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STABILIZATION OF NONLINEAR DYNAMICAL SYSTEMS

by

Chen-yao Ho

A THESIS SUBMITTED
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Thesis Director's signature:

Rui J. P. de Figueiredo

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CHAPTER I

INTRODUCTION

I-1. Introduction to the Research Topics

To design a system capable of a desired performance is one of the main objectives in most engineering problems. One of the common requirements of such a performance is that the system exhibit a stable behavior. The thesis is concerned with the development of design techniques for systems which do not possess stable solutions. More precisely, the thesis consists of two parts. The first part deals with the possibility of designing a control law to stabilize a class of nonlinear systems described by

$$\frac{dx(t)}{dt} = Ax(t) + bu(t) - bf(y(t),t) \quad (1-1)$$

$$y(t) = c^T x(t) \quad (1-2)$$

Where $x(t) = \text{col.}(x_1(t), x_2(t), \dots, x_n(t)) \in R^n$ represents the state of the system at time t , $u(t)$, $y(t)$ are respectively the real valued scalar input to and output from the system, A , b , c are real constant matrices of dimension $n \times n$, $n \times 1$, $n \times 1$ respectively; the superscript T denotes the transpose; and $f : R^2 \rightarrow R^1$ is a real valued nonlinear continuous function with respect to each of its arguments, and satisfies

$$0 \leq f(y(t),t)/y(t) \leq k, \quad \forall t, \forall y \quad (1-3a)$$

$$f(0,t) = 0, \quad \forall t \quad (1-3b)$$

The scheme representing the system (1-1)-(1-3) , and the nonlinear characteristic are shown in Figure 1 and Figure 2 respectively.

Throughout the rest of this thesis, for the sake of notational convenience, we will use the same symbol for functions and their values. The meaning being clear from the context, thus instead of (1-1)-(1-3), we will write

$$\frac{dx}{dt} = Ax + bu - bf(y,t) \quad (1-1a)$$

$$y = c^T x \quad (1-2a)$$

and

$$0 \leq f(y,t)/y \leq k \quad , \quad \forall y \quad , \quad \forall t \quad (1-3c)$$

$$f(0,t) = 0 \quad , \quad \forall t \quad (1-3d)$$

Systems (1-1)-(1-3) arise in control problems, after suitable idealization, when the nonlinearity of one element only is dominant and the remaining elements can be considered to be linear. In such cases, y is the deviation of the controlled element, and $f(y,t)$ is the nonlinearity of the power element.

The analysis of the stability of the system (1-1)-(1-3) has attracted wide attention in past decade [1] - [6], [36] . But the question of the synthesis of a stable system by designing a control law which insures the stability of the controlled system has remained relatively unexplored. Recently, several authors [7] - [12] attacked the problem by using a graphical approach or a mixed graphical and computer aided design approach. This thesis applies modern control theory

techniques to designing a control law to achieve desired performances.

The design techniques are also developed for certain classes of nonlinear systems of the form (1-1)-(1-3) having multiple-nonlinearities regardless of the complexity of pole-zero assignment.

The second part of the thesis investigates the possibility of stabilizing systems described by

$$\frac{dx}{dt} = g(x, u, p) \quad (1-4)$$

Where x , u are defined as before, p is a real parameter vector, which is unknown but satisfies certain constraints; and $g(.,.,.) = \text{col.}(g_1(.,.,.), \dots, g_n(.,.,.))$, $g_i(.,.,.)$ is a continuous function with respect to each of its arguments such that solutions of (1-4) exist and are unique.

The problem considered is to show the existence and the construction of a control law which stabilizes (1-4) for all parameters assuming values in an appropriate region.

This type of problem is of great theoretical and practical interest since in many concrete control systems, some parameters are given with only a certain degree of accuracy. For instance, the characteristics of a vacuum tube or a transistor provided by the manufacturers are given as average values over some set of samples of the product. Similar problems arise in the case of a system with parameters which vary slowly during its operation. Many other important technical questions involve the problems of the above type.

A parameter variation of a special class of nonlinear system has been analyzed by [13]. For large parameter variations of dominant-type

systems, [14] - [15] provide some approximation techniques and a graphical method. We develop the design techniques which are completely analytic and do not resort to a graphical procedure.

Attention has been given to the systems with unknown parameters which are linear time-invariant systems, linear time-varying systems and Luré systems. Different approaches are used for the above systems.

I-2. Outline of the Thesis

In Chapter II, we develop simple methods to stabilize the systems (1-1)-(1-3). The control law of interest is in the form of a linear combination of states, an output of a dynamic compensator. The compensator designed consists of a cascaded compensator and a compensator in the feedback loop. This special configuration can provide a scheme of assigning poles of the controlled systems without influencing the pattern of zeros. Results obtained are under the assumptions of complete controllability and complete observability of the system. In the latter part of this Chapter, we further relax the restriction of the complete observability of the system by requiring that the unobservable part of the system be asymptotically stable.

Chapter III represents an attempt to the generalization of the results obtained in Chapter II to multi-variable systems. As was pointed out in [16], complete pole-zero assignment for a multivariable system can not be done by using linear state feedback. We therefore restrict our attention to two subclasses of systems, i.e. systems with adjoints controls and systems having the property of simple-strong controllability.

We start with Chapter IV by assuming that the system parameters are unknown and satisfy certain constraints. The control law of interest is in the form of linear state feedback. In Chapter IV, attention has been given to the linear multi-variable time-invariant case in which the system

parameters are confined to a compact subset of the parameter space. In Chapter V, we consider the general linear time-varying systems in which the system parameters belong to the set of continuous and bounded functions. And in Chapter VI, a special class of nonlinear systems is considered in which the system parameters are again confined to a compact subset of the parameter space.

In Chapter VII, we conclude the thesis by stating the advantages and the limitations of the approaches developed, and suggest possible future research topics and approaches.

Examples are included to illustrate the results.

CHAPTER II

SIMPLE METHODS FOR STABILIZATION OF NONLINEAR FEEDBACK SYSTEMS

II-1. Introduction

In this chapter, we develop simple methods for designing a stable nonlinear feedback system which is described by (1-1)-(1-3) and is rewritten for handy reference :

$$\frac{dx}{dt} = Ax + bx - bf(y,t) , \quad (2-1)$$

$$y = c^T x , \quad (2-2)$$

where the notation is the same as before, and f is a real valued continuous function with respect to its arguments, and satisfies

$$0 \leq f(y(t),t)/y(t) \leq k , \quad \forall y,t, \quad (2-3a)$$

$$\text{and} \quad f(0,t) = 0 , \quad \forall t, \quad (2-3b)$$

by means of a control law which is either a linear combination of states or the output of a dynamic compensator. The scheme representing the system (2-1), (2-2) and the nonlinear characteristic (2-3) are shown in Figure 1 and Figure 2. The set of functions f satisfying (2-3a) will be said to lie in the sector $[0,k]$.

The absolute stability criterion for the system (2-1)-(2-3) has received wide attention in the past decade [1] , [2] , [17] - [20]. We state the results of [2] for its generality.

Theorem : For the system (2-1)-(2-3) to be absolutely stable in the sector $[0,k]$, it is sufficient that for all $\omega \geq 0$, the following inequality be satisfied

$$\operatorname{Re} G(j\omega) + 1/k \geq 0 \quad (2-4)$$

where $G(j\omega) = c^T(j\omega I - A)^{-1}b$.

Stabilization of the systems (2-1)-(2-3) through graphical methods has been studied in [7] - [10] in which Nyquist-like loci are used. Recently, [11], [12] use a mixed graphic and computer-aided design method to develop algorithms so that the desired compensator can be constructed. The development of algorithms in [11] is based on the maximization of the distance between the locus of $G(j\omega)$ and a vertical line passing through the point $(0, -\frac{1}{k})$ of the $\{ \operatorname{Re} G(j\omega), \operatorname{Im} G(j\omega) \}$ plane. While [12] minimizes the area of the region enclosed by that vertical line and the part of the locus of $G(j\omega)$ lying to the left of the said vertical, while [25] applies the linear observer concept to its design of a nonlinear compensator, the results can not be held for time-varying nonlinearity. It is the purpose of this chapter to use modern control techniques, and to avoid any graphical method for the stabilization, by use of linear components, of the systems (2-1)-(2-3).

It is shown that : 1) under the assumptions of complete controllability and observability of the systems, a control law which is a linear combination of states can always be found such that the controlled system satisfies the absolute stability criterion; 2) under the assumptions of complete controllability and observability of the systems, a compensator which consists of a cascaded part and a feedback part can always be found such that the controlled system will satisfy the absolute stability criterion; 3) results obtained in 1) and 2) are generalized to the systems which are not completely observable.

II-2. Stabilization of Nonlinear Systems by Linear State Feedback

In order to have a meaningful proposed problem, we first establish the complete controllability conditions and complete observability conditions for systems (2-1)-(2-3) :

Assertion II-1 System(2-1) is completely controllable if and only if the system

$$\frac{dx}{dt} = Ax + bu \quad (2-5)$$

is completely controllable.

Proof: If (2-5) is completely controllable, then for every $x(t_0) = x_0$, there exists a measurable function $\hat{u}(t)$ and $\tau > 0$ such that $x(\tau) = 0$. If we apply the control $\bar{u}(t) = \hat{u}(t) + f(y, t)$ to (2-1), we can drive the state x_0 to the origin in time τ , hence (2-1) is completely controllable.

Conversely, if (2-1) is completely controllable with $\bar{u}(t)$ as the desired control, then $\hat{u}(t) = \bar{u}(t) - f(y, t)$ will be the control for (2-5), which drives the same state to origin in same time τ , hence (2-5) is completely controllable. Q.E.D.

Assertion II-2 System (2-5), (2-2) is completely observable if and only if the system (2-1), (2-2) is completely observable.

Proof We first derive the conditions for complete observability of (2-1), (2-2).

Taking first (n-1)th derivative of (2-2), we have

$$\begin{aligned} y &= c^T x, \\ \frac{dy}{dt} &= c^T A x - c^T b f(y, t), \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

$$\frac{d^{n-1}y}{dt^{n-1}} = c^T A^{n-1} x - \sum_{i=1}^{n-1} c^T A^{n-2-i} b d^i f(y,t)/dt^i \quad ,$$

or $y = c^T x$

$$dy/dt + c^T b f(y,t) = c^T A x \quad , \quad (2-6)$$

$$\begin{array}{ccc} \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \end{array}$$

$$\frac{d^{n-1}y}{dt^{n-1}} + \sum_{i=1}^{n-1} c^T A^{n-2-i} b d^i f(y,t)/dt^i = c^T A^{n-1} x$$

Since $y(t)$ is measured over finite time interval, $dy/dt, \dots, d^{n-1}y/dt^{n-1}$ are known functions, so are

$$\frac{d^i f(y,t)}{dt^i} = F_i(y, \dots, d^i y/dt^i, f(y,t), \dots, \frac{\partial^i f(y,t)}{\partial t^i}, \dots, \frac{\partial^i f(y,t)}{\partial y^k \partial t^{i-k}}),$$

$i=0, \dots, n-1$, once $f(.,t)$ is known. Hence in order to determine $x(t)$ uniquely from (2-6), it is both necessary and sufficient that $\text{rank}(c, A^T c, \dots, (A^{n-1})^T c) = n$ which is exactly the necessary and sufficient conditions for complete observability of (2-1), (2-2). This completes the proof of the Assertion. Q.E.D.

Hereafter, we assume that systems (2-1)-(2-3) are completely controllable and completely observable.

To facilitate our design of a control law which is linear state feedback, we first transform the system (2-1), (2-2) into companion form [21], [22] by defining

$$x = K_C z \quad (2-7)$$

Then (2-1), (2-2) become

$$\frac{dz}{dt} = \bar{A}z + \bar{b}u - \bar{b}f(y,t) \quad (2-8)$$

$$y = \bar{c}^T z \quad , \quad (2-9)$$

where

$$\bar{A} = K_C^{-1} A K_C = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ -\bar{a}_1 & . & . & . & . & . & -\bar{a}_n \end{bmatrix} ,$$

$$\bar{b} = K_C^{-1} b = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} , \quad \bar{c}^T = c^T K_C = [\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n] ,$$

The transfer function of (2-1), (2-2), hence (2-8), (2-9) takes the form

$$G(s) = \bar{c}^T (sI - \bar{A})^{-1} \bar{b} = \frac{(\sum_{i=1}^n \bar{c}_i s^{i-1})}{s^n + \sum_{i=1}^n \bar{a}_i s^{i-1}} , \quad (2-10)$$

For this companion form of the system, we note that only the denominator of (2-10) can be influenced by the applied control. We intend here to show that there are values of \bar{a}_i , $i=1, \dots, n$, which will satisfy (2-4) regardless of the values of \bar{c}_i , $i=1, \dots, n$.

Assertion II-3 There exists a positive number λ_0 such that

$$\operatorname{Re} \left(\sum_{i=1}^n \bar{c}_i (j\omega)^{i-1} / (s - \lambda)^n \right) + 1/k \geq 0, \quad \forall \omega \text{ and } \forall \lambda \geq \lambda_0 \quad (2-11)$$

Proof: The numerator of (2-11) can be written as

$$\begin{aligned}
& k \operatorname{Re} \left(\sum_{i=1}^n \bar{c}_i (j\omega)^{i-1} (-j\omega + \lambda)^n + (\omega^2 + \lambda^2)^n \right) \\
= & k \operatorname{Re} \left(\bar{c}_n (-1)^n (j\omega)^{2n-1} + (C_1^n \bar{c}_n (-1)^{n-1} \lambda + \bar{c}_{n-1} (-1)^n) (j\omega)^{2n-2} \right. \\
& + (C_2^n \bar{c}_n (-1)^{n-2} \lambda^2 + C_1^n \bar{c}_{n-1} (-1)^{n-1} \lambda + \bar{c}_{n-2} (-1)^n (j\omega)^{2n-3} + \dots + \\
& + (-C_{n-1}^n \bar{c}_1 \lambda^{n-1} + \bar{c}_2 \lambda^n) (j\omega) + \lambda^n \bar{c}_1) + (\omega^{2n} + C_1^n \lambda^2 \omega^{2n-2} + C_2^n \lambda^4 \omega^{2n-4} + \\
& + \dots + C_{n-2}^n \lambda^{2n-4} \omega^4 + C_{n-1}^n \lambda^{2n-2} \omega^2 + \lambda^{2n}) \\
= & \omega^{2n} + \omega^{2n-2} (C_1^n \lambda^2 + k(j)^{2n-2} (C_1^n \bar{c}_n (-1)^{n-1} \lambda + \bar{c}_{n-1} (-1)^n) \\
& + \omega^{2n-4} (C_2^n \lambda^4 + k(j)^{2n-4} (C_3^n \bar{c}_n (-1)^{n-3} \lambda^3 + C_2^n \bar{c}_{n-1} (-1)^{n-2} \lambda^2 + C_1^n \bar{c}_{n-2} (-1)^{n-1} \lambda \\
& + \bar{c}_{n-3} (-1)^n + \dots + \omega^2 (C_{n-1}^n \lambda^{2n-2} - k C_{n-2}^n \bar{c}_1 \lambda^{n-2} + k C_{n-1}^n \bar{c}_2 \lambda^{n-1} - k \bar{c}_3 \lambda^n) \\
& + \lambda^{2n} + \lambda^n \bar{c}_1 k \quad (2-12)
\end{aligned}$$

Above, C_k^n denotes $n!/k!(n-k)!$.

Note that (2-12) is a polynomial in ω^2 and is greater than zero for all ω if all its coefficients are non-negative. This will be true if

$$\begin{aligned}
& C_1^n \lambda^2 + (C_1^n \bar{c}_n (-1)^{n-1} \lambda + \bar{c}_{n-1} (-1)^n) (j)^{2n-2} k \geq 0 \\
& C_2^n \lambda^4 + (C_3^n \bar{c}_n (-1)^{n-3} \lambda^3 + C_2^n \bar{c}_{n-1} (-1)^{n-2} \lambda^2 + C_1^n \bar{c}_{n-2} (-1)^{n-1} \lambda + \bar{c}_{n-3} (-1)^n) (j)^{2n-4} k \geq 0 \\
& \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
& C_{n-1}^n \lambda^{2n-2} - k C_{n-2}^n \bar{c}_1 \lambda^{n-2} + k C_{n-1}^n \bar{c}_2 \lambda^{n-1} - \bar{c}_3 \lambda^n \geq 0 \\
& \lambda^{2n} + k \lambda^n \bar{c}_1 \geq 0
\end{aligned} \quad (2-13)$$

Each of the inequalities of (2-13) is a polynomial in λ , and the

coefficient of the highest degree in λ is positive. Therefore, for sufficiently large $\lambda \geq 0$, all inequalities of (2-13) are positive, which in turn implies that (2-12) is positive for all w .

Let λ_i be the smallest positive number such that the i th inequality of (2-13) is positive. Then

$$\lambda_0 = \max(\lambda_1, \lambda_2, \dots, \lambda_n) \quad (2-14)$$

is the desired value for the assertion. Q.E.D.

Having obtained the value λ_0 which depends on \bar{c}_i , ($i=1, \dots, n$), we can design the control law as

$$\begin{aligned} u(t) &= \sum_{i=1}^n (\bar{a}_i - c_{i-1}^n \lambda^{n-i+1}) z_i \\ &= \langle \bar{a} - a^*, z \rangle, \quad \lambda \geq \lambda_0 \end{aligned} \quad (2-15)$$

where $a^* = \text{col.}(c_0^n \lambda^n, c_1^n \lambda^{n-1}, \dots, c_{n-1}^n \lambda)$, $\bar{a} = \text{col.}(\bar{a}_1, \dots, \bar{a}_n)$, and $\langle \cdot, \cdot \rangle$ is the inner product in Euclidean space. With respect to the original coordinate, we have

$$u(t) = \langle +a^* - \bar{a}, k_c^{-1} x \rangle \quad (2-16)$$

The design procedures are summarized as

Step 1 : Transform the given system into companion form.

Step 2 : Determine λ_0 from (2-13)

Step 3 : Obtain the control law as in (2-16)

II-3. Example

Consider the system described by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (u - f(y, t))$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad k = 3$$

Step 1 K_c can be found as

$$K_c = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix} \quad \tilde{c}^T = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

Step 2 From (2-13), we require

$$3\lambda^3 + k(6\lambda - 1) \geq 0$$

$$3\lambda^4 + k(-2\lambda^3 + 3\lambda^2 - 3) \geq 0$$

$$\text{and } \lambda^6 + k\lambda^3 \geq 0$$

For $k = 3$, we find $\lambda_0 = 2$. The corresponding \tilde{A} matrix is

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix}$$

Step 3 From (2-16)

$$u(t) = 0.25(7x_1 - 7x_2 - 20x_3)$$

II-4. Stabilization of Nonlinear Systems by Dynamic Compensation

In order to avoid the difficulty of measuring the possible inaccessible states of a system, we will develop a method of stabilizing a nonlinear system through measured outputs in this section. The scheme representing such dynamic compensators is shown in Figure 3, and consists

of the system itself, a cascaded compensator and a feedback compensator which are respectively described by

$$\text{The system : } \frac{dx}{dt} = Ax + by_3 - bf(y_2, t) \quad (2-17)$$

$$y_1 = c^T x \quad (2-18)$$

Cascaded compensator :

$$\frac{dz}{dt} = Dz + ey_1 \quad (2-19)$$

$$y_2 = q^T z \quad (2-20)$$

Feedback compensator :

$$y_3 = \sum_{i=1}^n h_i \frac{d^{i-1} y_2}{dt^{i-1}} \quad (2-21)$$

Without loss of generality, we assume that (A, b) , (D, e) are in companion form. And we assume $q^T = (1, 0, \dots, 0)$. By eliminating y_1 and y_3 , and defining $v = \text{col.}(x, z)$, we have

$$\frac{dv}{dt} = \begin{bmatrix} A & 0 \\ ec^T & D \end{bmatrix} v + \begin{bmatrix} b \\ 0 \end{bmatrix} \left(\sum_{i=1}^n h_i \frac{d^{i-1} y_2}{dt^{i-1}} \right) - \begin{bmatrix} b \\ 0 \end{bmatrix} f(y_2, t) \quad (2-22)$$

$$y_2 = [0, c]^T v \quad (2-23)$$

The transfer function with respect to the input $-f(y, t)$, and the output y_2 obtained from (2-22) and (2-23) is

$$\begin{aligned} G(s) &= [0, c]^T \begin{bmatrix} (sI-A)^{-1} & bc^T \left(\sum_{i=1}^n h_i s^{i-1} \right) \\ -ec^T & (sI-D)^{-1} \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \\ &= \frac{c^T (sI-A)^{-1} bc^T (sI-D)^{-1} e}{1 - c^T (sI-A)^{-1} bc^T (sI-D)^{-1} e \left(\sum_{i=1}^n h_i s^{i-1} \right)} \\ &= \frac{\sum_{i=1}^n c_i s^{i-1}}{(s^n + \sum_{i=1}^n a_i s^{i-1})(s^n + \sum_{i=1}^n d_i s^{i-1}) - \left(\sum_{i=1}^n c_i s^{i-1} \right) \left(\sum_{i=1}^n h_i s^{i-1} \right)}, \quad (2-24) \end{aligned}$$

We note that the numerator of (2-24) is not affected by the determination of compensator parameters, $d_i, h_i, i=1, \dots, n$. To show that the parameters $d_i, h_i, i=1, \dots, n$ can be chosen independently, so that the roots of the denominator of (2-24) can be arbitrarily placed, we assume that the denominator takes the form

$$s^{2n} + \sum_{i=1}^{2n} \alpha_i s^{i-1} \quad (2-25)$$

and compare the corresponding coefficients in s . We have the relation between $\alpha_i, i=1, \dots, 2n$ and $h_i, d_i, i=1, \dots, n$, as

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \alpha_{n+1}^{-a_1} \\ \alpha_{n+2}^{-a_2} \\ \vdots \\ \alpha_{2n}^{-a_n} \end{bmatrix} = \begin{bmatrix} c_1 & 0 & \dots & 0 & a_1 & 0 & \dots & 0 \\ c_2 & c_1 & 0 & \dots & 0 & a_2 & a_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n & c_{n-1} & \dots & \dots & c_1 & a_n & a_{n-1} & \dots & a_1 \\ 0 & c_n & \dots & \dots & c_2 & 1 & a_n & \dots & a_2 \\ 0 & 0 & c_n & \dots & c_3 & 0 & 1 & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & 1 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_n \\ 0 & \dots & \dots & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} -h_1 \\ -h_2 \\ \vdots \\ -h_n \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \quad (2-26)$$

Following [23], we find that $d_i, h_i, i=1, \dots, n$ can be uniquely determined by $\alpha_i, i=1, \dots, 2n$, if and only if

$$\text{rank} [c, cA^T, \dots, c(A^{n-1})^T] = n \quad (2-27)$$

i.e. the system is completely observable which is assumed at the very beginning.

In particular, $\alpha_i, i=1, \dots, 2n$ of (2-25) can be chosen as

$$\alpha_i = c_{i-1}^{2n} \lambda^{2n-i+1}, \quad \lambda \geq 0 \quad (2-28)$$

The characteristic equation of (2-24) becomes $(s+\lambda)^{2n}$. Then from Assertion II-3, a λ_0 can be found such that the whole system satisfies absolute stability criterion. For any $\lambda \geq \lambda_0$, and with (2-28), we can determine the compensator parameter from (2-26). The design procedures are summarized as :

Step 1 Determine λ_0 for

$$\operatorname{Re} \frac{\sum_{i=1}^n c_i s^{i-1}}{(s+\lambda)^{2n}} + 1/k \geq 0 \quad \text{for all } w$$

Step 2 For any $\lambda \geq \lambda_0$, determine $h_i, d_i, i=1, \dots, n$ from λ through (2-26).

II-5. Example

Suppose the system is described by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u-f(y,t)),$$

$$y = (0 \ 1) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad k=2,$$

The cascaded compensator and the feedback compensator are respectively described by

$$\begin{bmatrix} dz_1/dt \\ dz_2/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -d_1 & -d_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_1$$

$$y_2 = [1, 0] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and

$$u = h_1 y_2 + h_2 dy_2/dt$$

Then

$$G(s) = \frac{s}{(s^2 - 5s - 3)(s^2 + d_2 s + d_1) - s(h_1 + h_2 s)}$$

Step 1

$$\operatorname{Re} \frac{j\omega}{(j\omega + \lambda)^4} + 1/k \geq 0 \quad \forall \omega, \text{ if } \lambda \geq \lambda_0 = 2$$

Step 2

From (2-26), we have

$$d_1 = -0.3333 \lambda^4$$

$$d_2 = 5 + 4 \lambda$$

for all $\lambda \geq 2$

$$h_1 = 1.6667 \lambda^4 - 4 \lambda^3 - 12 \lambda - 15$$

$$h_2 = -0.3333 \lambda^4 - 6 \lambda^2 - 20 \lambda - 28$$

Note that the compensator so designed has poles in the right half plane.

II-6. Stabilization of Nonlinear Systems Which Are Not Completely

Observable

The assumption of complete observability for (2-1)-(2-3) is essential to the stabilization of such systems by using the methods considered in previous sections. Occasionally, we have systems that are not completely observable. We intend, in this section to extend the previously developed techniques to this class of systems.

First we introduce a coordinate transformation, which is dual to that for the controllability considered in [24], and we decompose the systems into observable and unobservable parts. Since (A, c) is not an observable pair, $\text{rank } (c, A^T c, \dots, (A^{n-1})^T c) = n_1 < n$. The construction of the transformation matrix can be obtained as: form an $n \times n_1$ matrix $T_1 = (c, A^T c, \dots, (A^{n_1-1})^T c)$ which has linearly independent column vectors. Then we supplement matrix T_1 by an $n \times n_2$ matrix T_2 which is obtained by taking any column vector c_1 with $\text{rank } (T_1, c_1) = n_1 + 1$, and $T_2 = (c_1, A^T c_1, \dots, (A^{n_2-1})^T c_1)$ with $\text{rank } (T_1, T_2) = n_1 + n_2$. Unless $n_1 + n_2 = n$, we have to find c_2 such that $\text{rank } (T_1, T_2, c_2) = n_1 + n_2 + 1$ and obtain $T_3 = (c_2, A^T c_2, \dots, (A^{n_3-1})^T c_2)$. Repeating this process, finally we have

$$K_T = (T_1, T_2, \dots, T_m) \quad \text{and} \quad \sum_{i=1}^m n_i = n \quad (2-29)$$

with $\det(K_T) \neq 0$

Now, we introduce the transformation $z = K_T^T x$ to (2-1)-(2-3) and obtain

$$\frac{dz}{dt} = \hat{A}z + \hat{b}(u - f(y, t)) \quad (2-30)$$

$$y = \hat{c}^T z \quad (2-31)$$

where

$$\hat{A} = K_T^T A (K_T^T)^{-1} = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

with

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{ii}^1 & -a_{ii}^2 & \dots & \dots & -a_{ii}^{n_i} \end{bmatrix} \quad i=1, \dots, m$$

$$A_{ij} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ -a_{ij}^1 & -a_{ij}^2 & \dots & \dots & -a_{ij}^{n_j} \end{bmatrix} \quad \begin{matrix} i, j=1, \dots, m \\ i \neq j \end{matrix}$$

and

$$\hat{c} = (\bar{c}, 0)z$$

where \bar{c} is an $n_1 \times 1$ vector.

Let us partition the states z into observable and unobservable parts

as $z = \text{col.}(z_1, z_2)$, and denote

$$A_1 = A_{11}$$

$$A_2 = \begin{bmatrix} A_{22} & 0 & 0 & \dots & 0 \\ A_{32} & A_{33} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m2} & \dots & \dots & \dots & A_{mm} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} A_{21} \\ A_{31} \\ \vdots \\ A_{m1} \end{bmatrix} \quad \hat{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_2 \end{bmatrix} \begin{matrix} \nearrow n_1 \\ \nwarrow n-n_1 \end{matrix}$$

System (2-30), (2-31) can be written as

$$\frac{dz_1}{dt} = A_1 z_1 + b_1 u - b_1 f(y, t) \quad (2-32)$$

$$\frac{dz_2}{dt} = A_3 z_1 + A_2 z_2 + b_2 u - b_2 f(y, t) \quad (2-33)$$

$$y = c^T z_1 \quad (2-34)$$

Note that the eigenvalues of (2-32), (2-33) are exactly the eigenvalues of

A_1 and A_2 , while A_1 is the system submatrix for the observable part of the system and A_2 is that for the unobservable part. The control input u which only involves observable parts, i.e. z_1 , can in no way affect the eigenvalues of A_2 , and it can affect only the eigenvalues of A_1 . This allows us to make the following assertion.[#]

Assertion II-4 System (2-30), (2-31), hence (2-32)-(2-34) can be stabilized only if the eigenvalues of the unobservable part have negative real parts.

With the assumption of satisfaction of Assertion II-4, the problem of stabilizing (2-32)-(2-34) can be reduced to that of (2-32), (2-34) which have the same forms as (2-1), (2-2) with lower order, and has been considered in previous section. Hence the previously developed design techniques can be directly applied to the unobservable systems once the decomposition of the states has been performed.

[#] Once we have stabilized the system (2-32), (2-34) by any of the above design techniques, we have $\lim_{t \rightarrow \infty} z_1(t) = 0$ which implies $\lim_{t \rightarrow \infty} y(t) = 0$ and $\lim_{t \rightarrow \infty} f(y(t), t) = 0$. By considering z_1 , u , $f(., t)$ as input to (2-33), together with the strict stability of A_2 , we can conclude that $\lim_{t \rightarrow \infty} z_2 = 0$ from the well-known theory of ordinary differential equations.

CHAPTER III

STABILIZATION OF SYSTEMS WITH MULTIPLE NONLINEARITIES

III-1. Introduction

In this Chapter, we attempt to generalize the results of Chapter II to a class of systems containing multiple nonlinearities. Several authors [3] - [6] have studied the absolute stability of such class of systems described by

$$\frac{dx}{dt} = Ax + Bu - BF(y,t) \quad (3-1)$$

$$y = Cx \quad (3-2)$$

where x is a n -vector representing states of the systems, u, y are m -vectors, and A, B, C are respectively $n \times n, n \times m, m \times n$ matrices; and $F(y,t) = \text{col.}(f_1(y_1,t), \dots, f_m(y_m,t))$ is assumed smooth enough to ensure the existence and uniqueness for the solutions of (3-1), (3-2).

In addition, it satisfies

$$0 \leq f_i(y_i,t)/y_i \leq k_i \quad , \quad \text{for all } t, y_i, \quad i=1, \dots, m \quad (3-3a)$$

$$\text{and} \quad f_i(0,t) = 0 \quad \forall t \quad , \quad i=1, \dots, m \quad (3-3b)$$

Above, we use again the same symbol for functions and their values for notational convenience; i.e., x for $x(t)$, y for $y(t)$ etc., as long as the meaning is clear from the context.

Theorem [5] System (3-1)-(3-3) is absolutely stable in the sector $[0, k_1]$, $i=1, \dots, m$, if the hermitian part of

$$C(j\omega - A)^{-1}B + K \quad (3-4)$$

is positive definite for all ω , where $K^{-1} = \text{diag.}(k_1, \dots, k_m)$

It was shown in [16], arbitrary pole-zero placement for multiple-input system can not be done by using linear state feedback. We therefore restrict our attention to two subclasses of the systems of the form (3-1)-(3-3) having multiple nonlinearities, i.e. the systems with adjoint controls [26], and the systems having the property of simple-strong controllability, despite the fact that systems with adjoint control are seldom encountered in a realistic case.

It is shown that a control law which is a linear combination of states always exists such that the controlled system (3-1)-(3-3) will satisfy absolute stability conditions for systems with adjoint controls. A method giving successive steps for constructing a control law for both subclasses is developed.

III-2. Stabilization of Systems with Adjoint Controls

In order to have a meaningful problem, we first establish the complete controllability conditions and complete observability conditions for the system (3-1)-(3-3).

Assertion III-1 The system (3-1)-(3-3) is completely controllable (observable) if and only if the system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx, \quad (3-5)$$

is completely controllable (observable).

Hereafter, we assume that system the (3-1)-(3-3) under consideration is completely controllable and completely observable.

Assume that A has distinct eigenvalues. Let b_k be the k th column of B , and $\{\lambda_{(k)}\}$ be the set of eigenvalues of A which may be influenced by u_k through b_k . The class of multi-input systems considered has the following properties :

$$\bigcup_{k=1}^m \{\lambda_{(k)}\} = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \text{set of eigenvalues of } A, \quad (3-6a)$$

$$\text{and } \{\lambda_{(i)}\} \cap \{\lambda_{(j)}\} = \phi, \quad i \neq j. \quad (3-6b)$$

Condition (3-6a) guarantees that the system is completely controllable, while condition (3-6b) indicates that each b_k influences a different set of eigenvalues of A . The class of systems satisfying (3-6a) and (3-6b) is referred to as systems with adjoint controls.

For this class of systems, we may represent (3-1)-(3-2) in terms of the canonical form for multi-inputs systems [27], [28] as

$$\frac{dx}{dt} = \bar{A} x + \bar{B} u - \bar{B} F(y, t) \quad (3-7)$$

$$y = \bar{C} x \quad (3-8)$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_m & 0 & \dots & \dots & 0 \\ 0 & \bar{A}_{m-1} & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & \dots & \dots & \dots & \bar{A}_1 \end{bmatrix} \quad (3-9)$$

and

$$\bar{B} = \begin{bmatrix} 0 & 0 & & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & \dots & & \vdots & 0 \\ 0 & 0 & & & 0 & 1 \\ \hline 0 & 0 & & & 0 & 0 \\ \vdots & \vdots & \dots & & \vdots & \vdots \\ \vdots & \vdots & \dots & & 0 & \vdots \\ 0 & 0 & & & 1 & 0 \\ \hline \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & \dots & & \vdots & \vdots \\ \vdots & \vdots & \dots & & \vdots & \vdots \\ \hline 0 & 0 & & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & \dots & & \vdots & \vdots \\ \vdots & 0 & \dots & & \vdots & \vdots \\ 0 & 1 & & & 0 & 0 \\ \hline 0 & 0 & & & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & \vdots & & & \vdots & \vdots \\ 1 & 0 & \dots & & 0 & 0 \end{bmatrix}$$

Each \bar{A}_i , $i=1, \dots, m$ is in companion form, corresponding to eigenvalues influenced by u_k . A typical submatrix \bar{A}_1 is given by

$$\bar{A}_i = \begin{pmatrix} 0 & 1 & . & . & . & . & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ -a_{i,1} & . & . & . & . & . & . & . & . & -a_{i,n_i} \end{pmatrix} ,$$

with the corresponding characteristic equation

$$s^{n_i} + \sum_{j=1}^{n_i} a_{i,j} s^{j-1} = 0 .$$

Since the system is completely controllable, we have

$$\sum_{i=1}^m n_i = n .$$

Every eigenvalue of A may be altered by changing the characteristic equation of the companion matrix it is associated with. Due to the special structure of (3-9)

$$(sI - \bar{A})^{-1} = \begin{bmatrix} (sI - \bar{A}_m)^{-1} & 0 & \dots & 0 \\ 0 & (sI - \bar{A}_{m-1})^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & (sI - \bar{A}_1)^{-1} \end{bmatrix},$$

where a typical submatrix $(sI - \bar{A}_i)^{-1}$ is of the form

$$(sI - \bar{A}_i)^{-1} = \frac{1}{\det(sI - \bar{A}_i)} \begin{bmatrix} z & z & \dots & \dots & 1 \\ z & z & \dots & \dots & s \\ \vdots & \vdots & \ddots & \ddots & s^2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z & z & \dots & \dots & s^{n_i-1} \end{bmatrix}, \quad (3-10)$$

where z in matrix (10) denotes those elements in the matrix which are not important in our discussion. The transfer function matrix corresponding to (3-7), (3-8) is

$$G(s) = \bar{C}(sI - \bar{A})^{-1} \bar{B} \quad (3-11)$$

$$= \begin{bmatrix} 0 & 0 & \dots & \frac{1}{\det(sI - \bar{A}_m)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n_m-1} \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{\det(sI - \bar{A}_2)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n_2-1} \end{bmatrix} & \dots & 0 \\ \frac{1}{\det(sI - \bar{A}_1)} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n_1-1} \end{bmatrix} & 0 & \dots & 0 \end{bmatrix}$$

Let elements of \bar{C} be given as

$$\bar{C} = \left[\begin{array}{c|c} \begin{matrix} 1 & & 1 \\ c_{m,0}, \dots, & c_{m,n_m-1} \\ \\ 2 & & 2 \\ c_{m,0}, \dots, & c_{m,n_m-1} \\ \vdots & & \vdots \\ m & & m \\ c_{m,0}, \dots, & c_{m,n_m-1} \end{matrix} & \begin{matrix} 1 & & 1 \\ c_{1,0}, \dots, & c_{1,n_1-1} \\ \\ 2 & & 2 \\ c_{1,0}, \dots, & c_{1,n_1-1} \\ \vdots & & \vdots \\ m & & m \\ c_{1,0}, \dots, & c_{1,n_1-1} \end{matrix} \end{array} \right]$$

Then

$$G(s) = \begin{pmatrix} \frac{\sum_{i=0}^{n_1-1} c_{1,i} s^i}{\det(sI - \bar{A}_1)} & \frac{\sum_{i=0}^{n_2-1} c_{2,i} s^i}{\det(sI - \bar{A}_2)} & \dots & \frac{\sum_{i=0}^{n_m-1} c_{m,i} s^i}{\det(sI - \bar{A}_m)} \\ \frac{\sum_{i=0}^{n_1-1} c_{1,i}^m s^i}{\det(sI - \bar{A}_1)} & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sum_{i=0}^{n_1-1} c_{1,i}^2 s^i}{\det(sI - \bar{A}_1)} & \dots & \dots & \frac{\sum_{i=0}^{n_m-1} c_{m,i}^m s^i}{\det(sI - \bar{A}_m)} \end{pmatrix}$$

$$= \begin{pmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & \dots & \dots & G_{mm}(s) \end{pmatrix},$$

where

$$G_{ij}(s) = \frac{\sum_{k=0}^{n_j-1} c_{j,k}^i s^k}{\det(sI - \bar{A}_j)}, \quad i, j=1, \dots, m.$$

Note that the numerator of $G_{ij}(s)$ depends upon $c_{j,k}^i$, $i, j=1, \dots, m$,

$k=0, \dots, n_j-1$, which are fixed for the given systems, and are not influenced by the assignment of eigenvalues of \bar{A}_j through linear state feedback.

Now we consider the control law of the form

$$u = \begin{bmatrix} -C_0^{n_1} \lambda^{n_1-a_{1,1}}, \dots, -C_{n_1-1}^{n_1} \lambda^{n_1-a_{1,n_1}} & \dots & 0 \\ 0 & -C_0^{n_2} \lambda^{n_2-a_{2,1}}, \dots, -C_{n_2-1}^{n_2} \lambda^{n_2-a_{2,n_2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & -C_0^{n_m} \lambda^{n_m-a_{m,1}}, \dots, -C_{n_m-1}^{n_m} \lambda^{n_m-a_{m,n_m}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (3-12)$$

where $\lambda > 0$ and $C_n^m = \frac{m!}{n!(m-n)!}$

With this applied control law (3-12), the controlled system has a transfer function matrix representation as

$$\hat{G}(s) = \begin{bmatrix} \hat{G}_{11}(s) & \hat{G}_{12}(s) & \dots & \hat{G}_{1m}(s) \\ \hat{G}_{21}(s) & \hat{G}_{22}(s) & \dots & \hat{G}_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{G}_{m1}(s) & \dots & \dots & \hat{G}_{mm}(s) \end{bmatrix}, \quad (3-13)$$

where

$$\hat{G}_{ij}(s) = \frac{\sum_{k=0}^{n_j-1} c_{j,k}^i s^k}{(s+\lambda)^{n_j}}, \quad \lambda > 0 \quad i, j=1, \dots, m. \quad (3-14)$$

Assertion III-2 For any $k_0 > 0$, there is an $\lambda_{ij} > 0$ such that

$$|\hat{G}_{ij}(w)| \leq k_0 \quad \text{for all } \lambda \geq \lambda_{ij}, \quad i, j=1, \dots, m$$

Proof: It is clear that $|\hat{G}_{ij}(w)|$ is bounded. Since it is continuous over $0 \leq w < \infty$ and $\lim_{w \rightarrow 0} |\hat{G}_{ij}(w)| < \infty$, $\lim_{w \rightarrow \infty} |\hat{G}_{ij}(w)| = 0$. To show that it is bounded by any specified value k_0 , we write (3-14) in the form

$$\hat{G}_{ij}(s) = \frac{\sum_{k=0}^{n_j-1} c_{j,k}^i \left(\frac{s}{\lambda}\right)^k}{\lambda^{n_j} \left(\frac{s}{\lambda} + 1\right)^{n_j}},$$

and define $\bar{w} = w/\lambda$

Then

$$\hat{G}_{ij}(w) = \frac{\sum_{k=0}^{n_j-1} c_{j,k}^i (j\bar{w})^k}{(j\bar{w} + 1)^{n_j}}. \quad (3-15)$$

As far as the range of w , $0 \leq w < \infty$, is concerned, there would be no difference in using w or \bar{w} as variable. Hereafter, we replace \bar{w} by w in (3-15). The coefficients of (3-15) are proportional to at least of order $(1/\lambda)$, and hence can be made as small as we wish. This completes the proof of the Assertion. Q.E.D.

For the absolute stability of (3-7), (3-8), by invoking the condition (3-4), we require that the following matrix be positive definite for all $w \geq 0$:

$$\bar{G}(jw) = \begin{bmatrix} 2\operatorname{Re} \hat{G}_{11}(jw) + 2/k_1 & \hat{G}_{12}(jw) + \hat{G}_{21}(jw) & \dots & \hat{G}_{1m}(jw) + \hat{G}_{m1}(jw) \\ \hat{G}_{12}(jw) + \hat{G}_{21}(jw) & 2\operatorname{Re} \hat{G}_{22}(jw) + 2/k_2 & \dots & \hat{G}_{2m}(jw) + \hat{G}_{m2}(jw) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{G}_{m1}(jw) + \hat{G}_{1m}(jw) & \dots & \dots & 2\operatorname{Re} \hat{G}_{mm}(jw) + 2/k_m \end{bmatrix} \quad (3-16)$$

for some suitable $\lambda > 0$.

To this end, we first state:

Definition III-1 An $m \times m$ matrix H with real diagonal elements and complex off-diagonal elements is called a dominant matrix if

$$h_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^m |h_{ij}| > 0 \quad i=1, \dots, m$$

It may happen that h_{ij} is a function of some parameters, we then further have

Definition III-2 An $m \times m$ matrix $H(w)$ with real diagonal elements and complex off-diagonal elements is called a uniformly dominant matrix, if there exists an $\mu > 0$ (independent of w) such that

$$h_{ii}(w) - \sum_{\substack{j=1 \\ j \neq i}}^m |h_{ij}(w)| \geq \mu > 0, \quad \text{for all } w, \quad i=1, \dots, m$$

An immediate consequence of the above definitions is the following

Assertion III-3 If $H(w)$ is hermitian and is uniformly dominant in w , then it is positive definite for all w

Proof Let \bar{z} be any complex $n \times 1$ vector, and consider

$$\begin{aligned} \bar{z}^{*T} H(w) \bar{z} &= \sum_{i=1}^m h_{ii}(w) \bar{z}_i^* \bar{z}_i + \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m h_{ij}(w) \bar{z}_i^* \bar{z}_j \\ &> \sum_{i=1}^m \bar{z}_i^* \bar{z}_i \left(\sum_{\substack{j=1 \\ j \neq i}}^m |h_{ij}(w)| \bar{z}_i^* \bar{z}_j \right) + \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m h_{ij}(w) \bar{z}_i^* \bar{z}_j \\ &\quad + \mu \sum_{i=1}^m \bar{z}_i^* \bar{z}_i \\ &\geq \sum_{i=1}^m |z_i|^2 > 0 \end{aligned} \quad \text{Q.E.D.}$$

Consider the off-diagonal elements of (3-16), and for specified $k_0 > 0$, we have

$$\left| \hat{G}_{ij}(w) + \hat{G}_{ji}(w) \right| \leq \left| \hat{G}_{ij}(w) \right| + \left| \hat{G}_{ji}(w) \right| \leq 2k_0, \quad i \neq j, \quad i, j=1, \dots, m$$

for suitably chosen $\lambda_{ij} > 0$ according to Assertion III-2. The diagonal elements, for suitably chosen $\lambda_{ij} > 0$, leads to

$$\begin{aligned} \frac{2}{k_i} + 2\operatorname{Re} \hat{G}_{ii}(jw) &\geq \frac{2}{k_i} - 2k_0 - 2 \hat{G}_{ii}(jw) + 2k_0 \\ &\geq \frac{2}{k_i} - 2k_0 \end{aligned}$$

Let $k_M = \max.(k_1, \dots, k_m)$, and k_0 can be chosen such that

$$k_0 \leq \frac{1}{mk_M} \quad (3-17)$$

For this k_0 , there are $\bar{\lambda}_{ij}$, $i, j=1, \dots, n$, such that $\left| \hat{G}_{ij}(jw) \right| < k_0$ for all $\lambda_{ij} \geq \bar{\lambda}_{ij}$. And (3-16) is a uniformly dominant matrix, hence is positive definite for all w .

$$\text{Let } \lambda^* = \max.(\bar{\lambda}_{ij}, \quad i, j=1, \dots, m) \quad (3-18)$$

Then the control law has the form (3-12) with $\lambda \geq \lambda^*$.

The design procedure is summarized below :

- Step 1: Transform the given system into canonical form;
Step 2: Specify k_0 according to (3-17);
Step 3: Find $\bar{\lambda}_{ij}$ according to Assertion III-2;
Step 4: Determine λ^* as in (3-18);
Step 5: Obtain control law as in (3-12) with any $\lambda \geq \lambda^*$.

We are now in the position to state the following

Assertion III-4 For the completely controllable and completely observable system of the form (3-1)-(3-3) with adjoint controls, a control law which is linear state feedback can always be found such that the controlled systems will satisfy the absolute stability criterion.

III-3. Example

Let us consider the system described by

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \\ dx_3/dt \\ dx_4/dt \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 - f_1(y_1, t) \\ u_2 - f_2(y_2, t) \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad k_1=1, \quad k_2=2 ;$$

Step 2: Let $k_0 = 0.2 < \frac{1}{4}$;

Steps 3,4:

$$\bar{G}(s) = \begin{bmatrix} 2 \operatorname{Re}\left(\frac{1}{(s+\lambda_{11})^2}\right) + 1/2 & \frac{1}{(s+\lambda_{12})^2} + \frac{-s}{(-s+\lambda_{21})^2} \\ \frac{1}{(-s+\lambda_{12})^2} + \frac{s}{(s+\lambda_{21})^2} & 2 \operatorname{Re}\left(\frac{s}{(s+\lambda_{22})^2}\right) + 1 \end{bmatrix}$$

we require

$$0.2 - \left| \frac{1}{(s+\lambda_{11})^2} \right| \geq 0 ,$$

$$0.2 - \left| \frac{s}{(s+\lambda_{22})^2} \right| \geq 0 ,$$

which are true if

$$\lambda_{11} \geq \bar{\lambda}_{11} = \sqrt{5} ,$$

$$\lambda_{22} \geq \bar{\lambda}_{22} = \sqrt{5} ,$$

similarly, we have

$$\lambda_{12} = \lambda_{21} = \sqrt{5}$$

let $\lambda^* = \sqrt{5}$

Step 5:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -7 & -2\sqrt{5}-3 & 0 & 0 \\ 0 & 0 & -10 & -2\sqrt{5}-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

then the system will satisfy the absolute stability criterion.

III-4. Stabilization of Simple-strongly Controllable Nonlinear Systems

In the last section, we have considered the systems with adjoint controls, in which each control input can only influence some of the system eigenvalues. In this section, we consider a class of systems in which every eigenvalue can be influenced by at least one control. The systems under consideration are assumed to have the property of simple-strong complete controllability which is defined as

Definition III-3 The system (3-1), (3-2) is said to be strongly completely controllable if it is completely controllable by each control separately, i.e. with all others set equal to zero.

Definition III-4 The system (3-1), (3-2) is said to be simple-strongly completely controllable if it is completely controllable by AT LEAST one control separately, with all others set to zero.

The simple-strong complete controllability is something in-between strong complete controllability and complete controllability.

Assume that system (3-1), (3-2) is simple-strongly completely controllable by u_k (if more than one, let u_k be one of them). Then (A, b_k) is a controllable pair, where b_k is the k th column of B . (3-1) can be written as

$$\frac{dx}{dt} = A x + \sum_{i=1}^m b_i u_i - BF(y, t) \quad (3-19)$$

Set $u_i = 0$, $i \neq k$, $i=1, \dots, m$, and transform (3-19) into the companion form

$$\frac{d\bar{x}}{dt} = \bar{A}\bar{x} + \bar{b}u_k - \bar{B}F(y,t), \quad y = \bar{C}\bar{x}, \quad (3-20)$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\bar{a}_1 & -\bar{a}_2 & \dots & \dots & -\bar{a}_n \end{bmatrix}, \quad (3-21)$$

Assume that the control u_k takes the form

$$u_k = - \left(\sum_{i=1}^n C_{i-1}^n \lambda^{n-i-1} -\bar{a}_i \right) \bar{x}_i, \quad \lambda > 0 \quad (3-22)$$

Then the transfer function matrix between the output y and input $-F(y,t)$ has a characteristic equation $(s+\lambda)^n$, and is denoted by

$$\begin{aligned} \bar{G}(s) &= \bar{C}(sI - \bar{A})^{-1} \bar{B} \\ &= \frac{1}{(s+\lambda)^n} \begin{bmatrix} N_{11}(s) & N_{12}(s) & \dots & N_{1m}(s) \\ N_{21}(s) & N_{22}(s) & \dots & N_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ N_{m1}(s) & \dots & \dots & N_{mm}(s) \end{bmatrix} \end{aligned}$$

where $N_{ij}(s)$ is the numerator of $\bar{G}_{ij}(s)$, and is a polynomial in s of degree at most $(n-1)$. And $N_{ij}(s)$ may contain λ of degree at most n . From the same argument of previous section, there is a $\bar{\lambda}_{ij} > 0$ such that

$$\left| \frac{N_{ij}(j\omega)}{(j\omega + \lambda)^n} \right| \leq k_{ij}^0 \quad \text{for all } \lambda \geq \bar{\lambda}_{ij} \quad (3-23)$$

where k_{ij}^0 can not be specified beforehand, since $\bar{G}_{ij}(s)$ may contain λ of the same degree in both numerator and denominator.

For the absolute stability of (3-1), (3-2), we invoke condition (3-4) and require $\text{Re } \bar{G}(s) + K$ be positive definite for all ω . Following the same lines as in the previous section, we further require

$$\sum_{j=1}^m k_{ij}^0 < \frac{1}{k_i} \quad , \quad i=1, \dots, m \quad (3-24)$$

Let

$$\lambda^* = \max.(\bar{\lambda}_{ij}, i, j=1, \dots, n) \quad (3-25)$$

Then the control law u_k takes the form (3-22) with $\lambda \geq \lambda^*$.

The design procedures are summarized as

Step 1: Choose u_k such that (A, b_k) is a controllable pair. Set

$$u_i = 0, i \neq k;$$

Step 2: Transform (A, b_k) into companion form;

Step 3: Find k_{ij}^0 from (3-23);

Step 4: Repeat step 3, if (3-24) fails to hold, or show that non-existence of such $\bar{\lambda}_{ij}$, $i, j=1, \dots, n$;

Step 5: Determine $\bar{\lambda}_{ij}$, hence λ^* ;

Step 6: Obtain control law u_k as in (3-22) with $\lambda \geq \lambda^*$.

Remark: The design procedures developed for systems which are simple-strongly completely controllable may fail in some cases because the values of k_{ij}^0 , $i, j=1, \dots, n$, can not be specified beforehand. But this particular structure of the characteristic equation for the chosen control type (3-22) will give a quick indication for the non-existence of such a λ^* for the design scheme.

CHAPTER IV
STABILIZATION OF MULTI-INPUT CONTROLLABLE LINEAR SYSTEMS
WITH UNKNOWN PARAMETERS

IV-1. Introduction

The stabilization of multi-input controllable linear systems with fixed system parameters has received wide attention in recent research in system theory [29] - [31]. Various techniques are developed to obtain the control law which stabilizes the systems and in the meantime achieves an optimality with respect to certain performance index. However, in many of the real-world control systems, some parameters are given with only a certain degree of accuracy. Similar problems arise in the case where the system parameters vary slowly or drift from the operating values. For these cases, it would be no longer valid to stabilize the systems with respect to a particular parameter value, or even with respect to the neighborhood of a particular parameter value. The parameters are initially unknown. The only constraint on parameters is that they belong to a compact set. We will show the existence of linear state feedback control law which stabilizes the system with possible parameter values in a compact set.

This type of problem has been studied in [14] - [15], in which the dominant type systems of scalar case are considered. The results are limited and depend on the determination of the stability region.

We start with the systems with parameters assuming two different values. Then the obtained results are generalized to the case where the system parameters assume values in a compact subset of the parameter space.

IV-2. Stabilization of Multi-input Controllable Linear System with
Parameters Assuming Two Different Values

The systems under consideration are described by

$$\frac{dx}{dt} = Ax + Bu \quad (4-1)$$

where the state x is a n vector, the control u is an m vector, and A, B are constant matrices of compatible order.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & & A_{2m} \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots \\ A_{m1} & \dots & \dots & A_{mm} \end{bmatrix} \quad (4-2a)$$

where

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ -a_{ii}^1 & -a_{ii}^2 & \dots & & -a_{ii}^{n_i} \end{bmatrix} \quad i=1, \dots, m$$

and

$$A_{ij} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ -a_{ij}^1 & -a_{ij}^2 & \dots & & -a_{ij}^{n_j} \end{bmatrix} \quad \begin{matrix} i, j = 1, \dots, m \\ i \neq j \end{matrix}$$

We assume that the parameters of A, B, assume two possible values $\Omega_1, \Omega_2 \in C_p$, where $\Omega_1 = \left\{ \hat{a}_{ij}^k, i, j=1, \dots, m, \hat{b}_i, i=1, \dots, m \right\}_{k=1, \dots, n_j}$ and $\Omega_2 = \left\{ \hat{a}_{ij}^k, i, j=1, \dots, m, \hat{b}_i, i=1, \dots, m \right\}_{k=1, \dots, n_j}$.

Mathematically, systems with parameters assuming two possible values are equivalent to two different systems, which are described respectively by

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \cdot & 0 & \vdots & & & \vdots \\ \vdots & & & & \vdots & \cdot & \cdot & \vdots & 0 & \dots & \dots & 0 \\ \vdots & & & & \vdots & \cdot & \cdot & \vdots & 0 & \dots & \dots & 0 \\ -\hat{a}_{11}^1 & -\hat{a}_{11}^2 & \dots & -\hat{a}_{11}^{n_1} & -\hat{a}_{12}^1 & -\hat{a}_{12}^2 & \dots & -\hat{a}_{12}^{n_2} & -\hat{a}_{1m}^1 & -\hat{a}_{1m}^2 & \dots & -\hat{a}_{1m}^{n_m} \\ 0 & \dots & \dots & 0 & 0 & 1 & \dots & 0 & 0 & \dots & \dots & 0 \\ \vdots & & & \vdots & 0 & 0 & 1 & \dots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & & \vdots & & & \vdots \\ 0 & & & 0 & \vdots & & & & 0 & & & \vdots \\ \vdots & & & \vdots & \vdots & & & & \vdots & & & \vdots \\ -\hat{a}_{21}^1 & -\hat{a}_{21}^2 & \dots & -\hat{a}_{21}^{n_1} & -\hat{a}_{22}^1 & -\hat{a}_{22}^2 & \dots & -\hat{a}_{22}^{n_2} & -\hat{a}_{2m}^1 & -\hat{a}_{2m}^2 & \dots & -\hat{a}_{2m}^{n_m} \\ \vdots & & & \vdots & \vdots & & & & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & & \vdots & & & \vdots \\ 0 & \dots & \dots & 0 & \vdots & & & & 0 & 1 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & & & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & & \vdots & 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & 0 & \vdots & & & & \vdots & & & \vdots \\ -\hat{a}_{m1}^1 & \dots & -\hat{a}_{m1}^{n_1} & -\hat{a}_{m2}^1 & \dots & -\hat{a}_{m2}^{n_2} & \dots & -\hat{a}_{mm}^1 & \dots & -\hat{a}_{mm}^{n_m} & \dots & -\hat{a}_{mm}^{n_m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hat{b}_1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \hat{b}_2 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \vdots \\ \vdots \\ \hat{b}_m \end{bmatrix} u \quad (4-3a)$$

$$H = \begin{bmatrix} \hat{b}_1 & 0 & \dots & \dots & 0 \\ 0 & \hat{b}_2 & & & \vdots \\ \vdots & & \hat{b}_3 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \hat{b}_m \end{bmatrix},$$

and

$$T = \begin{bmatrix} \hat{a}_{11}^1 & \dots & \hat{a}_{11}^{n_1} & \hat{a}_{12}^1 & \dots & \hat{a}_{12}^{n_2} & \dots & \hat{a}_{1m}^1 & \dots & \hat{a}_{1m}^{n_m} \\ \hat{a}_{21}^1 & \dots & \hat{a}_{21}^{n_1} & \hat{a}_{22}^1 & \dots & \hat{a}_{22}^{n_2} & \dots & \hat{a}_{2m}^1 & \dots & \hat{a}_{2m}^{n_m} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \hat{a}_{m1}^1 & \dots & \hat{a}_{m1}^{n_1} & \hat{a}_{m2}^1 & \dots & \hat{a}_{m2}^{n_2} & \dots & \hat{a}_{mm}^1 & \dots & \hat{a}_{mm}^{n_m} \end{bmatrix}.$$

Then (4-3a), and (4-3b) become respectively

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 \\ & & & 0 & \dots & 0 \\ & & & & 0 & 1 & 0 & \dots & 0 \\ & 0 & & 0 & 0 & 1 & \dots & 0 & \dots & 0 \\ & & & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ & & & 0 & 0 & \dots & \dots & 0 & & \vdots \\ & & & & & & & 0 & 1 & 0 & \dots & 0 \\ & & & & & & & 0 & 0 & 1 & \dots & 0 \\ & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & 0 & & & & & & & & & 0 & 0 & 1 & \dots & 0 \\ & & & & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & & & & & & 0 & 0 & \dots & \dots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} v, \quad (4-5a)$$

Define $\eta_i = \frac{\hat{b}_i}{\hat{b}_i}$, $i=1, \dots, m$, and assume $\eta_i \geq 0.5 + \xi_i$, where

ξ_i is an arbitrarily small number, $i=1, \dots, m$. Note that we have options to choose as to which system becomes (4-5a) or (4-5b) simply by interchanging the systems (4-3a) and (4-3b) at the very beginning.

To design a single control law v to stabilize both (4-5a) and (4-5b), we consider the control v of the form

$$v = \begin{bmatrix} -c_0^{n_1} \lambda^{n_1} & -c_1^{n_1} \lambda^{n_1-1} & \dots & -c_{n_1-1}^{n_1} \lambda & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & -c_0^{n_2} \lambda^{n_2} & \dots & -c_{n_2-1}^{n_2} \lambda & \dots & 0 \\ \vdots & & & \vdots & & \vdots & & \\ \vdots & & & \vdots & & \vdots & & \\ \vdots & & & \vdots & & \vdots & & \\ \vdots & & & \vdots & & \vdots & & \\ 0 & \dots & 0 & \dots & -c_0^{n_m} \lambda^{n_m} & -c_1^{n_m} \lambda^{n_m-1} & \dots & -c_{n_m-1}^{n_m} \lambda \end{bmatrix} x \quad (4-6)$$

$$= \hat{T} x$$

for some $\lambda \geq 0$. With (4-6), (4-5a) has a characteristic equation

$$\hat{G}(s) = \prod_{i=1}^m (s + \lambda)^{n_i} = (s + \lambda)^{\sum_{i=1}^m n_i} = (s + \lambda)^n, \quad (4-7)$$

which is a Hurwitz polynomial for any $\lambda \geq 0$. In order to show that the control v of the form (4-6) can stabilize (4-5b) for some $\lambda \geq 0$, we

first prove the following Assertion.

Assertion IV-1

$$\prod_{i=1}^m (s^{n_i} + \eta_i \sum_{j=1}^{n_i} c_{j-1}^{n_i} \lambda^{n_i-j+1} s^{j-1}) \quad \text{is a Hurwitz polynomial}$$

if $\eta_i \geq 0.5 + \varepsilon_i$, ε_i is an arbitrarily positive number, $i=1, \dots, m$ and $\lambda \geq 0$.

Proof Since $\prod_{i=1}^m (s^{n_i} + \eta_i \sum_{j=1}^{n_i} c_{j-1}^{n_i} \lambda^{n_i-j+1} s^{j-1})$ is a product of m

polynomials in s , and it is a Hurwitz polynomial if each polynomial

$(s^{n_i} + \eta_i \sum_{j=1}^{n_i} c_{j-1}^{n_i} \lambda^{n_i-j+1} s^{j-1})$, $i=1, \dots, m$ is a Hurwitz polynomial.

Consider the i th polynomial, and write it in a convenient form

$$\begin{aligned} & s^{n_i} + \eta_i \sum_{j=1}^{n_i} c_{j-1}^{n_i} \lambda^{n_i-j+1} s^{j-1} \\ &= \eta_i (s + \lambda)^{n_i} + (1 - \eta_i) s^{n_i} \end{aligned}$$

$$\text{Denote } \hat{H}_i(s) = \eta_i (s + \lambda)^{n_i} \quad (4-8)$$

$$\text{and } \hat{\hat{H}}_i(s) = (1 - \eta_i) s^{n_i} \quad (4-9)$$

and consider the contour Γ as shown in Figure 4.

Along Γ_1 , $|\hat{H}_1(jw)|^2 = \eta_1^2 (w^2 + \lambda^2)^{n_1}$, and $|\hat{\hat{H}}_1(jw)|^2 = (1 - \eta_1)^2 w^{2n_1}$,

we have $|\hat{H}_1(jw)| > |\hat{\hat{H}}_1(jw)|$ if $\eta_i \geq 0.5 + \epsilon_i$, $i=1, \dots, m$.

Along Γ_2 , it is obvious that $|\hat{H}_1(jw)| \geq |\hat{\hat{H}}_1(jw)|$.

Since the conditions of Rouché's theorem [33] are satisfied, we conclude that $\hat{H}_1(s)$ and $\hat{H}_1(s) + \hat{\hat{H}}_1(s)$ have the same number of zeros within Γ . Since all zeros of $\hat{H}_1(s)$ are located outside Γ , it must be true that all zeros of $\hat{H}_1(s) + \hat{\hat{H}}_1(s)$ are located outside Γ . This completes the proof of the Assertion. Q.E.D.

An immediate consequence of the Assertion VI-1 is that, for $\lambda = 1$

$$\Delta_\ell = \begin{bmatrix} \sum_{i=1}^m \eta_i c_1^{n_i} & \sum_{i=1}^m \eta_i c_3^{n_i} + \sum_{i=1}^m \sum_{j=1}^m \eta_i \eta_j c_2^{n_i} c_4^{n_j} & \dots & \dots & \dots \\ 1 & \sum_{i=1}^m \eta_i c_2^{n_i} + \sum_{i=1}^m \sum_{j=1}^m \eta_i \eta_j c_1^{n_i} c_3^{n_j} & \dots & \dots & \dots \\ & & \sum_{i=1}^m \eta_i c_1^{n_i} & & \dots \\ 0 & & & 1 & \dots \\ \vdots & & & & \ddots \\ \vdots & & & & & \ddots \\ \vdots & & & & & & \ddots \\ 0 & & & & & & & 0 \end{bmatrix} \geq 0 \quad (4-10a)$$

$\ell=1, \dots, n-1$

and

$$\Delta_n = \eta_1 \eta_2 \dots \eta_m \geq 0. \quad (4-10b)$$

Substituting (4-6) into (4-5b), and letting $\alpha_{ij}^k = \eta_i \hat{a}_{ij}^k - \hat{a}_{ij}^k$, $i, j=1, \dots, m$, $k=1, \dots, n_j$, we have

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -C_0^{n_1} \lambda^{n_1} + \alpha_{11}^1 \dots - C_{n_1-1}^{n_1} \lambda + \alpha_{11}^{n_1} & \alpha_{12}^1 & \alpha_{12}^2 & \dots & \alpha_{12}^{n_2} & \dots & \alpha_{1m}^1 & \alpha_{1m}^2 & \dots & \alpha_{1m}^{n_m} \\ & 0 & 1 & 0 & \dots & 0 & & \vdots \\ & 0 & 0 & 1 & \dots & 0 & \dots & \vdots \\ & \vdots & & & & & & \vdots \\ \alpha_{21}^1 & \alpha_{21}^2 & \dots & \alpha_{21}^{n_1} & -C_0^{n_2} \lambda^{n_2} + \alpha_{22}^1 \dots - C_{n_2-1}^{n_2} \lambda + \alpha_{22}^{n_2} & \dots & \alpha_{2m}^1 & \alpha_{2m}^2 & \dots & \alpha_{2m}^{n_m} \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & & 0 & 1 & 0 & \dots \\ \vdots & \vdots & & \vdots & & & 0 & 0 & 1 & \dots \\ \alpha_{m1}^1 & \alpha_{m1}^2 & \dots & \alpha_{m1}^{n_1} & \dots & \dots & -C_0^{n_m} \lambda^{n_m} + \alpha_{mm}^1 \dots - C_{n_m-1}^{n_m} \lambda + \alpha_{mm}^{n_m} \end{bmatrix}$$

(4-11)

$$= \tilde{A} x.$$

Expanding $\det(sI - \tilde{A})$, we have a polynomial in s

$$\hat{G}(s) = s^n + \sum_{i=1}^n \tilde{a}_{n-i+1} s^{i-1}, \quad (4-12)$$

which when subjected to the stability test of Routh-Hurwitz criterion takes the form

$$\bar{\Delta}_l = \begin{bmatrix} \tilde{a}_1 & \tilde{a}_3 & \cdot & \cdot & \cdot & \tilde{a}_{2l-1} \\ 1 & \tilde{a}_2 & \cdot & \cdot & \cdot & \tilde{a}_{2l-2} \\ 0 & \tilde{a}_1 & \cdot & & & \cdot \\ 0 & 1 & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \tilde{a}_l \end{bmatrix} \quad l = 1, \dots, n-1 \quad (4-13a)$$

and

$$\bar{\Delta}_n = \tilde{a}_n \quad . \quad (4-13b)$$

Expansion of the determinant of (4-13) yields

$$\bar{\Delta}_l = \gamma_{l,0} \lambda^{\frac{l(l+1)}{2}} + \gamma_{l,1} \lambda^{\frac{l(l+1)}{2}-1} + \dots + \gamma_{l,\frac{l(l+1)}{2}} \quad , l = 1, \dots, n, \quad (4-14)$$

where

$$\gamma_{l,0} = \Delta_l \quad .$$

The right hand sides of (4-14) are polynomials in λ of decreasing degree. The coefficients of the highest powers of λ are all positive,

since we have shown in (4-10) that $\Delta_\ell, \ell=1, \dots, n$ are positive if $\eta_i = 0.5 + \varepsilon_i, i=1, \dots, m$. Hence, there exists a $\bar{\lambda}_\ell > 0$ such that $\bar{\Delta}_\ell > 0$ for all $\lambda \geq \bar{\lambda}_\ell$.

let
$$\lambda^* = \max.(\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n) \quad (4-15)$$

Then control $v = \hat{T}x$ with any $\lambda \geq \lambda^*$. Hence $u = H^{-1}(\hat{T}+T)x$ with $\lambda \geq \lambda^*$ will stabilize both systems.

Assertion IV-2 For the multi-input controllable linear system (4-1) of the form (4-2) with parameters assuming two possible different values, there always exists a control law which is in the form of linear state feedback and which can stabilize the system having either parameter value, if

$$\eta_i \geq 0.5 + \varepsilon_i, \quad i=1, \dots, m. \quad (4-16)$$

Assertion IV can be easily generalized to the systems with parameters assuming any finite number of possible values. These results can be applied to systems containing a finite number of on-off control switches.

Assertion IV-3 For the multi-input controllable linear system (4-1) of the form (4-2) with parameters assuming any finite number of possible values, there always exists a control law which is in the form of linear state feedback and which can stabilize the system having any of the

parameter values, if for any possible combination of different parameter pair, we have

$$\eta_i^l \geq 0.5 + \varepsilon_i^l \quad \begin{matrix} i=1, \dots, m \\ l=1, \dots, \frac{\bar{N}(\bar{N}+1)}{2} \end{matrix} \quad (4-17)$$

where \bar{N} is the number of possible parameter values.

Proof For any two possible parameter values, we can find a λ_l^* , so that $u = H^{-1}(\hat{T}+T)x$ with $\lambda \geq \lambda_l^*$ will stabilize the system assuming either parameter value. Let them run through all possible pair of parameter values and take

$$\lambda^* = \max.(\lambda_1^*, \dots, \lambda_{\frac{\bar{N}(\bar{N}-1)}{2}}^*) \quad . \quad (4-18)$$

Then $u = H^{-1}(\hat{T}+T)x$ with $\lambda \geq \lambda^*$ is the desired control. Q.E.D.

IV-3. Stabilization of Multi-Input Controllable Linear Systems with Parameters Assuming Values in a Compact Subset of the Parameter Space

The results of IV-2 are now generalized to systems with parameters assuming values in a compact subset in the parameter space. The problem considered in this section is more difficult than that of previous section, because there is now an infinite set of Hurwitz inequalities.

Assertion IV-4 For the multi-input controllable linear systems (4-1) of the form (4-2) with parameters assuming values in C_p , there always exists

a control law which is linear state feedback and which can stabilize the system having any parameter values in C_p if $b_{im} \geq \varepsilon > 0$ or $b_{iM} \leq -\varepsilon < 0$ in C_p , where ε is an arbitrarily small number.

Proof Let us consider a particular element $\Omega_1 \in C_p$, having the specific values for b_i

$$\Omega_1 = \left\{ \begin{matrix} a_{ij}^{*k} \in A_{ij}^k, & i, j=1, \dots, m, & k=1, \dots, n_j; & b_i^* = b_{im}, & b_{im} > 0 \\ & & & b_{iM}, & b_{iM} < 0 \end{matrix} \right\}_{i=1, \dots, m}$$

and any other element $\Omega \in C_p$.

For this pair of parameter values, we have

$$\eta_i = \frac{b_i}{b_i^*} \geq 1 > 0.5 + \varepsilon_i, \quad i=1, \dots, m.$$

Then from Assertion IV-2, we have shown the existence of a $\lambda^* \geq 0$ and of a control law $u = H^{-1}(\hat{T}+T)x$ with $\lambda \geq \lambda^*$, which stabilizes the system having either of the parameter values. It is obvious that λ^* is dependent upon the choice of $\Omega \in C_p$. Since the determination of $\lambda^*(\Omega)$ involves only continuous operations, the dependence of λ^* on $\Omega \in C_p$ is continuous. Since C_p is compact by assumption, the existence of a $\bar{\lambda}^*$ such that the control law $u = H^{-1}(\hat{T}+T)x$ with $\lambda \geq \bar{\lambda}^*$ which stabilizes the systems having any of the parameter values in C_p is guaranteed, where

$$\bar{\lambda}^* = \sup_{\Omega \in C_p} (\lambda^*(\Omega))$$

This completes the proof of the Assertion. Q.E.D.

CHAPTER V

STABILIZATION OF LINEAR TIME-VARYING SYSTEMS WITH UNKNOWN PARAMETERS

V-1. Introduction

In Chapter IV, we have considered the problem of the stabilization of multi-input controllable linear systems. This chapter extends the results of Chapter IV to linear time-varying systems. Results obtained in Chapter IV are heavily dependent on the use of complex variable theory which is not valid for linear time-varying systems. A different method of analysis is developed for this case.

The time-varying coefficients of the system are initially unknown, but they belong to the set of continuous and bounded real valued functions. We will show the existence and the construction of a control law which is linear in state feedback and which stabilizes the system with all possible time-varying coefficient in the set. The approach used is similar to that of Chapter IV.

V-2. Stabilization of Linear Time-Varying Systems with Parameters

Assuming Two Different Time Functions

The linear time-varying systems considered are described by

$$\frac{dx(t)}{dt} = A(t)x(t) + b(t)u(t) \quad x(t_0) = x_0 \quad (5-1)$$

where state $x(t)$ is n -vector, control $u(t)$ is a scalar, and

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ -a_1(t) & -a_2(t) & \dots & \dots & -a_n(t) \end{bmatrix},$$

$$b(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0(t) \end{bmatrix},$$

where $a_i(t)$, $i=1, \dots, n$, $b_0(t)$ are bounded and continuous functions for $t \geq t_0$. Let

$$\mathcal{F}_M = \left\{ f(t) : \begin{array}{l} |f(t)| < M, \text{ and } f(t) \text{ is continuous for} \\ t \geq t_0 \end{array} \right\} \quad (5-2)$$

and

$$\mathcal{F} = \left\{ f(t) : \begin{array}{l} f(t) \text{ is bounded and is continuous for} \\ t \geq t_0 \end{array} \right\} \quad (5-3)$$

Obviously, $\mathcal{F}_M \subset \mathcal{F}$

The problem considered is to design a single control law of the form

$$u(t) = \sum_{i=1}^n q_i(t) x_i(t) = q^T(t) x(t) \quad (5-4)$$

where $q_i(t) \in \mathcal{F}$, $i=1, \dots, n$ and $q(t) = \text{col.}(q_1(t), \dots, q_n(t))$, such that the controlled system

$$\frac{dx(t)}{dt} = (A(t) + b(t) q^T(t)) x(t)$$

has asymptotically stable solutions for all $a_i(t)$, $i=1, \dots, n$, $b_0(t) \in \mathcal{F}_M$.

The above problem has the same important implications as for time-invariant systems. We assume that $|b_0(t)| \geq \varepsilon > 0$ or

$b_0(t) \leq -\varepsilon < 0$ for $t \geq t_0$, and that the parameters of $A(t)$, $b(t)$ assume two possible sets Ω_1 , Ω_2 of time functions in \mathcal{F}_M , i.e.

$$\Omega_1 = \left\{ \hat{a}_i(t), i=1, \dots, n, \hat{b}_0(t), \hat{a}_i(t), \hat{b}_0(t) \in \mathcal{F}_M, i=1, \dots, n \right\}$$

$$\text{and } \Omega_2 = \left\{ \hat{\hat{a}}_i(t), i=1, \dots, n, \hat{\hat{b}}_0(t), \hat{\hat{a}}_i(t), i=1, \dots, n, b_0(t) \in \mathcal{F}_M \right\}.$$

Mathematically, we have two different systems s_1 and s_2 described respectively by

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ -\hat{a}_1(t) & -\hat{a}_2(t) & \dots & \dots & -\hat{a}_n(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \hat{b}_0(t) \end{bmatrix} u(t) \quad (5-5a)$$

and

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \hat{\hat{a}}_1(t) & \hat{\hat{a}}_2(t) & \dots & \dots & \hat{\hat{a}}_n(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \hat{\hat{b}}_0(t) \end{bmatrix} u(t) \quad (5-5b)$$

To simplify the derivation, we make the change of control $u(t)$ by defining

$$\hat{b}_0(t) u(t) = v(t) + \sum_{i=1}^n \hat{a}_i(t) x_i(t) \quad (5-6)$$

Substituting (5-6), into (5-5), we have (5-5a), (5-5b) become respectively

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v(t) \quad (5-7a)$$

and

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & 1 \\ -\hat{a}_1(t) - \frac{\hat{b}_0(t)}{\hat{b}_0(t)} \hat{a}_1(t) & -\hat{a}_2(t) - \frac{\hat{b}_0(t)}{\hat{b}_0(t)} \hat{a}_2(t) & \dots & -\hat{a}_n(t) - \frac{\hat{b}_0(t)}{\hat{b}_0(t)} \hat{a}_n(t) \end{bmatrix} x$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\hat{b}_0(t)}{\hat{b}_0(t)} \end{bmatrix} v(t) \quad (5-7b)$$

We consider the control $v(t)$ of the form

$$v(t) = - \sum_{i=1}^n C_{i-1}^n \lambda^{n-i+1} x_i(t) \quad (5-8)$$

where $C_i^n = \frac{n!}{i!(n-i)!}$, and λ is a positive constant to be determined.

Expressing $u(t)$ in terms of $x(t)$, we have

$$u(t) = \frac{1}{\hat{b}_0(t)} \sum_{i=1}^n (\hat{a}_i(t) - C_{i-1}^n \lambda^{n-i+1}) x_i(t) \quad (5-9)$$

Substituting (5-8) into (5-7a), we obtain a linear time-invariant system, which has a characteristic equation of the form $\hat{G}(s) = (s+\lambda)^n$ and which is asymptotically stable for any $\lambda > 0$.

To show that the control $v(t)$ of the form (5-8) can stabilize (5-7b) for some $\lambda > 0$, we first prove the following Assertion

Assertion V-1 The matrix \bar{A} , where

$$\bar{A} = \begin{bmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ -\frac{\hat{b}_0(t)}{\hat{b}_0(t)} C_0^n \lambda^n & -\frac{\hat{b}_0(t)}{\hat{b}_0(t)} C_1^n \lambda^{n-1} & \dots & \dots & -\frac{\hat{b}_0(t)}{\hat{b}_0(t)} C_{n-1}^n \lambda \end{bmatrix} \quad (5-10)$$

is such that the real parts of all its eigenvalues are negative,
if

$$\begin{aligned} (1) \quad & \lambda > 0 \\ (2) \quad & \beta(t) = \frac{\hat{b}_0(t)}{\hat{b}_0(t)} \geq 1 \quad \forall t \geq t_0 \end{aligned}$$

Proof: $\det(\mu(t) I - \bar{A})$ can be expanded as

$$\begin{aligned} & \mu^n(t) + \beta(t) c_1^n \lambda \mu^{n-1}(t) + \beta(t) c_2^n \lambda^2 \mu^{n-2}(t) + \dots + \beta(t) \lambda^n \\ &= \beta(t) \left[(\mu(t) + \lambda)^n - \left(1 - \frac{1}{\beta(t)}\right) \mu^n(t) \right] \end{aligned}$$

The eigenvalues of \bar{A} are exactly the roots of the equation

$$\left[(\mu(t) + \lambda)^n - \left(1 - \frac{1}{\beta(t)}\right) \mu^n(t) \right] = 0$$

$$\text{or} \quad (\mu(t) + \lambda)^n = \phi(t) \mu^n(t) \quad (5-11)$$

where $0 \leq \phi(t) = \left(1 - \frac{1}{\beta(t)}\right) < 1$, $\forall t$

We will prove the Assertion by contradiction. Suppose that one eigenvalue of \bar{A} has the property, $\mu(t) = \psi(t) + j\sigma(t)$ with $\psi(t) > 0$ for some $t \geq t_0$. Then from (5-11), we have

$$\left[\psi(t) + j\sigma(t) + \lambda \right]^n = \phi(t) (\psi(t) + j\sigma(t))^n \quad (5-12)$$

Taking absolute values of both sides of (5-12), we have

$$\left[(\psi(t) + \lambda)^2 + \sigma^2(t) \right]^n = \phi^2(t) (\psi^2(t) + \sigma^2(t))^n$$

$$\text{or} \quad \left[(\psi(t) + \lambda)^2 + \sigma^2(t) \right] = \sqrt[n]{\phi^2(t)} (\psi^2(t) + \sigma^2(t))$$

$$\text{or} \quad \lambda^2 + 2\lambda\psi(t) = (\sqrt[n]{\phi^2(t)} - 1)(\psi^2(t) + \sigma^2(t)) \quad (5-13)$$

Since the right hand side of (5-13) is always negative for $\phi(t) < 1$, while the left hand side of (5-13) is positive for $\psi(t) \geq 0$ for some t , and $\lambda > 0$ which is impossible. This contradiction completes the proof. Q.E.D.

From (5-13), we have

$$\lambda^2 + 2\lambda\psi(t) \leq 0 \quad \forall t \geq t_0, \quad \lambda > 0 \quad (5-14)$$

$$\text{or} \quad -\psi(t) \geq \frac{1}{2}\lambda \quad \forall t \geq t_0, \quad \lambda > 0$$

which gives a lower bound for real part of the eigenvalues of \bar{A} in terms of λ . Hereafter, we assume the conditions of the Assertion V-1 are

satisfied for our systems, i.e. $\hat{\hat{b}}_0(t)/\hat{b}_0(t) = \beta(t) \geq 1 \quad \forall t \geq t_0$.

Substituting (5-8) into (5-7b), we obtain

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 & . & . & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 \\ -\hat{\hat{a}}_1(t) + \beta(t)(\hat{a}_1(t) - c_0^n \lambda) & . & . & . & . & . & -\hat{\hat{a}}_n(t) + \beta(t)(\hat{a}_n(t) - c_{n-1}^n \lambda) \end{bmatrix} x \quad (5-1)$$

which can be decomposed into two parts

$$\frac{dx(t)}{dt} = \bar{A}(t) x(t) + A_1(t) x(t) \quad (5-16)$$

$$A_1(t) = \begin{bmatrix} 0 & 0 & . & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & 0 \\ -\hat{\hat{a}}_1(t) + \beta(t)\hat{a}_1(t) & . & . & . & . & . & -\hat{\hat{a}}_n(t) + \beta(t)\hat{a}_n(t) \end{bmatrix}$$

and $\bar{A}(t)$ is defined as in (5-10).

Let $W(t, t_0)$ be the fundamental matrix of the system

$$\frac{dx(t)}{dt} = \bar{A}(t) x(t) , \quad x(t_0) = x_0 , \quad (5-17)$$

which is asymptotically stable for any $\lambda > 0$. It is well-known [34] that

$$\|W(t_0, t)\| \leq \theta e^{-\frac{\lambda}{2}(t-t_0)} \quad (5-18)$$

where θ is a constant, and $\| \dots \|$ denotes the Euclidean norm.

The solutions of (5-16) can be written in the Cauchy integral form as

$$\begin{aligned} x(t) &= W(t, t_0) x(t_0) + \int_{t_0}^t W(t, t_0) W^{-1}(\tau, t_0) A_1(\tau) x(\tau) d\tau \\ &= W(t, t_0) x(t_0) + \int_{t_0}^t W(t, \tau) A_1(\tau) x(\tau) d\tau \end{aligned} \quad (5-19)$$

Taking norm on both sides of (5-19), we have

$$\|x(t)\| \leq \theta e^{-\frac{\lambda}{2}(t-t_0)} \|x(t_0)\| + \int_{t_0}^t \theta e^{-\frac{\lambda}{2}(t-t_0)} \|A_1(\tau)\| \|x(\tau)\| d\tau \quad (5-20)$$

Since every elements of $A_1(t)$ belongs to \mathcal{F}_M , we then have

$$\|A_1(t)\| \leq M_1 \quad (5-21)$$

with (5-21), (5-20) becomes

$$\|x(t)\| e^{\frac{1}{2}\lambda t} \leq \theta e^{\frac{1}{2}\lambda t_0} \|x(t_0)\| + \int_{t_0}^t \theta M_1 e^{\frac{1}{2}\lambda \tau} \|x(\tau)\| d\tau$$

Using Gronwell's inequality, we further have

$$\|x(t)\| e^{\frac{1}{2}\lambda t} \leq \theta e^{\frac{1}{2}\lambda t_0} \|x(t_0)\| e^{\theta M_1(t-t_0)}$$

or

$$\|x(t)\| \leq \|x_0\| \theta e^{-\frac{1}{2}(\lambda - 2\theta M_1)(t-t_0)} \quad (5-22)$$

If λ is chosen such that

$$\lambda \geq 2\theta M_1 \quad , \quad (5-23)$$

then we can show $\lim_{t \rightarrow \infty} x(t) = 0$

The results obtained are summarized and stated as

Assertion V-2 For the linear time-varying system of the form (5-1) with parameters assuming two possible sets of time functions, there always exists a control law which is linear state feedback, and which can stabilize the systems having either parameter values if

$$\beta(t) \geq 1 \quad \forall t \geq t_0$$

The design procedures is summarized as :

- Step 1: Find θ , which is independent of λ ;
- Step 2: Find M_1 ;
- Step 3: Pick $\lambda \geq \lambda^* = 2\theta M_1$;
- Step 4: Obtain control law in (5-9) with $\lambda \geq \lambda^*$;

Assertion V-2 can be easily generalized to systems with parameters assuming any finite number of possible sets of time functions.

Assertion V-3 For the linear time-varying systems of the form (5-1) with parameters assuming \bar{N} possible sets of time functions, there always exists a control law which is linear state feedback and which stabilizes the systems having any set of time function if for any possible combination of two different sets of time functions, we have

$$\beta_l(t) \geq 1 \quad \forall t \geq t_0 \quad l=1, \dots, \frac{\bar{N}(\bar{N}+1)}{2}$$

Proof: Follow the same lines as in Assertion V-3, Hence the proof is omitted.

V-3. Example

Consider the system described by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_1(t) & -a_2(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0(t) \end{bmatrix} u(t)$$

where $\hat{a}_1 = \sin t$, $\hat{a}_2 = e^{-t}$, $\hat{b}_0(t) = 1 + e^{-2t}$

$$\hat{\hat{a}}_1 = \cos t \quad , \quad \hat{\hat{a}}_2 = e^{-2t} \quad , \quad \hat{\hat{b}}_0(t) = 1 + e^{-t} \quad , \quad \beta(t) = \frac{1 + e^{-t}}{1 + e^{-2t}} \geq$$

Step 1: Following [35] , we find

$$\theta = \sqrt{2} ;$$

Step 2: $M_1 = 4$;

Step 3: $\lambda^* = 2 \cdot \sqrt{2} \cdot 4 = 8\sqrt{2}$;

Step 4: $u = \frac{1}{1+e^{-2t}} \left[\sin t - \lambda^2, e^{-t-2\lambda} \right] \text{ with } \lambda \geq 8\sqrt{2} .$

V-4. Stabilization of Linear Time-Varying Systems with Parameters

Assuming Time Functions in \mathcal{F}_M

Assertion V-4 For linear time-varying systems of the form (5-1) with parameters assumign possible sets of time functions in \mathcal{F}_M , there always exists a control law which is linear state feedback and which stabilizes the systems having any set of time function in \mathcal{F}_M if $b_{0m} > 0$ or $b_{0M} < 0$ where

$$b_{0m} \leq b_0(t) \leq b_{0M} \quad \forall t \quad (5-24)$$

Proof: Let us consider a particular set of time functions in \mathcal{F}_M , which has a specific value for $b_0(t)$,

$$\Omega_1 = \left\{ a_i^*(t), i=1, \dots, n, \bar{b}_0 ; a_i^*(t) \in \mathcal{F}_M, i=1, \dots, n, \bar{b}_0 = b_{0m} \text{ if } b_{0m} > 0 \right. \\ \left. = b_{0M} \text{ if } b_{0M} < 0 \right\}$$

and any other $\Omega = \left\{ a_i(t), i=1, \dots, n, b_0(t); a_i(t), b_0(t) \in \mathcal{F}_M \right\}$

Since $\frac{b_0(t)}{b_0} \geq 1 \quad \forall t \geq t_0$, and from the Assertion V-2, we have shown the existence of an $\lambda^* > 0$ and a control law (5-9) with $\lambda \geq \lambda^*$, which stabilize the systems having either of the parameter values.

Actually, in the proof of Assertion V-2, we can easily show that the results hold for any $A(t)$ with $a_i(t) \in \mathcal{F}_M$, $i=1, \dots, n$. Hence the existence and the construction of control law can be obtained from

$$u(t) = \frac{1}{b_0} \left[\sum_{i=1}^n (a_i^*(t) - c_{i-1}^n \lambda^{n-i+1}) x_i(t) \right]$$

with $\lambda > \lambda^*$. Q.E.D.

CHAPTER VI

STABILIZATION OF NONLINEAR SYSTEMS WITH UNKNOWN PARAMETERS

VI-1. Introduction

In Chapter II, we have considered the problem of stabilization for a class of nonlinear systems (2-1)-(2-3) with known and fixed system parameters. The control law so obtained is guaranteed to stabilize the system having this parameter value. In this chapter, we intend to generalize the above problem to the case where the system parameters are initially unknown, but they belong to a compact subset \mathcal{Q}_p of the parameter space. We will show the existence of a control law which is in the form of linear state feedback and which stabilizes the systems with any possible parameters values in \mathcal{Q}_p .

[13] has studied the parameter variation for the system (2-1)-(2-3) by allowing some of the parameters to be unspecified, and obtained the relation between these parameters for absolute stability of the system.

It is well known that the stability of the system (2-1),(2-2) with $f(y,t) = 0$ for all y,t is a necessity for the absolute stability of system (2-1)-(2-3). It is therefore, a must to consider the problem of stabilization of the linear system (2-1),(2-2) with parameters assuming values in a compact subset of the parameter space before considering the corresponding problem for nonlinear systems.

In Chapter IV, we have shown the existence of a control law which is linear state feedback and which stabilizes the multi-input controllable linear systems with parameters assuming values in a compact subset of the parameter space. For the case $m = 1$, the results can be applied to this chapter. Based on this structure of the control law, we will show, by putting more restrictions on the parameters, the existence

of a control law for stabilizing nonlinear systems(2-1)-(2-3).

VI-2. Stabilization of Nonlinear Systems with Parameters Assuming Two Different Values

The systems under consideration are described by (2-1), (2-2) which are repeated for reference:

$$\frac{dx}{dt} = A x + b u - b f(y, t) \quad (6-1)$$

$$y = c^T x \quad (6-2)$$

where the state x is an n -vector, the control u , output y are scalars, and A , b are $n \times n$, $n \times 1$ matrices respectively, $f(y, t)$ is a continuous function with respect to each of its arguments and satisfies

$$0 \leq f(y, t)/y \leq k \quad \forall t, y, \quad (6-3a)$$

$$\text{and} \quad f(0, t) = 0, \quad \forall t \quad (6-3b)$$

We assume that the systems (6-1), (6-2) are given in companion form, i.e.

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & 1 \\ -a_1 & -a_2 & \dots & \dots & -a_n \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, \quad c = [c_1 \dots c_n]$$

where the parameters $\{a_i, c_i, i=1, \dots, n, b_0\} \in \mathcal{Q}_p$, which is defined as

$$\mathcal{Q}_p = \left\{ a_i, c_i, i=1, \dots, n, b_0 : a_i \in \mathcal{A}_i, c_i \in \bar{\mathcal{A}}_i, \mathcal{A}_i, \bar{\mathcal{A}}_i \text{ being compact subset of } \mathbb{R}, i=1, \dots, n, \text{ and } b_m \leq b_0 \leq b_M \right\}$$

We first assume that the system parameter assume two possible values Ω_1 and Ω_2 where $\Omega_1 = \{\hat{a}_i, \hat{c}_i, i=1, \dots, n, \hat{b}_0\}$ and $\Omega_2 = \{\hat{\hat{a}}_i, \hat{\hat{c}}_i, i=1, \dots, n, \hat{\hat{b}}_0\}$. Mathematically, we have two different systems described respectively by

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ -\hat{a}_1 & -\hat{a}_2 & \dots & \dots & -\hat{a}_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \hat{b}_0 \end{bmatrix} (u-f(y,t)) \quad (6-4a)$$

$$y = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_n) x$$

and

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ -\hat{\hat{a}}_1 & -\hat{\hat{a}}_2 & \dots & \dots & -\hat{\hat{a}}_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \hat{\hat{b}}_0 \end{bmatrix} (u-f(y,t)) \quad (6-4b)$$

$$y = (\hat{\hat{c}}_1, \hat{\hat{c}}_2, \dots, \hat{\hat{c}}_n) x$$

In order to have a meaningful problem, we assume $\hat{b}_0, \hat{\hat{b}}_0 \neq 0$. We intend to design a single control u of the form $u(t) = \sum_{i=1}^n q_i x_i$, $q_i, i=1, \dots, n$ are constant, such that both (6-4a) and (6-4b) will satisfy the absolute stability conditions.

To simplify our derivation, we introduce the change of input

$$v = - \sum_{i=1}^n \hat{\hat{a}}_i x_i + \hat{\hat{b}}_0 u \quad (6-5)$$

Substituting (6-5) into (6-4a) and (6-4b), we have respectively

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & 1 \\ -\hat{a}_1 + \frac{\hat{\hat{b}}_0}{\hat{b}_0} \hat{a}_1 & \dots & \dots & \dots & -\hat{a}_n - \frac{\hat{\hat{b}}_0}{\hat{b}_0} \hat{a}_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\hat{\hat{b}}_0}{\hat{b}_0} \end{bmatrix} (v-f(y,t)) \quad (6-6a)$$

and

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} (v-f(y,t)) , \quad (6-6b)$$

Attention is given to the case where $\hat{\hat{b}}_0 \cdot \hat{b}_0 > 0$, which implies $\alpha = \frac{\hat{\hat{b}}_0}{\hat{b}_0} \geq 1$, since otherwise, we can always interchange the two systems so that $\alpha \geq 1$.

To design a single control $v(t)$ to stabilize both (6-6a) and (6-6b), we consider the control v of the form

$$v(t) = - \sum_{i=1}^n C_{i-1}^n \lambda^{n-i+1} x_i , \quad (6-7)$$

where $C_1^n = n!/1!(n-1)!$, and $\lambda > 0$ to be determined .

with (6-7), (6-6b) has a characteristic equation with output y and input $-f(y,t)$ as

$$\hat{G}(s) = (s + \lambda)^n \quad (6-8)$$

which is a Hurwitz polynomial for any $\lambda > 0$. And in Chapter II, we have shown the existence of $\lambda_1^* > 0$, and a control law of the form (6-7) with $\lambda > \lambda_1^*$, such that

$$\operatorname{Re} \hat{G}(j\omega) + 1/k > 0 \quad \forall \omega$$

In order to show that the control $v(t)$ of the form (6-7) can stabilize (6-6a) for some $\lambda_2^* > 0$, we first prove the following assertion

Assertion VI-1 There exists a λ_2^* such that

$$\operatorname{Re} \left[\frac{\alpha \sum_{i=1}^n \hat{G}_i s^{i-1}}{s^n + \sum_{i=1}^n (\alpha C_{i-1}^n \lambda^{n-i+1}) s^{i-1}} \right] + \frac{1}{k} > 0 \quad \forall \omega$$

and all $\lambda > \lambda_2^*$, and

$1 \leq \alpha \leq \alpha_0$ where $\alpha_0 = \infty$ for $n=1, 2, 3$, and

$$\alpha_0 = \min. \left(\frac{2C_4^n}{2C_4^n - C_2^n}, \dots, \frac{2C_{n-4}^n}{2C_{n-4}^n - C_{\frac{n-1}{2}-2}^n} \right), \text{ for } n=4m, m=1,2,3\dots \quad (6-9a)$$

$$\text{or} \quad \alpha_0 = \min. \left(\frac{2C_4^n}{2C_4^n - C_2^n}, \dots, \frac{2C_{n-1}^n}{2C_{n-1}^n - C_{\frac{n-1}{2}-1}^n} \right), \text{ for } n=4m+1, m=1,2,3\dots \quad (6-9b)$$

$$\text{or} \quad \alpha_0 = \min. \left(\frac{2C_4^n}{2C_4^n - C_2^n}, \dots, \frac{2C_{n-2}^n}{2C_{n-2}^n - C_{\frac{n}{2}-1}^n} \right), \text{ for } n=4m+2, m=1,2,3\dots \quad (6-9c)$$

$$\text{or} \quad \alpha_0 = \min. \left(\frac{2C_4^n}{2C_4^n - C_2^n}, \dots, \frac{2C_{n-3}^n}{2C_{n-3}^n - C_{\frac{n-1}{2}-3}^n} \right), \text{ for } n=4m+3, m=1,2,3\dots \quad (6-9d)$$

where every entry of $\min.(\dots, \dots, \dots,)$ should be positive, it will not be considered for determining α_0 if it is negative. And $\alpha_0 = \infty$

if no entry appears in $\min.(\dots, \dots, \dots, \dots)$

Proof:

$$\operatorname{Re} \left[\frac{\alpha \sum_{i=1}^n \hat{c}_i s^{i-1}}{s^n + \sum_{i=1}^n C_{i-1}^n \lambda^{n-i+1} s^{i-1}} \right] + 1/k = \operatorname{Re} \left[\frac{\sum_{i=1}^n \hat{c}_i s^{i-1}}{(\frac{1}{\alpha} - 1) s^n + (s + \lambda)^n} \right] + 1/k$$

The numerator of the above equation is

$$\begin{aligned} k \operatorname{Re} \left\{ \frac{1}{\alpha} (-1)^n \hat{c}_n s^{2n-1} + \left[\left(\frac{1}{\alpha} \right) (-1)^n \hat{c}_{n-1} + (-1)^{n-1} \hat{c}_n C_1^n \right] s^{2n-2} + \left[\frac{1}{\alpha} (-1)^n \hat{c}_{n-2} \right. \right. \\ \left. \left. + (-1)^{n-1} \hat{c}_{n-1} C_1^n \lambda + (-1)^{n-2} \hat{c}_n C_2^n \lambda^2 \right] s^{2n-3} + \dots + \left[\hat{c}_5 \lambda^n - \hat{c}_4 C_{n-1}^n \lambda^{n-1} \right. \right. \\ \left. \left. + \hat{c}_3 C_{n-2}^n \lambda^{n-2} - \hat{c}_2 C_{n-3}^n \lambda^{n-3} + \hat{c}_1 C_{n-4}^n \lambda^{n-4} \right] s^4 + \left[\hat{c}_4 \lambda^n - \hat{c}_3 C_{n-1}^n \lambda^{n-1} \right. \right. \\ \left. \left. + \hat{c}_2 C_{n-2}^n \lambda^{n-2} - \hat{c}_1 C_{n-3}^n \lambda^{n-3} \right] s^3 + \left[\hat{c}_3 \lambda^n - \hat{c}_2 C_{n-1}^n \lambda^{n-1} + \hat{c}_1 C_{n-2}^n \lambda^{n-2} \right] s^2 \\ \left. + \left[\hat{c}_2 \lambda^n - \hat{c}_1 C_{n-1}^n \lambda^{n-1} \right] s + \hat{c}_1 \lambda^n \right\} + \frac{1}{\alpha^2} (-1)^n s^{2n} + \left[\frac{1}{\alpha} C_2^n (-1)^{n-2} + (C_1^n)^2 (-1)^{n-1} \right. \\ \left. + \frac{1}{\alpha} C_2^n (-1)^n \right] \lambda^2 s^{2n-2} + \left[\frac{1}{\alpha} C_4^n (-1)^{n-4} + C_1^n C_3^n (-1)^{n-3} + (C_2^n)^2 (-1)^{n-2} + C_1^n C_3^n (-1)^{n-1} \right. \\ \left. + \frac{1}{\alpha} C_4^n (-1)^n \right] \lambda^4 s^{2n-4} + \dots + \left[\frac{1}{\alpha} C_{n-4}^n - C_{n-1}^n C_{n-3}^n + (C_{n-2}^n)^2 - C_{n-1}^n C_{n-3}^n + \frac{1}{\alpha} C_{n-4}^n \right] . \end{aligned}$$

$$\begin{aligned}
& \cdot \lambda^{2n-4} s^4 + \left[\frac{1}{\alpha} C_{n-2}^n - (C_{n-1}^n)^2 + \frac{1}{\alpha} C_{n-2}^n \right] \lambda^{2n-2} s^2 + \lambda^{2n} \\
& = \frac{1}{\alpha^2} w^{2n} + \left[\left(\frac{2}{\alpha} C_2^n (-1)^{n-2} + (C_1^n)^2 (-1)^{n-1} \right) \lambda^2 + k \left(\frac{1}{\alpha} (-1)^n \hat{c}_{n-1} + (-1)^{n-1} \hat{c}_n C_1^n \lambda \right) \right] \cdot \\
& \cdot (jw)^{2n-2} + \left[\left((C_2^n)^2 (-1)^{n-2} + 2C_1^n C_3^n (-1)^{n-1} + 2\frac{1}{\alpha} C_4^n (-1)^n \right) \lambda^4 + k \left(\frac{1}{\alpha} (-1)^n \hat{c}_{n-3} \right. \right. \\
& + (-1)^{n-1} \hat{c}_{n-2} C_1^n \lambda + (-1)^{n-2} \hat{c}_{n-1} C_2^n \lambda^2 + (-1)^{n-3} \hat{c}_n C_3^n \lambda^3 \left. \right] (jw)^{2n-4} + \dots \\
& + \left[\left((C_2^n)^2 - 2C_{n-1}^n C_{n-3}^n + 2C_{n-4}^n \right) \lambda^{2n-4} + k \left(\hat{c}_5 \lambda^n - \hat{c}_4 C_{n-1}^n \lambda^{n-1} + \hat{c}_3 C_{n-2}^n \lambda^{n-2} \right. \right. \\
& \left. \left. - \hat{c}_2 C_{n-3}^n \lambda^{n-3} + \hat{c}_1 C_{n-4}^n \lambda^{n-4} \right) \right] w^4 + \left[\left((C_{n-1}^n)^2 - 2\frac{1}{\alpha} C_{n-2}^n \right) \lambda^{2n-2} \right. \\
& \left. + k \left(\hat{c}_3 \lambda^n - \hat{c}_2 C_{n-1}^n \lambda^{n-1} + \hat{c}_1 C_{n-2}^n \lambda^{n-2} \right) \right] w^2 + \lambda^{2n} + \hat{c}_1 k \lambda^n \quad (6-10)
\end{aligned}$$

(6-10) is greater than zero $\forall w$ if all of its coefficients are positive.

This will be true if

$$\left[-\frac{2}{\alpha} C_2^n + (C_1^n)^2 \right] \lambda^2 + k \left(-\frac{1}{\alpha} \hat{c}_{n-1} + \hat{c}_n C_1^n \lambda \right) > 0$$

$$\left[(C_2^n)^2 - 2C_1^n C_3^n + \frac{2}{\alpha} C_4^n \right] \lambda^4 + k \left(\frac{1}{\alpha} \hat{c}_{n-3} - \hat{c}_{n-2} C_1^n \lambda + \hat{c}_{n-1} C_2^n \lambda^2 - \hat{c}_n C_3^n \lambda^3 \right) > 0$$

$$\begin{array}{ccc}
 \vdots & \vdots & \\
 \vdots & \vdots & (6-11) \\
 \vdots & \vdots & \\
 \vdots & \vdots &
 \end{array}$$

$$\left[(C_{n-1}^n)^2 - \frac{2}{\alpha} C_{n-2}^n \right] \lambda^{2n-2} + k(\hat{c}_3 \lambda^n - \hat{c}_2 \lambda^{n-1} C_{n-1}^n + \frac{1}{\alpha} \hat{c}_1 \lambda^{n-2} C_{n-2}^n) > 0$$

$$\lambda^{2n} + k \hat{c}_1 \lambda^n > 0$$

For sufficiently large $\lambda > 0$, the signs of the inequalities of (6-11) are determined by the coefficients of the highest degree of λ in the inequalities. Hence every inequality of (6-11) is positive if α satisfies (6-9). The negative entry of $\min(\dots, \dots, \dots)$ of (6-9) implies the positive leading coefficients for $1 \leq \alpha < \infty$ in one of the inequalities of (6-11), hence it should not be considered in determining α'_0 . If no entry appears in $\min(\dots, \dots, \dots)$, this implies each inequality of (6-11) has a positive leading coefficients for all $1 \leq \alpha < \infty$. This completes the proof. Q.E.D.

Hereafter, we assume that $1 \leq \hat{b}_0/\hat{b}_0 \leq 0$, which is independent of λ . We are now in a position to prove the following assertion.

Assertion VI-2 There exists an $\lambda_3^* > 0$ such that

$$\operatorname{Re} \left[\frac{\alpha \sum_{i=1}^n \hat{c}_i s^{i-1}}{s^n + \sum_{i=1}^n (\hat{a}_i + \alpha C_{i-1}^n \lambda^{n-i+1}) s^{i-1}} \right] + 1/k > 0 \quad \forall w \quad (6-12)$$

for all $\lambda \geq \lambda_3^*$ and $1 \leq \alpha \leq \alpha_0$, and $\bar{a}_i = \hat{a}_i - \alpha \hat{\hat{a}}_i$, $i=1, \dots, n$

The numerator of (6-12) can be obtained as

$$\begin{aligned}
& \frac{1}{\alpha^2} w^{2n} + \left\{ \left[\frac{2}{\alpha} \left(\frac{\bar{a}_{n-1}}{\alpha} + c_2^n \lambda^2 \right) (-1)^{n-2} + \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right)^2 (-1)^{n-1} \right] + k \left[\frac{1}{\alpha} (-1)^n \hat{c}_{n-1} \right. \right. \\
& + \left. (-1)^{n-1} \hat{c}_n \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right) \right] \left. \right\} (jw)^{2n-2} + \left\{ \frac{2}{\alpha} (-1)^n \left(\frac{\bar{a}_{n-3}}{\alpha} + c_4^n \lambda^4 \right) + 2 \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right) \right. \\
& \cdot \left(\frac{\bar{a}_{n-2}}{\alpha} + c_3^n \lambda^3 \right) (-1)^{n-1} + (-1)^{n-2} \left(\frac{\bar{a}_{n-1}}{\alpha} + c_2^n \lambda^2 \right)^2 + k \left[\frac{1}{\alpha} (-1)^n \hat{c}_{n-3} + (-1)^{n-1} \hat{c}_{n-2} \right. \\
& \cdot \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right) + (-1)^{n-2} \hat{c}_{n-1} \left(\frac{\bar{a}_{n-1}}{\alpha} + c_2^n \lambda^2 \right) + (-1)^{n-3} \hat{c}_n \left(\frac{\bar{a}_{n-2}}{\alpha} + c_3^n \lambda^3 \right) \left. \right] \left. \right\} (jw)^{2n-4} \\
& + \dots + \left\{ \left[2 \left(\frac{\bar{a}_5}{\alpha} + c_{n-4}^n \lambda^{2n-4} \right) - 2 \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right) \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right) \right. \right. \\
& + \left. \left(\frac{\bar{a}_3}{\alpha} + c_{n-2}^n \lambda^{n-2} \right)^2 \right] + k \left[\hat{c}_5 \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right) - \hat{c}_4 \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right) + \hat{c}_3 \left(\frac{\bar{a}_3}{\alpha} \right. \right. \\
& + \left. c_{n-2}^n \lambda^{n-2} \right) - \hat{c}_2 \left(\frac{\bar{a}_4}{\alpha} + c_{n-3}^n \lambda^{n-3} \right) + \hat{c}_1 \left(\frac{\bar{a}_5}{\alpha} + c_{n-4}^n \lambda^{n-4} \right) \left. \right] \left. \right\} w^4 + \left\{ - \left[2 \left(\frac{\bar{a}_3}{\alpha} \right. \right. \right. \\
& + \left. c_{n-2}^n \lambda^{n-2} \right) - \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right)^2 \left. \right] + k \left[\hat{c}_3 \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right) - \hat{c}_2 \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right) \right. \\
& + \left. \hat{c}_1 \left(\frac{\bar{a}_3}{\alpha} + c_{n-2}^n \lambda^{n-2} \right) \right] \left. \right\} w^4 + \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right)^2 + \hat{c}_1 \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right) \quad (6-13)
\end{aligned}$$

(6-13) is greater than zero for all w , if all of its coefficients are positive. This will be true if

$$\begin{aligned}
 & -\frac{2}{\alpha} \left(\frac{\bar{a}_{n-1}}{\alpha} + c_2^n \lambda^2 \right) + \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right)^2 + k \left[-\frac{1}{\alpha} \hat{c}_{n-1} + \hat{c}_n \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right) \right] > 0 \\
 & \left(\frac{\bar{a}_{n-1}}{\alpha} + c_2^n \lambda^2 \right)^2 - 2 \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right) \left(\frac{\bar{a}_{n-2}}{\alpha} + c_3^n \lambda^3 \right) + \frac{2}{\alpha} \left(\frac{\bar{a}_{n-3}}{\alpha} + c_4^n \lambda^4 \right) \\
 & + k \left[\frac{1}{\alpha} \hat{c}_{n-3} - \hat{c}_{n-2} \left(\frac{\bar{a}_n}{\alpha} + c_1^n \lambda \right) + \hat{c}_{n-1} \left(\frac{\bar{a}_{n-1}}{\alpha} + c_2^n \lambda^2 \right) - \hat{c}_n \left(\frac{\bar{a}_{n-2}}{\alpha} + c_3^n \lambda^3 \right) \right] > 0 \\
 & \vdots \\
 & \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right)^2 - 2 \left(\frac{\bar{a}_3}{\alpha} + c_{n-2}^n \lambda^{2n-2} \right) + k \left[\hat{c}_3 \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right) - \hat{c}_2 \left(\frac{\bar{a}_2}{\alpha} + c_{n-1}^n \lambda^{n-1} \right) \right. \\
 & \left. + \hat{c}_1 \left(\frac{\bar{a}_3}{\alpha} + c_{n-2}^n \lambda^{n-2} \right) \right] > 0 \\
 & \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right)^2 + \hat{c}_1 \left(\frac{\bar{a}_1}{\alpha} + \lambda^n \right) > 0 \tag{6-14}
 \end{aligned}$$

Since for sufficiently large $\lambda > 0$, we examine the coefficients of the highest order of λ in each inequality of (6-14), and find that these coefficients are exactly the same as that in (6-11), the existence of λ_3^* such that (6-12) holds for any $\lambda \geq \lambda_3^*$ is guaranteed.

By choosing $\lambda^* = \max.(\lambda_1^*, \lambda_2^*, \lambda_3^*)$, the control $u(t)$ of the form

$$u(t) = \frac{1}{\hat{\hat{b}}_0} \left[\sum_{i=1}^n (\hat{a}_i - c_{i-1}^n \lambda^{n-i+1}) x_i \right] \quad \text{with } \lambda \geq \lambda^* \quad (6-15)$$

will stabilize both systems (6-4a), (6-4b).

The results are summarized as :

Assertion VI-3 For the nonlinear system of the form (6-1)-(6-3) with parameters assuming two possible different values, there always exists a control law of the form (6-15) which can stabilize the systems having either parameter values if

$$1 \leq \alpha \leq \alpha_0$$

The design procedures are summarized as :

- Step 1: Check if $1 \leq \hat{\hat{b}}_0/\hat{b}_0$ or $\hat{b}_0/\hat{\hat{b}}_0 \leq \alpha_0$;
- Step 2: Transform (6-1)-(6-3) into companion form, and change the input by (6-5);
- Step 3: Find λ_1^* , λ_2^* , λ_3^* ;
- Step 4: Let $\lambda^* = \max.(\lambda_1^*, \lambda_2^*, \lambda_3^*)$ and obtain control as in (6-15) with $\lambda > \lambda^*$.

The results are easily extended to the system with parameters assuming any finite number of possible values. The results are similar to previous chapters and the derivation is omitted.

VI-3. Example

Given the systems described by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} (u-f(y,t))$$

$$y = (c_1, c_2) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad k=1$$

with $\hat{a}_1 = -1$, $\hat{a}_2 = 1$, $\hat{b}_0 = 1$, $\hat{c}_1 = 1$, $\hat{c}_2 = -1$ and

$$\hat{\hat{a}}_1 = 2, \hat{\hat{a}}_2 = -1, \hat{\hat{b}}_0 = 2, \hat{\hat{c}}_1 = -1, \hat{\hat{c}}_2 = 0$$

Step 1: From (6-9), we have $\alpha_0 = \infty$ since $n=2$

Step 2: Change the input u by defining

$$v(t) = u(t) + x_1 - x_2$$

Then we have two different systems described by

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} (v-f(y,t)) \quad , \quad y = \begin{bmatrix} -1, 0 \end{bmatrix} x$$

and

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (v-f(y,t)) \quad , \quad y = (1, -1) x ;$$

Step 3: To determine λ_1^* , consider

$$\operatorname{Re} \left(\frac{s-1}{(s+\lambda)^2} \right) + 1, \text{ which is greater than zero for all}$$

$$\lambda \geq \lambda_1^* = 1, \quad \text{and consider}$$

$$\operatorname{Re} \left[\frac{-2s}{s^2 + (4+4\lambda)s + (-1+\lambda^2)} \right] + 1, \text{ which is greater than zero}$$

$$\text{for all } \lambda \geq \lambda_3^* = 1.$$

It is obvious that $\lambda_2^* = 1$

Let $\lambda^* = 2$

Step 4: $u(t) = -5x_1 - 3x_2$ will stabilize both systems.

VI-4. Stabilization of Nonlinear Systems with Parameters Assuming Values in A Compact Subset of the Parameter Space

The results of VI-2 are generalized to the systems with parameters assuming values in a compact subset in the parameter space.

Assertion VI-4 For the nonlinear systems of the form (6-1)-(6-3) with parameters assuming values in Ω_p , there always exists a control $u(t)$ of the form (6-15) which can stabilize the system having any parameter values in D_p , if

$$0 < \varepsilon \leq b_m \leq b_0 \leq \alpha_0 b_m$$

$$\text{or} \quad -\alpha_0 b_M \leq b_0 \leq -b_M \leq -\varepsilon < 0 \quad (6-16)$$

where ε is an arbitrarily small number

Proof: Let us consider a particular element $\Omega_1 \in \Omega_p$, having the specific values for b_0 ,

$$\Omega_1 = \left\{ a_i^*, c_i^*, i=1, \dots, n, b_0^* = b_m \text{ if } b_m > 0 \right. \\ \left. b_0^* = b_M \text{ if } b_M < 0 \right\}$$

and any other element $\Omega \in \Omega_p$ with b_0 satisfying (6-16).

For this pair of parameter values, we have

$$1 \leq \alpha = \frac{b_0}{b_0^*} \leq \alpha_0$$

then from Assertion VI-2, we have shown the existence of an $\lambda^* \geq 0$ and of a control of the form (6-15) with $\lambda \geq \lambda^*$, which stabilizes the system having either parameter values. It is obvious that λ^* depends upon the choice of $\Omega \in \mathcal{Q}_p$. Since the determination of $\lambda^*(\Omega)$ involves only continuous operations, the dependence of λ^* on $\Omega \in \mathcal{Q}_p$ is continuous. Since \mathcal{Q}_p is compact by assumption, the existence of a $\bar{\lambda}^*$ such that the control law of the form (6-15) with $\lambda \geq \bar{\lambda}^*$ which stabilizes the systems having any of the parameter values in \mathcal{Q}_p is guaranteed, where

$$\bar{\lambda}^* = \sup_{\Omega \in \mathcal{Q}_p} \lambda(\Omega)$$

This completes the proof of the Assertion. Q.E.D.

CHAPTER VII

CONCLUSIONS

VII-1. Conclusion

Problems of stabilization of nonlinear dynamic systems with fixed or unknown parameters are considered. In Chapter II, the methods for the stabilization of a scalar Lure system by means of a linear state feedback or dynamics compensators design of specific type are developed. The existence of such a control law is guaranteed under the assumptions of complete controllability and complete observability of the systems, in which the condition of complete observability can be further relaxed by requiring the unobservable part be asymptotically stable. Chapter III attempts to extend the results obtained in Chapter II to the systems containing multiple-nonlinearities. The effort has been successful for two subclasses of such systems, i.e. systems with adjoint controls and systems having the property of simple-strong complete controllability.

In many of the real concrete systems, the system parameters are given with a certain degree of accuracy, or the parameter values vary slowly during the operation. It is then necessary to consider the problems of stabilization of the system assuming all possible parameter values. The proof for existence of a linear state feedback control law for the stabilization of the class of multi-input controllable linear systems, the class of the general linear time-varying systems, and the class of Lure system is derived respectively in Chapter IV, Chapter V, and Chapter VI, in which the parameters of the systems are assumed to satisfy certain constraints.

VII-2. Discussions

In this thesis, the development of design techniques and the

proof of the existence of a control law are made use of some special structures of the control law. The advantage of this approach consists in reducing the determination of n feedback coefficients to that of only one, and which gives the results in a completely analytic fashion. It is possible to improve the results by assuming the structure of the control law with more than one undetermined coefficient. This, however, will definitely increase the complexity in the proof of our results.

The results obtained in this thesis consist in finding a real number $\bar{\lambda}^*$ such that any linear feedback control law with $\lambda \geq \lambda^*$ will achieve the desired goal. The selection of λ over the range $\infty > \lambda \geq \lambda^*$ seems arbitrary. However, λ has direct effect on the system response and performance. Large value of λ may have better system responses, in the meantime, it may increase the difficulty in realizing the control law. For a suitably chosen λ , the compromise between various factors affecting the system responses should be made. This may lead to a formulation of an optimal control problem.

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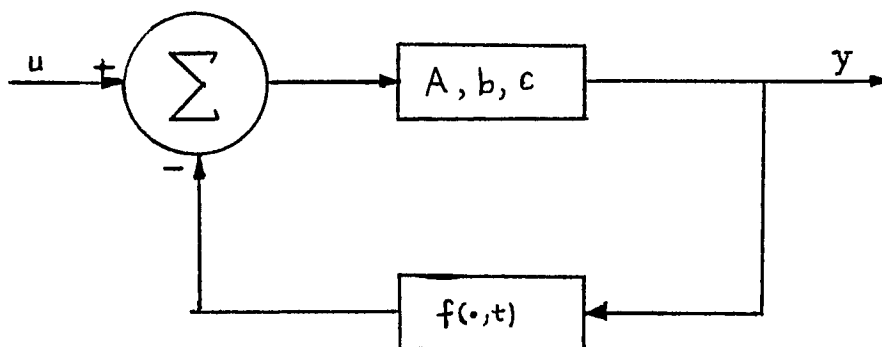


Figure 1 : Schematic representation of the nonlinear system considered

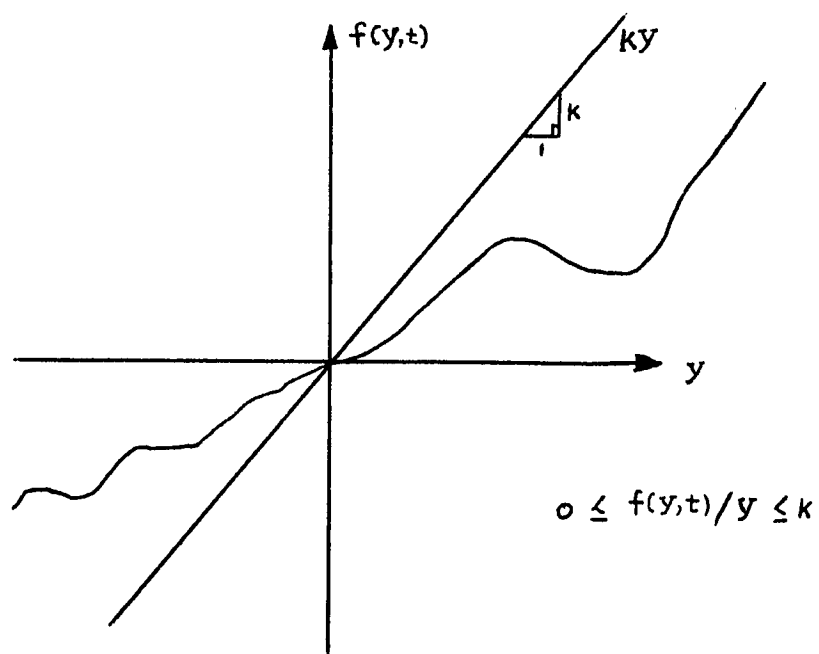


Figure 2 : Characteristic of nonlinear function $f(y, t)$

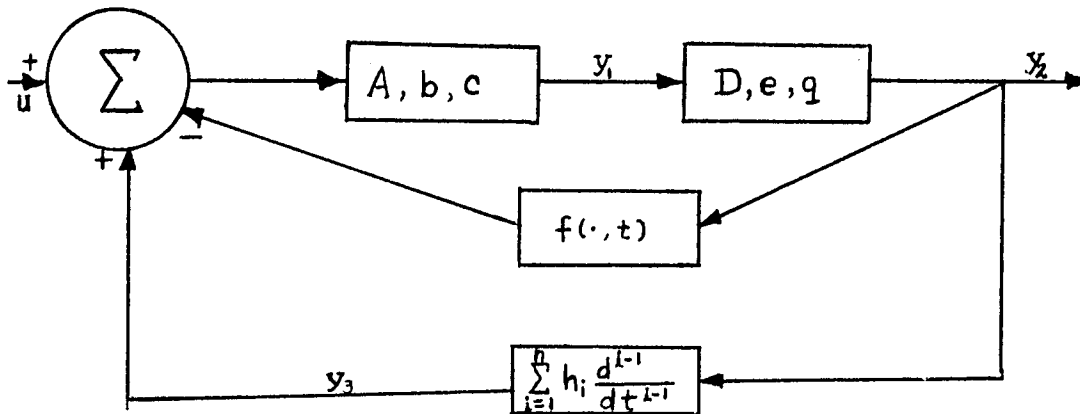


Figure 3 : Configuration for Compensator Design

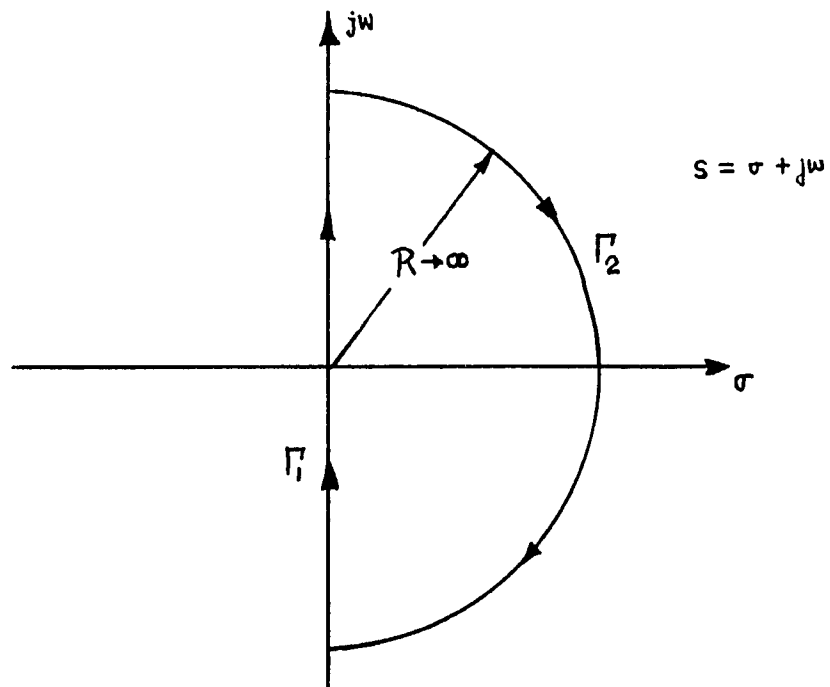


Figure 4 : The contour