Pseudo Power Scale Signatures: Frequency Domain Approach

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Abstract

In an earlier work, we introduced a new form of signal representation called the **pseudo power signature (PPS)** *that was essentially independent of the duration of the signal. The signatures were obtained based on the continuous wavelet transform, and were shown to be reliable and discriminating for classification purposes. In this paper, we take a fresh look at the problem of obtaining PPS by carrying out our analysis in the frequency domain. The main advantages of this approach over our earlier one, are that it is more versatile, permits the development of efficient computational algorithms, offers a solution to some unresolved uniqueness problems in our original formulation, and allows the study of the effect of the choice of analyzing wavelet to better aid the classification process.*

Keywords: classification, signatures, wavelets, speaker identification

1 Introduction and previous work

The classification of nonstationary signals of unknown duration is of great importance in areas like oil exploration, moving target detection, and pattern recognition. Consider the following classification problem (common in non-intrusive subsurface exploration):

Signals are obtained by propagating electromagnetic waves through several layers of different classes of materials. The goal is the determination of the various classes present and the thickness of each layer. The presence of a particular class is equated to the occurrence of an event.

In the current literature, there exist several classification schemes which use time-frequency representations to perform the above classification ([1],[2]). While each of these approaches works well for the specific problem motivating their formulation, their applicability to the classification problem under consideration, where each event may have an unknown time support, is limited. This drawback (owing to the signal length dependent nature) of conventional classification techniques using time-frequency distributions ([2]), led us to explore the possibility of obtaining representations which are intrinsically independent of time.

1.1 Notation

To simplify understanding, we first establish the notation followed in the paper. We have some wavelet $\psi(t) \in L^2(\Re)$ satisfying the admissibility condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} d\omega < \infty$$

and the family of its translations and dilations

$$\psi_{ab}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$

We have two function spaces that are used in addition to the $L^2(\Re)$ space:

$$\begin{aligned} \mathcal{H} &= \left\{ c(a,b) : C_\psi^{-1} \int_a \int_b |c(a,b)|^2 \frac{dadb}{a^2} < \infty \right\} \\ \mathcal{A} &= \left\{ s(a) : C_\psi^{-1} \int_a |s(a)|^2 \frac{da}{a^2} < \infty \right\} \end{aligned}$$

We know that $\mathcal{H} = \mathcal{A} \otimes L^2(\Re)$ and that the space of the continuous wavelet transforms (CWT) is a proper closed subspace, $\mathcal{M} \subset \mathcal{H}$. Moreover, the continuous wavelet transform is the map $\Gamma : L^2(\Re) \to \mathcal{H}$ defined by

$$c_\psi^x = \Gamma[x];\ x \in L^2(\Re) \tag{1}$$

$$c_\psi^x(a,b) = < x, \psi_{ab} >_{L^2(\Re)} \tag{2}$$

$$= \int x(t) \overline{\frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)} dt \tag{3}$$

The adjoint transformation, $\Gamma^* : \mathcal{H} \to L^2(\Re)$, has the definition

$$
\begin{aligned}
x^c &= \Gamma^*[c]; c \in \mathcal{H} \\
&= C_\psi^{-1} \int_a \int_b c(a,b) \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) \frac{dadb}{a^2}
\end{aligned}
$$

1.2 Pseudo Power Signatures

In our earlier work ([4],[5]), we introduced the concept of time-independent signal representations called **pseudo power signatures** (PPS) which had the following properties:

(1) They were independent of signal length, location, and magnitude.

(2) They were reliable and fairly robust.

(3) They were discriminating.

(4) They had few parameters and lent themselves to fast classification routines.

For $x \in L^2(\Re)$ with CWT, $c_\psi^x \in \mathcal{H}$, where ψ is an admissible wavelet, we approximated $c_\psi^x(a,b)$ by a separable element of the form

$$
c_\psi^x(a,b) \approx s_\psi^x(a) r_\psi^x(b)
$$

where $s_\psi^x \in \mathcal{A}$, and $r_\psi^x \in L^2(\Re)$. The normalized function s_ψ^x then corresponds to the PPS of x. We showed that these signatures essentially characterize the scale power distribution in a manner independent of time, and hence, are invariant to time shifts. An important consequence is that *pieces of signals* are all characterized by the *same signature*. In [5], we showed that the best separable approximation obtained is the element $s \otimes r \in \mathcal{H}$ that orthogonally projects onto $c_\psi^x \in \mathcal{M} \subset \mathcal{H}$. The problem of determining the PPS then was reduced to solving the following minimization problem **P**:

For a given $c_\psi^x \in M$, find the decomposition $s_\psi^x r_\psi^x \in \mathcal{H}$ that minimizes the index

$$J(s,r) = \| c_\psi^x - \mathcal{K}[s \otimes r] \|_{\mathcal{H}}^2 \tag{4}$$

where $\mathcal{K} : \mathcal{H} \to \mathcal{M}$ is the orthogonal projection operator defined as

$$\mathcal{K}\,[h]\,(a,b) = C_\psi^{-1} \int_\alpha \int_\beta \overline{c_\psi^{\psi_{ab}}(\alpha,\beta)} h(\alpha,\beta) \frac{d\beta d\alpha}{\alpha^2}, \;\; \forall\, h \in \mathcal{H} \tag{5}$$

At that point, there was no known result concerning the uniqueness of solution and the problem was regularized by adding a term $\lambda \parallel s \otimes r \parallel^2$ to the cost function. For analysis purposes, we set $\lambda = 1$, and presented a technique using the redundant wavelet transform to compute these signatures ([5]). While this approach was shown to work well for classification, it suffered from the following two drawbacks: (a) It was computationally expensive; (b) It did not yield the 'true' PPS due to the addition of the regularizing term. In this paper we address both of these issues.

1. First, we formulate the entire problem in the frequency domain, which helps to solve the regularized problem in a very efficient way.

2. Next, we use this same approach to present a strong argument that the minimization problem without the regularizing term has, in general, multiple solutions. However, we show that for band-limited signals, we can create PPS even for the non-regularized problem using a practical, and efficient algorithm.

3. Finally, we present a technique that permits the study of the effect of wavelet selection on the classification problem.

2 Frequency domain formulation

The frequency domain optimization of the cost function, $J(s,r)$, introduced in Eq.(4), is is based on the following (see [3]):

Lemma 2.1 *The orthogonal projector $\mathcal{K}$ defined in Eq (5) satisfies*

$$\mathcal{K} = \Gamma\Gamma^* \tag{6}$$

where Γ is the wavelet transform operator defined in Eq(1). Moreover, one has

$$\Gamma^*\Gamma = I_{L^2(\Re)} \tag{7}$$

Using this result the cost function in Eq(4) becomes $J(s,r) = \parallel \Gamma[x] - \Gamma\Gamma^*[s \otimes r] \parallel^2_{\mathcal{H}}$. Hence,

$$J(s,r) \quad = \quad \parallel x - \Gamma^*[s \otimes r] \parallel^2_{L^2(\Re)} \tag{8}$$

The map Γ^* has a well defined structure.

$$\Gamma^*[s \otimes r] \quad = \quad C_\psi^{-1} \int_a \int_b s(a) r(b) \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right) \frac{dadb}{a^2} \tag{9}$$

Let $x^{sr} = \Gamma^*[s \otimes r] \in L^2(\Re)$. For its Fourier transform it is possible to write

$$X^{sr}(\omega) \quad = \quad \int_\omega C_\psi^{-1} \int_a \int_b s(a) r(b) \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right) \frac{dadb}{a^2} e^{-j\omega t} dt \tag{10}$$

Interchanging the order of integrations and rearranging, one obtains

$$X^{sr}(\omega) = C_\psi^{-1} \int_a s(a)\sqrt{a}\Psi(a\omega)\frac{da}{a^2}R(\omega) \tag{11}$$

This last equation permits the definition of a map $\hat{U}$ as follows

$$\begin{aligned} \hat{U}[s](\omega) &= C_\psi^{-1} \int_a s(a)\sqrt{a}\Psi(a\omega)\frac{da}{a^2} &(12)\\ &= \left\langle s, \overline{\Psi_a} \right\rangle_{\mathcal{A}} &(13) \end{aligned}$$

where $\Psi_a = \sqrt{a}\Psi(a\omega)$. Using conventional manipulations, one can prove

Lemma 2.2 *The map $\hat{U}$ satisfies the following properties:*

1. $|\hat{U}[s](\omega)| \leq \| s \|_{\mathcal{A}},\ \forall\omega,\ \forall s \in \mathcal{A}$.

2. *For each $s \in \mathcal{A}$, the function $\hat{U}[s](\omega) \in L^2(\Re)$, and hence has a unique inverse Fourier transform given by*

$$\begin{aligned} U[s](t) &= C_\psi^{-1} \int_a s(a)\frac{1}{\sqrt{a}}\psi\left(\frac{t}{a}\right)\frac{da}{a^2} \\ &= \left\langle s, \overline{\psi_a} \right\rangle_{\mathcal{A}}, \quad \textit{where } \psi_a = \frac{1}{\sqrt{a}}\psi\left(\frac{t}{a}\right) \end{aligned}$$

3. *Under mild conditions ($\sqrt{t}\psi_a(t) \in L^2(\Re)$), for each value of time, $U[s](t)$ is a well defined inner product in $\mathcal{A}$, since $\psi_a \in \mathcal{A}$.*

2.1 Optimal signatures using frequency domain formulation

The determination of optimal signatures requires the minimization of the regularized cost function, introduced in Eq (8). This cost function, transformed to the frequency domain becomes

$$J(s,r) \;=\; \| X - \hat{U}[s]R \|^2_{L^2(\Re)} + \lambda \| s \|^2_{\mathcal{A}} \| R \|^2_{L^2(\Re)} \tag{14}$$

Using Calculus of Variations, we obtain the two necessary conditions for optimality. Taking variations with respect to $R(\omega)$ yields the first necessary condition

$$X(\omega)\overline{\hat{U}[s](\omega)} \;=\; \left(|\hat{U}[s](\omega)|^2 + \lambda \| s \|^2_{\mathcal{A}}\right) R(\omega) \tag{15}$$

Taking variations with respect to s gives the second condition. This process is somewhat more involved and requires the use of

$$\hat{U}[\delta s](\omega) \;=\; C_\psi^{-1} \int \delta s(a)\sqrt{a}\Psi(a\omega)\frac{da}{a^2}$$

The resulting expression is

$$\int X(\omega)\overline{R(\omega)\sqrt{a}\Psi(a\omega)}d\omega \;=\; \int \hat{U}[s](\omega)|R(\omega)|^2\sqrt{a}\overline{\Psi(a\omega)}d\omega + \lambda \| R \|^2_{L^2(\Re)} s(a) \tag{16}$$

The effect of the regularizing term in providing a unique solution is evident since it guarantees that the first necessary condition always has a unique solution for R for any non-zero $s \in \mathcal{A}$. Similarly, for any non-zero $R \in L^2(\Re)$, we have a unique solution for s from the second necessary condition.

Remark 2.3 *It is apparent that for a fixed $s \in \mathcal{A}$, the cost function is quadratic in the variable $R(\omega) \in L^2(\Re)$. Thus, we can establish the existence of a unique minimizing solution $R^s(\omega)$ from the* *necessary condition in Eq (15). Likewise, the second necessary condition can be used to establish the existence of a unique $s^R(a) \in \mathcal{A}$ for each choice of $R \in L^2(\Re)$. If s is constrained to a unit ball, then we can use the argument in [3] to prove the existence of a minimizing sequence.*

2.1.1 An efficient numerical algorithm to compute the PPS

For numerical computations it is standard procedure to restrict $s(a)$ to a subspace of the form

$$s(a) = \sum_k \sigma_k v_k(a) \tag{17}$$

Thus, determination of the signature $s(a)$ is equivalent to the determination of the vector $\sigma = col\{\sigma_k\}$. The continuous frequency domain is also discretized using the set $\{\omega_n;\ n = 1, 2, \ldots\}$. Instead of discretizing the necessary conditions it is more convenient to introduce the discretization in the formulation of the cost function and re-derive the necessary conditions as shown below. Assuming a discrete frequency set $\{\omega_n;\ n = 1, 2, \ldots\}$, the function $R(\omega)$ is replaced by the vector $R_d = col\{R(\omega_n)\}$. In a similar manner, the error function $E(\omega) = X(\omega) - \hat{U}[s](\omega)R(\omega)$ will be replaced by a vector E_d. The expression for $\hat{U}$ in Eq (12), becomes

$$\hat{U}[s](\omega_n) = \sum_k \sigma_k \hat{U}[v_k](\omega_n)$$

which can be expressed in compact form as a vector

$$col\{\hat{U}[s](\omega_n)\} = U_d\sigma \tag{18}$$

$$U_d(n,k) = \hat{U}[v_k](\omega_n) \tag{19}$$

Using the Hadamard product $\bullet$ (element-by-element matrix multiplication), we get the following compact representation:

$$E_d = X_d - (U_d\sigma) \bullet R_d \tag{20}$$

The discrete cost function is then

$$J_d = \| X_d - (U_d\sigma) \bullet R_d \|_{l^2}^2 + \lambda \| \sigma \|_{l^2}^2 \| R_d \|_{l^2}^2 \tag{21}$$

Applying variations to this cost function one obtains the two necessary conditions

$$X_d \bullet (\overline{U_d\sigma}) = [\overline{U_d\sigma} \bullet (U_d\sigma)] \bullet R_d + \lambda \| \sigma \|_{l_2}^2 R_d \tag{22}$$

$$U_d^*(X_d \bullet \overline{R_d}) = U_d^*[(U_d\sigma) \bullet |R_d|^2] + \lambda \| R_d \|_{l^2}^2 \sigma \tag{23}$$

Let $D_r = diag(|R_d|^2)$ be the diagonal matrix with the entries of $|R_d|^2$ along the main diagonal. One can easily see that

$$(U_d\sigma) \bullet |R_d|^2 = D_r U_d \sigma$$

Hence the second necessary condition becomes

$$U_d^*(X_d \bullet \overline{R_d}) = U_d^* D_r U_d \sigma + \lambda \| R_d \|_{l^2}^2 \sigma \tag{24}$$

The observations in remark 2.3 also apply to this discretized case. Hence, a similar argument can be used to establish the existence of a minimizing sequence when σ is constrained to the unit ball.

2.2 Existence of multiple signatures

The function $U[s](t)$ introduced in lemma 2.2 offers a good insight into the problem of uniqueness. Consider the following result:

Theorem 2.4 *If the wavelet $\psi(t)$ has compact support, it is possible to select an infinite number of functions $s(a)$ such that $U[s](t)$ has compact support.*

Proof

Assume that the wavelet $\psi(t)$ has support, $S_\psi \subset (0,\ T_\psi)$. It is clear that for any $\alpha > 1$ the support of $\psi(\alpha t)$ is also contained in the interval $(0,\ T_\psi)$.

In the expression for $U[s](t)$, suppose we make a change of the integration variable as $\alpha = \frac{1}{a}$. Letting $s^1(\alpha) = s(1/\alpha)$, (this is an $L^2(0,\ \infty)$ function) we can write

$$U[s](t) \quad = \quad \int_\alpha s^1(\alpha)\sqrt{\alpha}\psi(\alpha t)d\alpha$$

If $s^1(\alpha)$ is **any** *$L^2(0,\ \infty)$ function with support outside the interval $(0,\ T_\psi)$ then it follows that*

$$s^1(\alpha)\sqrt{\alpha}\psi(\alpha t) \quad = \quad 0; \quad \forall t > 1$$

and therefore $U[s](t) = 0;\ \forall t \ni [0\ \ 1]$. ∎

Let $x \in L^2(\Re)$ be an arbitrary signal with Fourier transform, $X(\omega)$. Consider $x^{sr}(t) = \Gamma^*[s \otimes r](t)$. If $U[s](t)$ has compact support, then $\hat{U}[s](\omega)$ is non zero on any interval of non zero length. Hence, for the non-regularized problem, for almost all frequencies one can write

$$R(\omega) = \frac{X(\omega)}{\hat{U}[s](\omega)}$$

Since one can build functions $\hat{U}[s](\omega)$ that are nonzero on any interval using infinitely many functions $s \in \mathcal{A}$ (from Th. 2.4), it would appear that there are infinitely many signatures producing a perfect match. The problem is that even if $\hat{U}[s](\omega)$ is always nonzero, one cannot assure that $R(\omega)$ will correspond to an $L^2(\Re)$ signal. Thus, even though this result is not a formal proof of the existence of multiple signatures, it gives a strong indication that there will exist cases with non-unique signatures. This has been borne out by experimental simulations.

3 Signatures for band-limited signals

For classification purposes it appears highly convenient not to use the regularization term and, thus, have a 'true' PPS. We determined that when signals are band-limited, this is indeed possible. Hence, it is useful to consider this class of signals in more detail. If the signal, $x(t)$, is band-limited, its Fourier transform $X(\omega)$ has compact support, Ω_x. In this case, one can always define R to be band-pass on the support of X

$$R(\omega) = 1; \ \forall \omega \in \Omega_x, \ \ and \ \ 0 \ \ elsewhere$$

The function, $R(\omega)$, is guaranteed to be an $L^2(\Re)$ signal. The determination of the signature requires the solution for $s(a)$ of the equation

$$X(\omega) = \hat{U}[s](\omega); \ \forall \omega \in \Omega_x$$

$$= C_\psi^{-1} \int_a s(a) \sqrt{a} \Psi(a\omega) \frac{da}{a^2}$$

Numerically, this equation is replaced by the set of equations

$$X(\omega_n) = C_\psi^{-1} \int_a s(a) \sqrt{a} \Psi(a\omega_n) \frac{da}{a^2}; \; n = 1, 2, \ldots N$$

A simple way to obtain a signature, $s(a)$, for this case is by selecting

$$s(a) = \sum_{k=1}^{N} \sigma_k \sqrt{a} \overline{\Psi(a\omega_k)}$$

The determination of the signature requires the solution of the linear equation

$$X_d = G_\psi \sigma \qquad (25)$$

where G_ψ is the Grammian matrix of the set of functions $\{\sqrt{a}\Psi(a\omega_k)\}$ and $\sigma = col\{\sigma_k\}$. This solution, however, is a vector of high dimension and not very convenient for fast classification methods. A more compact "matched signature" can be obtained by selecting a subset of frequencies and solving Eq. (25) using pseudo inverses. Experimental results show that this approach produces very discriminating signatures. Moreover, the wavelet affects only the Grammian matrix and its effect can be easily analyzed.

3.1 Simulation results and wavelet selection issues

The experimental results included here are preliminary. Because of its flexibility, there are yet unresolved issues in the frequency domain approach. Unless explicitly stated, the wavelet used is Daubechies' Db2. The first set of results explores the difference between the optimal signatures

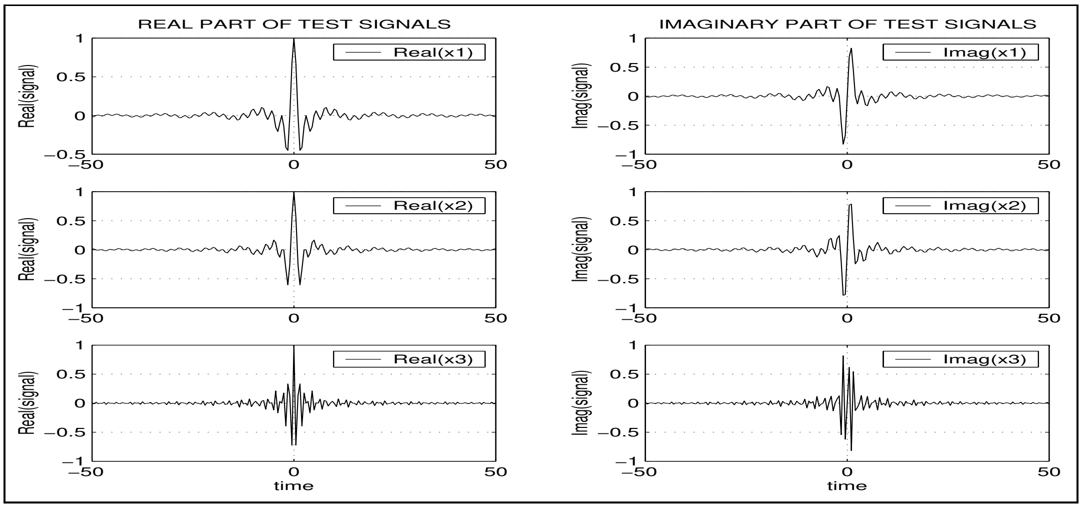

Figure 1: Complex Valued Test Signals

and the suboptimal signatures developed in the bandlimited approach. The signals used are chirp signals shown in Fig (1). Two of them, x_1, x_2, are very similar. The results in Fig (2) show very clearly that the optimal signatures are quite distinct, even for the signals that are similar. The suboptimal signatures are distinct but show less discrimination. The suboptimal approach has also been tested with speech signals. Figure 3 shows the signatures obtained by processing records of the letters **a,b,c** with 4096 points each for both a male and a female speaker. The discrimination capability of the signatures is apparent.

We also used this example to explore the issue of wavelet selection. The signatures in Fig (4) were obtained using the wavelet Db8. A tentative conclusion is that the difference in signatures is sharper for the higher order wavelet.

4 Conclusion and future work

In this paper, we have formulated an efficient frequency domain approach to determine PPS, and shown through an example using band-limited signals, the excellent discriminating capabilities of this class of signatures. Currently, we are in the process of studying the wavelet selection issues, and

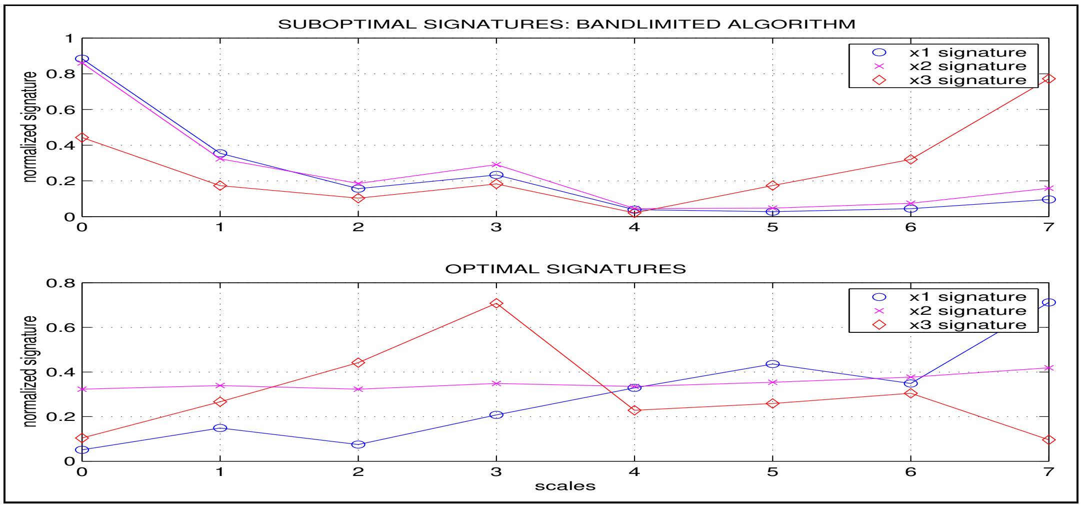

Figure 2: Signatures for Test Signals

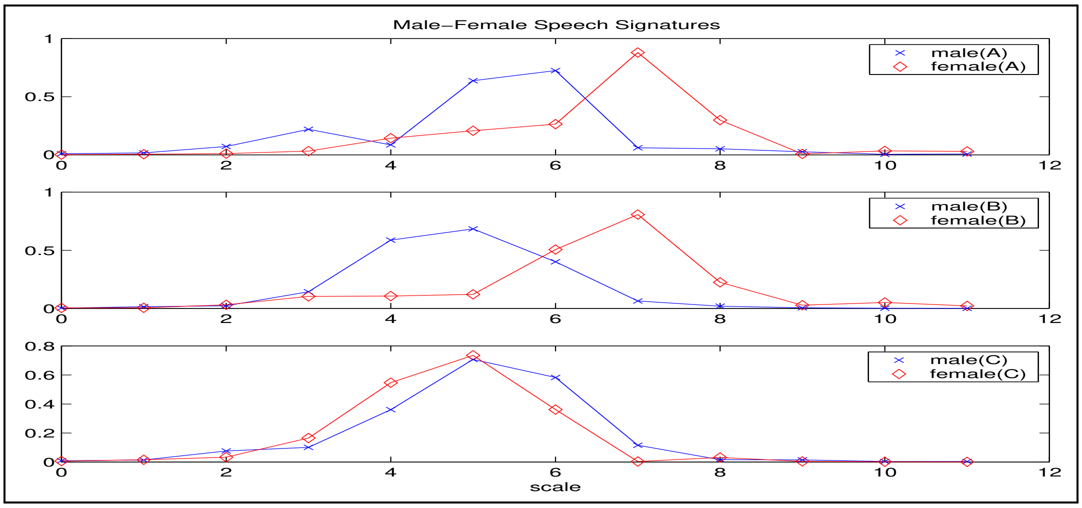

Figure 3: Signatures for Letters Using Db2

refining the computational algorithm. Due to the efficiency, reliability and discrimination afforded by these signatures, this technique has good potential in applications like speech recognition and target detection. The actual application in such a problem will be considered in a future paper.

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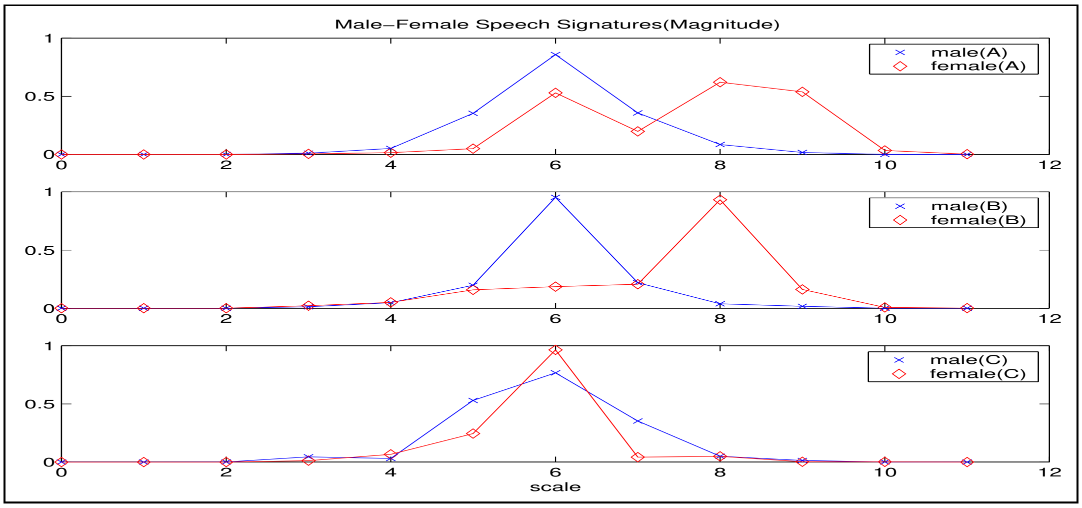

Figure 4: Signatures for Letters Using Db8

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