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The large scale geometry of strongly aperiodic subshifts of finite type

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ABSTRACT

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A subshift on a group G is a closed, G-invariant subset of A^G, for some finite set A. It is said to be of finite type if it is defined by a finite collection of "forbidden patterns" and to be strongly aperiodic if it has no points fixed by a nontrivial element of the group. We show that if G has at least two ends, then there are no strongly aperiodic subshifts of finite type on G (as was previously known for free groups). Additionally, we show that among torsion free, finitely presented groups, the property of having a strongly aperiodic subshift of finite type is invariant under quasi isometry.

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0.1 Introduction

Recall that a topological dynamical system is a pair (Ω, G) where G is a group acting by homeomorphisms on the compact space Ω. For instance, if A is a finite discrete set, then the group G acts (on the right) on the compact space A^G by homeomorpisms via

$$(\sigma \cdot h)(g) = \sigma(hg).$$

This action makes the pair (A^G, G) into a topological dynamical system called the right shift. When $G = \mathbb{Z}$, an element h of G acts on a biinfinite word $\sigma \in A^G$ by "shifting" it, whence the name. A closed, G-invariant subset of A^G is known as a subshift. To say that a subshift X codes a dynamical system (Ω, G) means that there exists a continuous G-equivariant surjection from X to Ω.

Subshifts of finite type (see [3, §2]). How would one construct a subshift? The simplest idea is to start with a closed set C of A^G and intersect its G-translates. The most important case of this construction arises when C is determined by finitely many coordinates.

Definition 0.1 Let A be a finite set and G a group. If S is a finite subset of G and L a subset of A^S, then the clopen set

$$\{\sigma \in A^G : \sigma|_S \in L\}$$

is known as a cylinder set. If C is a cylinder set, then the set X given by $\bigcap_{g\in G}(C \cdot g)$ is called a subshift of finite type. We say that X is defined on S. If F is a finite set, then $\alpha \in A^F$ is called a forbidden pattern for X if it is never equal to $(\sigma \cdot g)|_F$ for any $\sigma \in X$.

0.1.1 Strongly aperiodic subshifts of finite type.

We will be most interested in subshifts of finite type on which G acts freely. These are the only subshifts which may code free actions.

Definition 0.2 Let $X \subset A^G$ be a nonempty subshift, and σ a point of X. Then σ is said to be periodic if it has nontrivial stabilizer in G, and is said to be g-periodic for any $g \in \mathrm{Stab}_G\, \sigma$. If X contains no periodic points, then X is said to be strongly aperiodic.

It is not hard to see that, if G is equal to $\mathbb{Z}$, then it admits no strongly aperiodic subshifts of finite type (by convention, the empty subshift is not strongly aperiodic.) The idea is that if X is defined by forbidden patterns on $[-n, n]$, then we may find, inside some $\sigma_0 \in X$, a block of length longer than $2n$ occuring in at least two places. We can then repeatedly copy and paste the segment of σ_0 connecting these two blocks to obtain a periodic element σ of X (see Figure 3 in §0.2).

Wang tilings. The problem of finding a strongly aperiodic subshift of finite type on $\mathbb{Z}^2$ goes back to Wang [11]. Suppose we are given a finite set A of 1×1 square tiles, such that each edge of each tile is assigned some color. The tiling problem asks whether we may fill out the entire plane with copies of these tiles such that neighboring edges have the same color. In the simplest examples, a collection of tiles which succesfully tiles the plane can do so periodically. Wang conjectured that this is always the case—i.e., that A tiles periodically if it tiles at all. He observed that if his conjecture were true, then the tiling problem would be solvable (by Turing machines).

Wang's conjecture was disproved by his student Berger [1], who found a counterexample—i.e., a set A of tiles which can tile the plane, but cannot do so periodically. Since

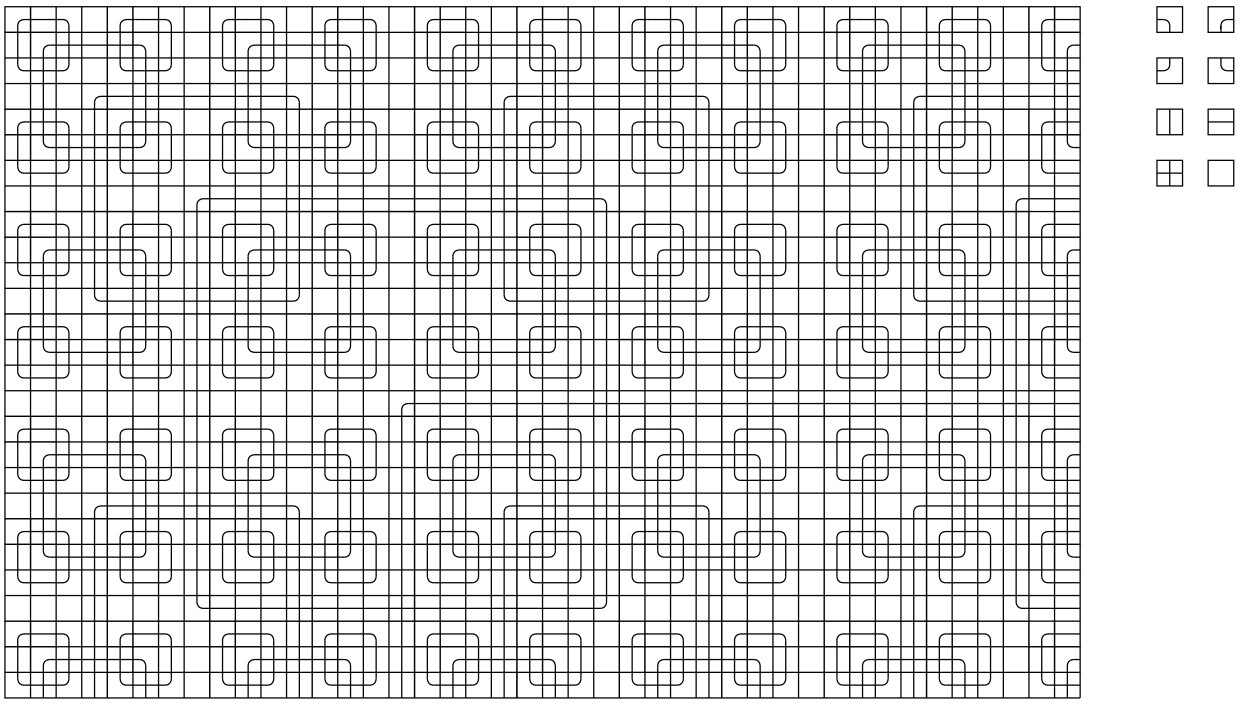

Figure 1 : An interesting tiling of the plane using the set A consisting of the eight tiles depicted at right. The orbit closure in $A^{\mathbb{Z}^2}$ of the given pattern is strongly aperiodic, and is coded by a subshift of finite type called the Robinson tiling, in which the tiles carry slightly more data.

then, many people have obtained tile sets with this property. Our favorite is the Robinson tiling [8], which codes the orbit closure of the pattern depicted in Figure 1. The reader has probably observed that if A cannot tile periodically, then we obtain a strongly aperiodic subshift of finite type inside $A^{\mathbb{Z}^2}$, where the forbidden patterns consist of pairs of adjacent tiles with non-matching edges.

0.1.2 Endedness and QI invariance

Some progress on the problem of finding strongly aperiodic subshifts of finite type has been made for other groups. In particular, higher dimensional free abelian groups [1], uniform lattices in higher rank simple Lie groups [6], and the integral Heisenberg group [9] are known to admit strongly aperiodic subshifts of finite type, and because there exist strongly aperiodic tilings of $\mathbb{H}^2$ [4], it seems likely that there are strongly aperiodic subshifts of finite type on the fundmental group of a closed surface. On the other hand, free groups, including $\mathbb{Z}$, are known not to admit strongly aperiodic subshifts of finite type [7, §3, Theorem 2.2].

Recall that the Cayley graph of a group G with respect to a generating set S is the graph whose vertex set is G, with an edge between vertices g and h whenever $gs = h$ for some s in S (Figure 2 depicts some Cayley graphs). The existence of strongly aperiodic subshifts of finite type on a finitely generated group G appears to be closely connected to the large scale geometry of its Cayley graph (with respect to an arbitrary finite generating set). In particular, the arguments which show that free groups have no such subshift are based on the fact that the Cayley graph of a free group may be disconnected by removing a sufficiently large ball around a point. The following definition captures this idea.

Definition 0.3 Let S be a finite generating set for a group G. The number of ends of G is defined to be the limit as n goes to infinity of the number of unbounded connected components of $G \setminus B_n$, where B_n is the ball of radius n around 1_G in the Cayley graph of G. It is understood that this limit is often infinite.

The number of ends of G is invariant under changing the generating set S. The point is that, if S' is some other finite generating set, then the Cayley graphs associated to S and S' are quasi isometric (Definition 0.4) and the the number of ends is a "QI invariant". Hopf discovered that the number of ends of a group is either $0, 1, 2,$ or ∞ [5], and Stallings [10] showed that a group has at least 2 ends if and only if it splits nontrivially as an amalgamated free product or HNN extension over a finite

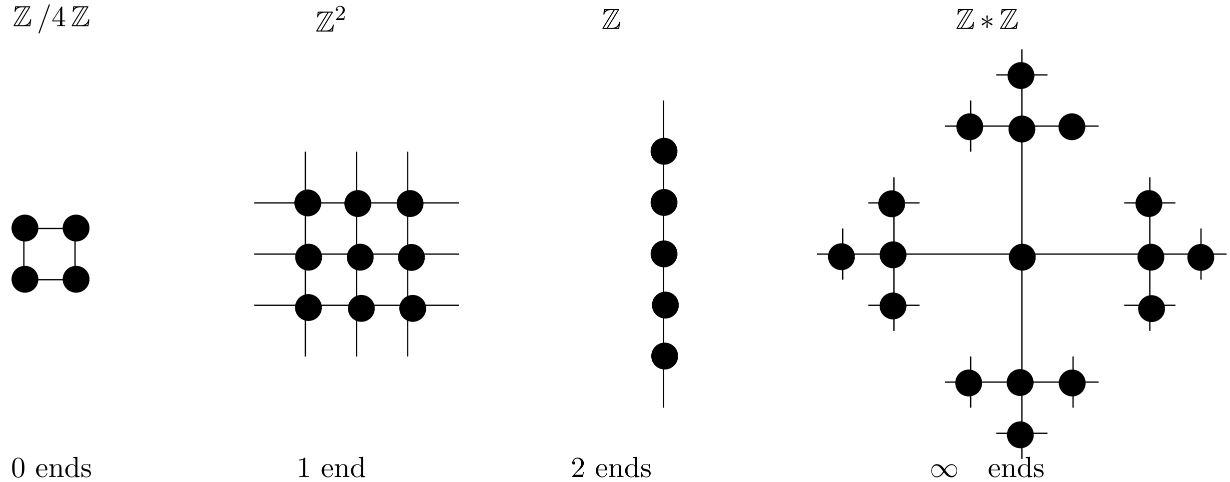

Figure 2 : Some Cayley graphs of groups with respect to their standard generating sets.

group (of course, G has 0 ends if and only if it is finite). Of the examples above, $\mathbb{Z}$ has 2 ends (as $\mathbb{Z} \cong \{1\}*_{\{1\}}$), higher rank free groups have infinitely many ends, and the Heisenberg group, fundamental groups of closed surfaces, and free abelian groups are all one ended. Our first main theorem, which will be proved in §0.4, is as follows.

Theorem 0.1

If G is a finitely generated group with at least 2 ends, then G does not admit a strongly aperiodic subshift of finite type.

This theorem gives one instance where the geometry of G constrains the existence of strongly aperiodic subshifts of finite type on G. Our other main theorem will show that having such a subshift is, in some sense, a geometric property. To make this precise, we will need the notion of a QI invariance, which was alluded to above.

Definition 0.4 A map $f : X \to Y$ between metric spaces X and Y is said to be an n-quasi isometric embedding if for any points $x_1, x_2 \in X$,

$$\frac{d(x_1, x_2)}{n} - n \leq d(f(x), f(y)) \leq nd(x_1, x_2) + n.$$

It is said to be n-quasi surjective if the n neighborhood of $f(X)$ equals all of Y. We say that f is a quasi isometry if (for some n) it is an n-quasi surjective n-quasi isometric embedding.

Two spaces are said to be quasi isometric if there exists a quasi isometry between them, and it is easily seen that this is an equivalence relation. This equivalence relation is interesting for Cayley graphs, which may be metrized by taking each edge to have length 1. As mentioned above, if S, S' are finite generating sets for a group G, then the Cayley graph of G with respect to S is quasi isometric to the Cayley graph of G with respect to S'. The following definition is the basis of the subject of geometric group theory.

Definition 0.5 We say that finitely generated groups G and H are quasi isometric if their Cayley graphs are quasi isometric. If η is an invariant of groups such that $\eta(G) = \eta(H)$ whenever G and H are quasi isometric, we say that η is a QI invariant.

As remarked above, the number of ends of a group is the prototypical QI invariant. For another example, finite presentation is a QI invariant—if G is finitely presented and G is quasi isometric to H, then H is finitely presented. Our (second) main theorem states that, for finitely presented groups, having a strongly aperiodic subshift of finite type is a QI-invariant.

Theorem 0.2

Let G and H be torsion free finitely presented groups, and suppose that G is quasi isometric to H. Then G admits a strongly aperiodic subshift of finite type if and only if H does.

Remark. We note that Carroll and Penland have shown independently that having a strongly aperiodic subshift of finite type is a commensurability invariant [2], even without assuming torsion freeness or finite presentation. Two groups G and H are said to be commensurable if some finite index subgroup of G is isomorphic to some finite index subgroup of H. If G and H are commensurable, then they are quasi isometric to each other, but there are many examples of pairs of groups which are quasi isometric but not commensurable.

The main step in the proof of Theorem 0.2 produces a subshift of finite type which parameterizes quasi isometries (of some fixed regularity) from G to H. This idea is closely related to the construction of Mozes [6] which codes the tiling of a symmetric space associated to some uniform lattice by a subshift of finite type over some other uniform lattice.

0.1.3 Subshifts and geometry.

Given a map $f : G \mapsto H$, and a point $\sigma \in A^H$, we define the pullback $f^*\sigma \in A^G$ to be the composition $g \mapsto \sigma(f(h))$. In order to prove our theorem, we would like to take a strongly aperiodic subshift of finite type X on a group H, together with a quasi isometry $f : G \to H$, and use this data to obtain a strongly aperiodic subshift of finite type on G. As a first attempt, we could try $f^*X = \{f^*\sigma : \sigma \in X\}$, but this is usually not even a subshift. The biggest problem is that the behavior of f on $B(n, g)$ will generally change as we vary $g \in G$, and so there is no way to know which patterns to forbid. In order to "smooth out" the bad behavior of any individual quasi isometry f, we would like to somehow pull X back under a broad class of quasi isometries.

More precisely, assuming that G and H are finitely presented, we will find a set $\mathbb{A}$ and a subshift of finite type $Y \subset \mathbb{A}^G$ such that points of Y parametrize pairs (f, σ) such that $f : G \to H$ is a quasi isometry satisfying some conditions and σ is a point of X. Furthermore, when $g \in G$ fixes the point of Y corresponding to a pair (f, σ), it will follow that $f(g)$ must fix σ (and thus Y is strongly aperiodic because X is.)

0.1.4 Organization.

The paper is organized as follows. Section 0.2 gives the proof of Theorem 0.1. Section 0.3 defines the notion of the derivative of an n-Lipschitz function on a finitely generated group, which will be crucial in proving 0.2. In this section, we prove Theorem 0.4, which states that the collection of derivatives of n-Lipschitz functions on a finitely presented group forms a subshift of finite type. Section 0.4 proves Theorem 0.2.

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0.2 Groups with at least two ends do not admit a strongly aperiodic subshift of finite type.

We now prove theorem 0.1.

Theorem 0.3

Let G be a finitely generated group with at least 2 ends. Let $X \subset A^G$ be a nonempty subshift of finite type. Then there exists $\sigma \in X$ and $g \in G$ not equal to 1_G such that $\sigma \cdot g = \sigma$.

An example: $G = \mathbb{Z}$. We begin by illustrating the proof in a special case (see Figure 3). Assume $G = \mathbb{Z}$ and $X \subset A^G$ a non empty subshift of finite type. Suppose that X is defined on $B = B_G(n, 1_G)$, meaning that to determine whether $\sigma \in A^G$ is an element of X, we just need to check that the set

$$\{\sigma \cdot g|_B : g \in \mathbb{Z}\}$$

contains no forbidden pattern. Since we assumed that X is nonempty, there exists $\sigma_0 \in X$. We observe that there must exist $m_1, m_2 \in \mathbb{Z}$ such that $m_2 - m_1 > 2n$ and $\sigma_0 \cdot m_1|_B = \sigma_0 \cdot m_2|_B$. We will find $\sigma \in X$ such that σ is $m_2 - m_1$-periodic, meaning that $\sigma \cdot (m_2 - m_1) = \sigma$.

Let $\mathcal{S} = \{m_1 - n, m_1 - n + 1, \ldots, m_2 - n - 1\}$ and $\mathcal{S}' = \mathcal{S} \cup (m_2 + B)$. Let $\mathfrak{m} : G \to \mathcal{S}$ be specified by $\mathfrak{m}(x) \equiv x \bmod m_2 - m_1$ for every $x \in G$. We define σ to be $x \mapsto \sigma_0(\mathfrak{m}(x))$. Manifestly, σ is $m_2 - m_1$-periodic. To show that σ is in X, we start with the following observations.

(a) On $\mathcal{S}'$, the functions σ and σ_0 agree. To see this, note that if $x \in \mathcal{S}$, then $\sigma(x) = \sigma_0(x)$ by definition, and if $x \in \mathcal{S}' \setminus \mathcal{S}$, then $x \in m_2 + B$, so $\mathfrak{m}(x) = x - (m_2 - m_1)$ and

$$\sigma(x) = \sigma_0(x - (m_2 - m_1)) = \sigma_0(x)$$

by our assumption that $\sigma_0 \cdot m_1|_B = \sigma_0 \cdot m_2|_B$.

(b) For all $x \in G$, there exists some $k \in \mathbb{Z}$ such that

$$k(m_2 - m_1) + x + B \subset \mathcal{S}'.$$

This follows from the fact that $\mathcal{S}'$ contains the n-neighborhood of $\{m_1, m_1 + 1, \ldots, m_2 - 1\}$ which is a complete set of coset representatives mod $m_2 - m_1$.

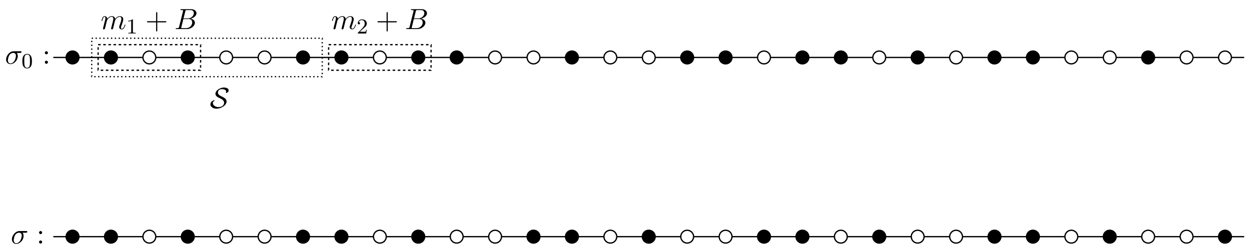

Figure 3 : Given σ_0, a point of X, a subshift of finite type on $\mathbb{Z}$, one can construct a periodic point σ of X by repeating the pattern found in σ_0 between two balls m_1+B and m_2+B on which σ_0 has the same behavior.

We now show $\sigma \in X$. Given $x \in G$, choose k as in (ii) above. Then by periodicity and (i),

$$\sigma \cdot x|_B = \sigma \cdot (k(m_2 - m_1) + x + B)|_B = \sigma_0 \cdot (k(m_2 - m_1) + x)|_B.$$

It follows that $\sigma \in X$.

The general case. From here on, we will assume that G is a group with at least 2 ends, so that for n sufficiently large, $\Gamma_S \setminus |B_G(n, 1_G)|$ has at least 2 unbounded connected components. The following definition will be crucial.

Definition 0.6 Let B_0, B_1, B_2 be finite subsets of G such that each $|B_i|$ is connected. We say that B_1 separates B_0 from B_2 when B_1 and B_2 lie in distinct connected components of $\Gamma_S \setminus |B_0|$.

For example, in $\mathbb{Z}$, $b + B(n,1)$ separates $a + B(n,1)$ from $c + B(n,1)$ when $c - b > 2n$ and $b - a > 2n$. The following lemma encodes some trivial observations about separation.

Lemma 0.1

Suppose that B_1 separates B_0 from B_2.

(a) *If $g \in G$, then gB_1 separates gB_0 from gB_2.*

(b) If $\mathcal{C}$ is an unbounded component of

$$\Gamma_S \setminus (|B_0| \cup |B_1| \cup |B_2|),$$

then $\mathcal{C}$ cannot touch both B_0 and B_2.

(c) If we also know that B_2 separates B_1 from some finite B_3, then it follows that B_0 and B_3 are separated by B_i if i is 1 or 2.

Proof 0.1 Part (a) is trivial.

To see part (b), observe that if $\mathcal{C}$ were an unbounded component which touched both B_0 and B_2, then we could find a path in $\mathcal{C}$ joining a vertex of B_0 to a vertex of B_2. Hence, B_0 and B_2 would lie in the same connected component of $\Gamma_S \setminus |B_1|$ (whichever one contains $\mathcal{C}$,) contrary to the definition.

To obtain part (c), we reason as follows. Because Γ_S is connected, there exists a path in Γ_S from B_3 to B_1, but any such path must go through B_2 because B_2 separates B_3 from B_1. Hence, B_2 and B_3 are in the same connected component of $\Gamma_S \setminus |B_1|$, and therefore B_3 and B_0 are in different connected components of $\Gamma_S \setminus |B_1|$ as desired since B_1 separates B_0 from B_2. The same argument shows that B_2 separates B_0 from B_3.

We now define the notion of an n-axial element $g \in G$. Intuitively (if not in reality,) left multiplication by such an element drags the Cayley graph of G along some axis.

Definition 0.7 Let n be a natural number. We say that $g \in G$ is n-axial if, for all integers $a < b < c$, we have that $g^b B_G(n, 1_G)$ separates $g^a B_G(n, 1_G)$ from $g^c B_G(n, 1_G)$.

In $\mathbb{Z}$, an element g is n-axial exactly when it has absolute value greater than $2n$. We now prove that every group with at least two ends has an n-axial element for sufficiently large n.

Lemma 0.2

Under our standing assumption that G is a finitely generated group with at least 2 ends, there exists some N_G such that for any $n \geq N_G$, there exists an n-axial $g \in G$.

Proof 0.2 (See the potentially deceptive Figure 4). Suppose n is large enough that $\Gamma_S \setminus |B_G(n, 1_G)|$ has at least two unbounded components, and write B for $B_G(n, 1_G)$. Choose $x, y \in G$ such that each has norm greater than $2n$ and x and y lie in distinct unbounded components of $\Gamma_S \setminus |B|$. Manifestly, B separates xB from yB, and B also separates $x^{-1}B$ from $y^{-1}B$ since S is assumed symmetric, so a path from x^{-1} to y^{-1} which did not pass through B could be reflected to get a path from y to x not passing through B. We will see that $x^{-1}y$ is n-axial.

Inductively define a biinfinite sequence B_i of finite subsets of G by setting $B_0 = B$ and $B_1 = x^{-1}B$, and mandating that $B_{i+2} = x^{-1}yB_i$ for all integers i. We know that B_0 separates $B_{-1} = y^{-1}B$ from B_1, and also that B_1 separates B_0 from $B_2 = x^{-1}yB$ (by translating xB, B, yB by x^{-1}.) Hence Lemma 0.1(a) gives us that B_i separates B_{i-1} from B_{i+1} for all i. But then part (c) of the lemma says that (in particular) B_{2b} separates B_{2a} from B_{2c} whenever $a < b < c$ are integers. I.e., $x^{-1}y$ is n-axial.

We are now finally ready to prove the theorem (Figure 5 illustrates the proof in the case where $G = \mathbb{Z} * \mathbb{Z}$). Choose n large enough that X is defined on n and there exists an n-axial $g_{ax} \in G$. Write B for $B_G(n, 1_G)$ and B^2 for $B_G(2n, 1_G)$ and let g be some power of g_{ax} such that $g^k B^2$ is always disjoint from B^2 for $k \neq 0$—such a g exists because B^2 only meets finitely many $g_{ax}^k B^2$. Since X is nonempty, there exists some $\sigma_0 \in X$. Pick distinct integers m_1 and m_2 such that $\sigma_0 \cdot g^{m_1}$ and $\sigma_0 \cdot g^{m_2}$ agree on B^2. If we wish to proceed as we did in the case $G = \mathbb{Z}$ must find a set of orbit

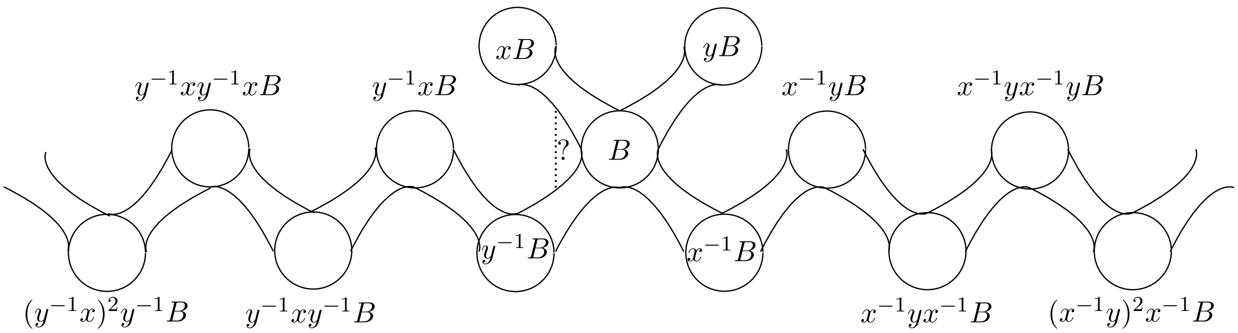

Figure 4 : Constructing an n-axial element. The question mark indicates one possible way the diagram can be misleading: it is possible that xB and $y^{-1}B$ are in the same connected component.

representatives $\mathcal{S} \subset G$ for the (left) action of $\langle g^{m_2-m_1} \rangle$ on G containing $g^{m_1}B$ and having properties analogous to the $\mathcal{S}$ we found for $\mathbb{Z}$. We define $\mathcal{S}$ as follows.

Definition 0.8 Let $B_k = g^{m_1+k(m_2-m_1)}B$ and let $\{\mathcal{C}_i\}$ consist of all connected components of $\Gamma_S \setminus \bigcup_{k\in\mathbb{Z}} B_k$. Note that Lemma 0.1 implies that each $\mathcal{C}_i$ touches at most two of the B_k, and these two must have consecutive k. We take $\mathcal{S}$ to be the union of

- B_0,
- the vertex sets of those $\mathcal{C}_i$ which touch only B_0 (and no other B_k,)
- and the vertex sets of those $\mathcal{C}_i$ which touch both B_0 and B_1.

The following lemma enumerates most of the neccessary properties of $\mathcal{S}$.

Lemma 0.3

In the situation of the above paragraph, there exists $\mathcal{S} \subset G$ such that the following conditions hold.

- *For any integer $k \neq 0$, we have $g^{k(m_2-m_1)}\mathcal{S} \cap \mathcal{S} = \emptyset$.*
- *For any $h \in G$, there exists an integer k such that h lies in $g^{k(m_2-m_1)}\mathcal{S}$.*

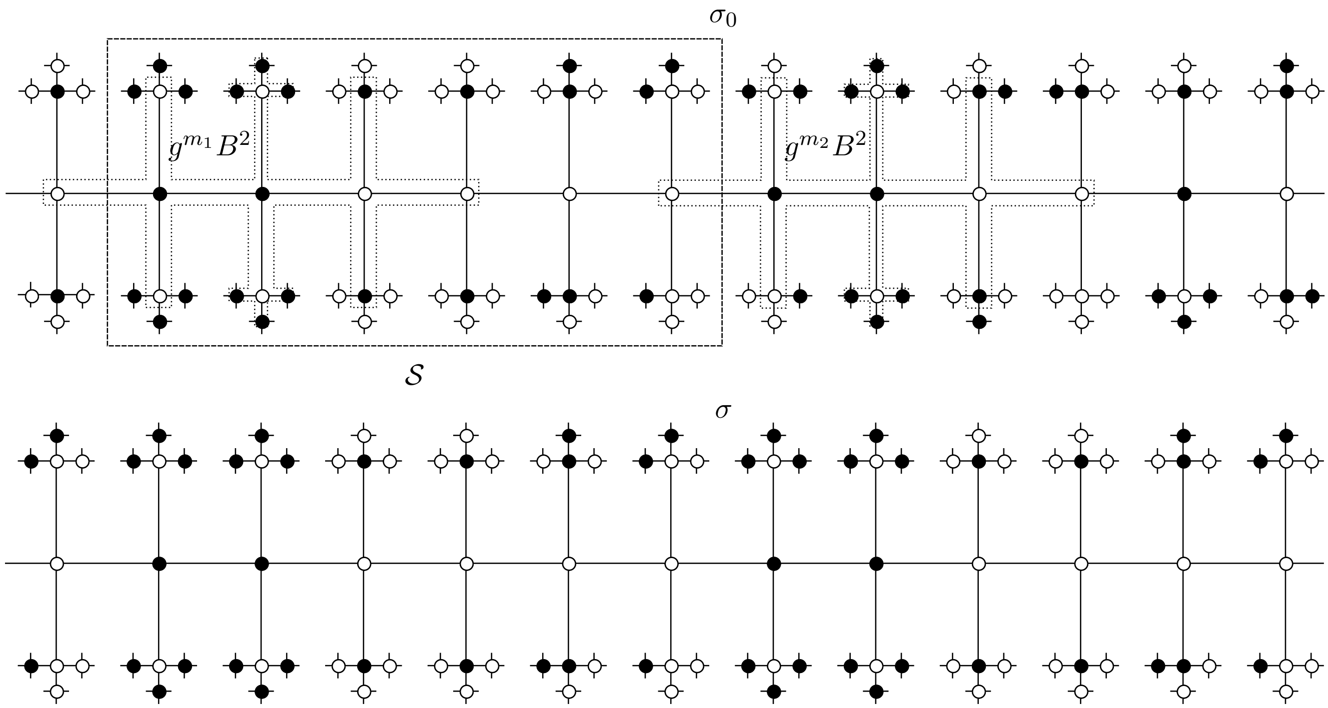

Figure 5 : If X a subshift of finite type on $\mathbb{Z} * \mathbb{Z}$, and $\sigma_0 \in X$, then σ_0 has the same behavior on two balls $g^{m_1}B^2$ and $g^{m_2}B^2$ whose radius is twice the defining radius of X. A periodic element σ of X is constructed by repeating the pattern realized by X on $\mathcal{S}$, a fundamental domain for the action of $g^{m_2-m_1}$ which contains a ball around g^{m_1}.

- *For any $h \in G$, there exists an integer k such that*

$$hB \subset \mathcal{S} \cup g^{m_1}B^2 \cup g^{m_2}B^2.$$

- *$\mathcal{S}$ contains $g^{m_1}B$.*

Proof 0.3 We now verify that $\mathcal{S}$ has the desired properties, in order.

- For a nonzero integer k, it is clear that $g^{k(m_1-m_2)}B_0$ (which is just B_k) will not meet $\mathcal{S}$. Similarly, if $\mathcal{C}_i$ touches just B_0, then $g^{k(m_1-m_2)}\mathcal{C}_i$ touches just B_k, and does not intersect $\mathcal{S}$. Finally, if some $\mathcal{C}_i$ touches B_0 and B_1, then $g^{k(m_1-m_2)}\mathcal{C}_i$ touches B_k and B_{k+1}, and hence does not intersect $\mathcal{S}$.

- Any element of G lies in some B_k or some $\mathcal{C}_i$. The translate $g^{-k(m_1-m_2)}B_k$ is equal to $B_1 \subset \mathcal{S}$. If $\mathcal{C}_i$ meets just B_k, then $g^{-k(m_1-m_2)}\mathcal{C}_i$ meets just B_0, and hence lies in $\mathcal{S}$. If $\mathcal{C}_i$ meets B_k and B_{k+1}, then $g^{-k(m_1-m_2)}\mathcal{C}_i$ meets B_0 and B_1, and is thus a subset of $\mathcal{S}$.

- If $x \in B_k$, then $g^{-k(m_2-m_1)}B \subset g^{m_1}B^2$. If x is in some $\mathcal{C}_i$ which touches just B_k, then any path of length n starting at x must either stay in $\mathcal{C}_i$ or go through B_k. Hence, $xB \subset \mathcal{C}_i \cup g^{m_1+k(m_2-m_1)}B^2$, so $g^{-k(m_2-m_1)}xB \subset g^{m_1}B^2 \cup \mathcal{S}$. If x lies in some $\mathcal{C}_i$ which touches B_k and B_{k+1}, then by the same logic, $g^{-k(m_2-m_1)}xB \subset g^{m_1}B^2 \cup \mathcal{S} \cup g^{m_2}\mathcal{S}$.

- By definition, $\mathcal{S}$ contains $g^{m_1}B$, which is B_0.

We now finish the proof of Theorem 0.3. Take $\mathcal{S}$ as in the lemma. For $x \in G$, define $\mathfrak{m}(x)$ to be the $g^{k(m_2-m_1)}$ translate of x which lies in $\mathcal{S}$. Define $\sigma(x) = \sigma_0(\mathfrak{m}(x))$, so that by definition $\sigma \cdot g^{m_2-m_1} = \sigma$. Let $\mathcal{S}' = \mathcal{S} \cup g^{m_1}B^2 \cup g^{m_2}B^2$. As in the $\mathbb{Z}$ case, we have that σ agrees with σ_0 on $\mathcal{S}'$ by the following case by case argument.

- If $x \in \mathcal{S}$, then $\sigma(x) = \sigma_0(x)$ by definition.

- If $x \in g^{m_1}B^2 \setminus \mathcal{S}$, then x lies in some $\mathcal{C}_i$ which touches B_0 (and possibly also B_{-1},) because there is a path of length at mst n from x to B_0, and this path cannot pass through any other B_k by our assumption that the g^kB^2 are all disjoint. It follows that either x or $g^{m_2-m_1}x$ lies in $\mathcal{S}$, so that we have either

$$\sigma(x) = \sigma_0(x)$$

by definition, or

$$\sigma(x) = \sigma_0(g^{m_2-m_1}x) = \sigma_0(x),$$

by our assumption that $\sigma_0 \cdot g^{m_1}$ and $\sigma_0 \cdot g^{m_2}$ agree on B^2.

- If $x \in g^{m_2}B^2 \setminus \mathcal{S}$, then we see similarly that x lies in some $\mathcal{C}_i$ which touches B_1 (and possibly also B_2,) and we can proceed in the same fashion.

We see that $\sigma \in X$ because for any $x \in G$, Lemma 0.3 gives us a k such that $g^{k(m_2-m_1)}xB \in \mathcal{S}'$, and then we have

$$\sigma \cdot x|_B = \sigma \cdot g^{-k(m_2-m_1)}x|_B = \sigma_0 \cdot g^{-k(m_2-m_1)}x|_B.$$

Since X is defined on B, this establishes the desired result. We already observed that σ is $g^{m_2-m_1}$ periodic, so we have proved the theorem.

0.3 Derivative subshifts.

In this section, we will exhibit a subshift of finite type which parametrizes n-Lipschitz functions from a finitely presented group G to a finitely generated group H, up to translation on H (Theorem 0.4). The idea is that an n-Lipschitz function f is determined, up to choice of $f(1)$, by its derivative (Definition 0.9), which is a bounded function from $G \times S$ to H whose value at (g, s) measures the difference between $f(g)$ and $f(gs)$. The set of such derivatives is shown to be a subshift of finite type when G is finitely presented, by showing that a function on $G \times S$ which looks like a derivative locally can be "integrated" to give a globally defined n-Lipschitz function. Of course, the condition of looking like a derivative locally will be encoded by a finite set of forbidden patterns. Note that similar subshifts have previously arisen in the literature. For example Gromov used a subshift parameterizing "integer 1-cocycles" to code the boundary of a hyperbolic group [3, §3].

Notation. Throughout this section, G will be a group generated by a finite symmetric set S, and H will be a group generated by a finite symmetric set T. As usual, fixing a finite generating set for a group endows it with a word metric.

Definition 0.9 Fix $n \in \mathbb{Z}$ and finitely generated groups G and H. We denote the set of n-Lipschitz functions from G to H by $\mathrm{Lip}_n(G, H)$. The derivative is the map $d : \mathrm{Lip}_n(G, H) \to (B_H(N, 1_H)^S)^G$ which takes $f \in \mathrm{Lip}_n(G, H)$ to

$$df : g \mapsto (s \mapsto f(g)^{-1} f(gs)).$$

We write $\langle df(g), s\rangle$ for $df(g)$ evaluated at s.

Observe that $f \in \mathrm{Lip}_n(G, H)$ is determined by $f(1)$ and df. We now state the primary theorem of this section.

Theorem 0.4

If G is finitely presented, then for any integer n and finitely generated group H, we have that

$$\{df : f \in \mathrm{Lip}_n(G, H)\} \subset (B_H(N, 1_H)^S)^G$$

is a subshift of finite type.

Proof of Theorem 0.4. Let X_n denote the set $\{df : f \in \mathrm{Lip}_n(G, \mathbb{Z})\}$, let $\mathbb{A}$ denote $B_{\mathbb{Z}}(n, 0)^S$, and let the natural number $K_G \geq 2$ be such that G is presented with respect to S by relators of length at most K_G. We wish to prove that X_n is a subshift of finite type, meaning that membership in X_n is determined by some finite list of local conditions. What sort of local conditions must derivatives satisfy? At least one is immediately obvious, namely we know, for any $g \in G$, that

$$\langle df(g), s\rangle = \langle df(gs), s^{-1}\rangle.$$

More generally, if some word $w = s_0 \dots s_k$ in S^* is a relation, then we must have that the telescoping product

$$\langle df(g), s_0\rangle \langle df(gs_0), s_1\rangle \dots \langle df(gs_0 \dots s_{k-1}), s_k\rangle$$

represents 1_H for any $g \in G$. For a fixed w, this represents a local condition on df, since the product depends only on the values taken by df in $B_G(|w|, 1)$. Since G is finitely presented, we might hope that X_n is defined by a finite set of conditions of this nature, and this is in fact the case. We begin by giving the expected notation for products like the above.

Definition 0.10 Let g be an element of G, let $w \in S^*$ be some word $s_0 s_1 \dots s_k$ (where $s_i \in S$,) and let σ be an element of $\mathbb{A}^G$. We define $\int_{g \cdot w} \sigma$ as the product

$$\langle \sigma(g), s_0 \rangle \langle \sigma(gs_0), s_1 \rangle \langle \sigma(gs_0 s_1), s_2 \rangle \dots \langle \sigma(gs_0 \dots s_{k-1}), s_k \rangle.$$

We now record some properties of this gadget.

Lemma 0.4

The integral has the following familiar properties.

- **Locality.** *The value of $\int_{g \cdot w} \sigma$ is determined by $\sigma|_{B_G(|w|, g)}$.*
- **Additivity.** *If $w_1, w_2 \in S^*$, and h is the image of w_1 in G, then*

$$\int_{g \cdot w_1} \sigma \int_{gh \cdot w_2} \sigma = \int_{g \cdot w_1 w_2} \sigma.$$

- **Fundamental theorem.** *Suppose $f \in \mathrm{Lip}_n(G, H)$. Then for $g \in G$ and $w \in S^*$, we have*

$$\int_{g \cdot w} df = f(g)^{-1} f(gw).$$

Proof 0.4 Locality and additivity follow immediately from Definition 0.10. The fundamental theorem follows from collapsing the telescoping product

$$\int_{g \cdot w} \sigma = \langle \sigma(g), s_0 \rangle \langle \sigma(gs_0), s_1 \rangle \langle \sigma(gs_0 s_1), s_2 \rangle \dots \langle \sigma(gs_0 \dots s_{k-1}), s_k \rangle$$

$$= (f(g)^{-1} f(gs_0))(f(gs_0)^{-1} f(gs_0 s_1))(f(gs_0 s_1)^{-1} f(gs_0 s_1 s_2)) \dots (f(gs_0 \dots s_{k-1})^{-1} f(gs_0 \dots s_k))$$

$$= f(g)^{-1} f(gw).$$

We will now proceed with the proof of theorem 0.4. Let Y_n consist of all $\sigma \in \mathbb{A}^G$ such that for any $g \in G$,

$$\int_{g \cdot w_1} \sigma = \int_{g \cdot w_2} \sigma$$

whenever the words $w_1, w_2 \in S^*$ are such that $|w_1|, |w_2| \leq K_G$ and w_1 and w_2 represent same element of G. The fundamental theorem (Lemma 0.4) shows that Y_n contains X_n and locality (Lemma 0.4) shows that Y_n is a subshift of finite type. To prove Theorem 0.4, it thus suffices to show that $X_n \supset Y_n$, i.e., that every element of Y_n is the derivative of some element of $\mathrm{Lip}_n(G, H)$.

Lemma 0.5

For any $\sigma \in Y_n$, the quantity

$$\int_{g \cdot w} \sigma$$

depends only on g and the value w represents in G.

Proof 0.5 Let w and w' be words representing the same element of G. Then there exists a homotopy

$$w = w_0, w_1, \ldots, w_{k-1}, w_k = w'$$

from w to w', meaning a sequence of words $w_i \in S^*$ such that each pair (w_i, w_{i+1}) has the form $(uvx, uv'x)$ where $u, v, v', x \in S^*$ are such that v and v' have length $\leq K_G$ and represent the same element of G. But then by repeated application of Lemma 0.4 we have that

$$\int_{g \cdot uvx} \sigma = \int_{g \cdot u} \sigma \int_{gu \cdot v} \sigma \int_{guv \cdot x} \sigma$$
$$= \int_{g \cdot u} \sigma \int_{gu \cdot v'} \sigma \int_{guv' \cdot x} \sigma = \int_{g \cdot uv'x} \sigma.$$

It follows that $\int_{g \cdot w} \sigma = \int_{g \cdot w'} \sigma$ as desired.

Given $\sigma \in Y_n$, we may now define a function $f \in \mathrm{Lip}_n(G, H)$ with derviative σ by taking $f(g)$ to be

$$\int_{1_g \cdot w} \sigma$$

for any w representing g (by Lemma 0.5, the choice of w is irrelevant). We can see that f is n-Lipschitz by the fact that for words w_1, w_2 representing $g_1, g_2 \in G$ respectively, we have

$$d(f(g_1), f(g_2)) = f(g_1)^{-1} f(g_2) = \left| \left(\int_{1_G \cdot w_1} \sigma \right)^{-1} \int_{1_G \cdot w_2} \sigma \right|_T$$
$$= \left| \int_{g_1 \cdot w_1^{-1} w_2} \sigma \right|_T = \left| \int_{g_1 \cdot w} \sigma \right|_T \leq n d(g_1, g_2)$$

for a geodesic word w representing $g_1^{-1} g_2$ (we have used the fact that $\int_{g_1 \cdot w_1^{-1}} \sigma$ is the inverse of $\int_{1_G \cdot w_1} \sigma$ for $\sigma \in Y_n$, which follows from Lemma 0.5.) We see that $df = \sigma$ by similar reasoning. Hence, we have shown that the subshift of finite type Y_n is equal to X_n, thus establishing Theorem 0.4.

0.4 Quasi isometries and subshifts.

In this section we prove Corollary 0.1 which says that having a strongly aperiodic subshift of finite type is a QI invariant—among torsion free, finitely presented groups. We begin with quasi isometric groups G and H satisfying our conditions, together with a subshift of finite type $X_H \subset A^H$. We will use this data to construct a subshift of finite type X_G on G which inherits certain properties of X_H (Theorem 0.6). In particular, Corollary 0.1 will follow from the fact that X_G will be strongly aperiodic when X_H is.

Constructing the "pull back subshift" X_G described above requires us to parametrize objects we call n-QI pairs (Definition 0.11,) which consist of a Lipschitz function $f : G \to H$ and a Lipschitz 2-sided quasi inverse $F : H \to G$. Just as we found a subshift of finite type on a finitely presented group G which parametrized Lipschitz functions (Theorem 0.4), so too we parameterize our n-QI pairs in terms of local data on G (Theorem 0.5). We then form X_G by adding some extra data coming from X_H. This means that a point of X_G encodes some n-QI pair (f, F) as well as a configuration $\sigma \in X_H$.

0.4.1 Parametrizing quasi isometries

Suppose that G and H are finitely presented groups equipped with fixed generating sets. It would be fortuitous if the collection of all derivatives df, as f ranges over n-quasi isometries $G \to H$, formed a subshift of finite type, but it seems unlikely that this is the case. Suppose that you are trying to determine whether some unknown function $f : G \to H$ is a quasi isometry, and you are told only which patterns df realizes on N-balls around points of G (for some fixed N). That is, you are given the set of patterns

$$\{(df) \cdot g|_{B_G(N, 1_G)} : g \in G\} \subset (B_H(n, 1_H)^S)^{B_G(N, 1_G)}$$

without knowning which patterns correspond to which g. It is of course easy to tell whether f is n-Lipschitz, and with a little work we can tell whether f is locally n-quasi surjective, in the sense that $B_H(2n, f(g))$ can always be covered by n-balls around points of $f(B_G(m, g))$ (for fixed m,) which is a neccessary condition for being a quasi isometry. Unfortunately, it seems difficult to verify global quasi injectivity (the property that f cannot take distant points too close to each other) based just on this data. Hence, if we wish to parameterize quasi isometries f, we will also need to encode data about a quasi inverse to f (see the following definition).

Definition 0.11 Let G and H be groups finitely generated by S and T respectively. Suppose given $f : G \to H$, $F : H \to G$, and $n \in \mathbb{N}$.

- If $|g^{-1} \cdot (F \circ f)(g)|_S \leq n$ for all $g \in G$, we say that F is a left n-quasi inverse to f and f is a right n-quasi inverse to F.
- We say that F and f are n-quasi inverses of each other if they are left (and hence right) n-quasi inverses of each other. (This exactly means that, in the uniform metric, $F \circ f$ and $f \circ F$ are within distance n of Id_G and Id_H respectively).
- We say that (f, F) is an n-QI pair if F and f are n-Lipschitz and are n-quasi inverses of each other.

The set of all n-QI pairs is denoted $\mathrm{QIP}_n(G, H)$.

The following lemma says that being part of a QI pair is the same thing as being a quasi isometry.

Lemma 0.6

Let G and H be groups, generated by finite sets S and T respectively. A function $f : G \to H$ is a quasi isometry if and only there exists a natural number n and a function $F : H \to G$ such that (f, F) is an n-QI pair.

Proof 0.6 Suppose $(f, F) \in \mathrm{QIP}_n(G, H)$. Then for any $x, y \in G$,

$$d(x, y) \leq d(x, (F\circ f)(x)) + d((F\circ f)(x), (F\circ f)(y)) + d((F\circ f)(y), y) \leq 2n + nd(f(x), f(y)),$$

and thus:

$$d(f(x), f(y)) \geq \frac{d(x, y)}{n} - 2.$$

Since f is Lipschitz, this implies that f is a quasi isometric embedding. But f is quasi surjective because for $h \in H$ we have $d(f(F(h)), h) \leq n$. Hence, f is a quasi isometry.

Conversely, suppose f is an N-quasi isometry. We will now define $F : H \to G$ such that (f, F) is a QI pair. Using N-quasi surjectivity, for each $h \in H$, choose an $F(h)$ in G such that $d(f(F(h)), h) \leq N$. Since f is an N-quasi isometric embedding, we know that for all $h_1, h_2 \in H$, we have

$$\frac{d(F(h_1), F(h_2))}{N} - N \leq d(f(F(h_1)), f(F(h_2))) \leq d(f(F(h_1)), h_1) + d(h_1, h_2) + d(h_2, f(F(h_2)))$$

$$\leq 2N + d(h_1, h_2).$$

Hence

$$d(F(h_1), F(h_2)) \leq N d(h_1, h_2) + 3N^2,$$

so F is $3N^2 + N$-Lipschitz (as 1 is the smallest positive distance in H). We can see that f is $2N$-Lipschitz because, since 1 is the smallest positive distance in G,

$$d(f(x), f(y)) \leq N d(x, y) + N \leq 2N d(x, y).$$

By definition, F is a right N-quasi inverse to f. To see that it is a left quasi inverse, note that for $g \in G$ we have

$$d(f((F \circ f)(g)), f(g)) = d((f \circ F)(f(g)), f(g)) \leq N.$$

It follows that

$$d((F \circ f)(g), g) \leq N d(f((F \circ f)(g)), f(g)) + N \leq N^2 + N.$$

So, taking n to be greater than each of $\{2N, 3N^2 + N, N, N^2 + N\}$, we have that (f, F) is an n-QI pair.

We now define the local data ℓ_{Ff} which will encode an n-QI pair (f, F). Although we will not immediately need the function Q_{Ff} which is also defined below, it seems logical to define it here.

Definition 0.12 (Local data.) To a pair $(f, F) \in \mathrm{QIP}_n(G, H)$, we associate the following two functions.

- The function $\ell_{Ff} : G \to B_G(nN + n, 1_G)^{B_H(N,1_H)}$ is given by setting $\ell_{Ff}(g)$ to be
$$\ell_{Ff}(g) : h \mapsto g^{-1}F(f(g)h)$$
for $g \in G$, where $N = 2n^2 + 1$.

- The function $Q_{Ff} : H \to H$ is given by $Q_{Ff}(h) = h^{-1}f(F(h))$.

We write $\langle \ell_{Ff}(g), h\rangle$ for $(\ell_{Ff}(g))(h)$.

We should think of $\ell_{Ff}(g)$ as telling us the values of F near $f(g)$, via the formula $F(f(g)h) = g\langle \ell_{Ff}(g), h\rangle$. The value $N = 2n^2 + 1$ has been chosen so that we see enough such values around each $f(g)$ that we can reconstruct F globally. We may now state our theorem.

Theorem 0.5

Suppose G, H are finitely presented groups generated by sets S and T respectively. Suppose that n is a natural number, then, writing $\mathbb{A}$ for the finite set

$$B_H(n, 1_H)^S \times B_G(n + nN, 1_G)^{B_H(N,1_H)},$$

we have that

$$\{(df, \ell_{Ff}) : (f, F) \in \mathrm{QIP}_n(G, H)\} \subset \mathbb{A}^G$$

is a subshift of finite type. (Here $N = 2n^2 + 1$ as in Definition 0.12).

Proof 0.7 We write K_G for the length of the longest relator of G in some fixed presentation for G with respect to S. Our strategy is as follows.

- First, we produce seven local conditions on $\mathbb{A}^G$,

- then we show that these conditions are satisfied by $(df, \ell_{Ff}) \in \mathbb{A}^G$ whenever (f, F) is an n-QI pair,
- and finally we show that if $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfies these conditions, we can construct an n-QI pair (f, F) such that $(df, \ell_{Ff}) = \sigma$.

In general, the second bullet point is easy, and the third is hard. The reader should think of $\sigma_\ell(g)$ as giving us a local guess about how to reconstruct F near $f(g)$ and the later conditions as telling us that these local guesses are compatible for different g. In practice, after stating a condition and proving that it is local, we will immediately verify that it holds for pairs (df, ℓ_{Ff}).

We now summarize the conditions (in order, because each will be defined for $\sigma \in \mathbb{A}^G$ which satisfy all the previous conditions). The reader should think of conditions **C**f, **C**$F\ell$, **NC**F, and **RC**F as "integrability conditions", saying that when we try to reconstruct F or f in two different ways, we get the same result. On the other hand, conditions **RI**, **L**F, and **LI** are "regularity conditions" on σ_ℓ, saying that at each point it looks like the restriction of an n-Lipschitz n-quasi inverse to a function from G to H with derivative σ_d.

- The first condition on $(\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ is **C**f (Definition 0.13,) which ensures that σ_d is the derivative of a unique n-Lipschitz function $f : G \to H$ such that $f(1_G) = 1_H$. (See Theorem 0.4).
- For σ which satisfies **C**f, we have a local guess for F near $f(g)$, namely, $h \mapsto g\langle\sigma_\ell(g), f(g)^{-1}h\rangle$. Condition **L**$F$ (Definition 0.14) says that this local guess is n-Lipschitz.
- Similarly, condition **RI** (Definition 0.14) says that our local guess for F is a local right n-quasi inverse to f.

Once σ satisfies $\mathbf{C}f$, $\mathbf{RI}$, and $\mathbf{L}F$, we will be able to construct, for each $g \in G$, a function $F_\sigma(fg; w)$ on words $w \in T^*$, which recovers $F(f(g)w)$ when σ is of the form (df, ℓ_{Ff}) for some pair (f, F) in $\mathrm{QIP}_n(G, H)$, as described in Definition 0.16, Figure 6, and Lemma 0.7. See the notes after Definition 0.16 for an explanation of our idiosyncratic notation. Three of the remaining four conditions on σ will be defined in terms of this gadget $F_\sigma(fg; w)$, which is "local" in the sense described by Lemma 0.8(b). These integrability conditions will say that, as g and w vary, $F_\sigma(fg; w)$ indeed behaves as if it were of the form $F(f(g)w)$ for some n-QI pair (f, F), in that $F_\sigma(fg_1; w_1) = F_\sigma(fg_2; w_2)$ for certain g_i and w_i such that $f(g_1)w_1 = f(g_2)w_2$.

- Condition $\mathbf{C}F\ell$ (Definition 0.17) will mandate a local compatibility between F_σ and σ_ℓ. In particular, it says that if $g_1, g_2 \in G$ are sufficiently close, and $f(g_1)w = f(g_2)h$ for some sufficiently small word w and $h \in B_H(n+1, 1_H)$, then

$$F_\sigma(fg_1; w) = g_2\langle \sigma_\ell(g_2), h\rangle.$$

- Along the same lines, condition $\mathbf{RC}F$ (Definition 0.18,) will say that $F_\sigma(fg; w_1) = F_\sigma(fg; w_2)$ for short words $w_1, w_2 \in T^*$ which define the same element of H. This (along with the previous condition) will imply that $w \mapsto F_\sigma(fg; w)$ descends to a well defined function on H (Lemma 0.11).

- Condition $\mathbf{NC}F$ (Definition 0.20) will say that $F_\sigma(fg; -)$ and $F_\sigma(fgs; -)$ are compatible whenever $s \in S$, meaning that for all $h \in H$,

$$F_\sigma(fg; f(g)^{-1}h) = F_\sigma(fgs; f(gs)^{-1}h).$$

  This in turn will imply that all of the $F_\sigma(fg; -)$ come from a single function $F : H \to G$, and we can construct F using, say, $F_\sigma(f1_G; -)$.

- Finally, condition **LI** (Definition 0.22) will confirm that our putative F is in fact a left n-quasi inverse to f, thus completing the proof of the theorem.

We now proceed with the plan outlined above: define local conditions, show that they are satisfied if σ arises from an n-QI pair, and show conversely that σ satisfying these conditions is of the form (df, ℓ_{Ff}) for an n-QI pair (f, F).

Defining local conditions. We now give the formal definitions of all these conditions, and verify that each is local and satisfied by (df, ℓ_{Ff}) for $(f, F) \in \mathrm{QIP}_n(G, H)$. Recall that K_G is the length of the longest relator of G.

Definition 0.13 (Consistency of f) We say that $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfies condition $\mathbf{C}f$ if for all $w_1, w_2 \in S^*$ which represent the same element of G and have length at most K_G, we have (for all $g \in G$) that

$$\int_{g \cdot w_1} \sigma_d = \int_{g \cdot w_2} \sigma_d.$$

Note that this condition already appeared in the proof of Theorem 0.4. See the remarks before Lemma 0.5

Locality of $\mathbf{C}f$. Recall that $\int_{g \cdot w} \sigma_d$ is the product of $\sigma_d(gv)$ as v ranges over prefixes of w. Hence, to check $\mathbf{C}f$, we only need to look at $\sigma|_{B_G(g, K_G)}$.

Condition $\mathbf{C}f$ is satisfied for σ arising from an n-QI pair. If $\sigma = (df, \ell_{Ff})$ for some $(f, F) \in \mathrm{QIP}_n(G, H)$, then by Lemma 0.4, σ satisfies $\mathbf{C}f$ because for $g \in G$ and $w_1, w_2 \in S^*$ representing the same element,

$$\int_{g \cdot w_1} \sigma_d = f(g)^{-1} f(g w_1) = f(g)^{-1} f(g w_2) = \int_{g \cdot w_2} \sigma_d.$$

Conversely, Lemma 0.5 says that if σ satisfies $\mathbf{C}f$, there is a unique f such that $f(1_G) = 1_H$ and $df = \sigma_d$ (namely, one defines $f(g)$ to be $\int_{1_G \cdot w} \sigma_d$ for any $w \in S^*$ which represents g). From now on, we will assume that σ satisfies $\mathbf{C}f$, and that f is the unique function such that $df = \sigma_d$ and $f(1_G) = 1_H$. In particular, we will use σ_d and df interchangeably. Additionally, we will write expressions like $\int_{g_1}^{g_2} df$, understanding this to be the well defined quantity $f(g_1)^{-1} f(g_2)$.

Defining condition LF. Recall that, for an n-QI pair (f, F), we have that the function

$$h \mapsto g\langle \ell_{Ff}(g), f(g)^{-1} h\rangle$$

represents F on $B_H(N, f(g))$. Condition $\mathbf{L}F$ will say that this function must be n-Lipschitz.

Definition 0.14 (Lipschitzness of F) We say that $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfies condition $\mathbf{L}F$ if, for all $h_1, h_2 \in B_H(N, 1_H)$, and all $g \in G$,

$$d_G(\langle \sigma_\ell(g), h_1\rangle, \langle \sigma_\ell(g), h_2\rangle) \leq n d_H(h_1, h_2).$$

Locality of LF. Condition $\mathbf{L}F$ is manifestly local, since verifying it at g only requires us to know $\sigma(g)$.

Condition LF is satified for σ arising from an n-QI pair. Suppose $(f, F) \in \mathrm{QIP}_n(G, H)$. Since F is n-Lipschitz, we see that

$$h \mapsto g\langle \ell_{Ff}(g), f(g)^{-1} h\rangle$$

is n-Lipschitz on $B_H(f(g), N)$. It follows that

$$h \mapsto \langle \ell_{Ff}(g), h\rangle$$

is n-Lipschitz on $B_H(N, 1_H)$ (i.e., (df, ℓ_{Ff}) satisfies $\mathbf{L}F$).

In a similar vein, the next condition will say that

$$h \mapsto g\langle \sigma_\ell(g), f(g)^{-1}h\rangle$$

is a local right n-quasi inverse to f.

Definition 0.15 (Local right inverse) We say that $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfies condition **RI** if for all $h \in B_H(N, 1_H)$, and all $g \in G$,

$$d_H\left(h \quad , \quad \int_g^{g\langle \sigma_\ell(g), h\rangle} df\right) \leq n.$$

Locality of RI. To verify condition **RI** at g, we need only know enough values of σ to compute the integral. Thus, it suffices to know $\sigma|_{B_G(nN+n,g)}$, so **RI** is local.

Condition RI is satisfied for σ arising from an n-QI pair. Given $(f, F) \in \mathrm{QIP}_n(G, H)$, we have that F is a local right n-quasi inverse to f, so that

$$(f \circ F)(f(g)h) = f(g) \int_g^{g\langle \ell_{Ff}(g), h\rangle} df$$

is within distance n of $f(g)h$. It follows that (df, ℓ_{Ff}) satisfies condition **RI**.

Developing a candidate for F Consider the following problem. We are given, for some $h \in H$, the values of $F(h)$, $h^{-1}f(F(h))$, and $\ell_{Ff}(F(h))$, where $(f, F) \in \mathrm{QIP}_n(G, H)$ is unknown to us. Additionally, we are given df in the $n + nN$ neighborhood of $F(h)$. From this data, we are asked to reconstruct $F(ht)$ for some $t \in T$. We can solve this probelm because ℓ_{Ff} tells us the values of F in the N-neighborhood of $f(F(h))$, and $d_H(f(F(h)), ht) \leq n + 1 \leq N$. Hence we have the ungainly formula

$$F(ht) = F(h)\langle \ell_{Ff}(F(h)), f(F(h))^{-1}ht\rangle$$

$$= F(h)\langle \ell_{Ff}(F(h)), Q^{-1}t\rangle$$

where Q is the known quantity $h^{-1}f(F(h))$.

Now suppose that we start with $\sigma \in \mathbb{A}^G$ which satisfies conditions **C**f, **RI**, and **L**F. If it happens that $\sigma = (df, \ell_{Ff})$ for some $(f, F) \in \mathrm{QIP}_n(G, H)$, then F should satisfy a recursion given by the formula above, and using this recursion we should be able to recover F. We now make this precise, working in terms of the reparameterizations $h \mapsto F(f(g)h)$ and $h \mapsto Q_{Ff}(f(g)h)$ instead directly in terms of F. Specifically, in the definition below (depicted in Figure 6), $F_\sigma(fg; v)$ is intended as a guess at the value of $F(f(g)v)$ and $Q_\sigma(fg; v)$ is a guess as to $Q_{Ff}(f(g)v)$, based on σ-data in some $O(|v|)$ neighborhood of g (recall Q_{Ff} from Definition 0.12). A careful reader will observe that we must do a little work to ensure that this definition is meaningful.

Definition 0.16 Given $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfying conditions **C**f, **RI**, and **L**F, and further given $g \in G$, we recursively define functions $F_\sigma(fg; -) : T^* \to G$ and $Q_\sigma(fg; -) : T^* \to B_H(n, 1_H)$ as follows, using ϵ to denote the empty word in T^*. We initialize $F_\sigma(fg; -)$ by setting

$$F_\sigma(fg; \epsilon) = g\langle \sigma_\ell(g), 1_H\rangle.$$

If $F_\sigma(fg; v)$ has been defined for some $v \in T^*$, we then define

$$Q_\sigma(fg; v) = (f(g)v)^{-1}f(F_\sigma(fg; v)),$$

and, for $t \in T$,

$$F_\sigma(fg; vt) = F_\sigma(fg; v)\langle \sigma_\ell(F_\sigma(fg; v)), Q_\sigma(fg; v)^{-1}t\rangle$$

In general, $F_\sigma(fg; v)$ is not determined by the image of v in H.

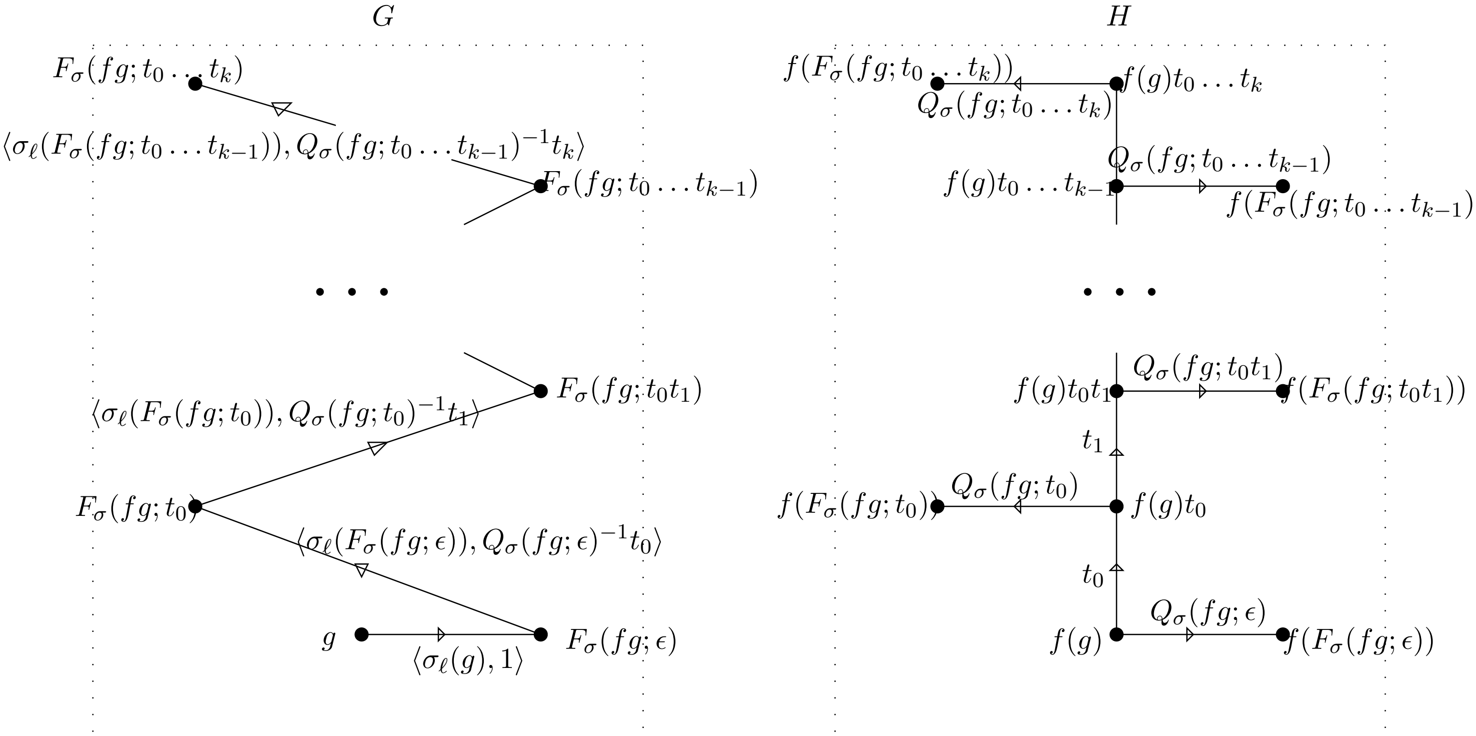

Figure 6 : The functions $F_\sigma(fg; -)$ and $Q_\sigma(fg; -)$ on words in T, are defined recursively. Roughly speaking, when σ arises from a QI-pair (f, F), the defining recursions for $F_\sigma(fg; t_0 \dots t_k)$ and $Q_\sigma(fg; t_0 \dots t_k)$ answer the question, "what are $F(f(g)vt)$ and $Q_{Ff}(f(g)vt)$ in terms of $F(f(g)v)$ and $Q_{Ff}(f(g)v)$?"

Remark on notation. Before showing that $F_\sigma(fg; -)$ is in fact well defined, we remark that $F_\sigma(fg; w)$ is determined by σ, g, and w, and the f is entirely decorative. Indeed, as we saw in the remarks following the definition of condition $\mathbf{C}f$, f is entirely determined by σ since we assume that $f(1_G) = 1_H$. We write $F_\sigma(fg; w)$ instead of something more natural like $F_\sigma(g; w)$ because $F_\sigma(fg; w)$ clearly aspires to be $F(f(g)w)$. Similar considerations apply for $Q_\sigma(fg; w)$. In writing this paper, we tried many different notations for these gadgets, and any other choice would make future equations totally opaque.

$F_\sigma(fg;-)$ **is well defined.** Observe that in order to even define $F_\sigma(fg;vt)$, we needed to know that $Q_\sigma(fg;v)$ was in $B_H(N-1,1_H)$, so that we could evaluate $\langle\sigma_\ell(F_\sigma(fg;v)),Q_\sigma(fg;v)^{-1}t\rangle$. In fact, the definition even promised that $Q_\sigma(fg;v)$ would always be in $B_H(n,1_H)$. We will now use **RI** to prove this inductively. First, condition **RI** directly implies that $Q_\sigma(\epsilon)\in B_H(n,1_H)$. Now, assume for induction that $Q_\sigma(v)$ is in $B_H(n,1_H)$, so that $F_\sigma(fg;vt)$ is well defined. Then it suffices to show that $Q_\sigma(fg;vt)$ is in $B_H(n,1_H)$. We have

$$|Q_\sigma(fg;vt)|_T = d_H(f(g)vt, f(F_\sigma(fg;vt)))$$

$$= d_H(f(F_\sigma(fg;v))^{-1}f(g)vt, f(F_\sigma(fg;v))^{-1}f(F_\sigma(fg;vt)))$$

$$= d_H\left(Q_\sigma(fg;v)^{-1}t, \int_{F_\sigma(fg;v)}^{F_\sigma(fg;vt)}\right)$$

$$= d_H\left(Q_\sigma(fg;v)^{-1}t, \int_{F_\sigma(fg;v)}^{F_\sigma(fg;v)\langle\sigma_\ell(F_\sigma(fg;v)),Q_\sigma(fg;v)^{-1}t\rangle}\right)\leq n,$$

as desired, by **RI**. The fact that we can define $F_\sigma(fg;v)$ (for σ satisfying some finite set of local conditions) constitutes most of the actual mathematical content of this proof, so we have been very careful here.

Properties of $F_\sigma(fg;-)$**.** Before defining the last four conditions, which will force the $F_\sigma(fg;v)$ to glue together into a well defined n-quasi inverse to f, we will prove three trivial lemmas about $F_\sigma(fg;-)$ and $Q_\sigma(fg;-)$. The first, Lemma 0.7, says that $F_\sigma(fg;w)$ and $Q_\sigma(fg;w)$ recover $F(f(g)w)$ and $Q_{Ff}(f(g)w)$ when $\sigma=(df,\ell_{Ff})$. The second, Lemma 0.8, says that to compute $g^{-1}F_\sigma(fg;w)$ and $Q_\sigma(fg;w)$, we only need to know σ on a ball of radius $O(|w|+1)$ around g. The third, Lemma 0.9 gives a condition for when we can expect two expressions of the form $F_\sigma(fg;w)$ to be equal. The reader is advised to skip the proofs.

Lemma 0.7

If $(f, F) \in \mathrm{QIP}_n(G, H)$, *and* $\sigma = (df, \ell_{Ff})$, *then for any* $w \in T^*$, *we have that* $F_\sigma(fg; w) = F(f(g)w)$ *and* $Q_\sigma(fg; w) = Q_{Ff}(f(g)w)$.

Proof 0.8 First observe that

$$F_\sigma(fg; \epsilon) = g\langle \ell_{Ff}(g), 1\rangle = F(f(g)).$$

To obtain the desired equalities, it now suffices to check that $w \mapsto F(f(g)w)$ and $w \mapsto Q_{Ff}(f(g)w)$ satisfy the defining recursion for $F_\sigma(fg; -)$ and $Q_\sigma(fg; -)$. We have, by 0.12,

$$Q_{Ff}(f(g)v) = (f(g)v)^{-1} F(f(g)v),$$

and

$$F(f(g)vt) = F(f(g)v)(F(f(g)v)^{-1} F(f(g)vt))$$

$$= F(f(g)v)\langle \sigma_\ell(F(f(g)v)), ((f \circ F)(f(g)v))^{-1} \cdot (f(g)vt)\rangle$$

$$= F(f(g)v)\langle \sigma_\ell(F(f(g)v)), Q(f(g)v)^{-1} t\rangle.$$

Lemma 0.8

(a) *For all* $g \in G$ *and* $w \in T^*$, *we have*

$$|g^{-1} F_\sigma(fg; w)|_T \leq n + (n^2 + n)|w|$$

(b) *Both* $g^{-1} F_\sigma(fg; w)$ *and* $Q_\sigma(fg; w)$ *can be computed from* $\sigma|_{B_G(n+(n^2+n)|w|, g)}$.

Proof 0.9 (a) Observe that by **RI**

$$g^{-1} F_\sigma(fg; \epsilon) = \langle \sigma_\ell(g), 1\rangle \in B_G(n, 1_G).$$

Assume for induction that $g^{-1} F_\sigma(fg; v) \in B_G(n + n|v|, 1_G)$. Then for $t \in T$,

$$|g^{-1} F_\sigma(fg; vt)| = |g^{-1} F_\sigma(fg; v)\langle \sigma_\ell(F_\sigma(fg; v)), Q_\sigma(fg; v)^{-1} t\rangle|$$

$\leq |g^{-1}F_\sigma(fg;v)| + |\langle \sigma_\ell(F_\sigma(fg;v)), Q_\sigma(fg;v)^{-1}t\rangle| \leq n + (n^2+n)|v| + (n^2+n)$

by $\mathbf{L}F$ and the fact that $|Q_\sigma(fg;v)t| \leq n+1$.

(b) We know that $g^{-1}F_\sigma(fg;\epsilon)$ is determined by $\sigma(g)$. Observe that

$$Q_\sigma(fg;\epsilon) = f(g)^{-1}f(g\langle\sigma_\ell(g),1\rangle) = \int_g^{F_\sigma(fg;\epsilon)} df$$

is determined by $\sigma|_{B_G(n)}$. Assume now, for induction, that for some $v \in T^*$ we know that $g^{-1}F_\sigma(fg;v)$ and $Q_\sigma(fg;v)$ are determined by $\sigma|_{B_G(n+(n^2+n)|v|,g)}$. Then for $t \in T$, we have

$$g^{-1}F_\sigma(fg;vt) = g^{-1}F_\sigma(fg;v)\langle\sigma_\ell(F_\sigma(fg;v)), Q_\sigma(fg;v)^{-1}t\rangle$$

so $g^{-1}F_\sigma(fg;vt)$ is determined by $\sigma|_{B_G(n+(n^2+n)|v|,g)}$ (by part (a) together with our inductive hypothesis). Now observe that

$$Q_\sigma(fg;vt) = (f(g)vt)^{-1}f(F_\sigma(fg;vt)) = (vt)^{-1}\int_g^{F_\sigma(fg;vt)} df,$$

so by part (a), we see that $Q_\sigma(fg;vt)$ is determined by $\sigma|_{B_G(n+(n^2+n)|vt|,g)}$.

Lemma 0.9

If $f(g_1)w_1 = f(g_2)w_2$ and $F_\sigma(fg_1;w_1) = F_\sigma(fg_2;w_2)$ for some $g_1, g_2 \in G$ and $w_1, w_2 \in T^*$, then for any $v \in T^*$,

$$F_\sigma(fg_1;w_1v) = F_\sigma(fg_2;w_2v).$$

Proof 0.10 It suffices to consider the case where $|v| = 1$ i.e., $v \in T$. Then we have

$$Q_\sigma(fg_1;w_1) = (f(g_1)w_1)^{-1}f(F_\sigma(fg_1,w_1))$$

$$= (f(g_2)w_2)^{-1}f(F_\sigma(fg_2,w_2)) = Q_\sigma(fg_2;w_2),$$

and hence

$$F_\sigma(fg_1;w_1v) = F_\sigma(fg_1;w_1)\langle\sigma_\ell(F_\sigma(fg_1;w_1)), Q_\sigma(fg_1;w_1)^{-1}v\rangle$$

$$= F_\sigma(fg_2;w_2)\langle\sigma_\ell(F_\sigma(fg_2;w_2)), Q_\sigma(fg_2;w_2)^{-1}v\rangle = F_\sigma(fg_2;w_2v).$$

Defining condition C$F\ell$. The following condition on σ will ensure that F_σ and σ_ℓ are locally compatible, in the following sense. Assume as usual that $\sigma = (df, \ell_{Ff})$ for some unknown $(f, F) \in \mathrm{QIP}_n(G, H)$. Suppose we have some $h_0 \in H$ which is close to both $f(g_1)$ and $f(g_2)$ for some $g_1, g_2 \in G$. Then we could try to find $F(h_0)$ by computing $F_\sigma(fg_1; w)$ (where w represents $f(g_1)^{-1}h_0$) or by computing $g_2\langle\sigma_\ell(g_2), h\rangle$ (where $h = f(g_2)^{-1}h_0$). We would like these calculations to give the same result so long as h and w are not too big. In fact, this is a local condition.

Definition 0.17 (Compatibility of F_σ with σ_ℓ) We say that $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfies condition **C**$F\ell$ if

$$F_\sigma(fg_1; w) = g_2\langle\sigma_\ell(g_2), h\rangle$$

whenever

- $g_1, g_2 \in G$ with $d_G(g_1, g_2) \leq 2n + 1$,
- $h \in B_H(n + 1, 1_H)$,
- and $w \in T^*$ with $\ell(w) \leq n$

are such that $f(g_1)w = f(g_2)h$.

Locality of C$F\ell$. To check the condition at g_1, we just have to compute, for various bounded values of h, w, and $g_1^{-1}g_2$, the value of

$$F_\sigma(fg_1; w)^{-1}g_2\langle\sigma_\ell(g_2, h)\rangle = (g_1^{-1}F_\sigma(fg_1; w))^{-1}(g_1^{-1}g_2\langle\sigma_\ell(g_2), h\rangle).$$

By Lemma 0.8, this calculation is local.

Condition $\mathbf{C}F\ell$ is satisfied for σ arising from an n-QI pair. Let $\sigma = (df, \ell_{Ff})$ for some $(f, F) \in \mathrm{QIP}_n$. Then if g_1, g_2, w, h are as in the statement of condition $\mathbf{C}F\ell$, we have $f(g_1)w = f(g_2)h$ in H and hence, using Lemma 0.7,

$$F_\sigma(fg_1; w) = F(f(g_1)w) = F(f(g_2)h) = g_2\langle \ell_{Ff}(g_2), h\rangle,$$

so σ satisfies $\mathbf{C}F\ell$.

The following lemma implies that if σ satisfies all conditions up through $\mathbf{C}F\ell$, then $F_\sigma(fg; uv)$ is determined by $F_\sigma(fg; u)$ together with the values taken by σ on a $O(|v|)$-neighborhood of $F_\sigma(fg; u)$.

Lemma 0.10

If σ satisfies conditions $\mathbf{C}f$, $\mathbf{RI}$, $\mathbf{L}F$, and $\mathbf{C}F\ell$, then for words $u, v \in T^*$ and $g \in G$, we have

$$F_\sigma(fg; uv) = F_\sigma(fF_\sigma(fg; u); Q^{-1}v)$$

where Q is any word of length at most n representing $Q_\sigma(fg; u)$.

Proof 0.11 We proceed by induction on v, so we must first handle the case where $v = \epsilon$. If u is also trivial, then the statement reduces to

$$g\langle \sigma_\ell(g), 1\rangle = F_\sigma(fF_\sigma(fg; \epsilon); Q^{-1}),$$

which is true by condition $\mathbf{C}F\ell$, applied with $g_1 = g$ and $g_2 = F_\sigma(fg; \epsilon)$. Hence, we may assume that $u = xt$ for some $x \in T^*$ and $t \in T$. We obtain

$$F_\sigma(fg; u) = F_\sigma(fg; x)\langle \sigma_\ell(F_\sigma(fg; x)), Q_\sigma(fg; x)^{-1}t\rangle$$

which is equal to $F_\sigma(fF_\sigma(fg; u); Q^{-1})$ by condition $\mathbf{C}F\ell$, applied with $g_1 = F_\sigma(fg; x)$ and $g_2 = F_\sigma(fg; u)$ (and thus $h = Q_\sigma(fg; x)^{-1}t$ and $w = Q^{-1}$).

We have now handled the base case (where v is empty,) so assume for an induction that we know $v = xt$ for some $x \in T^*$ and we know that $F_\sigma(fg; ux) = F_\sigma(fg; u)F_\sigma(fF_\sigma(fg; u); Q^{-1}x)$. The desired equality

$$F_\sigma(fg; uxt) = F_\sigma(fg; u)F_\sigma(fF_\sigma(fg; u); Q^{-1}xt)$$

then follows immediately from Lemma 0.9.

Defining condition RCF. The following condition says that, if we fix $g \in G$, then $F_\sigma(fg; -)$ should give the same answer on any two sufficiently short words which represent the same element of H. Together with Lemma 0.10, this property will imply that $F_\sigma(fg; -)$ always gives the same answer on words which represent the same element, regardless of length. This means that $F_\sigma(fg; -)$ descends to a function on H (Lemma 0.11).

Definition 0.18 (Relator consistency of F) Suppose $\sigma = (\sigma_d, \sigma_\ell) \in \mathbb{A}^G$ satisfies all of our previous conditions. We say that σ satisfies condition **RC**F if $F_\sigma(fg; w_1) = F_\sigma(fg; w_2)$ for all $g \in G$ and $w_1, w_2 \in T^*$ which represent the same element of H and have length less than or equal to $K_H + n$.

Locality of RCF. By Lemma 0.8, both sides of the equality

$$F_\sigma(fg; w_1) = F_\sigma(fg; w_2)$$

can be computed if we know σ on the ball of radius $n + (n^2 + n)(K_H + n)$ around g.

Condition RCF is satisfied for σ arising from an n-QI pair. If $\sigma = (df, \ell_{Ff})$, then for $g \in G$ and words $w_1, w_2 \in T^*$ representing the same element of H, we have, by Lemma 0.7,

$$F_\sigma(fg; w_1) = F(f(g)w_1) = F(f(g)w_2) = F_\sigma(fg; w_2).$$

We now show that if σ satisfies all of the previous conditions, then $F_\sigma(fg; w)$ is determined by the image of w in H, so that we may drop our cumbersome notation and write $F_{\sigma g}(h)$ for $F_\sigma(fg; w)$ where $w \in T^*$ is any word representing $f(g)^{-1}h$.

Lemma 0.11

If σ satisfies conditions $\mathbf{C}f$, $\mathbf{RI}$, $\mathbf{L}F$, $\mathbf{C}F\ell$, *and* $\mathbf{RC}F$, *and if $w, w' \in T^*$ represent the same element of H, then for any $g \in G$,*

$$F_\sigma(fg; w) = F_\sigma(fg; w'),$$

so that we may speak of $F_\sigma(fg; -)$ as a function on H.

Proof 0.12 Recall that there exists a homotopy $w = w_0, \ldots, w_k = w'$ from w to w', where each pair (w_i, w_{i+1}) is of the form $(uvx, uv'x)$ for some words $u, v, v', x \in T^*$ with v and v' having length at most K_H and representing the same element of H. Hence, it suffices to show that $F_\sigma(fg; uvx) = F_\sigma(fg; uv'x)$ for such pairs.

By Lemma 0.9, it suffices to show that $F_\sigma(fg; uv) = F_\sigma(fg; uv')$. Let Q be a word of length at most n representing $Q_\sigma(fg; u)$. By lemma 0.10 and assumption $\mathbf{RC}F$,

$$F_\sigma(fg; uv) = F_\sigma(fg; u)F_\sigma(fF_\sigma(fg; u); Q^{-1}v)$$

$$= F_\sigma(fg; u)F_\sigma(fF_\sigma(fg; u); Q^{-1}v') = F_\sigma(fg; uv'),$$

as desired.

Definition 0.19 If σ satisfies conditions $\mathbf{C}f$, $\mathbf{RI}$, $\mathbf{L}F$, $\mathbf{C}F\ell$, and $\mathbf{RC}F$, then for $g \in G$ and $h \in H$, we let $F_{\sigma g}(h)$ denote $F_\sigma(fg; w)$ for any word $w \in T^*$ representing $f(g)^{-1}h$. By lemma 0.11, the choice of w is irrelevant.

Defining condition NCF. By Lemma 0.7, we know that if $\sigma = (df, \ell_{Ff})$ for some $(f, F) \in \mathrm{QIP}_n(G, H)$, then $F_{\sigma g}$ is always the same function regardless of g, (in particular, $F_{\sigma g}(h) = F(h)$ for any $g \in G$ and $h \in H$). We will now define a local condition on σ which will imply that $F_{\sigma g_1}(h) = F_{\sigma g_2}(h)$ for any $g_1, g_2 \in G$ and $h \in H$ (Lemma 0.12). In particular, the condition will mandate that this equality must hold when g_1 and g_2 are neighbors in the Cayley graph of G.

Definition 0.20 (Neighbor consistency of F) Suppose σ satisfies conditions **C**f, **RI**, **L**F, **C**$F\ell$, and **RC**F. Then we say that σ satisfies condition **NC**F if for all $g \in G$ and $s \in S$, we have that $F_{\sigma gs}(h) = F_{\sigma g}(h)$ for all $h \in H$.

Locality of NCF. Suppose we knew that $F_{\sigma gs}(f(g))$ were equal to $F_{\sigma g}(f(g))$ for all $s \in S$. Then for $h \in H$, we would have that $F_{\sigma g}(h) = F_\sigma(fg; w)$ for some word w representing $f(g)^{-1}h$. But then if v were any word representing $f(gs)^{-1}f(g)$, we would have

$$F_\sigma(fgs; v) = F_{\sigma gs}(f(g)) = F_{\sigma g}(f(g)) = F_\sigma(fg; \epsilon),$$

and thus by Lemma 0.9,

$$F_{\sigma gs}(h) = F_\sigma(fgs; vw) = F_\sigma(fg; w) = F_{\sigma g}(h).$$

Thus, it suffices to check that for all $s \in S$ and $g \in G$, we have $F_{\sigma g}(f(g)) = F_{\sigma gs}(f(g))$, which is the same as checking that $F_\sigma(fg; \epsilon) = F_\sigma(fgs; v)$ for some word v of length at most n representing $f(gs)^{-1}f(g)$. Lemma 0.8 implies that we can do this if we know σ on the ball of radius $n + (n^2 + n)n$ around gs. It follows that **NC**F is a local condition.

Condition NCF is satisfied for σ arising from an n-QI pair. As remarked above, Lemma 0.7 says that $F_{\sigma g} = F$ when $\sigma = (df, \ell_{Ff})$ for some $(f, F) \in \mathrm{QIP}_n(G, H)$.

Hence, such σ satisfy condition $\mathbf{NC}F$.

The following lemma says that all of $F_{\sigma g}$ are equal, as expected, so we can define a global candidate quasi inverse $F_\sigma : H \to G$.

Lemma 0.12

If σ satisfies conditions $\mathbf{C}f$, $\mathbf{RI}$, $\mathbf{L}F$, $\mathbf{C}F\ell$, $\mathbf{RC}F$, and $\mathbf{NC}F$, then $F_{\sigma g_1}(h) = F_{\sigma g_2}(h)$ for any $g_1, g_2 \in G$ and $h \in H$.

Proof 0.13 Let $s_0, \ldots, s_k \in S$ be such that $g_2 = g_1 s_0 s_1 \ldots s_k$. Then

$$F_{\sigma g_1}(h) = F_{\sigma g_1 s_0}(h) = \ldots = F_{\sigma g_1 s_0 s_1 \ldots s_{k-1}}(h) = F_{\sigma g_2}(h).$$

Definition 0.21 If σ satisfies conditions $\mathbf{C}f$, $\mathbf{RI}$, $\mathbf{L}F$, $\mathbf{C}F\ell$, $\mathbf{RC}F$, and $\mathbf{NC}F$, then we define $F_\sigma : H \to G$ via

$$F_\sigma(h) = F_{\sigma g}(h)$$

for $h \in H$, and any $g \in G$ (by Lemma 0.12, the choice of g does not matter).

Defining condition LI. The reader may have noticed that although we know F_σ is n-Lipshitz by condition $\mathbf{L}F$ and is a right n-quasi inverse to f by $\mathbf{RI}$, we have not yet imposed any condition which would imply that F is a left n-quasi inverse to f. The following condition will ensure this.

Definition 0.22 (F is a left inverse) Suppose $\sigma \in \mathbb{A}^G$ satisfies conditions $\mathbf{C}f$, $\mathbf{RI}$, $\mathbf{L}F$, $\mathbf{C}F\ell$, $\mathbf{RC}F$, and $\mathbf{NC}F$. We say that σ satisfies condition $\mathbf{LI}$ if for all $g \in G$,

$$|\langle \sigma_\ell(g), 1 \rangle|_S \leq n.$$

Locality of LI. Manifestly, $\mathbf{LI}$ is a local condition.

Condition LI is satisfied for σ arising from an n-QI pair. Certainly, if $\sigma = (df, \ell_{Ff})$ for some $(f, F) \in \mathrm{QIP}_n(G, H)$, then

$$\langle \sigma_\ell(g), 1 \rangle = g^{-1}F(f(g)) \in B_G(n, 1_G)$$

because F is a left n-quasi inverse to f.

We now finish the proof of the theorem by showing that, if σ satisfies all of our conditions, then $(f, F_\sigma) \in \mathrm{QIP}_n(G, H)$ (and of course $\sigma = (df, \ell_{F_\sigma f})$).

Lemma 0.13

Suppose that $\sigma \in \mathbb{A}^G$ satisfies **C**f, **RI**, **L**F, **C**$F\ell$, **RC**F, **NC**F, *and* **LI**. *Then (f, F_σ) is an n-QI pair and $(df, \ell_{F_\sigma f}) = \sigma$.*

Proof 0.14 First we check that $\ell_{F_\sigma f} = \sigma_\ell$. By definition,

$$\langle \ell_{F_\sigma f}(g), h \rangle = g^{-1}F_\sigma(f(g)h)$$

$$= g^{-1}F_\sigma(fg; h) = \langle \sigma_\ell(g), h \rangle,$$

where the last equality follows from condition **C**$F\ell$. Now we must confirm that (f, F_σ) is an n-QI pair. Trivially, f is n-Lipschitz. By condition **RI**, the n neighborhood of $f(G)$ contains the N-neighborhood of $f(G)$, and thus is all of H. Condition **L**F says that $\sigma_\ell(g)$ is n-Lipschitz for any $g \in G$. But we know that $g\sigma_\ell(g)$ is F_σ restricted to $B_H(N, f(g))$, so it follows that F_σ is n-Lipschitz (since it is n-Lipschitz on each $B_H(N, f(g))$ and these cover H). It follows from **RI** that $(f(g)h)^{-1}f(g\langle \sigma_\ell(g), h \rangle)$ is within distance n of $f(g)h$. Thus, F_σ is a right n-quasi inverse to f. That it is a left n-quasi inverse follows from condition **LI**, in light of the fact that $F_\sigma(f(g)) = g\langle \sigma_\ell(g), 1 \rangle$.

In light of Lemma 0.13, we are done.

0.4.2 Subshifts on G from subshifts on H.

Given $\sigma \in A^H$ and a function $f : G \to H$, define the pullback $f^*\sigma$ of σ under f to be $\sigma \circ f$, as σ may be thought of as a function $H \to A$. In order to prove that having a strongly aperiodic subshift of finite type is a QI invariant, we now describe a procedure which takes two quasi isometric, finitely presented groups G and H, together with a subshift of finite type $X \subset A^H$, and produces a subshift $\widetilde{X_n} \subset A'^H$ which can be thought of as the simultaneous pullback of X under all n-Lipschitz $f : G \to H$ which have an n-Lipschitz two-sided n-quasi inverse. Theorem 0.6 will show that $\widetilde{X_n}$, called the pull back subshift, is of finite type. First, we require a definition.

Definition 0.23 Let H be a group, generated, as always, by a fixed finite set T.

- For $\sigma \in A^H$, define $B_n(\sigma) \in \left(A^{B_H(n,1)}\right)^H$ by setting $(B_n(\sigma))(h)$ to be $k \mapsto \sigma(hk)$.
- For a subshift X of A^H, define $B_nX \subset \left(A^{B_H(n,1)}\right)^H$ to be $\{B_n(\sigma) : \sigma \in X\}$.

The idea is that the value taken by $B_n(\sigma)$ at some $h \in H$ records the behavior of σ in the n-ball around h. This definition is useful because, if $f : G \to H$ is a quasi isometry, then $f^*\sigma$ may not see all the values of σ, but for sufficiently large n, $f^*B_n(\sigma)$ will. It is trivial to observe that if X is a subshift of finite type, then so is B_nX. Furthermore, if H has solvable word problem, then the defining forbidden patterns of B_nX can be computed from those of X. Given $\sigma_0 \in \left(A^{B_H(n,1)}\right)^H$, we will write $\langle \sigma_0(h), k\rangle$ to denote $(\sigma_0(h))(k)$.

Theorem 0.6

If G, H are finitely presented groups, A finite, and $X \subset A^H$ a subshift of finite type, then the set $\widetilde{X_n}$ defined as

$$\{(df, \ell_{Ff}, f^*B_n(\sigma)) : (f, F) \in \mathrm{QIP}_n(G, H), \sigma \in X\}$$

is a subshift of finite type on G. Furthermore, if some point $(df, \ell_{Ff}, f^*B_n(\sigma))$ of this subshift is periodic under $g_\pi \in G$, then σ is periodic under $f(g_\pi)f(1_G)^{-1}$.

Before proving the theorem, we remark that $f(g_\pi)f(1_G)^{-1}$ is not a typo for $f(1_G)^{-1}f(g_\pi)$, and in particular, this period is not determined by g and df as one might expect. An equivalent conclusion would be that $\sigma \cdot f(1)$ (also an element of X) is $f(1)^{-1}f(g_\pi)$-periodic. Either way we produce a periodic element of X from a periodic element of $\widetilde{X_n}$ as long as $f(1)$ is different from $f(g_\pi)$.

Proof 0.15 First, we will show that the given subshift is of finite type, then address the question of periodicity. Theorem 0.5 already tells us that the set of (df, ℓ_{Ff}) such that (f, F) is an n-QI pair is a subshift of finite type. Hence, we must find local conditions on a triple $(df, \ell_{Ff}, \sigma_0)$ which are satisfied exactly when σ_0 is of the form $f^*B_n(\sigma)$ for some $\sigma \in X$. How shall we recover σ from σ_0? Observe that since f is n-quasi surjective, every $h \in H$ has the form $h = f(g)k$ for some $g \in G$ and $k \in B_H(n, 1_H)$, and so we must have

$$\sigma(h) = \sigma(f(g)k) = \langle \sigma_0(g), k \rangle.$$

This potentially overdetermines $\sigma(h)$ because g and k are not unique. However, if $g_0, g_1 \in G$ and $k_0, k_1 \in B_H(n, 1_H)$ are such that $f(g_0)k_0 = f(g_1)k_1$, then

$$d(g_0, g_1) \leq 2n + d(F \circ f(g_0), F \circ f(g_1))$$

$$\leq 2n + nd(f(g_0), f(g_1)) \leq 2n + 2n^2.$$

This yields that it is a local condition to mandate that

$$\langle \sigma_0(g_0), k_0 \rangle = \langle \sigma_0(g_1), k_1 \rangle$$

whenever $f(g_0)k_0 = f(g_1)k_1$. But we have seen that this condition is sufficient to ensure that σ_0 is of the form $f^*B_n(\sigma)$. Hence, the set of triples $(df, \ell_{Ff}, f^*B_n(\sigma))$ is of finite type as desired.

To show the result about periodicity, suppose that $(f, F) \in \mathrm{QIP}_n(G, H)$ and $\sigma \in X$, and that $(df, \ell_{Ff}, f^*B_n(\sigma))$ is g_π-periodic. This means in particular that $df \cdot g_\pi = df$ and $f^*B_n(\sigma) = f^*B_n(\sigma) \cdot g_\pi$. It follows that for any $g \in G$,

$$\begin{aligned} f(g_\pi g) &= f(g_\pi) \int_{g_\pi}^{g_\pi g} df = f(g_\pi) \int_1^g (df \cdot g_\pi) \\ &= f(g_\pi) \int_1^g df = f(g_\pi) f(1)^{-1} f(g), \end{aligned} \tag{1}$$

so if $h \in H$ is equal to $f(g)k$ for $g \in G$ and $k \in B_H(n, 1_H)$, then

$$\begin{aligned} (\sigma \cdot f(g_\pi)f(1)^{-1})(h) &= \sigma(f(g_\pi)f(1)^{-1}h) = \sigma(f(g_\pi)f(1)^{-1}f(g)k) \\ &= \sigma(f(g_\pi g)k) = \langle f^*B_n(\sigma)(g_\pi g), k\rangle = \langle f^*B_n(\sigma)(g), k\rangle \\ &= \sigma(f(g)k) = \sigma(h). \end{aligned}$$

Since every element of h can be written as such an $f(g)k$ (because f is n-quasi surjective), this establishes that σ is $f(g_\pi)f(1)^{-1}$-periodic.

We will now see that under the additional hypothesis of torsion-freeness, this theorem implies that having a strongly aperiodic subshift of finite type is a QI invariant.

Corollary 0.1

Let G, H be torsion free finitely presented groups, and suppose that G and H are quasi isometric. Then G admits a strongly aperiodic subshift of finite type if and only if H does.

Proof 0.16 Suppose that X_H is a strongly aperiodic subshift of finite type. For some n, we have $\mathrm{QIP}_n(G, H)$ non empty. Let X_G be the subshift

$$\{(df, \ell_{Ff}, f^*B_n(\sigma)) : (f, F) \in \mathrm{QIP}_n(G, H); \sigma \in X_H\}$$

which is a subshift of finite type by Theorem 0.6. We will show that X_G is strongly aperiodic.

If some point $(df, \ell_{Ff}, f^*B_n(\sigma))$ of X_G is periodic, say under $g_\pi \in G \setminus \{1\}$, then Theorem 0.6 implies that σ is $f(g_\pi)f(1)^{-1}$ periodic. Since X_H is strongly aperiodic, we must have that $f(g_\pi)f(1)^{-1} = 1$, i.e., $f(g_\pi) = f(1)$. But our initial data is also periodic under any power of g_π, hence $f(g_\pi^k) = f(1)$ for all natural numbers k by equation (1) in the proof of Theorem 0.6. We conclude in particular that $f(g_\pi^k)$ does not depend on k. This is impossible because $|f(g_\pi^k)| \to \infty$ (because $|g_\pi^k| \to \infty$ by torsion-freeness, and f is a quasi isometry, so the image under f of an unbounded sequence must be unbounded). It follows that X_G is strongly aperiodic.

We have shown that G admits a strongly aperiodic subshift of finite type if H does. The converse follows by symmetry.

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