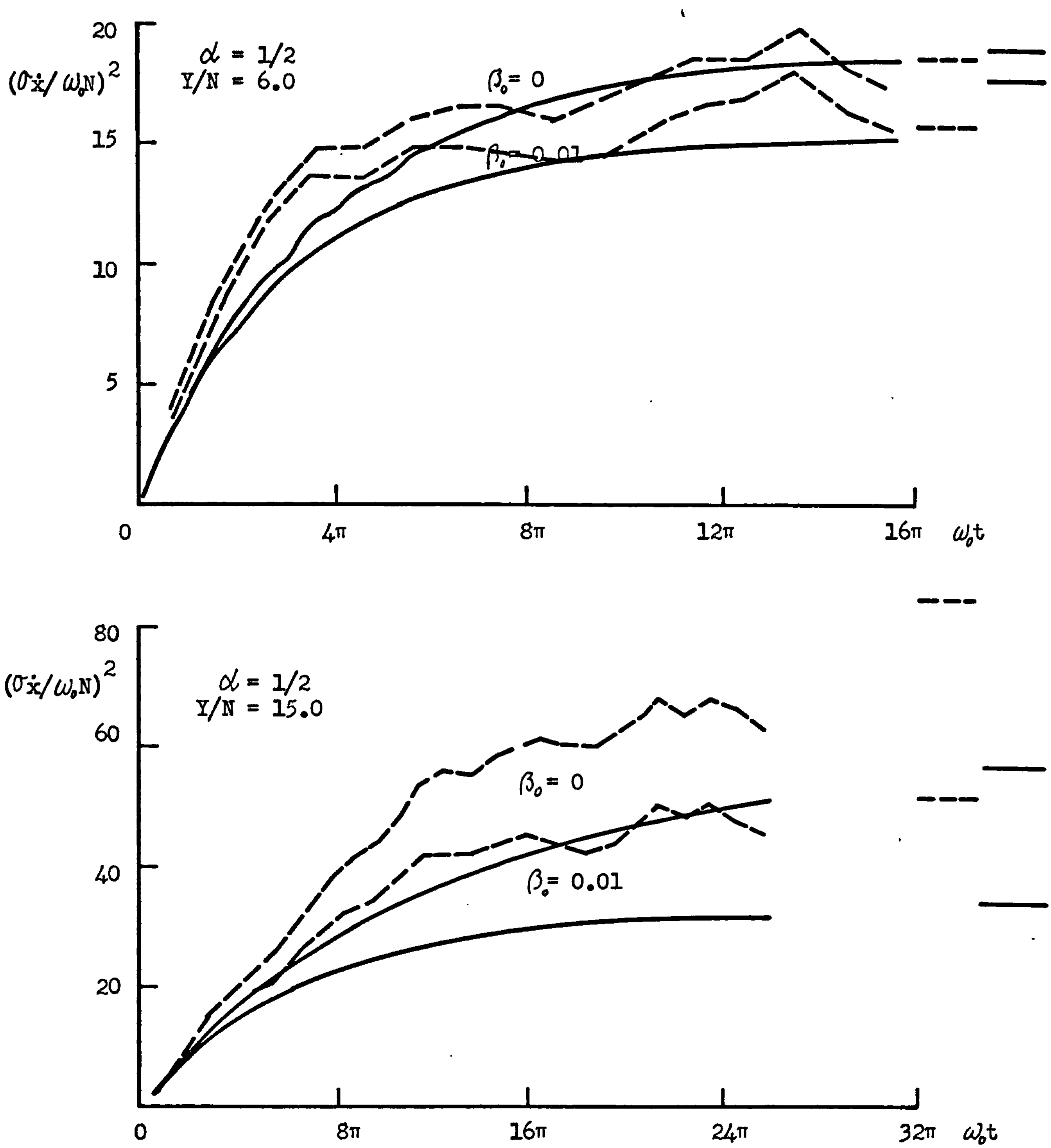


Fig.19.2 Mean-square Velocity
of the Two-mode 3rd Order System

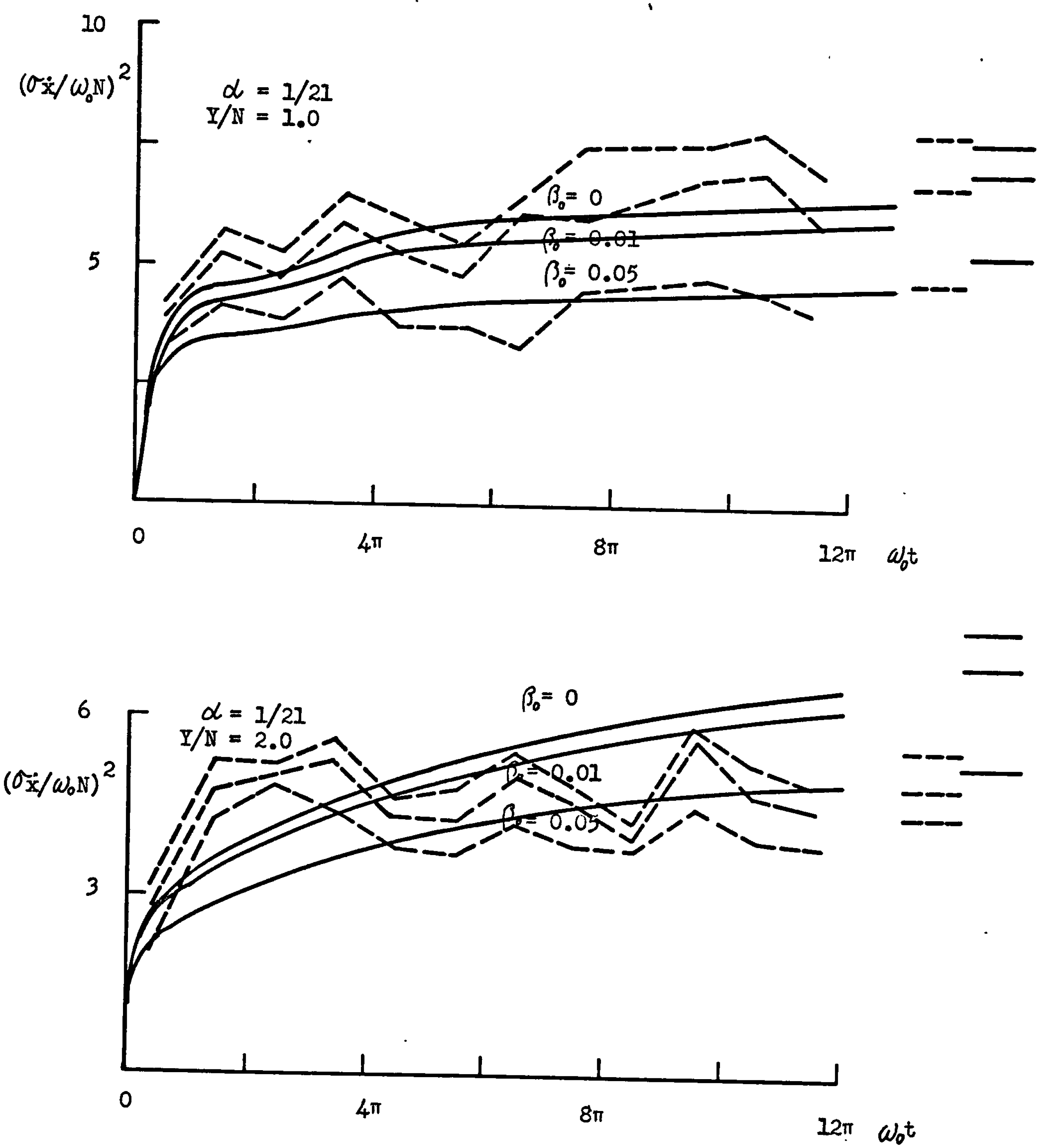


Fig.19.3 Mean-square Velocity
of the Two-mode 3rd Order System

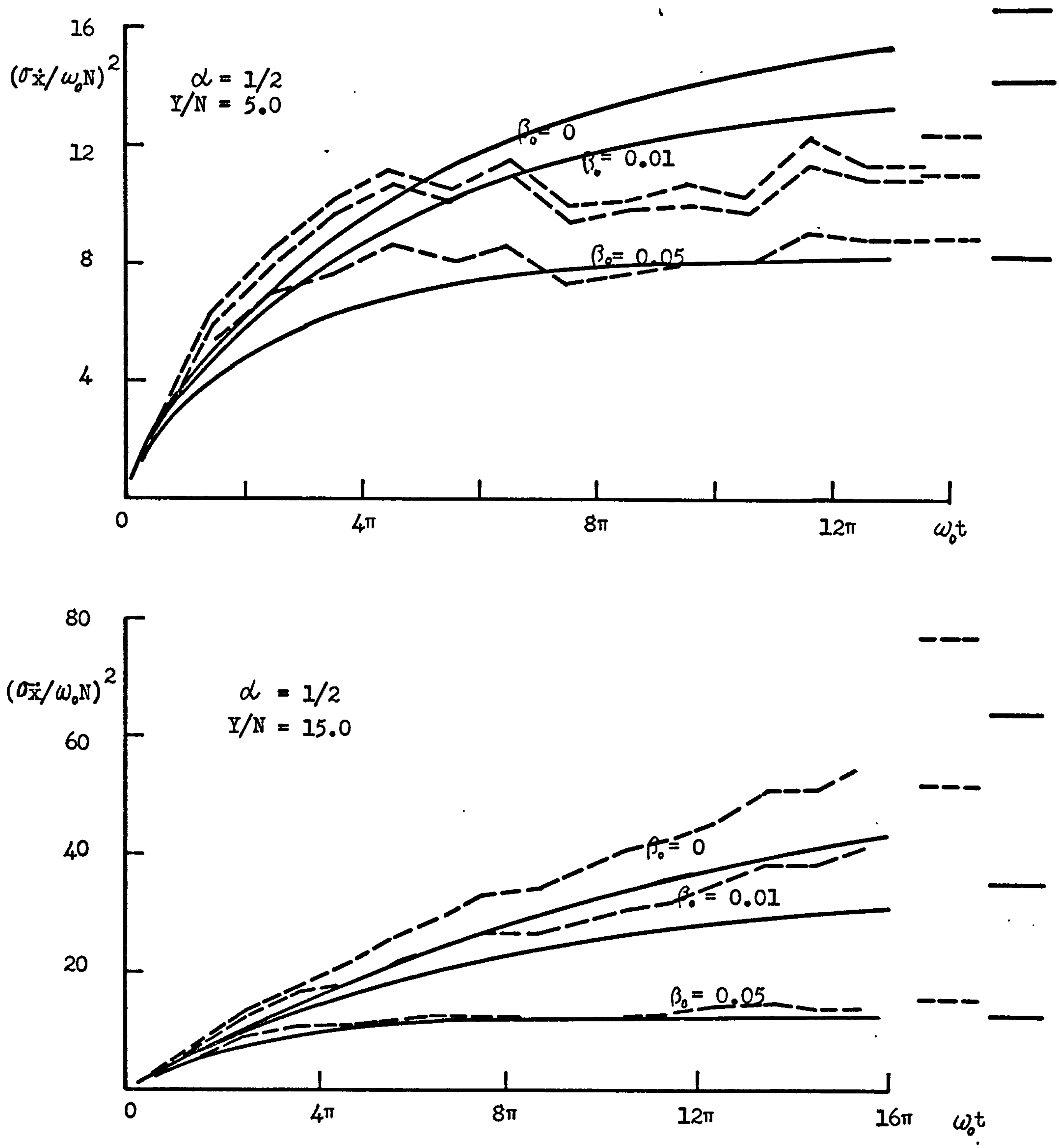


Fig.19.4 Mean-square Velocity of the Two-mode 3rd Order System