SIAM J. APPL. ALGEBRA GEOMETRY
Vol. 1, pp. 328–351

Strong μ-Bases for Rational Tensor Product Surfaces and Extraneous Factors Associated to Bad Base Points and Anomalies at Infinity*

Li-Yong Shen[†] and Ron Goldman[‡]

Abstract. We investigate conditions under which the resultant of a μ-basis for a rational tensor product surface is the implicit equation of the surface without any extraneous factors. In this case, we also derive a formula for the implicit degree of the rational surface based only on the bidegree of the rational parametrization and the bidegrees of the elements of the μ-basis without any knowledge of the number or multiplicities of the base points, assuming only that all the base points are local complete intersections. We conclude that in this case the implicit degree of a rational surface of bidegree (m, n) is at most mn, so the rational surface must have at least mn base points counting multiplicity. When the resultant of a μ-basis generates extraneous factors, we show how to predict and compute these extraneous factors from either the existence of bad base points or anomalies occurring in the parametrization at infinity. Examples are provided to flesh out the theory.

Key words. implicitization, strong μ-basis, resultant, base point, infinity, extraneous factor

AMS subject classifications. 68U05, 68Q25

DOI. 10.1137/16M1091952

1. Introduction.

Implicitization—computing the implicit equation of an algebraic surface from a rational parametrization of the surface—is an important computational task in computer graphics and geometric modeling. Determining whether a point lies inside, outside, or on a closed surface, algorithms for surface-surface intersection and procedures for ray tracing surfaces may all require converting a surface from rational parametric to implicit algebraic form.

For rational planar curves, finding the implicit equation from a rational parametrization can be done in several ways: one can use Gröbner bases, resultants, moving lines (syzygies), moving conics, or μ-bases. Among the most efficient of these techniques are μ-bases, since μ-bases permit us to compute with polynomials of lowest possible degree [6, 14, 21].

In order to recall the notion of a μ-basis for rational planar curves, consider a rational planar curve expressed in homogeneous coordinates (x, y, w) with homogeneous parameters (s, u),

$$\mathbf{P}(s, u) = (a(s, u), b(s, u), c(s, u)),$$

where $\gcd(a, b, c) = 1$ and the parametrization is proper. The syzygy module of $a(s, u)$, $b(s, u)$, $c(s, u)$ is known to be a free module with two generators [11, 14]. A μ-basis for the rational

*Received by the editors August 31, 2016; accepted for publication (in revised form) February 23, 2017; published electronically July 18, 2017.
http://www.siam.org/journals/siaga/1/M109195.html

[†]School of Mathematical Sciences, University of Chinese Academy of Sciences, 100049 Beijing, China (lyshen@ucas.ac.cn).
[‡]Computer Science Department, Rice University, Houston, TX 77005 (rng@rice.edu).

curve $\mathbf{P}(s,u)$ is a pair of moving lines

$$\begin{aligned}\mathbf{p}(s,u) &= (p_1(s,u), p_2(s,u), p_3(s,u)),\\ \mathbf{q}(s,u) &= (q_1(s,u), q_2(s,u), q_3(s,u))\end{aligned}$$

that follow the curve (i.e., syzygies) and that form a basis for the syzygy module of $a(s,u), b(s,u), c(s,u)$, which is a free module with rank two.

These μ-bases for rational planar curves also satisfy the following three important properties:

- The degrees of $\mathbf{p}, \mathbf{q}$ are unique, and $\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathbf{P})$;
- $\mathbf{p} \times \mathbf{q} = \kappa \mathbf{P}$ for some constant $\kappa \neq 0$;
- $F(x,y,w) \equiv \mathrm{Res}(\mathbf{p} \cdot (x,y,w), \mathbf{q} \cdot (x,y,w)) = 0$ is the implicit equation of $\mathbf{P}(s,u)$.

Thus we can recover both the parametric equations and the implicit equation of a rational planar curve from a μ-basis for the curve, since the curve is the intersection of these two moving lines. Moreover, there are fast algorithms for computing a μ-basis for a rational planar curve based on Gaussian elimination [6, 15].

The notion of a μ-basis extends from rational curves to rational surfaces [4, 7, 8]. Consider a rational surface expressed in homogeneous coordinates (x,y,z,w) with affine parameters (s,t),

$$\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)),$$

where $\gcd(a,b,c,d) = 1$ and the parametrization is proper. A μ-basis for the rational surface $\mathbf{P}(s,t)$ is a collection of three moving planes

$$\begin{aligned}\mathbf{p}(s,t) &= (p_1(s,t), p_2(s,t), p_3(s,t), p_4(s,t)),\\ \mathbf{q}(s,t) &= (q_1(s,t), q_2(s,t), q_3(s,t), q_4(s,t)),\\ \mathbf{r}(s,t) &= (r_1(s,t), r_2(s,t), r_3(s,t), r_4(s,t))\end{aligned}$$

that follow the surface (i.e., syzygies) and that form a basis for the syzygy module of $a(s,t)$, $b(s,t), c(s,t), d(s,t)$, which is known to be a free module with three generators [4]. In contrast to μ-bases for rational planar curves, μ-bases for rational surfaces satisfy the following three important properties:

- $\deg(\mathbf{p}) + \deg(\mathbf{q}) + \deg(\mathbf{r}) \geq \deg(\mathbf{P})$, where $\deg(\cdot)$ can be total degree or bidegree;
- $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \kappa \mathbf{P}(s,t)$ for some constant $\kappa \neq 0$, where $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ denotes the outer product of $\mathbf{p}, \mathbf{q}, \mathbf{r}$;
- $\mathrm{Res}(\mathbf{p} \cdot (x,y,z,w), \mathbf{q} \cdot (x,y,z,w), \mathbf{r} \cdot (x,y,z,w)) = F(x,y,z,w)E(x,y,z,w)$, where $F(x,y,z,w) = 0$ is the implicit equation of $\mathbf{P}(s,t)$ and $E(x,y,z,w)$ is an extraneous factor whenever $\deg(E) > 0$.

Notice that if we begin with a parametrization of a rational surface with homogeneous parameters

$$\mathbf{P}((s,u),(t,v)) = (a((s,u),(t,v)), b((s,u),(t,v)), c((s,u),(t,v)), d((s,u),(t,v))),$$

the syzygy module of $a((s,u),(t,v)), b((s,u),(t,v)), c((s,u),(t,v)), d((s,u),(t,v))$ need not be a free module. This fact leads us to always starting our discussion on parametric surfaces with affine parameters.

There are several important differences between μ-bases for rational curves and μ-bases for rational surfaces [4, 6, 7, 8, 14]:

1. *affine vs. homogeneous parametrization.*
   • For rational curves, the μ-bases are defined relative to homogeneous parameters;
   • for rational surfaces, the μ-bases are defined relative to affine parameters.
2. *outer product.*
   • The outer product of the elements of a μ-basis for the rational curve retrieves the parametrization of the curve relative to homogeneous parameters, i.e.,
   $$\mathbf{p}(s,u) \times \mathbf{q}(s,u) = \kappa \mathbf{P}(s,u)$$
   for some constant $\kappa \neq 0$;
   • the outer product of the elements of a μ-basis for the rational surface retrieves the parametrization of the surface relative only to affine parameters, i.e.,
   $$[\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)] = \kappa \mathbf{P}(s,t)$$
   for some constant $\kappa \neq 0$. But for homogeneous parameters $((s,u),(t,v))$, we have
   $$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = \kappa(u,v)\mathbf{P}((s,u),(t,v)),$$
   where $\kappa(u,v)$ may have positive degree in u,v.
3. *base points.*
   • μ-bases for rational curves are not affected by base points, since we can remove base points by cancelling common factors;
   • μ-bases for rational surfaces are affected by base points, since base points are often intrinsic and cannot be cancelled.
4. *uniqueness of degree.*
   • The degrees of the elements of a μ-basis for a rational curve are unique;
   • the degrees of the elements of a μ-basis for a rational surface are not unique (see section 4).
5. *degree formula.*
   • μ-bases for rational curves satisfy the degree formula
   $$\deg(\mathbf{p}) + \deg(\mathbf{q}) = \deg(\mathbf{P});$$
   • μ-bases for rational surfaces satisfy the degree formula
   $$\deg(\mathbf{p}) + \deg(\mathbf{q}) + \deg(\mathbf{r}) \geq \deg(\mathbf{P}).$$
6. *implicit equation.*
   • μ-bases for rational curves satisfy the resultant formula $\mathrm{Res}(\mathbf{p}\cdot(x,y,w), \mathbf{q}\cdot(x,y,z)) = F(x,y,w)$, where $F(x,y,w) = 0$ represents the implicit equation of $\mathbf{P}(s,u)$ with no extraneous factors;
   • μ-bases for rational surfaces satisfy the resultant formula $\mathrm{Res}(\mathbf{p} \cdot (x,y,z,w), \mathbf{q} \cdot$

$(x, y, z, w), \mathbf{r} \cdot (x, y, z, w)) = F(x, y, z, w)E(x, y, z, w)$, where $F(x, y, z, w) = 0$ represents the implicit equation of $\mathbf{P}(s, t)$ and $E(x, y, z, w)$ is an extraneous factor whenever $\deg(E) > 0$. Moreover, there are degenerate cases (see Table 4.1 and Example A.2) where $\mathrm{Res}(\mathbf{p} \cdot (x, y, z, w), \mathbf{q} \cdot (x, y, z, w), \mathbf{r} \cdot (x, y, z, w)) \equiv 0$, and therefore the implicit equation cannot be recovered directly from the resultant of this μ-basis.

7. *computation.*
   - μ-bases for rational curves can be computed quickly using Gaussian elimination;
   - μ-bases for rational surfaces can be computed much more slowly using either Gröbner bases or polynomial matrix factorization [15].

We reiterate the last two points as follows: while there are simple, fast algorithms for computing μ-bases for rational planar curves based on Gaussian elimination [6], algorithms for computing μ-bases for general rational surfaces are not simple, and the computed μ-bases often have unnecessarily high degrees [15]. Fast, efficient algorithms for computing μ-bases for surfaces are known only for rational ruled surfaces [7, 22], quadric surfaces [5], surfaces of revolution [24], and cyclides [20]. Moreover, although the resultant of a μ-basis for a rational surface is guaranteed to contain the implicit equation as a factor, this resultant may also contain extraneous factors.

After quadric surfaces, tensor product surfaces are the most common curved surfaces that appear in computer graphics and geometric modeling. Therefore we focus our attention on μ-bases for rational tensor product surfaces.

One of the primary goals of this paper is to determine conditions which guarantee that the resultant of a μ-basis is the implicit equation of a rational tensor product surface with no extraneous factors. Thus in these cases we can use the resultant of a μ-basis to find the implicit equation without needing to remove extraneous factors. Moreover, when the resultant of a μ-basis does contain extraneous factors, we show how to predict and compute all these extraneous factors.

Several other authors have investigated the structure of the resultant of a μ-basis of a rational surface using subtle algebraic techniques; see, for example, Theorem 4.1 of [3] and Proposition 7 of [2]. For a rational surface $\mathbf{P}(s, t, u)$ with parameters in complex projective space $\mathbb{P}^2(\mathbb{C})$, the authors of [3] prove that under certain assumptions, the resultant of a μ-basis of $\mathbf{P}(s, t)$ is a power of the implicit equation. But without some of these assumptions, they leave as a conjecture (see Conjecture 5.1 of [3]) the structure of the extraneous factors associated to the base points of the parametrization. This conjecture is proved in [2] using the theories of symmetric algebras and Rees algebras. Some additional advanced algebraic concepts are also used in these proofs such as MacRae's invariant and Koszul syzygies. In contrast, in our paper we address similar problems (see Theorems 3.4 and 3.10) with simpler techniques accessible to a wider audience; in addition, our computations are much easier to implement.

However, extraneous factors can be introduced not only by base points but also by homogeneous parameters corresponding to infinity. By analyzing the extraneous factor of the homogeneous outer product of a μ-basis, we identify for the first time other extraneous factors of the resultant including ruled surfaces (see Theorem 4.1). Therefore we can now implicitize a parametrized surface by computing the resultant of a μ-basis and removing all of the extraneous factors.

We proceed in the following fashion. In section 2 we provide the basic background, definitions, and notation that we shall use throughout this paper. In section 3 we give a sufficient condition for the existence of μ-bases for rational tensor product surfaces, whose resultants are the implicit equations of the surfaces without any extraneous factors. For these surfaces, we also derive a formula for the implicit degree of the rational surface based only on the bidegree of the rational parametrization and the bidegrees of the elements of the μ-basis. In addition we derive a similar formula for the number of base points with their multiplicities. We also discuss how the resultant of a μ-basis can be the implicit equation without any extraneous factors of a rational tensor product surface even if the sufficient condition in section 3 is not satisfied, and we provide examples to clarify this anomaly. In section 4 we investigate extraneous factors due to anomalies at infinity. We conclude in section 5 with a brief summary of our main results along with a conjecture and our plans for future research.

2. Preliminaries.

A rational tensor product surface $\mathcal{P}$ of bidegree (m,n) can be represented by homogeneous parametric equations

$$\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)), \tag{2.1}$$

where

$$a(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}s^i t^j, \quad b(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{i,j}s^i t^j,$$
$$c(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}s^i t^j, \quad d(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} d_{i,j}s^i t^j$$

are polynomials in $\mathbb{R}[s,t]$ and $\gcd(a,b,c,d) = 1$. We drop the trivial case where $\mathbf{P}(s,t)$ defines a plane; i.e., we shall assume that a,b,c,d are linearly independent. We also assume that the rational surface $\mathbf{P}(s,t)$ is properly parametrized, i.e., the map

$$(s,t) \to \left(\frac{a(s,t)}{d(s,t)}, \frac{b(s,t)}{d(s,t)}, \frac{c(s,t)}{d(s,t)}\right)$$

is birational.

2.1. Base points.

Homogeneous parameters are necessary in the discussion of base points. Considering the parameters in complex projective space $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, we can write $\mathbf{P}(s,t)$ with homogeneous parameters $\mathbf{P}((s,u),(t,v))$. A *base point* of a rational parametrization $\mathbf{P}((s,u),(t,v))$ is a parameter pair $((s_0,u_0),(t_0,v_0)) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that $\mathbf{P}((s_0,u_0),(t_0,v_0)) = (0,0,0,0)$. A base point $((s_0,u_0),(t_0,v_0))$ is a *local complete intersection* if the ideal $\langle a,b,c,d\rangle$ is generated by two polynomials in a neighborhood of $((s_0,u_0),(t_0,v_0))$.

Base points affect the implicit degree of a rational surface. Let $D_{\mathbf{P}}$ denote the implicit degree of a rational surface defined by a parametrization $\mathbf{P}(s,t)$. For a bidegree (m,n) parametrization $\mathbf{P}(s,t)$ of a rational tensor product surface $\mathcal{P}$ with l base points in general position, each with multiplicity $d_i, i = 1,\dots,l$, the generic formula for the degree of the implicit equation is [27]

$$D_{\mathbf{P}} = 2mn - \sum_{i=1}^{l} d_i. \tag{2.2}$$

Some bases points are simpler than others. In particular, base points that are local complete intersections are simpler than base points that are not local complete intersections [2, 3, 11]. The importance of this distinction between simple and complicated base points will become evident in subsequent sections.

2.2. Moving planes. A *moving plane* of bidegree (σ_1, σ_2) has the implicit form

$$L(s,t,x,y,z,w) \equiv \sum_{i=0}^{\sigma}(A_i x + B_i y + C_i z + D_i w)\gamma_i(s,t) = 0, \tag{2.3}$$

where $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j} \in \mathbb{R}$ and $\gamma_i(s,t), i = 1, \ldots, \sigma$, are polynomials with bidegree at most (σ_1, σ_2). We call $\gamma_i(s,t)$ *blending functions* of the moving plane. If the blending functions are $s^i t^j, i = 0, \ldots, \sigma_1, j = 0, \ldots, \sigma_2$, then $\sigma = (\sigma_1 + 1)(\sigma_2 + 1)$. For each parameter pair (s,t), (2.3) is the implicit equation of a plane in $\mathbb{R}^3$. A moving plane is also sometimes written in parametric form as the vector

$$\mathbf{L}(s,t) = (A(s,t), B(s,t), C(s,t), D(s,t))$$

by extracting the coefficients of (2.3) with respect to x, y, z, w, i.e., $L(s,t,x,y,z,w) = \mathbf{L}(s,t)\cdot\mathbf{X}$ where $\mathbf{X} = (x,y,z,w)$.

A moving plane $\mathbf{L}(s,t)$ is said to *follow* the surface $\mathbf{P}(s,t)$ if

$$\mathbf{L}(s,t) \cdot \mathbf{P}(s,t) = A(s,t)a(s,t) + B(s,t)b(s,t) + C(s,t)c(s,t) + D(s,t)d(s,t) \equiv 0. \tag{2.4}$$

Thus $\mathbf{L}(s,t)$ follows the surface $\mathbf{P}(s,t)$ if $\mathbf{L}(s,t)$ is a syzygy of $\mathbf{P}(s,t)$. The syzygy module of $\mathbf{P}(s,t)$ is known to be a free module with three generators [4].

Three moving planes $\mathbf{p}(s,t), \mathbf{q}(s,t)$, and $\mathbf{r}(s,t)$ are said to form a *μ-basis* of $\mathbf{P}(s,t)$ if $\mathbf{p}(s,t), \mathbf{q}(s,t)$, and $\mathbf{r}(s,t)$ are a basis for the syzygy module of $\mathbf{P}(s,t)$. Chen, Cox, and Liu [4] show that if $\mathbf{p}(s,t), \mathbf{q}(s,t)$, and $\mathbf{r}(s,t)$ are a μ-basis for the syzygy module of $\mathbf{P}(s,t)$, then

$$[\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)] = \kappa\mathbf{P}(s,t), \quad \kappa \neq 0, \tag{2.5}$$

where $[\cdot]$ denotes the outer product defined by

$$[\mathbf{p},\mathbf{q},\mathbf{r}] = \left(\begin{vmatrix} p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \\ r_2 & r_3 & r_4 \end{vmatrix}, - \begin{vmatrix} p_1 & p_3 & p_4 \\ q_1 & q_3 & q_4 \\ r_1 & r_3 & r_4 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 & p_4 \\ q_1 & q_2 & q_4 \\ r_1 & r_2 & r_4 \end{vmatrix}, - \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \right).$$

We will also be interested in (2.5) in the homogeneous setting. Introducing the homogeneous form of the tensor parameters, we see that (2.5) becomes

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = \kappa(u,v)\mathbf{P}((s,u),(t,v)). \tag{2.6}$$

By comparing the degrees of the homogeneous parameters, it follows that $\kappa(u,v) = \lambda u^i v^j$ for a constant $\lambda \neq 0$, where

$$(i,j) = \text{bideg}(\mathbf{p}) + \text{bideg}(\mathbf{q}) + \text{bideg}(\mathbf{r}) - \text{bideg}(\mathbf{P}).$$

Thus when the parameter pair $((s_0,u_0),(t_0,v_0))$ is not a base point of the surface $\mathbf{P}(s,t)$, the vectors $\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0))$ are linearly independent, since their outer product is not zero. But at base points $((s_0,u_0),(t_0,v_0))$, their outer product

$$[\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}(s(s_0,u_0),(t_0,v_0))] = \lambda u_0^i v_0^j \mathbf{P}((s_0,u_0),(t_0,v_0)) = 0,$$

so at base points the vectors $\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0))$ are linearly dependent, i.e.,

$$\operatorname{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0))) \leq 2.$$

Lemma 2.1. *Let* $\mathbf{p},\mathbf{q},\mathbf{r}$ *be a* μ*-basis for the surface* $\mathbf{P}(s,t)$*, and let* $((s_0,u_0),(t_0,v_0))$ *be a base point of the surface* $\mathbf{P}(s,t)$*. Then* $((s_0,u_0),(t_0,v_0))$ *is a local complete intersection if and only if* $\operatorname{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\ \mathbf{r}((s_0,u_0),(t_0,v_0))) = 2$.

Proof. This lemma follows by Lemma 3.2 of [4] and also from the argument given in Case 2 of Remark 5.1 of [3]. ■

Our goal is to recover the implicit equation $F(x,y,z,w) = 0$ of a rational surface $\mathbf{P}(s,t)$ from its μ-basis. When $\mathbf{X} = (x,y,z,w)$ is a point on the surface $\mathbf{P}(s,t)$,

$$\operatorname{Res}(\mathbf{p}(s,t)\cdot\mathbf{X},\mathbf{q}(s,t)\cdot\mathbf{X},\mathbf{r}(s,t)\cdot\mathbf{X}) = 0,$$

since $\mathbf{X} = \mathbf{P}(s,t)$ is a common root of $\mathbf{p}(s,t)\cdot\mathbf{X},\mathbf{q}(s,t)\cdot\mathbf{X},\mathbf{r}(s,t)\cdot\mathbf{X}$. Therefore $F(\mathbf{X})$ is a factor of $\operatorname{Res}(\mathbf{p}(s,t)\cdot\mathbf{X},\mathbf{q}(s,t)\cdot\mathbf{X},\mathbf{r}(s,t)\cdot\mathbf{X})$. However, this resultant may also contain extraneous factors—that is, in general,

$$\operatorname{Res}(\mathbf{p}(s,t)\cdot\mathbf{X},\mathbf{q}(s,t)\cdot\mathbf{X},\mathbf{r}(s,t)\cdot\mathbf{X}) = F(\mathbf{X})E(\mathbf{X}),$$

where $E(\mathbf{X})$ is an extraneous factor if $\deg(E(\mathbf{X})) > 0$ [7, 4, 26]. These extraneous factors may be due to bad base points (i.e., base points that are not local complete intersections) or to extraneous points at infinity (see section 4). But before we go on to develop conditions which guarantee that this resultant contains no extraneous factors, we need to say a few words about resultants for three bivariate polynomials.

2.3. The resultant of three polynomials.

Here we review some basic properties of the resultant of three bivariate polynomials. A resultant is a polynomial in the coefficients of a set of n polynomials in $n-1$ variables (or n homogeneous variables) that vanishes whenever the given polynomials have a common root (see Chapter 3 of [12]).

To compute the degree of the resultant, we need to know the area of certain Newton polygons. We begin by recalling some definitions leading to the notion of a Newton polygon. For a polynomial $f(s,t) = \sum_i \sum_j f_{ij} s^i t^j$, each monomial corresponds to a lattice point (i,j) in $\mathbb{Z}^2 \subset \mathbb{R}^2$. All of these lattice points form a lattice support S, called the *parametric support* of $f(s,t)$. For any lattice set $S \subset \mathbb{Z}^2$, the Newton polygon $NP(S)$ is the convex hull of S in the Euclidean plane.

The Minkowski sum of $NP(S_1)$ and $NP(S_2)$ is defined by

$$NP(S_1) \oplus NP(S_2) = \{s_1 + s_2 \mid s_1 \in NP(S_1), s_2 \in NP(S_2)\}.$$

The Minkowski sum is a rectangle if both $NP(S_1)$ and $NP(S_2)$ are rectangles. In particular, if $NP(S_1)$ and $NP(S_2)$ are rectangles with lower left and upper right corner points $\{(0,0),(\sigma_{11},\sigma_{12})\}$ and $\{(0,0),(\sigma_{21},\sigma_{22})\}$, then $NP(S_1)\oplus NP(S_2)$ is also a rectangle with lower left and upper right corner points $\{(0,0),(\sigma_{11}+\sigma_{21},\sigma_{12}+\sigma_{22})\}$.

Consider three bivariate polynomials $f(s,t), g(s,t), h(s,t)$. Their resultant $\mathrm{Res}(f,g,h)$ with respect to s,t is a polynomial in the coefficients of f,g,h. The degree of the resultant in the coefficients of f is given by the formula (see Chapter 7 of [12])

$$\deg_f(\mathrm{Res}(f,g,h)) = AR(NP(S_g)\oplus NP(S_h)) - AR(NP(S_g)) - AR(NP(S_h)), \tag{2.7}$$

where S_f, S_g, and S_h are lattice sets consisting of all the monomials of $f(s,t), g(s,t)$, and $h(s,t)$, and $AR(P)$ is the area of the polygon P. Similar formulas hold for g and h

$$\begin{aligned}\deg_g(\mathrm{Res}(f,g,h)) &= AR(NP(S_f)\oplus NP(S_h)) - AR(NP(S_f)) - AR(NP(S_h)),\\ \deg_h(\mathrm{Res}(f,g,h)) &= AR(NP(S_f)\oplus NP(S_g)) - AR(NP(S_f)) - AR(NP(S_g)).\end{aligned}$$

We can consider three moving planes $\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}$ as polynomials in s,t with coefficients that are linear in x,y,z,w. So the degree in x,y,z,w of the resultant $\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X})$ is equal to

$$\begin{aligned}&\deg(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}))\\ &= \deg_{\mathbf{p}(s,t)\cdot\mathbf{X}}(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}))\\ &\quad + \deg_{\mathbf{q}(s,t)\cdot\mathbf{X}}(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}))\\ &\quad + \deg_{\mathbf{r}(s,t)\cdot\mathbf{X}}(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X})).\end{aligned} \tag{2.8}$$

Suppose $\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}$ are dense polynomials with bidegrees $(\sigma_{11},\sigma_{12})$, $(\sigma_{21},\sigma_{22})$, $(\sigma_{31},\sigma_{32})$. Since the corresponding Newton polygons are rectangles, we have

$$\begin{aligned}&AR(NP(S_{\mathbf{q}\cdot\mathbf{X}})) = \sigma_{21}\sigma_{22},\\ &AR(NP(S_{\mathbf{r}\cdot\mathbf{X}})) = \sigma_{31}\sigma_{32},\\ &AR(NP(S_{\mathbf{q}\cdot\mathbf{X}})\oplus NP(S_{\mathbf{r}\cdot\mathbf{X}})) = (\sigma_{21}+\sigma_{31})(\sigma_{22}+\sigma_{32}).\end{aligned}$$

Hence by (2.7),

$$\begin{aligned}&\deg_{\mathbf{p}\cdot\mathbf{X}}(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}))\\ &\quad = (\sigma_{21}+\sigma_{31})(\sigma_{22}+\sigma_{32}) - \sigma_{21}\sigma_{22} - \sigma_{31}\sigma_{32}\\ &\quad = \sigma_{31}\sigma_{22} + \sigma_{32}\sigma_{21}.\end{aligned}$$

Similarly,

$$\begin{aligned}\deg_{\mathbf{q}\cdot\mathbf{X}}(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X})) &= \sigma_{11}\sigma_{32} + \sigma_{12}\sigma_{31},\\ \deg_{\mathbf{r}\cdot\mathbf{X}}(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X})) &= \sigma_{11}\sigma_{22} + \sigma_{12}\sigma_{21}.\end{aligned}$$

Therefore (2.8) simplifies to

$$\begin{aligned}&\deg(\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X}, \mathbf{q}(s,t)\cdot\mathbf{X}, \mathbf{r}(s,t)\cdot\mathbf{X}))\\ &= (\sigma_{11}\sigma_{22}+\sigma_{12}\sigma_{21}) + (\sigma_{11}\sigma_{32}+\sigma_{12}\sigma_{31}) + (\sigma_{31}\sigma_{22}+\sigma_{32}\sigma_{21}).\end{aligned} \tag{2.9}$$

The are many papers and books on the theory and computation of resultants for a variety of different types of polynomial systems, e.g., [13] (see Chapters 3 and 7), [18] (see Chapters 3, 8, and 13), and [16, 17, 25].

We close this section by presenting a degenerate case for the resultant computation.

Lemma 2.2. *Let* $\mathbf{p},\mathbf{q},\mathbf{r}$ *be a* μ*-basis for the surface* $\mathbf{P}(s,t)$*. Then* $\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X},\mathbf{q}(s,t)\cdot\mathbf{X},\mathbf{r}(s,t)\cdot\mathbf{X})\equiv 0$ *if* $\mathrm{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0)))=0$.

Proof. The three polynomials $\mathbf{p}((s,u),(t,v))\cdot\mathbf{X},\mathbf{q}((s,u),(t,v))\cdot\mathbf{X},\mathbf{r}((s,u),(t,v))\cdot\mathbf{X}$ vanish simultaneously at $(s_0,u_0),(t_0,v_0)$ since $\mathrm{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0)))=0$. Thus for every $\mathbf{X}=(x,y,z,w)$, there are parameters where the three polynomials have a common root. Therefore the resultant must vanish for all values of $\mathbf{X}$, so the resultant must be identically zero. ∎

3. Strong μ-bases.

For some rational surfaces $\mathbf{P}(s,t)$, there are μ-bases whose resultant is exactly equal to the implicit equation $F(x,y,z,w)$ of the surface without extraneous factors.

Definition 3.1. *A* μ*-basis* $\mathbf{p}(s,t),\mathbf{q}(s,t),\mathbf{r}(s,t)$ *for a rational tensor product surface* $\mathbf{P}(s,t)$ *is* algebraically strong *if*

$$\mathrm{Res}(\mathbf{p}(s,t)\cdot\mathbf{X},\mathbf{q}(s,t)\cdot\mathbf{X},\mathbf{r}(s,t)\cdot\mathbf{X}))=F(x,y,z,w), \tag{3.1}$$

where $F(x,y,z,w)=0$ *is the implicit equation of the surface* $\mathbf{P}(s,t)$*.*

Not much is known about the degrees of the elements of a μ-basis. From (2.5) it follows that for tensor product surfaces with tensor product μ-bases,

$$\mathrm{bideg}(\mathbf{p})+\mathrm{bideg}(\mathbf{q})+\mathrm{bideg}(\mathbf{r})\geq\mathrm{bideg}(\mathbf{P}). \tag{3.2}$$

Notice that equality need not hold, since there can be cancellation in (2.5) of high degree terms. When equality does hold or, equivalently, when $\kappa(u,v)=\kappa$ is a nonzero constant in (2.6), we have the following definition.

Definition 3.2. *A* μ*-basis* $\mathbf{p}(s,t),\mathbf{q}(s,t),\mathbf{r}(s,t)$ *for a rational tensor product surface* $\mathbf{P}(s,t)$ *is* parametrically strong *if*

$$\mathrm{bideg}(\mathbf{p})+\mathrm{bideg}(\mathbf{q})+\mathrm{bideg}(\mathbf{r})=\mathrm{bideg}(\mathbf{P}) \tag{3.3}$$

or, equivalently, if

$$[\mathbf{p}((s,u),(t,v)),\mathbf{q}((s,u),(t,v)),\mathbf{r}((s,u),(t,v))]=\kappa\mathbf{P}((s,u),(t,v))$$

for some constant $\kappa\neq 0$*.*

Definition 3.3. *A* μ*-basis* $\mathbf{p}(s,t),\mathbf{q}(s,t),\mathbf{r}(s,t)$ *for a rational tensor product surface* $\mathbf{P}(s,t)$ *is* strong *if the* μ*-basis is algebraically strong and parametrically strong.*

Note that the definition of a parametrically strong μ-basis is analogous to that of a *strong* μ*-basis,* which is first introduced for a rational parametrization with parameters in complex projective space $\mathbb{P}^2(\mathbb{C})$; see section 5.1 of [10].

Theorem 3.4. *If all of the base points of* $\mathbf{P}(s,t)$ *are local complete intersections, then a parametrically strong* μ*-basis is also algebraically strong.*

Proof. Consider a rational tensor product surface $\mathbf{P}(s,t)$ with a parametrically strong μ-basis $\mathbf{p}(s,t),\mathbf{q}(s,t)$, $\mathbf{r}(s,t)$, and let $F(x,y,z,w)=0$ be the implicit equation of $\mathbf{P}(s,t)$.

Let $\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))$ be homogenizations of $\mathbf{p}(s,t)$, $\mathbf{q}(s,t)$, $\mathbf{r}(s,t)$, and let $\mathbf{X} = (x,y,z,w)$. Consider the system of equations

$$\left\{\begin{array}{l} \mathbf{p}((s,u),(t,v)) \cdot \mathbf{X} = 0, \\ \mathbf{q}((s,u),(t,v)) \cdot \mathbf{X} = 0, \\ \mathbf{r}((s,u),(t,v)) \cdot \mathbf{X} = 0. \end{array}\right. \tag{3.4}$$

We first study the solutions of (3.4). For any fixed parameters $((s_0,u_0),(t_0,v_0))$, (3.4) has a common root at those points $\mathbf{X} = \mathbf{X}_0$ that are simultaneously perpendicular to $\mathbf{P}((s_0,u_0),(t_0,v_0)), \mathbf{q}((s_0,u_0),(t_0,v_0))$, $\mathbf{r}((s_0,u_0),(t_0,v_0))$.

Suppose $((s_0,u_0),(t_0,v_0))$ is not a base point of $\mathbf{P}((s,u),(t,v))$. Since $\mathbf{p},\mathbf{q},\mathbf{r}$ form a parametrically strong μ-basis, we have

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = \kappa \mathbf{P}((s,u),(t,v)), \quad \kappa \neq 0.$$

Therefore $\mathrm{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)), \mathbf{q}((s_0,u_0),(t_0,v_0)), \mathbf{r}((s_0,u_0),(t_0,v_0))) = 3$. Thus the only vectors simultaneously perpendicular to $\mathbf{p},\mathbf{q},\mathbf{r}$ are in the direction of $\mathbf{P}(s,t)$, so such values are $\mathbf{X}_0 = \lambda \mathbf{P}((s_0,u_0),(t_0,v_0))$ for constants $\lambda \neq 0$. Hence $\mathbf{X}_0$ must lie on the surface $\mathbf{P}(s,t)$, so $F(\mathbf{X}_0) = 0$.

Suppose, however, that $BP = ((s_0,u_0),(t_0,v_0))$ is a base point of $\mathbf{P}(s,t)$. Then by assumption, $((s_0,u_0),(t_0,v_0))$ is a local complete intersection. Therefore by Lemma 2.1 the solution of (3.4) is a line determined by two planes, since $\mathrm{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)), \mathbf{q}((s_0,u_0),(t_0,v_0)), \mathbf{r}((s_0,u_0),(t_0,v_0))) = 2$. We denote the line associated with the base point BP by L_{BP}.

Therefore, the solutions of (3.4) are given by

$$\{\mathbf{X} \mid F(\mathbf{X}) = 0\} \cup \{L_{BP} \mid \mathbf{P}(BP) = \mathbf{0}\}. \tag{3.5}$$

We will soon show that the lines L_{BP} all lie on the surface $F(x,y,z,w) = 0$.

To prove that the μ-basis $\mathbf{p},\mathbf{q},\mathbf{r}$ is algebraically strong, we will now prove that the resultant on the left-hand side of (3.1) does not vanish identically and that $F(\mathbf{X})$ vanishes on the lines L_{BP}. To show that the resultant does not vanish identically, observe that $\{\mathbf{X} \mid F(\mathbf{X}) = 0\} \cup \{L_{BP} \mid \mathbf{P}(BP) = \mathbf{0}\}$ is a proper subvariety of $\mathbb{P}^3(\mathbb{C})$. Hence there are points $\mathbf{X}$ that are not solutions of (3.4). Therefore at these points $\mathbf{X}$, we have $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}) \neq 0$, so the resultant has zero locus of pure codimension 1. Since L_{BP} has codimension 2 in $\mathbb{P}^3(\mathbb{C})$ and the number of base points is finite, the zero locus of $\{L_{BP} \mid \mathbf{P}(BP) = \mathbf{0}\}$ must lie in $\{\mathbf{X} \mid F(\mathbf{X}) = 0\}$.

Finally, we have shown that the resultant vanishes on $\{\mathbf{X} = \mathbf{P}((s,u),(t,v)) \mid \mathbf{P}((s,u),(t,v)) \neq (0,0,0,0)\}$, which is a Zariski dense subset of $\{\mathbf{X} \mid F(\mathbf{X}) = 0\}$. Hence since the resultant and the implicit equation are both polynomials in $\mathbf{X}$, the resultant vanishes on $\{\mathbf{X} \mid F(\mathbf{X}) = 0\}$.

Moreover, the resultant vanishes nowhere else because the only solutions of (3.4) are given by the set of points in (3.5), and we have shown that the lines L_{BP} in (3.5) all lie on the surface defined by the implicit equation. Since $F(x,y,z,w)$ is irreducible and $\mathbf{P}(s,t)$ is a proper parametrization, it follows that

$$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}) = C_0 F(\mathbf{X}),$$

where C_0 is a nonzero constant. ∎

Corollary 3.5. *A base point that is a local complete intersection does not induce any extraneous factors.*

Corollary 3.6. *Consider a rational tensor product surface $\mathbf{P}(s,t)$, where $\mathrm{bideg}(\mathbf{P}) = (m,n)$. Suppose that the base points of $\mathbf{P}(s,t)$ are local complete intersections and $\mathbf{P}(s,t)$ has a parametrically strong μ-basis $\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)$ with bidegrees $(\sigma_{11},\sigma_{12})$, $(\sigma_{21},\sigma_{22})$, $(\sigma_{31},\sigma_{32})$. Let $D_{\mathbf{P}}$ denote the algebraic degree of $\mathbf{P}(s,t)$. Then*

1. $D_{\mathbf{P}} = \deg(\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}))$,
2. $m = \sigma_{11}+\sigma_{21}+\sigma_{31}$ *and* $n = \sigma_{12}+\sigma_{22}+\sigma_{32}$.

   If the Newton polygons of $\mathbf{P}(s,t), \mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)$ are dense, then

3. $D_{\mathbf{P}} = mn - \sigma_{11}\sigma_{12} - \sigma_{21}\sigma_{22} - \sigma_{31}\sigma_{32}$,
4. $D_{\mathbf{P}} = AR(NP(S_{\mathbf{P}\cdot\mathbf{X}})) - AR(NP(S_{\mathbf{p}\cdot\mathbf{X}})) - AR(NP(S_{\mathbf{q}\cdot\mathbf{X}})) - AR(NP(S_{\mathbf{r}\cdot\mathbf{X}}))$.

Proof. Since $\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)$ is a parametrically strong μ-basis, part 1 follows from Theorem 3.4. By (3.3) we have $m = \sigma_{11}+\sigma_{21}+\sigma_{31}$ and $n = \sigma_{12}+\sigma_{22}+\sigma_{32}$. Then by Theorem 3.4 and formula (2.9),

$$D_{\mathbf{P}} = \deg(\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X})) = (\sigma_{11}\sigma_{22}+\sigma_{12}\sigma_{21}) + (\sigma_{11}\sigma_{32}+\sigma_{12}\sigma_{31}) + (\sigma_{31}\sigma_{22}+\sigma_{32}\sigma_{21}).$$

Therefore $mn = (\sigma_{11}+\sigma_{21}+\sigma_{31})(\sigma_{12}+\sigma_{22}+\sigma_{32}) = D_{\mathbf{P}} + \sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22} + \sigma_{31}\sigma_{32}$.

Finally, part 4 follows immediately from part 3, since if the Newton polygons of $\mathbf{P}, \mathbf{p}, \mathbf{q}, \mathbf{r}$ are dense, then these polygons are rectangles whose areas are given by the products of their bidegrees. ■

Corollary 3.7. *Consider a rational tensor product surface $\mathbf{P}(s,t)$, where $\mathrm{bideg}(\mathbf{P}) = (m,n)$, that has exactly l base points, each of which is a local complete intersection and with multiplicity d_i, $i = 1,\dots,l$. Suppose too that $\mathbf{P}(s,t)$ has a parametrically strong μ-basis $\mathbf{p}(s,t), \mathbf{q}(s,t)$, $\mathbf{r}(s,t)$ with bidegrees $(\sigma_{11},\sigma_{12})$, $(\sigma_{21},\sigma_{22})$, $(\sigma_{31},\sigma_{32})$. If the Newton polygons of $\mathbf{P}(s,t), \mathbf{p}(s,t)$, $\mathbf{q}(s,t)$, $\mathbf{r}(s,t)$ are dense, then*

1. $\sum_{i=1}^{l} d_i = mn + \sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22} + \sigma_{31}\sigma_{32}$. *Thus $\mathbf{P}(s,t)$ must have at least mn base points counting multiplicity.*
2. $\sum_{i=1}^{l} d_i = AR(NP(S_{\mathbf{P}\cdot\mathbf{X}})) + AR(NP(S_{\mathbf{p}\cdot\mathbf{X}})) + AR(NP(S_{\mathbf{q}\cdot\mathbf{X}})) + AR(NP(S_{\mathbf{r}\cdot\mathbf{X}}))$.

Proof. This result follows immediately from Corollary 3.6 and formula (2.2). ■

Example 3.8. Here we give an example of a strong μ-basis. Consider the following rational tensor product surface with bidegree $(2,2)$:

$$\mathbf{P}(s,t) = \begin{pmatrix} s^2t - \frac{147\,st^2}{47} + \frac{275\,st}{47} - \frac{348\,s^2}{47} + 2\,t^2 + \frac{348\,s}{47} + 2\,t \\ -s^2t^2 - s^2t + \frac{82\,st^2}{47} - \frac{169\,st}{94} + \frac{441\,s^2}{94} + 2\,t^2 - \frac{159\,s}{94} - t - 3 \\ 2\,s^2t^2 - 4 - \frac{170\,st^2}{47} - \frac{26\,st}{47} - \frac{460\,s^2}{47} + 2\,t^2 + \frac{648\,s}{47} - 2\,t \\ -3 + 2\,s^2t - \frac{102\,st^2}{47} + \frac{185\,st}{94} - \frac{129\,s^2}{94} + t^2 + \frac{411\,s}{94} - 2\,t \end{pmatrix}^T.$$

There is a μ-basis formed by a triple of moving planes that follow this surface with bidegrees $(1,0), (0,1), (1,1)$,

$$\mathbf{p} = (-\tfrac{167\,s}{329} + 1, \tfrac{1263\,s}{329} - \tfrac{771}{329}, \tfrac{1263\,s}{658} + \tfrac{339}{658}, \tfrac{715\,s}{329} + \tfrac{545}{329}),$$

$$\mathbf{q} = (\tfrac{908}{57} - \tfrac{202\,t}{19}, \tfrac{426}{19} + 2\,t, \tfrac{43}{19} + t, -\tfrac{1450}{57} + \tfrac{290\,t}{19}),$$

$$\mathbf{r} = (-\tfrac{7849\,st}{1652} + \tfrac{4867841}{47082} - \tfrac{1570802\,t}{23541}, \tfrac{9000447}{62776} + s + \tfrac{3507\,t}{236}, \tfrac{1058669}{125552} + \tfrac{9501\,s}{3304},$$
$$-\tfrac{29118679}{188328} - \tfrac{7058\,s}{413} + \tfrac{4883915\,t}{47082}).$$

Indeed, we can easily verify that $[\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)] = -\frac{14157425}{109858} \cdot \mathbf{P}(s,t)$. This μ-basis is parametrically strong, since $(1,0)+(0,1)+(1,1) = (2,2)$. Moreover, $\mathbf{P}(s,t)$ has five simple base points $(1,0), (0,-1), (-1,2), (2,3), (-2,-2)$, which are all local complete intersections. Therefore this μ-basis is strong by Theorem 3.4, and by Corollary 3.6 the implicit degree of this surface is three.

Note that the converse of Theorem 3.4 is not true: a μ-basis may be algebraically strong even if the μ-basis is not parametrically strong (see Example 3.9).

Example 3.9. Consider the following rational tensor product surface with bidegree $(2,2)$:

$$\mathbf{P}(s,t) = \begin{pmatrix} 2s^2t^2 + 3 - \frac{50st^2}{3} + \frac{118st}{3} - s^2 - t^2 - 2s + 2t \\ -2s^2t^2 - 1 + \frac{70st^2}{3} - \frac{167st}{3} - t^2 + s - 2t \\ s^2t^2 + 2st^2 - 8st - 2t^2 - 2t \\ 2s^2t^2 + 2 - \frac{28st^2}{3} + \frac{59st}{3} - s^2 - 2t^2 - s \end{pmatrix}^T.$$

The implicit degree of this surface is four. There is a triple of moving planes that follow $\mathbf{P}(s,t)$ with bidegrees $(1,0), (1,1), (1,1)$,

$$\begin{aligned} \mathbf{p} &= \left(\tfrac{215\,s}{59} + \tfrac{330}{59}, s + \tfrac{18}{59}, 2\,s + \tfrac{312}{59}, -\tfrac{215\,s}{59} - \tfrac{486}{59}\right), \\ \mathbf{q} &= (-st + \tfrac{295}{36} + \tfrac{2365\,t}{3}, \tfrac{295}{12} + \tfrac{295\,s}{36} - \tfrac{3991\,t}{54}, \\ &\qquad -\tfrac{9263}{54} + \tfrac{295\,s}{18} + \tfrac{18313\,t}{18}, st - \tfrac{148457\,t}{108}), \\ \mathbf{r} &= (-\tfrac{10}{3} - 215\,t, -8 - \tfrac{7}{3}\,s + \tfrac{215\,t}{9}, \\ &\qquad \tfrac{472}{9} - \tfrac{14}{3}\,s - \tfrac{860\,t}{3}, 1 + \tfrac{3440\,t}{9}). \end{aligned}$$

The sum of the bidegrees of these three moving planes is $(3,2)$, which is greater than $(2,2)$. One can check that these three moving planes satisfy (2.5) and (3.1); hence, they form an algebraically strong μ-basis of $\mathbf{P}(s,t)$, but this μ-basis is not parametrically strong.

$\mathbf{P}(s,t)$ can have a base point that is not a local complete intersection, i.e., $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r}) \le 1$ at this point. Omitting the degenerate case where $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r}) = 0$, we will show that when $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r}) = 1$, no μ-basis can be algebraically strong even if the μ-basis is parametrically strong. That is, bad base points always introduce extraneous factors into the resultant.

Theorem 3.10. *Consider a rational tensor product surface* $\mathbf{P}(s,t)$ *with a base point* $((s_0,u_0),(t_0,v_0))$ *such that* $\mathrm{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0)))=1$. *Then* $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})$ *contains the linear factor* $\mathbf{p}((s_0,u_0),(t_0,v_0))\cdot\mathbf{X}$ *as an extraneous factor.*

Proof. Since $\mathrm{Rank}(\mathbf{p}((s_0,u_0),(t_0,v_0)),\mathbf{q}((s_0,u_0),(t_0,v_0)),\mathbf{r}((s_0,u_0),(t_0,v_0)))=1$, we can assume without loss of generality that $\mathbf{p}((s_0,u_0),(t_0,v_0))\neq\mathbf{0}$, Therefore the system of equations (3.4) has an additional solution P_{BP}, where P_{BP} is the plane $\mathbf{p}((s_0,u_0),(t_0,v_0))\cdot\mathbf{X}=0$ associated to the base point BP. The plane P_{BP} cannot lie in the surface $\mathbf{P}(s,t)$, since by assumption $\mathbf{P}(s,t)$ is nonplanar. Therefore the linear factor P_{BP} must be an extraneous factor of the resultant $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})$. ■

Remark 3.11. Each extraneous factor associated to a bad base point in Theorem 3.10 may appear to some power. This power can be computed by taking the difference between the degree and the multiplicity of the base point (see [2] for similar results on rational triangular surfaces). Precisely,

$$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})=F(x,y,z,w)\prod_{BP}(P_{BP})^{e_{BP}-d_{BP}},$$

where d_{BP} is the degree and e_{BP} is the Hilbert–Samuel multiplicity of the base point BP; for further details, see [2].

Example 3.12. Here we give an example of Theorem 3.10. Consider the following rational tensor product surface with bidegree $(2,2)$:

$$\mathbf{P}(s,t)=\begin{pmatrix} 2s^2t^2+2s^2t-2st^2+27s^2-31st+2t^2 \\ s^2t^2+2st^2+24s^2-25st-2t^2 \\ -2s^2t^2-s^2t+st^2-24s^2+24st+2t^2 \\ -2s^2t+st^2+st \end{pmatrix}^T.$$

$\mathbf{P}(s,t)$ has two simple base points $(1,1)$ and $(2,3)$ and a double base point at $(0,0)$ that is not a local complete intersection. A parametrically strong μ-basis is

$$\begin{aligned}
\mathbf{p}&=(16\,s,-4\,s-17,14\,s-17,9\,s-17),\\
\mathbf{q}&=(t,4\,t+17,3\,t+17,-9\,t+17),\\
\mathbf{r}&=(\,st+88\,s,-18\,st-112\,s-80,-13\,s-80,54\,st-54\,t-80),
\end{aligned}$$

since

$$\mathrm{bideg}(\mathbf{p})+\mathrm{bideg}(\mathbf{q})+\mathrm{bideg}(\mathbf{r})=(1,0)+(0,1)+(1,1)=(2,2)=\mathrm{bideg}(\mathbf{P}).$$

However this μ-basis is not algebraically strong, since $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})=27(y+z+w)(32w^2-349wx+100wy-265wz-8x^2+123xy+122xz+42y^2+182yz+149z^2)$.

It is easy to check that $\mathrm{Rank}(\mathbf{p}(0,0),\mathbf{q}(0,0),\mathbf{r}(0,0))=1$. The base point $(0,0)$ that is not a local complete intersection generates the plane $\mathbf{p}(0,0)\cdot\mathbf{X}=(0,-17,-17,-17)\cdot(x,y,z,w)=-17(y+z+w)=0$, which is exactly the extraneous factor in the resultant up to a nonzero scalar.

Notice that the μ-basis $\mathbf{p}, \mathbf{q}, \mathbf{r}$ of $\mathbf{P}(s,t)$ is parametrically strong, but this μ-basis is not algebraically strong.

Remark 3.13. The base points are usually not easy to compute explicitly. Alternatively, we can find the extraneous factors associated with the base points by finding the base polynomials of a zero-dimensional variety. Let $m_i, i = 1, \ldots, 18$, be all of the 2×2 minors of the matrix $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$. Suppose the Gröbner basis of $S_{BP} = \{a, b, c, d, m_{i=1,\ldots,18}\}$ is $\{f(s), g(s,t)\}$. Then $\mathrm{Res}(\mathbf{p}(s,t) \cdot \mathbf{X}, f(s), g(s,t))$ are the real extraneous factors (possibly with powers) associated to the base points determined by $\{f(s), g(s,t)\}$. Some techniques for solving for this zero-dimensional variety can be found in [1, 9, 19].

Rational ruled surfaces always have an algebraically strong μ-basis. For a ruled surface $\mathbf{P}(s,t)$ with bidegree $(m,1)$ and implicit degree $D_{\mathbf{P}}$, there exists a μ-basis formed by three moving planes $\mathbf{p}(s), \mathbf{q}(s), \mathbf{r}(s,t)$ with bidegrees $(\mu, 0)$, $(D_{\mathbf{P}} - \mu, 0)$, $(\mu_1, 1)$, where $\mu \le [D_{\mathbf{P}}/2], \mu_1 < D_{\mathbf{P}} - \mu$ (see [7]).

Proposition 3.14. *The μ-basis $\{\mathbf{p}(s), \mathbf{q}(s), \mathbf{r}(s,t)\}$ of a ruled surface is an algebraically strong μ-basis.*

Proof. We already have three moving planes $\mathbf{p}, \mathbf{q}, \mathbf{r}$ such that $[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \kappa \mathbf{P}, \kappa \neq 0$. Moreover, by (2.9), $\deg(\mathrm{Res}(\mathbf{p}(s) \cdot \mathbf{X}, \mathbf{q}(s) \cdot \mathbf{X}, \mathbf{r}(s,t) \cdot \mathbf{X})) = 0 + \mu + (D_{\mathbf{P}} - \mu) = D_{\mathbf{P}}$ is the implicit degree, so $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is an algebraically strong μ-basis. ■

Note that $\mu + D_{\mathbf{P}} - \mu + \mu_1 = D_{\mathbf{P}} + \mu_1$ can often be greater than m (see the example at the end of section 3 of [7]), so an algebraically strong μ-basis for a rational ruled surface need not be a parametrically strong μ-basis.

4. Extraneous factors associated to infinity. A parametrically strong μ-basis is also algebraically strong if all the base points of $\mathbf{P}(s,t)$ are local complete intersections (see Theorem 3.4). However, there are extraneous factors associated to the base points that are not local complete intersections (see Theorem 3.10). In this section, we identify the other extraneous factors that appear in the resultant of a μ-basis that is not parametrically strong, i.e., when $\deg(\kappa(u,v)) > 0$ in (2.6). In this case, we have

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = \lambda u^i v^j \mathbf{P}((s,u),(t,v)), \quad i + j > 0. \tag{4.1}$$

Recall the system of equations

$$\begin{cases} \mathbf{p}((s,u),(t,v)) \cdot \mathbf{X} = 0, \\ \mathbf{q}((s,u),(t,v)) \cdot \mathbf{X} = 0, \\ \mathbf{r}((s,u),(t,v)) \cdot \mathbf{X} = 0. \end{cases} \tag{4.2}$$

We will investigate extraneous factors by considering the right-hand side of (4.1). If the right-hand side of (4.1) is not zero at $((s,u),(t,v))$, then (4.2) has a common root only at points $\mathbf{X}$ that lie on the surface defined by $\mathbf{P}(s,t)$ (see the similar discussion in the proof of Theorem 3.4). If the right-hand side of (4.1) is zero induced by $\mathbf{P}((s,u),(t,v)) = 0$, then we can identify the extraneous factor associated to the base point $((s,u),(t,v))$ in a manner similar to Theorem 3.10.

The final case occurs when the right-hand side of (4.1) is zero at $((s,u),(t,v))$, i.e., when $uv=0$ but $\mathbf{P}((s,u),(t,v))\neq 0$. In this case, just as with base points we may be able to find vectors simultaneously perpendicular to $\mathbf{p}((s,u),(t,v))$, $\mathbf{q}((s,u),(t,v))$, $\mathbf{r}((s,u),(t,v))$ other than multiples of $\mathbf{P}((s,u),(t,v))$, since $\mathbf{p}((s,u),(t,v))$, $\mathbf{q}((s,u),(t,v))$, $\mathbf{r}((s,u),(t,v))$ are linearly dependent at $uv=0$. We call the solutions of (4.2) in this case *candidate factors associated to infinity,* which may be extraneous factors. Notice that the $uv=0$ reduces to three possible cases: $(u=0, v\neq 0)$, $(v=0, u\neq 0)$, $(u=v=0)$ (see Example 4.4). The case $(u\neq 0, v=0)$ is analogous to the case $(u=0, v\neq 0)$, so we shall only treat one of these two cases.

Let EF = Extraneous Factor, and let $\mathbb{P}^3(\mathbb{R})$ be real homogeneous projective space. Suppose the moving plane $\mathbf{p}((s,u),(t,v))$ has bidegree $(\sigma_{11},\sigma_{12})$. Then $\mathbf{p}((s,u),(t,v))|_{u=v=0} = s^{\sigma_{11}}t^{\sigma_{12}}\mathbf{p}_0$, $\mathbf{p}_0\in\mathbb{P}^3(\mathbb{R})$. For convenience, we write $\mathbf{p}((s,u),(t,v))|_{u=v=0}\in\mathbb{P}^3(\mathbb{R})$, since we consider the vector in homogeneous projective space. Similarly, $\mathbf{p}((s,u),(t,v))|_{u=0,v\neq 0}\in\mathbb{P}^3(\mathbb{R})$ means that $\mathbf{p}((s,u),(t,v))|_{u=0,v\neq 0} = s^{\sigma_{11}}\alpha(t,v)\mathbf{p}_0, \mathbf{p}_0\in\mathbb{P}^3(\mathbb{R})$ and $\deg(\alpha(t,v)) = \sigma_{12}$. Without loss of generality, we assume that $\mathrm{Rank}(\mathbf{p}((s,u),(t,v)),\mathbf{q}((s,u),(t,v))) = 2$ if $\mathrm{Rank}(\mathbf{p}((s,u),(t,v)),\mathbf{q}((s,u),(t,v)),\mathbf{r}((s,u),(t,v))) = 2$ and $\mathrm{Rank}(\mathbf{p}((s,u),(t,v))) = 1$ if $\mathrm{Rank}(\mathbf{p}((s,u),(t,v)),\mathbf{q}((s,u),(t,v)),\mathbf{r}((s,u),(t,v))) = 1$.

Theorem 4.1. *Consider a rational tensor product surface $\mathbf{P}(s,t)$ with bidegree (m,n). If the μ-basis is not parametrically strong, then the possible extraneous factors associated to infinity are listed in Table 4.1, possibly raised to positive powers.*

Table 4.1
Extraneous factors associated to infinity.

$\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})$	$u=v=0$	$u=0, v\neq 0$
0	$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})\equiv 0$	$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})\equiv 0$
1	EF – a plane: $\mathbf{p}\vert_{u=v=0}\cdot\mathbf{X}$. See Example 4.4.	EF – a plane: $\mathbf{p}\vert_{u=0,v\neq 0}\cdot\mathbf{X}$ if $\mathbf{p}\vert_{u=0,v\neq 0}\in\mathbb{P}^3(\mathbb{R})$. See Example A.1.
		$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})\equiv 0$ if $\mathbf{p}\vert_{u=0,v\neq 0}\notin\mathbb{P}^3(\mathbb{R})$. See Example A.2.
2	No EF. See Example 4.3.	No EF if $\mathbf{p}\vert_{u=0,v\neq 0},\mathbf{q}\vert_{u=0,v\neq 0}\in\mathbb{P}^3(\mathbb{R})$. See Example 4.3.
		EF – a plane: $\mathbf{p}\vert_{u=0,v\neq 0}\cdot\mathbf{X}$ if $\mathbf{p}\vert_{u=0,v\neq 0}\in\mathbb{P}^3(\mathbb{R}),\mathbf{q}\vert_{u=0,v\neq 0}\notin\mathbb{P}^3(\mathbb{R})$. See Example 4.4.
		EF – a ruled surface: $\mathbf{p}\vert_{u=0,v\neq 0}\cdot\mathbf{X}=0\cap\mathbf{q}\vert_{u=0,v\neq 0}\cdot\mathbf{X}=0$ if $\mathbf{p}\vert_{u=0,v\neq 0}\notin\mathbb{P}^3(\mathbb{R}),\mathbf{q}\vert_{u=0,v\neq 0}\notin\mathbb{P}^3(\mathbb{R})$. See Example A.3.

Proof. We prove this theorem by examining each of the cases in Table 4.1.
Case I: The rank of the μ-basis is zero.
The results follow by Lemma 2.2.
Case II: The rank of the μ-basis is one.
- At $u=v=0$: The solution $\mathbf{p}|_{u=v=0}\cdot\mathbf{X}=0$ of (4.2) is then a plane. This plane cannot

lie in the surface $\mathbf{P}(s,t)$, since by assumption $\mathbf{P}(s,t)$ is nonplanar. Therefore the linear factor $\mathbf{p}|_{u=v=0}\cdot\mathbf{X}$ must be an extraneous factor of the resultant $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})$.

• At $u=0, v\neq 0$:

- If $\mathbf{p}|_{u=0,v\neq 0}$ defines a constant plane, then the solution $\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}=0$ of (4.2) is a plane. Therefore the linear factor $\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}$ must be an extraneous factor of the resultant. In Table 4.1, we express the condition that $\mathbf{p}|_{u=0,v\neq 0}$ defines a constant plane by writing $\mathbf{p}|_{u=0,v\neq 0}\in\mathbb{P}^3(\mathbb{R})$.

- If $\mathbf{p}|_{u=0,v\neq 0}$ defines a moving plane, then the solution $\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}=0$ of (4.2) is a moving plane with a homogeneous parameter (t,v). In this case, $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})$ must be identically zero, since this resultant vanishes on a three-dimensional subset given by this moving plane. In Table 4.1, we express the condition that $\mathbf{p}|_{u=0,v\neq 0}$ defines a moving plane by writing $\mathbf{p}|_{u=0,v\neq 0}\notin\mathbb{P}^3(\mathbb{R})$.

Case III: the rank of the μ-basis is two.

• At $u=v=0$: The solution $\mathbf{p}|_{u=v=0}\cdot\mathbf{X}=0\cap\mathbf{q}|_{u=v=0}\cdot\mathbf{X}=0$ of (4.2) is a line. The line has dimension 1, so this line must belong to a surface (the surface $\mathcal{P}$ or some surface from another extraneous factor). Thus this line does not generate a new extraneous factor.

• At $u=0, v\neq 0$:

- If $\mathbf{p}|_{u=0,v\neq 0}$, $\mathbf{q}|_{u=0,v\neq 0}$ define two constant planes, then the solution $\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}=0\cap\mathbf{q}|_{u=0,v\neq 0}\cdot\mathbf{X}=0$ of (4.2) is again a line. So once again, this line will not generate a new extraneous factor.

- If $\mathbf{p}|_{u=0,v\neq 0}$ defines a constant plane, while $\mathbf{q}|_{u=0,v\neq 0}$ defines a moving plane, then the solution $\{\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}=0\cap\mathbf{q}|_{u=0,v\neq 0}\cdot\mathbf{X}=0\}=\{\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}=0\}$ of (4.2) is a plane. In this case, the linear factor $\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}$ must be an extraneous factor of the resultant.

- If $\mathbf{p}|_{u=0,v\neq 0}$, $\mathbf{q}|_{u=0,v\neq 0}$ define two moving planes, then the solution $\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}=0\cap\mathbf{q}|_{u=0,v\neq 0}\cdot\mathbf{X}=0$ of (4.2) is a ruled surface, since the surface is generated by two moving planes with the same parameter (v,t) [22]. The parametrization of this ruled surface can be solved directly, and its implicit equation can be computed quickly using a univariate resultant [23]. This ruled surface is either an extraneous factor of the resultant or exactly the surface $\mathcal{P}$. ■

Corollary 4.2. *Consider a surface* $\mathbf{P}(s,t)$ *with bidegree* (m,n)*. The extraneous factors of the resultant of a* μ*-basis for* $\mathbf{P}(s,t)$ *are ruled surfaces.*

Proof. By Theorem 3.10 the extraneous factors associated to base points are planes. By Theorem 4.1 the extraneous factors associated to infinities are planes and ruled surfaces. ■

Now we give two examples to illustrate some cases in Theorem 4.1.

Example 4.3. (continuation of Example 3.9). The μ-basis of $\mathbf{P}(s,t)$ is not parametrically strong. By direct computation, we find that $\kappa(u,v)=(1505/9)u$. Dehomogenizing at the infinity corresponding to $u=0$, the μ-basis reduces to

$$\begin{aligned}\mathbf{p}((u,s),(v,t))|_{u=0}&=\left(\tfrac{215}{59},1,2,-\tfrac{215}{59}\right),\\ \mathbf{q}((u,s),(v,t))|_{u=0}&=\left(-t,\tfrac{295v}{36},\tfrac{295v}{18},t\right),\\ \mathbf{r}((u,s),(v,t))|_{u=0}&=\left(0,-\tfrac{7}{3},-\tfrac{14}{3},0\right).\end{aligned}$$

There are two constant planes and $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u=0}=2$. Hence the intersection of the elements of the μ-basis at $u = 0$ is the line

$$\mathbf{X}_u = (\alpha, -2\beta, \beta, \alpha).$$

By Theorem 4.1, this line cannot generate a new extraneous factor.

Actually this μ-basis is algebraically strong, since the implicit equation is $F(x, y, z, w) \equiv \mathrm{Res}(\mathbf{p} \cdot \mathbf{X}, \mathbf{q} \cdot \mathbf{X}, \mathbf{r} \cdot \mathbf{X}) = 0$ without any extraneous factor. We can check that the extraneous vector $\mathbf{X}_u$ lies on the algebraic surface corresponding to $\mathbf{P}(s, t)$, i.e., $F(\mathbf{X}_u) = 0$. We omit the long expression for $F(x, y, z, w)$ since this expression is the sum of 35 monomials.

Example 4.4. For the same surface as in Example 3.8, there exists a μ-basis formed by a triple of moving planes that follow this surface with bidegrees $(1, 1), (1, 1), (1, 1)$,

$$\begin{aligned}\mathbf{p} = (&-\tfrac{7849\,st}{413} + \tfrac{9759223}{23541} - \tfrac{6283208\,t}{23541} - \tfrac{167\,s}{329}, \tfrac{421292427}{737618} + \tfrac{2579\,s}{329} + \tfrac{3507\,t}{59}, \tfrac{7216783}{210748} + \tfrac{260532\,s}{19411},\\ &-\tfrac{1364912243}{2212854} - \tfrac{1284719\,s}{19411} + \tfrac{9767830\,t}{23541}),\end{aligned}$$

$$\begin{aligned}\mathbf{q} = (&-\tfrac{7849\,st}{413} + \tfrac{10110686}{23541} - \tfrac{6533486\,t}{23541}, \tfrac{9352323}{15694} + 4\,s + \tfrac{3625\,t}{59}, \tfrac{1129705}{31388} + \tfrac{9501\,s}{826} + t,\\ &-\tfrac{30316379}{47082} - \tfrac{28232\,s}{413} + \tfrac{10127140\,t}{23541}),\end{aligned}$$

$$\begin{aligned}\mathbf{r} = (&-\tfrac{7849\,st}{1652} + \tfrac{4867841}{47082} - \tfrac{1570802\,t}{23541}, \tfrac{9000447}{62776} + s + \tfrac{3507\,t}{236}, \tfrac{1058669}{125552} + \tfrac{9501\,s}{3304},\\ &-\tfrac{29118679}{188328} - \tfrac{7058\,s}{413} + \tfrac{4883915\,t}{47082}).\end{aligned}$$

We can check that $[\mathbf{p}(s, t), \mathbf{q}(s, t), \mathbf{r}(s, t)] = -\frac{14157425}{109858} \cdot \mathbf{P}(s, t)$, so $\mathbf{p}, \mathbf{q}, \mathbf{r}$ form an affine μ-basis. But with homogeneous parameters, we have

$$[\mathbf{p}((s, u), (t, v)), \mathbf{q}((s, u), (t, v)), \mathbf{r}((s, u), (t, v))] = -\frac{14157425}{109858} \cdot u \cdot v \cdot \mathbf{P}((s, u), (v, t)).$$

Consider the candidate factor associated to $u = v = 0$. We find $\mathrm{Rank}(\mathbf{p}, \mathbf{q}, \mathbf{r})|_{u=v=0} = 1$, and the intersection of the elements of the μ-basis at $u = v = 0$ is $-\frac{7849}{1652}x = 0$. Therefore by Theorem 4.1, x must be an extraneous factor of the resultant of the μ-basis.

In a similar manner, we deal with the cases $(u = 0, v \neq 0)$ and $(u \neq 0, v = 0)$. In these cases we find two additional extraneous factors. We find that $\mathrm{Rank}(\mathbf{p}, \mathbf{q}, \mathbf{r})|_{u=0,v\neq 0} = 2$, and there is a constant plane at $(u = 0, v \neq 0)$: $2526\,y + 1430\,w - 334\,x + 1263\,z$, which must be an extraneous factor by Theorem 4.1. We also find $\mathrm{Rank}(\mathbf{p}, \mathbf{q}, \mathbf{r})|_{u\neq 0,v=0} = 2$ and again get a constant plane at $(u \neq 0, v = 0)$ as an extraneous factor: $38\,y + 290\,w - 202\,x + 19\,z$.

Finally, computing the resultant of $\mathbf{p} \cdot \mathbf{X}, \mathbf{q} \cdot \mathbf{X}, \mathbf{r} \cdot \mathbf{X}$ we get

$$\begin{aligned}\mathrm{Res}(\mathbf{p} \cdot \mathbf{X}, \mathbf{q} \cdot \mathbf{X}, \mathbf{r} \cdot \mathbf{X}) = x(2526\,y + 1430\,w - 334\,x + 1263\,z)(38\,y + 290\,w - 202\,x\\ +19\,z)F(x, y, z, w),\end{aligned}$$

where $F(x, y, z, w) = 0$ is the implicit equation of $\mathbf{P}(s, t)$ with degree three. Thus the resultant has exactly the extraneous factors identified by Theorem 4.1.

Examples 3.8 and 4.4 together show that the same rational tensor product surface can have two different μ-bases with different bidegrees: one parametrically and algebraically strong and the other neither parametrically nor algebraically strong! Thus the term *strong* depends not only on the surface parametrization but also on the choice of the μ-basis. Although we can identify the extraneous factors when we try to find the implicit equation by the resultant of any μ-basis for a rational parametrization $\mathbf{P}(s,t)$, we prefer to use μ-bases with lower degrees.

Remark 4.5. Extraneous factors due to anomalies at infinity may appear to some power in the resultant. We defer to the appendix examples to illustrate the existence of these powers as well as further examples corresponding to some additional cases in Table 4.1, since these examples have rather long expressions.

5. Conclusion.

We have presented a sufficient condition which guarantees that the resultant of a μ-basis for a rational tensor product surface generates the implicit equation of the surface with no extraneous factors. In this case, we have derived formulas for both the implicit degree and the number of base points counting multiplicity of the rational surface based only on the bidegree of the rational parametrization and the bidegrees of the elements of the μ-basis, provided that all of the base points are local complete intersections. The parametrization is assumed proper to simplify the presentation. With only minor modifications, our results remain valid when the parametrization is not proper. For improper parametrizations, we need only replace $F(x,y,z,w)$ with $F(x,y,z,w)^{\deg(\mathfrak{m})}$ in our implicitization formulas, where $\deg(\mathfrak{m})$ is the mapping degree of $\mathfrak{m} : (s,t) \mapsto \mathbf{P}(s,t)$. We have concentrated on rational tensor product surfaces, but similar results hold for rational triangular surfaces—that is, rational surfaces of fixed total degree. The proofs are much the same. Further discussion regarding the effects of base points of rational triangular surfaces can be found in [2].

As we have seen in the introduction, μ-bases for rational surfaces have many properties that are qualitatively different from the characteristic properties of μ-bases for rational curves. The analogous notion of μ-bases for rational surfaces seems to be strong μ-bases. Like μ-bases for rational curves, strong μ-bases for rational surfaces are bases for the syzygy module with respect to homogeneous parametrizations; in addition, the degrees of their elements sum to the degree of the parametrization, and their resultant generates the implicit equation of the surface with no extraneous factors.

For parametrizations that have no strong μ-bases, we identify all the extraneous factors coming from bad base points or anomalies at infinity. Therefore we can implicitize a rational parametrization by computing the resultant of a μ-basis for the surface and removing all of the extraneous factors. The power of an extraneous factor associated to a bad base point can be determined by analyzing the degree and the multiplicity of this base point. For the power of an extraneous factor associated to an infinity, we propose the following conjecture based on computational experiments.

CONJECTURE. *Consider a rational tensor product surface* $\mathbf{P}(s,t)$ *with* μ*-basis* $\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)$. *Suppose that in* (2.6) $\kappa(u,v) = \lambda u^i v^j$ *or, equivalently, that*

$$(i,j) = \mathrm{bideg}(\mathbf{p}) + \mathrm{bideg}(\mathbf{q}) + \mathrm{bideg}(\mathbf{r}) - \mathrm{bideg}(\mathbf{P}).$$

Then the power of a planar extraneous factor associated to an infinity is

1. *greater than or equal to* i *for* $u = 0, v \neq 0$;
2. *greater than or equal to* j *for* $u \neq 0, v = 0$;
3. *greater than or equal to* $\max(i,j)$ *for* $u = v = 0$.

Fast algorithms for computing μ-bases for rational planar curves are available based on Gaussian elimination [6], but current algorithms for computing μ-bases for general rational tensor product surfaces are neither simple nor fast [15]. In the future, we plan to take advantage of the results in this paper to develop fast algorithms to compute strong μ-bases for rational tensor product surfaces of moderate bidegree based only on solving a simple system of linear equations.

Finally, we have seen that there are degenerate cases (see Table 4.1 and Example A.2) where the resultant of a μ-basis is identically zero. Nevertheless, we hope to develop algorithms for finding the implicit equation from the resultant matrix by either computing a perturbation or examining maximal minors.

Appendix A. More examples.

Here we give additional examples to illustrate more cases given in Theorem 4.1.

Example A.1. Consider the following rational tensor product surface with bidegree $(1, 2)$:

$$\mathbf{P}(s,t) = \begin{pmatrix} -15\,st^2 - t^2 - 12\,s + 18\,t + 6 \\ -6\,st^2 + 3\,st - 6\,t^2 - 6\,s + 18\,t + 12 \\ -st^2 + 2\,st - 15\,t^2 - 4\,t \\ 2\,st^2 + 2\,t^2 + 9\,t \end{pmatrix}^T .$$

We shall see that this parametrization actually defines a quadric surface. There exists a μ-basis $\mathbf{p}, \mathbf{q}, \mathbf{r}$ with bidegrees $(1, 0)$, $(1, 1)$, $(1, 1)$,

$$\begin{aligned} &\mathbf{p} = (s - 2, -2\,s + 1, 3\,s, 3\,s + 2), \\ &\mathbf{q} = (s - 2 + t, -2\,s + 1 - 2\,t, 3\,s + t, 3\,s + 4 + 2\,t), \\ &\mathbf{r} = (st + s - 2\,t - 2, -2\,st - 2\,s + 1, 3\,st + 3\,s - 3, 3\,st + 3\,s - t + 2). \end{aligned}$$

The outer product of this μ-basis with homogeneous parameters is

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = u \cdot \mathbf{P}.$$

Since $\mathrm{Rank}(\mathbf{p}, \mathbf{q}, \mathbf{r})|_{u=0, v\neq 0} = 1$ and $\mathbf{p}|_{u=0,v\neq 0} \cdot \mathbf{X} = (x - 2\,y + 3\,z + 3\,w)$ defines a constant plane, by Theorem 4.1 this plane is an extraneous factor.

We compute and find that

$$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}) = -\left(6\,w^2 + 2\,wy - 6\,wz - 3\,xz + 6\,yz - 3\,z^2\right)(x - 2\,y + 3\,z + 3\,w)^2$$

, which has only one extraneous factor $\mathbf{p}|_{u=0,v\neq 0} \cdot \mathbf{X}$ but with power two.

Example A.2. Using the same surface as in Example 3.8 (continued), there exists a μ-basis formed by a triple of moving planes that follow this surface with bidegrees $(2, 1), (1, 1)$, $(1, 1)$,

$$\begin{aligned} \mathbf{p} = (&\tfrac{227665285\,s}{2212854} + 1 - \tfrac{7849\,s^2t}{1652} - \tfrac{1570802\,st}{23541}, \tfrac{434347593\,s}{2950472} - \tfrac{771}{329} + s^2 + \tfrac{3507\,st}{236}, \tfrac{61084027\,s}{5900944} + \tfrac{339}{658} \\ &+ \tfrac{9501\,s^2}{3304}, -\tfrac{192763079\,s}{1264488} + \tfrac{545}{329} - \tfrac{7058\,s^2}{413} + \tfrac{4883915\,st}{47082}), \end{aligned}$$

$$\begin{aligned}\mathbf{q} = (&-\tfrac{7849\,st}{413} + \tfrac{10110686}{23541} - \tfrac{6533486\,t}{23541}, \tfrac{9352323}{15694} + 4\,s + \tfrac{3625\,t}{59}, \tfrac{1129705}{31388} + \tfrac{9501\,s}{826} + t, -\tfrac{30316379}{47082}\\ &- \tfrac{28232\,s}{413} + \tfrac{10127140\,t}{23541}),\end{aligned}$$

$$\begin{aligned}\mathbf{r} = (&-\tfrac{7849\,st}{1652} + \tfrac{4867841}{47082} - \tfrac{1570802\,t}{23541}, \tfrac{9000447}{62776} + s + \tfrac{3507\,t}{236}, \tfrac{1058669}{125552} + \tfrac{9501\,s}{3304}, -\tfrac{29118679}{188328} - \tfrac{7058\,s}{413}\\ &+ \tfrac{4883915\,t}{47082}).\end{aligned}$$

Note that $\mathbf{p}$ is updated compared to Example 4.4, and we can check that

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = -\tfrac{14157425}{109858} \cdot u^2 \cdot v \cdot \mathbf{P}((s,u),(v,t)).$$

Consider the case $(u = 0, v \neq 0)$. Then $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u=0,v\neq 0} = 1$ and $\mathbf{p}|_{u=0,v\neq 0} = (-\frac{7849\,t}{1652}, v, \frac{9501\,v}{3304}, -\frac{7058\,v}{413})$, which is not a constant plane. Therefore $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X})$ must be identically zero by Theorem 4.1, and one can check that this is indeed the case.

Example A.3. Consider the following rational tensor product surface with bidegree $(2,2)$:

$$\mathbf{P}(s,t) = \begin{pmatrix} s^2t^2 - st^2 + s^2 + 3\,st - t - 1 \\ -s^2t^2 - 2\,s^2t + 2\,s^2 - 6\,st - 2\,t^2 - 3\,s - t + 1 \\ 2\,s^2t^2 - s^2t - st^2 + 4\,st - t^2 - s + 1 \\ -s^2t - s^2 + 4\,st + 2\,t^2 + s + 2\,t \end{pmatrix}^T.$$

There exists a μ-basis $\mathbf{p},\mathbf{q},\mathbf{r}$ with bidegrees $(2,1),(2,1),(3,1)$,

$$\begin{aligned}\mathbf{p} = (&-\tfrac{112659099585841}{147112579955996} + \tfrac{6391715757435\,s}{73556289977998} - \tfrac{910311484805789003\,t}{2860604117244434222} + \tfrac{2940859272149877\,st}{2624407447013158} - \tfrac{2175535271480337\,s^2t}{2624407447013158}\\ &+ \tfrac{36974491989765\,s^2}{73556289977998}, \tfrac{241726141275593\,s^2t}{2624407447013158} - \tfrac{59427628533091\,s^2}{147112579955996} - \tfrac{1615968228153149\,st}{2624407447013158} + \tfrac{203620929149845\,s}{147112579955996}\\ &- \tfrac{4035248230815347\,t}{2624407447013158} - \tfrac{109016083489817}{147112579955996}, \tfrac{10128305880071\,s^2}{147112579955996} - \tfrac{232399828586567\,s}{147112579955996} - \tfrac{2761866032702699\,st}{2624407447013158}\\ &+ \tfrac{1208630706377965\,s^2t}{2624407447013158} - \tfrac{910754024006}{36778144988999} + \tfrac{1022502795710357\,t}{1312203723506579}, -\tfrac{289128935613883}{147112579955996} + \tfrac{271636798223555\,s}{73556289977998}\\ &- \tfrac{1506372717552495\,t}{1312203723506579} - \tfrac{670365173321455925\,st}{2860604117244434222} + \tfrac{228605517610220389\,s^2t}{572120823448868444} - \tfrac{11226568271663\,s^2}{36778144988999}),\end{aligned}$$

$$\begin{aligned}\mathbf{q} = (&\tfrac{306560287480}{132615890667} + \tfrac{158217080885\,s^2}{44205296889} - \tfrac{163718239418\,s}{44205296889} - \tfrac{365709496322\,t}{132615890667}, \tfrac{591962272910}{132615890667} + \tfrac{539019499336\,s}{132615890667}\\ &- \tfrac{94036898\,s^2}{109690563}, -\tfrac{285401985430}{132615890667} - \tfrac{202743552088\,s^2}{132615890667} + \tfrac{193904083676\,s}{132615890667}, \tfrac{266406532034}{132615890667} + \tfrac{182854748161\,s^2t}{132615890667}\\ &+ \tfrac{82423341097\,s^2}{44205296889} - \tfrac{182854748161\,st}{132615890667} + \tfrac{459323228281\,s}{44205296889}),\end{aligned}$$

$$\begin{aligned}\mathbf{r} = (&\tfrac{1520990193969753041}{406405640657522844} + \tfrac{993760889520400979\,s^2}{169335683607301185} - \tfrac{4253178067574892923\,s}{677342734429204740} - \tfrac{365709496322\,t}{132615890667} - \tfrac{13155129\,s^3}{15322660},\\ &\tfrac{1352880557740585821}{2032028203287614220} + \tfrac{1677118875051875986\,s}{508007050821903555} - \tfrac{680889077509976\,s^2}{420187800514395} + \tfrac{1461681\,s^3}{15322660}, -\tfrac{1480963651889273404}{508007050821903555}\\ &- \tfrac{3297864407092638083\,s^2}{1016014101643807110} + \tfrac{7429495283557616381\,s}{2032028203287614220} + \tfrac{1461681\,s^3}{3064532}, \tfrac{776505713817068375 3}{2032028203287614220} + \tfrac{182854748161\,s^2t}{132615890667}\\ &+ \tfrac{8106465337592851 57\,s^2}{677342734429204740} - \tfrac{182854748161\,st}{132615890667} + \tfrac{7490351954986220323\,s}{677342734429204740} - \tfrac{10231767\,s^3}{15322660}).\end{aligned}$$

The outer product of this μ-basis with homogeneous parameters is

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = \tfrac{35940983000614765385429}{48607977963178534308140} \cdot u^5 \cdot v \cdot \mathbf{P}.$$

By Theorem 4.1 we can analyze the extraneous factors in accordance with Table 4.1.

1. $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u=v=0}=2$; $\mathbf{p}|_{u=v=0}$ and $\mathbf{q}|_{u=v=0}$ are two constant vectors; hence this case does not induce an extraneous factor.

2. $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u=0,v\neq 0}=2$; $\mathbf{q}|_{u=0,v\neq 0}$ defines a moving plane and $\mathbf{r}|_{u=0,v\neq 0}$ defines a constant plane. Therefore $\mathbf{r}|_{u=0,v\neq 0}\cdot\mathbf{X}=7w+9x-y-5z$ is an extraneous factor.

3. $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u\neq 0,v=0}=2$; $\mathbf{p}|_{u\neq 0,v=0}$ and $\mathbf{q}|_{u\neq 0,v=0}$ define two moving planes. Therefore these two moving planes generate a ruled surface as an extraneous factor. This ruled surface is

$$\begin{aligned}
&\mathrm{Res}_{(s,u)}(\mathbf{p}|_{u\neq0,v=0}\cdot\mathbf{X},\mathbf{q}|_{u\neq0,v=0}\cdot\mathbf{X}) = \tfrac{4121987986422976887265729231681}{98627420059871087137950666299303081917128}(161458094790\,w^4\\
&+100615556757\,w^3\,x+323894469147\,w^3\,y-119380596594\,w^3\,z+649631883482\,w^2\,x^2\\
&+148992698510\,w^2\,x\,y-481429080698\,w^2\,x\,z+144170492858\,w^2\,y^2-86169892276\,w^2\,y\,z\\
&+6642117360\,w^2\,z^2+329726325024\,w\,x^3+261705836744\,w\,x^2\,y-228975148088\,w\,x^2\,z\\
&-55103668136\,w\,x\,y^2-199077858416\,w\,x\,y\,z+8632081704\,w\,x\,z^2+125038002888\,x^4\\
&-27786222864\,x^3\,y-138931114320\,x^3\,z+1543679048\,x^2\,y^2+15436790480\,x^2\,y\,z\\
&+38591976200\,x^2\,z^2)=E(x,y,z,w)_{u\neq0,v=0}.
\end{aligned}$$

Finally, we compute and find that (up to a nonzero scalar)

$$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X})=(7w+9x-y-5z)^5E(x,y,z,w)_{u\neq0,v=0}F(x,y,z,w),$$

which contains the two extraneous factors we have identified.

Example A.4. Here we give an example where the ruled surface associated to infinity is exactly the given surface $\mathcal{P}$. Consider the following rational tensor product surface with bidegree $(1,4)$:

$$\mathbf{P}(s,t)=\begin{pmatrix} -2\,st^3+4\,t^2\\ -t^4s-2\,st^3+t^4+2\,st^2+2\,t^3+2\,st-t^2-4\,t\\ t^4s+st^3-t^4+st^2+2\,t^3+2\,st-5\,t^2-6\,t\\ 2\,t^4s-t^4-st^2-3\,t^3-st+3\,t^2+t\end{pmatrix}^T.$$

This parametrization $\mathbf{P}(s,t)$ defines a ruled surface and has five simple base points $(0,0)$, $(-1,0),(1,2),(2,1),(-2,\ -1)$. By Theorem 3.10 there are no extraneous factors associated to these base points. Therefore this parametrization defines a surface with algebraic degree $8-5=3$. In fact, this surface has implicit equation

$$\begin{aligned}
F(x,y,z,w)&=48\,xw^2+32\,yw^2+32\,zw^2-183\,wx^2+42\,xwy-142\,xwz+56\,wy^2\\
&+32\,ywz-24\,wz^2+167\,x^3-174\,x^2y+158\,x^2z+6\,xy^2-114\,xyz+48\,xz^2\\
&+24\,y^3+4\,y^2z-16\,yz^2+4\,z^3=0.
\end{aligned}$$

There exists an algebraically strong μ-basis $\mathbf{p}_a,\mathbf{q}_a,\mathbf{r}_a$ for this ruled surface with bidegrees $(0,1)$, $(0,2)$, $(1,1)$. To find a μ-basis having extraneous factors, we construct a new μ-basis

$\mathbf{p} = \mathbf{p}_a, \mathbf{q} = \mathbf{r}_a + s^2\mathbf{q}_a, \mathbf{r} = \mathbf{q}_a + s\mathbf{p}_a$ with bidegrees $(0,1), (2,2), (1,2)$. Explicitly,

$$\begin{aligned}
\mathbf{p} &= (-5/2\,t + 23/2, t-8, t+4, -8),\\
\mathbf{q} &= (s^2t^2 - 3\,s^2t + 6\,s^2 - 28\,st + 72\,s + t - 11, 2\,s^2t - 4\,s^2 + 16\,st - 48\,s - 10\,t, 2\,s^2 + 24\,s\\
&\quad - 2\,t, 2\,s^2t - 4\,s^2 + 8\,st - 48\,s - 8\,t),\\
\mathbf{r} &= (t^2 - 3\,t + 6 - 5/2\,st + 23/2\,s, st - 8\,s + 2\,t - 4, st + 4\,s + 2, -4 + 2\,t - 8\,s).
\end{aligned}$$

Moreover,

$$[\mathbf{p}((s,u),(t,v)), \mathbf{q}((s,u),(t,v)), \mathbf{r}((s,u),(t,v))] = -8 \cdot u^2 \cdot v \cdot \mathbf{P}((s,u),(v,t)).$$

Now by Theorem 4.1 we can use Table 4.1 to analyze the extraneous factors of $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X})$.

1. $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u=v=0} = 2$; $\mathbf{p}|_{u=v=0}$ and $\mathbf{q}|_{u=v=0}$ are two constant vectors; hence this case does not induce an extraneous factor.

2. $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u\neq 0,v=0} = 2$; $\mathbf{p}|_{v=0}$ and $\mathbf{q}|_{v=0}$ are two constant vectors; hence this case does not induce an extraneous factor either.

3. $\mathrm{Rank}(\mathbf{p},\mathbf{q},\mathbf{r})|_{u=0,v\neq 0} = 2$; $\mathbf{p}|_{u=0}$ and $\mathbf{q}|_{u=0}$ define two moving planes. The corresponding ruled surface is

$$\begin{aligned}
\mathrm{Res}_{(s,u)}(\mathbf{p}|_{u=0,v\neq 0}\cdot\mathbf{X}, \mathbf{q}|_{u=0,v\neq 0}\cdot\mathbf{X}) &= \tfrac{1}{2}(48\,xw^2 + 32\,yw^2 + 32\,zw^2 - 183\,wx^2\\
&\quad + 42\,xwy - 142\,xwz + 56\,wy^2 + 32\,ywz - 24\,wz^2 + 167\,x^3 - 174\,x^2y + 158\,x^2z\\
&\quad + 6\,xy^2 - 114\,xyz + 48\,xz^2 + 24\,y^3 + 4\,y^2z - 16\,yz^2 + 4\,z^3),
\end{aligned}$$

which is exactly equal to $F(x,y,z,w)$. Computing the resultant, we find that $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}) = \frac{1}{4}F(x,y,z,w)^2$.

By (2.9) the resultant degree of polynomials with bidegrees $(0,1)$, $(2,2)$, $(1,2)$ should be 9. Notice that formula (2.9) holds for polynomials having dense Newton polygons with generic coefficients. However, the polynomials $\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}$ are not generic but are specially constructed and so fail to satisfy formula (2.9). Actually, the true resultant can be computed by $\mathrm{Res}(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}) = \mathrm{Res}_s(\mathrm{Res}_t(\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}), \mathrm{Res}_t(\mathbf{p}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}))$, whose degree is 6.

Example A.5. We give one more example to show the powers of the extraneous factors. For the same surface as in Example 3.8, there exists a μ-basis formed by a triple of moving planes that follow this surface with bidegrees $(1,1), (2,1)$, $(1,1)$,

$$\begin{aligned}
\mathbf{p} &= \Big(-\tfrac{7849\,st}{413} + \tfrac{9759223}{23541} - \tfrac{6283208\,t}{23541} - \tfrac{167\,s}{329}, \tfrac{421292427}{737618} + \tfrac{2579\,s}{329} + \tfrac{3507\,t}{59}, \tfrac{7216783}{210748} + \tfrac{260532\,s}{19411}, -\tfrac{1364912243}{2212854}\\
&\quad - \tfrac{1284719\,s}{19411} + \tfrac{9767830\,t}{23541}\Big),
\end{aligned}$$

$$\begin{aligned}
\mathbf{q} &= \Big(-\tfrac{202\,t}{19} + \tfrac{908}{57} - \tfrac{7849\,s^2t}{826} + \tfrac{4867841\,s}{23541} - \tfrac{3141604\,st}{23541}, 2\,t + \tfrac{426}{19} + \tfrac{9000447\,s}{31388} + 2\,s^2 + \tfrac{3507\,st}{118},\\
&\quad t + \tfrac{43}{19} + \tfrac{1058669\,s}{62776} + \tfrac{9501\,s^2}{1652}, \tfrac{290\,t}{19} - \tfrac{1450}{57} - \tfrac{29118679\,s}{94164} - \tfrac{14116\,s^2}{413} + \tfrac{4883915\,st}{23541}\Big),\\
\mathbf{r} &= \Big(-\tfrac{7849\,st}{1652} + \tfrac{4867841}{47082} - \tfrac{1570802\,t}{23541}, \tfrac{9000447}{62776} + s + \tfrac{3507\,t}{236}, \tfrac{1058669}{125552} + \tfrac{9501\,s}{3304}, -\tfrac{29118679}{188328} - \tfrac{7058\,s}{413} + \tfrac{4883915\,t}{47082}\Big).
\end{aligned}$$

Note that $\mathbf{q}$ is updated compared to Example 4.4, and we can check that

$$[\mathbf{p}((s,u),(t,v)),\mathbf{q}((s,u),(t,v)),\mathbf{r}((s,u),(t,v))] = -\tfrac{14157425}{109858}\cdot u^2\cdot v\cdot \mathbf{P}((s,u),(v,t)).$$

Consider the candidate factors associated to $u = v = 0$, $(u = 0, v \neq 0)$, and $(u \neq 0, v = 0)$. Similar to Example 4.4, we find three extraneous factors x, $2526\,y + 1430\,w - 334\,x + 1263\,z$, and $38\,y + 290\,w - 202\,x + 19\,z$.

Finally, computing the resultant of $\mathbf{p}\cdot\mathbf{X}, \mathbf{q}\cdot\mathbf{X}, \mathbf{r}\cdot\mathbf{X}$ we get

$$\mathrm{Res}(\mathbf{p}\cdot\mathbf{X},\mathbf{q}\cdot\mathbf{X},\mathbf{r}\cdot\mathbf{X}) = x^2(2526\,y+1430\,w-334\,x+1263\,z)^2(38\,y+290\,w-202\,x+19\,z)F(x,y,z,w),$$

where $F(x, y, z, w) = 0$ is the implicit equation of $\mathbf{P}(s,t)$ with degree three. Notice that some extraneous factors have powers greater than one.

Acknowledgments. The authors thank Professor David Cox, Professor Laurent Busé, and Professor Falai Chen for their generous assistance, which helped us to understand the important role of base points in the analysis of the resultants corresponding to μ-bases.

REFERENCES

[1] C. Brand and M. Sagraloff, *On the complexity of solving zero-dimensional polynomial systems via projection*, in Proceedings of the ACM International Symposium on Symbolic and Algebraic Computation (ISSAC '16), ACM, New York, 2016, pp. 151–158, https://doi.org/10.1145/2930889.2930934.

[2] L. Busé, M. Chardin, and J.-P. Jouanolou, *Torsion of the symmetric algebra and implicitization*, Proc. Amer. Math. Soc., 137 (2009), pp. 1855–1865, https://doi.org/10.1090/S0002-9939-09-09550-1.

[3] L. Busé, D. A. Cox, and C. D'Andrea, *Implicitization of surfaces in $\mathbb{P}^3$ in the presence of base points*, J/ Algebra Appl., 2 (2003), pp. 189–214, https://doi.org/10.1142/S0219498803000489.

[4] F. Chen, D. A. Cox, and Y. Liu, *The μ-basis and implicitization of a rational parametric surface*, J. Symbolic Comput., 39 (2005), pp. 689–706, https://doi.org/10.1016/j.jsc.2005.01.003.

[5] F. Chen, L. Shen, and J. Deng, *Implicitization and parametrization of quadratic and cubic surfaces by μ-bases*, Computing, 79 (2007), pp. 131–142, https://doi.org/10.1007/s00607-006-0192-0.

[6] F. Chen and W. Wang, *The μ-basis of a planar rational curve—properties and computation*, Graph. Models, 64 (2002), pp. 368–381, https://doi.org/10.1016/S1077-3169(02)00017-5.

[7] F. Chen and W. Wang, *Revisiting the μ-basis of a rational ruled surface*, J. Symbolic Comput., 36 (2003), pp. 699–716, https://doi.org/10.1016/S0747-7171(03)00064-6.

[8] F. Chen, J. Zheng, and T. W. Sederberg, *The μ-basis of a rational ruled surface*, Comput. Aided Geom. Design, 18 (2001), pp. 61–72, https://doi.org/10.1016/S0167-8396(01)00012-7.

[9] J.-S. Cheng, X.-S. Gao, and L. Guo, *Root isolation of zero-dimensional polynomial systems with linear univariate representation*, J. Symbolic Comput., 47 (2012), pp. 843–858, https://doi.org/10.1016/j.jsc.2011.12.011.

[10] D. A. Cox, *Equations of parametric curves and surfaces via syzygies*, in Symbolic Computation: Solving Equations in Algebra, Geometry, and Engineering, Contemp. Math. 286, AMS, Providence, RI, 2001, pp. 1–20, https://doi.org/10.1090/conm/286.

[11] D. A. Cox, *Curves, surfaces and syzygies*, in Topics in Algebraic Geometry and Geometric Modeling, Contemp. Math. 334, AMS, Providence, RI, 2004, pp. 131–150, https://doi.org/10.1090/conm/334.

[12] D. A. Cox, J. Little, and D. O'shea, *Using Algebraic Geometry*, 2nd ed., Grad. Texts Math. 185, Springer New York, 2005, https://doi.org/10.1007/b138611.

[13] D. A. Cox, J. B. Little, and D. O'Shea, *Using Algebraic Geometry*, Grad. Texts Math. 185, Springer-Verlag, New York, 1998, https://doi.org/10.1007/978-1-4757-6911-1.

[14] D. A. Cox, T. W. Sederberg, and F. Chen, *The moving line ideal basis of planar rational curves*, Comput. Aided Geom. Design, 15 (1998), pp. 803–827, https://doi.org/10.1016/S0167-8396(98)00014-4.

[15] J. Deng, F. Chen, and L. Shen, *Computing μ-bases of rational curves and surfaces using polynomial matrix factorization*, in Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation, (ISSAC '05), ACM New York, 2005, pp. 132–139, https://doi.org/10.1145/1073884.1073904.

[16] A. Dickenstein and I. Z. Emiris, *Multihomogeneous resultant formulae by means of complexes*, J. Symbolic Comput., 36 (2003), pp. 317–42, https://doi.org/10.1016/S0747-7171(03)00086-5.

[17] I. Z. Emiris and A. Mantzaflaris, *Multihomogeneous resultant formulae for systems with scaled support*, J. Symbolic Comput., 47 (2012), pp. 820–842, https://doi.org/10.1016/j.jsc.2011.12.010.

[18] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional determinants*, Math. Theory Appl., Springer Science+Business Media, New York, 1994, https://doi.org/10.1007/978-0-8176-4771-1.

[19] A. Hashemi and D. Lazard, *Sharper complexity bounds for zero-dimensional Gröbner bases and polynomial system solving*, Internat. J. Algebra Comput., 21 (2011), pp. 703–713, https://doi.org/10.1142/S0218196711006364.

[20] X. Jia, *Role of moving planes and moving spheres following Dupin cyclides*, Comput. Aided Geom. Design, 31 (2014), pp. 168–181, https://doi.org/10.1016/j.cagd.2014.02.006.

[21] T. W. Sederberg and F. Chen, *Implicitization using moving curves and surfaces*, in Proceedings of the 22nd Annual Conference on Computer Graphics and Interactive Techniques (SIGGRAPH '95), ACM, New York, 1995, pp. 301–308, https://doi.org/10.1145/218380.218460.

[22] L.-Y. Shen, *Computing μ-bases from algebraic ruled surfaces*, Comput. Aided Geom. Design, 46 (2016), pp. 125–130, https://doi.org/10.1016/j.cagd.2016.07.001.

[23] L.-Y. Shen and C.-M. Yuan, *Implicitization using univariate resultants*, J. Syst. Sci. Complex., 23 (2010), pp. 804–814, https://doi.org/10.1007/s11424-010-7218-6.

[24] X. Shi and R. Goldman, *Implicitizing rational surfaces of revolution using μ-bases*, Comput. Aided Geom. Design, 29 (2012), pp. 348–362, https://doi.org/10.1016/j.cagd.2012.02.003.

[25] X. Shi, X. Jia, and R. Goldman, *Using a bihomogeneous resultant to find the singularities of rational space curves*, J. Symbolic Comput., 53 (2013), pp. 1–25, https://doi.org/10.1016/j.jsc.2012.09.005.

[26] X. Shi, X. Wang, and R. Goldman, *Using μ-bases to implicitize rational surfaces with a pair of orthogonal directrices*, Comput. Aided Geom. Design, 29 (2012), pp. 541–554, https://doi.org/10.1016/j.cagd.2012.03.026.

[27] J. Zheng, T. W. Sederberg, E.-W. Chionh, and D. A. Cox, *Implicitizing rational surfaces with base points using the method of moving surfaces*, in Topics in Algebraic Geometry and Geometric Modeling. Papers from the Workshop on Algebraic Geometry and Geometric Modeling held at Vilnius University, Vilnius, July 29–August 2, 2002, Contemp. Math. 334, R. Goldman and R. Krasauskas, eds., AMS, Providence, RI, 2003, pp. 151–168, https://doi.org/10.1090/conm/334.