THE RICE INSTITUTE

CONCERNING ENTIRE FUNCTIONS OF GENUS ZERO HAVING REAL, POSITIVE ZEROS

by

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A THESIS

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Approved,
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CHAPTER I*

1. Description of Surfaces and Statement of Problem. The symmetric "semi-cosinic" surface $\mathcal{F}$ covering the w-plane is determined by the sequence of real numbers $a_K (K = 1, 2, \dots)$, with $a_1 > 0$, $a_{2m\pm 1} > a_{2m}$. $\mathcal{F}$ is composed of the sheets $S_1, S_2, \dots, S_K, \dots$; S_1 is a replica of the w-plane cut along the positive real axis from $w = a_1$ to $w = \infty$; $S_K (K > 1)$ is a replica of the w-plane cut along the real axis except for the interval between a_{K-1} and a_K. S_1 and S_2 are joined along their cuts from a_1 to $+\infty$, forming a first order branch point over $w = a_1$; S_K and S_{K+1} are joined along their cuts from a_K to $(-1)^{K+1}\infty$. Note that in the particular case $a_K = (-1)^{K+1}$, $\mathcal{F}$ is the Riemann surface of the function $z = (\arccos w)^2$; therefore, any symmetric semi-cosinic surface, being topologically equivalent to this one, is open and simply-connected.

MacLane, [3],** proves the following theorem:

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* The author wishes to express his appreciation to Professor G. R. MacLane for the suggestion of the problem treated in this paper.

** The numbers in brackets refer to the references listed at the end of this paper.

Theorem II: The symmetric semi-cosinic surface $\mathcal{F}$ is always parabolic; it is the (1-1) image of the z-plane by an entire function

$$W = F(z) = \int_0^z f(t)\,dt$$

where $f(z) = e^{-\delta z}\prod_1^\infty \left(1 - \frac{z}{b_K}\right), \ 0 < b_1 < b_2 < \cdots, \ \sum \frac{1}{b_K} < \infty, \ \delta \geq 0,$

the branch point over $W = a_K$ corresponding to $z = b_K$, and $F(0) = 0 \in S_1$.

We shall be concerned with properties of $F(z)$ for some suitably chosen functions $f(z)$; in particular, we are interested in the relation between the distribution of the zeros of $f(z)$ and the distribution of the branch points of $F(z)$. In each case to be treated $\delta = 0$; and, thus, $f(z)$ reduces to the canonical product.

2. Preliminary Theorems. It is convenient to list here several theorems which will be valuable in the sequel. They will be referred to by the number given here.

Theorem 1). Given an entire function $g(z)$ of order $< 1/2$, we can find a sequence of circles of indefinitely increasing radii described about the origin as center on which the minimum modulus of $g(z)$ tends to infinity. ([5, p. 127])

We say that the zeros of $g(z)$ accumulate in a direction φ if the angle $\varphi-\epsilon < \arg z < \varphi+\epsilon$ contains an infinity of zeros of $g(z)$, however small $\epsilon>0$. Then, we have

Theorem 2). If $g(z)$ is entire and of finite order, the zeros of $g'(z)$ may accumulate only in those directions which are directions of accumulation of the zeros of $g(z)$. (Biernacki, [2, pp. 530-535]).

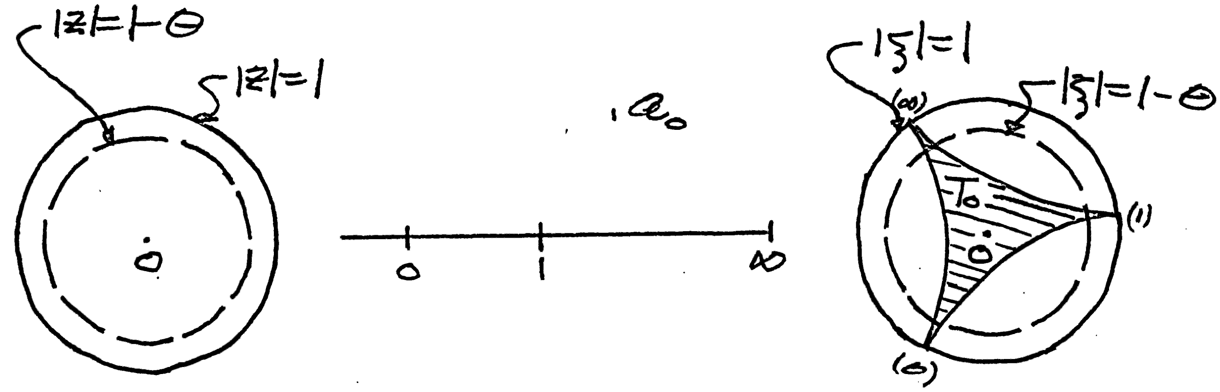
Theorem 3). Let $f(z)$ be an analytic function of $z=re^{i\theta}$, regular in the region D between two straight lines making an angle $\frac{\pi}{\alpha}$ at the origin, and on the lines themselves. Suppose that $|f(z)| \leq M$ on the lines, and that, as $r \to \infty$, $f(z)=O(e^{r^\beta})$, $\beta<\alpha$, uniformly in the angle. Then, $|f(z)| \leq M$ holds throughout the region D. (Phragmén-Lindelöf, [4, p. 177]).

We shall also need an extension of Picard's theorem due to Bieberbach, ([1.2, pp. 175-178]); as this theorem is not as well known as the preceding ones, an outline of its proof will be given below. First, we must obtain an extension of Schottky's theorem due to Bohr and Landau.

Theorem 4). If $f(z)=a_0+a_1z+\cdots$ is regular in $|z|<R$ and in this domain does not take the values zero and one, then

$$\left|\log|f(z)|\right| = O\left(\frac{K}{1-|z|}\right) \quad \text{, where } K = K(a_0).$$

Proof



z-plane   w-plane ζ-plane

Let $w = f(z) = a_0 + a_1 z + \cdots$

Denote by $\zeta = \varphi(w)$ the modular function which maps the universal covering surface of the w-plane with the points 0, 1, and ∞ removed onto $|\zeta| < 1$ such that $\varphi(a_0) = 0$. Let $F(z) = \varphi(f(z))$ with $F(0) = 0$. By the monodromy theorem, $F(z)$ is defined, single-valued, and holomorphic for $|z| < 1$. Since $F(0) = 0$ and $|F(z)| < 1$, we have by Schwarz's lemma,

$$|F(z)| \leq z$$

Hence, the circle $\{|z| < 1-\theta\}$ is mapped onto a subset of

$\{|\zeta| \le 1-\theta\}$.

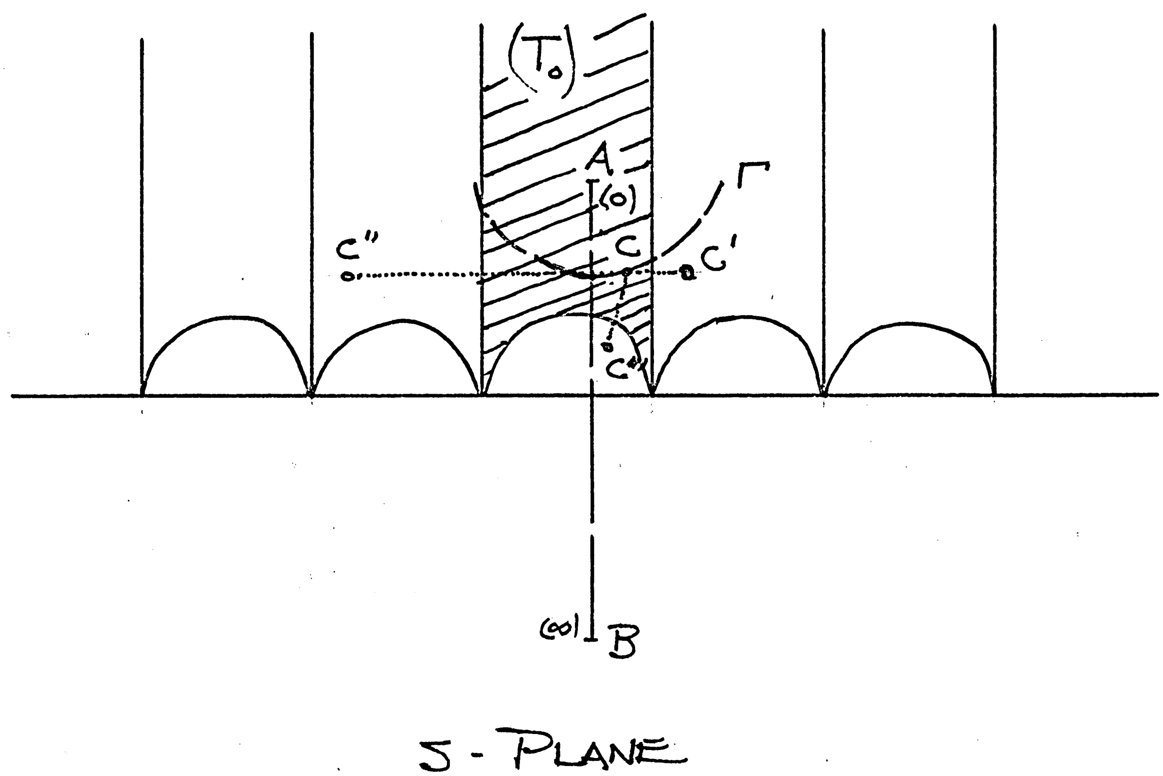
As $f(z) = \bar{\varphi}'(F(z))$, for $|z| < 1-\theta$,

$$|f(z)| \le \max_{|\zeta| \le 1-\theta} |\bar{\varphi}'(\zeta)|.$$

Now, there are many triangles in the circle $|\zeta| < 1$ which cut the region $\{|\zeta| \le 1-\theta\}$. However, it will be sufficient to consider the one triangle containing the origin:

To show this, it will be sufficient to show that any point of this triangle will be carried further from the origin by any of the inversions which are used to generate the modular function.

Map the circle $|\zeta| \le 1$ onto the half-plane $\mathcal{I}\mathit{s} \ge 0$:

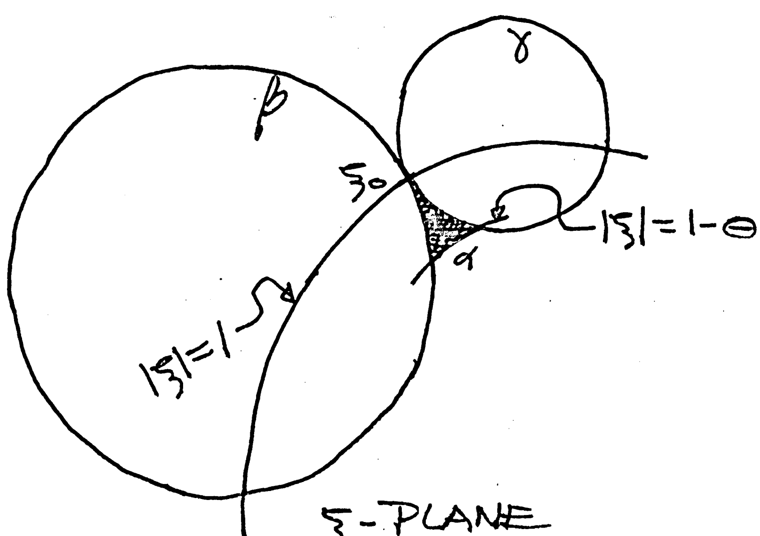


S - PLANE

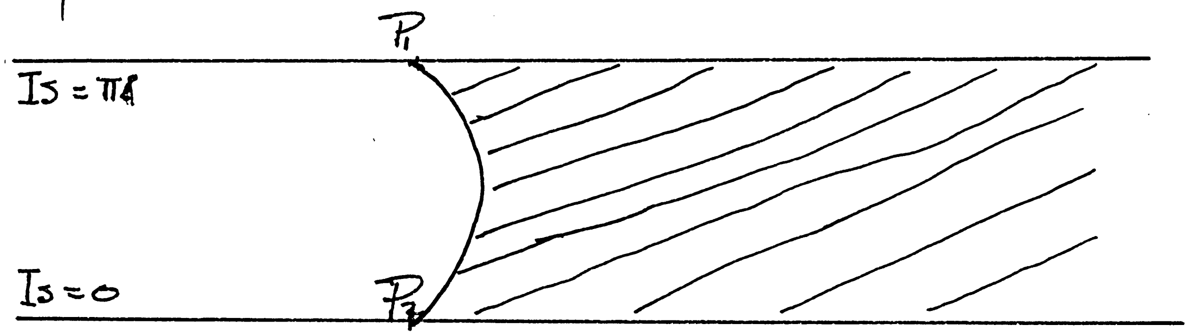
Now, the point $\zeta = \infty$ goes into the conjugate of the point into which $\zeta = 0$ is transformed; hence, the circles $|\zeta| =$ constant pass into circles orthogonal to AB, all of the circles having finite radius except that for $|\zeta| = 1$. Consider the point C belonging to the triangle and its three nearest inverse points, C', C'', and C'''; it is obvious that these points are outside the circle Γ through C orthogonal to AB. Further reflections carry the homographs farther away from Γ; thus, on the ζ-plane, farther from the center.

q.e.d.

So, let us consider the intersection of T_0 and $\{|\zeta| \leq 1-\theta\}$. By the nature of the mapping, it is apparent that $\max\limits_{\{|\zeta|\leq 1-\theta\}\cap T_0} |\bar{\Phi}'(\zeta)|$ is assumed for some ζ belonging to the arc α. Thus, we need be concerned only with the portion of T_0 between $|\zeta| = 1-\theta$ and $|\zeta| = 1$:



Mapping the exterior of the two circular regions bounded by β and γ onto the strip $0 \leq I s \leq \pi$:

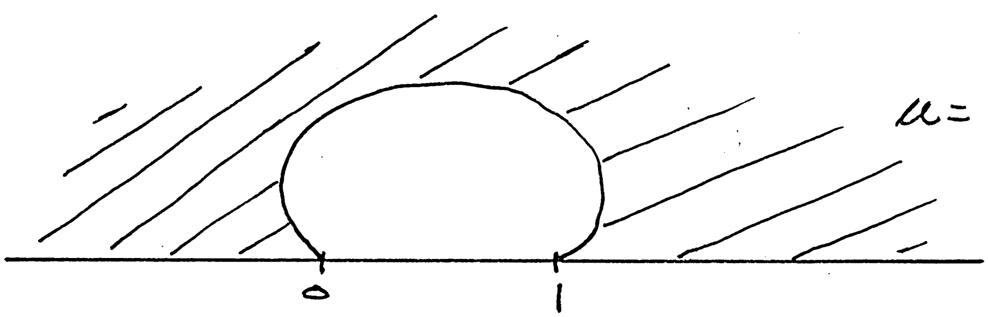


s-PLANE

This transformation is given by

$$s = \frac{L(\xi)}{\xi - \xi_0} \, , \quad L(\xi) = a\xi + b.$$

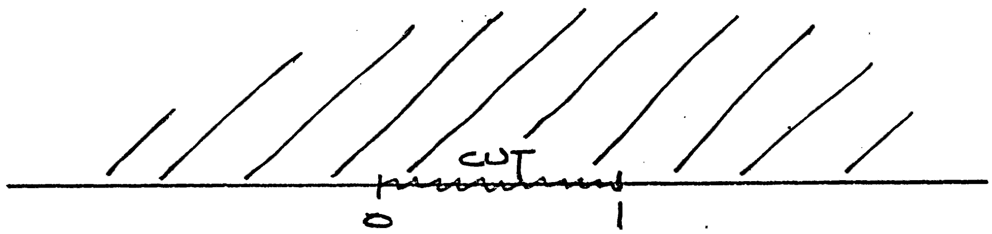
Then, mapping this region on a half-plane such that $\infty \longleftrightarrow \infty$, $P_1 \longleftrightarrow 0$, and $P_2 \longleftrightarrow 1$:



$$u = k + ce^{s}$$

u - PLANE

Then, again onto a half plane, preserving ∞ , 0 , and 1 :



W - PLANE

The real axis must be preserved; hence, by the reflection principle, this last transformation is, about infinity, of the form

$$W = du + e + \frac{f}{u} + \cdots$$
$$= du + O(1).$$

We have, by this series of mappings, developed the form of the inverse of the modular function about ? .

Thus,

$$W = du + O(1)$$
$$= d'e^{s} + O(1)$$
$$= d'e^{\frac{L(\zeta)}{\zeta-\zeta_0}} + O(1)$$
$$= d'e^{\frac{L(\zeta)}{\zeta-\zeta_0}} + O(1)$$
$$= d'e^{\frac{L(\zeta)}{\zeta-\zeta_0}}(1+\eta), \quad \eta \to 0$$

AS $\zeta \to \zeta_0$.

As $\zeta \to \zeta_0$,

$$|w| = |\varphi^{-1}(\zeta)|$$
$$= O\left(e^{\frac{K_1}{|\zeta-\zeta_0|}}\right)$$
$$= O\left(e^{\frac{K_1}{1-\theta}}\right), \text{ where } K_1 = K_1(a_0).$$

Hence, for $|z| < 1-\theta$, $|f(z)| = O\left(e^{\frac{K_1}{1-\theta}}\right)$

Now, $\frac{1}{f(z)}$ is regular and does not take the values 0 or 1 for $|z| < 1$. Thus, for $|z| < 1-\theta$,

$$\left|\frac{1}{f(z)}\right| = O\left(e^{\frac{K_2}{1-\theta}}\right), \quad K_2 = K_2\left(\tfrac{1}{a_0}\right)$$

$$\therefore, \quad \bigl|\log|f(z)|\bigr| = O\left(\frac{K}{1-\theta}\right), \quad K = K(a_0).$$

q.e.d.

A more explicit estimate on K may be obtained; for this, see Bieberbach, [1.1 , p. 225]. However, this estimate is sufficient for our purposes.

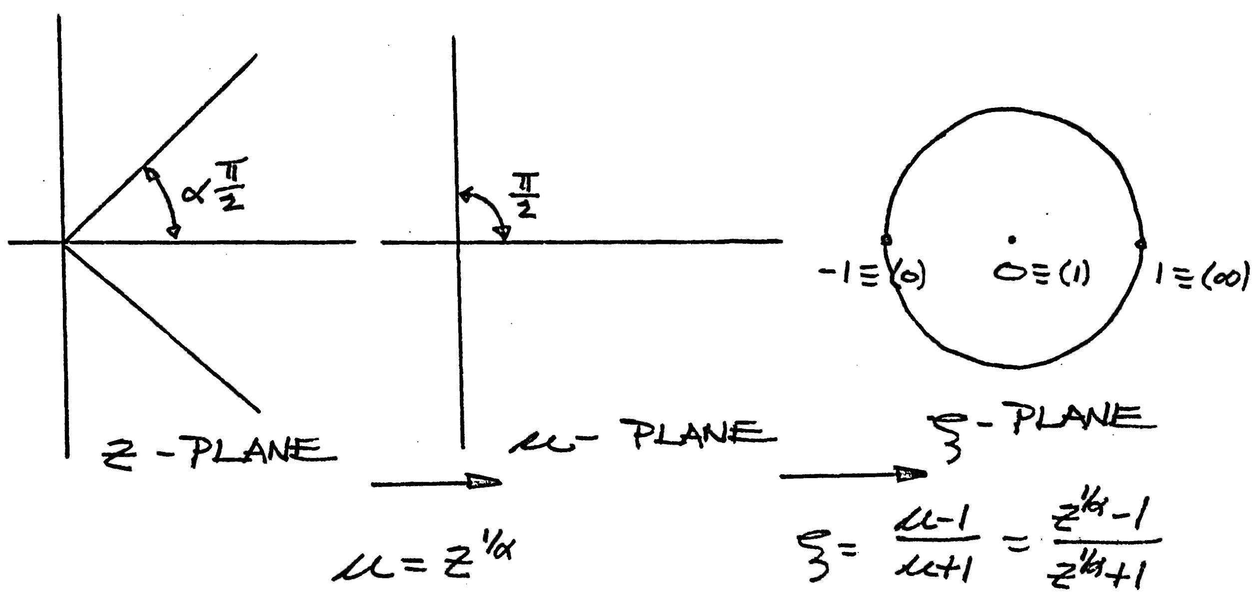
Only that part of Bieberbach's theorem that ~~shall~~ will prove useful later will be given here; actually the theorem is somewhat more general.

Theorem 5). There exists no entire function of finite order

<u>$\rho > 1/2$ such that all the roots of $f(z) - a = 0$ and $f(z) - b = 0$, $a \neq b$, accumulate in a single direction.</u> (The theorem is not true for $\rho = 1/2$).

<u>Proof</u>

It is obviously sufficient to take $a = 0$ and $b = 1$.



First, let us consider $g(z)$, regular in $\{|\mathrm{ARG}\, z| < \alpha\frac{\pi}{2}\}$ and $\neq 0, 1$ in this region.

Let $\quad \varphi(\zeta) = \varphi\left(\frac{z^{1/\alpha}-1}{z^{1/\alpha}+1}\right) = g(z)$

Then, $\varphi(\zeta)$ is regular for $|\zeta| < 1$, and $\neq 0, 1$ there.

Now,
$$\varphi(\zeta) = a_0 + a_1\zeta + \cdots, \quad |\zeta| < 1.$$

By the Schottky-Landau-Bohr theorem,

$$|\log|\varphi(\zeta)|| < \frac{K(a_0)}{1-|\zeta|}$$

or,

$$|\log|g(z)|| < \frac{K(a_0)}{1-|\zeta|}, \quad \zeta = \frac{z^{1/\alpha}-1}{z^{1/\alpha}+1}$$

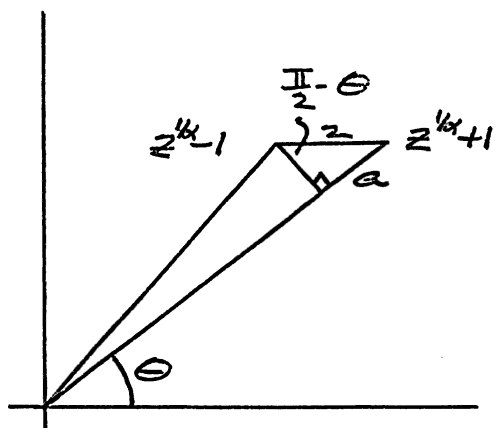
But,

$$1-|\zeta| = 1 - \left|\frac{z^{1/\alpha}-1}{z^{1/\alpha}+1}\right|$$

$$= \frac{|z^{1/\alpha}+1|-|z^{1/\alpha}-1|}{|z^{1/\alpha}+1|}$$

$$\geq \frac{C}{|z|^{1/\alpha}}, \quad C = C(\epsilon),$$

FOR $|z| > Z_0$, $|\mathrm{ARG}\, z| \leq \alpha(\frac{\pi}{2}-\epsilon)$.

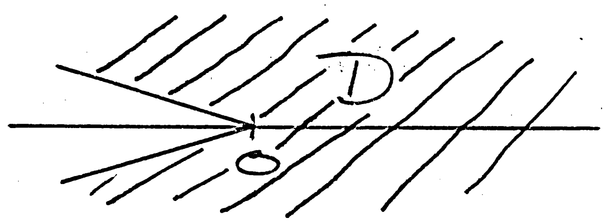


FOR θ FIXED,
$|z^{1/\alpha}+1|-|z^{1/\alpha}-1| \sim a$
$a = 2\sin(\frac{\pi}{2}-\theta)$
$\therefore$, FOR $\theta \leq \frac{\pi}{2}-\epsilon$, $a \geq 2C(\epsilon) > 0$

Therefore, for $|z|$ sufficiently large, $|\mathrm{ARG}\, z| \leq \alpha(\frac{\pi}{2}-\epsilon)$,

$$|\log|g(z)|| < K'(a_0)\cdot|z|^{1/\alpha}.$$

Now consider $f(z)$, a function such that the roots of $f(z)=0$ and $f(z)-1=0$ accumulate in a single direction. It is sufficient to consider this direction as the negative real axis.



Consider the domain D:

$$|\operatorname{ARG} z| < \alpha \frac{\pi}{2}, \alpha < 2.$$

As all of the roots of $f(z)=0$ and $f(z)-1=0$ accumulate about $\operatorname{ARG} z = \pi$, for $|z|$ sufficiently large, there exists $A = A(\alpha)$ such that

$$|f(z)| < e^{A|z|^{1/\alpha}}$$

If ρ_D is "the order of $f(z)$ in D", $\rho_D \leq 1/\alpha$.
As this is true for all $\alpha < 2$, $\rho_D \leq 1/2$.
It remains to show that the order of $f(z) \leq \frac{1}{2}$. This may be shown in general; however, for all cases considered in this thesis, the maximum modulus is assumed in the portion of the plane already considered; so the proof will be omitted.

CHAPTER II

1. Order of $F(z)$. Consider the sequence $a_m = \int_0^{b_m} f(x)\,dx = F(b_m)$. The problem to be considered is this: what is the relation between the distribution of the a_m's and the order of $F(z)$? The results obtained are the following:

Theorem 6). If $a_m = O(1)$, then $\rho \geq 1/2$.

Proof

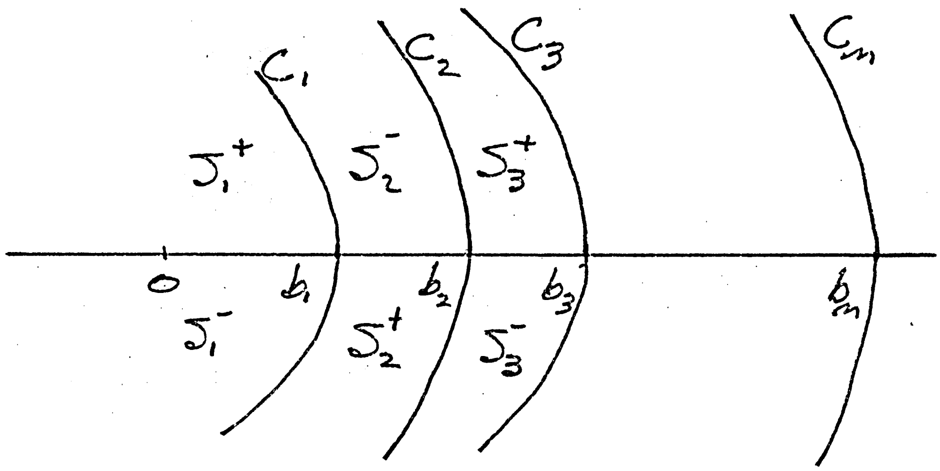
Consider $F(x) = \int_0^x f(x)\,dx$. If $b_K \leq x \leq b_{K+1}$, K even, then $a_K \leq F(x) \leq a_{K+1}$. Thus, $F(z)$ is bounded on the positive real axis. By theorem 1), $\rho \geq \frac{1}{2}$.

q.e.d.

Theorem 7). If $\underline{\lim}\, a_{2m-1} - \overline{\lim}\, a_{2m} > 0$, then $\rho \leq 1/2$.

Proof

Let us consider the fundamental regions of the mapping; MacLane ([3 , p. 112]) has shown them to be as shown below:



z-PLANE

The curve C_n, $n=1,2,\cdots$, is symmetric about the x-axis. The real axis plus the curves C_n form the real paths of $F(z)$, with C_n being mapped into the interval between a_n and $(-1)^{n+1}\infty$.

Consider the interval $(\overline{\lim}\, a_{2n}, \underline{\lim}\, a_{2n-1}) = I$. The only direction of accumulation of zeros of $F(z)-\gamma = 0$, $\gamma \in I$, is the positive real axis. Hence, by theorem 5), $\rho \leq \frac{1}{2}$.

q.e.d.

<u>Corollary: If $a_n = O(1)$, and $\underline{\lim}\, a_{2n-1} - \overline{\lim}\, a_{2n} > 0$, then $\rho = \frac{1}{2}$.</u>

2. <u>Estimate on the Minimum Size of $|a_{n+1} - a_n|$</u>. Only functions of order less than one will be considered here. The Paley-Wiener theorem will be used to obtain some

rough estimates.

Let $f(z) = \prod_1^\infty \left(1 - \frac{z}{b_n}\right)$, $\rho < 1$.

Let $G(z) = f(z)f(-z)$. The order of $G(z)$ is also ρ. Therefore, for $\delta > 0$, there exists $R_\delta > 0$ such that

$$|G(z)| < e^{\delta|z|}, \quad |z| > R_\delta.$$

First, $G(z) \notin L_1 \cup L_2$, where L_p denotes the class of functions $g(x)$ such that $\int_{-\infty}^{\infty} |g(x)|^p dx$ exists in the sense of Lebesgue and is finite. For, if $G(z) \in L_1 \cup L_2$, then the Fourier transform $S(u; G)$ would exist. But, by the Paley-Wiener theorem,

$$S(u; G) = 0, \quad |u| \geq \delta$$

As this is true for all $\delta > 0$,

$$S(u; G) \equiv 0$$

Therefore, $G(z) \equiv 0$. But $G(0) = 1$, by hypothesis. Thus, we have a contradiction.

q.e.d.

Since $G(z) = G(-z)$, $G(z)$ does not belong to $L_1(0,\infty)$ or $L_2(0,\infty)$ as well; i.e.,

$$\sum_1^\infty \int_{b_n}^{b_{n+1}} |f(x)f(-x)|\,dx = \infty.$$

Hence, if $A_n = \max\limits_{x\in[b_n, b_{n+1}]} |f(-x)|$,

$$\sum_1^\infty A_n \int_{b_n}^{b_{n+1}} |f(x)|\,dx = \sum_1^\infty A_n |a_{n+1} - a_n| = \infty .$$

Now, as our functions have real positive zeros, the maximum modulus is assumed along the negative axis; thus, for $x > x_\epsilon$,

$$|f(-x)| < e^{x^{\rho+\epsilon}}$$

Therefore, $A_n \leq e^{b_{n+1}^{\rho+\epsilon}}$

So, $$\sum_1^\infty e^{b_{n+1}^{\rho+\epsilon}} |a_{n+1} - a_n| = \infty .$$

Hence, there exist infinitely many n_i such that

$$|a_{n_i+1} - a_{n_i}| > e^{-b_{n_i+1}^{\rho+\epsilon'}}, \quad \epsilon' > \epsilon .$$

Stating this result in theorem form, we have

Theorem 8). If $f(z) = \prod_1^\infty \left(1 - \frac{z}{b_n}\right),\ 0 < b_n \uparrow \infty$, is of order $\rho < 1$ and $a_n = \int_0^{b_n} f(x)\,dx$, then, for any $\epsilon > 0$, there exist infinitely many n_i for which

$$|a_{n_i+1} - a_{n_i}| > e^{-b_{n_i+1}^{\rho+\epsilon}} .$$

CHAPTER III

1. In this chapter actual numerical calculations will be made for asymptotic values for $|a_{n+1} - a_n|$ when $b_n = n^\alpha,\ \alpha > 1$; $b_n = e^{kn},\ K > 0$; AND $b_n = \underline{kn}$. It will prove convenient to introduce a function $h(t)$ which is equivalent to the inverse of the counting function $n(t)$ for the zeros of $f(z)$. Consider

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{b_n}\right),\ 0 < b_n \uparrow \infty,\ \sum \frac{1}{b_n} < \infty .$$

Let $h(t)$ be a non-decreasing function with continuous first derivative such that $h(n) = b_n$. It is obvious that $h^{-1}(t)$, the inverse of $h(t)$, is equal to $n(t)$ when $t = b_n, n = 1, 2, \ldots$, and $h^{-1}(t) \uparrow \infty$ in such a way that $h^{-1}(t) \leq n$ for $b_{n-1} < t \leq b_n$; hence, it is asymptotically equivalent to $n(t)$. Now, as b_n is positive, $\sum \frac{1}{b_n} < \infty$ implies

1) $\dfrac{n(t)}{t} \to 0$ AS $t \to \infty$

2) $\displaystyle\int^\infty \frac{n(t)}{t^2}\,dt < \infty$.

Analogously,

$$\sum_1^N \frac{1}{b_\nu} = \int_{1-0}^{N+0} \frac{d[t]}{h(t)} = \frac{N}{h(N)} + \int_{1-0}^{N+0} [t]\,\frac{h'(t)\,dt}{h^2(t)}$$

Now, $\displaystyle\sum_1^\infty \frac{1}{b_\nu} < \infty$, and both terms on the right are positive, Hence,

$$\int^{\infty} t \frac{h'(t)dt}{h^2(t)} < \infty \quad \cancel{\text{and}}$$

Also, $\frac{n}{b_n} \to 0$. This gives $\frac{t}{h(t)} \to 0$ as $t \to \infty$.

Now, let $b_{n-1} \leq x \leq b_n$.

$$|f(x)| = \left(\frac{x}{b_{n-1}} - 1\right)\left(1 - \frac{x}{b_n}\right) g_1(x) g_2(x)$$, where

$$g_1(x) = \prod_1^{n-2} \left(\frac{x}{b_k} - 1\right) \quad , \quad g_2(x) = \prod_{n+1}^{\infty} \left(1 - \frac{x}{b_k}\right) .$$

$$\log g_1(x) = \sum_1^{n-2} \log\left(\frac{x}{b_k} - 1\right) = \int_{1-0}^{n-2+0} \log\left(\frac{x}{h(t)} - 1\right) d[t]$$

$$= [t] \log\left(\frac{x}{h(t)} - 1\right) \Big]_{1-0}^{n-2+0} + x \int_1^{n-2} [t] \frac{h'(t)dt}{h(t)\{x - h(t)\}}$$

$$= (n-2) \log\left(\frac{x}{b_{n-2}} - 1\right) + x \int_1^{n-2} [t] \frac{h'(t)dt}{h(t)\{x - h(t)\}}$$

$$\log g_2(x) = \sum_{n+1}^{\infty} \log\left(1 - \frac{x}{b_k}\right) = \int_{n+1-0}^{\infty} \log\left(1 - \frac{x}{h(t)}\right) d[t]$$

$$\log g_2(x) = [t]\log\left(1-\frac{x}{h(t)}\right)\Big]_{m+1-0}^{\infty} - x\int_{m+1}^{\infty}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}}$$

$$= -m\log\left(1-\frac{x}{b_{m+1}}\right) - x\int_{m+1}^{\infty}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}},$$

As $\lim_{t=\infty}[t]\log\left(1-\frac{x}{h(t)}\right)=0.$

Therefore,

$$|f(x)| = \left(\frac{x}{b_{m-1}}-1\right)\left(1-\frac{x}{b_m}\right)\left(\frac{x}{b_{m-2}}-1\right)^{m-2}\left(1-\frac{x}{b_{m+1}}\right)^{m}\cdot$$

$$\cdot\ \mathrm{EXP}\left\{x\int_1^{m-2}[t]\frac{h'(t)dt}{h(t)\{x-h(t)\}} - x\int_{m+1}^{\infty}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}}\right\}$$

2. Let $b_m = m^\alpha$, $\alpha>1$, and take $h(t)=t^\alpha$. Neither of the integrals above can be integrated in a finite number of steps for non-rational α, but we can obtain estimates which differ from them by a bounded factor.

As $[t] = t-(t-[t]-\frac{1}{2})-\frac{1}{2}$,

$$\int_1^{m-2}[t]\frac{h'(t)dt}{h(t)\{x-h(t)\}} = \alpha\int_1^{m-2}[t]\frac{dt}{t(x-t^\alpha)}$$

$$= \alpha\int_1^{m-2}\frac{dt}{x-t^\alpha} - \frac{\alpha}{2}\int_1^{m-2}\frac{dt}{t(x-t^\alpha)} - \alpha\int_1^{m-2}(t-[t]-\tfrac{1}{2})\frac{dt}{t(x-t^\alpha)}$$

$$\int_1^{n-2} \frac{dt}{x - t^\alpha} = \frac{x^{\frac{1}{\alpha}-1}}{\alpha} \int_{\frac{1}{x}}^{\frac{(n-2)^\alpha}{x}} \frac{u^{\frac{1}{\alpha}-1}\,du}{1-u}$$

$$= \frac{x^{\frac{1}{\alpha}-1}}{\alpha} \int_0^{\frac{(n-2)^\alpha}{x}} \frac{u^{\frac{1}{\alpha}-1}}{1-u} - \frac{x^{\frac{1}{\alpha}-1}}{\alpha} \int_0^{\frac{1}{x}} \frac{u^{\frac{1}{\alpha}-1}}{1-u}\,du$$

Now, $$\int_0^{\frac{1}{x}} u^{\frac{1}{\alpha}-1}\,du < \int_0^{\frac{1}{x}} \frac{u^{\frac{1}{\alpha}-1}}{1-u}\,du < \frac{1}{1-\frac{1}{x}} \int_0^{\frac{1}{x}} u^{\frac{1}{\alpha}-1}\,du$$

$$\alpha x^{-\frac{1}{\alpha}} < \int_0^{\frac{1}{x}} \frac{u^{\frac{1}{\alpha}-1}}{1-u}\,du < \frac{\alpha x^{-1/\alpha}}{1-\frac{1}{x}}$$

Let $F(\beta,t) = \int_0^t \frac{v^\beta}{1-v}\,dv,\ 0 \le t < 1$. The properties of $F(\beta,t)$ will be discussed later. Then,

$$\int_1^{n-2} \frac{dt}{x - t^\alpha} = \frac{x^{\frac{1}{\alpha}-1}}{\alpha} F\left(\frac{1}{\alpha}-1, \frac{(n-2)^\alpha}{x}\right) + O\left(\frac{1}{x}\right)$$

$$\int_1^{n-2} \frac{dt}{t(x-t^\alpha)} = \frac{1}{\alpha x} \int_{\frac{1}{x}}^{\frac{(n-2)^\alpha}{x}} \frac{du}{u(1-u)} \quad , \quad (t^\alpha = ux)$$

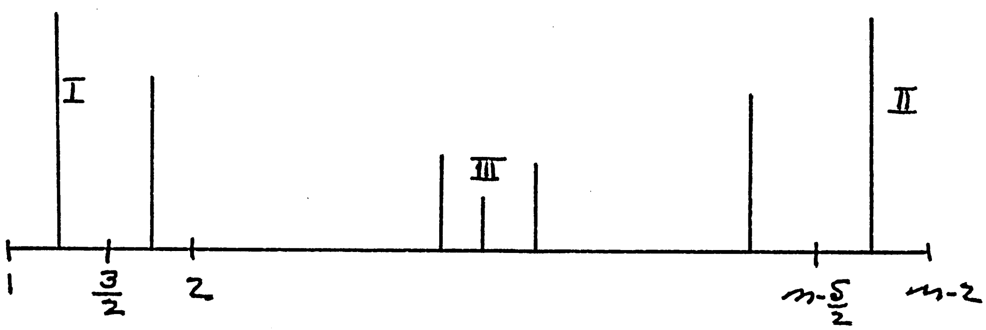
$$= \frac{1}{\alpha x} \int_{\frac{x}{(n-2)^\alpha}}^{x} \frac{dv}{v-1} \quad , \quad \left(u = \frac{1}{v}\right)$$

$$= \frac{1}{\alpha x}\left[\log(x-1) - \log\left(\frac{x}{(n-2)^\alpha} - 1\right)\right]$$

We wish to show that the remaining term is $O\left(\frac{1}{n^\alpha}\right) = O\left(\frac{1}{x}\right)$.

$$\int_1^{n-2} \left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{t(x-t^\alpha)} = \left\{\int_1^{3/2} + \int_{3/2}^{2} + \cdots + \int_{n-5/2}^{n-2}\right\}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{t(x-t^\alpha)}.$$

Note that the sign of the integrand alternates and that the magnitude of the integral varies as shown below:



Thus,
$$\left| \int_1^{n-2} \left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{t(x-t^\alpha)} \right| < I + II + III$$

But each of these is $O\left(\frac{1}{n^\alpha}\right)$ by the mean value theorem; therefore,

$$\int_1^{n-2} \left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{t(x-t^\alpha)} = O\left(\frac{1}{n^\alpha}\right) = O\left(\frac{1}{x}\right)$$

Combining the results,

$$\int_1^{n-2} [t]\frac{dt}{t(x-t^\alpha)} = \frac{x^{\frac{1}{\alpha}-1}}{\alpha} F\left(\frac{1}{\alpha}-1, \frac{(n-2)^\alpha}{x}\right) - \frac{1}{2\alpha x}\log(x-1)$$
$$+ \frac{1}{2\alpha x}\log\left(\frac{x}{(n-2)^\alpha}-1\right) + O\left(\frac{1}{x}\right).$$

Hence, $$\log g(x) = (m-3/2)\log\left(\frac{x}{(m-2)^\alpha}-1\right) + x^{\frac{1}{\alpha}}F\left(\frac{1}{\alpha}-1, \frac{(m-2)^\alpha}{x}\right) - \frac{1}{2}\log(x-1) + O(1) \quad .$$

$$\int_{m+1}^{\infty}[t]\frac{h'(t)\,dt}{h(t)\{h(t)-x\}} = \alpha\int_{m+1}^{\infty}\frac{dt}{t^\alpha-x} - \frac{\alpha}{2}\int_{m+1}^{\infty}\frac{dt}{t(t^\alpha-x)} - \alpha\int_{m+1}^{\infty}(t-[t]-\tfrac{1}{2})\frac{dt}{t(t^\alpha-x)}$$

$$\int_{m+1}^{\infty}\frac{dt}{t^\alpha-x} = \frac{x^{\frac{1}{\alpha}-1}}{\alpha}\int_{\frac{(m+1)^\alpha}{x}}^{\infty}\frac{u^{\frac{1}{\alpha}-1}}{1-u}\,du \quad , \quad (t^\alpha = ux)$$

$$= \frac{x^{\frac{1}{\alpha}-1}}{\alpha}\int_{0}^{\frac{x}{(m+1)^\alpha}}\frac{v^{-1/\alpha}\,dv}{1-v} \quad , \quad \left(u = \frac{1}{v}\right)$$

$$= \frac{x^{\frac{1}{\alpha}-1}}{\alpha}F\left(-\frac{1}{\alpha}, \frac{x}{(m+1)^\alpha}\right)$$

$$\int_{m+1}^{\infty}\frac{dt}{t(t^\alpha-x)} = \frac{1}{\alpha x}\int_{\frac{(m+1)^\alpha}{x}}^{\infty}\frac{du}{u(u-1)} = \frac{1}{\alpha x}\int_{0}^{\frac{x}{(m+1)^\alpha}}\frac{dv}{1-v}$$

$$= -\frac{1}{\alpha x}\log\left(1-\frac{x}{(m+1)^\alpha}\right)$$

$$\int_{m+1}^{\infty}(t-[t]-\tfrac{1}{2})\frac{dt}{t(t^\alpha-x)} = \left\{\int_{m+1}^{m+3/2} + \int_{m+3/2}^{m+2} + \cdots\right\}(t-[t]-\tfrac{1}{2})\frac{dt}{t(t^\alpha-x)}$$

Thus, $$\left| \int_{n+1}^{\infty} (t-[t]-\tfrac{1}{2}) \frac{dt}{t(t^{\alpha}-x)} \right| < \left| \int_{n+1}^{n+3/2} \right| = O\left(\frac{1}{n^{\alpha}}\right)$$

Substituting,

$$\log g_2(x) = -(n+\tfrac{1}{2}) \log\left(1-\frac{x}{(n+1)^{\alpha}}\right) - x^{1/\alpha} F\left(-\tfrac{1}{\alpha}, \frac{x}{(n+1)^{\alpha}}\right) + O(1).$$

Now, to study $F(\beta,t)$:

$$F(\beta,t) = \int_0^t \frac{v^{\beta}}{1-v}\,dv = \int_0^t \frac{dv}{1-v} - \int_0^t \frac{1-v^{\beta}}{1-v}\,dv$$

$$= -\log(1-t) - \int_0^1 \frac{1-v^{\beta}}{1-v}\,dv + \int_t^1 \frac{1-v^{\beta}}{1-v}\,dv$$

As $\lim\limits_{v\to 1} \frac{1-v^{\beta}}{1-v} = \beta$, $\int_t^1 \frac{1-v^{\beta}}{1-v}\,dv = O(1-t)$

Setting $-\int_0^1 \frac{1-v^{\beta}}{1-v}\,dv = C_{\beta}$, we have

$$F(\beta,t) = C_{\beta} - \log(1-t) + O(1-t)$$

Letting $C_1 = C_{\frac{1}{\alpha}-1}$, $C_2 = C_{1/\alpha}$, and noting that $x^{\frac{1}{\alpha}}\left(1-\frac{(n-2)^{\alpha}}{x}\right) = O(1)$ and $x^{1/\alpha}\left(1-\frac{x}{(n+1)^{\alpha}}\right) = O(1)$,

$$\log g_1(x) = (n-\tfrac{3}{2}) \log\left(\frac{x}{(n-2)^{\alpha}}-1\right) - x^{\frac{1}{\alpha}} \log\left(1-\frac{(n-2)^{\alpha}}{x}\right)$$
$$+ C_1 n - \tfrac{1}{2}\log(x-1) + O(1).$$

$$\log g_2(x) = -\left[(n+\tfrac{1}{2}) - x^{1/\alpha}\right] \log\left(1-\frac{x}{(n+1)^{\alpha}}\right) - C_2 n + O(1).$$

Since $(n-1)^\alpha \le x \le n^\alpha$, $x = (n-\theta)^\alpha$, $0 \le \theta \le 1$.

$$\begin{aligned}
(n-\tfrac{3}{2})\log\left(\frac{x}{(n-2)^\alpha}-1\right) &= (n-\tfrac{3}{2})\log\left[\left(\frac{n-\theta}{n-2}\right)^\alpha-1\right] \\
&= (n-\tfrac{3}{2})\log\left[\frac{\alpha(2-\theta)}{n}+O\left(\frac{1}{n^2}\right)\right] \\
&= -(n-\tfrac{3}{2})\log n + (n-\tfrac{3}{2})\log\alpha(2-\theta) + O(1)
\end{aligned}$$

$$\begin{aligned}
-x^{\frac{1}{\alpha}}\log\left(1-\frac{(n-2)^\alpha}{x}\right) &= -(n-\theta)\log\left[1-\left(\frac{n-2}{n-\theta}\right)^\alpha\right] \\
&= (n-\theta)\log n - (n-\theta)\log\alpha(2-\theta) + O(1)
\end{aligned}$$

$$\begin{aligned}
(x^{\frac{1}{\alpha}}-n-\tfrac{1}{2})\log\left(1-\frac{x}{(n+1)^\alpha}\right) &= -(\theta+\tfrac{1}{2})\log\left[\frac{\alpha(\theta+1)}{n}+O\left(\frac{1}{n^2}\right)\right] \\
&= (\theta+\tfrac{1}{2})\log n - (\theta+\tfrac{1}{2})\log\alpha(\theta+1) + O(1)
\end{aligned}$$

Substituting and adding,

$$\log g_1(x)g_2(x) = (C_1-C_2)n - \tfrac{1}{2}\log(x-1) + 2\log n + O(1)$$

Using the symbol $\asymp$ as follows:

$A \asymp B$ if and only if there $K > 1$ such that $\frac{1}{K}A < B < KA$, we have

$$g_1(x)g_2(x) \asymp n^{-\frac{\alpha}{2}+2} e^{(C_1-C_2)n}$$

$$\begin{aligned}
|a_n - a_{n-1}| = \int_{b_{n-1}}^{b_n} |f(x)|dx &= \int_{(n-1)^\alpha}^{n^\alpha} \left(\frac{x}{(n-1)^\alpha}-1\right)\left(1-\frac{x}{n^\alpha}\right) g_1(x)g_2(x)\,dx \\
&\asymp n^{-\frac{\alpha}{2}+2} e^{(C_1-C_2)n} \int_{(n-1)^\alpha}^{n^\alpha} \left(\frac{x}{(n-1)^\alpha}-1\right)\left(1-\frac{x}{n^\alpha}\right)dx
\end{aligned}$$

As $$\int_a^b \left(\frac{x}{a}-1\right)\left(1-\frac{x}{b}\right)dx = \frac{(b-a)^3}{6ab},$$

$$\int_{(n-1)^\alpha}^{n^\alpha} \left(\frac{x}{(n-1)^\alpha}-1\right)\left(1-\frac{x}{n^\alpha}\right)dx = \frac{[n^\alpha-(n-1)^\alpha]^3}{6n^\alpha(n-1)^\alpha}$$

$$= [n^\alpha-(n-1)^\alpha]\left\{\frac{1}{6}\left(\frac{n}{n-1}\right)^\alpha+\frac{1}{6}\left(\frac{n-1}{n}\right)^\alpha-\frac{1}{3}\right\}$$

$$= [n^\alpha-(n-1)^\alpha]\left\{\frac{\alpha}{6}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\right\}$$

$$= \frac{\alpha^2}{6}\frac{1}{n^2}\alpha n^{\alpha-1}\left\{1+O\left(\frac{1}{n}\right)\right\}$$

$$= \frac{\alpha^3}{6}n^{\alpha-3}\left[1+O\left(\frac{1}{n^2}\right)\right]$$

Therefore,

$$|a_n-a_{n-1}| \simeq n^{\frac{\alpha}{2}-1}e^{K_\alpha n}$$, where

$$K_\alpha = C_{\frac{1}{\alpha}-1}-C_{-\frac{1}{\alpha}} = \int_0^1 \frac{u^{\frac{1}{\alpha}-1}-u^{-\frac{1}{\alpha}}}{1-u}\,du$$

We may write K_α in terms of the beta function:

$$K_\alpha = \lim_{\epsilon\downarrow 0}\left\{B\left(\tfrac{1}{\alpha},\epsilon\right)-B\left(1-\tfrac{1}{\alpha},\epsilon\right)\right\}$$

Collecting the results,

<u>Theorem 9)</u>. <u>Let</u> $f(z)=\prod_1^\infty\left(1-\frac{z}{b_n}\right)$, $b_n = n^\alpha,\ \alpha>1$.

Let $a_n = \int_0^{b_n} f(x)\,dx$. Then,

$$|a_n - a_{n-1}| \simeq n^{\frac{\alpha}{2}-1} e^{K_\alpha n}$$

where $$K_\alpha = \lim_{\epsilon \downarrow 0} \left\{ B\left(\tfrac{1}{\alpha},\epsilon\right) - B\left(1-\tfrac{1}{\alpha},\epsilon\right) \right\}$$.

From the general theorem of Chapter II, we can determine only the existence of an infinite sequence of indices n_i such that $|a_{n_i} - a_{n_i-1}| > e^{-n_i^{\alpha\rho+\epsilon}}$. As $\rho = \frac{1}{\alpha}$,

$$|a_{n_i} - a_{n_i-1}| > e^{-n_i^{(1+\epsilon)}}$$

It is apparent that this result is not so exact as the one obtained above.

3. Let $b_n = \alpha^n = e^{Kn}$, $K>0$. . Take $h(t)=e^{Kt}$ and consider $e^{K(n-1)} \le x \le e^{Kn}$.

As $\frac{h'(t)}{h(t)} = K$, and $[t] = t - \left(t-[t]-\tfrac{1}{2}\right) - \tfrac{1}{2}$,

$$\frac{1}{k}\int_1^{m-2}[t]\frac{h'(t)dt}{h(t)\{x-h(t)\}} = \int_1^{m-2} t\frac{dt}{x-e^{kt}} - \int_1^{m-2}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{x-e^{kt}}$$
$$-\frac{1}{2}\int_1^{m-2}\frac{dt}{x-e^{kt}}$$

And,

$$\frac{1}{k}\int_{m+1}^{\infty}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}} = \int_{m+1}^{\infty} t\frac{dt}{e^{kt}-x} - \int_{m+1}^{\infty}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{e^{kt}-x} - \frac{1}{2}\int_{m+1}^{\infty}\frac{dt}{e^{kt}-x}$$

Now,

$$\int_1^{m-2}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{x-e^{kt}} = \int_1^{3/2} + \int_{3/2}^{2} + \cdots + \int_{m-5/2}^{m-2}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{x-e^{kt}}$$

But the terms on the right form an increasing sequence of alternating sign; therefore,

$$\int_1^{m-2}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{x-e^{kt}} = O\left(\int_{m-5/2}^{m-2}\frac{dt}{x-e^{kt}}\right) = O\left(\frac{1}{e^{km}}\right),$$

by the mean value theorem.

In like manner,

$$\int_{m+1}^{\infty}\left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{e^{kt}-x} = \int_{m+1}^{m+3/2} + \int_{m+3/2}^{m+2} + \cdots \quad \left(t-[t]-\tfrac{1}{2}\right)\frac{dt}{e^{kt}-x}$$

$$= O\left(\frac{1}{e^{km}}\right)$$

$$\int_1^{n-2} \frac{dt}{x-e^{kt}} = \frac{1}{k}\int_k^{k(n-2)} \frac{d\tau}{x-e^{\tau}} = \frac{1}{xk}\left[(n-3)k - \log\frac{x-e^{(n-2)k}}{x-e^{k}}\right]$$

$$\int_{n+1}^{\infty} \frac{dt}{e^{kt}-x} = \frac{1}{k}\int_{(n+1)k}^{\infty} \frac{d\tau}{e^{\tau}-x} = \frac{1}{xk}\left[(n+1)k - \log(e^{(n+1)k}-x)\right]$$

$$\int_{n+1}^{\infty} t\,\frac{dt}{e^{kt}-x} = \frac{1}{k^2}\int_{(n+1)k}^{\infty} \tau\,\frac{d\tau}{e^{\tau}-x} \quad ,$$

$$\frac{\tau}{e^{\tau}-x} = \frac{\tau}{e^{\tau}}\,\frac{1}{1-\frac{x}{e^{\tau}}} = \sum_{m=0}^{\infty} x^m\,\tau e^{-(m+1)\tau}$$

Integrating,

$$\int \frac{\tau}{e^{\tau}-x}\,d\tau = \sum_{m=0}^{\infty} x^m\left\{-\frac{\tau e^{-(m+1)\tau}}{m+1} - \frac{e^{-(m+1)\tau}}{(m+1)^2}\right\}$$

$$= -\frac{\tau}{x}\sum_{1}^{\infty}\frac{x^m}{m e^{m\tau}} - \frac{1}{x}\sum_{0}^{\infty}\frac{x^{m+1}}{(m+1)^2 e^{(m+1)\tau}}$$

$$= \frac{\tau}{x}\log\left(1-\frac{x}{e^{\tau}}\right) - \frac{1}{x}\sum_{1}^{\infty}\frac{x^m}{m^2 e^{m\tau}}$$

Thus,

$$\int_{n+1}^{\infty} t\,\frac{dt}{e^{kt}-x} = \frac{1}{k^2}\left[\frac{\tau}{x}\log\left(1-\frac{x}{\tau}\right) - \frac{1}{x}\sum_{1}^{\infty}\frac{x^m}{m^2 e^{m\tau}}\right]_{(n+1)k}^{\infty}$$

$$= -\frac{n+1}{k}\log\left(1-\frac{x}{e^{(n+1)k}}\right) + O\left(\frac{1}{x}\right),$$

Since $\sum_1^\infty \frac{x^m}{m^2 e^{m(m+1)k}} < \sum_1^\infty \frac{1}{m^2} = \frac{\pi^2}{6}$.

Similarly,

$$\int_1^{m-2} t \frac{dt}{x-e^{kt}} = \frac{(m-2)^2}{2x} - \frac{m-2}{kx}\log\left(1-\frac{e^{(m-2)k}}{x}\right)$$
$$+ O\left(\frac{1}{x}\right).$$

Thus,

$$x\int_1^{m-2}[t]\frac{h'(t)dt}{h(t)\{x-h(t)\}} - x\int_{m+1}^{\infty}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}} = \frac{k(m-2)^2}{2} + \tfrac{1}{2}k(m+1)$$
$$-(m-2)\log\left(1-\frac{e^{(m-2)k}}{x}\right) + (m+1)\log\left(1-\frac{x}{e^{(m+1)k}}\right)$$
$$-\tfrac{1}{2}k(m-3) + \tfrac{1}{2}\log\frac{x-e^{(m-2)k}}{x-e^{k}} - \tfrac{1}{2}\log\left(e^{(m+1)k}-x\right)$$
$$+ O(1).$$

Simplyfying,

$$-x\int_1^{m-2}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}} - x\int_{m+1}^{\infty}[t]\frac{h'(t)dt}{h(t)\{h(t)-x\}} = -k\left(\frac{m^2}{2}+4m\right)$$
$$-\left(m-\tfrac{5}{2}\right)\log\left(x-e^{(m-2)k}\right) + \left(m-\tfrac{5}{2}\right)\log x$$
$$+\left(m+\tfrac{1}{2}\right)\log\left(e^{(m+1)k}-x\right) + O(1), \text{ AS } \log\left(x-e^{k}\right)$$
$$= \log x + O(1).$$

Substituting,

$$\log g_1(x) g_2(x) = (m-2)\log\left(\frac{x}{e^{(m-2)K}}-1\right) - m\log\left(1-\frac{x}{e^{(m+1)K}}\right)$$
$$- K\left(\frac{m^2}{2}+4m\right) + (m-5/2)\log x$$
$$-(m-5/2)\log\left(x-e^{(m-2)K}\right) + (m+\tfrac{1}{2})\log\left(e^{(m+1)K}-x\right) + O(1).$$

$$= -K\left(\frac{m^2}{2}-m\right) + (m-5/2)\log x + \tfrac{1}{2}\log\left(x-e^{(m-2)K}\right)$$
$$+ \tfrac{1}{2}\log\left(e^{(m+1)K}-x\right) + O(1).$$

As $e^{(m-1)K} \le x \le e^{mK}$, $x = e^{(m-\theta)K}$, $0 \le \theta \le 1$.

$$x - e^{(m-2)K} = e^{(m-\theta)K} - e^{(m-2)K} = K(m-\theta-m+2)e^{(m-\theta')K}$$
$$= K(2-\theta)e^{(m-\theta')K}, \quad |\theta'| < 2$$

Thus, $\log\left(x - e^{(m-2)K}\right) = mK + O(1)$

Similarly, $\log\left(e^{(m+1)K} - x\right) = mK + O(1)$

This gives $\log g_1(x) g_2(x) = -K\left(\frac{m^2}{2} - 2m\right) + (m-5/2)\log x + O(1)$

or,

$$g_1(x) g_2(x) \asymp x^{m-5/2} e^{-\frac{Km^2}{2}+2Km}$$
$$\asymp x^m e^{-K/2\,(m^2+m)}$$

Hence, $$\int_{e^{(n-1)k}}^{e^{nk}} |f(x)|dx \asymp e^{-\frac{k}{2}(n^2+n)} \int_{e^{(n-1)k}}^{e^{nk}} x^n\left(\frac{x}{e^{(n-1)k}}-1\right)\left(1-\frac{x}{e^{nk}}\right)dx$$

$$\int_{e^{(n-1)k}}^{e^{nk}} x^n\left(\frac{x}{e^{(n-1)k}}-1\right)\left(1-\frac{x}{e^{nk}}\right)dx = e^{kn^2}\left[\frac{e^{(n+1)k}}{(n+2)(n+3)} - \frac{e^{nk}}{(n+1)(n+2)} + \frac{e^{-k}}{(n+1)(n+2)} - \frac{e^{-2k}}{(n+2)(n+3)}\right]$$

Now the first two terms in the parenthesis are asymptotically equal to $\dfrac{(e^{k}-1)e^{nk}}{n^2}$ and the last two are $O\left(\frac{1}{n^2}\right)$.

Therefore, $$|a_n - a_{n-1}| = \int_{e^{(n-1)k}}^{e^{nk}} |f(x)|dx \asymp \frac{1}{n^2} e^{\frac{1}{2}kn(n+1)}$$

<u>Theorem 10)</u>. <u>If $f(z) = \prod_1^{\infty}(1 - z/b_n)$</u>, <u>$b_n = e^{kn}$</u>, <u>$k>0$</u>, <u>and $a_n = \int_0^{b_n} f(x)dx$</u>, <u>then</u>

$$|a_n - a_{n-1}| \asymp \frac{1}{n^2} e^{\frac{1}{2}kn(n+1)}.$$

We would have discovered only that there are infinitely many $|a_n - a_{n-1}|$ greater than $e^{-e^{Kn\epsilon}}$ from Theorem 8).

4. Let $b_n = \underline{|n}$, and consider $b_{n-1} \le x \le b_n$.

$$|f(x)| = \left(\frac{x}{\underline{|n-1}} - 1\right)\left(1 - \frac{x}{\underline{|n}}\right) g_1(x) g_2(x) \text{, WHERE}$$

$$g_1(x) = \prod_1^{n-2}\left(\frac{x}{\underline{|k}} - 1\right) \quad , \quad g_2(x) = \prod_{n+1}^{\infty}\left(1 - \frac{x}{\underline{|k}}\right)$$

Consider $g_1(x)$:

$$\frac{x}{\underline{|k}} - 1 = \frac{x}{\underline{|k}}\left(1 - \frac{\underline{|k}}{x}\right)$$

Now,

$$\left(1 - \frac{1}{n-1}\right)\left(1 - \frac{1}{(n-1)(n-2)}\right) \cdots \left(1 - \frac{1}{(n-1)\cdots 2}\right) \le \prod_1^{n-2}\left(1 - \frac{\underline{|k}}{x}\right) \le$$

$$\le \left(1 - \frac{1}{n(n-1)}\right)\left(1 - \frac{1}{n(n-1)(n-2)}\right) \cdots \left(1 - \frac{1}{n \cdots 2}\right)$$

Thus,
$$\left(1 - \frac{1}{n-1}\right)\left(1 - \frac{1}{(n-1)(n-2)}\right)^{n-3} \le \prod_1^{n-2}\left(1 - \frac{\underline{|k}}{x}\right) \le \left(1 - \frac{1}{n(n-1)}\right)\left(1 - \frac{1}{n \cdots 2}\right)^{n-3}$$

$$1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right) \le \prod_1^{n-2}\left(1 - \frac{\underline{|k}}{x}\right) \le 1 - \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)$$

Therefore,

$$g_1(x) = \prod_1^{n-2}\left(\frac{x}{\underline{|k}} - 1\right) \sim \frac{x^{n-2}}{\prod_1^{n-2} \underline{|k}}$$

Also, $\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n^2}\right)\cdots < g_2(x) < 1$.

Hence, $g_2(x) \sim 1$; giving, $|f(x)| \sim \left(\frac{x}{\underline{|n-1}}-1\right)\left(1-\frac{x}{\underline{|n}}\right)\frac{x^{n-2}}{\prod_1^{n-2}\underline{|k}}$

$$\int_{\underline{|n-1}}^{\underline{|n}} x^{n-2}\left(\frac{x}{\underline{|n-1}}-1\right)\left(1-\frac{x}{\underline{|n}}\right)dx = -\frac{(\underline{|n})^{n+1}-(\underline{|n-1})^{n+1}}{(n+1)\underline{|n}\cdot\underline{|n-1}} + \frac{(\underline{|n})^{n}-(\underline{|n-1})^{n}}{n\underline{|n-1}}$$

$$+\frac{(\underline{|n})^{n}-(\underline{|n-1})^{n}}{n\underline{|n}} - \frac{(\underline{|n})^{n-1}-(\underline{|n-1})^{n-1}}{n-1}$$

$$= (\underline{|n})^{n-1}\cdot\frac{n^2-2n-1}{(n-1)n(n+1)} + (\underline{|n-1})^{n-1}\cdot\frac{n^2+1}{(n-1)n^2(n+1)}$$

$$= A_n$$

Therefore, $|a_n - a_{n-1}| = \int_{\underline{|n-1}}^{\underline{|n}} |f(x)|dx \sim \frac{A_n}{\prod_1^{n-2}\underline{|k}}$

Theorem 11). If $f(z) = \prod_1^{\infty}(1-z/b_n)$, $b_n = \underline{|n}$, and $a_n = \int_0^{b_n} f(x)dx$, then

$$|a_n - a_{n-1}| \sim \frac{A_n}{\prod_1^{n-2}\underline{|k}}$$, where

A_n is given immediately above.

Again, the general theorem would not have led to a very accurate estimate.

REFERENCES

[1.1] Bierberbach, L. Funktionentheorie, 2nd edition, Volume II, Leipzig.

[1.2] ________________ "Über eine Vertiefung des Picardschen Satzes bei ganzen Funktionen endlicher Ordnung," Mathematische Zeitschrift, Vol. III, pp. 175-190.

[2] Biernacki, M. "Sur la théorie des fonctions entières," Cracow Académie des Sciences, Bull. Internat. Classe des Sci. Math. et Nat., Series A, 1929, pp. 530-590.

[3] MacLane, G. R. Concerning the Uniformization of certain Riemann Surfaces allied to the Inverse-Cosine and Inverse-Gamma Surfaces, Trans. Amer. Math. Soc., v. 62, pp. 99-113; or, Thesis, Rice Institute, 1946.

[4] Titchmarsh, E. C. Theory of Functions, 2nd edition, 1939.

[5] Valiron, G. General Theory of Integral Functions, Toulouse, 1923.

[6] Paley, R.E.A.C., and N. Wiener. Fourier Transforms in the Complex Domain, New York, 1934.