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FINITE ELEMENT ANALYSIS OF BEAMS ON ELASTIC FOUNDATIONS

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RICE UNIVERSITY

FINITE ELEMENT ANALYSIS OF BEAMS ON ELASTIC FOUNDATIONS

by

QIDAO ZHANG

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

MASTER OF SCIENCE

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November, 1985

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ABSTRACT

When an elastic foundation is incapable of exerting tensile reaction on the beam it supports, Winkler foundation models cannot be used throughout the whole span of the beam. An iterative procedure is used to produce the more realistic solutions. Two-parameter foundation model is employed. It is more accurate than one-parameter (Winkler) model, and is simpler than continuum foundation model. Both C^1 and C^2 continuity elements are displayed. Numerical tests show that the element based on the C^2 continuity polynomial can give more accurate results with fewer elements.

ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to Dr. John E. Akin, Thesis Director, for his guidance and assistance in all phases of this research.

He also wishes to express hearty thanks to Dr. Y. C. Angel and Dr. John E. Merwin for serving on this thesis committee.

LIST OF FIGURES

LIST OF TABLES

NOMENCLATURE

a_i _____ coefficient of polynomial expansion;
{c}---- element force vector;
E ----- elastic modulus of structure;
E_s----- elastic modulus of foundation;
{**F**}---- system force vector;
[H]---- interpolation function;
K_w ____ modulus of Winkler foundation;
K_s ---- modulus of second-parameter foundation;
[**K**]---- system stiffness matrix;
[K]---- element stiffness matrix;
[K_1]--- beam stiffness matrix;
[K_2]--- geometric stiffness matrix;
[K_3]--- second-parameter foundation stiffness matrrix;
[K_4]--- Winkler foundation stiffness matrix;
L_e _____ length of element;
M ----- point moment;
N ----- axial force;
P ----- point load;
q ----- distributed load;
Q_i----- nodal vertical forces;
{u}---- element nodal displacement vector;
{**U**}---- system nodal displacement vector;
w ----- vertical deflection;

TABLE OF CONTENTS

1. INTRODUCTION

1.1 Generalization

The stresses in a statically indeterminate structure are influenced by the deformation of the foundation while the pressure distribution on the foundation is affected by the relative stiffnesses of the structure and the foundation medium. To allow for this structure-foundation interaction the powerful finite element method is ideally suited.

This problem has been studied by many authors (4, 5, 8, 17, 18, 19, 22). Most of those works use Winkler hypothesis, and assume that the soil adheres to the beam, i. e., the separation between beam and soil is not allowed. This is not true for many physical cases. For instance, when a beam or a beam-column rests on the soil foundation with some type of load on it, some parts of the beam might be lifted up. Because of soils lacking both adhesive and cohesive properties, gaps occur in those region (see Fig. 3.10). The method presented in this thesis represents the foundation by a one-dimensional line finite element. The foundation is assumed to be of the two-parameter type. Also, the separation between structure and soil foundation is allowed when tension develops. The location of those regions is solved by recycling the solutions.

1.2 Governing Differential Equations

We cut out of the beam an infinitely small element bounded by two verticals a distance dx apart shown in Fig. 1.1, where $Q_v(x)$ is the transverse shear force, M(x) is the bending moment q(x) is the general load, N is the axial load, and p(x) is the general foundation reactions. From the equilibrium equations and moment-curvature equation of elementary beam theory, we obtain

$$dQ_v(x)/dx = q(x) - p(x) \tag{1.1}$$

$$dM(x)/dx = Q_v(x) - N\, dw(x)/dx \tag{1.2}$$

$$EI\, d^2w(x)/dx^2 = M(x) \tag{1.3}$$

in which the bending stiffness EI is assumed to be a constant and the transverse deformation is neglected. The normal shear Q_N acting in the plane of the section normal to the deflection line can be obtained as

$$Q_N = Q_v \cos\theta - N \sin\theta \tag{1.4}$$

and for this, making the usual assumption that since θ is generally small,

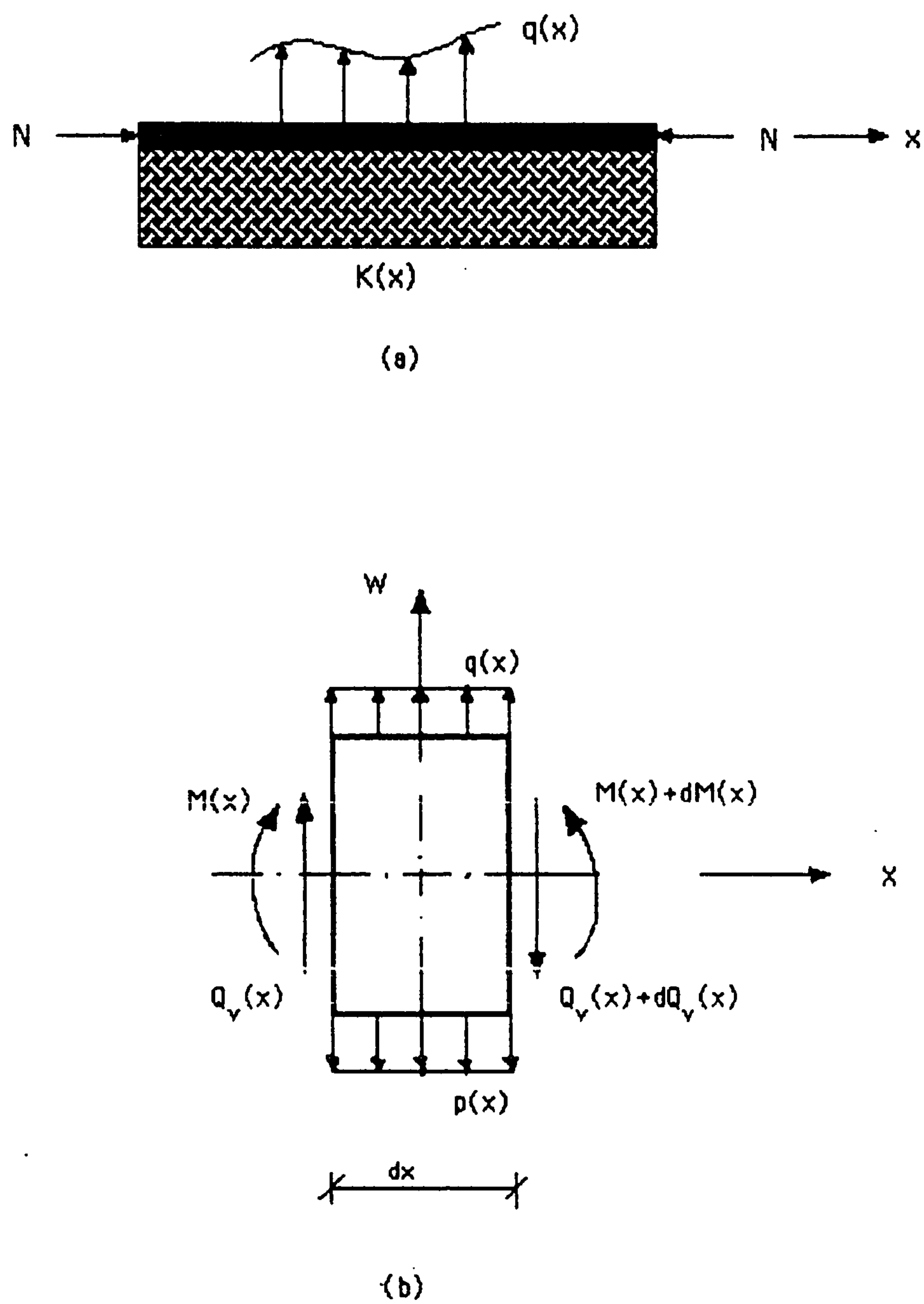


Fig. 1.1 Equilibrium of an Infinitesimal Element. (a) A beam on an elastic foundation with axial loading;(b)An infinitesimal element.

$\cos\theta = 1$, and $\sin\theta = \tan\theta = \theta = dw/dx$, thus

$$Q_N = Q_v - N\, dw(x)/dx = dM(x)/dx\,. \tag{1.5}$$

Substituting Eq.(1.3) into Eq. (1.2), then differentiating with respect to x, we obtain the differential equation of beam on elastic foundation as

$$EI\, d^4w/dx^4 + N\, d^2w/dx^2 - k_s\, d^2w/dx^2 + k_w w = q(x) \tag{1.6}$$

in which k_w and k_s are Winkler modulus and second-parameter respectively. When the axial force N is compression, the differential equation can be obtained simply by changing the sign of the axial force N in Eq. (1.6). When N and k_s are equal to zero, Eq. (1.6) becomes

$$EI\, d^4w/dx^4 + kw = q(x)\ . \tag{1.7}$$

This is the differential equation of ordinary beam on Winkler elastic foundation which is well known.

1.3 Equations for Foundation Modulii

The two foundation parameters k_w and k_s are evaluated by Vlasov's foundation (7). These equations apply to a foundation of infinite depth.

$$k_w = \frac{E_o b}{2(1-\mu_o^2)} \frac{r}{A} \tag{1.8}$$

$$k_s = \frac{E_o b}{4(1+\mu_o)} \frac{A}{r} \tag{1.9}$$

$$A = \left[\frac{2D(1-\mu_o^2)}{E_o b}\right]^{1/3} \tag{1.10}$$

$$E_o = \frac{E_s}{1-\mu_s^2} \tag{1.11}$$

For plane strain problem, according to elementary thin-plate theory, EI used in plane stress problem should be replaced by

$$D = \frac{E b h^3}{12(1-\mu^2)} \tag{1.12}$$

in which b is the width of the beam and h is its depth. Coefficient r depends on the properties of the foundation. In following examples, we let r=1. For sandy clay foundation, we choose that the foundation elastic modulus is 45.4N/mm^2, and the Poisson ratio of it is 0.21.

2. LITERATURE SURVEY

2.1 Solution of Beam on Elastic Foundation

The analysis of beams, beam-columns and plates on elastic foundations is widespread in engineering. Hetenyi (2) extensively develops the classical differential equation approaches. Miranda and Nair (19) adopt the method of initial parameters to express the solution of the beams on elastic foundation differential equation in terms of four special functions that are associated with the deflection, slope, moment and shear, respectively. These special functions possess an interesting property. The derivatives of these functions can be related back to the original functions, leading to substantial simplification in the solution of boundary value problem.

In recent years, finite element approachs have been used extensively in the analysis of beams on an elastic foundation (3, 4, 5, 16, 18, 21). Most of these works use the Winkler hypothesis. Thus, the foundation acts as if it consisted of infinitely many closely spaced linear springs. Bowles (18) formulates a stiffness matrix by combining a conventional beam element with discrete soil springs at the end of the beam. The degree of accuracy using this element is highly dependent on the number of elements modeled. Ting (5) derives the stiffness and flexibility matices from the exact solution of the differential equation. Some authors (3) use the cubic Hermitian polynomial (C^1 element) to approximate the beam on elastic foundation.

The Winkler model is very simple. But interactions between springs are not considered, so it does not accurately represent the characteristics of many practical foundations. For some problems a continuous medium model is more accurate, but it is difficult to obtain an exact solution with this model and is expensive to obtain a numerical result by finite element methods.

2.2 Foundation Models

Fig. 2.1 shows the action of an elastic foundation. When unloaded, the beam axis and the x-axis coincide. As a result of a line load $q(x)$ on the upper surface the beam deflects, causing the foundation to resist with a line load $p(x)$, whose units may be taken as N/mm. Various foundation models define $p(x)$ in various ways.

Winkler Foundation (2).--This foundation model has been used for a century. It assumes that the foundation applies only a reaction $p(x)$ normal to the beam, and that $p(x)$ is directly proportional to the beam deflection $w(x)$:

$$p(x) = kw(x). \tag{2.1}$$

The Winkler foundation modulus k has units N/mm/mm. Effectively, this

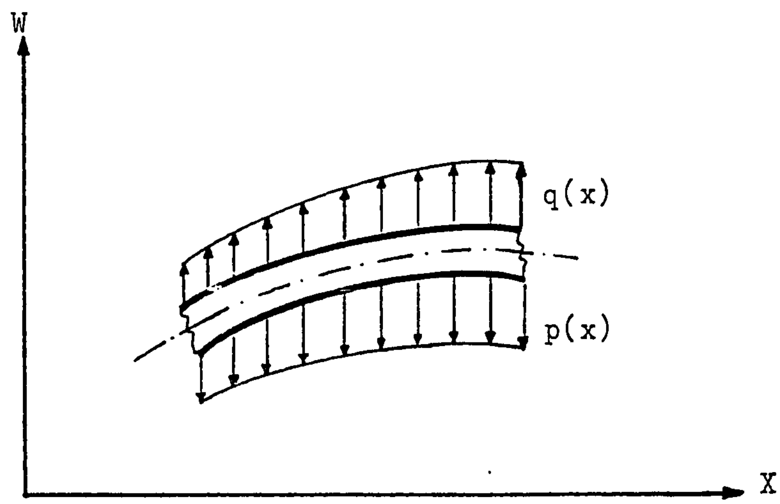


Fig. 2.1 Portion of Deflected Beam on Elastic Foundation

foundation is a row of closely-packed linear springs. There is only one foundation parameter in Eq. (2.1). To improve the Winkler model some authors assumed interactions between the springs and added a second parameter to Eq. (2.1). In the following we outline four of these two-parameter models.

Filonenko-Borodich Foundation.--Filonnenko-Borodich assumed that the top ends of the springs are connected to an elastic membrane that is stretched by a constant tension T. He obtained

$$p(x) = kw(x) - T\, d^2w(x)/dx^2 . \tag{2.2}$$

Pasternak Foundation--Pasternak introduced shear interactions between the springs. He assumed that the top ends of the springs are connected to an incompressible layer that resists only transverse shear deformation, and obtained

$$p(x) = kw(x) - k_G\, d^2w(x)/dx^2 . \tag{2.3}$$

in which k_G = a parameter of the shear layer.

Generalized Foundation--This model assumes that at each point of contact there is not only a pressure but also a moment applied to the beam by the foundation. The moment is assumed to be proportional to the angle of rotation. Thus, action of the foundation per unit length of the beam is taken as

$$p(x) = kw(x) \tag{2.4a}$$

$$m(x) = k_m \, dw(x)/dx \tag{2.4b}$$

in which k and k_m are the two modulii of the foundation. From m(x) we can obtain an equivalent line load. This can be done in the same way that twisting moment M_{xy} on the edge of a plate is standard in elementary thin-plate theory. Thus, Eqs. (2.4) can be replaced by

$$p(x) = kw(x) - k_m \, d^2w(x)/dx^2 \tag{2.5}$$

without any moment load from the foundation.

Vlasov Foundation (7) -- Some authors did not start from Winkler foundation but regarded the foundation as a semi-infinite elastic medium. This approach is mathematically complicated, so simplifying assumptions were introduced (7). Vlasov obtained the foundation reaction

$$p(x) = kw(x) - 2t \, d^2w(x)/dx^2 \tag{2.6}$$

as well as formulas for determining the parameters k and t in terms of elastic constants and dimensions of the beam and foundation.

Remarks-- Mathematically, Eqs. 2.2, 2.3, 2.5 and 2.6 are equivalent. The only difference is the definition of the parameters. When we solve the problems mathematically we need not pay attention to this difference, so we rewrite these equations in the form

$$p(x) = kw(x) - k_s\, d^2w(x)/dx^2 \tag{2.7}$$

in which k is the first parameter (Winkler's modulus); and k_s is the second parameter.

3. THE FINITE ELEMENT MODEL

3.1 Introduction

In classical continuum mechanics, as we discussed in CH.1, the physical problem is usually described by a set of differential or partial differential equations with proper boundary conditions, or by the extremum (in most cases, the minimum) of a variational principle, if it exists. Today the finite element method has become the most popular method for solving such equations. The method coupled with developments in computer technology has successfully been applied to the solution of steady and transient problems in linear and non-linear regions for one-, two-, and three-dimensional domains. It can easily handle discontinuous geometrical shapes as well as material discontinuities.

For the mathematical point of view the finite element method is based on integral formulations. Modern finite element integral formulations are obtained by two different procedures: variational formulations and weighted residual formulations. All of these techniques use the same bookkeeping operations to generate the final assembly of algebraic equations that must be solved for the unknown nodal parameters. Many physical problems have variational formulations that result in quadratic forms. These in turn yield algebraic equations for the system which are symmetric and positive definite. Another important practical advantage of variational formulations is that they often have error bound theorems associated with them.

There is a increasing emphasis on the various weighted residual techniques that can generate an integral formulation directly from the origional differential equations. The weighted residual method starts with the governing differential equation like

$$\mathcal{L}(u) = Q \tag{3-1}$$

and avoids the often tedious search for a mathematically equivalent variational statement. In these methods, an approximate solution is substituted into the differential equation. Since the approximate solution does not satisfy the equation, a residual or error term results. Suppose that u^* is an approximate solution to eq. (3-1). Substitution gives

$$\mathcal{L}(u^*) - Q = R \neq 0 \tag{3-2}$$

since u^* does not satisfy the equation. The weighted residual method requires that

$$\int_\Omega RW \, d\Omega = 0 \ . \tag{3-3}$$

The residual R is multiplied by a weighting function W, and the integral of the product is required to be zero. The number of weighting functions equals the number of unknown coefficients in the approximate solution. There are several choices for the weighting functions, and some of the more popular are the collocation method, subdomain method, Galerkin's method, and least square method, etc.. To obtain the Galerkin criterion one selects

$$W = u^* \tag{3-4}$$

while for a least squares criterion

$$W = \partial R/\partial u^* \tag{3-5}$$

gives the desired result. Similarly, selecting the Dirac delta function gives a point collocation procedure; i.e.

$$W = \delta \, . \tag{3-6}$$

For both variational and weighted residual formulations the following restrictions are now generally accepted as means for establishing convergence of the finite element model as the mesh refinement increase [9]:

1. (A necessary criterion) The element interpolation functions must be capable of modelling any constant values of the dependent variable or its derivatives, to the order present in the defining integral statement, in the

limit as the element size decreases.

2. (A sufficient criterion) The element shape functions should be chosen so that at element interface the dependent variable and its derivatives, of one order less than those occurring in the defining integral statement, are continuous.

Interpolation functions are used for the approximate solution, u^*, and the weighting function, W, and result in a set of algebraic equations that can be solved for the unknown coefficients in the approximate solution. In following two sections and CH. 5, we are going to employ both cubic and fifth order Hermite interpolation functions.

3.2 Cubic Elements (C^1 continuity)

The practical application of the finite element method depends on the use of various interpolation functions and their derivatives. Most of the interpolation functions for C^0 and C^1 continuity elements are well known. The C^0 functions are continuous across an inter-element boundary while the C^1 functions also have their first derivatives continuous across the boundary. Here we display a C^1 continuity element. This element is obtained by using Hermite interpolation polynomial and nodal variables that include derivatives as well. The geometrical nodes, function H and the Jacobian matrix [J] remain identical to those of the linear element (C^0 continuity element).

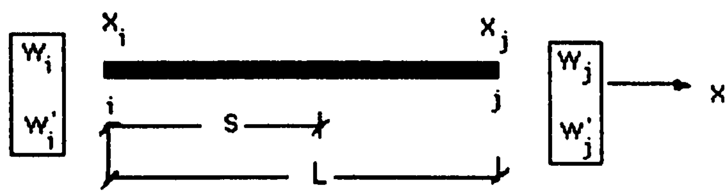
Note that a global derivative has been selected as a degree of freedom. Since there are two nodes with two degree of freedom each, the interpolation function has four constants. Thus, it is a cubic polynomial. The element is shown in Fig. 3.1 along with the interpolation functions and their global derivatives. Fig. 3.2 shows such a set of cubics. In order to obtain the element properties, we consider the typical element shown in Fig. 3.3, where 1 and 2 are local node numbers. Consider the cubic polynomial for deflection, w,

$$w(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \ . \tag{3.7}$$

If w(x) and its derivatives are evaluated at the node coordinates x_1 and x_2, we obtain

$$\begin{aligned}
w_1 &= a_1 + a_2x_1 + a_3{x_1}^2 + a_4{x_1}^3 \\
w'_1 &= a_2 + 2a_3x_1 + 3a_4{x_1}^2 \\
w_2 &= a_1 + a_2x_2 + a_3{x_2}^2 + a_4{x_2}^3 \\
w'_2 &= a_2 + 2a_3x_2 + 3a_4{x_2}^2 \ .
\end{aligned} \tag{3.8}$$

Equations (3.8) can be used to solve for the a_i (i = 1, 2, 3, 4) in terms of the four nodal values w_i and w'_i (i = 1, 2). If we define the generalized nodal displacement vector U by



$$w(x) = H^e(x)\, w^e$$

$$w^e = [\; w_i \quad w'_i \quad w_j \quad w'_j \;]$$

$$H^e(x) = [(1-3B^2+2B^3) \quad L(B-2B^2+B^3) \quad (3B^2-2B^3) \quad L(B^3-B^2)]$$

$$H'(x) = [(6B^2-6B)/L \quad (1-4B+3B^2) \quad (6B-6B^2)/L \quad (3B^2-2B)]$$

$$H''(x) = [(12B-6)/L^2 \quad (6B-4)/L \quad (6-6B)/L^2 \quad (6B-2)/L]$$

$$B = S/L, \quad S = X-X_i, \quad L = X_j - X_i .$$

Fig. 3.1 A One-Dimensional Hermite Cubic Element

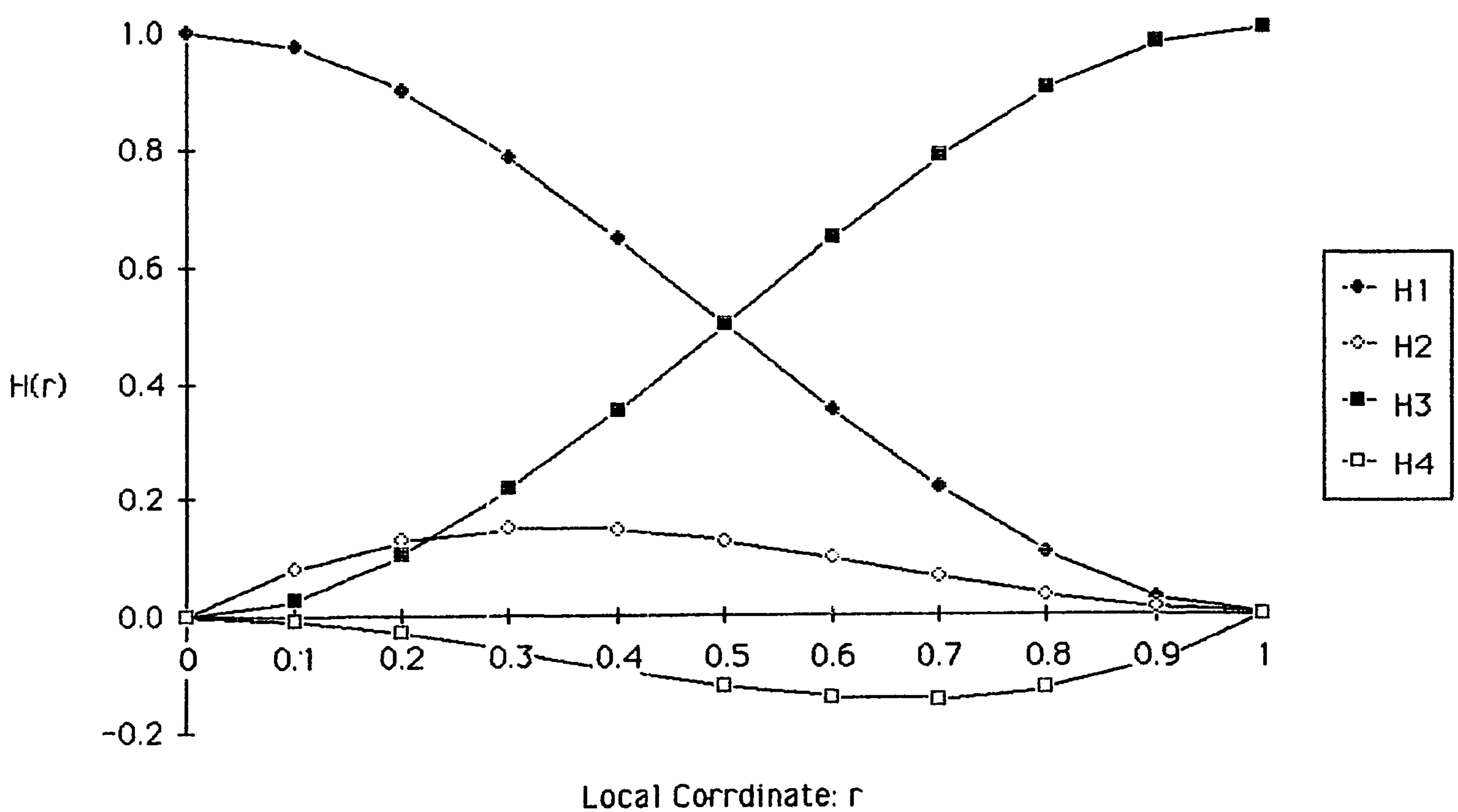


Fig. 3.2 Cubic Hermite Interpolation

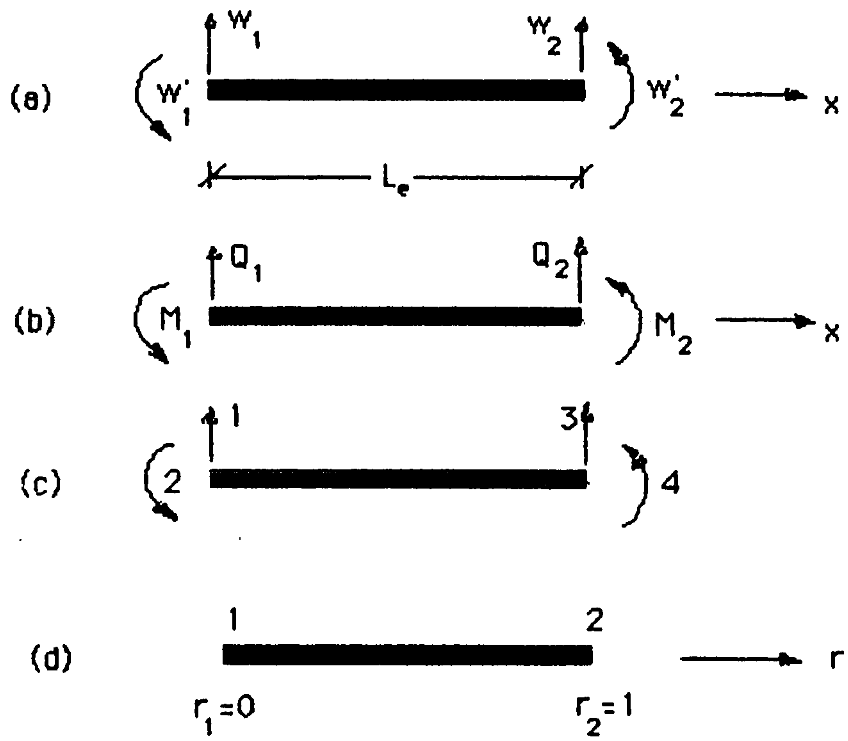


Fig. 3.3 Element Nodal Parameters: (a)Nodal Displacements; (b)Nodal Loads; (c) Degree of Freedom; (d) Local Coordinates.

$$U^T = [w_1 \;\; w'_1 \;\; w_2 \;\; w'_2] = [u_1 \;\; u_2 \;\; u_3 \;\; u_4] \tag{3.9}$$

then w(x) can be written in a form where appropriate interpolation functions are exhibited. In this case the functions turn out to be third-order Hermitian interpolation functions, from which we obtain

$$w(x) = \sum_{i=1}^{4} u_i H_i (x) \tag{3.10}$$

where the H_i are given in Fig. 3.1. In terms of nodal quantities, Eq. (3.4) can be written as

$$w(x) = H_1 w_1 + H_2 w'_1 + H_3 w_2 + H_4 w'_2 \tag{3.11}$$

or

$$w(x) = H_1 u_1 + H_2 u_2 + H_3 u_3 + H_4 u_4 \; . \tag{3.12}$$

In matrix notation, it becomes

$$w(x) = [H] \{U\} \; . \tag{3.13}$$

The error of using Eq. (3.13) with a ξ within r_1 to r_2 is given by

$$E(r) = w^4(\xi) \, [L(r)]^2 / 4\,! \tag{3.14}$$

where

$$L(r) = (r - r_1)(r - r_2) \dots (r - r_m)\,. \tag{3.15}$$

Thus, if $w(\xi)$ is a polynomial of order 2m-1 or less, this interpolation is exact for certain load conditions.

3.3 Fifth Order Element (C^2 Continuity)

We now have three variables per node: w_i, w'_i, w''_i. The corresponding generalized parameters are u_1, u_2, and u_3 respectively. For a two nodes line element, the equation of deflection is

$$w(x) = H_1u_1 + H_2u_2 + H_3u_3 + H_4u_4 + H_5u_5 + H_6u_6\,. \tag{3.16}$$

The shape function in local coordinates are

$$\begin{aligned}
H_1 &= 1 - 10r^3 + 15r^4 - 6r^5 \\
H_2 &= (r - 6r^3 + 8r^4 - 3r^5)\,L_e \\
H_3 &= (r^2 - 3r^3 + 3r^4 - r^5)\,L_e^2/2 \qquad\qquad (3.17) \\
H_4 &= 10r^3 - 15r^4 + 6r^5 \\
H_5 &= (7r^4 - 3r^5 - 4r^3)\,L
\end{aligned}$$

$$H_6 = (r^3 - 2r^4 + r^5)\, L_e^{\,2}/2$$

where $L_e = x_2 - x_1$ is the length of the element, and $x = rL_e$. The first three functions are shown in Fig. 3.4. The error in this case for a ξ within r_1 to r_2 is

$$E(r) = w^6(\xi)\, [L(r)]^3/6\,! \; . \qquad (3.18)$$

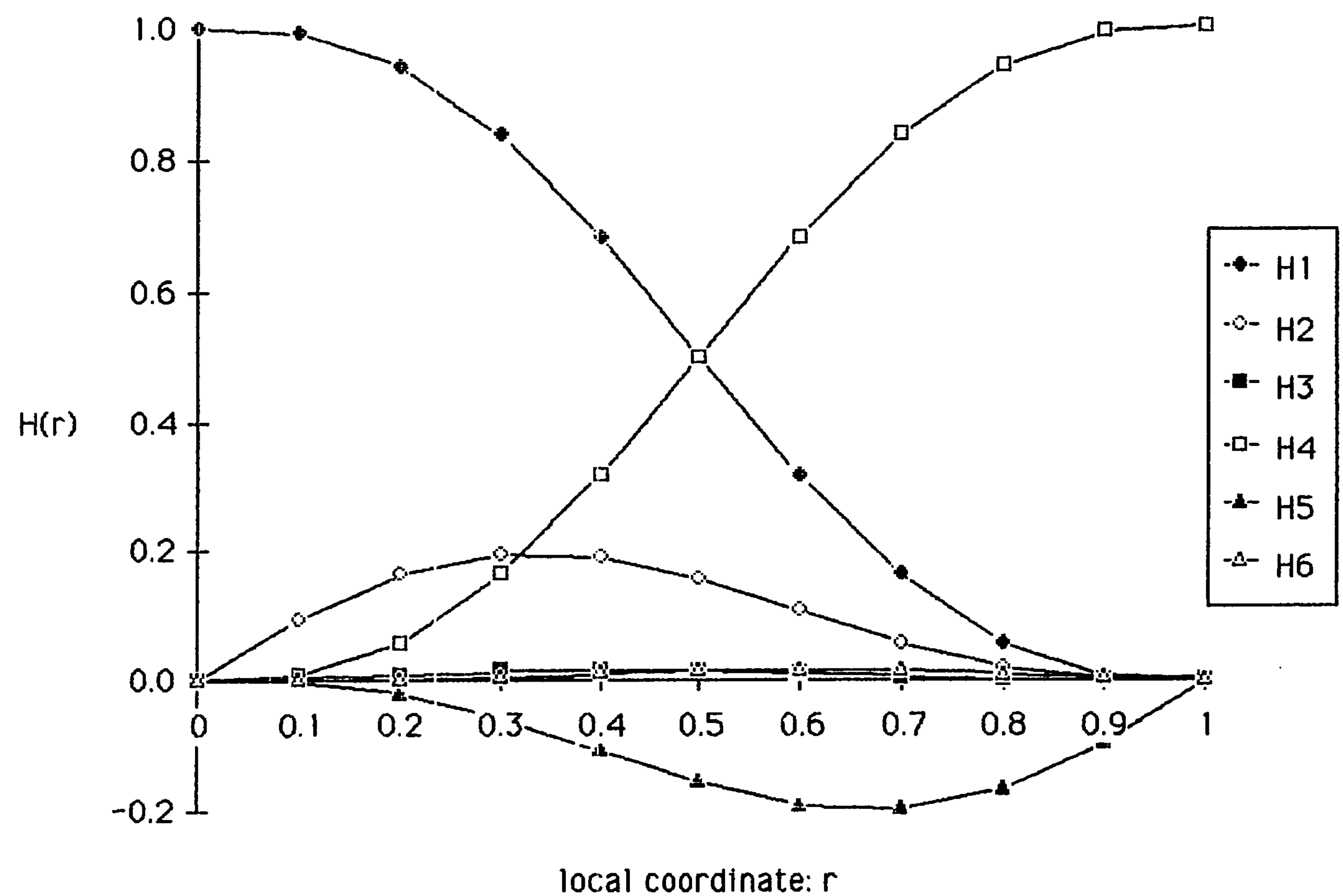


Fig. 3.4 Fifth Order Hermite Interpolation

3.4 Stiffness Matrix

This derivation is limited to a one-dimensional beams on a two-parameter foundation with the axial loads. The differential equation for deflection, w, of a beam on an elastic foundation with uniform cross section has been obtained in CH.1 [Eq. (1.6)]. This is mathematically equivalent to the problem of finding the extremum of functional

$$\Pi = \int_L [EI(w'')^2/2 + N(w')^2/2 - k_s(w')^2/2 + k_w w^2/2 - q(x)w]dx \qquad (3.19)$$

for all smooth functions w satisfying the boundary conditions. In a finite element formulation the assumed interpolation functions used to represent w should be such that w and w' are continuous. This assures that π is defined and that we can write π as a sum of contributions from the elements into which the region is divided. Therefore, using Eq. (3.19), we have

$$\Pi = \sum_{i=1}^{n-1} \int_{x_i}^{x_i+1} [EI(w'')^2/2+N(w')^2/2-K_s(w')^2/2+K_w w^2/2-qw] \qquad (3.20)$$

where n denotes the number of nodes (i.e., there are n-1 elements). Eq. (3.20) can also be written in matrix form with w of each element expressed in terms of generalized coordinates and interpolation functions, as

$$\Pi = \sum(\{u\}^T[k_1]\{u\}/2 + \{u\}^T[k_2]\{u\}/2 + \{u\}^T[k_3]\{u\}/2 + \{u\}^T[k_4]\{u\}/2 - \{u\}^T\{c\}) \tag{3.21}$$

or

$$\Pi = \sum(\{u\}^T [K] \{u\}/2 - \{u\}^T \{c\}) \tag{3.22}$$

where [K] is the total element stiffness matrix

$$[K] = [k_1] + [k_2] + [k_3] + [k_4] \tag{3.23}$$

where $[k_1]$ is the beam stiffness matrix which involves the beam flexural stiffness EI; $[k_2]$ is the stiffness matrix which involves the axial load N, we call it "geometric stiffness matrix"; $[k_3]$ is the second-parameter foundation stiffness matrix; and $[k_4]$ is the Winkler foundation stiffness matrix. They are defined by the relations

$$\{u\}^T [k_1] \{u\} = \int EI(w'')^2\, dx \tag{3.24a}$$

$$\{u\}^T [k_2] \{u\} = \int -N(w')^2\, dx \tag{3.24b}$$

$$\{u\}^T [k_3] \{u\} = \int -k_s(w')^2\, dx \tag{3.24c}$$

$$\{u\}^T [k_4] \{u\} = \int k_w w^2 \, dx \tag{3.24d}$$

$$\{u\}^T \{c\} = \int q(x) w \, dx \tag{3.24e}$$

where the integrations are carried out over the element and where u is defined by Eq. (3.9).

For equilibrium, minimizing the functional π, i.e., $d\pi/du = 0$, we have

$$[K]\{u\} = \{c\}. \tag{3.25}$$

Assembling the elements stiffness matrices by "bookkeeping" according the connectivity of elements, the system equation can be obtained as

$$[\mathbf{K}]\{\mathbf{U}\} = \{\mathbf{F}\}. \tag{3.26}$$

This is equivalent to the differential equation obtained in CH. I .

By using the numerical integration and cubic interpolation on Eq. (3.24), one obtains the numerical form equivalent to element stiffness matrices in Fig. 3.5. If we employ the fifth order interpolation functions, the (6*6) matrices are obtained.

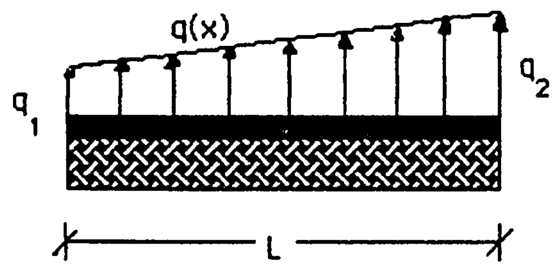
$$[k_1] = \frac{EI}{L^3}\begin{bmatrix} 12 & & & \text{Sym.} \\ 6L & 4L^2 & & \\ -12 & -6L & 12 & \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$[k_2] = \frac{-N}{30L}\begin{bmatrix} 36 & & & \text{Sym.} \\ 3L & 4L^2 & & \\ -36 & -3L & 36 & \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

$$[K_3] = \frac{-k_s}{30L}\begin{bmatrix} 36 & & & \text{Sym.} \\ 3L & 4L^2 & & \\ -36 & -3L & 36 & \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$$

$$[K_4] = \frac{k_w}{420}\begin{bmatrix} 156 & & & \text{Sym.} \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Fig. 3.5 Element Stiffness Matrices for Cubic Interpolation and Constant Properties.



(a)

$$\{C\} = \begin{Bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} q_1 L/2 + 3(q_2 - q_1)L/20 \\ q_1 L^2/12 + (q_2 - q_1)L^2/30 \\ q_1 L/2 + 7(q_2 - q_1)L/20 \\ -q_1 L^2/12 - (q_2 - q_1)L^2/20 \end{Bmatrix}$$

(b)

Fig. 3.6 (a)Elastically-supported beam element with a trapezoidal line load; (b)Typical element load vector for a cubic element.

3.5 Member Force Recovery

We can write the system equilibrium equation [Eq. (3.25)] in the form

$$(\Sigma\,[K]\,)\,\{u\} = (\Sigma\,\{c\}\,) + (\Sigma\,\{p\}\,) \tag{3.27}$$

where $\{F\} = (\Sigma\{c\}) + (\Sigma\{p\})$. The first term of right-hand side denotes the distributed load resultants, and the second denotes concentrated joint forces and moments. An approximation is to assume that each element is also in equilibrium, so

$$[K]\,\{u\} = \{c\} + \{p\}\,. \tag{3.28}$$

Thus, we approximate the member joint forces as

$$\{p\} = [K]\,\{u\} - \{c\} \tag{3.29}$$

the matrices [K] and {c} must be stored for latter use in computing the final member actions. Assume [K] is a sum of beam and foundation effects shown in Eq. (3.23). We want to know the reactions from the foundation, and the beam internal effects, from Eq. (3.28) we have

$$\{p\} = ([k_1] + [k_2] + [k_3] + [k_4])\,\{u\} - \{c\} \tag{3.30}$$

and set $\{c_f\} = -([k_3] + [k_4])\{u\}$, here c_f are the reactions from the foundation, we obtain

$$\{p\} = ([k_1] + [k_2])\{u\} - \{c\} - \{c_f\}. \qquad (3.31)$$

After $\{u\}$ has been computed the foundation reactions and member forces are recovered for each element.

Suppose a beam on elastic foundation shown in Fig. 3.7 loaded by a concentrated force in midspan. The numerical values of the parameters are: $K_w = 6.12 N/mm$; $E = 5200 N/mm^2$; $I = 3.413(10^7) mm^4$; $L = 5500 mm$; $P = N$. The deflection curve in this case is symmetric. Member forces recovery are computed for each element of the 4-element half span shown in Fig. 3.8. Both system and element equilibriums are satisfied. The flowchart of this procedure is in Fig. 3.9.

3.6 The Treatment for Gaps

Fig. 3.10 shows the situations of separations between structures and soil foundations. We may simply call them gaps. Sections AB in both schematics are lifted up and the structure-foundation interaction no longer exists in those sections. So Winkler model cannot be used throughout the whole span of the beam. What we are intrerested in is the locations of zero-deflection points from which we can find the regions of gaps. In

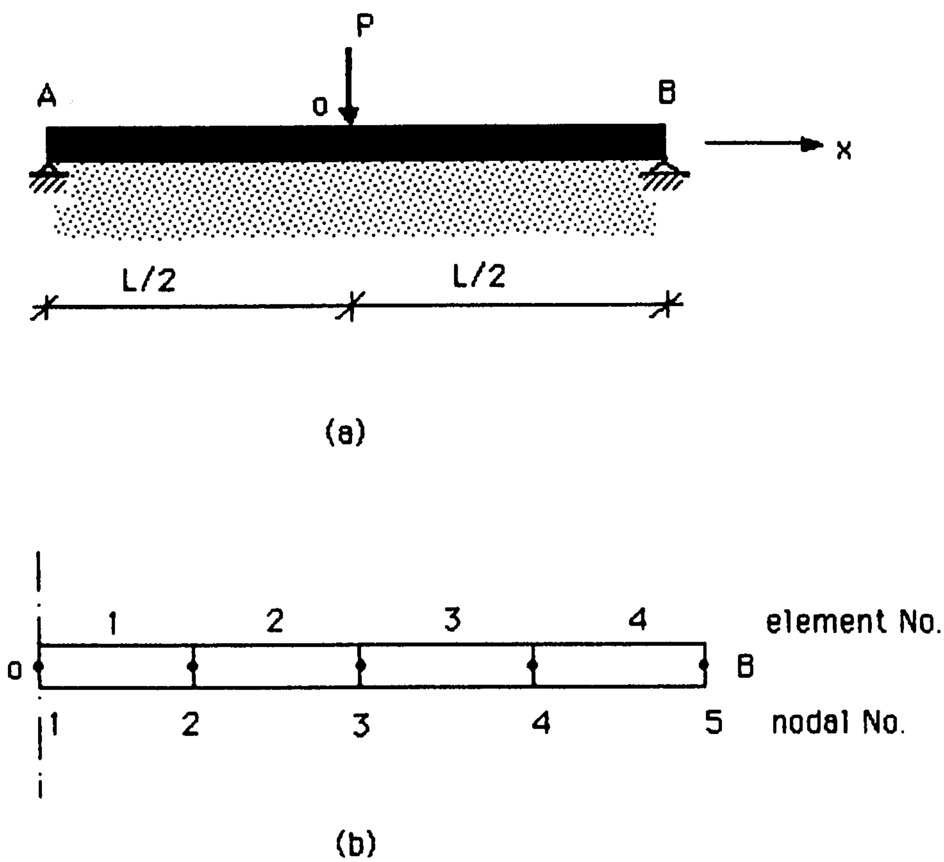


Fig. 3.7 Example for force recovery. (a) Simply supported beam under a concentrated load at midspan; (b) Elements mesh.

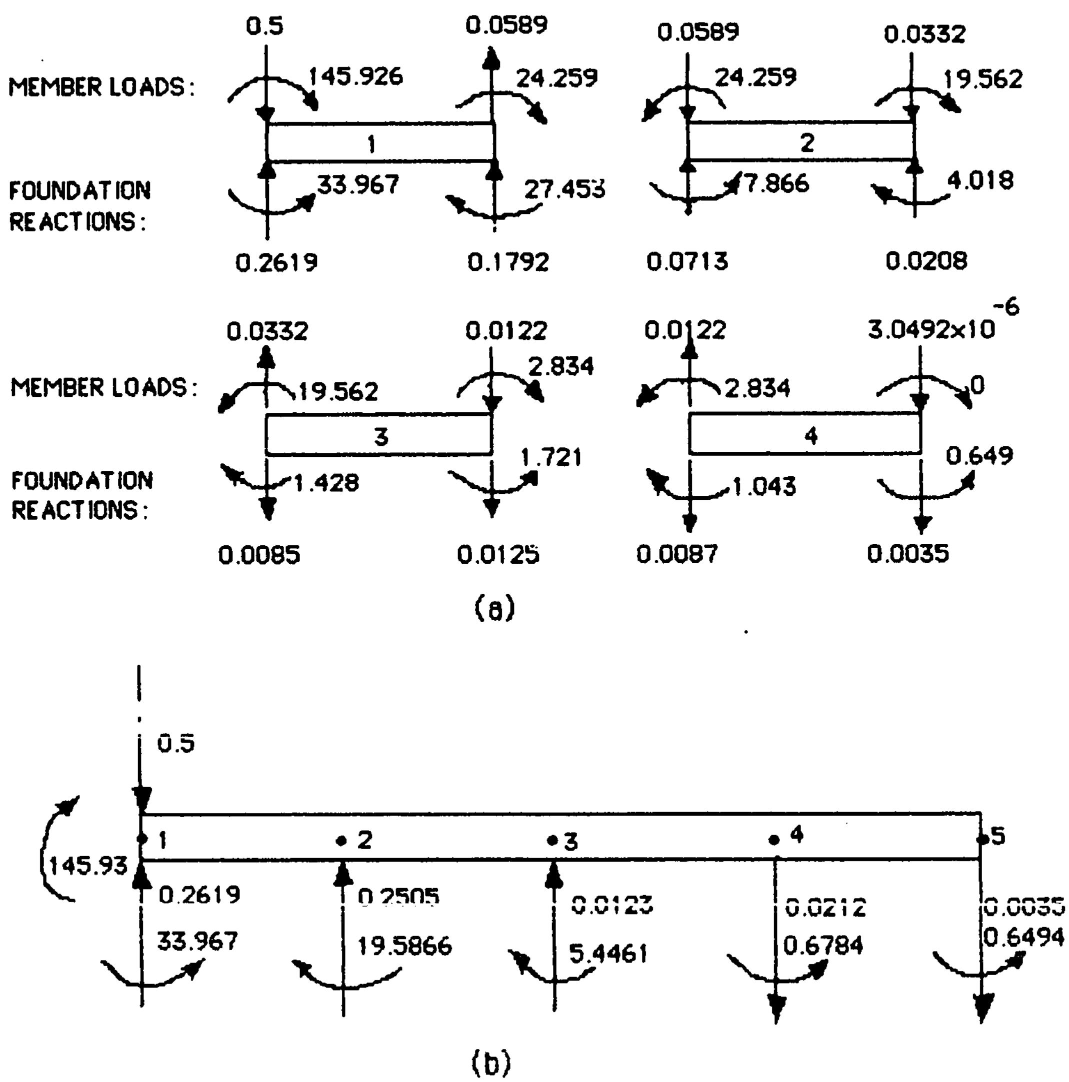


Fig. 3.8 Member Forces Recovery Example Solution. (a) Elements Equilibrium; (b) System Equilibrium(half of beam).

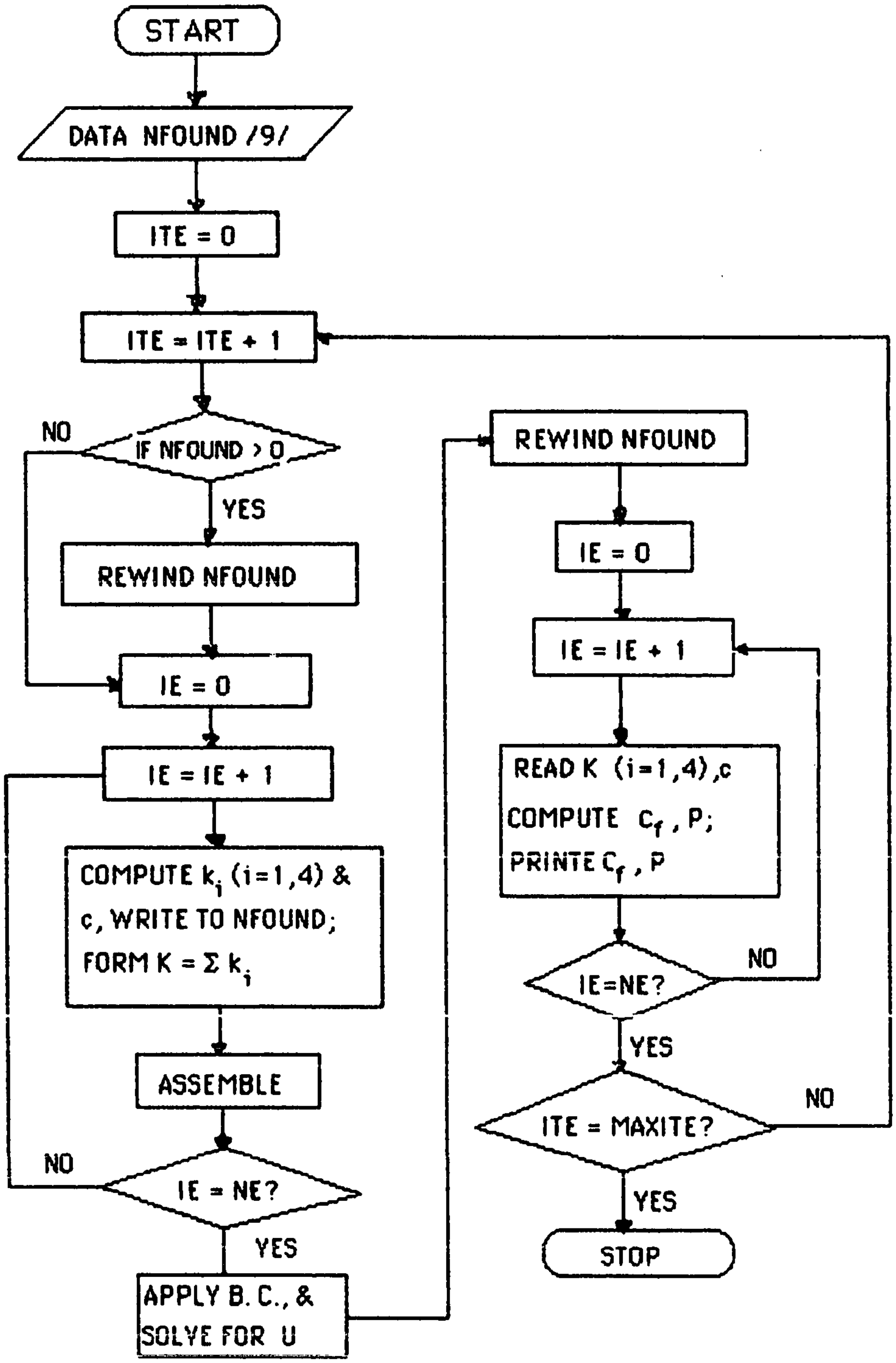
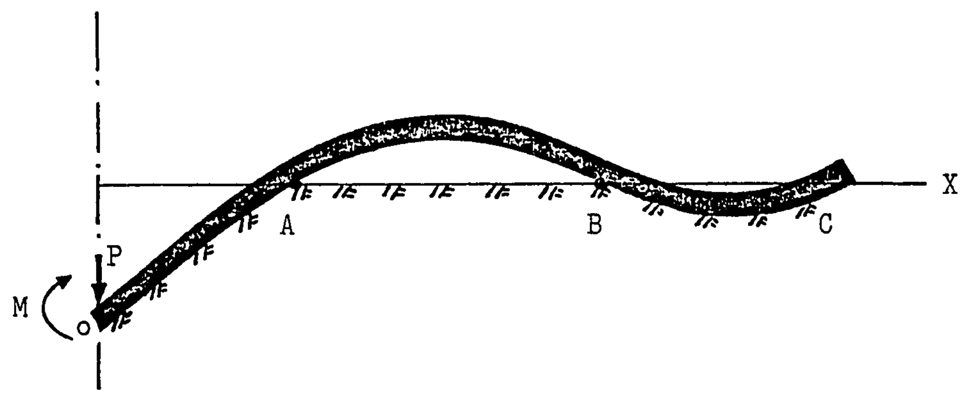
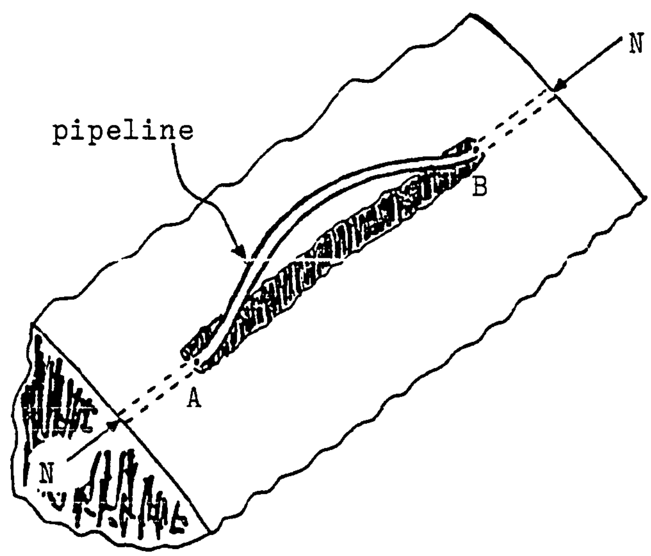


Fig. 3.9 Flowchart for Force Recovery



(a) Schematic of Beam on an Soil Foundation



(b) Schematic of Buried Pipeline

Fig. 3.10 Separations Between Structure and Foundation zero-deflection points A and B, four continuity conditions are required, i.e.,

(1) continuity of deflection

$$W_{Aleft} = W_{Aright} = 0 ; \qquad (3.32)$$

(2) continuity of slope; (3) continuity of moment; and (4) continuity of shear force. The differential equation for the gap section is

$$EI\ d^4W/dX^4 = q(X) \qquad (3.33)$$

and for the sections where beam contacts with the foundation Eq. (1.6) is employed. Now we have two kinds of differential equations. To solve the problem the finite element procedure is employed. In all cases, the six-point Gaussian quadrature is used for numerical integration. Testing each Gauss point, we can find the zero-deflection points. Two types of elements, one resting on the foundation and another one lifting off the foundation, are established. By using the iterative procedure, solution is repeated until nodal coordinates of zero-deflection points remain unchanged. Then, the deflection of beam on the whole length can be found. This means we can find the gaps, if they exist. By recycling the solution in the iterative steps, the effect of foundation is ignored in the sections with gaps. Usually, a quickly converged solution can be obtained. During this procedure the foundation modulii, k_w and k_s, are not constant but depend on location. Thus, the usual form in Fig. 3.5 is no longer valid. The most practical way to treat partial contact over the element is to numerically integrate the contributions to [k_3] and [k_4]. Using Gaussian quadratures also simplifies the programming required to implement both the C^2 and C^1 beam elements and their associated load resultants. Fig. 3.11 gives the flowchart of this procedure. To distinguish from the Winkler foundation model, I simply call the procedure a no-tension foundation model.

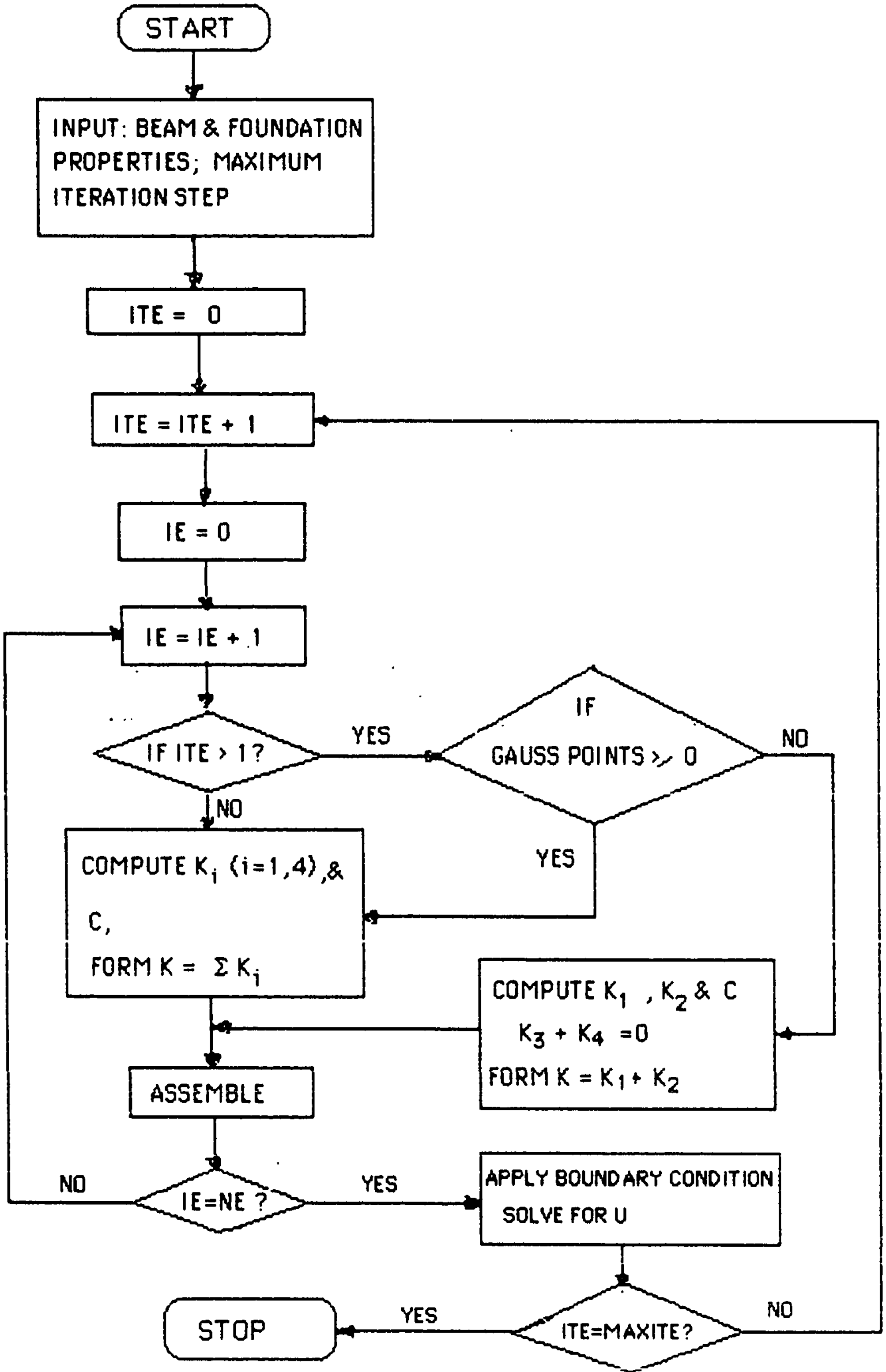


Fig. 3.11 Flowchart for Recycling solution

4. NUMERICAL EXAMPLES

4.1 Infinite Beam with Concentrated Load

Consider a long beam resting on the elastic foundation loaded by a concentrated force at the middle point of the beam as shown in Fig. 4.1. The numerical values for the parameters are: Young's modulus E = 9100N/mm^2; Winkler foundation modulus K_w = 4.0N/mm^2; the second-parameter of foundation K_s = 6*10^{5}N; P = 20000N; L = 18050mm. Since it is symmetry, Fig. 4.1 shows only half of the beam.

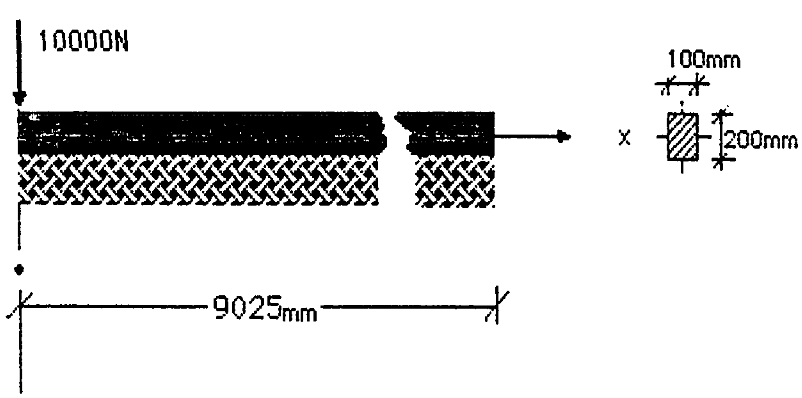


Fig. 4.1 Half of the infinite beam resting on the elastic foundation.

One-parameter solutions by using both C^1 and C^2 continuity element models are compared with the analytic infinite beam solution in Table 1 through Table 4. The C^2 continuity element model gives more accurate results than those obtained by using the C^1 continuity element model. Even the 5-element solution of C^2-model is better than those of 20-element by C^1-model. Table 5 shows the two- parameter solutions.

From the numerical results of Table 1 to Table 4, we can see that the deflection and the rotation at the far end point are very close to zero. It is reasonable that we add boundary condations w=0 and dw/dx=0 at the far end point. Then, the problem was resolved under the conditions by the iterative procedure with the consideration of no-tension foundation model, i.e., set k_w=0 in the sections where the beam lifted off from the foundation. Fig. 4.2 shows the comparision of deflections of Winkler model solution and the no-tension foundation model solution (CH. 3.6). The iterative solution converges within five steps. If this problem is solved with free boundary conditions at far end, solutions will converge after nine steps. Also the deflection at that point will be lifted up very high. This is impossible for a physical long beam resting on the elastic foundation. The weight of the beam is not considered here. The self-weight is included in the second example. So the curves of deflection look more realistic.

Table 1: Deflections --Winkler model solution

NODE	INFINITE SOLUTION	C^1 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 5 ELEMENTS
1	2.83271	2.83191	2.83271	2.83322
2	2.31280	2.31207	2.31280	
3	1.40045	1.39989	1.40045	
4	0.63310	0.63274	0.63310	
5	0.15856	0.15839	0.15856	0.15855
6	-0.06184	-0.06188	-0.06184	
7	-0.12171	-0.12169	-0.12171	
8	-0.10507	-0.10502	-0.10507	
9	-0.06614	-0.06609	-0.06614	-0.06614
10	-0.03139	-0.03136	-0.03139	
11	-0.00905	-0.00903	-0.00905	
12	0.00183	0.00183	0.00183	
13	0.00516	0.00516	0.00516	0.00517
14	0.00474	0.00475	0.00475	
15	0.00311	0.00311	0.00311	
16	0.00154	0.00155	0.00155	
17	0.00050	0.00050	0.00050	0.00050
18	-0.00004	-0.00004	-0.00004	
19	-0.00022	-0.00024	-0.00024	
20	-0.00021	-0.00029	-0.00029	
21	-0.00015	-0.00029	-0.00029	-0.00029

Table 2: Rotations --Winkler model solution

NODE	INFINITE SOLUTION	C^1 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 5 ELEMENTS
1	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	-1.8838E-03	-1.8835E-03	-1.8838E-03	
3	-1.9705E-03	-1.9700E-03	-1.9705E-03	
4	-1.3837E-03	-1.3832E-03	-1.3837E-03	
5	-7.3866E-04	-7.3830E-04	-7.3866E-04	-7.3961E-04
6	-2.7501E-04	-2.7479E-04	-2.7501E-04	
7	-2.2010E-05	-2.1911E-05	-2.2010E-05	
8	7.5887E-05	7.5903E-05	7.5887E-05	
9	8.7296E-05	8.7270E-05	8.7296E-05	8.6601E-05
10	6.4021E-05	6.3984E-05	6.4021E-05	
11	3.5571E-05	3.5540E-05	3.5572E-05	
12	1.4183E-05	1.4165E-05	1.4185E-05	
13	2.0423E-06	2.0363E-06	2.0460E-06	2.2501E-06
14	-2.9646E-06	-2.9613E-06	-2.9587E-06	
15	-3.8356E-06	-3.8278E-06	-3.8289E-06	
16	-2.9459E-06	-2.9405E-06	-2.9427E-06	
17	-1.7020E-06	-1.7093E-06	-1.7114E-06	-1.7380E-06
18	-7.2082E-07	-7.5545E-07	-7.5693E-07	
19	-1.4187E-07	-2.1991E-07	-2.2086E-07	
20	1.1085E-07	-1.7793E-08	-1.8473E-08	
21	1.6697E-07	1.1096E-08	1.0464E-08	1.1007E-08

Table 3: Moments --Winkler model solution

NODE	INFINITE SOLUTION	C^1 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 5 ELEMENTS
1	4412730.00	4412340.00	4412730.00	4412540.00
2	1013010.00	1012710.00	1013010.00	
3	-527428.00	-527529.00	-527428.00	
4	-916038.00	-915960.00	-916038.00	
5	-768508.00	-768336.00	-768508.00	-768600.00
6	-474423.00	-474241.00	-474423.00	
7	-219861.00	-219722.00	-219861.00	
8	-59351.80	-59270.80	-59351.70	
9	16990.50	17022.30	16990.50	17000.00
10	39118.70	39119.10	39118.30	
11	34808.50	34794.20	34807.30	
12	22341.40	22323.90	22339.20	
13	10850.60	10835.80	10847.70	10853.30
14	3314.86	3305.32	3312.37	
15	-435.06	-437.11	-434.14	
16	-1647.29	-1638.13	-1637.68	
17	-1566.64	-1540.58	-1541.25	-1543.01
18	-1046.29	-998.32	-999.17	
19	-531.00	-463.97	-464.52	
20	-179.14	-116.64	-116.82	
21	3.60	0.00	0.00	0.00

Table 4: Shears --Winkler model solution

NODE	INFINITE SOLUTION	C^1 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 20 ELEMENTS	C^2 CONTINUITY 5 ELEMENTS
1	1.0000E+04	1.0000E+04	1.0000E+04	1.0000E+04
2	5.2301E+03	5.2297E+03	5.2301E+03	
3	1.8743E+03	1.8738E+03	1.8743E+03	
4	7.9534E+01	7.9188E+01	7.9534E+01	
5	-5.9091E+02	-5.9103E+02	-5.9091E+02	-5.9088E+02
6	-6.4671E+02	-6.4667E+02	-6.4671E+02	
7	-4.6396E+02	-4.6383E+02	-4.6396E+02	
8	-2.5272E+02	-2.5259E+02	-2.5272E+02	
9	-9.7484E+01	-9.7392E+01	-9.7483E+01	-9.7510E+01
10	-1.1080E+01	-1.1029E+01	-1.1079E+01	
11	2.3471E+01	2.3489E+01	2.3473E+01	
12	2.8536E+01	2.8535E+01	2.8538E+01	
13	2.1408E+01	2.1398E+01	2 1409E+01	2.1420E+01
14	1.2130E+01	1.2116E+01	1.2127E+01	
15	4.9891E+00	4.9691E+00	4.9766E+00	
16	8.5600E-01	8.2531E-01	8.2926E-01	
17	-8.9895E-01	-9.4316E-01	-9.4189E-01	9.4341E-01
18	-1.2482E+00	-1.2985E+00	-1.2987E+00	
19	-9.8233E-01	-1.0091E+00	-1.0099E+00	
20	-5.7862E-01	-5.1769E-01	-5.1839E-01	
21	-2.5196E-01	0.0000E+00	0.0000E+00	0.0000E+00

Table 5: Two-parameter elastic foundation model solutions by using C^2 continuity element

NODE	DEFLECTION	ROTATION	MOMENT	SHEAR
1	2.5939E+00	0.0000E+00	4.0408E+06	1.0000E+04
2	2.1300E+00	-1.6598E-03	8.2619E+05	5.6260E+03
3	1.3355E+00	-1.7023E-03	-4.7536E+05	2.4967E+03
4	6.7245E-01	-1.2026E-03	-7.5800E+05	7.1903E+02
5	2.5251E-01	-6.7616E-04	-6.2180E+05	-7.9831E+01
6	3.8859E-02	-2.9982E-04	-3.8961E+05	-3.1725E+02
7	-4.2803E-02	-8.6168E-05	-1.9605E+05	-2.9924E+02
8	-5.6620E-02	9.7698E-06	-7.3533E+04	-2.0306E+02
9	-4.4059E-02	3.8227E-05	-1.1242E+04	-1.1031E+02
10	-2.6654E-02	3.6008E-05	1.2540E+04	-4.6662E+01
11	-1.2914E-02	2.4417E-05	1.6541E+04	-1.1749E+01
12	-4.5198E-03	1.3256E-05	1.2859E+04	3.2247E+00
13	-4.0769E-04	5.6042E-06	7.7736E+03	7.1527E+00
14	1.0625E-03	1.4098E-06	3.7629E+03	6.2783E+00
15	1.2249E-03	-3.8716E-07	1.3152E+03	4.0932E+00
16	9.1071E-04	-8.5841E-07	1.2034E+02	2.1347E+00
17	5.3456E-04	-7.5756E-07	-2.9892E+02	8.3766E-01
18	2.4810E-04	-5.0868E-07	-3.3007E+02	1.4840E-01
19	6.8533E-05	-3.0161E-07	-2.1586E+02	-1.2325E-01
20	-3.8985E-05	-1.9138E-07	-8.4153E+01	-1.4246E-01
21	-1.1691E-04	-1.6451E-07	0.0000E+00	0.0000E+00

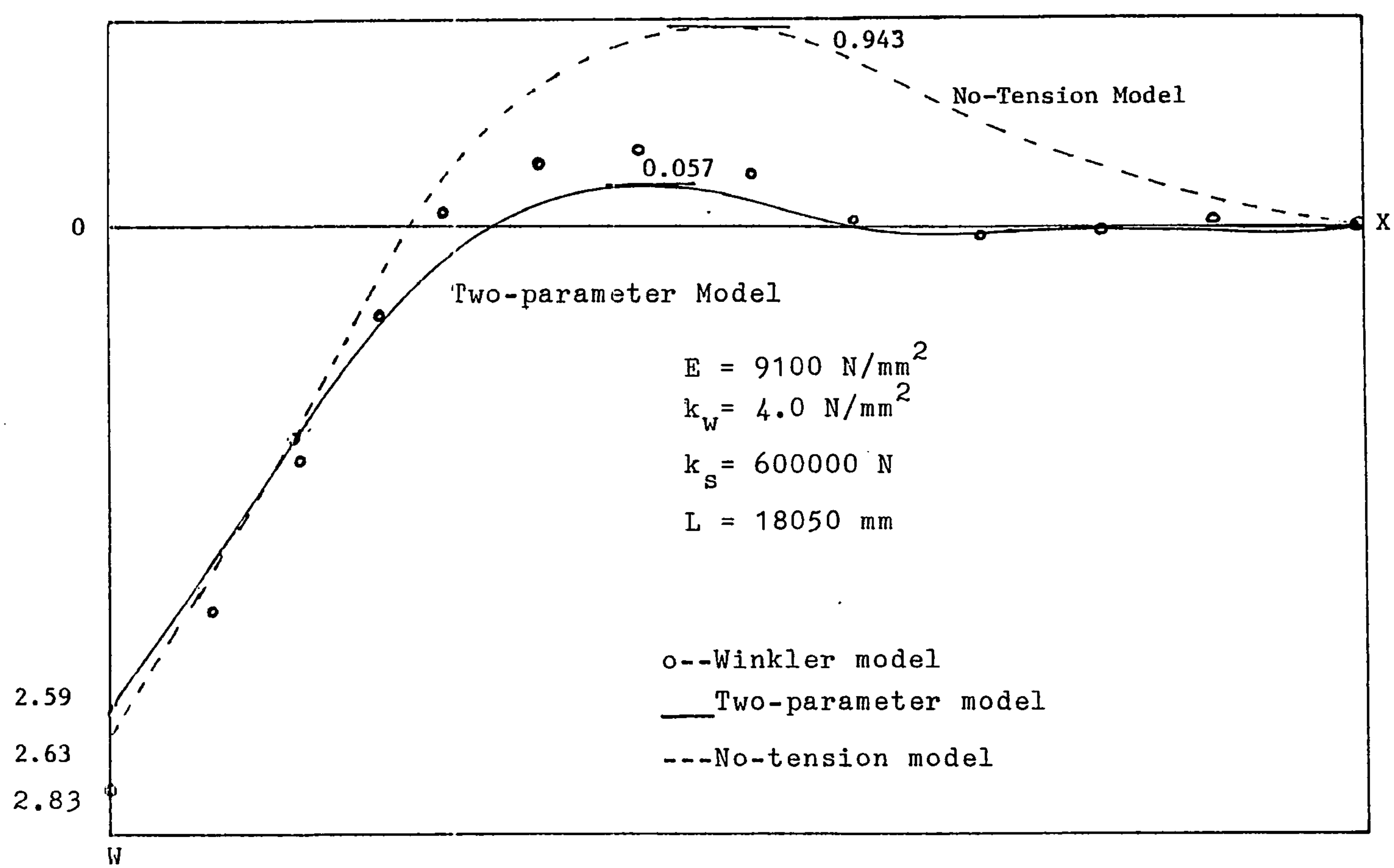


Fig. 4.2 Deflections.--Two-parameter Foundation Model with the Boundary Conditions W=0 and dW/dX=0 at far end.

4.2 Infinite Beam with Uniform Load

This example is the same as example 1 except for the consideration of the beam weight which we take as the uniform load on the beam shown in Fig. 4.3. Fig. 4.4 shows the comparisions of Winkler model solutions and the no-tension foundation model solutions for deflections. The iterative procedure used for the no-tension foundation model converges within nine steps.

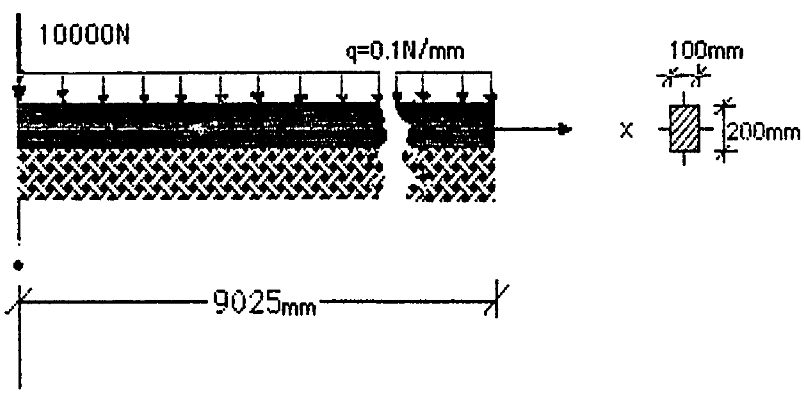


Fig. 4.3 Half of the infinite beam resting on the elastic foundation including the uniform load.

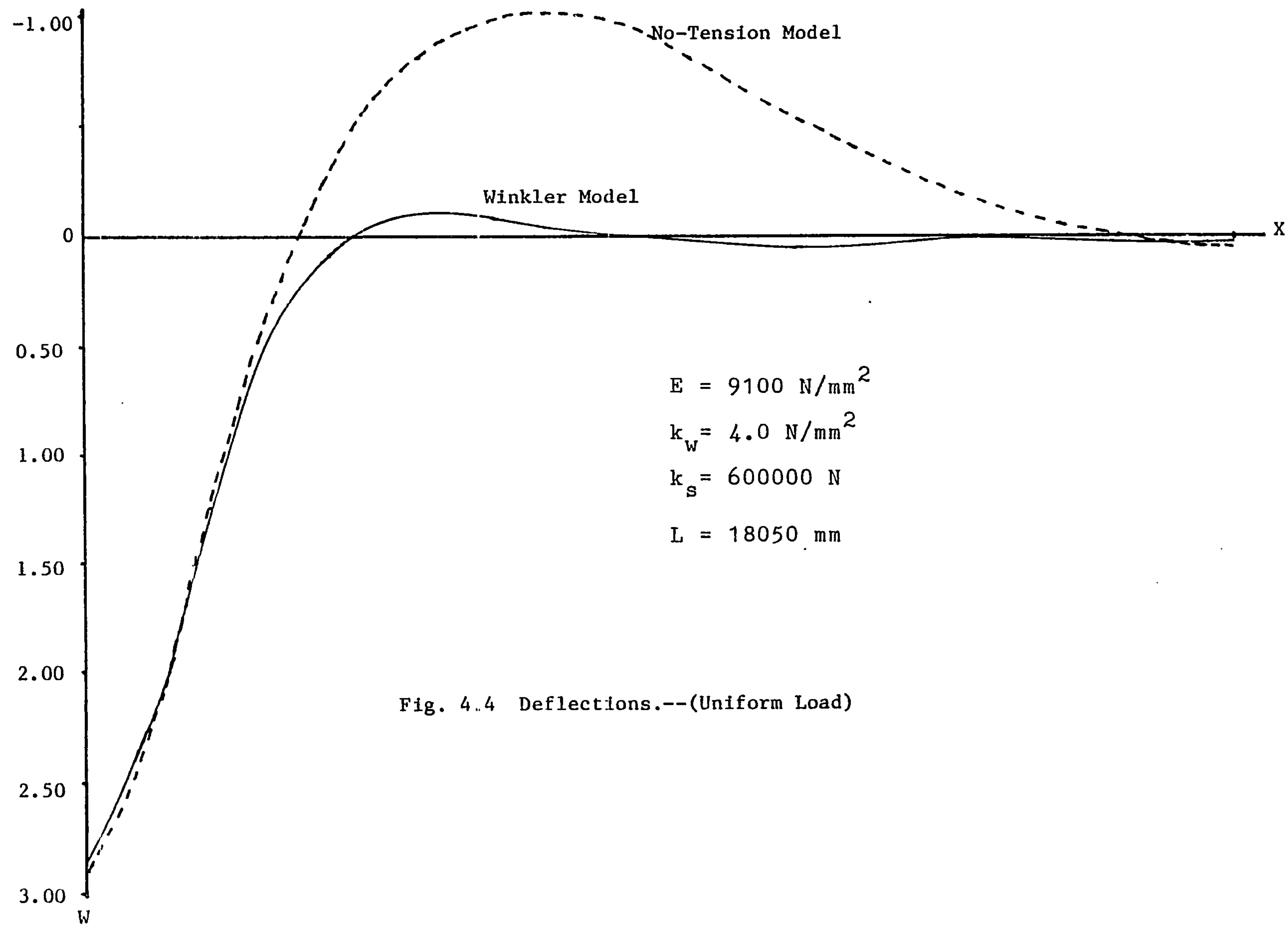


Fig. 4.4 Deflections.--(Uniform Load)

4.3 Finite Beam with Axial Load

Consider a more complex problem which includes concentrated forces and moments, as well as axial forces as shown in Fig. 4.5. The beam also has different cross sections and different foundation modulii. These make the problem more complicated to obtain the analytic solutions, but it is easier to get the finite element solutions. Since it is a symmetric problem, we only need to compute half of the beam. A ten-element mesh is used. The solutions converge within four iterations. Deflections and moments are shown in Fig. 4.6, and 4.7 respectively. This example may represent a non-uniform foundation beam supporting three columns. The inclusion of the axial load N also leads to buckling eigenvalue problems where N plays the role of the eigenvalue. Additional detail for such problems will not be discussed here.

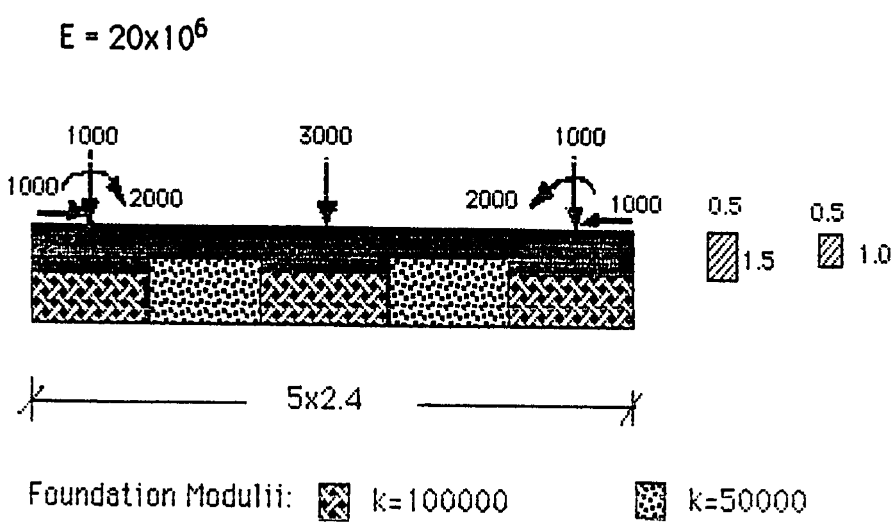


Fig. 4.5 Beam on elastic foundation with axial load and non-uniform foundation.

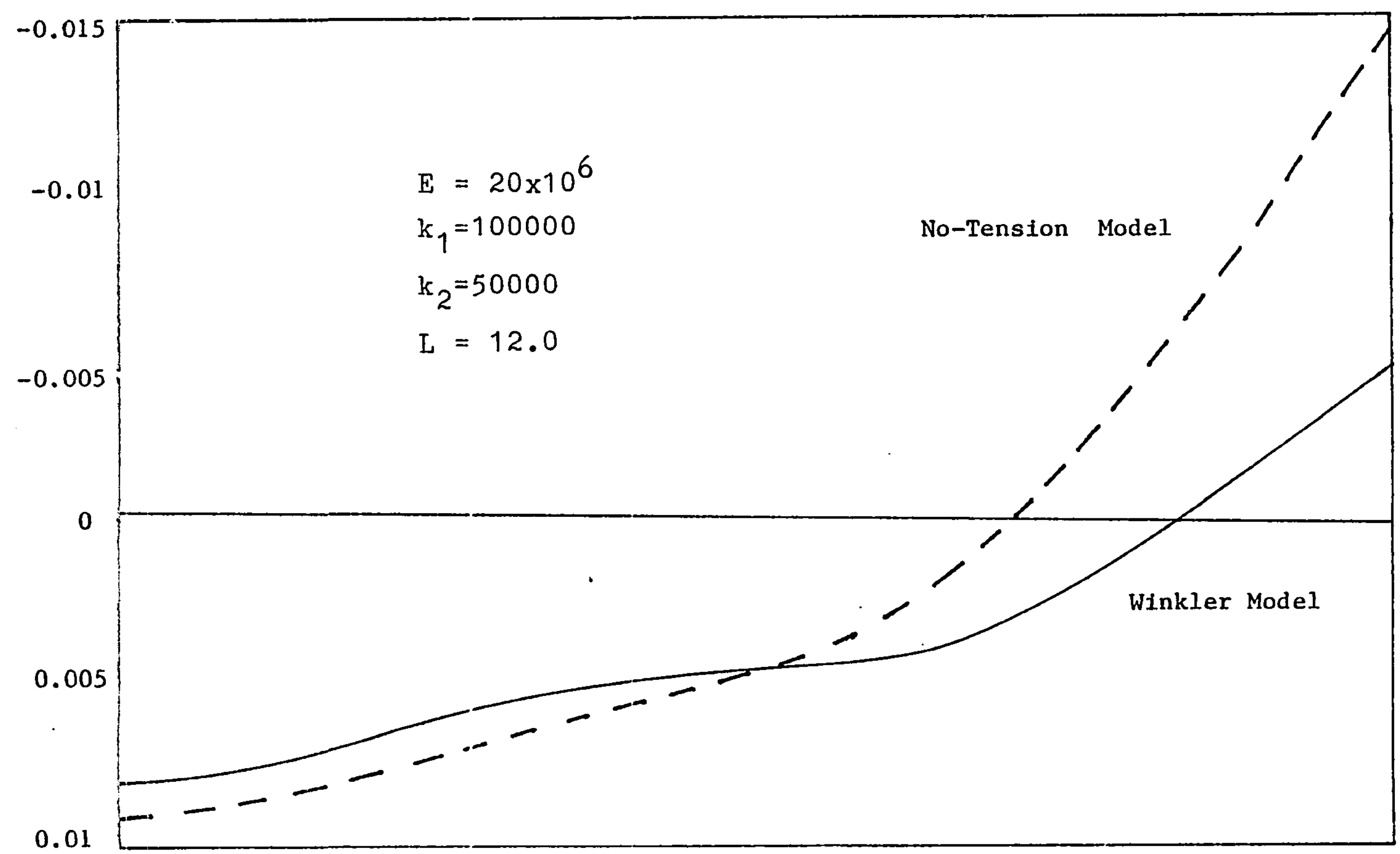


Fig. 4.6 Deflections.

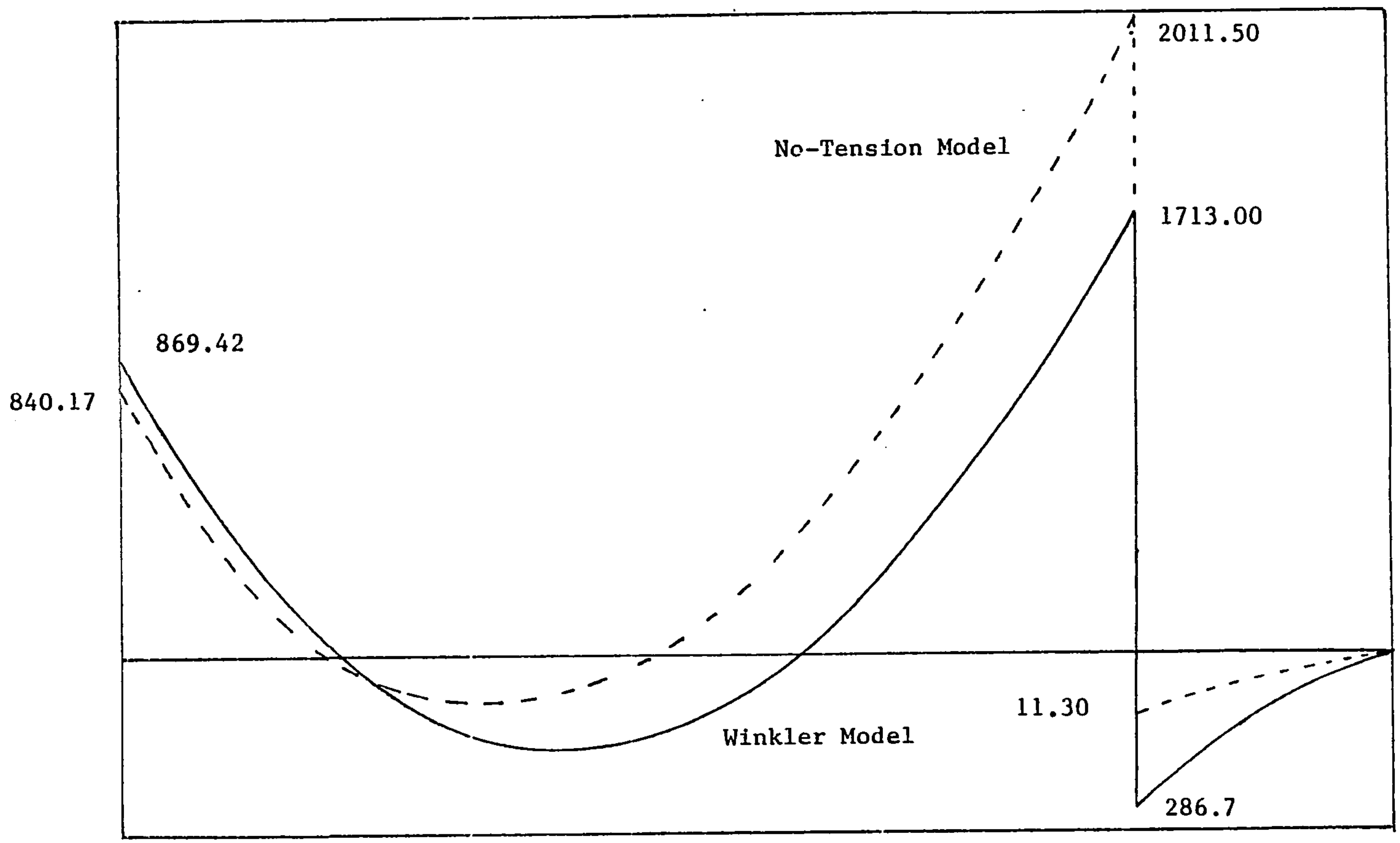


Fig. 4.7 Moments.

5. CONCLUSIONS

Two kinds of elements presented in this paper can be used to analyze beams resting on one- or two- parameter elastic foundation. Elements based on a cubic displacement function can give reasonable results for deflections, rotations, and bending moments by a moderately fine element mesh. A very fine mesh is needed to obtain good predictions for transverse shear force. Elements based on a fifth order displacement function can give exact solutions at most nodal points for deflections, rotations, bending moments, as well as transverse shear forces. Even using very small number of elements, the accuracy is still higher than that obtained by using a cubic element model with much larger number of elements.

When the second-parameter k_s is not very large the beam can be analyzed as if it rests on Winkler foundation. When k_s is large, especially when k_s is close to $(4KEI)^{1/2}$, the error caused by ignoring k_s may be appreciable. The appropriate value of k_s, either from experiment or from formulas that use known foundation data, is a topic that requires a great deal more study.

The iteration procedure has been found very efficient for solving the beam on elastic foundation problems with the consideration of gaps (no-tension foundation). Even for the more complex problems the solutions were obtained requiring a small number of elements and within only a few iterations.

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