A Canonical Covariance-Based Method for Generalized Joint Signal Representations

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Abstract— **Generalized joint signal representations extend the scope of joint time-frequency representations to a richer class of nonstationary signals. Cohen's marginal-based generalized approach is canonical from a distributional viewpoint, whereas, in some other applications, for example, in a signal detection framework, a covariance-based formulation is needed and/or more attractive. In this note, we present a canonical covariance-based recipe for generating generalized joint signal representations. The method is highlighted by its simple characterization and interpretation, and naturally extends the concept of the corresponding linear representations.**

I. Introduction

Recognizing the limitations of time-frequency representations (TFRs), generalized joint signal representations which analyze signals in terms of physical quantities other than time and frequency have recently been investigated by a number of authors [1], [2], [3], [4], [5], [6], [7]. For example, joint time-scale representations analyze signals in terms of time and scale content [1], [2].

In existing literature, the construction of joint signal representations has been based on two main approaches. Cohen's pioneering method of constructing bilinear TFRs interprets the TFRs as quasi-energy distributions which satisfy certain marginal constraints analogous to probability distributions. The other main approach is to consider arbitrary quadratic forms in the signal, parameterized by variables of interest, and then to impose certain covariance constraints to characterize a certain class of joint signal representations. For example, the affine class of time-scale representations proposed by Rioul and Flandrin [1] and the Bertrands [2] characterizes representations which are covariant to time-shifts and scalings. Such covariance properties are important in situations in which signals of interest undergo certain unitary transformations, for example time-frequency shifts and scalings. In particular, the use of TFRs and TSRs in signal detection crucially depends on such properties [8].

Cohen has recently extended his original marginal-based method to joint representations which analyze signal energy in terms of arbitrary variables [3], [4]. A similar approach was proposed by Baraniuk [5] and was shown to be equivalent to Cohen's [7]. Cohen's marginal-based approach seems fairly complete and is canonical from a distributional viewpoint because the representations measure the distribution of signal energy as a function of the variables. Some results on a covariance-based generalization have also been recently reported by Hlawatsch *et. al* [9], [10]. However, more work needs to be done before the covariance-based theory is complete.

In this letter we present a canonical covariance-based method for generating generalized joint signal representations. Our theory is similar in principle to that proposed in [9], [10] but has a much simpler, direct form that makes it conceptually more attractive. Some differences between our approach and that presented in [9], [10] will be discussed.

II. Preliminaries

We assume that the signals of interest belong to a closed subspace $\mathcal{H}$ of the Hilbert space of finite-energy signals $L^2(\mathbb{R})$. Let $G \subset \mathbb{R}^N$ be a parameter set and let $\{\mathcal{U}_g\}_{g\in G}$ be a family of unitary operators defined on $\mathcal{H}$; that is, for any $g \in G$, $\mathcal{U}_g : \mathcal{H} \to \mathcal{H}$ and $\langle \mathcal{U}_g s, \mathcal{U}_g s\rangle = \langle s, s\rangle$ for all $s \in \mathcal{H}$ where $\langle \cdot, \cdot\rangle$ denotes the inner product defined on $\mathcal{H}$. For a given $g \in G$, each "coordinate" represents a variable or physical quantity of interest.

The family of unitary operators $\{\mathcal{U}_g\}$ represents signal transformations that are of interest to us; for example, time-frequency shifts or time-shifts and scalings for $N = 2$. In many cases, the following constraints can be naturally imposed on the family $\{\mathcal{U}_g\}$:

1. The mapping $g \mapsto \mathcal{U}_g$ is one-to-one.
2. The family $\{\mathcal{U}_g\}$ is closed under composition (up to a phase factor[1]); that is, for any $a, b \in G$ there exists a $c \in G$ such that $\mathcal{U}_a\mathcal{U}_b = \mathcal{U}_c$.
3. The effect of any mapping $\mathcal{U}_g$ can be reversed; that is, for each $g_1 \in G$ there exists a $g_2 \in G$ such that $\mathcal{U}_{g_2}\mathcal{U}_{g_1} = \mathcal{I}$, where $\mathcal{I}$ is the identity operator on $\mathcal{H}$.
4. The operators $\{\mathcal{U}_g\}$ are associative; that is, $\mathcal{U}_a(\mathcal{U}_b\mathcal{U}_c) = (\mathcal{U}_a\mathcal{U}_b)\mathcal{U}_c$ for all $a, b, c \in G$.

In other words, the family of operators $\{\mathcal{U}_g\}$ forms a group (modulo a phase factor) with composition as the group operation. This in turn implies that the set G is itself a group with group operation $\bullet$ defined by (1) below; that is: 1) $a, b \in G \Rightarrow a\bullet b \in G$, 2) $a\bullet(b\bullet c) = (a\bullet b)\bullet c$ for all $a, b, c \in G$, 3) there exists an identity element $\theta \in G$ such that $\theta \bullet a = a \bullet \theta = a$ for all $a \in G$, and 4) for each $a \in G$ there exists an $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = \theta$. With the above constraints, $\{\mathcal{U}_g\}$ is a unitary representation of

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[1] In certain cases $\mathcal{U}_a\mathcal{U}_b = e^{j\psi(a,b)}\mathcal{U}_c$ but since we are mainly interested in quadratic signal representations, this phase factor will not be an issue. Even for linear representations such a phase factor is inconsequential in most cases.

G on $\mathcal{H}$ and this will be assumed throughout the rest of the letter; that is,

$$U_a \mathcal{U}_b = \mathcal{U}_{a \bullet b} \quad \text{(within a phase factor)}. \tag{1}$$

Although we have considered arbitrary group representations of the form $\{\mathcal{U}_g\}$, it is worth noting that in most cases of interest, the operator $\mathcal{U}_g$ will be a composition of N unitary operators that are themselves unitary representations of one-parameter groups.[2] The reason is that in most cases, we associate individual variables of interest, like time, frequency or scale, with operators and then construct the joint representations.

III. A Canonical Covariance-Based Method

Suppose that $\{\mathcal{U}_g\}$ is a group of unitary operators and we are interested in bilinear (quadratic) signal representations which are covariant to the unitary transformation $\mathcal{U}_g$ in the sense that

$$(P\mathcal{U}_g s)(a) = (Ps)(g^{-1} \bullet a) \quad \text{for all} \quad a, g \in G \ , \tag{2}$$

where the signal s belongs to $\mathcal{H}$ and the quadratic signal representation is denoted by the operator P which maps the signal into the space of (possibly) complex-valued functions defined on G.[3] Recall that each "coordinate" of an element a of G represents a variable or quantity of interest. The following theorem provides a simple characterization of all bilinear signal representations covariant to $\{\mathcal{U}_g\}$.

Theorem. For any bilinear signal representation P satisfying (2) there exists a linear operator $\mathcal{K}_P : \mathcal{H} \to \mathcal{H}$ such that

$$(Ps)(a) = \langle \mathcal{K}_P \mathcal{U}_{a^{-1}} s, \mathcal{U}_{a^{-1}} s \rangle \quad , \text{ for all } s \in \mathcal{H} \ , a \in G \ . \tag{3}$$

Conversely, any linear operator $\mathcal{K}_P : \mathcal{H} \to \mathcal{H}$ defines a bilinear signal representation via (3) which satisfies the covariance relation (2).

Proof. First suppose that P is defined by (3). Then, we have

$$\begin{aligned}(P\mathcal{U}_g s)(a) &= \langle \mathcal{K}_P \mathcal{U}_{a^{-1}} \mathcal{U}_g s, \mathcal{U}_{a^{-1}} \mathcal{U}_g s \rangle \\ &= \langle \mathcal{K}_P \mathcal{U}_{(g^{-1} \bullet a)^{-1}} s, \mathcal{U}_{(g^{-1} \bullet a)^{-1}} s \rangle \\ &= (Ps)(g^{-1} \bullet a) \end{aligned} \tag{4}$$

and thus P satisfies (2). Conversely, suppose that P is an arbitrary bilinear representation that satisfies (2). It follows that there exists a family of linear operators $\{\mathcal{K}_a\}$ such that $(Ps)(a) = \langle \mathcal{K}_a s, s \rangle$, $a \in G$. Now, (2) implies that

$$\langle \mathcal{K}_a \mathcal{U}_g s, \mathcal{U}_g s \rangle = \langle \mathcal{K}_{g^{-1} \bullet a} s, s \rangle \tag{5}$$

for all $s \in \mathcal{H}$ and for all $a, g \in G$. By setting $a = \theta$ and substituting g for g^{-1}, (5) yields

$$(Ps)(g) = \langle \mathcal{K}_g s, s \rangle = \langle \mathcal{K}_\theta \mathcal{U}_{g^{-1}} s, \mathcal{U}_{g^{-1}} s \rangle \tag{6}$$

which completes the proof.[4]

The covariance properties of the representations are determined by $\mathcal{U}_a$ in (3) and all the other properties are completely determined by the linear operator $\mathcal{K}_P$.[5] The choice of the operator $\mathcal{K}_P$ in controlling the properties of the representation is completely equivalent to the choice of the "kernel" in Cohen's and affine classes.

Further insight into the interpretation of (3) can be gained by using the singular value decomposition of the operator $\mathcal{K}_P$ (if it is compact [13]):[6]

$$\begin{aligned}(Ps)(a) &= \sum_k \sigma_k \langle \mathcal{U}_{a^{-1}} s, v_k \rangle \langle u_k, \mathcal{U}_{a^{-1}} s \rangle \\ &= \sum_k \sigma_k \langle s, \mathcal{U}_a v_k \rangle \langle \mathcal{U}_a u_k, s \rangle \end{aligned} \tag{7}$$

which implies that the value of Ps at a particular value of a is completely determined by the projection of $\mathcal{U}_{a^{-1}} s$ onto the singular vectors, u_k's and v_k's, and the singular values σ_k's. If $\mathcal{K}_P$ is Hermitian, then

$$(Ps)(a) = \sum_k \lambda_k |\langle s, \mathcal{U}_a u_k \rangle|^2 \tag{8}$$

where the λ_k's are the eigenvalues and the u_k's are the eigenfunctions. In particular, (8) implies that the resulting representation is real-valued. Moreover, if $\mathcal{K}_P$ is rank-1 then (8) reduces to $(Ps)(a) = \lambda |\langle s, \mathcal{U}_a u \rangle|^2$ which is a generalization of the spectrogram/scalogram, where u is analogous to the analyzing window or the mother wavelet. Thus, any arbitrary bilinear representation (corresponding to a compact operator) can be thought of as a higher-rank extension of the energetic version of the corresponding linear representation $\langle s, \mathcal{U}_a u \rangle$.

IV. Examples

We now illustrate our method by applying Theorem 1 to some well-known classes of joint signal representations.

Cohen's class of bilinear TFRs. Let $\mathcal{H} = L^2(\mathbb{R})$ and let $G = \mathbb{R}^2$ with the group operation defined by $(x_1, y_1) \bullet (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$; $(x, y)^{-1} = (-x, -y)$.[7] For $(\tau, \nu) \in \mathbb{R}^2$, define the time-shift and frequency-shift operators as $(\mathcal{T}_\tau s)(x) = s(x - \tau)$ and $(\mathcal{F}_\nu s)(x) = e^{j2\pi\nu x} s(x)$, respectively. Let $a = (t, f)$ and define the time-frequency-shift operators $\{\mathcal{U}_{(t,f)}\}$ as $\mathcal{U}_{(t,f)} = \mathcal{F}_\nu \mathcal{T}_\tau$ which satisfy the group composition law (1). Using (3) in Theorem 1, the class of representations covariant to time-frequency shifts is characterized by

$$\begin{aligned}(Ps)(t,f) &= \langle \mathcal{K}_P \mathcal{U}_{(t,f)^{-1}} s, \mathcal{U}_{(t,f)^{-1}} s \rangle \\ &= \langle \mathcal{K}_P \mathcal{F}_{-f} \mathcal{T}_{-t} s, \mathcal{F}_{-f} \mathcal{T}_{-t} s \rangle \end{aligned}$$

[2] To be more precise, the underlying group G is an exponential Lie group [11].

[3] Related group theoretic and covariance-based arguments (coadjoint representations and the method of orbits [11]) are used in [2] to derive analogues of the Wigner distribution for the affine group, and in [12] to define wideband ambiguity functions.

[4] We note that in [2], the Bertrands arrive at a specialized version of (6) for signal representations covariant to time-shifts and scalings.

[5] Operators of the form $\mathcal{U}_g \mathcal{K} \mathcal{U}_g^{-1}$ are discussed in [12] with regard to covariance properties of phase space functions defined on G.

[6] Similar expansions for Cohen's class of time-frequency representations are discussed in [14] and [15].

[7] G can be thought of as a 2D subgroup of the Weyl-Heisenberg group [16].

$$
\begin{aligned}
&= \int\int K_P(u_2,u_1)s(u_1+t)s^*(u_2+t) \\
&\quad e^{-j2\pi f(u_1-u_2)}du_1du_2 \\
&= \int\int \Phi(u,\tau)s\left(t+u+\frac{\tau}{2}\right) \\
&\quad s^*\left(t+u-\frac{\tau}{2}\right)e^{-j2\pi f\tau}dud\tau \qquad (9)
\end{aligned}
$$

where K_P is the kernel corresponding to the operator $\mathcal{K}_P$, and $\Phi(u,\tau) = K_P(u-\tau/2, u+\tau/2)$. We note that (9) is a familiar expression for Cohen's class [17], and that this operator-based characterization of Cohen's class is also used in [14].

Affine class of bilinear TSRs. Let $\mathcal{H} = L^2(\mathbb{R})$ and $G = \mathbb{R} \times (0,\infty)$, and let the group operation be defined by $(t_1,c_1)\bullet(t_2,c_2) = (t_1+c_1t_2, c_1c_2)$ (affine group); $(t,c)^{-1} = (-t/c, 1/c)$. Define the dilation operator as $(\mathcal{D}_c s)(x) = \frac{1}{\sqrt{c}}s(x/c)$ and the time-shift and scaling operator as $\mathcal{U}_{(t,c)} = \mathcal{T}_t\mathcal{D}_c$ which satisfies the composition group law (1). Using (3) in Theorem 1, the class of bilinear representations covariant to time-shifts and scalings is characterized by

$$
\begin{aligned}
(Ps)(t,c) &= \langle \mathcal{K}_P\mathcal{U}_{(t,c)^{-1}}s, \mathcal{U}_{(t,c)^{-1}}s\rangle \\
&= \langle \mathcal{K}_P\mathcal{T}_{-t/c}\mathcal{D}_{1/c}s, \mathcal{T}_{-t/c}\mathcal{D}_{1/c}s\rangle \\
&= c\int\int\int K_P(u_2,u_1)W_s\left(\frac{c(u_1+u_2)}{2}+t, f\right) \\
&\quad e^{j2\pi fc(u_1-u_2)}dfdu_1du_2 \\
&= \int\int \Pi\left(\frac{u-t}{c}, fc\right)W_s(u,f)dudf \qquad (10)
\end{aligned}
$$

where W_s is the Wigner distribution of s defined by $W_s(t,f) = \int s(t+\tau/2)s^*(t-\tau/2)e^{-j2\pi f\tau}d\tau$ and the kernel Π is related to K_P as $\Pi(u,f) = \int K_P(u+\tau/2, u-\tau/2)e^{-j2\pi f\tau}d\tau$. Note that (10) is a familiar characterization of the affine class [1].

Other covariance-based classes, like the hyperbolic class [6] covariant to scalings and hyperbolic time-shifts, can be characterized in a similar way (within a remapping of coordinates).

V. Conclusion

Cohen's marginal-based recipe for constructing generalized joint signal representations, although canonical from a distributional viewpoint, is not adequate to characterize the effect of certain unitary transformations on signals; a covariance-based method is needed in such situations (for example, in signal detection scenarios). In this paper, we have presented a simple technique for generating joint representations having arbitrary group covariance properties with respect to given unitary signal transformations.

Our method, although similar in principle to that presented in [9], [10], is simpler and more direct. In particular, in [9], [10] generalized *time-frequency* representations are considered which necessarily involve a remapping of coordinates that makes their characterization more unwieldy and complicated.[8]

It is worth mentioning that in the marginal-based approach of Cohen's, the covariance properties are difficult to analyze in general, and in the covariance-based method, the marginal properties become nontrivial to characterize. Some preliminary results on such issues have been recently reported [18], [10] but still more work needs to be done to completely bridge the gap between the methodologies of the two approaches.

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[8] In many applications of covariance-based signal representations (signal detection, for example), such a remapping is unnecessary.