THE THEOREM OF JORDAN ON PLANE CURVES

A Thesis

Presented to the Faculty of

The Rice Institute in partial fulfillment of

the requirements for

the degree of Master of Arts

by

Anthony Andrew Aucoin

7-3-36 Mathematics Department.

THE THEOREM OF JORDAN ON PLANE CURVES

The Theorem of Jordan states that a simple closed Jordan curve divides the plane into two regions, the interior and the exterior. The interior region and the exterior region are respectively, the bounded and the unbounded regions into which the plane is divided. It follows that a polygonal line or a Jordan curve joining an interior and an exterior point cuts the curve in at least one point.

A simple Jordan curve is represented as

$$x = x(t),\ y = y(t),$$

continuous for $0 \leq t \leq 1$, where $(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2 > 0$ for $0 \leq t_1, t_2 \leq 1$, and $t_1 \neq t_2$. A simple closed Jordan curve is represented by

$$x = x(t),\ y = y(t)$$

continuous for $0 \leq t \leq 1$, where $x(o) = x(1)$, $y(o) = y(1)$ but otherwise $(x(t_1) - x(t_2))^2 + (y(t_1) - y(t_2))^2 > 0$ for $0 \leq t_1, t_2 \leq 1$ and $t_1 \neq t_2$.

Since most of the proofs considered depend upon the Theorem of Jordan for the simple polygon, a proof of that special case, due to H. Hahn* will be given. The proof depends upon a system of axioms which are eight in number. The preliminary theorems upon which the Theorem of Jordan depends will be quoted without proof. The axioms involve the point as an undefined element and an order relation as the undefined relation. The axioms and theorems are as follows:

Axiom 1. There exist at least two distinct points.

Axiom 2. If the points A, B, and C lie in the order

---

* H. Hahn, Über die Anordnungssätze der Geometrie, Monatsheft fur Mathematik und Physik, vol 19, pp. 289-303.

(A,B,C) they also lie in the order (C,B,A).

Axiom 3. If the points A,B, and C lie in the order (A,B,C) they do not lie in the order (B,C,A).

Axiom 4. If the points A, B, and C lie in the order (A,B,C) then the point A is not identical with the point C.

Axiom 5. If the points A and B are distinct, there exists a point C in the order (A,B,C).

Definition, A straight line AB will consist of the points A and B and all points X in the orders (X,A,B), (A,X,B) and (A,B,X).

Definition. The segment AB will be the point set consisting of the points A and B and the points X in the order (A,X,B). A and B are the end points of the segment. The points X are inner points.

Axiom 6. If the points C and D lie on the line AB, the point A lies on the line CD.

Theorem 1. Two distinct points determine only one line.

See ax. 3 Axiom 7. Given three points, the three points A, B, and C do not lie in all the orders (A,B,C),(B,C,A), (C,A,B).

Definition. If three points A, B, and C do not lie on a line they are said to form a triangle. The points A, B, and C are the vertices; the segments AB, BC, and CA are the sides.

Axiom 8. If the three points A, B, and C form a triangle, the point D lies in the order (B,C,D), the point E in the order (C,E,A), then F exists in the order (A,F,B) and lies on the line thru D and E.

Theorem 2. There is no line which meets all three sides of a triangle in inner points.

Theorem 3. Let n distinct points lie on a line. Then we may designate them be $A_1, A_2, A_3, \ldots\ldots, A_n$ so that they lie in the order $(A_1, A_2, \ldots\ldots, A_n)$. They also lie in the order $(A_n, A_{n-1}, \ldots\ldots, A_1)$.

Definition. Given two points A and B. The points X in the order (A,B,X) constitute the prolongation of the segment AB beyond B. The points X in the order (X,A,B) constitute the prolongation of AB beyond A.

Theorem 4. Each segment and each prolongation of a segment contains infinitely many points.

Definition. The points A, B, and C form a triangle. We mean by the plane ABC the point set consisting of the points of the segments AB, BC, CA together with those points that are collinear with any two points of these segments.

?

Theorem 5. Three non-collinear points determine a plane.

Theorem 6. If two points of a line belong to a plane, all points of the line belong to the plane.

Theorem 7. Thru a point in the plane pass infinitely many distinct lines lying wholly in the plane.

Theorem 8. Let A, B, and C form a triangle. Then each lies in the plane ABC. Then any line in the plane ABC which passes thru an inner point of the segment AB, passes thru a second point of the triangle ABC.

Theorem 9. A point on a line separates the line into two subsets, each of which is a half-line.

Theorem 10. By n points in the order $(A_1, A_2, \ldots\ldots, A_n)$ the line is divided into n + 1 subsets consisting of the n - 1 sets determined by the inner points of the segments A_iA_{i+1} $(i = 1, 2, \ldots, n-1)$ and the prolongation of A_1A_n from A_1 and from A_n.

Theorem 11. The plane is divided into two parts by a line.

Theorem 12. Two points not on a line b in a plane c and lying in the same half-plane determined by b may be joined by a segment not meeting b.

Theorem 13. Two intersecting lines in a plane divide the plane into four parts.

Definition. Let O be a point and h and k two half-lines going from O. The point set consisting of the point O and the two half-lines h and k will be called the angle (h,k). O is the vertex of the angle and h and k are the sides.

Theorem 14. Any angle lying in a plane divides the plane into two parts.

Theorem 15. Let A and B be two points, one on each side of an angle. All points of the segment AB are interior points (Def?) of the angle. The segment joining two interior points of an angle lies within the angle.

Theorem 16. Thru a vertex O of angles and a point C interior to an adjacent angle draw a line. Then all points of this line lie outside the angles (excepting the point O).

Theorem 17. The segment joining two points exterior to an angle which does not pass thru the vertex, cuts the angle in two points or not at all.

Theorem 18. Let A, B, and C form a triangle. The lines AB, BC, and CA divide the plane into seven parts.

Theorem 19. A triangle divides the plane into two parts.

Theorem 20. Let D and E be points on different sides of a triangle. Then all inner points of the segment DE are interior to the triangle. The segment joining two inner points of a triangle lies within the triangle.

Theorem 21. If the segment joining two exterior points of a triangle does not pass thru a vertex, it cuts the triangle in two points or not at all.

Theorem 22. Each half-line going out from an inner point of a triangle cuts the triangle in only one point.

Theorem 23. Each line which contains an interior point of a triangle, contains two points of the triangle.

Theorem 24. If there are n points in a plane there is a line such that the n points lie in the same half-plane determined by this line.

Definition. Consider n segments. They form a polygonal line when an end point of the first is identical with one end point of the second; the other end point of the second is identical with an end point of the third; and finally, the other end point of the n - 1 th., is identical with an end point of the nth.

Definition. A polygon is a polygonal line both end points of which are identical. The segments of the polygonal line are called the sides; the end points of the sides, the vertices. A polygon will be called simple if:

1) No interior point of a side belongs to any other side, and

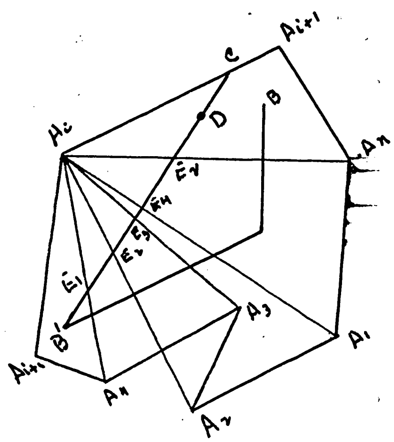
2) Each vertex belongs to only two sides.

Definition. A point P of a polygon (or Jordan curve) is accessible from a point O not on the polygon (or the Jordan curve) if there exists a polygonal line or Jordan curve joining P and O and having no point of the polygon (or the Jordan curve) other than P. The principle of accessibility is fundamental in this proof.

Lemma 1. If C, a point of a polygon side A_iA_{i+1} is accessible from a point B not on the polygon, then the points A_i A_{i+1} are accessible from B.

Proof: Let C be a point of the polygon side A_iA_{i+1}. Let B'C be the last segment of the broken line BC. We draw from A_i straight lines to all the remaining vertices of the polygon. These lines cut the segment B'C in at most a finite number of points. Let $E_1, E_2, \ldots, E_\nu$ be the points labeled so that they fall inside the segment B'C in the order $(B', E_1, E_2, \ldots, E_\nu, C)$. Let D be an inner point of the segment $E_\nu C$. We shall show that the segment DA_i does not cut the side of the polygon and from this our lemma follows.

where is D if there were no E_ν?



We show now that all vertices of the polygon different from A_i fall outside the triangle A_iCD. The segment A_iC does not contain a vertex as otherwise the polygon would not be simple. The segment CD does not contain a vertex since it is a part of a broken line which does not meet the polygon. The segment A_iD does not, for if so, D would be the point E_ν and would not be inner pt on the segment CE_ν. There is no vertex A_k inside the triangle A_iCD for if so, by Theorem 23, the line A_iA_k would cut the segment CD and then E_ν would not be the last point in the sense from B' to C in which lines from A_i to the other vertices cut the segment B'C.

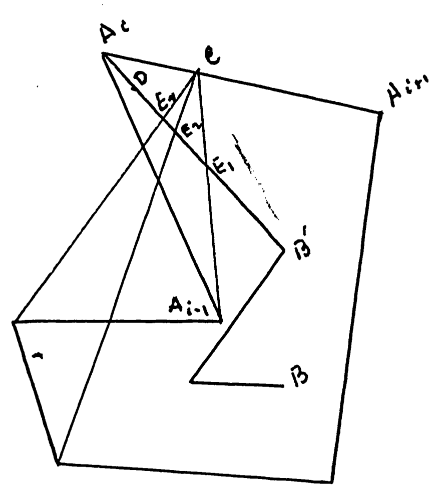
If now A_kA_{k+1} is a side of the polygon not terminating at A_i, it can not contain a vertex of the triangle A_iCD. It can not go thru A_i or C for if so the polygon would not be simple. It can not go thru D because D lies on the broken line BC which contains no points of the polygon (except C). According to Theorem 21, it has, therefore, two points or none at all in common with the triangle A_iCD. If this segment A_kA_{k+1} has a point on the interior of the triangle it must cut two sides by Theorem 23. Since it does not cut A_iC and CD, it does not cut A_iD. Hence any side of the polygon A_kA_{k+1} not ending in A_i does not cut A_iD. The segments $A_{i-1}A_i$ and A_iA_{i+1} do not cut the segment A_iD except in the point A_i as otherwise they would coincide with A_iD since two distinct lines can meet in at most one point. If this did happen it would contradict our choice of the point D. Hence the broken line from B to B', the segment B'D and the segment DA_i joins B and A_i and does not meet the polygon except in A_i.

Lemma 2. If a vertex A_i of a simple polygon is accessible from a point B not on the polygon, then any point C of the sides $A_{i-1}A_i$ or A_iA_{i+1} is accessible from B.

Proof.

Case 1. The last segment $B'A_i$ of the broken line which connects B with A_i lies inside the angle $A_{i-1}A_iA_{i+1}$.

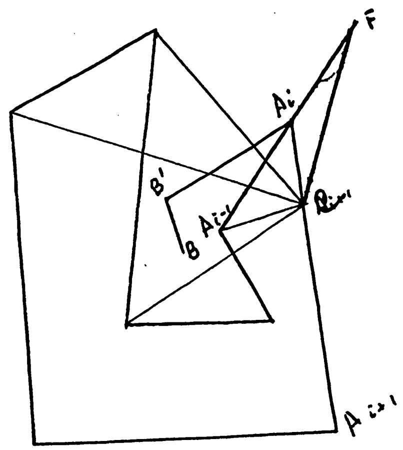
Suppose the point lies on the segment A_iA_{i+1}. We draw from C straight lines to all the vertices of the polygon. Call the points of intersection of these lines with the inner points of the segment $B'A$, E_1, E_2, ... , E_ν, and choose again the inner point D of the segment $B'A$ so that the segment DA_i contains none of these points E_ν, i.e., on the segment $E_\nu A_i$. See p6



We show that if A_k is any vertex distinct from A_i it is not in or on the triangle A_iCD. A_k is not on the side A_iC as then the polygon would not be simple. It is not on A_iD as then the broken line BA would meet the polygon in a point other than A_i contrary to hypothesis. It is not on DC for if so, D would be E_ν and would not be on the segment A_iE_v. The vertex A_k is not inside the triangle for if so the line CA_k would, by Theorem 23, cut the segment A_iD and E_v would not be the last point in the sense from B' to A_i in which lines from C to the other vertices cut the segment $B'A_i$.

A side of the polygon A_kA_{k+1}, not terminating in A_i cannot contain a vertex of the triangle A_iCD. It can not contain A_i or C since then the polygon would not be simple. It could not contain D since the segment $B'A_i$ is part of the brokenline BA_i which has no point on the polygon (except A_i). According to Theorem 21, it has, therefore, two points or none at all in common with the triangle A_iCD. If this segment has a point on the interior of the triangle it must cut two sides by Theorem 23. Since it does not cut A_iC and A_iD it does not cut CD. Hence any side of the polygon not ending in A_i does not cut CD. By Theorem 15 the sides $A_{i-1}A_i$ and A_iA_{i+1} could not cut DC since $B'A_i$ and hence DA_i is interior to the angle $A_{i-1}A_iA_{i+1}$. Hence the broken line from B to B', the segment $B'D$ and the segment DC joins B and C and does not meet the polygon except in C.

Case 2. The segment $B'A_i$ lies outside of the angle $A_{i-1}A_iA_{i+1}$.

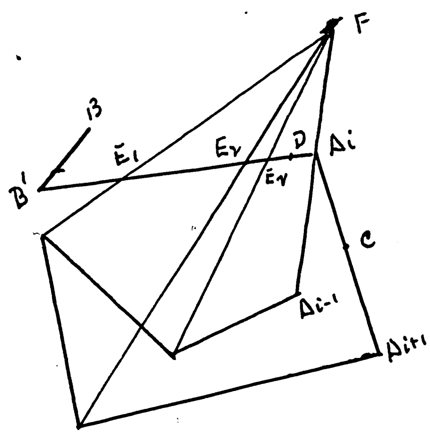


Let C be a point on A A. We consider a point F on the prolongation of $A_{i-1}A_i$ so that the segment A_iF neither is cut by a polygon side nor by one of the lines joining C with the polygon vertices with the exception of the point A_i.

No vertex A_k distinct from A_i can lie in or on the triangle A_iCF. A_k could not lie on A_iC since the polygon is simple.

It could not be on A_iF or FC because of the choice of F. A_k could not be interior to the triangle A_iCF for if so, the segment CA_k would cut A_iF contrary to our hypothesis on F.

A side of the polygon A_kA_{k+1} not terminating in A_i could not contain a vertex of the triangle A_iCF. It could not contain A_i or C since the polygon is simple. Because of the choice of F it could not contain F. According to Theorem 21, it has two points or none at all in common with the triangle A_iCF. If the segment A_kA_{k+1} has a point on the interior of the triangle it must cut two sides by Theorem 23. Since it could not cut A_iC or A_iF it does not cut CF. The segment $A_{i-1}A_i$ does not meet CF since F is on the prolongation of $A_{i-1}A_i$. Obviously CF does not meet A_iA_{i+1}. Hence CF has no point in common with the polygon except C.



Now choose the point D on the segment $B'A_i$ so that the segment DA_i is not cut by any of the lines drawn from F to the vertices of the polygon except in the point A_i. We show that no polygon side not ending in A_i cuts the segment DF. A vertex of the polygon A_k different from A_i can not lie on or in the triangle A_iDF. The segments A_iF does not contain A_k because of the choice of the point F. A_k is not on A_iD since A_iD is part of the broken line BA_i which does not meet the polygon. FD does not contain A_k for if so $D = E_v$. This would contradict our choice of D.

If A_kA_{k+1} is a side of the polygon not terminating in A_i it can not contain a vertex of the polygon. It can not contain A_i obviously, nor F by the choice of F. It can not contain D since

$B'A_i$ and hence DA_i has no point on the polygon. According to Theorem 21 it has two points on the triangle A_iDF or none at all. If this segment A_kA_{k+1} has a point on the interior of the triangle it must cut two sides by Theorem 23. Since it does not cut A_iF nor DA_i it can not cut FD. $A_{i-1}A_i$ and A_iA_{i+1} do not cut FD since F was chosen outside of the angle $A_{i-1}A_iA_{i+1}$. Hence the broken line from B to B', the segment B'D, the segment DF, and the segment FC taken in order join B to C without meeting the polygon.

Theorem 25. Every point of a simple polygon is accessible from every point of the plane.

Proof. Consider an arbitrary point B of the plane, not on the polygon. Join B to a point B' of the polygon by a straight line. This line will meet the polygon in a finite number of points. There is a first point E, in the sense from B to B', which the line BB' has in common with the polygon.* If the point E is a point of the side A_iA_{i+1}, then by Lemma 1, since E is accessible from B, A_i is accessible from B. By Lemma 2, every point of the side A_iA_{i+1} is accessible from B. If E is a vertex we apply Lemma 2 directly. By applying Lemmas 1 and 2, we have that every point of the polygon is accessible from every point of the plane.

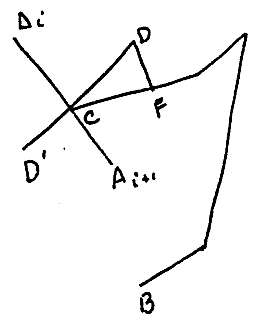
Theorem 26. A simple polygon divides the plane into two regions at most.

Proof. Let C be an arbitrary inner point of the polygon side A_iA_{i+1}. We draw thru C a line distinct from A_iA_{i+1}, and we can choose on it two points D and D' lying in the distinct half-planes determined by the line A_iA_{i+1} so that neither of the two segments CD and CD' is cut by the polygon side. Any arbitrary point B not

---

* E may be the point B'.

belonging to the polygon is now joined to C by a broken line not cutting the polygon except in C. Let B'C be the last of these segments. B' lies with D or D' in the same half-plane determined by the line $A_i A_{i+1}$; suppose with D. On the segment B'C choose F so



that the segment FC is not cut by any of the lines joining D to the vertices of the polygon.

All vertices of the polygon A_k lie outside of the triangle CDF. The segments CD and CF do not contain A_k because of the choice of D and F. DF does not contain A_k for if so, DA would meet CB' in F which would contradict our choice of F. For the same reason there is no vertex A_k inside the triangle CDF.

A side of the polygon different from $A_i A_{i+1}$ could not contain a vertex of the triangle CDF. It could not contain C since the polygon is simple. It could not contain D or F because of the choice of these points. As before, if $A_k A_{k+1}$ distinct from $A_i A_{i+1}$, has a point on the interior of the triangle it must cut two sides. It can not cut CD or CF, hence it can not cut DF. $A_i A_{i+1}$ can not cut DF because D and F lie in the same half-plane determined by $A_i A_{i+1}$. Hence B and D may be joined by a broken line not meeting the polygon. Then every arbitrary point of the plane not belonging to the polygon may be joined either with D or with D' by a broken line not meeting the polygon. The polygon then, divides the plane into two regions at most.

Lemma 3. Let B be an arbitrary point of the plane joined to a point C by a broken line lying in the plane, which does not cut the polygon. Then these two points may be joined by a broken line such that the prolongations of any of its segments do not pass thru a vertex of the polygon.

Proof. Let $BB_1B_2.......B_nC$ be the given broken line. Draw from B lines to all vertices of the polygon. These lines have at most a finite number of points in common with the segment B_1B_2. We can then choose C_1 on the segment B_1C_2, distinct from these intersection points and such that the segment B_1C_1 contains none of these points. By the method used above we can show that the segment BC_1 does not cut the polygon. The broken line $BC_1B_2....B_nC$ does not cut the polygon and the segment BC_1 belongs to a line which passes thru no vertex of the polygon. We continue in this manner with the other segments of the broken line.

Theorem 27. A simple polygon divides the plane into two regions at least.

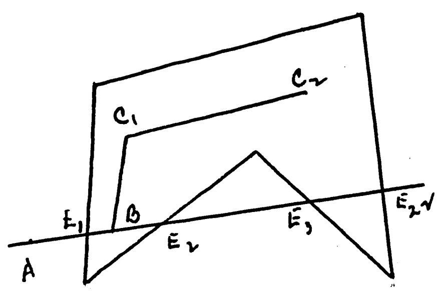
Proof. Consider a given straight line which passes thru no vertex of the polygon. It follows that two vertices A_i and A_{i+1} lie on the same or different sides of this line according as the segment A_iA_{i+1} is not or is cut by the line. This line either does not cut the polygon or cuts it in an even number of points.

Consider a given angle whose vertex does not lie on the polygon and whose sides pass thru no vertex of the polygon. If the points A_i and A_{i+1} are divided by the angle, then the segment A_iA_{i+1} contains a point of the angle; if not, it contains two or no points of the angle. It follows from this that the angle has no points or an even number of points in common with the polygon.

From an arbitrary point in the plane draw a straight line which passes thru no vertex of the polygon but which cuts the polygon. Let the intersection points of this line and the polygon be $E_1, E_2, ..., E_{2\nu}$. Let them have the ordering $(E_1, E_2, ..., E_{2\nu})$.

Choose on this line a point A so that we have the ordering $(AE_1, E_2, \ldots, E_{2\nu})$. One or both of the half lines into which our line is divided by A contains no points of the polygon.

Choose a point B on the same line such that on each of the two half-lines determined by B there are an odd number of intersection points E of the polygon. Let C be a point which may be joined to B by a broken line which does not cut the polygon. To show that every ray drawn from C must cut the polygon.



We can choose the broken line joining B and C, by the preceeding Lemma, so that the prolongations of its segments have no vertices of the polygon. Let $BC_1C_2\ldots C_nB$ C be such a broken line.

Consider the angle ABC_1.* The vertex of this angle does not lie on the polygon and its sides pass thru no vertex of the polygon. One of the sides of this angle contains an odd number of intersection points with the polygon. Its second side must therefore contain an odd number of intersections, and since the segment BC_1 contains none, the prolongation of the segment BC_1 contains this odd number of intersections.

We now consider the angle formed by the segments BC_1 and C_1C_2 and their prolongations. By the same reasoning we have that the prolongation of C_1C_2 contains an odd number of points of the polygon. We have finally that the prolongation of the segment C_nC from C has an odd number of intersection points with the polygon.

Draw now from C an arbitrary half-line. If it goes thru a vertex it has a point in common with the polygon. If it does not

---

* If A, B and C_1 are collinear the case is trivial.

go thru a vertex then it may be treated as above by considering the angle which it forms with the prolongation of $C_n C$ from C. This arbitrary half-line then contains an odd number of intersection points with the polygon*.

We have then that the point A may not be joined to the point B by a broken line not cutting the polygon since there is at least one half-line drawn from A which does not cut the polygon.

Since we have proved also that the polygon does not divide the plane into more than two regions, we have that it divides the plane into exactly two regions.

In this paper we shall consider the proofs of the Theorem of Jordan as given by Schoenflies, Kerekjarto, de la Vallee Poussin, Alexander, and Veblen, as well as the converse theorem and a criticism of the proof of de la Vallee Poussin, both by Schoenflies. Of these proofs, two are outstanding. The one by Kerekjarto because of its simplicity, and the one by Schoenflies because of its elegance and further because the method used leads to the converse theorem.

The proof of Schoenflies ** depends upon the point set theory. The proof depends upon the property of isolation, i.e., that an arc H of a simple closed Jordan curve may be enclosed in a generalized polygon such that C(H) is on the exterior of the polygon. The Jordan curve will be considered as the 1-1 and continuous image of a circle. The polygons used are generalized. They consist

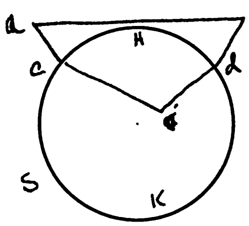
- -

* In case these two half-lines do not form an angle, they are either coincident in which case our conclusion follows; or together they form a straight line, and since this line must have an even number of intersections with the polygon and one half-line has an odd number, the other must also.

** A. Schoenflies, Uber das eineindeutig und Stetige Abbild des Kreises (Jordancurve), Jahresbericht der Deutschen Mathematiker-Vereinigung, Vol 33, pp. 147-157.

of two broken lines, the corresponding end points of which approach the same point while the number of sides increases indefinitely. We wish to show that if the Theorem of Jordan holds for an ordinary polygon composed of a finite number of segments, that it holds for a generalized polygon. Consider such a generalized polygon with the limit points P_1 and P_2. About P_1 and P_2 we may place squares of arbitrarily small side length. These squares combine with the generalized polygon to form an ordinary polygon*. Now any point of the plane not on the generalized polygon may be placed on the interior or the exterior of the ordinary polygon by choosing the side of the squares small enough. Any two points P and Q separated by every such ordinary polygon are separated by the generalized polygon. This follows since any polygonal line joining P and Q will either pass thru P_1 or be at a finite distance from it. If P_1 is on the polygonal line then the generalized polygon separates P and Q. If P_1 is not on the polygonal line we choose the square about P_1 so small that all points of it are exterior to the square. Then as the polygonal line cuts the ordinary polygon and not points of the square it cuts the generalized polygon**. We may show by the method of Theorem 30, that only two regions are formed.

Let S be a circle and let c and d be two distinct points on it. They determine two circular arcs, consisting of points (exclusive



of c and d) which we will call $H = \{h\}$ and $K = \{k\}$. Hence

$$S = H + K + \{c, d\},$$

where, by $\{c,d\}$ is meant the two points c and d. For

---

* There will be two possible boundaries near each point P_1 and P_2. We may take either one.

** Similar conclusions hold for P_2).

the sets H and K, the following hold:

1. Every limit point of points h is either itself a point of H or is one of the two points c and d. The closed set $\overline{H}$ consists of the set H and the points c and d. Similarly for K.

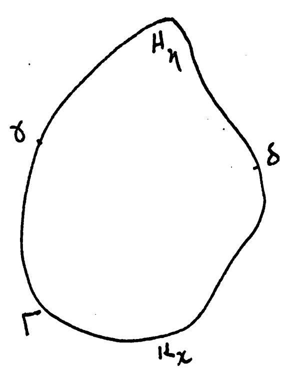
2. If i is any inner point and a is any outer point of the circle*, one can connect i and a with c and d** such that a polygon P is formed for which all points of H are inner points and therefore belong to I(P), and all points of K are outer points and therefore belong to O(P).

We say, therefore, that the polygon P isolates the point set H from the point set K (or $\overline{K}$). In a similar manner we can isolate the point set K from the point set H (or $\overline{H}$) by an analogous polygon. We call H and K isolated point sets.

We show that the above elementary properties of the circle hold for a general Jordan curve.

If Γ is the image circle (i.e., the image of S) and γ and δ are the image points of c and d, and if $H_\eta = \{\eta\}$ and $K_\chi = \{\chi\}$ are the image sets of H and K, then one has

$$\Gamma = H_\eta + K_\chi + \{\gamma, \delta\}.$$



On account of the constant relation between S and Γ (this relation is a 1-1 and continuous correspondence), Γ possesses property (1) directly, i.e., every limit point of points η is either itself a point η or one of the points γ, δ. It remains to show that the point sets H_η and K_χ are

---

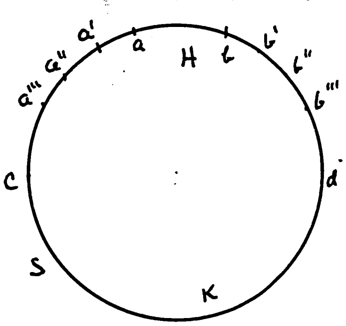
* Assuming that the circle divides the plane.

** This is the property of accessibility which will be developed for a general Jordan curve.

isolated from each other. We must show that we can enclose the set H_η in a polygon P of which γ and δ are vertices, while the set K_χ lies wholly outside of it.

If a and b are two points of the circular arc $\overline{H}$ and (ab) the closed arc determined by them, let α and β be the image points of a and b and $(\alpha\beta)$ the image set of the set (ab). Let the distance of the set $(\alpha\beta)$ from the closed set $\overline{K}_\chi$ $(=(\gamma\delta))$, which is the mimimum distance for the points of both sets, be 2ϵ. This is a non-zero distance because of the 1-1 and continuous property of the correspondence. If one surrounds each point of $(\alpha\beta)$ with a square of constant side direction and of side length $\sigma = \frac{1}{2}\epsilon$, then the whole set $\overline{K}_\chi$ lies outside these squares. By the Heine-Borel Theorem, a finite set of these squares exists such that every point of $(\alpha\beta)$ is an inner point of at least one square. These squares have one outer edge polygon. Let this outer edge polygon be $R_{\alpha\beta} = R$. Then every point of the arc belongs to $I(R_{\alpha\beta})$. In general the set $\overline{K}_\chi$ lies outside all squares, and in particular it lies outside $R_{\alpha\beta}$.

Let a, a', a'', ... be a sequence of points of H not interior to (ab). They are arranged in the order a, a', a'', a''', ... going from a to, but not including c. Similarly, the points b, b', b'', b''', ... form a sequence of points of H not interior to (ab) arranged in the order b, b', b'', ... going from b to, but not including d. The image points of the sequences of a's and b's are respectively α, α', ... β, β', ...and because the correspondence relating them is 1-1 and continuous, the α's and β's bear the same relation to α and γ and β and δ on Γ as the a's and b's do to a and c and b and d on S.



We wish to determine an edge polygon R' for ($\alpha'\beta'$) as was done for the set ($\alpha\beta$). The sets ($\alpha\alpha'$) and ($\beta\beta'$) are similar to the set ($\alpha\beta$). For these sets there are determined, as was done for ($\alpha\beta$), edge polygons P and O respectively. They overlap R and the edge of this overlapping configuration is designated as R'. The set ($\alpha'\beta'$) belongs to I(R') and the whole set K_χ to O(R'). In a similar manner we construct for the set ($\alpha''\beta''$) the edge polygon R''. This is done by constructing edhe polygons P' and O' for the sets ($\alpha'\alpha''$) and ($\beta'\beta''$). They project thru R' and form with R', the outer edge of R'', i.e., form the edge polygon R''. P' and O' satisfy the condition (to be proved later) that they both lie outside of R and hence do not cross R. We continue in this manner and obtain a series of polygons

R, R', R'',.......

which are made up of the polygon R R and the polygons

P, P', P'',...... and O, O', O'',........ .

We must show that the limit of these edge polygons forms an isolating polygon for the set H_1 and therefore its interior contai H_1 and the set K_χ belongs to its exterior.

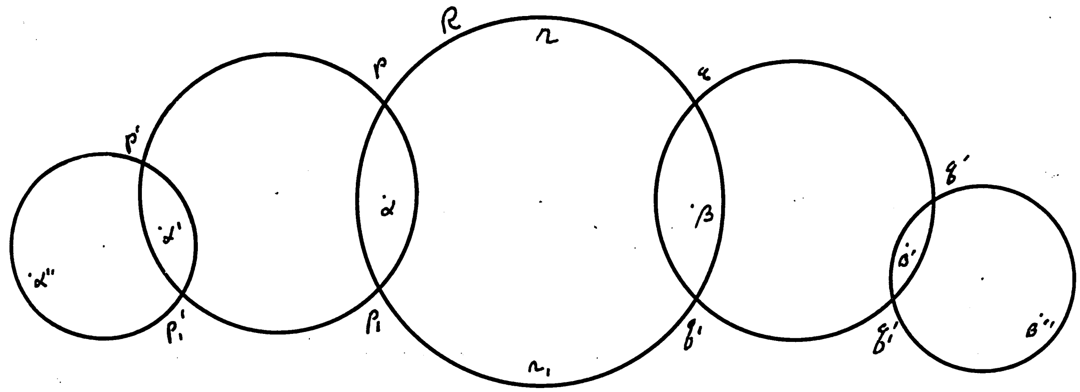
We have the following constructing properties, which will be justified later:

(1) The set $\overline{K}_\chi$ (= ($\gamma\delta$)) lies outside of each and every edge polygon.

(2) The polygons P and O, as is the case with P' and O' etc., lie outside of each other.

(3) Just as P' and O' lie outside of R, so P'' and O'' lie outside of R', etc.

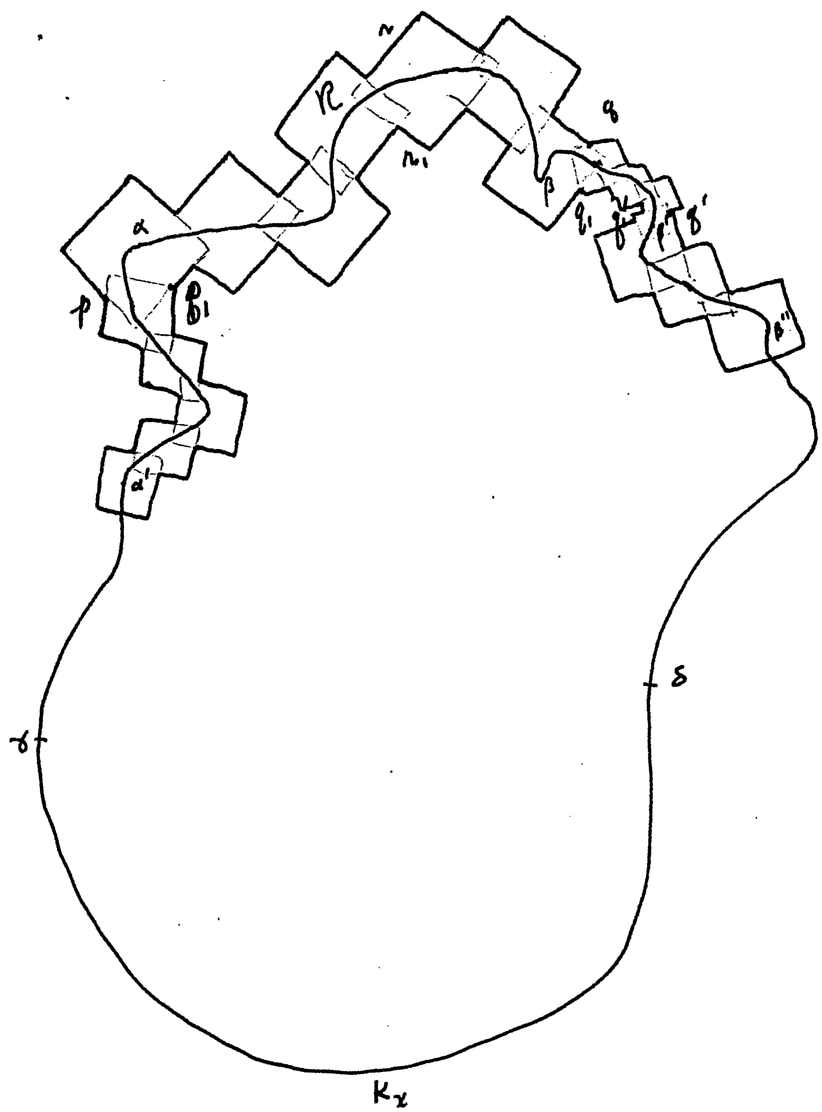
On the basis of these three properties we establish the theorem. The edge polygon R' is formed out of the overlapping of the polygons P and O with R. Now P, O, and R are each edge polygons formed by the outer boundary of a finite set of squares. R' will be the outer boundary of these three sets of squares. Now the polygons P and O lie outside of one another by property (1) and hence to R there belongs two well defined segments, r joining p and q and r_1 joining p_1 and q_1 along R where p and p_1 are the points of intersection of R and P, and q and q_1 are the points of intersection of R and O. Two such segments exist, and in case more than two exist, i.e., say P intersects R in more than two points the outer segment is chosen. A schematic drawing is shown below.



Now the edge polygon R'' also contains two such segments as noted above which will be designated as $r' = p'pqq'$ and $r_1' = p_1'p_1q_1q_1'$. To these segments belong the segments $r = pq$ and $r_1 = p_1q_1$ respectively as parts. The polygon R' consists of points of the outer boundary of R, P, and O. Likewise R'' contains similarly determined points of R', P', and O'; therefore also of R, P, O, P', and O' and to the segments determined by these points, belong the segments r and r_1, since R does not overlap P' or O' by property (3). We know likewise that r' and r_1' are subsets of r'' and r_1''. In such a manner we arrive at two increasing series of segments

$$r,\ r',\ r'',\ \ldots.\ \text{and}\ r_1,\ r_1',\ r_1'',\ \ldots\ldots\ldots,$$

and their combination sets,



α
p
β_1
α'
R
n
n_1
q
β
q_1
p'
q'
β''
γ
δ
K_x

$$u = S(r^{(\nu)}) \text{ and } u_1 = S(r_1^{(\nu)})$$

are such that as ν increases, the segments approach $R^{(\nu)}$.

We consider now the region which is determined by the increasing edge polygon. Every point of $I(R)$ is also a point of $I(R')$. Every point of $I(R')$ is also a point of $I(R'')$, etc. We set therefore,

$$I(R^{(\nu)}) = I_\nu \quad \text{and} \quad G = S(I_\nu).$$

The combination set G describes the above region.

It follows as in the preceeding, that every point of H_η is a point of G, while every point of $\overline{K}_\chi$ can belong at most to the boundary of G. The boundary of G is considered as being a combination of the boundaries of the regions

$$I(R),\ I(P^{(\nu)}),\ \text{and } I(O^{(\nu)}).$$

Each point of the boundary of G is thereby a limit point of points which belong to these regions (since they are closed). If the points lie in only a finite number of the sets $I(R)$, $I(P^{(\nu)})$, $I(O^{(\nu)})$, the limit point falls on a segment of the polygonal lines u or u_1, since any finite number of the above sets is bounded by the segments u or u_1. If however, they lie in an infinite number of the sets $I(R)$, $I(P^{(\nu)})$, $I(O^{(\nu)})$, then the diameter of the polygons $P^{(\nu)}$ and $O^{(\nu)}$ approaches zero with $1/\nu$, for suppose in particular, that the diameter of $P^{(\nu)}$ does not approach zero. The diameter of $P^{(\nu)}$ depends upon σ. If $\sigma \to 0$ the diameter of $P^{(\nu)} \to 0$. The diameter of $P^{(\nu)}$ remains $> M > 0$ if and only if σ does not approach zero. Since $P^{(\nu)}$ does not approach zero, let the greatest lower bound of σ be $\mathfrak{J} > 0$. Since $\alpha, \alpha', \alpha'', \ldots\ldots$ form a sequence of points approaching γ, for ν sufficiently large, $> \nu_0$, the arc length from α^ν to γ will be $< \mathfrak{J}/4$. Consequently, there is a point of Γ between α^ν and $\alpha^{\nu+1}$, the center of a square, part of the boundary of which will be contained in the boundary of $P^{(\nu)}$, and $P^{(\nu)}$ will contain the point γ on its interior, i.e., a point of $\bar{K}_\chi$, which violates property (1). A similar argument shows that the diameter of $O^{(\nu)} \to 0$ with $1/\nu$.

Since the diameters of $P^{(\nu)}$ and $O^{(\nu)}$ approach zero, each point of $P^{(\nu)}$ and $O^{(\nu)}$ (for ν sufficiently large) may be replaced by any other point of the polygonal region to which it belongs. In particular, each point may be replaced by a point of Γ. As was pointed out before, limit points of points of Γ belong either to Γ itself, which case is not relevent here, or they fall on one of the points γ or δ. These points, γ and δ, are however, limit points of point sets which belong to u and u_1, and consequently belong to the boundary of the region G.

The boundary of the region G is thus a generalized polygon U. It represents an isolating polygon for the set H_η and hence we have proved that we may isolate each subset H_η of the image circle.

We must now establish the three construction properties used above. In order to prove that P and O, for example, lie outside of each other, we must show that they have no points in common and that one does not lie within the other. The first is proved by an appropriate choice of the side square length σ. The second is proved below.

Now if a function is continuous in the interval r.......s, one can determine for this an interval-length $\varphi > 0$, so that in every sub-interval $\tau < \varphi$, the oscillation of the function is less than a preassigned bound w. Hence, given w, one may determine a sub-arc (lm) of (ab) such that if λ', μ' are two points of the image set $(\lambda\mu)$ of (lm), it follows that for the diameter of the set $(\lambda\mu)$,

$$(a)\ \beta(\lambda\mu) < w,$$

and for the distance of the points λ', μ'

$$(b)\ \rho(\lambda'\mu') < w.$$

We have the following considerations:

(1) The bound w may have such a value that for the diameter of the image set $\bar{K}_\chi$ $(=(\gamma\delta))$ the relation

$$(c)\ \beta(\gamma\delta) > 2w$$

holds*. Furthermore, each of two adjacent points of the points a, a', a'',....... , b, b', b'',..... of the circle shall determine an arc, as that determined by (lm); in particular, the arc (ab) shall determine an arc of this type.**

(2) Let k, l, m, n be four consecutive points of the set of a's or b's or both. They divide the circle into four circular arcs

(kl), (lm), (mn), and (kcdn).

The image sets $(\lambda\mu)$ and $(\chi\gamma\delta\nu)$ of the sets (lm) and (kcdn) are new sets. If 2ς is their distance apart, then $2\varsigma > 0$ because of the 1-1 and continuous relation between S and Γ, while for the diameter of $(\chi\gamma\delta\nu)$ the relation

$$(d)\ \beta(\chi\gamma\delta\nu) \geq \beta(\gamma\delta) > 2w$$

holds. Also the distance ρ of the set $(\lambda\mu)$ from each and every subset of $(\chi\gamma\delta\nu)$ satisfies the relation

$$\rho \geq 2\varsigma.$$

The lower bound of ς for all sets $(\lambda\mu)$ coming under consideration is obviously zero.

We construct now around the set $(\lambda\mu)$ an edge polygon $R_{\lambda\mu}$ by using squares of side length σ which satisfy the relations

---

* This is no essential restriction.

** See fig. p. 17

$$(e)\ \sigma < \tfrac{1}{2} s;\ \sigma < \tfrac{1}{2}\omega.$$

Becauee of the relation $\sigma < \frac{1}{2} s$, the covering squares for the set $(\lambda\mu)$, and therefore also the associated edge polygon $R_{\lambda\mu}$, have no points in common with the set $(\chi\gamma\delta\nu)$. It is not sufficient to conclude however, that the set $(\chi\gamma\delta\nu)$ lies outside the edge polygon. These squares can determine a multiply connected region, and then inside $R_{\lambda\mu}$ there is at least one region which lies outside all these squares. The set $(\chi\gamma\delta\nu)$ can belong to it. Since the set $(\chi\gamma\delta\nu)$ is connected, it belongs to such a region wholly or not at all, and this is true of each of its subsets. If it belongs to such a region, the relation

$$\beta\{R_{\lambda\mu}\} > \beta\ (\chi\gamma\delta\nu) > 2w$$

necessarily holds.

One the other hand we have also

$$\beta\{R_{\lambda\mu}\} < \beta\ (\lambda\mu + 2\sigma) < w + 2\sigma < 2w.$$

This is a contradiction. The set $(\chi\gamma\delta\nu)$ lies, therefore, outside the polygon $R_{\lambda\mu}$, and on account of the relation (e), it holds for every edge polygon composed of squares of side length σ that surrounds some subset of $(\chi\gamma\delta\nu)$. In particular, the set $(\gamma\delta)$ (which is a subset of each and every set $(\chi\gamma\delta\nu)$) lies outside each and every edge polygon.

Now the set $(\gamma\delta)$ lies outside each and every edge polygon surrounding the set $(\lambda\mu)$. Now as was noted above, (lm) is determined by any two consecutive a's or b's or both. Its image set is determined by the two corresponding α's or β's or both and they are necessarily consecutive. In particular $(\lambda\mu)$ may be $(\alpha\beta)$, $(\alpha\alpha')$, $(\beta\beta')$, $(\alpha'\alpha'')$,.... and $(\gamma\delta)$ will lie outside the edge polygons covering these sets, i.e., outside the polygons R, P, O, P', O',..... . Hence K_χ lies outside the polygons R, P, O,..

This is property (1).

Of two such polygons as those considered in the preceeding paragraph, one may say that the one surrounding a set $(\lambda\mu)$, the other a subset of $(\chi\gamma\delta\nu)$. The set $(\lambda\mu)$ may be taken to be $(\alpha\beta)$ and the subset of $(\chi\gamma\delta\nu)$ may be taken to be $(\alpha'\alpha'')$ or $(\beta'\beta'')$ and by a proper choice of σ and w, the polygon surrounding $(\alpha\beta)$ and those surrounding $(\alpha'\alpha'')$ or $(\beta'\beta'')$ will lie outside of one another, i.e., R and P', and R and O' lie outside of one another and have no points in common. Continuing in this manner we deduce properties (2) and (3).

With the aid of the isolating polygon U we prove the Theorem of Jordan. First, as the side of the squares σ becomes smaller the polygonal lines u and u_1 become closer and closer to the set H_η. Secondly, as the set H_η was isolated from the set K_χ by a polygon U, we can isolate the set K_χ from the set H_η by a polygon B. The polygons U and B have only the points γ, δ in common, because of the choice of σ, the length of the side of the square surrounding each point of the sets H_η and K_χ. Let ξ and ξ_1 be the polygonal lines of B corresponding to the polygonal lines u and u_1 of U.

The polygons U and B divide the plane into four regions. Two are the polygons I(U) and I(B), the third comes from u_1 and ξ_1, the fourth, the exterior of u and ξ.

If g is determined by u_1 and ξ_1, it depends upon the size of σ. $\sigma^{(\nu)}$ takes on a sequence of values $\{\sigma^{(\nu)}\}$ as $\sigma \to 0$. $g^{(\nu)}$ is the region to which the values of $\sigma^{(\nu)}$ correspond. The combination set

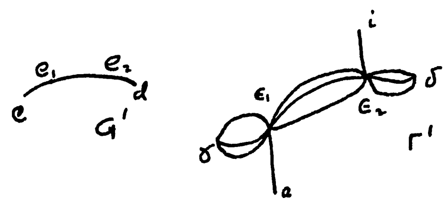
$$h = G\{g^{(\nu)}\}$$

of all these regions determines a region, all of whose limit points belong to the image set Γ. Hence the image set Γ determines in the plane at least two regions.

In order to show that the plane is divided into only two regions by Γ, we need the auxiliary theorem, that the image set of a circular arc does not divide the plane. If G' is the circular arc, c and d its end points and Γ' its image, with γ and δ the image points of c and d, then we have as before,

$$G' = H + \{c, d\}, \quad \Gamma' = H_1 + \{\gamma, \delta\} .$$

Assuming that Γ' determines a regional division of the plane,



let i and a be two points belonging to different regions. From a, draw a straight line to Γ', meeting it in ϵ_1. This line may meet Γ' in more than one point, but let ϵ_1 be the first point of meeting in the sense from a to ϵ_1. Draw a similar line from i meeting Γ' in ϵ_2. ϵ_1 and ϵ_2 on Γ' correspond to the points e_1 and e_2 on G'.* The points e_1 and e_2 divide the arc G' into three sub-arcs

$$(ce_1), \ (e_1 e_2) \text{ and } (e_2 d).$$

We now enclose their image sets in an isolating polygon. First we isolate the set $(\epsilon_1 \epsilon_2)$ from the sets $(\gamma \epsilon_1 a)$ and $(i \epsilon_2 \delta)$; then we isolate $(\gamma \epsilon_1)$ from the polygon just obtained and the sets $(a \epsilon_1)$ and $(i \epsilon_2 \delta)$, and finally we isolate $(\epsilon_2 \delta)$ from these polygons and from $(a \epsilon_1)$ and $(i \epsilon_2)$. The points i and a lie outside all three polygons and hence may be joined. This is a contradiction, hence the theorem.

We now show that Γ divides the plane into only two regions. If F is any region into which the plane is divided, let φ be its boundary.** Suppose φ is not identical with Γ. Then φ is a

---

* The drawing of these lines is necessary since the sets we later isolate must be simply connected.

**We make the convention that the boundary, eg., of the points $x^2+y^2>a^2$ is the circle $x^2+y^2=a^2$, as it is for the points $x^2+y^2<a^2$.

connected subset of Γ and is the image of a circular arc and hence does not divide the plane by the above auxiliary theorem. Since any regional division of the plane has Γ as its boundary, divides the plane into exactly two regions.

We have seen that the sets H_η and K_χ may be enclosed in polygons U and B respectively, such that they have no point in common except γ and δ. Further, these polygons may be made arbitrarily close to Γ by a choice of the side length σ. Hence any point P of the plane not on Γ will be, by a proper choice of σ, in the third region determined by u_1 and ξ_1 or in the fourth region determined by u and ξ. Hence it may be joined to u or u_1 depending upon which of the two regions P is in. Hence P may be joined to γ or δ by a Jordan curve which has no point on Γ except γ or δ. Since γ and δ were any two distinct points of Γ, we are lead to the fact that the points of a Jordan curve are accessible from any point of the plane. It is noted that the points γ and δ are actually limit points of points which are accessible from P by polygonal lines.

We now turn to the proof by Kerekjarto* which is by far the simplest of the proofs considered, but unfortunately it relies too much upon intuition. The details of the proof are included here. Kerekjarto omits those details which are obviously true and may be easily justified (there is one exception to this, which will be pointed out later). He assumes no credit for originality since he states that the proof is a combination of the simpler parts of the simpler existing proofs. He also assumes the theorem for a simple polygon composed of a finite number of segments.

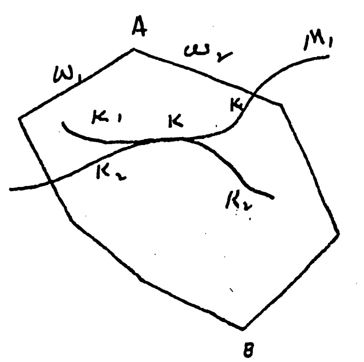
---

* B. de Kerekjarto, Demonstration Elementaire du Theoreme de Jordan sur les Curbes Planes, Acta Litterarum Ac Scientiarum, vol 5, (1930-1932) pp. 56-59.

Kerekjarto uses the theorem, as does Schoenflies, that a simple arc does not divide the plane.

Lemma 4. Let K_1 and K_2 be two bounded continua in the plane, of which the common part K is a continuum.* If two points of the plane A and B which do not belong to $K_1 + K_2$, are separated neither by K_1 nor K_2, then they are not separated by $K_1 + K_2$.
Proof. Let w_1 and w_2 be two lines which join A and B such that w_1 does not meet K_1 and w_2 does not meet K_2. We may suppose that w_1 and w_2 have no other points in common except their extremities A and B**.

The polygon $\Pi = w_1 + w_2$ divides the plane into two domains. Suppose that the contimuum K, which has no point on Π, is found on the interior of Π ***. The points of K situated on Π and on its exterior form a set M_1. The set M_1 is closed. It is null, or contains a finite number of points in which case it is closed, or it contains an infinitude of points. The set M_1 is bounded



as it is a subset of K_1 which is bounded. If M_1 has an infinitude of points it has a limit point by the Bolzano-Weierstrass Theorem. If P is such a limit point suppose that M_1 is not closed and P is not a point of M_1. Then P is an interior point of Π. We have then an interval about P which contains no points of M_1. This is a contradiction.

The set M_1 is at a positive distance $> \delta > 0$ from K_2 and from w_1.

---

* These sets are assumed to be one dimensional.

** In case w_1 and w_2 have other points in common we may replace A and B by two points A', and B' common to w_1 and w_2 such that the part A'B' of w_1 does not meet K_1 and A'B' of w_2 does not meet K_2.

*** This is no restriction.

M_1 is at a positive distance from w_1 since w_1 does not meet K_1. It is at a positive distance from K_2 for if not K_2 and K_1 would have a point in common other that K and hence the points in common would not form a continuum.

About each point of M place a square of side length $< \delta/4$, with constant side direction. By the Heine-Borel Theorem, a finite number of such squares exist, and the sides of these squares form a polygon. This polygon forms with π and its interior a polygonal domain. Let π' be the boundary of that domain, part of which is formed by the whole line w_1. The other part of π' is a line which joins A and B without meeting $K_1 + K_2$.

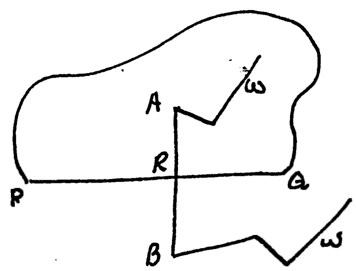
Theorem 28. A simple arc does not divide the plane.

Proof. Let A and B be two points which are not on the simple arc $\widehat{PQ}$. We divide the arc $\widehat{PQ}$ into a finite number of consecutive arcs $\widehat{P_0P_1}$, $\widehat{P_1P_2}$, ..., $\widehat{P_{n-1}P_n}$ ($P_0 = P$, $P_n = Q$) such that the diameters of the arcs $\widehat{P_kP_{k+1}}$ are less thanthe distance from the points A or B to the arc $\widehat{PQ}$. We wish to show that each of the subarcs does not separate A and B.

Let σ be the smaller of the distances of A and of B to the arc $\widehat{PQ}$. Consider any one of the arcs $\widehat{P_kP_{k+1}}$. About each point of it place a square of side length $< \sigma/4$ and of constant side direction. Again applying the Heine-Borel Theorem, we have a finite number of such squares strictly covering $\widehat{P_kP_{k+1}}$ and the outer boundary of this finite number of squares forms a polygonal region in which the points A and B do not lie. Since this polygon divides the plane into two domains, and since A and B are exterior to it, they may be joined by a polygonal line not meeting this polygon, and hence not meeting $\widehat{P_kP_{k+1}}$. Hence each of the arcs $\widehat{P_kP_{k+1}}$ does not separate the points A and B. Applying the auxiliary theorem above successively to the arcs $K_1 = \widehat{P_0 P_1}$ and $K_2 = \widehat{P_1 P_2}$*; then to the arcs $K_1 = \widehat{P_0 P_2}$ and $K_2 = \widehat{P_2 P_3}$, etc., we have the theorem.

Theorem 29. Let $\overline{PQ}$ be a straight line segment and let $\widehat{PQ}$ be a simple arc which has no points in common with $\overline{PQ}$ except the points P and Q. The simple closed curve $j = \overline{PQ} + \widehat{PQ}$ divides the plane into two domains at least.

Proof. Let ARB be a segment perpendicular to $\overline{PQ}$, drawn from a point R interior to $\overline{PQ}$, such that ARB has no point on the arc $\widehat{PQ}$.



Suppose AwB is a polygonal line joining A and B which does not meet the curve j. AwB and ARB form a polygon π.

Suppose that P is interior to π. Then about P there is a neighborhood, all points of which are interior to π. In traversing $\overline{PQ}$ from P to Q we come to a first point of $\overline{PQ}$ on π. Let this point be K. Now K is on ARB and $\overline{PQ}$ or on AwB and $\overline{PQ}$. Since AwB does not cut j, in particular it does not cut $\overline{PQ}$. Hence K is on ARB, and therefore $K \equiv R$. Then all points of the segment $\overline{PR}$ are interior to π (Except the point R).

Suppose now that Q is interior to π. By an argument similar to that above, we have that the segment $\overline{QR}$ is interior to π except the point R which is on π. Since π divides the plane, there are points of $\overline{PR}$ and points of $\overline{QR}$ which are on opposite sides of the boundary of π, i.e., points of $\overline{PR}$ which are interior and points of $\overline{QR}$ which are exterior to π. This is a contradiction of the conclusion that $\overline{PR}$ and $\overline{QR}$ are interior to π excepting the point R. Hence if P is interior to π, Q is exterior to π. By interchanging P and Q we have similar results.

---

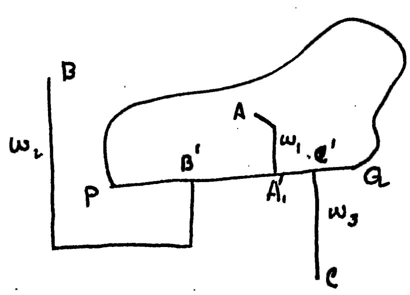
* Here, one point is considered as a contimuum.

P and Q are then separated by the polygon Π. Then the arc $\widehat{PQ}$ cuts Π, since Π divides the plane. Now $\widehat{PQ}$ does not cut ARB, so it must cut AmB, contrary to hypothesis. Hence A and B are two points separated by j. Hence the theorem.

Theorem 30. The curve $j = \overline{PQ} + \widehat{PQ}$ divides the plane into two domains at most.

Proof. Let A, B, and C be three arbitrary points of the plane not situated on j. By Theorem 28 the arc $\widehat{PQ}$ does not divide the plane and hence does not separate any two points of the plane not on $\widehat{PQ}$.

As any point of a straight setment is accessible from any point of the plane we may join A, B, and C to points A', B', and C' of the segment $\overline{PQ}$ by lines w_1, w_2, and w_3 respectively, which do not meet the arc $\widehat{PQ}$. Two at least of the lines w_1, w_2, and w_3 must meet on the same side of $\overline{PQ}$. Suppose that w_2, w_3 end at B', C' respectively on the same side of $\overline{PQ}$. The points B' and C' are



distinct from P and Q since w_2 and w_3 do not meet the arc $\widehat{PQ}$. Let σ be the distance of the points of the arc $\widehat{PQ}$ from the point of $\overline{PQ}$ between B' and C'. We may join w_2 and w_3 then, by a line parallel to, and at a distance $\frac{1}{2}\sigma$ from, $\overline{PQ}$, which will have no point on the arc $\widehat{PQ}$, or on the line $\overline{PQ}$ and hence no point on j. Therefore for any three points of the plane not on j, at least two of them may be joined by a line not meeting j. Hence j divides the plane into not more than two domains.

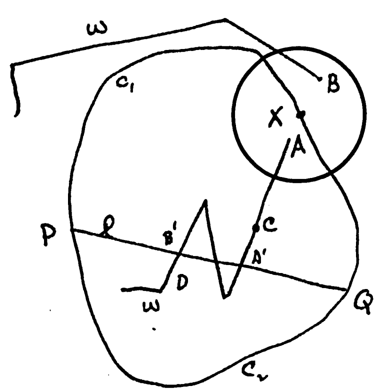
We understand by the interior and by the exterior of a curve $j = \overline{PQ} + \widehat{PQ}$, the bounded domain and the unbounded domain respectively, determined by the curve j.

Theorem 31. A simple arbitrary closed curve j divides the plane into two domains at least.

Proof. Consider any two distinct points M and N of the curve j, and draw the segment $\overline{MN}$. If the segment $\overline{MN}$ has a point A which is not a point of j then there is an interval about A free of points of j for otherwise A would be a limit point of points of j and hence a point of j since j is closed. Consider now the maximum interval about A which is free of points of j. The end points of this interval are points of the curve j for if not they are neither limit points of the curve and the above interval would not be the maximum interval of the segment free of points of j. This segment $\overline{PQ}$ (= l) has its end points P and Q on the curve but no other point.

If every point of the segment $\overline{MN}$ is a point of j then j contains a linear element. The segment $\overline{MN}$ contains no other point of j as otherwise the curve would not be simple. The theorem then reduces to the preceeding case.

Let c_1 and c_2 be the two arcs of j into which j is divided by the two points P and Q. For the curves $j_1 = c_1 + l$ and $j_2 = c_2 + l$, Theorems 29 and 30 are valid. About a point X of c_1 draw a



circle which has no point on j_2. Kerekjarto now assumes that two points A and B exist within this circle such that A is an interior point and B an exterior point of the curve $j_1 = c_1 + l$. The existence of these two points is not easily verified, even though it seems intuitively obvious. We may however use the principle of accessibility as developed by Schoenflies and show the existence of these points. If, as Schoenflies shows, the point X of c_1 about which the above circle is drawn is accessible from a point of each of the two regions into which the plane is divided by j_1, then the circle about X will contain points of both regions. We then have a point A interior and a point B exterior to j_1 and within the circle about X.

We wish to show that the points A and B are separated by j. If they are not let w be a line joining A and B without meeting j. w must meet l since A and B are separated by $j_1 = c_1 + l$. Consider the line AwB, letting A' and B' be the first and last point of meeting with l in the sense of going from A to B along w. A first and last point of meeting exist since the line w is composed of a finite number of segments and hence can meet l in only a finite number of points.* Let C and D be two points near respectively to A' and B' on the parts AA' and BB' of w. On the one hand the points C and D are found on opposite sides of l and hence are separated by $j_2 = c_2 + l$ by Theorem 29. On the other hand, the parts AC and BD of the line w and the straight segment AB together form a line joining C and D without meeting $j_2 = c_2 + l$, since w does not meet j by hypothesis. This is a contradiction. Hence there exists two points separated by j, i.e., j divides the plane into two domains at least.

Theorem 32. A simple arbitrary closed curve j divides the plane into two domains at most.

Proof. By Theorem 31 a simple closed curve j divides the plane into at least two domains. Hence there exista a point R not on j which is separated from the point at infinity. Then any line thru R meets the curve j in two points at least. In traversing this line in each direction we come to a first point on this line

---

* If w meets l in only one point, A' and B' are identical.

which is on j. If these points are P and Q, the segment PRQ ($=l$) has no points on j except P and Q. Every point of this segment, excepting P and Q, is separated from the point at infinity.

Let c_1 and c_2 be the arcs of j determined by the points P and Q and let $j_1 = c_1 + l$ and $j_2 = c_2 + l$. We show that the arc c_1 is exterior to j_2. If c_1 is not exterior to j_2 we may join a point of l, (exclusive of P and Q), to the point at infinity without meeting c_2 (by Theorem 38) and since c_1 is interior to c_2+l, without meeting c_1. This is a contradiction of the assumption that R is separated from the point at infinity by j. Hence c_1 is exterior to j_2, and by a change of subscripts, c_2 is exterior to j_1.

We understand by the interior of j, the sum of the interiors of j_1 and of j_2, and the points of l, differing from P and Q. Each interior point of j_1 may be joined, thru a point of l, to a point in the interior of j_2. (The same may be said with a change of subscripts). Then any two points interior to j may be joined by a line interior to j without meeting j.

The exterior of j is defined as the common part of the exteriors of j_1 and j_2. Let A and B be two points on the exterior of j. They are separated neither by j_1 nor by j_2. The curves j_1 and j_2 are two bounded continua of which the common part is a contimuum l. By the auxiliary theorem (Lemma 4), we may join A and B by a line which has no points on $j_1 + j_2$, i.e., no point on j. Hence all points of the plane not on j belong either to the interior or the exterior of j. Hence j divides the plane into two domains at most.

Alexander's treatment of the Theorem of Jordan* involves chains, which are a sort of generalized polygon. Alexander

---

* J.W. Alexander, <u>A Proof of Jordan's Theorem about a Simple Closed Curve</u>, Annals of Mathematics, vol 21 pp. 180-184.

assumes the theorem for the triangle and apparently does so for the convex polygon, although his statements concerning the two sides of a chain may be justified without the assumption of the polygon case.

The theorem on the simple arc is used by Alexander. He uses also Lemma 4 of Kerekjarto in a slightly different form. Kerekjarto gives Alexander credit for this lemma.

A chain will be any sort of generalized polygon consisting of a finite number of non-intersecting edges (which may be line segments or rays), and vertices (the end points of the rays), where at each vertex there end an even number of edges. A chain need not be connected.

Suppose we have a chain whose edges are all segments. Then if two vertices, Y and Z, may be joined by a broken line made up of elements of the chain, they may also be joined by a second broken line which has no edges in common with the first. For if we remove from the chain the edges of the first broken line, there will still remain an even number of edges abutting at every vertex except Y and Z where there will now remain an odd number. But, within each connected group of edges and vertices, the total number of times that edges abut on vertices is equal to twice the number of edges and is therefore even. Hence the vertices Y and Z still belong to the same connected piece and may be joined by a broken line.

A simple illustration of a chain would be a pair of broken lines connecting the same two points Y and Z, and having a finite number of points in common.

A chain k, like a simple polygon, has two sides, although the sides are not in general connected regions. We may determine them as follows. Complete the lines to which the edges of the chain k belong and thus obtain a system of lines which subdivide one another into a finite number of line segments and rays $b_1, b_2, b_3, \ldots\ldots, b_n$, while they subdivide the plane into a finite number of convex regions $a_1, a_2, a_3, \ldots\ldots, a_m$. Now, the boundaries of the regions a_i are chains made up of sets of elements b_j and their end points. Out of the symbols for the elements in these sets, we shall form the expression

$$(1) \quad a_i = b_{i_1} + b_{i_2} + b_{i_3} + \ldots\ldots + b_{i_k} \qquad (i = 1, 2, 3, \ldots\ldots, m)$$

which shall be used to designate the boundaries of the various cells a_i. The expressions(1) will be combined by adding corresponding members, collecting terms, and reducing all coefficients modulo 2. In this way, we can obtain new combinations defining new chains whose edges can be read off from the right-hand members.

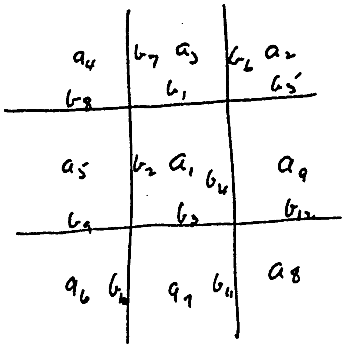
We use the following theorems: any chain, such as k composed of elements b_j and their end points, can be derived from elementary chanis (1) in two and only two ways,

$$(2) \quad \textstyle\sum_1 a_i = k$$

and

$$(2') \quad \textstyle\sum_2 a_i = k$$

and that each of the regions a_i occurs in one and only one of the combinations. For example:



$$a_1 = b_1 + b_2 + b_3 + b_4 = k$$

$$\begin{aligned} & a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \\ &= b_5 + b_6 + b_6 + b_1 + b_7 + b_7 + b_8 + b_8 + b_2 + b_9 \\ &+ b_9 + b_{10} + b_{10} + b_3 + b_{11} + b_{11} + b_{12} + b_{12} + b_4 + b_5 \\ &= 2b_5 + 2b_6 + 2b_7 + 2b_8 + 2b_9 + 2b_{10} + 2b_{11} + 2b_{12} + b_1 + b_2 + b_3 + b_4 \\ &= b_1 + b_2 + b_3 + b_4 = k \end{aligned}$$

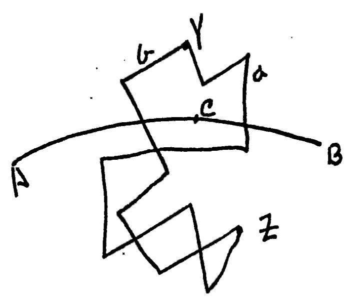
Therefore the points of the plane fall into two classes according as they belong to the interior or boundary of a region occuring in the first combination, or of a region occurring in the second. These two classes of points will be called the sides of the chain k.

Suppose we have two chains k_1 and k_2. Then the set of points which are on given sides both of the chain k_1 and the chain k_2, may be subdivided into a finite number of convex regions. Therefore, the set is bounded by a chain composed of the sum, modulo 2, of the boundaries of the convex regions. By combining this chain with the chain k_1, we obtain a new chain k_3.

A region is a set of points, each of which is interior to a triangle inclosing only points of the set, while any two may be joined by an arc made up of points of the set. The first condition is satisfied by the complement of any closed set. When the second condition is satisfied, two points, Y and Z, of the region may also be connected within the region by a broken line which may be so chosen as to have only a finite number of points in common with any preassigned finite system of lines. To show this property consider any point P, of the arc joining Y and Z. We may place about P a triangle which incloses only points of the region. Therefore, within this triangle, we may find a sub-arc containing the point P such that any two points of this sub-arc may be joined by a broken line of the required type and such that the point P is not an end point of the sub-arc unless it is an end point of the arc YZ itself. Since the whole arc is covered by these sub-arcs, it may be covered by a finite number of them by the Heine-Borel Theorem. We may therefore construct a broken line connecting the points Y and Z by piecing together a series of broken lines running from one sub-segment to an adjacent one, and so chosen that no two of them have more than a finite number of points in common. The broken line thus obtained, which may cross itself a finite number of times, has a finite number of segments, each of which cannot meet a pre-assigned finite system of lines in more than a finite number of points.

Lemma 5. Let ACB be a simple arc passing thru a point C and ending at the points A and B, and let Y and Z be any two points of the plane not on the arc ACB. Then, if the points Y and Z are not separated by either of the sub-arcs AC or CB, neither are they separated by the arc ACB itself.

Proof. The points Y and Z may be connected by a pair of broken lines a and b such that a does not meet the arc AC and b does not meet the arc CB. The broken line b may be chosen so that it meets the broken line a in at most a finite number of points and hence may be combined with a to form a chain k. Now consider such points of the arc CB as are either on the chain k, (i.e., on the broken line a), or on the opposite side of the chain k from the point C. Each of these points may be enclosed within a triangle which neither meets nor encloses a point of the arc AC or of the broken line b, and since the set of all such points is closed, they may all be enclosed within a finite number of these triangles by the Heine-Borel Theorem.



We add, modulo 2, to the chain k, the boundaries of the finite set of convex regions made up of points which are both interior to one of the triangles and on the opposite side of the chain k from the point C. We thus obtain a new chain k', which still contains the broken line b, as well as a supplementary piece a', made up of segments which neither meet nor end on the arc ABC. Therefore, the points Y and Z may be joined by a broken line which does not meet the arc ACB.

Theorem 33. The points of the plane not on a simple arc AB do not form more than one connected region.

Proof. We wish to show that any two points, Y and Z, not on the arc AB, may be joined by a broken line which does not meet the arc AB.

About any point C, of the arc AB, we may place a triangle with respect to which Y and Z are exterior points, since Y and Z are not on the arc AB. By remaining within this triangle, we may find a sub-arc of the arc AB which does not separate the points Y and Z, which contains the point C, and which ends at the point C only when C is one of the points A or B. The arc AB may thus be covered by a set of overlapping sub-arcs, and consequently by a finite set of overlapping sub-arcs, such that no one of them separates the points Y and Z. But the end points of this last set subdivide the arc AB into a still smaller finite set of non-overlapping sub-arcs. Therefore, since the arc AB may be built up by piecing together these sub-arcs, it cannot separate the points Y and Z by Lemma 5.

Theorem 34. The points of the plane not on a simple closed curve do not form more than two connected regions.

Proof. We wish to show that given any three points X, Y, and Z, not on the curve, two of them at least, may always be connected by a broken line which does not meet the curve.

Let A, B, and C be any three distinct points of the curve. Then the points X and Y, Y and Z, and Z and X, by Theorem33, may be joined by three broken lines a, b, and c respectively, which do not meet the arcs CAB, ABC, and BCA respectively. Moreover, the broken lines a, b, and c may be so chosen that no two of them have more than a finite number of points in common, so that they may be combined to form a chain k.

Now of the three points A, B, and C, two at least, must be on the same side of the chain k, and we may assume without loss of generality that the points B and C are.

We now have that the points X and Y are joined by a pair of broken lines a and bc which do not meet each other in more than a finite number of points and such that the line a does not meet the arc CAB and bc does not meet the arc BC. These broken lines form a chain k.

Consider the points of the arc BC which are either on the chain k (i.e., on the broken line a), or on the opposite side of the chain k from the point C. Each of these points may be enclosed within a triangle which neither meets nor encloses a point of the arc CAB or the broken line bc, and since the set of all such points is closed they may all be enclosed within a finite number of these triangles by the Heine-Borel Theorem.

We add modulo 2 to the chain k, the boundary of the finite set of convex regions made up of points which are both interior to one of the triangles and on the opposite side of the chain k from the point C. We thus obtain a new chain k', which still contains the broken line bc, as well as a supplementary piece a', made up of segments which neither meet nor end on the arc ACB, Therefore, the points X and Y may be joined by a broken line which does not meet the arc ACB. Since ACB is the closed curve, the theorem follows.

Theorem 35. The points of the plane not on a simple closed curve form at least two connected regions.

Proof. Choose any two points, A and B, on the curve and denote by AB and BA respectively, the two arcs of the curve bounded by these points. Then any line l, which separates the points A and B meets the arcs AB and BA in two closed sets of points. If this were not so, either there exists a point P on l such that every circle about P contains a point of the arc AB, P excepted, or, about any point P' of l we may draw a circle small enough so that none of these circles contain a point of the arc AB. The same may be said of the arc BA. In the first case, the point P would be a point of the arc AB since the curve is a closed set of points. In the second case, we should have the arc AB divided into two sets whose distance apart is finite. This would contradict the fact that the curve is a simple closed Jordan curve. The sets of points in which l meets AB and BA are then closed.

Now every point of the first set is interior to some interval of the line l which contains no points of the arc BA, for if not, every interval of l about a point P of AB would contain a point of BA. Then P would be a limit point of points of BA and hence a point of BA since the curve is a closed set of points. Then AB and BA would not be distinct sets of points. In fact, the curve would not be simple. By the Heine-Borel Theorem, the entire set of points may be covered by a finite number of intervals. By combining intervals when necessary, we may arrange them so that no two overlap or are contiguous. We shall prove that the end points of this last set of intervals, i, are not all within the same region by showing that, however we may connect them in pairs by a system of broken lines, one or more of the broken lines will always meet the curve.

Consider such a system of broken lines, assuming, as we may, that no one of them meets the line l, or another broken line of the system in more than a finite number of points. Then the system of lines may be combined with the intervals i to form a chain k. Moreover, if we add to the chain k the boundary of one side of the line l (i.e., the line l itself), we shall obtain a second chain k' made up of the broken lines combined with the segments s of the line l, complementary to the intervals i. By the definition of the sides of a chain it is clear that one or the other of the chains k and k' separates the points A and B.

Now, if the chain k separates the points A and B, it surely meets the arc BA. But the arc BA can not meet the intervals i and must therefore meet one of the broken lines of the system. Similarly if the net k' separates the points A and B, the other arc AB must meet one of the broken lines, since it can not meet the segments s. Therefore, in either case, the curve meets one of the broken lines, proving that the ends of the intervals i do not all belong to the same region.

Veblen's treatment * of the Theorem of Jordan is geometrical and non-metric in character. The theorem is assumed for the case of the triangle. It is based on a paper by Veblen on order relation**. Whose? Theorem 46 is not valid as is pointed out by Veblen himself. As does Kerekjarto, Veblen proves the therorm for the case of a simple closed curve which has a linear arc and then goes to the more general curve.

* O. Veblen, Theory on Plane Curves in Non-Metric Analysis Situs, American Mathematical Society Transactions, vol. 6 pp. 83-98.

Definition. A triangular region is the interior of a triangle. A geometrical limit point of a set of points, [X],* in a plane is a point P such that every triangular region including P includes a point X, distinct from P. A triangular region including a point is called a neighborhood of the point.

Definition. A region is a set of points, any two of which are points of at least one broken line composed entirely of points of the set. An interior point of a region R, is one that can be surrounded by a triangle containing only points of R. Consequently, an interior point of R is a geometrical limit point of no set of points that does not contain points of R. A frontier point of a region R is a point or geometrical limit point of R not an interior point, i.e., it is a limit point of points of R and of points not points of R. An exterior point of R or a point exterior to R is any point neither an interior nor a frontier point of R. The frontier or boundary of a region is a set of all frontier points. An open region contains no frontier points. A Closed region contains all its frontier points.

Definition. Simple curves, closed or unclosed, are composed of sets of points subject to certain conditions as follows:

A. Linear order. Among the points of a set of points $\{P\}$ there exists a relation $\odot$, which we may read precedes, such that,

1. $\{P\}$ contains at least two points.

2. If P_1 and P_2 are any two distinct points of $\{P\}$ then either $P_1 \odot P_2$ or $P_2 \odot P_1$.

3. If $P_1 \odot P_2$ then not $P_2 \odot P_1$.**

---

* If the set is ordered we use $\{X\}$ instead of $[X]$.

** From this it follows that if $P_1 \odot P_2$, then $P_1 \neq P_2$.

4. If $P_1 \oslash P_2$ and $P_2 \oslash P_3$, then $P_1 \oslash P_3$.

B. Ordinal continuity.

1. If P_1 and P_3 are any two distinct points of $\{P\}$, such that $P_1 \oslash P_3$, then there is a point P_2 of $\{P\}$ such that $P_1 \oslash P_2$ and $P_2 \oslash P_3$.

2. If every point of $\{P\}$ belongs to $[P_1]$ or $[P_2]$, two infinite subsets of $\{P\}$ such that for every P_1 and P_2, $P_1 \oslash P_2$, then there is a point P' such that for every P_1 and P_2 distinct from P', $P_1 \oslash P'$ and $P' \oslash P_2$.

C. Geometrical continuity.

1. Let P_0 be any point of $\{P\}$ for which there is an infinity of points P' such that $P' \oslash P_0$. Denote the set of all such points by $[P']$; then for every triangular region t, including P_0, there is a point of $[P']$, P'_t such that t includes all points of $[P']$ for which $P'_t \oslash P'$.

2. Let P_0 be any point of $\{P\}$ for which there is an infinity of points P'' such that $P_0 \oslash P''$. Denote the set of all such points by $[P'']$; then for every triangular region t, including P_0 there is a point of $[P'']$, P''_t such that t includes all points of $[P'']$ for which $P'' \oslash P''_t$.

Definition. By the term arc or arc of curve is meant a set of points $\{P\}$ satisfying conditions A, B, and C and including two points P_1, P_2 such that every point P, distinct from P_1 and P_2, satisfies the further conditions that $P_1 \oslash P$ and $P \oslash P_2$. The arc is said to join P_1 and P_2 which are called its end-points.

Definition. A simple closed curve j, is a set of points $\{J\}$, consisting of two arcs joining two points J_1 and J_2 but having no points in common except J_1 and J_2.

Theorem 36. Any two points of j may be taken as the points J_1 and J_2 in the above definition.

Definition. A simple unclosed curve is a set of points $\{C\} = c$ that satisfies conditions A, B, and C and also the following:

D. If C is any point of the curve, no point except C is a limit point (in the geometrical sense of definition above) both of the set of all points C' such that $C' \circledcirc C$ and of the set of all points C'' such that $C \circledcirc C''$.

Definition. A relation satisfying conditions A, B, and C is called a sense. A sense in which $P_1 \circledcirc P_2$ is said to be from P_1 to P_2

Theorem 37. From one point to another upon a simple unclosed curve there is one and but one sense, while upon a simple closed curve there are two and but two senses.

Definition. If with respect to any sense on a curve $P_1 \circledcirc P_2$ and $P_2 \circledcirc P_3$, P_2 is between P_1 and P_3 in that sense. The set of points between P_1 and P_3 in the given sense is called a segment $P_1P_2P_3$ whose end points are P_1 and P_3. The segment and its end-points together constitute an arc or interval of the curve. On a simple unclosed curve, if $P_1 \circledcirc P_2 \circledcirc P_3$, P_2 is said to separate P_1 and P_3. On any simple curve if $P_1 \circledcirc P_2 \circledcirc P_3 \circledcirc P_4$, P_1 and P_3 are said to separate and be separated by P_2 and P_4. If a set $[P_\nu]$ $(\nu = 1, 2, \ldots\ldots)$ is such that $P_\nu \circledcirc P_{\nu+1}$, the points P_ν are said to be in the order along the curve $P_1 P_2 P_3 \ldots\ldots\ldots P_\nu P_{\nu+1} \ldots\ldots$. A point P_0 is the first of a set $[P]$ if $P_0 \circledcirc P$ for every $P \neq P_0$; P_1 is the last of the set $[P]$ if $P \circledcirc P_1$ for every $P \neq P_1$.

Definition. A geometrically closed set of points is a set that includes all its geometrical limit points.

Theorem 38. If $[P]$ is any geometrically closed set of points, and a any arc that does not have a point incommon with $[P]$ then (1) there exists a finite set of triangles $\{t_n\}$ such that every point of a is interior to at least one t_n and every point of [P] is exterior to every t_n, and (2) the two end points A_1, A_2 of a can be joined by a broken line not meeting [P].

Proof. (1) If A is any point of a there is a triangle t, including A but not including any point of [P]; otherwise A would be a limit point of [P] and hence a point of [P], as [P] is closed. Place such a triangle about every point of a. By condition C each of these triangles determines an arc i, of a, which lies entirely within t and includes the point A to which t belongs. By the Heine-Borel Theorem there exists a finite subset $[i_n]$ of the arcs i such that every point of i belongs to one arc i_n. The finite set of triangles t_n which determine these arcs i_n are those required by the theorem.

(2) The end points of the arcs i_n constitute a finite set of points which we take as ordered by the sense of a from A_1 to A_2. The broken line (this broken line need not be simple) joining these points taken in order is such that each side lies within a triangle t and therefore can not meet [P].

Corollary. If [P] is any geometrically set of points and Q_0 a point not of [P], then Q_0 and the set of points, Q, that can be joined to Q_0 by arcs not meeting [P] constitute an open set.

Theorem 39. About any point of a segment of a simple curve there is a triangle which includes no points of the curve not on the segment.

Definition. A finitely closed set of points is a geometrically closed set of which every infinite subset possesses a geometrical limit point. A finitely closed set, every point of which is a geometrical limit point, is a finitely perfect set. A finitely perfect set of points which cannot consist entirely of two closed subsets is called a coherent set of points.

Theorem 40. A closed curve or an arc of curve is a finitely perfect set of points which can not consist entirely of two subsets, each of which includes all its limit points. In other words a closed curve or an arc of curve is a coherent set of points.

Theorem 41. If every point of a coherent set of points [A] is on a simple curve c, closed or unclosed, then [A] is an interval of c.

Corollary. If every point of an arc, a, is on a simple curve, c, then a is an interval of c.

Theorem 42. If c is any simple curve, any triangle t, of the plane includes points not on c.

Definition. Let P be an interior point of a region R, and B a point of the boundary b of R. An arc of a curve a, whose end points are P and B approaches B from P thru R if every interval of a, one of whose end points is B, contains interior points of R. The approach is one-sided if, besides the above condition, the arc a, contains no points exterior to R. The approach is simple if all the points of a, except B, are interior points of R.

An arc a' departs from a point B' of b to a point Q exterior to R if every interval of a' with B' as an end point contains points exterior to R. The departure is one-sided if, besides the above condition, the arc a' contains no points interior to R. The departure is simple if all the points of a' except B' are exterior to R.

A curve c crosses the boundary in a point B if, with respect to a fixed sense, B is between two points c_1, c_2 of c, c_1 interior and c_2 exterior to R, in such a way that the arc

c_1B approaches B through R and Bc_2 departs from B to c_2.

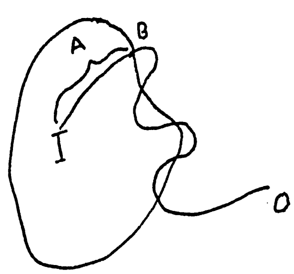
A curve c crosses the boundary b ina pair of points BB' if, with respect to a certain sense, one arc BB' of c is composed entirely of boundary points and if there are two points c_1, c_2 of c such that c_1 is interior to R and an arc c_1B of c approaches B from c_1 while c_2 is exterior to R and an arc $B'c_2$ departs from B' to c_2.

The crossing of a boundary is simple is both the approach and departure at the point B or a point pair BB' are simple.

The crossing of a straight line by a curve is a special case of the definition just given. A curve is said to cross a segment AB if the curve crosses the line AB in a point or a pair of points.

Theorem 43. Any simple curve joining an interior point of a region to an exterior point crosses the boundary in a point or a pair of points.

Proof. Let I be the interior, O the exterior point, and a any arc of the curve from I to O. Let $\{A\}$ be the set of all points, A, of the arc a such that every point following I and preceding C(A) is an interior or boundary point of the region. There are such points by Condition C. By the Ordinal Continuity of a, the set $\{A\}$ has a first forward bound B, i.e., a first point in the sense from I to O that follows every point of $\{A\}$ except possibly B itself.



The arc BO of a departs from B to O as otherwise every arc BB' of BO would contain only interior or boundary points of the region and thus B would not be a bound of $\{A\}$.* Two cases can now occur. Either B is approached from I by the arc IB of a, in which case our conclusion follows, or there are points A' of $\{A\}$ such that the arcs A'B include only boundary points. In the last case the set of all points A', must have a first forward bound B' in the sense from B to I. The point B' is evidently a boundary point and is approached from I by the arc of a, IB'. Thus in the second case, the boundary is crossed by the pair of points B'B.

Theorem 44. If a simple closed curve crosses a side of a polygon (simple or not) in one point or point pair, it must pass thru a vertex or cross the same or another side in another point or point pair.

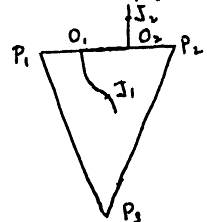
Proof. Let the polygon be $P_1 P_2 \ldots P_n$ and let the curve j cross it in a point of $P_1 P_2$. If there is another crossing on the segment $P_1 P_2$, or if j passes thru a vertex $P_1, P_2, P_3, \ldots, P_n$, the theorem is verified. Now P_1, P_2, P_3 may either be collinear or not. In the first case the original crossing is on $P_1 P_2$ and if the second crossing is on $P_2 P_3$ the theorem is verified. If there is not a crossing on $P_2 P_3$ we consider $P_1 P_3 P_4$ if not collinear or if so $P_1 P_4 P_5$, etc. In case $P_1 P_2 P_3$ are not collinear there must be a point J_1 of j and a point O_1 common to j and $P_1 P_2$ such that in a certain sense on j, the arc $J_1 O_1$ of j approaches O_1 thru the region on one side of $P_1 P_2$; likewise there must be a point J_2 of j on the opposite side of $P_1 P_2$ from J_1 and a point O_2 common to $P_1 P_2$ and j such that in the same sense the arc $O_2 J_2$ departs from O_2 to J_2.** Moreover the points J_1 and J_2 may be so chosen that

---

* This is not exactly correct. It is sufficient to have one arc BB' of BO containing only interior or boundary points, and then B would not be a bound of $\{A\}$. It follows that if there is one such arc there is an infinitude of them.

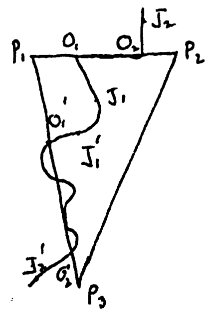
** O_1 and O_2 may be identical.

one and only one of them lies within the triangle $P_1P_2P_3$. Since j crosses P_1P_2 only once, O_1 and O_2 are on the same arc of j with end points J_1, J_2. The other arc a of j with end points J_1, J_2 must



by Theorem 45, cross the boundary of the triangle $P_1P_2P_3$ and since it does not cross a vertex, must either cross P_2P_3, in which case the theorem is verified, or must cross P_1P_3.

If a crosses P_1P_3, let O_1' be the first point in the sense



from J_1 to J_2 in which a meets P_1P_3 and O_2' the last such point. Upon the arcs J_1O_1' and $O_2'J_2$ there must be two points of a, J_1' and J_1' on opposite sides of P_1P_3 such that in opposite senese along j the arcs $J_1'O_1'$ and $J_2'O_2'$ approach O_1' and O_2' from opposite sides of P_1P_3.

In case $P_1P_3P_4$ are non-collinear, J_1' and J_2' may be so chosen that one is interior and the other exterior to the triangle $P_1P_3P_4$. Whether $P_1P_3P_4$ are collinear or not we proceed as with $P_1P_2P_3$. If $P_1P_3P_4$ are not collinear the curve must, by Theorem 43, cross P_3P_4, in which case the theorem is verified, or must cross P_1P_4 in which case we are lead to the consideration of $P_1P_4P_5$. Continuing in this manner we are lead by a finite number of steps to $P_1P_{n-1}P_n$ in which the theorem is verified if not having been verified before.

Corollary 1. If j_1 is a simple closed curve having an arc which is a linear interval J_1J_2, and if the segment J_1J_2 is crossed by a simple closed curve j_2 in one point or a point pair, then either J_1J_2 is crossed in another point or point pair or the non-linear arc J_1J_2 of j_1 has a point in common with j_2.

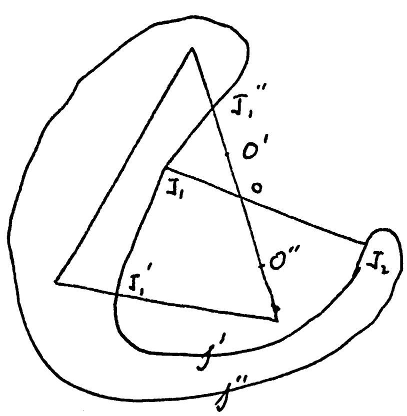
Proof. In case J_1J_2 were not crossed more than once and the other arc $J_1 J_2$ of j_1 did not meet j_2, by Theorem 38, J_1 and J_2 could be joined by a broken line not meeting j_2, since j_2 is a closed set and the non-linear arc J_1J_2 has no point on j_2. We have then a contradiction of Theorem 44 since j_2 meets the linear arc $J_1 J_2$ and does not meet the constructed polygonal line which forms with the linear arc a polygon, one side of which is cut by the closed curve j_2.

Corollary 2. Any simple closed curve j_1 having a linear arc J_1 J_2 decomposes the plane into at least two regions.

Proof. Let PQ be a linear segment crossing the linear arc $J_1 J_2$ in a point O. Then the region composed of all points that can be joined to P by polygonal lines not meeting j_1 is by Theorem 44, separated from the region similarly connected with Q.

Lemma 6. Any simple closed curve j decomposes the plane in which it lies into at least two regions.

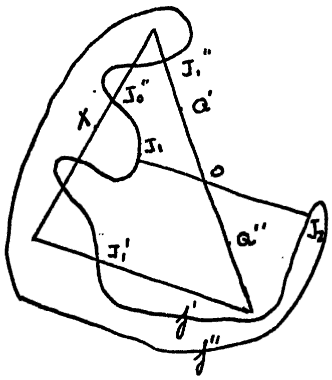
Proof. Let J_1 and J_2 be two points of j such that the linear segment $J_1 J_2$ has no point in common with j . Such points J_1, J_2 exist, for if a is any line joining two points of j, it either has an interval free of j points and whose end points are the required points J_1, J_2 or its points with j constitute a single arc of j by the Corollary to Theorem 41. In the latter case any line a' joining a point of



j on a to a point of j not on a evidently has the required points J_1, J_2.

Let t be a triangle about J_1 such that one of its sides meets the linear segment $J_1 J_2$ in a point O. Let Q' and Q'' be two points of this side separated by O and such that the linear interval Q'Q'' contains no point of j. The existence of these points depends on the theorem that j is a geometrical perfect set.

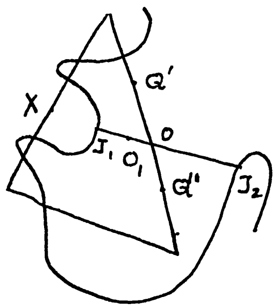
J_1 and J_2 decompose j into two segments which, with the linear interval $J_1 J_2$ constitute two closed curves j' and J''. Assign the notation so that the first point J_1' after Q'' in the sense $Q'\ O\ Q''$ in which the boundary of t meets j shall be a point of j'. It follows that the first point J_1'' after Q' in the sense $Q''OQ'$ in which the boundary of t meets j is a point of j'',



for if it were a point of j', the closed curve composed of the boundary of t from J_1'' to J_1' in the sense $Q'OQ''$ and the arc common to j and j' between J_1' and J_1'' would cross the linear segment $J_1 J_2$ of j'' simply in O and would meet j'' in no other point. This would contradict Corollary 1, Theorem 44.

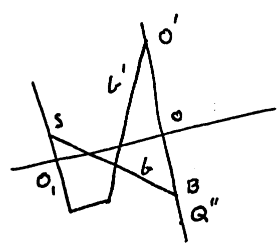
Thus J_1'' is a point of j''. Let J_0'' be the first point after J_1' in the sense $Q'OQ''$ in which the boundary of t meets j''. By the continuity of j, there exists a segment of the boundary of t just preceeding J_0'' in the sense $Q'OQ''$ and containing no point of j' or j''. Let X be any point of this segment. The broken line b'' composed of the boundary of t in the sense $Q'\ O\ Q''$ from Q'' to X does not meet j''. Likewise X is joined to Q' by a simple curve c' composed of the linear segment XJ_0'', the common part of j'' and j' from J_0'' to J_1'' (it may happen that $J_0'' = J_1''$ and in this case c' is a broken line), and the part of the boundary of t from J_1'' to Q' in the sense $Q'OQ''$. Thus c' can not meet j', and , applying Theorem 38, c' can be replaced by a broken line b' joining X to Q' without meeting j. We now extablish the lemma by showing that X can not be joined to O by a broken line not meeting j.

In the sense from X to O any such broken line would meet the linear segments J_1J_2 and $Q'Q''$ in some first point O_1. If O_1 were on

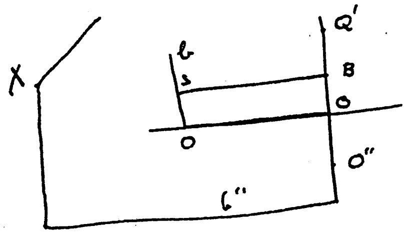


J_1J_2, some pointss preceeding O_1 in the sense from X to O could be joined to a point B of $Q'Q''$ by a segment not meeting j' or j''. Call b the resulting broken line from X to B. In case O were not on J_1J_2 it would be on $Q'Q''$ and different from O, and b would be the broken line from X to $O_1 = B$.

If B were on the same side of J_1J_2 with Q'' then the polygon



composed of b + b' and BQ' would be crossed by j' in O and would meet j in no other point, contradicting Theorem 44. If B were on the opposite side of the line J_1J_2 from Q'', the polygon composed of b and



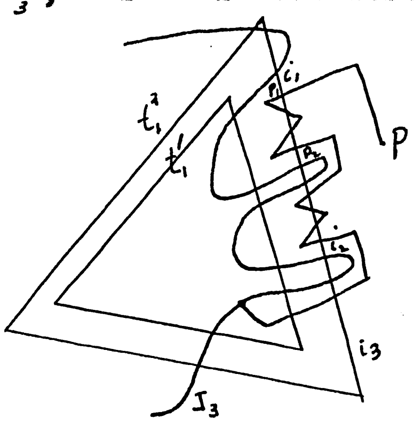
b'' and BQ'' would be crossed by j'' in O and would meet j'' in no other point. X and O are therefore two points that cannot be joined by a broken line mot meeting j.

Definition. A point C of a curve c is finitely accessible from a point P not on c if there is a broken line from C to P not meeting c except in C.

Lemma 7. If P is a point not on a simple closed curve j, and J_1 and J_3 are any two points of j finitely accessible from P or limit points of points finitely accessible from P, then there exists a pair of points J_2 and J_4 finitely accessible from P that separate J_1 and J_3.

Proof. Let t_1 be a triangle about J_1 not including J_3, and t_3 a triangle about J_3 not including any point of t_1. By condition C of the definition of j there is a segment of j including J_1 and lying wholly within t_1; by Theorem 39 there is a triangle t_1' about J_1 within t_1 and including no point of j not on this segment. Thus every segment of j with end points on t_1 which meets t_1' must include J_1. Similarly there is within t_3 a triangle t_3' such that every segment of j with end points on t_3 which meets t_3' must include J_3.

Let J_1' be a point within t_1' finitely accessible from P and J_3' a point within t_3' finitely accessible from P. The points J_1' and J_3' are thus joined by a broken line b, meeting j only in J_1' and J_3', which without loss of generality may be supposed to be simple.



On this broken line let P_1 be the first point in the sense from J_3' to J_1' in which it meets the boundary of t_1. P_1 lies on an an interval i_1 of the boundary of t_1 containing no points of j but such that its end points are points of j. Let P_2 be the last point in the sense from J_3' to J_1' in which b meets the interval i_1. In case P_2 is distinct from P_1 replace the portion of the broken line from P_1 to P_2 by the portion of i_1 from P_1 to P_2, calling the new broken line b_1. If b_1 crosses the interval i_1, J_2 and J_4 are the end points of the interval. If b_1 does not cross i_1 there must be some point P_3 beyond P_2 in the sense from J_3' to J_1' in which b_1 meets the boundary of t_1. The point P_3 must lies on an interval i of the boundary of t analogous to i. Proceed with i_3 as with i_1. Since J_1' is inside t_1 and J_3' outside t_1 and since b_1 has but a finite number of sides, we must by repeating the above process come to a first interval i_k in which the boundary of

t_1 is crossed by a reduced broken line b_k from J_3' to J_1' in a point P_k or a pair of points $P_k P_{k+1}$. The end points of the j-point free interval i_k of the boundary of t_1 are now shown to be the required J_2 and J_4.

We prove first that J_2 and J_4 separate J_3' and J_1'. If this were not so, let the simple closed curve formed by b_k and the arc $J_1'J_3'$ of j not including J_2 and J_4 be denoted by j_k. Also let j_i denote the simple cloded curve formed by i_k and the arc $J_2 J_4$ of j not including J_3' and J_1'. The simple closed curve j_k would cross the arc i_k of j_i in the point P_k or point pair $P_k P_{k+1}$ and would meet j_i in no other point, contrary to Corollary 1, Theorem 44.

Hence J_1' and J_3' are on different arcs of j with end points J_2 and J_4. But by the construction of the triangle t_1', J_1 must be on the same arc with J_1' and by the construction of t_3', J_3 must be on the same arc with J_3'. Hence $J_2 J_4$ separate $J_1 J_3$.

Theorem 45. The set of points of a simple curve j finitely accessible from a point P not on j is everywhere dense on j.

Proof. Denote by [J'] the set of points of j which are either finitely accessible from P or are limit points of the set of finitely accessible points. The Theorem amounts to showing that [J'] is identical with j. If J_0 is any point of j which does not belong to [J'] it would lie on an ar c of j free of points J' and having two points of [J'] as end points. This would contradict Lemma 7.

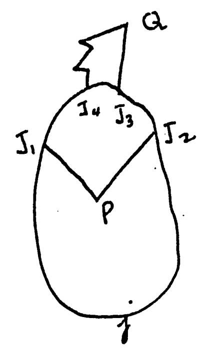
Lemma 8. Any simple closed curve of which one arc is a linear interval decomposes its plane into two open regions.

Proof. In Corollary 2, Theorem 44 it was shown that j decomposes the plane into at least two regions. The regions are open because a supposed frontier point of the set of points [P] not on j could if not itself a point of j, be surrounded by a triangle not meeting a point of j, and containing points of [P]. It would, therefore be an interior point of [P] , contrary to hypothesis.

By Theorem 45, every point O of the straight arc j is finitely accessible from any point of the plane. Thus if there were three distinct regions there would be three segments meeting in O and one lying in each of the three regions. But as two of these must lie on the same side of the straight segment of j, they could be joined by a straight segment not meeting j, contrary to the hypothesis that the three regions are separated from one another by j. Hence j decomposes the plane into two and only two regions.

Theorem 46. Every simple closed curve, j, decomposes its plane into two open regions.

Proof. By Lemma 6, the curve j decomposes the plane into at least two regions. The regions are open by the reasoning employed in Lemma 8. Let P be any point not on j and let PJ_1 and PJ_2 be two linear intervals meeting only in J_1 and J_2. The points J_1 and J_2 exist because j is a perfect set of points. It would then follow that P would be a point of j. Let Q be any point not on j and not in the same region with P and let J_3 be a point on j such that the linear segment QJ_3 does not meet j and such that J_3 is distinct from J_1 and J_2. Then QJ_3 does not meet PJ_1 or PJ_2 and Q can, by



Theorem 45, be joined by a broken line not meeting PJ_1, PJ_2, QJ_3 or j except in J_4, to a point J_4 of j in the order $J_1J_2J_3J_4$. The broken line J_1PJ_2, the points between J_2 and J_3 in the sense $J_1J_2J_3$, the broken line J_3QJ_4, and the points between J_4 and J_1 in the sense $J_1J_2J_3$ constitute a simple closed curve j' of the type which we have proved to decompose the plane into two and only regions. The points of the segments J_1J_2, J_3J_4 in the sense $J_1J_2J_3$

are not points of j' and must lie both in the same region or in opposite regions with respect to j'. If they were in the same region a point in the region not containing the segments $J_1 J_2$ and $J_3 J_4$ could by Theorem 45 be joined by broken lines not meeting j to P and Q, thus contradicting the hypothesis that P and Q are in different regions.

Having shown that the arcs $J_1 J_2$ and $J_3 J_4$ (in the fixed sense $J_1 J_2 J_3$) are in opposite regions with respect to j' we are ready to complete the proof that j does not decompose the plane into more than two regions. A point R in a supposed third region could be joined by Theorem 45, by a broken line not meeting j except in its end points to a point J_5 of $J_1 J_2$ and by a similar broken line to a point J_6 of $J_3 J_4$. Since R would not be in the same region with P or Q these broken lines would not meet the broken line part of j'. Thus we should have two points J_5 and J_6 in opposite regions with respect to j' joined by a broken line not meeting j', contrary to Lemma 8. Hence j decomposes the plane into not more than two, and therefore into exactly two, open regions.

The proof by de la Vallee Poussin* depends upon rings which are constructed by two polygons P_1 and P_2 such that P_1 is strictly interior to P_2. The ring is the portion of the plane exterior to P_1 and interior to P_2. This proof is not valid as was pointed out by Schoenflies who shows how the errors may be corrected. The Theorem of Jordan for the polygon is assumed by de la Vallee Poussin.

A link shall be a region of the plane, d'un seul tenant**

---

* de la Vallee Poussin, Cours D'Analyse Infinitesimale, vol. 1, (1914) pp. 374-379.

**i.e., connected.

bounded by an exterior polygonal contour without multiple points. It may have a number of interior regions, i.e., it may not be simply connected. If it has interior regions the boundary of these regions is the same as the boundary of the exterior.

If we have links (1), (2),.......,(n) such that each two consecutive links have points or parts in common, and two non-consecutive links do not toutch, the links constitute an open chain. Links (1) and (n) have no points in common.

If (1) and (n) toutch, they become consecutive and if all other conditions above hold, the links constitute a closed chain.

The domain containing all the points of an open chain, is as the link itself, d'un seul tenant, bounded by a polygonal contour and having interior regions.

A region of the plane is interior to a link or to an open chain if it is enclosed in their exterior contour, neglecting the interior regions.

A chain is regular if a link may not be enclosed in another link, nor in a group of two other links, taken in the chain and necessarily consecutive.

Theorem 47. If a closed chain (1),(2),....(n) is irregular, it is interior to a chain of at most four of its links. Proof. Since the chain is irregular, it contains a link interior to one or to two others. If interior to two links, let these two links be (1) and (2). If the interior link, (3) for example, is contiguous to the group of two links, all the following links (4),.....(n—1), which neither toutch (1) nor (2), are enclosed with (3). If a link (k), non contiguous with (1) and (2), which is enclosed in (1) and (2), all the chain (4),.....(n—1) which contains (k) is enclosed with (k), for the same reason. All the chain is then interior to (n),(1),(2),(3).

Theorem 48. Any interior region of an open chain is interior to a group of two consecutive links.

Proof. We prove the theorem by induction. The theorem is evident for a chain of two links. Suppose the theorem true for $k-1$. Let C be the exterior contour of the chain (1),(2),....(k − 1). Add the link (k). This last link is composed of a part (k') interior to C and a part (k'') exterior to C, each of which may be de plusieurs tenants*. The addition of (k') can only reduce the number of interior regions, for which the theorem is already verified.

We must now consider (k''). Suppose that an interior region is produced between C and the frontier of (k''). We show that it is interior to the group (k − 1), (k) and hence that any line L going from a point of an interior region to the point at infinity, without cutting (k), cuts (k − 1).

The line L goes out of an interior region by the frontier of C, of which the two extremities touching (k), are two points A and B of (k − 1). The line L divides the domain interior to C into at least two pieces which closes the interior region so as to separate A and B. Then (k − 1), which is all in the interior of C, is not all in the same piece and it is divided by L.

Theorem 59. The exterior contour of a regular open chain is unique and touches all the links, while the contour of an interior region touches at most four links.

Proof. The links which touch the exterior contour form themselves an open chain. That property persists if one cuts in the interior

---

* i.e., multiply connected.

of the chain (in order to make the interior regions) all the links which do not touch at the boundary, if there are any. Then, by Theorem 48, all links removed are interior to two links of the boundary, such that, if so the chain will be irregular. Then the exterior contour of a regular open chain touches all the links.

On the other hand, an interior region, being interior to two consecutive links, may only be bounded by themselves and the two contiguous links, the only ones which will be exterior to the first two.

Theorem 50. In a regular open chain of five links at least, an interior region may not touch at the same time the two extreme links.

Proof. If an interior region touches the two extreme links it touches all links and hence more than four, which contradicts Theorem 49.

We turn now to a consideration of closed chains. Let (1),(2),(3),.....,(n) be a regular closed chain of five links at least. We wish to replace the group of the first three links by another group of three which is simplier and has the same exterior contour.

Let C be the exterior contour of the group (1),(2),(3) and D the domain interior to C. Since (1) and (3) do not touch and each is exterior to the others (since the chain is regular), we may divide D by two transversals into three parts (I),(II),(III), such that the first contains all of (1) without touching (3), the third contains all (3) without touching (1), and the second contains the remainder.

We have thus constructed a new regular closed chain (I),(II),(III),(4),....,(n) of the same number of links as the first; but (I) and (II) touch only along a common frontier d'un seul tenant, which we may call by its extremities, ab.

In order to open that chain it is sufficient to consider the transversal ab as a cut disjointing (I) and (II). To that effect, suppose that there are two sides to that line, the one a'b' serving as the frontier of (I); the other a''b'' serving as the frontier of (II); considering the line ab itself as exterior to the chain. Then (I) and (II) are the two extreme links and the chain is open.

With the aid of the above considerations we establish the following theorem:

Theorem 51. A regular closed chain of five links or less constitutes a ring, interior to a polygon P_1 and exterior to a polygon P_2 contained in the interior of the first. Beyond the region that it encircles, the ring may contain certain interior regions. These are interior to one or two consecutive links and touch four at most. On the contrary, the regions interior and exterior to the ring, touch all links, the one by P_1 and the other by P_2.

Proof. Let P be the exterior contour of the above open chain. That contour has the two frontiers a'b' and a''b''. Make a circuit of P keeping the interior on the left. As a'b' and a''b'' are then traversed in the opposite sense, the circuit then is composed of a'b', a''b'', and the two polygons P_1 and P_2 joined respectively a' to a'' and b'' to b' and each passing by all the intermediate links of the cahin (I),.....(II); then each passes also by all the links (1),(2),......,(n), for a and b are on (2) and one passes from (III) to (4) only by (3), and from (I) to (n) only by (1).

Supress now the cut ab, which amounts to joining a'b' and a''b''; the chain as well as P_1 and P_2 are closed, and it is contained between the two polygonal lines of which one is, consequently, interior to the other. We give to the hoop comprised between them the name of the ring.

The chain (I),(II),.... differs from (1),(2),.... only by the suppression of the interior regions of the group (1),(2),(3). Replace these interior regions and apply Theorem 48 to the other part of the chain opened by the cut.

Theorem 52. A simple closed contour* may be enclosed in a regular closed chain of which the links have diameters as small as we please.

Proof. Consider $\delta > 0$, less than $\frac{1}{4}$ the diameter of the contour. Suppose the contour is described by (x,y) when t varies from t_1 to T. Divide the contour t_1T into pieces by the points $t_1, t_2, \ldots, t_n, T$ such that the diametersof each piece will be $< \delta/2$. Let δ' be the smallest of the distances from two non-consecutive pieces. Cover the curve with a network of rectangular meshes formed by parallels to the coordinat axes such that the diameter of each mesh is $<\delta'/2$. The meshes which touch the same part of the curve corresponding to each (t_i, t_{i+1}) form a link. The links (1),(2),..(n) are constructed respectively on $t_1t_2, t_2t_3, \ldots, t_nT$.

These links form a closed chain, for two consecutive links (t_{k-1}) and (t_k) have in common the point t_k. The diameter of a link does not surpass $\delta/2 + \delta'$ hence δ, for δ' may not surpass the diameter $\delta/2$, the diameter of a link. Finally the chain is regular, for otherwise it will be interior to four links and its diameter will be less than $4\,\delta$, hence less than that of the curve. That is evidently impossible since the curve is contained in the chain.

---

*i.e., a Jordan curve.

The interior region surrounded by the ring touches the most separated links, and hence is of greater diameter than $4\delta - 2\delta = 2\delta$. On the contrary, as the interior regions of the ring itself are interior to two links, they are of diameter $< 2\delta$. Then in order to construct the ring it is sufficient to cut out all the meshes of the net which are meet by the curve and all the interior regions except one, which is of diameter $> 2\delta$ and which constitutes the interior region of the ring.

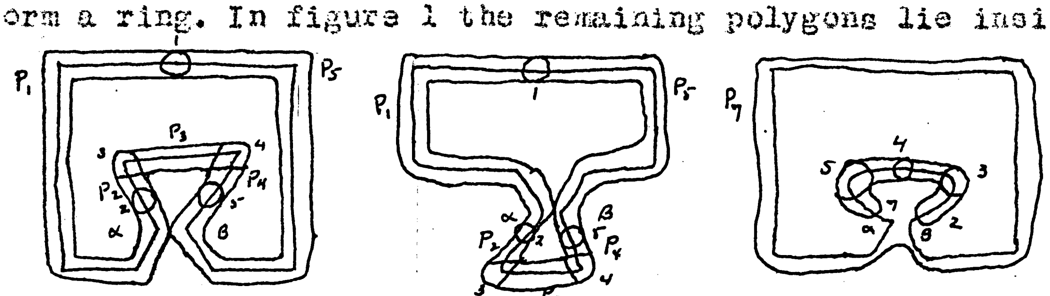
Theorem 53. All closed curves without multiple points decompose the plane into two regions, one interior and one exterior to the curve.

Proof. Let p be a point not situated on the curve and δ its distance from the curve. Construct a ring of which all the points will be at a distance $< \delta$ from the curve; the point p, being excluded from the ring, falls on the interior or the exterior of the ring. We say in the first case that it is interior to the curve and in the second that it is exterior.

That distinction depends in no manner upon the constructive properties of the ring, but depends upon the curve. In fact, if the point p is not surrounded by the ring, it is possible to trace a polygonal line starting from p and departing to the point at infinity without meeting the curve. That possibility disappears as soon as the point is surrounded by the ring. In fact, all polygonal lines going from p to the point at infinity should cut the chain and, consequently, cut a link in two, eg., (1), between the two points t_1 and t_2 which unite it to its adgacent one. The piece $t_1 t_2$, which is situated entirely with in a link, passes then from one to the other part of the link determined by the line which cuts the link and must meet the line.

The set of points interior to the ring and the set of points exterior to the ring constitute respectively the interior and the exterior regions of the curve. The interior region may not be reduced to zero, for one may always construct a ring. Finally, two interior or two exterior points may always be joined by a polygonal line which does not meet the curve, for one may construct a ring containing or excluding the two points and then a line which joins them does not meet the curve.

Schoenflies * points out that the method of de la Vallee Poussin in constructing the ring is not without fault. In dividing the curve into n arcs and placing the polygons $P_1, P_2, \ldots, P_n$ about them, there is no assurance that a ring is constructed. The Polygons may be arranged as in figure 1, 2, or 3, with the points α and β on two consecutive subarcs such that their distance apart is always $< \delta'/2$. In figures 1 and 2 we see that the polygons P_1 and P_5 form a ring. In figure 1 the remaining polygons lie inside



1 2 3

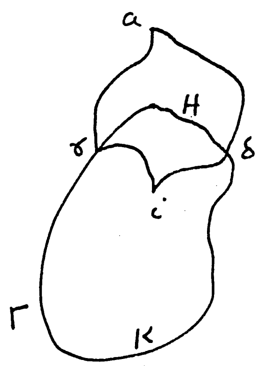
this ring. In figure 2 the remaining polygons lie outside the ring. In figure 3 one of the polygons encloses all the others. It is evident in these cases that the plane is divided into more than two parts by the ring. The polygons in question may be formed by method of Schoenflies and this difficulty is overcome.

The method of Schoenflies leads to the converse of the

---

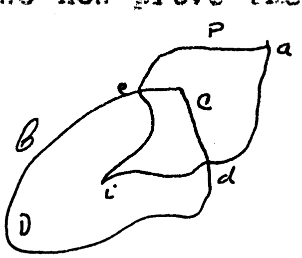
* A Schoenflies, Zusatz (Bemerjungen zu den Beweisen von C Jordan und Ch. J. de la Vallee Poussin), Jahresbericht der Deutschen, Mathematiker - Vereinigung, vol 33, pp. 157-160.

Theorem of Jordan*. The Jordan Theorem as proved by Schoenflies, states that the 1-1 and continuous image of a circle is a closed curve which divides the plane, and that each of its points is accessible from both the inner and outer regions. The converse Theorem is that any simple closed curve, which divides the plane into two regions, and whose points are accessible from points of each of the regions into which the plane is divided by the curve, is a 1-1 and continuous image of a circle. In the proof of Jordan's Theorem, Schoenflies shows that the property of isolation may be substituted for the property of accessibility. The property of accessibility means that any point P in the plane not on a simple closed Jordan curve, may be joined to any point Q of the curve, by a Jordan curve which has no point on the curve except Q. The property is isolation is defined as follows: Consider a



simple closed Jordan curve Γ, and two points γ and δ on it. These two points determine two point sets H and K. If a is any outer and i any inner point, and the points a and i are joined by open Jordan curves such that they meet Γ only in the points γ and δ, then these four open Jordan curves form an isolating polygon, which isolates the set H from the set K.

We now prove the converse of Jordan's Theorem, using the



property of isolation. Let $\mathcal{C}$ be a closed curve and let c and d be two points on it. Let the regions which it determines be I = I($\mathcal{C}$) and A = A($\mathcal{C}$). The paths which

---

* A. Schoenflies, Umkehrung des Jordanschen Kurvensatzes, Jahresbericht der Deutschen Mathematkker - Vereinigung, vol 33, pp. 160-165.

lead to c and d from an outer point a and an inner point i, form a polygon P, that encloses a certain subset C of $\mathcal{C}$. If we therefore set up the relation,

$$\mathcal{C} = \{c, d\} + C + D$$

then every point of D belongs to the exterior of P, and hence to A(P). Since the points of the curve are accessible we may pick the points in the outer and the point in the inner region arbitrarily close to the curve, and in fact, the paths themselves, so that all points of the polygon are arbitrarily close to the curve . If we make the sets C or D closed we increase them by the points c and d.

The polygon P is an isolating polygon and isolates the set C from D. A similar polygon O exists for the set D and is is such that the two polygons have only the points c and d in common, but otherwise lie outside of one another.

Let Γ be a circle. On the curve $\mathcal{C}$ choose two points c and d do that their distance apart is the maximum of the didtances of pairs of points. The points c and d are made to correspond to the end points γ and δ of the diameter of the circle Γ. We shall further let the sets C and D correspond to the inner points of the circular arcs determined by γ and δ. For this proof it is necessary to consider the sets closed. The above mentioned sets shall be designated as

$$\mathcal{C}_1 , \mathcal{C}_2 \text{ and } \Gamma_1 , \Gamma_2 .$$

For $\mathcal{C}_1$ and $\mathcal{C}_2$ we imagine isolating polygons P_1 and P_2 such that they lie outside one another excepting for the points c and d. The set contained in each of these polygons shall be divided by the insertion of proper division points, so that the corresponding subsets on the circle, as well as on the curve, become gradually small. On the arc $\mathscr{b}_1$ choose a point c_1 so that it is equidistant from c and d. In case there is more than one such point, we shall choose c_1 as that point which is greatest equidistant from c and d. We choose, similarly, a point c_2 on the arc $\mathscr{b}_2$ and make the points c_1 and c_2 correspond to the middle points γ_1 and γ_2 of the circular arcs Γ_1 and Γ_2. By these points (c_1 and c_2) $\mathscr{b}_1$ and $\mathscr{b}_2$ are divided into the arcs $\mathscr{b}_{11}$, $\mathscr{b}_{12}$ and $\mathscr{b}_{21}$, $\mathscr{b}_{22}$ corresponding to the sets Γ_{11}, Γ_{12} and Γ_{21}, Γ_{22}. This process is continued.

Finally, we surround each set $\mathscr{b}_{ik}$ with an isolating polygon P_{ik}. We shall have (by the isolating property and the method of determining the isolating polygons), that, first, P_{11} and P_{12} are inclosed by P_1, just as P_{21} and P_{22} are inclosed by P_2, and secondly, that each two of the polygons P_{ik} (i.e., P_{ik} and P_{ki}) lie outside of one another.

On each set $\mathscr{b}_{ik}$ we determine further a point c_{ik} as was done above, and make it correspond to the middle point γ_{ik} of the circular arc Γ_{ik}. The indices ik of the middle points are the dame as those of $\mathscr{b}_{ik}$. We proceed to the curve arcs
the circular arcs

$$\mathscr{b}_i,\ \mathscr{b}_{ik},\ \text{------},\ \mathscr{b}_{[ik\ldots m]},\ \mathscr{b}_{[ik\ldots m]n},\ \text{---}$$

the isolating polygons,

$$P_i,\ P_{ik},\ \text{-----},\ P_{[ik\ldots m]},\ P_{[ik\ldots m]n},\ \text{---}$$

and the points,

$$c_i, c_{ik},\ \text{----},\ c_{[ik\ldots m]},\ \text{---and}\quad \gamma_i,\ \gamma_{ik},\ \text{----},\ \gamma_{[ik\ldots m]},\ \text{---}$$

For these the following hold:

1. Each of the polygons $P_{ikl\ldots}$ is an isolating polygon for the curve arc $\mathscr{b}_{ikl\ldots}$ with the proper index.

2. Each polygon $P_{[ik\ldots m]n}$ lies outside the polygon $P_{[ik\ldots m]}$ with the same index group $[ik\ldots m]$.

3. The polygons $P_{[ik\ldots m]}$ of each and every index group lie entirely outside of one another.

The points $c_{ikl\ldots}$, the curve arcs $\mathcal{C}_{ikl\ldots}$, which these points determine, and their isolating polygons around them, possess the same ordering (according to the subscripts) as the circular arc points $\gamma_{ikl\ldots}$ and the circular arcs $\Gamma_{ikl\ldots}$, on which they lie. If we can then prove that the breadth of the curve arc $\mathcal{C}_{ikl\ldots}$ with increasing index number, becomes uniformly infinitely small, we show thereby that every point of the curve $\mathcal{C}$ by the ordering is in 1-1 and continuous correspondence with the points of the circle.

We show now the lemma that there are not infinitely many curve arcs whose breadth is $> \sigma$. Each arc whose breadth is $> \sigma$, has a subarc whose end points are exactly a distance σ apart. If now

(A) $\mathcal{C}', \mathcal{C}'', \mathcal{C}''', \ldots\ldots$

are such curve arcs, let

(B) $c', c'', c''', \ldots\ldots$ and $d', d'', d''', \ldots\ldots$

be their end points. Further let $\mathcal{C}_\omega$ be the limiting configuration determined by them. $\mathcal{C}_\omega$ is an arc of the curve all points of which are limit points of the arcs (A). The limit points of (B) likewise belong to this set. $\mathcal{C}_\omega$ is thus a connected set.

We assume now that there is only one pair of limit points present; namely c_ω and d_ω. Their distance apart is obviously σ. If we therefore put

$$\mathcal{C}_\omega = (c_\omega, d_\omega) + C_\omega,$$

then C_ω be comes $\mathcal{C}_\omega$ by the addition of c_ω and d_ω. For the curve itself, we have the following equation,

$$\mathcal{C} = (c_\omega, d_\omega) + C_\omega + D_\omega,$$

and of course all the points of the sets (A) belong to the set D, since all the sets C^i, C^j which determine $\mathcal{C}_\omega$ are distinct (for $i \neq j$), except possibly for end points. Since each of the sets

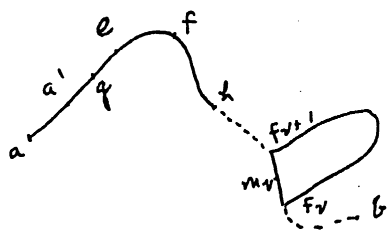
$$\mathcal{C}'', \mathcal{C}''', \ldots\ldots\ldots$$

is distinct from the $\mathcal{C}'$, one can surround $\mathcal{C}'$ by an isolating polygon P' so that all of these sets are excluded, and therefore no point of $\mathcal{C}_\omega$ lies inside of P'. If we pass therefore from D_ω to the closed set which contains it, this set contains all the points of $\mathcal{C}_\omega$ since all points of $\mathcal{C}_\omega$ are limit points of the arcs (A) which belong to D_ω. The sets C_ω and D_ω could not therefore be isolated from each other in contradiction to our assumption.

In the above we have assumed that one and only one pair of points c_ω and d_ω exist. If this is not the case, then we can select from the sets (A), a subset for which it is the case, and reach the same conclusion as before.

We now show that no arcs of the curve exist whose diameter is $> \sigma$, and none exist in which no division points lie. If this is shown, the curve $\mathcal{C}$ is filled with division points.

If (ef) is any arc containing no division point $c_{ihk\ldots}$, we consider any one of the index groups $[ihk\ldots]$ to which (ef) belongs i.e., e and f are division points. Let a and b be those division points next to e and f. The next division will cause a point to fall in the arc (ab). It may fall between a and e, i.e., at a'. In the next division, a division point falls in the arc (a'b). It can either fall in the arc (a'e) or in the arc (fb). In the first case the arcs in which the division point falls, change with each other infinitely often. In the second case only a finite number of division points fall in one of the arcs. We treat the second case first. Let (fb) be the arc in which infinitely many



division points fall. In the arc (ae) only a finite number fall. Let g be the last of these*. Further if f_1 is the division point of the arc (gb), f_2 that of (gf_1), etc., and if h is a limit point of the sequence $\{f_\nu\}$, then the arc (gh) is also free from division points. Now the diameter of the arc $(f_\nu f_{\nu+1})$ can not become infinitely small, since $\{f_\nu\} \to h$ and if the diameter of $(f_\nu f_{\nu+1}) \to 0$, then for $\nu > \nu_0$, the distance from h to f_ν will be less than the distance from g to f_ν and a division point will fall in the arc (gh). Hence $r > 0$ exists, such that for $\nu > \nu_0$, the diameter

$$\beta(f_\nu f_{\nu+1}) > r .$$

If now m_ν is the center of the straight line $\overline{f_\nu f_{\nu+1}}$, and q_ν the point of the arc $(f_\nu f_{\nu+1})$ which from m_ν has the greatest distance, then a number μ exists such that for each $\nu > \mu$,

$$\beta(f_\nu q_\nu) > \tfrac{1}{2} r \text{ and } \quad \beta(f_{\nu+1} q_\nu) > \tfrac{1}{2} r .$$

The arcs $(f_\nu q_\nu)$ form an infinite set of arch of $\mathcal{C}$, the breadth of all being $> \frac{1}{2} r$. This contradicts the above lemma. Hence the theorem.

In the first case, when two division points $f_\nu, f_{\nu+1}$ fall in the same arc, the treatment is the same as above. $f_{\nu+1}$ is replaced by f_ν, e_ν by $g_{\nu+1}$, $e_{\nu+1}$ by $g_{\nu+2}$. If f_ν is such a division point of (bf), that the next division point (it is called $g_{\nu+1}$), falls in (ae), the division point of the arc $(f_\nu g_{\nu+1})$ again has a maximum and equal distance from

---

* g may be identical with e.

f_ν and $g_{\nu+1}$. According as the point $g_{\nu+2}$ falls in (ae) or the point $f_{\nu+2}$ falls in (bf), one concludes that the arc $(g_{\nu+1}g_{\nu+2})$ or the arc $(f_{\nu+1}f_{\nu+2})$ has a breadth which remains greater than $k > 0$. The theorem follows as above.

We have then for the curve $\mathcal{C}$ that each subarc into which $\mathcal{C}$ is divided, is subdivided by other division points and approach zero in diameter. Each group of polygons about these subarcs determines one and only one point, since the diameter of the arcs approach zero. Any point of which is a point of division corresponds to one and only one point of the circle. Likewise such points of the circle correspond to one and only one point of $\mathcal{C}$. It is necessary to show that any point which is not a point of division also corresponds to one and only one point of the circle. The points of division are everywhere dense since between any two there is a third. Further, any point not a point of division is determined by a group of nested polygons. Such a point is then a limit point of points of division. The corresponding points of division on the circle approach a limit (unique), and these two limit points correspond. Similar conclusions hold in the case of points of the circle which are not points of division. Since the arcs of division of $\mathcal{C}$ approach zero uniformly in length, we have that the correspondence between $\mathcal{C}$ and the circle is 1-1 and continuous.