VARIETIES OF KLEINIAN GROUPS*

by Irwin Kra

As one of very few (if not the unique one among the) participants at a conference on singularities who works with Kleinian groups, my biggest contribution to this volume might be an expository paper concerning some problems of current interest in the general area of Kleinian groups. I will hence describe here some new results concerning the variety of homomorphisms of a Kleinian group into the Möbius group as well as discuss several interesting open problems.

§1. *Introduction*

Let G be the group of Möbius transformations; that is, G consists of mappings g of the complex sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where $\mathbf{C}$ = the complex numbers, of the form

$$(1.1)\quad g(z) = (az+b)(cz+d)^{-1},\ z \in \hat{\mathbf{C}},\ \{a,b,c,d\} \in \mathbf{C}^4,\ ad-bc = 1.$$

Let Γ be a subgroup of G. We say that Γ is *discontinuous* at $z \in \hat{\mathbf{C}}$ provided

(a) $\Gamma_z = \{\gamma \in \Gamma; \gamma z = z\}$ is a finite group, and

(b) there is a neighborhood U of z such that $\gamma U \cap U = \varnothing$ for $\gamma \in \Gamma - \Gamma_z$, and $\gamma U = U$ for $\gamma \in \Gamma_z$.

Let $\Omega = \Omega(\Gamma)$ denote the *region of discontinuity* of Γ; that is, the set of points $z \in \hat{\mathbf{C}}$ such that Γ is discontinuous at z. It is an immediate consequence of the definition that Ω is an open (not necessarily connected) subset of $\hat{\mathbf{C}}$. We call Γ a *Kleinian* group provided

(a) $\Omega \neq \varnothing$ (Ω is hence dense in $\hat{\mathbf{C}}$), and

(b) the *limit set* $\Lambda = \Lambda(\Gamma) = \hat{\mathbf{C}} - \Omega(\Gamma)$ consists of more than two points (Λ is hence always a perfect closed subset of $\hat{\mathbf{C}}$).

Remarks. (1) The groups Γ where Λ consists of two or fewer points are called *elementary* groups. They are well known. See Ford [*6*, Chapter VI].

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(2) Since G is a Lie group, it makes sense to discuss *discrete* subgroups Γ of G. Every Kleinian group is discrete (hence countable). However, the converse is not true. The Picard group consisting of mappings g (see (1.1)) with a, b, c, d Gaussian integers (that is, complex numbers of the form $m + n\sqrt{-1}$, with m and n integers) is discrete but not discontinuous at any point of $\hat{\mathbf{C}}$.

(3) The concept of discontinuity introduced above is valid for any group of motions on a topological space.

An element $g \in G - \{1\}$ is called *parabolic* if $\operatorname{trace}^2 g = (a+d)^2 = 4$, it is called *elliptic* if $\operatorname{trace}^2 g$ is real and in the interval $[0,4)$, and *loxodromic* otherwise (a loxodromic element with real trace is also called *hyperbolic*). A parabolic element has precisely one fixed point in $\hat{\mathbf{C}}$; all others have two. Elliptic elements of infinite order cannot appear in discrete groups (hence also not in Kleinian groups).

If Γ is a Kleinian group, then Ω/Γ can be given a unique conformal structure such that the natural projection mapping

$$\pi: \Omega \to \Omega/\Gamma \tag{1.2}$$

is holomorphic. It is ramified over the points z with Γ_z non-trivial (Γ_z is always a finite cyclic group). Thus Ω/Γ is a countable union of Riemann surfaces.

By a *puncture* on a Riemann surface S we mean a domain $D \subset S$ such that D is conformally equivalent to

$$\{z \in \mathbf{C};\ 0 < |z| \leqq 1\}$$

with $z = 0$ not on S. Every puncture D on Ω/Γ is determined by a parabolic element $A \in \Gamma$ (see Ahlfors [*1*]), in the sense that there exist a closed conformal disc $U \subset \Omega$ and a parabolic element $A \in \Gamma$ such that $\pi(U) = D$, and two points z_1 and z_2 in U are equivalent under Γ if and only if $z_1 = A^n z_2$ for some integer n. We shall say that A is the *parabolic element determined by the puncture D on* Ω/Γ. It is quite clear that Ω/Γ can be imbedded into a smallest union of surfaces $\overline{\Omega/\Gamma}$ such that each component of $\overline{\Omega/\Gamma}$ has no punctures.

§2. *The Main Problem*

From now on we assume that Γ is a finitely generated Kleinian group (with region of discontinuity Ω and limit set Λ); in fact, let $\gamma_1, \cdots, \gamma_r$ be a fixed set of generators for Γ. By Ahlfors's finiteness theorem [*1*], $\overline{\Omega/\Gamma}$ is a finite union of compact Riemann surfaces, the natural projection π

of (1.2) is ramified over finitely many points, and $\overline{\Omega/\Gamma} - \Omega/\Gamma$ consists of finitely many points (points of ramification order ∞).

Let $\mathrm{Hom}(\Gamma, G)$ denote the affine algebraic variety (see Bers [4]) of homomorphisms of Γ into the Möbius group G. A homomorphism $\chi \in \mathrm{Hom}(\Gamma, G)$ is called *parabolic* if

$$\chi(\gamma) \text{ is parabolic or the identity} \tag{2.1}$$

for every parabolic element $\gamma \in \Gamma$ determined by a puncture on Ω/Γ. The affine algebraic variety of parabolic homomorphisms ((2.1) is, of course, equivalent to the statement $\mathrm{trace}^2 \chi(\gamma) = 4$) of Γ into G is denoted by $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$. We are, of course, viewing $\mathrm{Hom}(\Gamma, G)$ as a subvariety of G^r, via the map

$$\mathrm{Hom}(\Gamma, G) \ni \chi \mapsto (\chi(\gamma_1), \cdots, \chi(\gamma_r)) \in G^r.$$

Example 1. Let D be a domain in $\hat{\mathbf{C}}$ bounded by $2p$ ($p \geqq 2$) disjoint simple closed curves (usually circles) $C_1, C_1', \cdots, C_p, C_p'$. For $j = 1, \cdots, p$, let $A_j \in G$ be such that $A_j(C_j) = C_j'$ and $A_j(D) \cap D = \varnothing$. Let Γ be the group generated by $A_1, \cdots, A_p$. The group Γ is called a *Schottky* group of genus p. It is easy to show that $\Omega(\Gamma)$ is connected (of infinite connectivity) and that $\Omega(\Gamma)/\Gamma$ is a compact Riemann surface of genus p. Furthermore, Γ is a purely loxodromic free group on the p generators $A_1, \cdots, A_p$. Thus, we see that

$$\mathrm{Hom}(\Gamma, G) \cong G^p.$$

Example 2. A Kleinian group F is called *Fuchsian* if there is a circle C in the complex sphere $\hat{\mathbf{C}}$ such that F leaves the interior of C invariant. By conjugation we may always assume that C is the extended real axis $\mathbf{R} \cup \{\infty\}$. For such a group $\Lambda \subset C$. It is called of the *first kind* whenever $\Lambda = C$; of the *second kind* otherwise. Let F be the covering group of a compact Riemann surface S of genus $p \geqq 2$. It is well known that F may be realized as a Fuchsian group that is generated by $2p$ elements $A_1, B_1, \cdots, A_p, B_p$ with defining relation

$$A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_p B_p A_p^{-1} B_p^{-1} = 1. \tag{2.2}$$

It is quite clear that we may use (2.2) to define a complex analytic mapping

$$P: G^{2p} \to G.$$

Thus

$$\mathrm{Hom}(F, G) = \{g \in G^{2p}; P(g) = 1\}.$$

Note that for every Fuchsian group F of the first kind, the region of discontinuity consists of precisely two simply connected invariant components: namely the upper half plane U and the lower half plane L. Furthermore U/F is anti-conformally equivalent to L/F. For Fuchsian groups F of the second kind, Ω/F consists of a single Riemann surface (Ω is connected) with an anti-conformal involution J. In this case U/F consists of a single Riemann surface with border $\Omega \cap (\mathbf{R} \cup \{\infty\})/F$.

Remark. If F is a group of conformal automorphisms of U (hence $F \subset G$), then F is discrete if and only if it acts discontinuously on U (that is, if and only if $U \subset \Omega(\Gamma)$).

Problem. Describe the variety $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$. Some particular questions include:

(2.1) Is the identity a manifold point of $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$? What does a neighborhood of the identity look like?

(2.2) Is the set of isomorphisms, $\mathrm{Isom}_{\mathrm{par}}(\Gamma, G)$, in $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$ connected? More specifically, is the set of isomorphisms of Γ onto Kleinian groups, K-$\mathrm{Isom}_{\mathrm{par}}(\Gamma, G)$, connected? Does the boundary of this set contain only discrete subgroups of G?

To begin to talk about such problems, we must discuss one of the two most important tools available in this area: quasiconformal mappings.

§3. *Quasiconformal Stability*

Let $w: \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a homeomorphism of the Riemann sphere. The automorphism w is said to be *quasiconformal* if it possesses measurable, locally square integrable derivatives that satisfy the *Beltrami equation*

$$w_{\bar{z}} = \mu w_z, \tag{3.1}$$

where μ has L^∞ (supremum) norm less than one. The basic result of Ahlfors-Bers [2] establishes a canonical one-to-one correspondence between the open unit ball in $L^\infty(\hat{\mathbf{C}})$ and normalized (fixing $0, 1, \infty$) quasiconformal automorphisms of $\hat{\mathbf{C}}$; that is, for every $\mu \in L^\infty(\hat{\mathbf{C}})$ with

$$\| \mu \| = \sup\{ |\mu(z)|; z \in \hat{\mathbf{C}}\} < 1$$

there is a unique quasiconformal automorphism $w = w^\mu$ that fixes $0, 1, \infty$ and satisfies the Beltrami equation (3.1). Furthermore, every other quasiconformal w that satisfies the same Beltrami equation is given by

$$w = g \circ w^\mu, \quad g \in G. \tag{3.2}$$

A quasiconformal automorphism w is *compatible* with the Kleinian group Γ provided $w\Gamma w^{-1} \subset G$. We denote by $M(\Gamma) = M(\Gamma, \hat{\mathbf{C}})$ the set of *Beltrami coefficients* for Γ; that is, those $\mu \in L^\infty(\hat{\mathbf{C}})$ that satisfy

$$\mu(\gamma z)\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z), \text{ a.e. } z \in \hat{\mathbf{C}}, \text{ all } \gamma \in \Gamma,$$

and

$$\|\mu\| < 1.$$

It is routine to verify that a quasiconformal automorphism w is compatible with Γ if and only if in the decomposition (3.2), $\mu \in M(\Gamma)$. If w is compatible with Γ, then the mapping

$$\Gamma \ni \gamma \mapsto \chi(\gamma) = w \circ \gamma \circ w^{-1} \in G$$

defines an isomorphism of Γ onto the Kleinian group $w\Gamma w^{-1}$ called a *quasiconformal deformation* of Γ. We thus have a well defined holomorphic mapping

$$\chi\colon G \times M(\Gamma) \to \mathrm{Hom}_{\mathrm{par}}(\Gamma, G). \tag{3.3}$$

(The mapping (3.3) is, of course, not one-to-one.) Following Bers [*4*], the group Γ is called *(quasiconformally) stable* if there exists a neighborhood V of the identity in $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$ such that every $\chi \in V$ is a quasiconformal deformation.

Problems. (3.1) What are necessary and sufficient conditions for stability?

(3.2) Is a quasiconformal deformation of a stable group again stable?

(3.3) If Γ_0 is a normal subgroup of finite index in Γ and if Γ_0 is stable, is Γ also stable?

Remark. I have very recently been able to answer the last of the above problems in the affirmative. The details will appear elsewhere.

To study this general question of quasiconformal stability we must turn to

§4. *Quadratic Differentials*

Recall that Ω is the region of discontinuity of the finitely generated Kleinian group Γ. A holomorphic function ϕ on Ω is called an *automorphic form of weight* -4 (quadratic differential) if

$$\phi(\gamma z)\gamma'(z)^2 = \phi(z), \text{ all } \gamma \in \Gamma,\ z \in \Omega.$$

The function ϕ is a *cusp form* if it satisfies one (and hence both) of the following equivalent conditions

$$\|\phi\|_1 = \iint_{\Omega/\Gamma} |\phi(z)dz \wedge d\bar{z}| < \infty$$

and

$$\|\phi\|_\infty = \sup\{\lambda^{-2}(z)|\phi(z)|; z \in \Omega\} < \infty,$$

where $\lambda(z)|dz|$ is the Poincaré metric on Ω (the unique complete Riemannian metric on Ω of constant negative curvature -4). We denote by $A(\Omega)$ the space of cusp forms for Γ.

Write

$$\Omega/\Gamma = \bigcup_{i=1}^{m} S_i,$$

where each S_i is a compact Riemann surface of genus $p_i \geqq 0$ with $q_i \geqq 0$ punctures, and the map

$$\pi: \pi^{-1}(S_i) \to S_i$$

is ramified over $r_i \geqq 0$ points. Let $n_i = q_i + r_i$. Then it is a well known consequence of the Riemann-Roch theorem that

$$\dim A(\Omega) = \sum_{i=1}^{m} (3p_i - 3 + n_i).$$

In [*4*], Bers obtained the following sufficient condition for stability.

If the variety $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$ *is locally irreducible at the identity and has dimension* $\dim A(\Omega) + 3$ *there, then* Γ *is stable and the identity is a regular point of* $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$.

The proof actually yields more. One constructs a map

$$\Phi: G \times A(\Omega)_1 \to \mathrm{Hom}_{\mathrm{par}}(\Gamma, G) \tag{4.1}$$

as follows:
(Here $A(\Omega)_1$ is the open unit ball in $A(\Omega)$ with $\|\cdot\|_\infty$ norm.)

Fof $\phi \in A(\Omega)$, define

$$\mu = \begin{cases} \lambda^{-2}\bar{\phi} & \text{on } \Omega \\ 0 & \text{on } \Lambda \end{cases}.$$

(**Remark.** It is not known if there are any finitely generated Kleinian groups Γ whose limit set Λ has positive two dimensional Lebesgue measure.)

For $(g, \phi) \in G \times A(\Omega)_1$, we define $\Phi(g, \phi)$ as conjugation of Γ by $g \circ w^\mu$. One shows that Φ covers a neighborhood of the identity in $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$.

To obtain a sufficient condition for stability that does not require information on the local properties of the variety $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$ at the identity we must turn to

§5. *Eichler Cohomology for Kleinian Groups*

The Lie group G acts on the right (by the *Eichler action*) on the vector space Π of quadratic polynomials via

$$p\gamma(z) = p(\gamma z)\gamma'(z)^{-1}, \ p \in \Pi, \gamma \in G, z \in \mathbf{C}.$$

One can thus define the first cohomology group $H^1(\Gamma, \Pi)$ for any (finitely generated Kleinian) subgroup Γ of G, as follows: A *cocycle* is a mapping $\chi: \Gamma \to \Pi$ such that

$$\chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1)\gamma_2 + \chi(\gamma_2), \ (\gamma_1, \gamma_2) \in \Gamma^2.$$

If $p \in \Pi$, then its *coboundary* is the cocycle

$$\chi(\gamma) = p\gamma - p, \ \gamma \in \Gamma.$$

The *cohomology group* $H^1(\Gamma, \Pi)$ is the group of cocycles factored by the group of coboundaries. A cohomology class $\chi \in H^1(\Gamma, \Pi)$ is called *parabolic* if for every parabolic element $A \in \Gamma$ that is determined by a puncture on Ω/Γ, we have

$$\chi \,|\, (A) = \{0\}.$$

The subgroup of parabolic cohomology classes is denoted by $H^1_{\mathrm{par}}(\Gamma, \Pi)$.

For $\phi \in A(\Omega)$, we define (see Ahlfors [*1*])

$$h(z) = \frac{z(z-1)}{2\pi i} \iint_\Omega \frac{\lambda^{-2}(\zeta)\overline{\phi(\zeta)}}{\zeta(\zeta - 1)(\zeta - z)} d\zeta \wedge d\bar{\zeta}.$$

It is then easy to check that for $\gamma \in \Gamma$,

$$p(\gamma)(z) = h(\gamma z)\gamma'(z)^{-1} - h(z), \ z \in \mathbf{C},$$

defines an element of Π. Hence h defines a cohomology class $\beta^*\phi \in H^1_{\mathrm{par}}(\Gamma, \Pi)$. We have thus defined the injective map (see Ahlfors [*1*]),

$$\beta^*: A(\Omega) \to H^1_{\mathrm{par}}(\Gamma, \Pi).$$

Remark. The map β^* need not be surjective. For a complete description of $H^1_{\mathrm{par}}(\Gamma, \Pi)$ see, for example, Kra [*9*], [*10*].

In terms of the above concepts, Gardiner and Kra [*7*] obtained:
The group Γ *is stable provided* β^* *is surjective.*

As before, the map Φ of (4.1) covers, under this hypothesis, a neighborhood of the identity in $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$.

Problems. (5.1) Does stability imply surjectivity of the map β^*?

(5.2) If β is surjective for a group Γ, is it also surjective for a quasiconformal deformation of Γ? (This is the linear version of Problem (3.2).)

The first of the above problems is intimately connected with another interesting problem. The G-module Π of quadratic polynomials with the Eichler action is G-isomorphic to the Lie algebra $\mathscr{G}(\cong \mathrm{Sl}(2, \mathbf{C}))$ of G with adjoint action. Thus $H^1(\Gamma, \Pi)$ is canonically isomorphic to $H^1(\Gamma, \mathscr{G})$. If β is a one parameter family of homomorphisms of Γ into G with $\beta(0) =$ identity, then

$$c(\gamma) = \lim_{t \to 0} \frac{\gamma^{-1} \circ \beta(t)\gamma - 1}{t}, \ \gamma \in \Gamma,$$

defines a $\mathscr{G}$-cocycle for Γ (hence via the isomorphism mentioned above a Π-cocycle).

Problem. (5.3) Does every Π-cocycle for Γ arise in the above manner?

As a consequence of the above criterion for stability as well as the results of Kra [*9*], [*10*] on cohomology of Kleinian groups, we conclude at once that Schottky and Fuchsian groups (all groups are assumed to be finitely generated) are stable, as are Kleinian groups with two invariant components. However, *degenerate* groups (Ω connected and simply connected) are not stable.

§6. *Quasiconformal Deformation Spaces*

Let Δ be an invariant union of components of $\Omega(\Gamma)$, with Γ a finitely generated Kleinian group. Let $M(\Gamma, \Delta)$ denote the space of Beltrami coefficients for Γ that vanish outside Δ. This is (an infinite dimensional) Banach manifold. Each $\mu \in M(\Gamma, \Delta)$ determines an isomorphism χ_μ of Γ into G as follows:

$$\chi_\mu(\gamma) = w^\mu \circ \gamma \circ (w^\mu)^{-1}, \quad \gamma \in \Gamma. \tag{6.1}$$

An element $\mu \in M(\Gamma, \Delta)$ is called a *trivial* if χ_μ is the identity isomorphism. The set of trivial Beltrami coefficients is denoted by $M_0(\Gamma, \Delta)$. The set $M_0(\Gamma, \Delta)$ acts as a group of *right translations and biholomorphic self-mappings* of $M(\Gamma, \Delta)$ by

$$M(\Gamma,\Delta) \times M_0(\Gamma,\Delta) \ni (\nu,\mu) \mapsto \nu\mu \in M(\Gamma,\Delta),$$

where

$$w^{\nu\mu} = w^{\nu}w^{\mu}.$$

The *quasiconformal deformation space of* Γ *(with support in* Δ*)* is

$$T(\Gamma,\Delta) = M(\Gamma,\Delta)/M_0(\Gamma,\Delta)$$

endowed with the quotient topology.

If F is a finitely generated Fuchsian group of the first kind operating on the upper half plane U, then $T(F,U)$ is the usual *Teichmüller space* (see Bers [*3*]).

It is known (Maskit [*19*], Kra [*14*] and Bers [*5*], Marden [*16*], under restrictive hypotheses) that $T(\Gamma,\Delta)$ is actually a finite dimensional complex analytic manifold of the same dimension as

$$A(\Delta) = \{\phi \in A(\Omega);\ \phi \text{ vanishes outside } \Delta\}.$$

To describe the deformation space $T(\Gamma,\Delta)$ more precisely, let Δ_j, $j = 1,\cdots,m$, be a maximal collection of non-equivalent components of Δ. For each j, let Γ_j be the stability subgroup of Δ_j; that is,

$$\Gamma_j = \{\gamma \in \Gamma;\ \gamma\Delta_j = \Delta_j\}.$$

By Ahlfors's finiteness theorem [*1*], Γ_j is again a Kleinian group, and (see Kra [*14*])

$$T(\Gamma,\Delta) \cong \prod_{j=1}^{m} T(\Gamma_j,\Delta_j).$$

Thus, to study the structure of these deformation spaces, it suffices to assume that Δ is an invariant component of the group Γ. Since Λ contains more than two points, the holomorphic universal covering space of Δ is conformally equivalent to the upper half plane U. We choose a holomorphic universal covering map

$$\rho\colon U \to \Delta,$$

and let F be the *Fuchsian model of* Γ; that is,

$$F = \{f \in G; fU = U \text{ and } \rho \circ f = \gamma \circ \rho \text{ for some } \gamma \in \Gamma\}.$$

Since $\Delta/\Gamma \cong U/F$, it is easy to see that F is a finitely generated Fuchsian group of the first kind. One shows (see Maskit [*19*] or Kra [*14*]) that

$$T(\Gamma,\Delta) \cong T(F,U)/\mathrm{cov}\,\rho,$$

where $\operatorname{cov}\rho$ is a group of biholomorphic automorphisms of $T(F, U)$. Furthermore, the group $\operatorname{cov}\rho$ acts discontinuously and fixed point freely on $T(F, U)$. Since $T(F, U)$ is simply connected, it is the holomorphic universal covering space of $T(\Gamma, \Delta)$ and $\operatorname{cov}\rho$ is isomorphic to the fundamental group of $T(\Gamma, \Delta)$.

To discuss some open problems we return to the general situation (arbitrary Γ and Δ). We use (6.1) to define a holomorphic mapping

$$\Phi\colon T(\Gamma, \Delta) \to \operatorname{Hom}_{\mathrm{par}}(\Gamma, G).$$

This mapping is one-to-one and of maximal rank.

Problems. (6.1) Is $\Phi(T(\Gamma, \Delta))$ a submanifold of $\operatorname{Hom}_{\mathrm{par}}(\Gamma, G)$?

(6.2) Describe the boundary of $\Phi(T(\Gamma, \Delta))$ in $\operatorname{Hom}_{\mathrm{par}}(\Gamma, \Delta)$. In particular, does every boundary point represent a discrete subgroup of G?

(6.3) We have stated that $T(\Gamma, \Delta)$ is a d-dimensional complex analytic manifold ($d = \dim A(\Delta)$). Is it biholomorphically equivalent to a submanifold of $\mathbf{C}^d$?

For Fuchsian groups F of the first kind the answer to Problem 6.3 is known. The realization of $T(F, U)$ as a submanifold of $A(L)$ ($L =$ lower half plane) is one of the central results of Teichmüller space theory (Bers [*3*]).

For a class of "nice" Kleinian groups (see also §8), Maskit obtains a positive solution in a forthcoming paper. The general problem is, however, still open.

§7. *Holomorphic Deformations*

We assume that the finitely generated Kleinian group Γ has a simply connected invariant component D. Fix a point $z_0 \in D$. For every $\phi \in A(D)$, let $f = f_\phi$ be the unique solution to the *Schwarzian equation*

$$\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \phi$$

normalized so that

$$f(z) = (z - z_0) + O(|z - z_0|^3), \ z \to z_0.$$

From the Cayley identity, one concludes that for each $\gamma \in \Gamma$ there is a $\chi_\phi(\gamma) \in G$ so that

$$f \circ \gamma = \chi_\phi(\gamma) \circ f.$$

It is easy to check that the mapping

$$\Gamma \ni \gamma \mapsto \chi_\phi(\gamma) \in G$$

is a parabolic homomorphism, and that we have constructed an injective (see Kra [13]) holomorphic mapping

$$\Psi_D: A(D) \to \mathrm{Hom}_{\mathrm{par}}(\Gamma, G). \tag{7.1}$$

It follows from Gardiner-Kra [7] that a neighborhood of the identity in $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$ is biholomorphically equivalent to a neighborhood of $(1,0,0)$ in

$$G \times A(D) \times A(D)$$

(the first two components are mapped into $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$ via the map Φ of (4.1), and the last via the map Ψ of (7.1)). In some sense $\Phi(G \times A(D))$ should always be "orthogonal" to $\Psi_D(A(D))$. But I do not know how to formulate this precisely.

Let us specialize to a finitely generated Fuchsian group F of the first kind operating on the upper half plane U (thus also on the lower half plane L). Even in this very classical case many questions remain.

Problems. (7.1) Let $\phi \in A(U)$. Find necessary and sufficient conditions for $\chi_\phi(F)$ to be Kleinian.

A simpler problem has been solved (Kra [11], [12]):

The following are equivalent for $\phi \in A(U)$:

(a) *The group* $\chi_\phi(F)$ *acts discontinuously on* $f_\phi(U)$,

(b) *The mapping* f_ϕ *is an unbranched unramified covering, and*

(c) $f_\phi(U) \neq \hat{\mathbf{C}}$.

Maskit [17] has shown that the above does not tell the whole story. For some groups F (for example, covering groups of compact surfaces) there are $\phi \in A(U)$ for which $\chi_\phi(F)$ is Kleinian (even Fuchsian and isomorphic to F) but $f_\phi(U) = \hat{\mathbf{C}}$.

(7.2) Does the intersection $\Psi_U(A(U)) \times \Psi_L(A(L))$ consist only of the identity?

Again, this is a kind of "orthogonality" condition. A partial result has been obtained by Kra-Maskit [15]:

If $f_1: U \to \hat{\mathbf{C}}$ *and* $f_2: L \to \hat{\mathbf{C}}$ *are two holomorphic universal covering mappings of subdomains in* $\hat{\mathbf{C}}$ *such that for each* $\gamma \in F$, *there is a* $\chi(\gamma)$ *belonging to a Kleinian group* Γ *with*

$$f_j \circ \gamma = \chi(\gamma) \circ f_j, \quad \textit{for } j = 1,2,$$

then there is a $g \in G$ *such that*

$$g|U = f_1 \quad \textit{and} \quad g|L = f_2,$$

whenever f_1 *and* f_2 *are schlicht. In general* $\chi(F)$ *is Fuchsian or a* $\mathbf{Z}_2$-*extension of a Fuchsian group.*

§8. *Combination Theorems*

The basic idea of Klein's combination theorem is quite simple. One starts with two Kleinian groups Γ_1 and Γ_2 satisfying certain algebraic and geometric conditions. and one concludes that the group Γ generated by Γ_1 and Γ_2 is again Kleinian. Furthermore, the structure of $\Omega(\Gamma)/\Gamma$ is read off from the structure of $\Omega(\Gamma_1)/\Gamma_1$ and $\Omega(\Gamma_2)/\Gamma_2$.

In Klein's original theorem, Γ is the free product of Γ_1 and Γ_2. Maskit [*20*] has generalized this theorem considerably. He studies two cases. In the first, Γ is the free product of Γ_1 and Γ_2 with amalgamated subgroup H (usually cyclic). In the second, Γ_2 is cyclic and the generator of Γ_2 conjugates a (usually cyclic) subgroup H_1 of Γ_1 into a subgroup H_2 of Γ_1.

Starting with a set of building blocks (consisting of elementary groups, Kleinian groups with two invariant components [this includes Fuchsian groups], degenerate groups and Schottky groups) and applying the above combination theorems one obtains a set of "nice" Kleinian groups. In a forthcoming paper Maskit investigates alternate descriptions for the set of "nice" Kleinian groups, and establishes necessary and sufficient conditions for stability of such groups.

It is also possible to obtain information on cohomology and cusp forms for Γ from the corresponding information for Γ_1 and Γ_2. This point of view will be pursued by this author in a subsequent paper on stability of Kleinian groups.

§9. *The Upper Half Space*

Let Δ be an invariant union of components for a Kleinian group Γ. By a *fundamental domain* ω for Γ in Δ we mean an open subset $\omega \subset \Delta$ such that no two points of ω are equivalent under Γ, and every point in Δ is equivalent under Γ to a point of the closure of ω. Using Ahlfors's finiteness theorem [*1*], and classical constructions for Fuchsian groups (see, for example, Ford [*6*, Chapter III]), it is easy to show that for Γ finitely generated we can construct an ω that consists of a union of finitely many non-Euclidean polygons in Δ each with finitely many sides. As a matter of fact:

If Δ is an invariant component of a Kleinian group Γ, then Γ is finitely generated if and only if Γ has a fundamental domain in Δ that is a finitely sided non-Euclidean polygon.

Every Kleinian group may be viewed as a group of motions of the upper half space

$$\mathscr{H} = \{(z,t);\, z \in \hat{\mathbf{C}},\ t \in \mathbf{R},\ t > 0\}.$$

Every $g \in G$ is a product of inversions in circles, which can be extended to inversions in half spheres in $\mathscr{H}$ that surmount the circles. In this way g acts as a conformal map (and an isometry in the non-Euclidean metric) of $\mathscr{H}$. The formula for the action of g on $\mathscr{H}$ is hardly ever used, but it is given as follows: If $g(z) = (az + b)(cz + d)^{-1}$, and if

$$g(z,t) = (z',t'),$$

then

$$z' = \frac{(az+b)(\overline{cz+d}) + a\bar{c}t^2}{|cz+d|^2 + |c|^2t^2}, \quad t' = \frac{t}{|cz+d|^2 + |c|^2t^2}.$$

If $\Gamma \subset G$ is arbitrary, then Γ acts discontinuously on $\mathscr{H}$ if and only if Γ is discrete.

Furthermore, fundamental domains and fundamental polyhedra for Γ in $\mathscr{H}$ are defined as in the two-dimensional case. Not all finitely generated Γ admit finitely sided fundamental polyhedra in $\mathscr{H}$ (degenerate groups are a counterexample, see Greenberg [*8*]). For discrete Γ, $\mathscr{H}/\Gamma$ is always a three dimensional manifold. It has a boundary if and only if Γ is Kleinian. In this case the boundary is $\Omega(\Gamma)/\Gamma$. This three dimensional point of view was already familiar to Poincaré. However, until recently, it provided very little new information. Among the recent important results of this approach are the following:

If Γ and Γ' are two discrete subgroups of G, with $\mathscr{H}/\Gamma$ and $\mathscr{H}/\Gamma'$ having finite non-Euclidean volume, then every isomorphism

$$\chi: \Gamma \to \Gamma'$$

is conjugation by a diffeomorphism g with g or $\bar{g} \in G$.

The above is a deep result of Mostow [*21*]. (See also Marden [*16*].) It does not, of course, apply to Kleinian groups (these do not have finite non-Euclidean volume). Recently Marden [*16*] has obtained the following:

If Γ is a torsion free Kleinian group that admits a finitely sided fundamental polyhedron, then Γ is stable and the image $\Phi(T(\Gamma, \Omega(\Gamma)))$ is a submanifold of $\mathrm{Hom}_{\mathrm{par}}(\Gamma, G)$.

§10. *Isomorphisms of Kleinian Groups*

There is a vast body of unpublished work by W. Fenchel and J. Nielsen that goes under the title "Discontinuous groups of non-Euclidean motions."

It is intimately related to the problems discussed in this paper, in particular Problem 2.2.

Let F be a Fuchsian group operating on U. If $\gamma \in F$ is loxodromic, then A_γ, *the axis of* γ, is the non-Euclidean straight line in U joining the fixed points of γ. An isomorphism χ between two Fuchsian groups F_1 and F_2 is called *allowable* if

(a) for any pair of loxodromic elements γ_1 and $\gamma_2 \in F_1$, A_{γ_1} and A_{γ_2} intersect if and only if $A_{\chi(\gamma_1)}$ and $A_{\chi(\gamma_2)}$ intersect, and

(b) for any triple $\gamma_1, \gamma_2, \gamma_3$ of loxodromic elements of F_1, A_{γ_1} separates A_{γ_2} and A_{γ_3} if and only if $A_{\chi(\gamma_1)}$ separates $A_{\chi(\gamma_2)}$ and $A_{\chi(\gamma_3)}$.

Maskit [*18*] obtained the following special case of a Fenchel-Nielsen result:

Let F_1 and F_2 be finitely generated Fuchsian groups acting on U. Let $\chi: F_1 \to F_2$ be a type preserving allowable isomorphism. Then there is a quasiconformal automorphism w of U such that $w \circ \gamma = \chi(\gamma) \circ w$, for all $\gamma \in F_1$.

The isomorphism χ is called *type-preserving* whenever $\text{trace}^2\gamma = \text{trace}^2\chi(\gamma)$ for every $\gamma \in F_1$ such that γ or $\chi(\gamma)$ are non-loxodromic.

The above shows that every allowable type-preserving isomorphism of a finitely generated Fuchsian group F onto another such group "is" in the Teichmüller space of F (which is known to be connected). Thus we have a partial answer to Problem 2.2 in the special case of Fuchsian groups. The general problem is open. However (see Maskit [*19*]) the following special case is known:

Let Γ be a finitely generated Kleinian group. If $w: \Omega(\Gamma) \to \Omega(\Gamma)$ is a quasiconformal automorphism satisfying

$$w \circ \gamma = \gamma \circ w \textit{ for all } \gamma \in \Gamma,$$

then there is a (global) quasiconformal automorphism W of $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that

$$W \circ \gamma = \gamma \circ W, \textit{ for all } \gamma \in \Gamma,$$

and $W \mid \Omega(\Gamma) = w$.

Not every isomorphism between Kleinian groups can be induced by a global quasiconformal automorphism of $\hat{\mathbf{C}}$. Let Γ be a degenerate group. Let

$$\rho: U \to \Omega(\Gamma)$$

be a holomorphic universal covering map, and let F be the Fuchsian model of Γ. The map ρ induces an isomorphism χ of F onto Γ via

$$\rho \circ \gamma = \chi(\gamma) \circ \rho, \; \gamma \in F.$$

It is obvious that ρ cannot be extended to be a quasiconformal automorphism of $\hat{\mathbf{C}}$.

Remark. A. Marden has also investigated this general area, and has obtained new proofs of many results of Fenchel-Nielsen.

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