A Set of Convolution Identities Relating the Blocks of Two Dixon Resultant Matrices

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Abstract

Resultants for bivariate polynomials are often represented by the determinants of very big matrices. Properly grouping the entries of these matrices into blocks is a very effective tool for studying the properties of these resultants. Here we derive a set of convolution identities relating the blocks of two Dixon bivariate resultant representations.

1 Introduction

For three bivariate polynomials of bidegree (m, n), [Dixon 1908] describes three distinct homogeneous determinant representations for the resultant. The first formulation, generated by Sylvester's dialytic method, is a $6mn \times 6mn$ determinant. All the entries of this determinant are of degree 1 in the coefficients of the original three polynomials. The second formulation, generated by Cayley's determinant device, is a $2mn \times 2mn$ determinant whose entries are of degree 3 in the coefficients of the three original polynomials. The third formulation, obtained from a combination of Cayley's determinant device and Sylvester's dialytic method, is a $3mn \times 3mn$ determinant. The entries of this $3mn \times 3mn$ determinant are of degree 2 in the coefficients of the three generating polynomials.

In general, these determinant expressions are very big. Properly grouping the entries of these matrices into blocks is a very effective tool for studying the properties of these large expressions. The block structures of these resultant matrices and transformation matrices relating these resultants have been studied in [Chionh *et al* 1998a]. Based on these block structures, hybrids of these resultant formulations have also been constructed [Zhang *et al* 1998].

In this paper, we investigate a remarkable set of convolution identities relating the blocks in the Sylvester and the mixed Cayley-Sylvester resultants. These identities first appear in [Chionh *et al* 1998a] as a side effect, obtained while computing the blocks of the Cayley resultant. Using these identities, one can speed up the computation of the entries of the Cayley resultant and the mixed Cayley-Sylvester resultant [Chionh *et al* 1998a]. Here we present a novel interpretation and a new, more direct proof of these convolution identities by viewing the columns of these determinantal expressions as polynomials.

2 Notation

Throughout this paper we adopt the following notation. Let $p_1(s,t), \cdots, p_k(s,t)$ be a collection of bivariate polynomials of bidegree (α, β). Then their coefficient matrix is denoted by $[p_1, \cdots, p_k]^C$, where the $j-$th column consists of the coefficients of p_j, $0 \le j \le k$. That is,

$$\begin{bmatrix} p_1 & \cdots & p_k \end{bmatrix}_{1\times k} = \begin{bmatrix} 1 & t^1 & \cdots & t^\beta & \cdots & s^\alpha & s^\alpha t^1 & \cdots & s^\alpha t^\beta \end{bmatrix} \cdot [p_1, \cdots, p_k]^C .$$

For example, the coefficient matrix of the four polynomials

$$p_1(s,t) = \sum_{i=0}^{1}\sum_{j=0}^{2} a_{i,j} s^i t^j, \qquad p_2(s,t) = \sum_{i=0}^{1}\sum_{j=0}^{2} b_{i,j} s^i t^j,$$
$$p_3(s,t) = \sum_{i=0}^{1}\sum_{j=0}^{2} c_{i,j} s^i t^j, \qquad p_4(s,t) = \sum_{i=0}^{1}\sum_{j=0}^{2} d_{i,j} s^i t^j,$$

is

$$[p_1, p_2, p_3, p_4]^C = \begin{bmatrix} a_{0,0} & b_{0,0} & c_{0,0} & d_{0,0} \\ a_{0,1} & b_{0,1} & c_{0,1} & d_{0,1} \\ a_{0,2} & b_{0,2} & c_{0,2} & d_{0,2} \\ a_{1,0} & b_{1,0} & c_{1,0} & d_{1,0} \\ a_{1,1} & b_{1,1} & c_{1,1} & d_{1,1} \\ a_{1,2} & b_{1,2} & c_{1,2} & d_{1,2} \end{bmatrix}.$$

3 The Sylvester and Mixed Cayley-Sylvester Resultants

Consider three bivariate polynomials of bidegree (m, n)

$$f(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}s^i t^j, \qquad g(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{i,j}s^i t^j, \qquad h(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}s^i t^j.$$

In this section, we outline the construction and explain the block structure of the Sylvester resultant and the mixed Cayley-Sylvester resultant for three such bivariate polynomials of bidegree (m, n).

3.1 The Sylvester Resultant

Let $L = [\,f \quad g \quad h\,]$. The Sylvester resultant is the determinant of the coefficient matrix

$$S_{m,n} = \left[L,\ \cdots,\ t^{n-1}L,\ \cdots,\ s^{2m-1}L,\ \cdots,\ s^{2m-1}t^{n-1}L\right]^C.$$

The matrix $S_{m,n}$ is of size $6mn \times 6mn$. Grouping the coefficients of $S_{m,n}$ into blocks, we can write $S_{m,n}$ as

$$S_{m,n} = \begin{bmatrix} S_0 & & & & & \\ \vdots & \ddots & & & & \\ S_{m-1} & \cdots & S_0 & & & \\ S_m & \cdots & S_1 & S_0 & & \\ & \ddots & \vdots & \vdots & \ddots & \\ & & S_m & S_{m-1} & \cdots & S_0 \\ & & & S_m & \cdots & S_1 \\ & & & & \ddots & \vdots \\ & & & & & S_m \end{bmatrix},$$

where S_i, $0 \le i \le m$, denotes the block of the coefficients of $s^i, \cdots, s^i t^{2n-1}$ in the polynomials $L,\ \cdots,\ t^{n-1}L$. Let

$$f_i = \sum_{j=0}^{n} a_{i,j}t^j, \qquad g_i = \sum_{j=0}^{n} b_{i,j}t^j, \qquad h_i = \sum_{j=0}^{n} c_{i,j}t^j.$$

Then

$$S_i = \left[f_i,\ g_i,\ h_i,\ \cdots,\ t^{n-1}f_i,\ t^{n-1}g_i,\ t^{n-1}h_i\right]^C.$$

Each matrix S_i is of size $2n \times 3n$.

3.2 The Mixed Cayley-Sylvester Resultant

Let

$$\phi(g,h) = \frac{\begin{vmatrix} g(s,t) & h(s,t) \\ g(s,\beta) & h(s,\beta) \end{vmatrix}}{\beta - t} = \sum_{v=0}^{n-1} \overline{f}_v(s,t)\beta^v,$$

$$\phi(h,f) = \frac{\begin{vmatrix} h(s,t) & f(s,t) \\ h(s,\beta) & f(s,\beta) \end{vmatrix}}{\beta - t} = \sum_{v=0}^{n-1} \overline{g}_v(s,t)\beta^v,$$

$$\phi(f,g) = \frac{\begin{vmatrix} f(s,t) & g(s,t) \\ f(s,\beta) & g(s,\beta) \end{vmatrix}}{\beta - t} = \sum_{v=0}^{n-1} \overline{h}_v(s,t)\beta^v,$$

where $\overline{f}_v(s,t), \overline{g}_v(s,t), \overline{h}_v(s,t)$, $0 \le v \le n-1$, are polynomials of degree $2m$ in s and $n-1$ in t. Let $\overline{L}_v = [\,\overline{f}_v(s,t) \quad \overline{g}_v(s,t) \quad \overline{h}_v(s,t)\,]$, and let

$$M_{m,n} = \left[\overline{L}_0, \cdots, \overline{L}_{n-1}, \cdots, s^{m-1}\overline{L}_0, \cdots, s^{m-1}\overline{L}_{n-1}\right]^C.$$

The mixed Cayley-Sylvester resultant is the determinant of $M_{m,n}$. The matrix $M_{m,n}$ is of size $3mn \times 3mn$. Grouping the coefficients of $M_{m,n}$ into blocks, we can write $M_{m,n}$ as

$$M_{m,n} = \begin{bmatrix} M_0 & & \\ \vdots & \ddots & \\ M_{m-1} & \cdots & M_0 \\ \vdots & \vdots & \vdots \\ M_{2m} & \cdots & M_m \\ & \ddots & \vdots \\ & & M_{2m} \end{bmatrix},$$

where M_i, $0 \le i \le 2m$, consists of the coefficients of $s^i, \cdots, s^i t^{n-1}$ in the polynomials $\overline{L}_0, \cdots, \overline{L}_{n-1}$. Let $\overline{f}_{v,i}$, $0 \le i \le 2m$, be the coefficient of s^i in the bivariate polynomials $\overline{f}_v(s,t)$. Then $\overline{f}_{v,i}$ is a univariate polynomial of degree $n-1$ in t. Similarly, let $\overline{g}_{v,i}$, $\overline{h}_{v,i}$ be the coefficients of s^i in $\overline{g}_v(s,t)$ and $\overline{h}_v(s,t)$ respectively. Then

$$M_i = \left[\overline{f}_{0,i}, \overline{g}_{0,i}, \overline{h}_{0,i}, \cdots, \overline{f}_{n-1,i}, \overline{g}_{n-1,i}, \overline{h}_{n-1,i}\right]^C.$$

Each matrix M_i is of size $n \times 3n$.

4 The Convolution Identities

In this section, we are going to prove the following convolution identities:

$$\sum_{u+v=i} S_u \cdot M_v^T = 0, \qquad 0 \le i \le 3m. \tag{1}$$

We shall proceed in the following manner. Note that

$$
\begin{aligned}
& f(s,t)\cdot\phi(g,h)+g(s,t)\cdot\phi(h,f)+h(s,t)\cdot\phi(f,g) \\
= & \frac{f(s,t)\begin{vmatrix} g(s,t) & h(s,t) \\ g(s,\beta) & h(s,\beta)\end{vmatrix}+g(s,t)\begin{vmatrix} h(s,t) & f(s,t) \\ h(s,\beta) & f(s,\beta)\end{vmatrix}+h(s,t)\begin{vmatrix} f(s,t) & g(s,t) \\ f(s,\beta) & g(s,\beta)\end{vmatrix}}{\beta-t} \\
= & \begin{vmatrix} f(s,t) & g(s,t) & h(s,t) \\ f(s,t) & g(s,t) & h(s,t) \\ f(s,\beta) & g(s,\beta) & h(s,\beta)\end{vmatrix} / (\beta-t) \\
\equiv & \ 0.
\end{aligned}
\tag{2}
$$

To prove the convolution identities (1), we will interpret the columns of the matrices on the left hand side of Equation (1) as the coefficients of the monomials on the left hand side of Equation (2). But first we need some preliminary observations.

4.1 Interleaving in S_u and M_v

By construction,

$$S_u = \left[f_u, g_u, h_u, \cdots, t^{n-1}f_u, t^{n-1}g_u, t^{n-1}h_u\right]^C.$$

Let

$$S_u^f = \left[f_u, \cdots, t^{n-1}f_u\right]^C, \quad S_u^g = \left[g_u, \cdots, t^{n-1}g_u\right]^C, \quad S_u^h = \left[h_u, \cdots, t^{n-1}h_u\right]^C.$$

Then S_u is the matrix generated by interleaving S_u^f, S_u^g, S_u^h column by column.

Similarly by construction,

$$M_v = \left[\overline{f}_{0,v}, \overline{g}_{0,v}, \overline{h}_{0,v}, \cdots, \overline{f}_{n-1,v}, \overline{f}_{n-1,v}, \overline{f}_{n-1,v}\right]^C.$$

Let

$$M_v^{\overline{f}} = \left[\overline{f}_{0,v}, \cdots, \overline{f}_{n-1,v}\right]^C, \quad M_v^{\overline{g}} = \left[\overline{g}_{0,v}, \cdots, \overline{g}_{n-1,v}\right]^C, \quad M_v^{\overline{h}} = \left[\overline{h}_{0,v}, \cdots, \overline{h}_{n-1,v}\right]^C.$$

Then M_v is the matrix generated by interleaving $M_v^{\overline{f}}, M_v^{\overline{g}}, M_v^{\overline{h}}$ column by column.

To compute the product $S_u \cdot M_v^T$, let us examine $M_v^{\overline{f}}, M_v^{\overline{g}}, M_v^{\overline{h}}$ in more detail.

$$
\begin{aligned}
\phi(g,h) &= \frac{\begin{vmatrix} g(s,t) & h(s,t) \\ g(s,\beta) & h(s,\beta)\end{vmatrix}}{\beta-t} \\
&= \left(\sum_{k=0}^{m}\sum_{l=0}^{m} g_k(t)s^k\cdot h_l(\beta)s^l - \sum_{k=0}^{m}\sum_{l=0}^{m} g_k(\beta)s^k\cdot h_l(t)s^l\right)/(\beta-t)
\end{aligned}
$$

$$= \sum_{k=0}^{m}\sum_{l=0}^{m} \frac{\begin{vmatrix} g_k(t) & h_l(t) \\ g_k(\beta) & h_l(\beta) \end{vmatrix}}{\beta - t} \cdot s^{k+l}$$

$$= \sum_{v=0}^{2m} \left(\sum_{k+l=v} \frac{\begin{vmatrix} g_k(t) & h_l(t) \\ g_k(\beta) & h_l(\beta) \end{vmatrix}}{\beta - t} \right) \cdot s^v. \tag{3}$$

But the coefficient matrix of $\begin{vmatrix} g_k(t) & h_l(t) \\ g_k(\beta) & h_l(\beta) \end{vmatrix} / (\beta - t)$ on the right hand side of Equation (3) is the Bezout resultant matrix of g_k and h_l [De Montaudouin & Tiller 1984]. Since $M_v^{\overline{f}}$, $0 \le v \le 2m$, denotes the coefficients of s^v in $\phi(g,h)$, $M_v^{\overline{f}}$ is a sum of Bezout matrices:

$$M_v^{\overline{f}} = \sum_{k+l=v} Bezout(g_k, h_l). \tag{4}$$

Now recall that Bezout matrices are symmetric [Goldman *et al* 1984; Chionh *et al* 1998b], so $M_v^{\overline{f}}$ is symmetric. Similarly, $M_v^{\overline{g}}$, $M_v^{\overline{h}}$ are symmetric. Since M_v is generated by interleaving $M_v^{\overline{f}}, M_v^{\overline{g}}, M_v^{\overline{h}}$ column by column, M_v^T is the matrix generated by interleaving $M_v^{\overline{f}}, M_v^{\overline{g}}, M_v^{\overline{h}}$ row by row. Therefore

$$\begin{aligned} & S_u \cdot M_v^T \\ = \; & \Big(\text{interleaving } S_u^f, S_u^g, S_u^h \text{ column by column}\Big) \cdot \Big(\text{interleaving } M_v^{\overline{f}}, M_v^{\overline{g}}, M_v^{\overline{h}} \text{ row by row}\Big) \\ = \; & S_u^f \cdot M_v^{\overline{f}} + S_u^g \cdot M_v^{\overline{g}} + S_u^h \cdot M_v^{\overline{h}}. \end{aligned} \tag{5}$$

4.2 The Convolution Identities

To prove the convolution identities (1), we now investigate the right hand side of Equation (5). By Equation (4),

$$\begin{aligned} S_u^f \cdot M_v^{\overline{f}} &= S_u^f \cdot \sum_{k+l=v} Bezout(g_k, h_l) \\ &= \Big[f_u, \cdots, t^{n-1} f_u\Big]^C \cdot \sum_{k+l=v} Bezout(g_k, h_l) \\ &= \sum_{k+l=v} \Big[f_u, \cdots, t^{n-1} f_u\Big]^C \cdot Bezout(g_k, h_l). \end{aligned} \tag{6}$$

Recall that each column in a Bezout matrix of order n represents a polynomial of degree $n-1$ [Goldman *et al* 1984; Sederberg *et al* 1997]. Let the polynomials represented by the columns of $Bezout(g_k, h_l)$ be $p_j^{k,l}(t)$, $0 \le j \le n-1$. Then, by Equation (6), we have

$$S_u^f \cdot M_v^{\overline{f}} = \sum_{k+l=v} \Big[f_u, \cdots, t^{n-1} f_u\Big]^C \cdot \Big[p_0^{k,l}, \cdots, p_{n-1}^{k,l}\Big]^C. \tag{7}$$

Let us compute the product $[f_u, \cdots, t^{n-1} f_u]^C \cdot \left[p_0^{k,l}, \cdots, p_{n-1}^{k,l}\right]^C$. By construction,

$$\begin{aligned}
& \left[1 \quad \cdots \quad t^{2n-1}\right]_{1\times 2n} \cdot \left[f_u, \cdots, t^{n-1} f_u\right]^C \cdot \left[p_0^{k,l}, \cdots, p_{n-1}^{k,l}\right]^C \\
= \; & \left[f_u \quad \cdots \quad t^{n-1} f_u\right]_{1\times n} \cdot \left[p_0^{k,l}, \cdots, p_{n-1}^{k,l}\right]^C \\
= \; & f_u \cdot \left[1 \quad \cdots \quad t^{n-1}\right]_{1\times n} \cdot \left[p_0^{k,l}, \cdots, p_{n-1}^{k,l}\right]^C \\
= \; & f_u \cdot \left[p_0^{k,l} \quad \cdots \quad p_{n-1}^{k,l}\right]_{1\times n} \\
= \; & \left[f_u \cdot p_0^{k,l} \quad \cdots \quad f_u \cdot p_{n-1}^{k,l}\right]_{1\times n} \\
= \; & \left[1 \quad \cdots \quad t^{2n-1}\right]_{1\times 2n} \cdot \left[f_u \cdot p_0^{k,l}, \cdots, f_u \cdot p_{n-1}^{k,l}\right]^C . \qquad (8)
\end{aligned}$$

It follows from Equation (8) that

$$\left[f_u, \cdots, t^{n-1} f_u\right]^C \cdot \left[p_0^{k,l}, \cdots, p_{n-1}^{k,l}\right]^C = \left[f_u \cdot p_0^{k,l}, \cdots, f_u \cdot p_{n-1}^{k,l}\right]^C . \qquad (9)$$

Therefore, by Equations (7) and (9)

$$S_u^f \cdot M_v^{\overline{f}} \;=\; \left[f_u \cdot \sum_{k+l=v} p_0^{k,l}, \cdots, f_u \cdot \sum_{k+l=v} p_{n-1}^{k,l}\right]^C . \qquad (10)$$

On the other hand, by definition

$$\begin{vmatrix} g_k(t) & h_l(t) \\ g_k(\beta) & h_l(\beta) \end{vmatrix} / (\beta - t) = \sum_{j=0}^{n-1} p_j^{k,l}(t)\beta^j .$$

Thus by Equation (3),

$$\phi(g,h) = \sum_{j=0}^{n-1} \sum_{v=0}^{2m} \left(\sum_{k+l=v} p_j^{k,l}(t) \right) s^v \beta^j ,$$

so

$$\sum_{k+l=v} p_j^{k,l}(t)$$

is the coefficient of $s^v \beta^j$ in $\phi(g,h)$. Therefore,

$$\sum_{u+v=i} f_u \cdot \sum_{k+l=v} p_j^{k,l}$$

denotes the coefficient of $s^i \beta^j$ in $f(s,t) \cdot \phi(g,h)$. Hence from Equation (10), we see that the $(j+1)$st column of

$$\sum_{u+v=i} S_v^f \cdot M_v^{\overline{f}}$$

is precisely the coefficient of $s^i \beta^j$ in $f(s,t) \cdot \phi(g,h)$.

Similar results hold for $S_u^g \cdot M_v^{\overline{g}}$, and $S_u^h \cdot M_v^{\overline{h}}$: the $(j+1)$st columns of

$$\sum_{u+v=i} S_u^g \cdot M_v^{\overline{g}} \qquad \text{and} \qquad \sum_{u+v=i} S_u^h \cdot M_v^{\overline{h}}$$

denote the coefficients of $s^i\beta^j$ in $g(s,t)\cdot\phi(h,f)$ and $h(s,t)\cdot\phi(f,g)$ respectively. Hence, by Equation (5), the polynomial represented by the $(j+1)$st column of $\sum_{u+v=i} S_u \cdot M_v^T$ is exactly the coefficient of $s^i\beta^j$ in $f(s,t)\cdot\phi(g,h)+g(s,t)\cdot\phi(h,f)+h(s,t)\cdot\phi(f,g)$.

But, by Equation (2), we know that

$$f(s,t)\cdot\phi(g,h)+g(s,t)\cdot\phi(h,f)+h(s,t)\cdot\phi(f,g)\equiv 0;$$

therefore

$$\sum_{u+v=i} S_u \cdot M_v^T = 0, \qquad 0 \le i \le 3m.$$

5 Conclusions and Open Questions

We have successfully interpreted the columns of the products of the blocks of the Sylvester resultant and the mixed Cayley-Sylvester resultant as the coefficients of certain polynomials. We then used this interpretation to prove the convolution identities. One natural question to ask is "can we find similar relations between the blocks of other resultant matrices?"

By inspecting the Sylvester matrix, we see that the resultant of three bivariate polynomials of bidegree (m,n) is homogeneous of degree $6mn$ in the coefficients of the three polynomials. We also know that there are resultant matrices of size $6mn\times 6mn$, $3mn\times 3mn$, $2mn\times 2mn$, with homogeneous entries of degree 1, 2, and 3 respectively [Dixon 1908]. Is there a resultant expression represented by an $mn\times mn$ determinant with homogeneous entries of degree 6 in the coefficients of the three original polynomials? What are the polynomials that might generate this very compact $mn\times mn$ determinant? What are the relationships between these polynomials and the blocks of the three known resultant matrices? These are a few of the questions we would like to answer at some future time.

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