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Positive Lyapunov Exponent for Ergodic Schrödinger Operators

by

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ABSTRACT

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The discrete Schrödinger equation describes the behavior of a 1-dimensional quantum particle in a disordered medium. The Lyapunov exponent $L(E)$ describes the exponential behavior of solutions at an energy E. Positivity of the Lyapunov exponent in a set of energies is an indication of absence of transport for the Schrödinger equation.

In this thesis, I will discuss methods based on multiscale analysis to prove positive Lyapunov exponent for ergodic Schrödinger operators. As an application, I prove positive Lyapunov exponent for operators whose potential is given by evaluating an analytic sampling function along the orbit of a skew-shift on a high dimensional torus.

The first method is based only on ergodicity, but needs to eliminate a small set of energies. The second method uses recurrence properties of the skew-shift combined with analyticity to prove a result for all energies.

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Last but not least, let me thank my parents.

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CHAPTER 1

Introduction

The discrete one dimensional Schrödinger equation models the motion of a quantum particle in a disordered environment. Here the environment is described through a bounded function $V : \mathbb{Z} \to \mathbb{R}$ called the *potential*. The quantum particle is described through its wave function $\psi \in \ell^2(\mathbb{Z})$ and evolves according to

$$\mathrm{i}\frac{\partial}{\partial t}\psi(t) = H\psi(t), \tag{1.1}$$

where $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is the Schrödinger operator

$$(Hu)(n) = u(n+1) + u(n-1) + V(n)u(n). \tag{1.2}$$

The interpretation is as follows: if ψ is normalized so that

$$\|\psi\|^2 = \sum_{n\in\mathbb{Z}} |\psi(n)|^2 = 1, \tag{1.3}$$

then for $A \subseteq \mathbb{Z}$ the probability to find the particle in A is given by

$$\sum_{n\in A} |\psi(n)|^2. \tag{1.4}$$

Since (1.1) is a differential equation on the infinite dimensional space $\ell^2(\mathbb{Z})$, understanding its dynamics might seem a hopeless endeavor. However, there is at least one situation where it is simple. Suppose $\psi \in \ell^2(\mathbb{Z})$ is an eigenfunction of H for the eigenvalue E, that is $H\psi = E\psi$. Then we can write down the time evolution as

$$\psi(t) = \mathrm{e}^{-\mathrm{i}Et}\psi. \tag{1.5}$$

In this thesis, I will try to answer a question related to the existence of eigenfunctions for operators of the form (1.2) for the potential V given by

$$V(n) = g(\alpha n^K), \tag{1.6}$$

where $g : \mathbb{T} \to \mathbb{R}$ is an analytic function, α an irrational number, and $K \geq 2$ an integer.

1.1 Numerical evidence

In this section, I will discuss numerical computations, which suggest an answer to whether there should be eigenvalues of H for the potential defined in (1.6) or not. Of course the study of $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is impossible numerically, since it is defined on an infinite dimensional space. However, it is possible to restrict the domain of definition to $\ell^2([1, N])$ and consider the equation there. Denote by $H_{[1,N]}$ the restriction of H to $\ell^2([1, N])$. We have that $H_{[1,N]}$ is given by the matrix

$$H_{[1,N]} = \begin{pmatrix} V(1) & 1 & & & & \\ 1 & V(2) & 1 & & & \\ & 1 & V(3) & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & V(N-1) & 1 \\ & & & & 1 & V(N) \end{pmatrix}. \tag{1.7}$$

Of course concluding statements from the restriction $H_{[1,N]}$ to H is non-trivial. For example, $H_{[1,N]}$ always has exactly N eigenvalues even when H may not have any (take $V \equiv 0$).

The main problem passing from $H_{[1,N]}$ to H is that there could be non-trivial interactions near the boundary points 1 and N. We will thus take for now the standpoint, that if the eigenfunctions of $H_{[1,N]}$ do not vanish near the boundary, then probably H will have no eigenvalues, and if they do, H will have eigenvalues. Of course this is only a heuristic statement, and in no way rigorous mathematics.

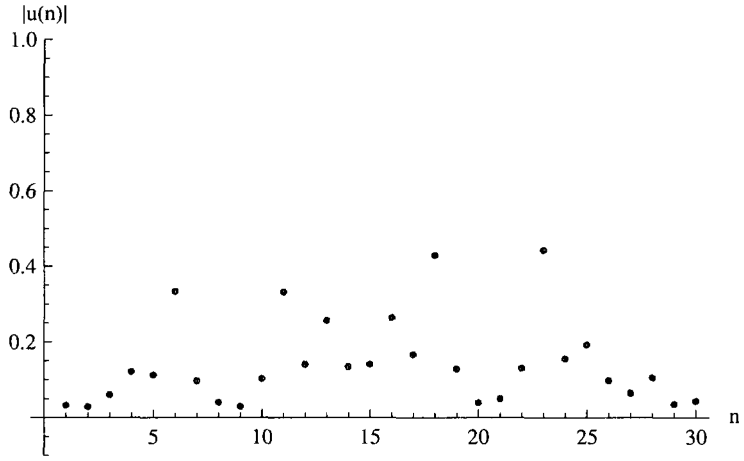

Figure 1.1: Quasi-periodic

I will now present the computations for $N = 30$ and the potential

$$V(n) = 2\lambda \cos(2\pi\alpha n^K), \tag{1.8}$$

where $\lambda = 0.9$, $\alpha = \sqrt{2} - 1$, and $K = 1$ or $K = 2$. One of the resulting eigenfunctions is shown for $K = 1$ in Figure 1.1 and for $K = 2$ in Figure 1.2. I will explain how these numerical computations were done in Appendix A.

In the *quasi-periodic* case, that is $K = 1$, one sees that the eigenfunction looks like uniformly distributed noise at the 30 values. This means, that one should expect that there are no eigenfunctions. In fact, in the case $K = 1$ the potential defined in (1.8) corresponds to the *Almost–Mathieu operator.*

For this operator, it is known that the spectrum is absolutely continuous for $0 \leq \lambda < 1$, which in particular implies the absence of eigenfunctions. This can for example be found in the work of Jitomirskaya [23] and references therein.

Let me also remark here that in the regime of $\lambda > 0$ large enough and $K = 1$, the operator H has an orthonormal basis of exponentially decaying eigenfunctions as was shown by Jitomirskaya [23] and Bourgain and Goldstein [10].

We will see shortly that the cases $K \geq 2$ are very different from the quasi-periodic one. For this reason, the quasi-periodic case will not concern us any further.

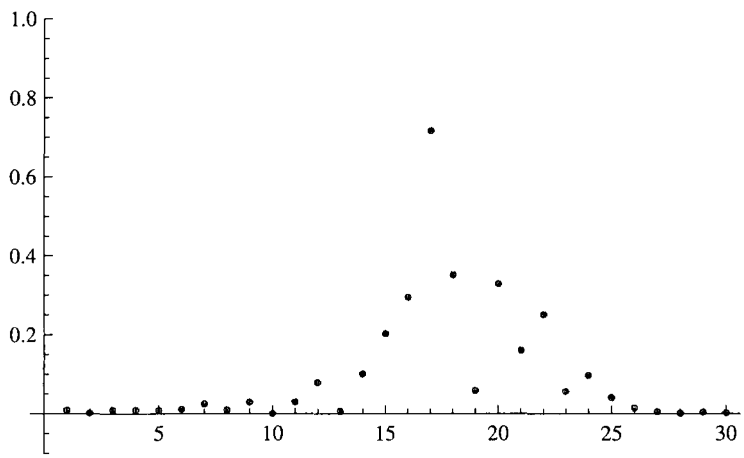

Figure 1.2: Skew-shift

In the case of $K = 2$, the eigenfunction is shown in Figure 1.2. One sees that it looks like a localized bump in $14 \leq n \leq 25$. By the previously given heuristics, one might expect that the operator H has eigenfunctions. One might even expect H to have an orthonormal basis of eigenfunctions. By taking a closer look at the numerics, one can even expect that the eigenfunctions decay exponentially away from their maximum. In fact trying to prove this was my main motivation for the work in this thesis. Let me finally remark here that for $K \geq 3$, the images of the eigenfunction will look as in the case $K = 2$. The same holds for any $\lambda > 0$, as long as one chooses N large enough. Let me summarize this in

Problem 1.1. *Let $K \geq 2$, $\lambda > 0$, and α irrational. Show that the Schrödinger operator with potential* (1.8) *has an orthonormal basis consisting of exponentially decaying eigenfunctions.*

Let me furthermore point out that the existence of an orthonormal basis of eigenfunctions has an interesting consequence for the dynamics of (1.1). In fact, if this holds, then by the *RAGE Theorem* for any normalized initial condition $\psi(0) \in \ell^2(\mathbb{Z})$ and any $\varepsilon > 0$, there exists an $N \geq 1$ such that for all $t > 0$

$$\sum_{n=-N}^{N} |\psi(t,n)|^2 \geq 1 - \varepsilon. \tag{1.9}$$

This means that the solution stays localized on the sites between $-N$ and N.

Let me now discuss a second observation, which can be obtained through numerics, but won't concern us in the main body of this thesis. If one computes the spectrum of the operator H, one also sees a transition at $K = 2$.

For $K = 1$, one sees that $\sigma(H)$ is a Cantor set and for $K \geq 2$ that $\sigma(H)$ is an interval. Again the behavior in the quasi-periodic case $K = 1$ has been extensively studied, and $\sigma(H)$ is known to be a Cantor set in a sequence of cases. The case $K \geq 2$ is completely open. So I record

Problem 1.2. *Show that the spectrum of H with potential defined by* (1.8) *is an interval for $K \geq 2$.*

1.2 Positive Lyapunov exponent and the existence of eigenfunctions

I will now begin to introduce the setting of ergodic Schrödinger operators in which I will work. Let (Ω, μ) be a probability space and $T : \Omega \to \Omega$ an invertible ergodic transformation. Given a bounded measurable function $f : \Omega \to \mathbb{R}$ called the *sampling function*, we introduce for $\omega \in \Omega$ the potential

$$V_\omega(n) = f(T^n \omega). \tag{1.10}$$

We then define a family of Schrödinger operators $\{H_\omega\}_{\omega \in \Omega}$ by

$$\begin{aligned} &H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ &H_\omega u(n) = u(n+1) + u(n-1) + V_\omega(n) u(n). \end{aligned} \tag{1.11}$$

Introduce the Lyapunov exponent

$$(1.12) \qquad L(E) = \lim_{N \to \infty} \frac{1}{N} \int_\Omega \log \left\| \prod_{n=1}^{N} \begin{pmatrix} E - V_\omega(N-n) & -1 \\ 1 & 0 \end{pmatrix} \right\| d\mu(\omega).$$

For quasi-periodic operators ($K = 1$), that is $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $T_\alpha \omega = \omega + \alpha$ (mod 1) (α irrational), and a real analytic sampling function, it is known that a uniformly positive Lyapunov exponent implies that the operator H has an orthonormal basis consisting of exponentially decaying eigenfunctions for almost every (α, ω). Results of this form were proved by Jitomirskaya [23] for the Almost–Mathieu operator and by Bourgain and Goldstein [10] for general analytic sampling functions.

Motivated by this, I will focus instead of Problem 1.1 on

Problem 1.3. *Let $K \geq 2$, $\lambda > 0$, and α irrational. Show that the Lyapunov exponent $L(E)$ associated with the potential* (1.6) *is uniformly positive in E.*

Let me now discuss how to study the potential defined in (1.6) in the ergodic setting. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle. For $K \geq 1$ and an irrational α, introduce the *skew-shift*

$$(1.13) \qquad \begin{aligned} &T_{\alpha,K} : \mathbb{T}^K \to \mathbb{T}^K \\ &(T_{\alpha,K}\underline{\omega})_k = \begin{cases} \omega_1 + \alpha, & k = 1; \\ \omega_k + \omega_{k-1}, & 2 \leq k \leq K. \end{cases} \end{aligned}$$

One can then show, that

$$(T_{\alpha,K}^{n}\underline{\omega})_K = \frac{\alpha}{K!}n^K + \dots, \tag{1.14}$$

where $\dots$ stands for a degree $K-1$ polynomial in n. For $g : \mathbb{T} \to \mathbb{R}$ continuous, define $f(\underline{\omega}) = g(\omega_K)$. One can check that the potential from (1.6) is equal to the ergodic potential

$$V_{\underline{\omega}}(n) = f(T_{K!\alpha,K}^{n}\underline{\omega}) \tag{1.15}$$

for some $\underline{\omega} \in \mathbb{T}^K$. It is furthermore known that the map $T_{\alpha,K}$ is uniquely ergodic and minimal.

1.3 Results

I will now present three results towards the solution of Problem 1.3. These results will be proved in this thesis. Further results are stated in the next section.

Using that $V_{\underline{\omega}}$ defined in (1.15) depends on K parameters $\omega_1, \dots, \omega_K$, one can show that these potentials converge to independent identically distributed random variables as $K \to \infty$. Combining this with a continuity result on the Lyapunov exponent of Avila and Damanik [1], one obtains the following result.

Theorem 1.4. *Assume $g : \mathbb{T} \to \mathbb{R}$ is non-constant and define the potential as in* (1.15). *There exists a constant $\gamma > 0$ such that*

$$(1.16) \qquad |\{E : \quad L_K(E) < \gamma\}| \to 0$$

as $K \to \infty$.

The details of the proof can be found in Chapter 4. The main problem with this result is that it is not quantitative, so we cannot specify a large K such that

$$(1.17) \qquad |\{E : \quad L_K(E) < \gamma\}| \leq \frac{1}{100}$$

using these methods. The goal of the next two results is to present methods that answer this type of question under additional assumptions.

In fact, we will work in the large coupling regime. Introduce for $g : \mathbb{T} \to \mathbb{R}$, $\lambda > 0$, $K \geq 2$, α irrational, and $\underline{\omega} \in \mathbb{T}^K$ a family of potentials by

$$(1.18) \qquad V(n) = \lambda g((T^n_{\alpha,K}\underline{\omega})_K).$$

We will now denote the Lyapunov exponent by $L_{\lambda,K,\alpha}(E)$, to emphasize that it depends on all three quantities. The first result we will show is a variant of one of the results from my work [28] and the proof is based on $T_{\alpha,K}$ being an ergodic transformation.

Theorem 1.5. *Let g be real analytic and α irrational. For $\varepsilon > 0$, there exists $\lambda_0(K, \varepsilon) > 0$ such that*

$$|\{E : \quad L_{\lambda,K,\alpha}(E) < \frac{1}{100}\log(\lambda)\}| < \varepsilon \tag{1.19}$$

for $\lambda > \lambda_0$.

The assumption on g being real analytic could be considerably weakened in this theorem. However, the proof of the last result of this thesis will use this in an essential way. We will furthermore need to require a Diophantine assumption on α. We will write $\alpha \in DC(c)$ if

$$\operatorname{dist}(n\alpha, \mathbb{Z}) \geq \frac{c}{n^2} \tag{1.20}$$

for all integers $n \neq 0$. The result is

Theorem 1.6. *Let g be a trigonometric polynomial of degree d and $\alpha \in DC(c)$. Then for $\lambda > \lambda_0(c, K, d)$, we have*

$$L_{\lambda,K,\alpha}(E) \geq \frac{1}{100}\log(\lambda) \tag{1.21}$$

for all E.

It is clear that this is the strongest statement of the three in terms of conclusions, since no energies need to be eliminated. However also the assumptions are stronger and the proof is more complicated. In particular, it takes up around half of this thesis. In the case $K = 2$, Theorem 1.6 is due to Bourgain, Goldstein, and Schlag [12].

The proofs of Theorem 1.5 and Theorem 1.6 split into two parts. First one can show that the eigenfunctions corresponding to finite sized boxes are localized. This is done in Chapter 5.

Then knowing that the eigenfunctions, or more exactly, the matrix elements of the resolvent, decay exponentially for H_Λ, where Λ is a length N_0 interval, one extends this to a sequence of larger and larger scales

$$N_0 \ll N_1 \ll N_2 \ll N_3 \ll \dots . \tag{1.22}$$

For this reason this process is known as *multiscale analysis*. The main problem with this approach is that to pass from information on $H_{[1,N_j]}$ to information on $H_{[1,N_{j+1}]}$ one needs weak assumptions on $H_{[1,N_{j+1}]}$.

In the proof of Theorem 1.5, this information is obtained through the process of *energy elimination*. For Theorem 1.6, one uses quantitative recurrence results combined with complex analysis methods to do this. For this reason, one needs more restrictive assumptions to prove Theorem 1.6.

1.4 Further results

In this section, I wish to discuss further results that can be proved about the potential (1.15). Both of them concern the region of small coupling. First, I will state the following result, which can be shown by combining the methods of Bourgain from [3] with my results from [26] and [28].

Theorem 1.7. *Given $\delta > 0$ and $\tau > 0$, there exists $\gamma > 0$ such that for λ small enough and some α, we have*

$$|\{E \in [-2+\delta, -\delta] \cup [\delta, 2-\delta] : \quad L_{\lambda,2,\alpha}(E) < \gamma\lambda^2\}| < \tau. \tag{1.23}$$

It should be noted here that this is the only result available in the case $K = 2$ for small coupling. Second, I will state the other main result from [28].

Theorem 1.8. *Let f be an analytic function. Given $\lambda > 0$ small enough and $\varepsilon > 0$, there exists $\gamma > 0$ and $K_0 \geq 1$ such that for $K \geq K_0$, we have for any irrational α*

$$|\{E : \quad L_{\lambda,K,\alpha}(E) < \gamma\}| < \varepsilon. \tag{1.24}$$

The proof of this theorem is similar to the one of Theorem 1.5, except that the initial condition is proved differently. See Sections 11 and 12 in [28].

1.5 Further questions

In this section, I wish to mention some related problems to the ones discussed previously.

Problem 1.9. *Assume the conclusion of Problem 1.3. Show the conclusion of Problem 1.1.*

It will be necessary to place further restrictions on the sampling function f and on α. This problem is partly solved in the case $K = 2$ by Bourgain, Goldstein, and Schlag [12] and Bourgain [5]. That further assumptions are needed follows for example from the work of Boshernitzan and Damanik [13]. There are obvious extensions of this problem. For example it would be interesting to prove *dynamical localization* for these models.

Another question one could ask is, why only consider potentials of the form (1.6), and not consider more general potentials

$$V(n) = g(\alpha n^{\rho}), \tag{1.25}$$

where $\rho > 0$ and $g : \mathbb{T} \to \mathbb{R}$ is continuous? So, we have now replaced the integer power K by a real number ρ. It now follows from my work [27], that the study of the Lyapunov exponent reduces to studying the Lyapunov exponent for the family of potentials

$$V_{\beta}(n) = g(\beta n^{r}), \tag{1.26}$$

where $r = \lfloor \rho \rfloor$ and $\beta \in [0, 1]$. Hence, it is sufficient to study the Lyapunov exponent for the potential defined in (1.6). However, let me take the opportunity to state the main open problem for the potentials of the form (1.25).

Problem 1.10. *Assume we know that the Lyapunov exponent for the potential* (1.25) *is uniformly positive on the spectrum. Can one conclude that H has an orthonormal basis of exponentially decaying eigenfunctions?*

In order to tackle this problem, one should probably insert a parameter θ into the definition (1.25). Both $V(n) = g(\alpha n^{\rho} + \theta)$ and $V(n) = g(\alpha(n + \theta)^{\rho})$ seem natural choices.

Last I wish to point out the the analogue of Problem 1.2 in the case ρ not an integer was solved in [25].

CHAPTER 2

Background from Ergodic Theory

The goal of this chapter is to review some basic results from ergodic theory. We will have three main goals: Introduce the notion of ergodicity, discuss some properties of the skew-shift, discuss the ergodic theorems and their consequences. As an introduction to these matters, I can recommend the books by Brin and Stuck [14] and Walters [37].

It should be noted that although the ergodic theorems tell us that averages converge, they do not tell us how quickly. Obtaining quantitative results is a harder problem and will be presented for the skew-shift in Chapter 10 and 11.

2.1 Definitions

In this section, (Ω, μ) will be a probability space. Associated with it comes a σ-algebra, and we will always assume that all functions are measurable with respect to it. We furthermore, recall that we have for $A \subseteq \Omega$ that $\mu(A) \in [0, 1]$, since $\mu(\Omega) = 1$. We will be interested in properties of invertible maps $T : \Omega \to \Omega$. Our first definition is

Definition 2.1. *An invertible map $T : \Omega \to \Omega$ is called measure preserving if $\mu(TA) = \mu(A)$ holds for every set $A \subseteq \Omega$.*

For simplicity, I restrict myself to the case, where $T : \Omega \to \Omega$ is invertible. This restriction is not really needed and not common in the literature. We now continue with the definition of ergodicity.

Definition 2.2. *A measure preserving transformation $T : \Omega \to \Omega$ is called ergodic if, for any $B \subseteq \Omega$,*

$$TB = B \tag{2.1}$$

implies $\mu(B) \in \{0, 1\}$.

We remark that this is equivalent to the condition that if $f : \Omega \to \mathbb{C}$ is measurable, then $f \circ T = f$ implies that f is almost surely constant. Ergodicity will be an important concept in the following, since it will allow us to ensure that various uniform distribution properties hold.

Let us now turn to more topological notions, namely: unique ergodicity and minimality. Let us begin by changing the setting. We will now assume that Ω is a compact metric space, and that $T : \Omega \to \Omega$ is a homeomorphism.

Definition 2.3. *$T : \Omega \to \Omega$ is called minimal if, for every $\omega \in \Omega$, its orbit*

$$\mathcal{O}_T(\omega) = \{T^n\omega : \quad n \in \mathbb{Z}\} \tag{2.2}$$

is dense in Ω.

This is important, since it implies that various things will be independent of ω. Let us now turn to unique ergodicity.

Definition 2.4. *$T : \Omega \to \Omega$ is called uniquely ergodic if, for every continuous function $f : \Omega \to \mathbb{R}$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) \tag{2.3}$$

exists uniformly in $\omega \in \Omega$.

One can show that uniquely ergodic transformations are ergodic, and that the above average converges to $\int f d\mu$ for a unique probability measure μ.

2.2 The Skew-Shift

We will now discuss the main example in this thesis of an ergodic transformation: the skew-shift. Let $K \geq 2$ and $\Omega = \mathbb{T}^K$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle. As usual, we equip $\mathbb{T}$ with the topology induced by the distance $d(x,y) = \mathrm{dist}((x-y),\mathbb{Z})$. It easily follows that the product $\Omega = \mathbb{T}^K$ will be a compact metric space. For $\alpha \in \mathbb{R}$, we introduce the map

$$T_{\alpha,K} : \mathbb{T}^K \to \mathbb{T}^K$$

$$(T_{\alpha,K}\underline{\omega})_k = \begin{cases} \omega_1 + \alpha, & k = 1; \\ \omega_k + \omega_{k-1}, & 2 \leq k \leq K. \end{cases} \tag{2.4}$$

One can show that T is a homeomorphism and preserves the Lebesgue measure on Ω. We even have

Theorem 2.5. *Let $\alpha \in \mathbb{R}$ be irrational. The map $T_{\alpha,K}$ is uniquely ergodic and minimal.*

Let us now make the connection between the skew-shift and degree K polynomials, which we already mentioned in the introduction. Consider a degree K polynomial

$$P(n) = \sum_{k=0}^{K} \alpha_k n^k, \quad \alpha_K \neq 0. \tag{2.5}$$

Introduce a sequence of polynomials for $1 \leq j \leq K$ by

$$P_K(n) = P(n), \quad P_j(n) = P_{j+1}(n+1) - P_{j+1}(n). \tag{2.6}$$

We observe

Lemma 2.6. *For $1 \leq j \leq K$, P_j is a degree j polynomial.*

Proof. The claim is clear for $j = K$. Let us now show the claim for j assuming $j+1$. Then we have that $P_{j+1}(n) = \sum_{k=0}^{j+1} \beta_k n^k$ for some β_k. We may compute

$$\begin{aligned} P_{j+1}(n+1) - P_{j+1}(n) &= \sum_{k=0}^{j+1} (\beta_k (n+1)^k - \beta_k n^k) \\ &= \sum_{k=0}^{j+1} \beta_k \left(\sum_{l=0}^{k} \binom{k}{l} n^l - n^k \right). \end{aligned}$$

This implies the claim. □

Continuing the computation from the proof, one finds

$$P_{j+1}(n+1) - P_{j+1}(n) = \sum_{l=0}^{j} \left(\sum_{t=1}^{j} \binom{k}{t+l} \beta_{l+t} \right) n^l$$

for $P_{j+1}(n) = \sum_{k=0}^{j+1} \beta_k n^k$. Using the procedure in the other way, we obtain that the skew-shift coordinates are given by degree $k+1$ polynomials. We furthermore remark that the above lemma is known as *Weyl differencing.* We summarize this in

Lemma 2.7. *Given a polynomial P as in (2.5), we have for $\omega_k = P_k(0)$ and $\alpha = P_1(1) - P_1(0)$ that*

$$(T^n_{\alpha,K} \underline{\omega})_k = P_k(n). \tag{2.7}$$

Through this computation one can show that $\alpha = K!\alpha_K$. Introduce $x_n(\underline{\omega}) = (T_\alpha^n \underline{\omega})_K$ for $0 \leq n \leq K-1$. We then have

Lemma 2.8. *The map*

$$\mathbb{T}^K \ni \underline{\omega} \mapsto \{x_n(\underline{\omega})\}_{n=0}^{K-1} \in \mathbb{T}^K \tag{2.8}$$

is invertible and preserves the Lebesgue measure.

Proof. One can show that

$$\begin{pmatrix} x_0(\underline{\omega}) \\ \vdots \\ x_{K-1}(\underline{\omega}) \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_K \end{pmatrix},$$

where A is a triangular matrix with 1 on the diagonal. This implies the claim by the change of variable formula. $\square$

We will use this to show that the skew-shift converges to independent identically distributed random variables as K gets large. We will make this precise in Chapter 4.

2.3 The ergodic theorems

In this section, we state the ergodic theorems. Their statement can be given informally in words as: *Time averages are equal to space averages.* We begin with the mean ergodic theorem, which talks about convergence in the L^2 norm, and then proceed to the Birkhoff ergodic theorem, which talks about pointwise convergence.

Theorem 2.9. *Let f be a function in $L^2(\Omega, \mu)$ and $T : \Omega \to \Omega$ an ergodic transformation. The averages*

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n \omega) \tag{2.9}$$

converge to $\int_\Omega f(\omega) d\mu(\omega)$ in $L^2(\Omega, \mu)$.

The Birkhoff ergodic theorem tells us

Theorem 2.10. *Let $f \in L^1(\Omega, \mu)$ and $T : \Omega \to \Omega$ an ergodic transformation. For almost every ω, we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n \omega) = \int_\Omega f(\omega) d\mu(\omega). \tag{2.10}$$

Despite their similar nature, the Birkhoff ergodic theorem is somewhat deeper than the mean one. We will now use the mean ergodic theorem to answer the following question: Given a good set $\Omega_g \subseteq \Omega$ and an integer $K \geq 1$, can we choose a large set of ω such that we have

$$T^{lK} \omega \in \Omega_g$$

for a set of l with density close to $\mu(\Omega_g)$? The following lemma does exactly this.

Lemma 2.11. *Let $\Omega_g \subseteq \Omega$, $0 < \kappa < 1$, $K \geq 1$. Then there exists $\Omega_0 \subseteq \Omega$ such that for $\omega \in \Omega_0$, there is a sequence $L_t = L_t(\omega) \to \infty$ such that*

$$\frac{1}{L_t}\#\{0 \leq l \leq L_t - 1 : \quad T^{lK}\omega \in \Omega_g\} \geq \kappa\mu(\Omega_g) \tag{2.11}$$

and $\mu(\Omega_0) > 0$.

Proof. Letting $f = \chi_{\Omega_0}$ in the mean ergodic theorem, we find that

$$\lim_{N\to\infty} \int_\Omega \left| \frac{1}{N}\#\{0 \leq n \leq N-1 : \quad T^n\omega \in \Omega_g\} - \mu(\Omega_g) \right|^2 d\mu(\omega) = 0.$$

Thus, we obtain

$$\lim_{N\to\infty} \mu(\{\omega : \quad \frac{1}{N}\#\{0 \leq n \leq N-1 : \quad T^n\omega \in \Omega_g\} < \kappa\mu(\Omega_g)\}) = 0.$$

We thus may find a set Ω_1 of positive measure, such that for each $\omega \in \Omega_1$, there is a sequence $N_t = N_t(\omega)$ going to ∞ such that

$$\frac{1}{N_t}\#\{0 \leq n \leq N_t - 1 : \quad T^n\omega \in \Omega_g\} \geq \kappa\mu(\Omega_g).$$

For each $\omega \in \Omega_1$, we may find an $0 \leq s = s(\omega) \leq K-1$ such that $N_t \pmod K = s$ for infinitely many t. Introduce

$$\Omega_0 = \{T^{-s(\omega)}\omega : \quad \omega \in \Omega_1\},$$

and choose for $\omega \in \Omega_0$ the sequence $L_t = \frac{N_t}{K}$, for the N_t with $N_t \pmod{K} = s$. The claim now follows by construction. $\square$

CHAPTER 3

The Lyapunov Exponent

The goal of this chapter is to introduce the Lyapunov exponent for ergodic Schrödinger operators. As usual, we let (Ω, μ) be a probability space, $T : \Omega \to \Omega$ an invertible ergodic transformation, and $f : \Omega \to \mathbb{R}$ a bounded and measurable function. For $\omega \in \Omega$, we define the potential

$$V_\omega(n) = f(T^n \omega). \tag{3.1}$$

We consider the Schrödinger operator

$$\begin{aligned} &H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ &H_\omega u(n) = u(n+1) + u(n-1) + V_\omega(n) u(n). \end{aligned} \tag{3.2}$$

Here and in the following $\ell^2(\mathbb{Z})$ denotes the Hilbert space of square summable sequences $u : \mathbb{Z} \to \mathbb{C}$ with scalar product

$$\langle u, v \rangle = \sum_{n \in \mathbb{Z}} \overline{u(n)} v(n) \tag{3.3}$$

for $u, v \in \ell^2(\mathbb{Z})$. Let me finally remark that H_ω is a bounded and self-adjoint operator. In particular, its spectrum $\sigma(H_\omega)$ is a compact subset of the real line.

Background on the topics of this chapter can be found in the review articles by Damanik [17], Jitomirskaya [24], and in Chapter 5 of Teschl's book [36].

3.1 Definition and basic properties

Introduce the transfer matrices $A_\omega(E, N)$ of H_ω by

$$A_\omega(E, N) = \prod_{n=1}^{N} \begin{pmatrix} E - V_\omega(N-n) & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

If $u : \mathbb{Z} \to \mathbb{C}$ solves $H_\omega u = Eu$ interpreted as a difference equation, then we have that

$$\begin{pmatrix} u(N+1) \\ u(N) \end{pmatrix} = A_\omega(E, N) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}. \tag{3.5}$$

The Lyapunov exponent $L(E)$ describes the exponential growth of norms of $A_\omega(E, N)$. Let $\|.\|$ be a matrix norm that is submultiplicative $\|A \cdot B\| \leq \|A\| \cdot \|B\|$. A convenient example is given by the Hilbert–Schmidt norm

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2. \tag{3.6}$$

Our first lemma is

Lemma 3.1. *For every E, we can define the Lyapunov exponent $L(E)$ by*

$$\begin{aligned} L(E) &= \lim_{N\to\infty} \frac{1}{N} \int_\Omega \log \|A_\omega(E, N)\| d\mu(\omega) \\ &= \inf_{N\geq 1} \frac{1}{N} \int_\Omega \log \|A_\omega(E, N)\| d\mu(\omega). \end{aligned} \tag{3.7}$$

Proof. Fix E and introduce the sequence

$$\alpha_N = \int_\Omega \log \|A_\omega(E, N)\| d\mu(\omega).$$

Since T is measure preserving, one easily shows that $\alpha_{N+M} \leq \alpha_N + \alpha_M$. The claim now follows, since α_N is a subadditive sequence. $\square$

By the subadditive ergodic theorem, we furthermore obtain

Lemma 3.2. *Let $E \in \mathbb{R}$. There exists a set $\Omega_1 = \Omega_1(E) \subseteq \Omega$ such that*

$\mu(\Omega_1) = 1$ *and for* $\omega \in \Omega_1$

$$L(E) = \lim_{N\to\infty} \frac{1}{N} \log \|A_\omega(E,N)\|. \tag{3.8}$$

Last, I add

Lemma 3.3. *The definition* (3.7) *is independent of the matrix norm.*

Proof. If $\|.\|_1$ is another norm on the 2×2 matrices, there exists a $C_1 > 1$ such that

$$\frac{1}{C_1}\|A\|_1 \le \|A\| \le C_1\|A\|_1$$

for all matrices A, since they form a finite dimensional vector space. Taking logarithms and dividing by N, we thus obtain

$$\frac{1}{N}\log\|A_\omega(E,N)\| = \frac{1}{N}\log\|A_\omega(E,N)\|_1 + o(1),$$

and independence of the norm follows. □

3.2 Subharmonic functions and consequences

In this section, we recall basic properties of subharmonic functions and show that the Lyapunov exponent is subharmonic. More information on subharmonic functions can be found in Chapter 7 of Levin's book [30].

Let $G \subseteq \mathbb{C}$ be an open set and $u : G \to \mathbb{R} \cup \{-\infty\}$ be a function. u is called *upper semi-continuous* if, for every $z_\infty \in G$, we have that

$$\limsup_{z \to z_\infty} u(z) \leq u(z_\infty). \tag{3.9}$$

A function u is *submean* if, for $z \in G$ and sufficiently small $r > 0$, we have

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\vartheta}) d\vartheta. \tag{3.10}$$

u is called *subharmonic* if it is both upper semicontinuous and submean. $u : G \to \mathbb{R} \cup \{\infty\}$ is *superharmonic*, if $-u$ is subharmonic. A function that is both sub- and superharmonic is called *harmonic*. If $f : G \to \mathbb{C}$ is an analytic function, then $|f|$ is a harmonic function. If $u : G \to \mathbb{R}$ is a harmonic function which is bounded away from 0, then $\log |u|$ is a harmonic function.

Lemma 3.4. *Let u_n be a sequence of subharmonic functions, then*

$$u(z) = \inf_{n \geq 1} u_n(z) \tag{3.11}$$

is a subharmonic function.

We begin by observing

Lemma 3.5. *The function*

$$E \mapsto \frac{1}{N} \int_\Omega \log \|A_\omega(E, N)\|_{\mathrm{HS}} d\mu(\omega) \tag{3.12}$$

is harmonic.

Proof. This follows from the entries of $A_\omega(E, N)$ being analytic functions in E, and thus $\|A_\omega(E, N)\|_{\mathrm{HS}}$ being a harmonic function. The logarithm takes this harmonic function to a harmonic function, since $\|A_\omega(E, N)\|_{\mathrm{HS}} \geq 1$. Integration and dividing by N preserves harmonicity. $\square$

We next note the following lemma due to Craig and Simon [16].

Lemma 3.6. *The function $L(E)$ is subharmonic in E.*

Proof. This follows from the infimum of subharmonic functions being a subharmonic function. $\square$

3.3 Individual Lyapunov exponents

Introduce for every $\omega \in \Omega$ and $E \in \mathbb{R}$ the upper Lyapunov exponent

$$\overline{L}(E, \omega) = \limsup_{N \to \infty} \frac{1}{N} \log \|A_\omega(E, N)\|. \tag{3.13}$$

It should be noted that $\overline{L}$ is no longer necessarily subharmonic. We have already seen that the limit exists for fixed E and almost every ω in Lemma 3.2. The following lemma makes a statement for all energies E.

Lemma 3.7. *There exists a set $\Omega_{\mathrm{CS}} \subseteq \Omega$ satisfying $\mu(\Omega_{\mathrm{CS}}) = 1$ and for $\omega \in \Omega_{\mathrm{CS}}$, we have, for every $E \in \mathbb{R}$,*

$$\overline{L}(E, \omega) \leq L(E). \tag{3.14}$$

Proof. See Craig and Simon [16]. □

3.4 The connection to solutions

In this section, I wish to make the connection between the Lyapunov exponent and solutions of $H_\omega u = Eu$ and the Green's function. Denote by $c_\omega(E,n)$ and $s_\omega(E,n)$ the solutions of $H_\omega u = Eu$ satisfying the initial conditions

$$(3.15) \qquad c_\omega(E,1) = 1 = s_\omega(E,0), \quad c_\omega(E,0) = 0 = s_\omega(E,1).$$

We call c_ω the *cosine type solution* and s_ω the *sine type solution.* An important role will be played by the observation

$$(3.16) \qquad A_\omega(E,n) = \begin{pmatrix} c_\omega(E,n) & s_\omega(E,n) \\ c_\omega(E,n-1) & s_\omega(E,n-1) \end{pmatrix},$$

which allows us to pass from information on the transfer matrices to information on the solutions.

For a subinterval $\Lambda \subseteq \mathbb{Z}$, we denote by $H_{\omega,\Lambda}$ the restriction of H_ω to $\ell^2(\Lambda)$. Through a computation, one can show that

$$(3.17) \quad c_\omega(E,n) = \det(E - H_{\omega,[1,n-1]}), \quad s_\omega(E,n) = \det(E - H_{\omega,[2,n-1]}).$$

For $x, y \in \Lambda$ and $E \in \mathbb{R}$, we introduce the Green's function as

$$G_{\omega,\Lambda}(E, x, y) = \langle e_x, (H_{\omega,\Lambda} - E)^{-1} e_y \rangle. \tag{3.18}$$

Here $\{e_x\}_{x \in \mathbb{Z}}$ is the standard basis of $\ell^2(\mathbb{Z})$ given by

$$e_x(n) = \begin{cases} 1, & x = n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

We have

Lemma 3.8. *For $a < x < y < b$, we have*

$$G_{\omega,[a,b]}(E, x, y) = \frac{\det(H_{\omega,[a,x-1]} - E) \det(H_{\omega,[y+1,b]} - E)}{\det(H_{\omega,[a,b]} - E)}. \tag{3.20}$$

Proof. This follows from Cramer's rule. $\square$

Combining this with the previous discussion, we see how to pass from information on the transfer matrices to information on the Green's function and back. We begin with

Lemma 3.9. *Let $2 \leq k_0 \leq N$ and $\gamma > 0$. Assume for $\Lambda \in \{[0, N], [1, N]\}$ and $k \in \{k_0 - 1, k_0\}$ that*

$$|G_\Lambda(E, k, N)| \leq \mathrm{e}^{-\gamma N} \tag{3.21}$$

then

$$\frac{1}{N}\log\|A(E,N)\| \geq \gamma - \frac{\log(\sqrt{2})}{N}. \tag{3.22}$$

Proof. We can use the previous discussion to conclude

$$|G_{[0,N]}(E,x,N)| = \left|\frac{c(E,x)}{c(E,N+1)}\right|, \quad |G_{[1,N]}(E,x,N)| = \left|\frac{s(E,x)}{s(E,N+1)}\right|.$$

By $\det(A(E,k_0)) = 1$, we conclude

$$\min(|c(E,k_0)|,|c(E,k_0-1)|,|s(E,k_0)|,|s(E,k_0-1)|) \geq \frac{1}{\sqrt{2}}.$$

The claim follows through some computations. □

The following lemma provides a converse to the previous one. For this think of $\kappa_1 \approx \mathrm{e}^{\gamma N}$ and of $\kappa_2 \approx \mathrm{e}^{\gamma \max(n,N-n)}$.

Lemma 3.10. *Assume that*

$$\|A_\omega(E,N)\| \geq \kappa_1, \quad \max(\|A_\omega(E,n)\|,\|A_{T^n\omega}(E,N-n)\|) \leq \kappa_2. \tag{3.23}$$

Then there exists

$$\Lambda \in \{[0,N],[1,N],[0,N-1],[1,N-1]\} \tag{3.24}$$

such that for $\Lambda = [a, b]$

$$\sup_{x \in \{a,b\}} |G_{\omega,\Lambda}(E, n, x)| \leq \frac{\kappa_2}{\kappa_1}. \tag{3.25}$$

Proof. This is a consequence of Lemma 3.8 and a quick computation, using a similar lower bound as used in the last lemma. □

3.5 The integrated density of states

In this section, we will introduce another quantity associated with an ergodic family of Schrödinger operators $H_\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$. We denote by $H_{\omega,[1,N]}$ the restriction of H_ω to $\ell^2([1, N])$. We will write

$$\mathrm{tr}(P_{(-\infty,E]}(H_{\omega,[1,N]}) \tag{3.26}$$

for the number of eigenvalues of $H_{\omega,[1,N]}$ which are less or equal to E. The *integrated density of states* is defined as the limit

$$\mathcal{N}(E) = \lim_{N\to\infty} \frac{1}{N} \int_\Omega \mathrm{tr}(P_{(-\infty,E]}(H_{\omega,[1,N]}) d\mu(\omega). \tag{3.27}$$

It is known that the limit exists. The relation between the integrated density of states and the Lyapunov exponent is given by *Thouless formula*, which tells us

$$L(E) = \int \log |t - E| d\mathcal{N}(t). \tag{3.28}$$

This formula implies that the integrated density of states is log Hölder continuous

$$|\mathcal{N}(E + \varepsilon) - \mathcal{N}(E)| \leq \frac{C}{\log(\frac{1}{\varepsilon})} \tag{3.29}$$

for some constant $C > 0$ and $0 < \varepsilon < \frac{1}{2}$.

CHAPTER 4

A continuity result and its consequences

The goal of this section will be to prove Theorem 1.4 from the introduction. The proof will have three essential ingredients: Furstenberg's proof that the Lyapunov exponent is positive for random potential, see [21]; the L^1 continuity of the Lyapunov exponent shown by Avila and Damanik [1]; the view point of identifying ergodic Schrödinger operators as measures on the space of potentials and using weak * convergence as done in [27].

4.1 Measures on the space of potentials

Choose a constant $C_1 > 0$ such that $\|f\|_\infty < C_1$. Let $\mathcal{V} = [-C_1, C_1]^{\mathbb{Z}}$ equipped with the topology of pointwise convergence. Then $\mathcal{V}$ is a compact metric space. The map

$$V : \Omega \ni \omega \mapsto \{V_\omega(n)\}_{n\in\mathbb{Z}} \in \mathcal{V} \tag{4.1}$$

is measurable. We may thus introduce a measure β on $\mathcal{V}$ by

$$\beta(A) = \mu(\{\omega : V_\omega \in A\}) \tag{4.2}$$

for all Borel sets $A \subseteq \mathcal{V}$. One can show that the measure β will be ergodic with respect to the shift $SV(n) = V(n+1)$ on $\mathcal{V}$.

We have gained two things here: first, that β is now a Borel measure on a compact metric space, and second, a natural notion of convergence: *weak * convergence.* We recall that $\beta_n \to \beta$ in the weak * topology, if for every continuous function $f : \mathcal{V} \to \mathbb{C}$ we have

$$\int f d\beta_n \to \int f d\beta. \tag{4.3}$$

We may define a Lyapunov exponent

$$\gamma_\beta(E) = \lim_{N\to\infty} \frac{1}{N} \int_{\mathcal{V}} \log \|A_V(E,N)\| d\beta(V), \tag{4.4}$$

which agrees with $L(E)$ if β was constructed as above. We denote by $\mathcal{M}$ the set of all ergodic measures on $\mathcal{V}$.

We will be interested in two particular measures. First the measures β_K

introduced as the pushforward of the Lebesgue measure on $\Omega = \mathbb{T}^K$ under the map $\underline{\omega} \mapsto V_{\underline{\omega}}$, where the potential is given as in (1.15). Second, define a measure $\tilde{\nu}$ on $[-C_1, C_1]$ by

$$\tilde{\nu}(A) = |\{x \in \mathbb{T} : \quad g(x) \in A\}|. \tag{4.5}$$

We then define a measure $\nu = \tilde{\nu}^{\otimes \mathbb{Z}}$. This measure corresponds to independent identically distributed random variables.

We will now show that $\beta_K \to \nu$ as $K \to \infty$. We follow the strategy used to prove Proposition 4.1 in [27]. For this, we first need the following lemma.

Lemma 4.1. *For ν_K, ν probability measures on $\mathcal{V}$, we have*

$$\nu_K \longrightarrow \nu \tag{4.6}$$

*in the weak * topology if and only if*

$$\lim_{K \to \infty} \int f d\nu_K = \int f d\nu \tag{4.7}$$

for every continuous $f : \mathcal{V} \longrightarrow \mathbb{R}$, which only depends on finitely many elements of $\mathcal{V}$.

Proof. One direction is obvious, for the other use that $\mathcal{V}$ is compact and thus any continuous function $g : \mathcal{V} \longrightarrow \mathbb{R}$ is uniformly continuous. □

We now come to

Proposition 4.2. *We have that*

$$\lim_{K\to\infty} \beta_K = \nu. \tag{4.8}$$

Proof. By the previous lemma, it suffices to check that

$$\lim_{K\to\infty} \int f d\beta_K = \int f d\nu$$

for functions f which only depend on $V(1), \dots, V(N)$ where $N \geq 1$. However, by Lemma 2.8 we even obtain that

$$\int f d\beta_K = \int f d\nu$$

as long as $K \geq N$. This finishes the proof. $\square$

Furthermore from Furstenberg's result from [21], we understand the Lyapunov exponent γ_ν.

Proposition 4.3. *There exists $\gamma_0 > 0$ such that for every $E \in \mathbb{R}$*

$$\gamma_\nu(E) \geq \gamma_0. \tag{4.9}$$

In the next section, we will explain how to exploit continuity of the map $\beta \mapsto \gamma_\beta$ to obtain positivity for β_K.

4.2 The Avila–Damanik continuity result

The result of Avila and Damanik [1] says in our notation

Proposition 4.4. *The map*

$$\begin{aligned} \mathcal{M} &\to L^1(-C_1-3, C_1+3) \\ \beta &\mapsto \gamma_\beta \end{aligned} \tag{4.10}$$

is continuous.

We note that

$$\sigma_{\text{ess}}(\Delta + V) \subseteq (-C_1-3, C_1+3) \tag{4.11}$$

for every $V \in \mathcal{V}$ and that also the Lyapunov exponent is strictly positive outside this set. As a consequence, we obtain

Proof of Theorem 1.4. Let γ_0 be as in Proposition 4.3. Choose $\gamma = \frac{1}{2}\gamma_0$. The result now follows from the previous proposition and that $\beta_K \to \nu$. □

Let us now discuss the proof of Proposition 4.4. We first note that the quantity

$$\frac{1}{N}\int_{\mathcal{V}} \log \|A_V(E,N)\| d\beta(V) \tag{4.12}$$

depends continuously on β and is still harmonic in E. This can be seen in the same way as Lemma 3.5. The argument giving subharmonicity of the Lyapunov exponent $L(E)$ in E now gives

Lemma 4.5. *The map $\beta \mapsto \gamma_\beta(E)$ is upper semi-continuous for every $E \in \mathbb{R}$. Furthermore, extending the map to the upper half plane $\mathbb{C}_+ = \{z : \quad \mathrm{Im}(z) > 0\}$, we have that $\beta \mapsto \gamma_\beta(z)$ is continuous for $z \in \mathbb{C}_+$.*

Let now $\beta_n \to \beta$ in the weak * topology. The last lemma then tells us that

$$\limsup_{n\to\infty} \max(0, \gamma_{\beta_n}(E) - \gamma_\beta(E)) = 0$$

for every $E \in \mathbb{R}$. As a direct consequence, we obtain for a bounded interval $[-C, C]$ that

$$\limsup_{n\to\infty} \int_{-C}^{C} \max(0, \gamma_{\beta_n}(E) - \gamma_\beta(E)) dE = 0. \tag{4.13}$$

Here, we used dominated convergence and that the Lyapunov exponent is uniformly bounded on such an interval.

It now remains to prove the opposite bound. For this, we conclude from the fact that γ is harmonic in E and that $\gamma_{\beta_n}(z) \to \gamma_\beta(z)$ for $\mathrm{Im}(z) > 0$ that

$$\lim_{n\to\infty} \int_{-C}^{C} \gamma_{\beta_n}(E) dE = \int_{-C}^{C} \gamma_\beta(E) dE.$$

This finishes the proof of Proposition 4.4. For details see [1].

CHAPTER 5

Large coupling

In this section, we will discuss how to obtain an initial condition at large coupling. The approach is robust and will yield a result, which is good enough to be used with both versions of the multiscale analysis we develop.

The proof essentially splits into two parts. First one shows that if one is outside of the specturm, then the assumptions hold. This is accomplished by the Combes–Thomas estimate. Second one shows that at large coupling the restrictions to a finite interval have no spectrum near any fixed energy with high probability.

5.1 The Combes–Thomas estimate

We first recall the Combes–Thomas estimate (see [15])

Proposition 5.1. *Let $\Lambda \subseteq \mathbb{Z}$, $V : \Lambda \to \mathbb{R}$ be a bounded sequence, and $H : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be defined by its action on $u \in \ell^2(\Lambda)$ by*

$$Hu(n) = u(n+1) + u(n-1) + V(n)u(n) \tag{5.1}$$

for $n \in \Lambda$ (where we set $u(n) = 0$ for $n \notin \Lambda$). Assume that $\operatorname{dist}(\sigma(H), E) > \delta$. *Let*

$$\gamma = \frac{1}{2}\log(1 + \frac{\delta}{4}), \quad K = \frac{1}{\gamma}\log(\frac{4}{\delta}). \tag{5.2}$$

Then for $k, l \in \Lambda$, $|k - l| \geq K$, the estimate

$$|G(E, k, l)| \leq \frac{1}{2}\mathrm{e}^{-\gamma|k-l|} \tag{5.3}$$

holds.

Our proof follows the treatment of Teschl (see Lemma 2.5 in [36]). Introduce the operator $P_\beta : \ell^2(\Lambda) \to \ell^2(\Lambda)$ by

$$(P_\beta u)(n) = \mathrm{e}^{\beta n}u(n). \tag{5.4}$$

Define $Q_\beta = P_\beta^{-1} H P_\beta - H$. One computes

$$(Q_\beta u)(n) = (\mathrm{e}^{\beta} - 1)u(n+1) + (\mathrm{e}^{-\beta} - 1)u(n-1). \tag{5.5}$$

We furthermore note that $P_\beta^* = P_\beta$ and $\|Q_\beta\| \leq 2(\mathrm{e}^{|\beta|} - 1)$. We now come to a preliminary lemma.

Lemma 5.2. *Let* $\beta > 0$. *Assume* $\mathrm{dist}(\sigma(H), E) > 2(\mathrm{e}^{\beta} - 1)$. *Then for* $l > k$ *we have that*

$$|G(E,k,l)| \leq \mathrm{e}^{-\beta(l-k)} \frac{1}{\mathrm{dist}(E, \sigma(H)) - 2(\mathrm{e}^{|\beta|} - 1)}. \tag{5.6}$$

Proof. We have

$$G(E,k,l) = \langle e_k, (H-E)^{-1} e_l \rangle = \mathrm{e}^{-\beta(l-k)} \langle e_k, P_{-\beta}(H-E)^{-1} P_\beta e_l \rangle,$$

where we used $e_k = \mathrm{e}^{-k\beta} P_\beta e_k$ in the last equality. A quick computation furthermore shows

$$P_{-\beta}(H-E)^{-1} P_\beta = (H - E + Q_\beta)^{-1}.$$

From the resolvent equation, one obtains

$$(H - E + Q_\beta)^{-1} = (H-E)^{-1} \cdot (\mathbb{I} - (H-E)^{-1} Q_\beta)^{-1}.$$

By assumption, we have $\|(H-E)^{-1} Q_\beta\| < 1$. Thus, the right hand side is well defined, and (5.6) follows from a computation. $\square$

We are now ready for

Proof of Proposition 5.1. Choose $\beta = \log(1 + \frac{\delta}{4}) = 2\gamma$ in (5.6). The claim now follows through some computations. $\square$

5.2 The Łojasiewicz inequality

The goal of this section is to show that analytic functions on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ satisfy the following nondegeneracy condition. This result is due to Łojasiewicz [29]; see also Malgrange [31].

Definition 5.3. *Let (Ω, μ) be a probability space, and $f : \Omega \to \mathbb{R}$ a measurable function. We call f non-degenerate if there are $F, \alpha > 0$ such that for every $E \in \mathbb{R}$ and $\varepsilon > 0$, we have that*

$$\mu(\{\omega : \quad |f(\omega) - E| \leq \varepsilon\}) \leq F\varepsilon^{\alpha}. \tag{5.7}$$

We begin with the following lemma:

Lemma 5.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be real analytic. Then there are $F > 0$ and $\alpha > 0$ such that*

$$|\{\omega \in \mathbb{T} : \quad |f(\omega)| \leq \varepsilon\}| \leq F\varepsilon^{\alpha}. \tag{5.8}$$

Proof. Let $x_1, \ldots, x_N$ be the finitely many zeros of f in $[0,1]$ counted with multiplicity. Define

$$g(x) = \left(\prod_{j=1}^{N} \frac{1}{x - x_j}\right) \cdot f(x).$$

g will again be analytic and have no zeros in $[0,1]$. Thus $C = \min_{x\in[0,1]} |g(x)| > 0$. Now

$$|f(x)| \geq C \left(\min_{1\leq j\leq N} |x - x_j| \right)^N ,$$

which implies the claim with $F = 2NC^{-1/N}$ and $\alpha = \frac{1}{N}$. □

This is a version of our theorem for a single E. An adaptation of the argument using that the maximal number of zeros of $\omega \mapsto f(\omega) - E$ will be bounded, and these depend continuously on E, shows that analytic functions are non-degenerate. We conclude

Theorem 5.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be real analytic. Then f is non-degenerate with respect to the Lebesgue measure.*

5.3 The initial condition

In order to state our result, we need to introduce a bit of notation.

Definition 5.6. *An interval $\Lambda \subseteq \mathbb{Z}$ is called $(\gamma, \mathcal{E})$-good if*

$$|G_\Lambda(E, x, y)| \leq \frac{1}{2} \mathrm{e}^{-\gamma|x-y|} \tag{5.9}$$

for $E \in \mathcal{E}$ and $x, y \in \Lambda$ with $|x - y| \geq \frac{|\Lambda|}{10}$. Otherwise, Λ is called $(\gamma, \mathcal{E})$-bad.

We note that this definition is stronger than anything we will require later. We have

Proposition 5.7. *Assume that f is non-degenerate in the sense of Definition 5.3. Let $E_0 \in \mathbb{R}$, $\sigma > 0$, and introduce*

$$K = \left\lfloor \frac{\sigma \lambda^{\alpha/2}}{F} \right\rfloor \tag{5.10}$$

$$\gamma = \frac{1}{5} \log(\lambda) \tag{5.11}$$

$$\mathcal{E} = [E_0 - 1, E_0 + 1]. \tag{5.12}$$

Assume that λ is sufficiently large. Then there exists a set $\Omega_1 \subseteq \Omega$ such that the following properties hold.

(i) $\mu(\Omega_1) \geq 1 - \sigma$.

(ii) For $\omega \in \Omega_1$

$$[0, K-1] \text{ is } (\gamma, \mathcal{E}) - \text{good for } H_\omega. \tag{5.13}$$

(iii) For $\omega \in \Omega_1$ and $E \in \mathcal{E}$, we have

$$\|(H_{\omega,[0,K-1]} - E)^{-1}\| \leq \frac{2}{\sqrt{\lambda}}. \tag{5.14}$$

We start by observing the following lemma.

Lemma 5.8. *Let f be a non-degenerate function, $K \geq 1$, $B > 0$. Then for*

$E \in \mathbb{R}$, the set

$$A_{K,B}(E) = \{\omega \in \Omega : \quad |f(T^k\omega) - E| \geq B,\, k = 0, \dots, K-1\} \tag{5.15}$$

has measure

$$\mu(A_{K,B}(E)) \geq 1 - B^\alpha F K. \tag{5.16}$$

Proof. By (5.7), the set

$$A_B(E) = \{\omega \in \Omega : \quad |f(\omega) - E| < B\}$$

has measure $\mu(A_B(E)) \leq B^\alpha F$. Since T is measure preserving, the set

$$A = \Omega \backslash \left(\bigcup_{k=0}^{K-1} T^{-k} A_B(E) \right)$$

satisfies $\mu(A) \geq 1 - B^\alpha F K$. The claim follows by noting $A \subseteq A_{K,B}(E)$. □

This implies

Lemma 5.9. *Let (Ω, μ, T, f) be as above. Let $E_0 \in \mathbb{R}$ and $\sigma > 0$. Introduce*

$$K(\lambda) = \left\lfloor \frac{\sigma \lambda^{\alpha/2}}{F} \right\rfloor. \tag{5.17}$$

Then there is a set A of measure $\mu(A) \geq 1 - \sigma$ such that for $\omega \in A$, we have *that*

$$(5.18) \qquad |\lambda f(T^k\omega) - E_0| > \sqrt{\lambda},$$

for $k = 0, \ldots, K(\lambda) - 1$.

Proof. Letting $B = \frac{1}{\sqrt{\lambda}}$ in the previous lemma, we obtain that the set $A_{K,B}(\frac{1}{\lambda}E_0)$ has measure $\mu(A_{K,B}(E)) \geq 1 - \frac{FK}{\lambda^{\alpha/2}}$. We have $\mu(A_{K,B}(E)) \geq 1 - \sigma$ by (5.17). □

We reformulate this again as

Lemma 5.10. *Let K be as in the previous lemma and $\lambda > 0$ large enough. There exists a subset $\Omega_1 \subseteq \Omega$ of measure $\mu(\Omega_1) \geq 1 - \sigma$ such that for $E \in \mathcal{E} = [E_0 - 1, E_0 + 1]$, we have for $\omega \in \Omega_1$*

$$(5.19) \qquad \operatorname{dist}(E, \sigma(H_{\omega,[0,K-1]})) > \frac{1}{2}\sqrt{\lambda}.$$

Proof. Since $|E - E_0| \leq 1$, $H = \Delta + V_\omega$ with $\|\Delta\| \leq 2$, we obtain

$$\begin{aligned} \operatorname{dist}(E, \sigma(H_{\omega,[0,K-1]})) &\geq \operatorname{dist}(E_0, \sigma(H_{\omega,[0,K-1]})) - 1 \\ &\geq \operatorname{dist}(E, \{\lambda f(T^k\omega)\}_{k=0}^{K-1}) - 3 \geq \sqrt{\lambda} - 3 \end{aligned}$$

for $\omega \in A$. The claim follows by also assuming $\lambda \geq 36$. □

We are now ready for

Proof of Proposition 5.7. Let Ω_1 be as in the last lemma. The claim now follows by the Combes–Thomas estimate (Proposition 5.1) and that $\|(H_{\omega,[0,K-1]} - E)^{-1}\| = \operatorname{dist}(E, \sigma(H_{\omega,[0,K-1]}))^{-1}$. $\square$

CHAPTER 6

A multiscale analysis based on ergodicity

The goal of this chapter is to discuss parts of the results from [28], which provide methods to prove positive Lyapunov exponent only based on ergodic properties of the underlying transformation. The main problem here is that ergodicity does not provide quantitative recurrence properties.

Furthermore, for general ergodic operators, there is no available mechanism to prove a priori resolvent estimates e.g., Wegner estimates. So, we will have to work without them, and replace them by energy elimination. The results of this chapter are of similar flavor as results of Bourgain [3], [4].

6.1 Abstract results that imply positive Lyapunov exponent

In this section, we will state a result that is independent of the ergodic setting. Let e_x be the standard basis of $\ell^2(\mathbb{Z})$, and denote the Green's function for $E \in \mathbb{R}$ and $x, y \in \Lambda$ an interval in $\mathbb{Z}$ by

$$G_\Lambda(E, x, y) = \langle e_x, (H_\Lambda - E)^{-1} e_y \rangle. \tag{6.1}$$

We will quantify the decay of the Green's function using the following notion.

Definition 6.1. *For $a \in \mathbb{Z}$ and $K \geq 1$, $[a-K, a+K]$ is called $(\gamma, \mathcal{E})$-good if*

$$|G_{[a-K,a+K]}(E, a, a \pm K)| \leq \frac{1}{2} \mathrm{e}^{-\gamma K} \tag{6.2}$$

for $E \in \mathcal{E}$. Otherwise, $[a-K, a+K]$ is called $(\gamma, \mathcal{E})$-bad.

We are now ready to state our first result.

Theorem 6.2. *Given $0 < \sigma \leq \frac{1}{4}$, $K \geq 1$, $\gamma > 0$, $L \geq 1$, and $\mathcal{E} \subseteq \mathbb{R}$ an interval, assume that*

$$\#\{1 \leq l \leq L : [(l-1)K + 1, (l+1)K - 1] \textit{ is } (\gamma, K, \mathcal{E}) - \textit{bad}\} \leq \sigma L, \tag{6.3}$$

and the following inequalities

$$\gamma \cdot K \geq \max\left(\frac{1}{\sigma}, \frac{25}{\sigma} \ln\left(|\mathcal{E}|^{-1}\right)\right) \tag{6.4}$$

$$\frac{1}{K^3} e^{\frac{8}{75}\sigma\gamma K} \geq \frac{2^{17} e^3}{\sigma^4}. \tag{6.5}$$

Then, there is $\mathcal{E}_0 \subseteq \mathcal{E}$ *such that*

$$|\mathcal{E}_0| \geq (1 - e^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}| \tag{6.6}$$

and for $E \in \mathcal{E}_0$*, we have that*

$$\frac{1}{LK} \log \left\| \prod_{n=1}^{LK} \begin{pmatrix} E - V(LK - n) & -1 \\ 1 & 0 \end{pmatrix} \right\| \geq e^{-8\sigma} e^{-\frac{1}{99}} \gamma - \frac{\sqrt{2}}{LK}. \tag{6.7}$$

The construction of the set $\mathcal{E}_0$ as the spectrum of restrictions of $H_{[1,N]}$ implies that generally $\mathcal{E}_0$ will approach a dense set in $\sigma(H)$ as $N \to \infty$. The proof of this theorem will be given in Sections 6.3 to 6.5.

6.2 Results for ergodic Schrödinger operators

We will now pass to the ergodic setting. It follows from the ergodic theorem that (6.3) is roughly equivalent to

$$\mu(\{\omega : \quad [1, 2K-1] \text{ is } (\gamma, \mathcal{E}) - \text{bad for } H_\omega\}) \leq \sigma. \tag{6.8}$$

In particular, this condition is now independent of N. Thus, one can hope to obtain the conclusion of the previous theorem for all sufficiently large N. In order to exploit this, we recall that we introduced the Lyapunov exponent $L(E)$ in (3.7) by

$$L(E) = \lim_{N \to \infty} \frac{1}{N} \int_{\Omega} \log \left\| \prod_{n=1}^{N} \begin{pmatrix} E - V_\omega(N-n) & -1 \\ 1 & 0 \end{pmatrix} \right\| d\mu(\omega). \tag{6.9}$$

We will show

Theorem 6.3. *Given* $0 < \sigma \leq \frac{1}{4}$, $K \geq 1$, $\gamma > 0$, $L \geq 1$, *and* $\mathcal{E} \subseteq \mathbb{R}$ *an interval, assume the inequalities* (6.4), (6.5), *and the initial condition* (6.8). *Then there is* $\mathcal{E}_0 \subseteq \mathcal{E}$ *such that*

$$|\mathcal{E}_0| \geq (1 - \mathrm{e}^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}| \tag{6.10}$$

and for $E \in \mathcal{E}_0$, *we have that*

$$L(E) \geq \mathrm{e}^{-8\sigma}\mathrm{e}^{-\frac{1}{99}}\gamma. \tag{6.11}$$

This theorem is in some sense a corollary of Theorem 6.2, since it follows by combining it with results from ergodic theory. The proof is given in Section 6.6.

6.3 The multiscale step

Let $\{V(n)\}_{n=0}^{N-1}$ be any real valued sequence of N numbers. Define H as the corresponding Schrödinger operator on $\ell^2([0, N-1])$ and denote by H_Λ the restrictions to intervals $\Lambda \subseteq [0, N-1]$. This generality is mainly used to simplify the notation, and to make clear when ergodicity enters.

We now start by defining our basic notion of a good sequence $\{V(n)\}_{n=0}^{N-1}$.

Definition 6.4. *Let $\delta > 0$, $0 < \sigma \leq \frac{1}{4}$, $\mathcal{E} \subseteq \mathbb{R}$ an interval, and $L \geq 1$.*

A sequence $\{V(n)\}_{n=0}^{N-1}$ is called $(\delta, \sigma, L, \mathcal{E})$-critical if there are integers

$$0 \leq k_0 < k_1 < k_2 < k_3 < \cdots < k_L < k_{L+1} \leq N-1, \tag{6.12}$$

and a set $\mathcal{L} \subseteq [1, L]$ such that

$$\frac{\#\mathcal{L}}{L} \leq \sigma, \tag{6.13}$$

and for $l \notin \mathcal{L}$, we have that

$$|G_{[k_{l-1}+1, k_{l+1}-1]}(E, k_l, k_{l\pm 1} \mp 1)| \leq \frac{1}{2}\mathrm{e}^{-\delta} \tag{6.14}$$

for $E \in \mathcal{E}$.

In order to state the next theorem, we have to explain a division of $\mathcal{E} = [E_0, E_1]$ into Q intervals of length $\approx \mathrm{e}^{-\sigma\delta}$. Introduce $Q = \lceil (E_1 - E_0)\mathrm{e}^{\sigma\delta} \rceil$, and

$$\mathcal{E}_q = \left[E_0 + q\frac{E_1 - E_0}{Q}, E_0 + (q+1)\frac{E_1 - E_0}{Q}\right], \tag{6.15}$$

for $q = 0, \dots, Q-1$. If

$$E_1 - E_0 \geq \mathrm{e}^{-\sigma\delta} \tag{6.16}$$

holds, we have that

$$(E_1 - E_0)\mathrm{e}^{\sigma\delta} \leq Q \leq 2(E_1 - E_0)\mathrm{e}^{\sigma\delta} \tag{6.17}$$

and for all q

$$\frac{1}{2}\mathrm{e}^{-\sigma\delta} \leq |\mathcal{E}_q| \leq \mathrm{e}^{-\sigma\delta}. \tag{6.18}$$

The main result of this section will be

Theorem 6.5. *Assume that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, L, \mathcal{E})$-critical, $M \geq 3$,*

$$\frac{\sigma L}{M} \geq 2, \tag{6.19}$$

and $\sigma \leq \frac{1}{4}$. Introduce

$$\tilde{\sigma} = \frac{1}{2}\sigma \tag{6.20}$$

and

$$\tilde{\delta} = (1 - 2\sigma) M \delta. \tag{6.21}$$

Then there exists a set $\mathcal{Q} \subseteq [0, Q-1]$ *and* $\tilde{L} \geq 1$ *such that*

$$\#\mathcal{Q} \leq \frac{2^{15}}{\tilde{\sigma}} \left(\frac{(M+1)}{\sigma} \cdot \frac{N}{L} \right)^3 \tag{6.22}$$

and

$$(1 - 2\sigma) \frac{L}{M+1} \leq \tilde{L} \leq \frac{L}{M+1} \tag{6.23}$$

and for $q \notin \mathcal{Q}$, *we have that* $\{V(n)\}_{n=0}^{N-1}$ *is also* $(\tilde{\delta}, \tilde{\sigma}, \tilde{L}, \mathcal{E}_q)$*-critical.*

We observe that in our case $L \gtrsim N$, so (6.19) will be satisfied for all large enough N. The rest of this section is spent proving the above theorem.

We will now describe how we choose the sequence $\tilde{k}_l$ given the integer $M \geq 1$ from Theorem 6.5. This will be the sequence we check Definition 6.4 with. First pick

$$\tilde{k}_0 = k_0. \tag{6.24}$$

Now assume that we are given $\tilde{k}_s = k_{l_s}$ for $0 \leq s \leq j$, then we choose

$\tilde{k}_{j+1} = k_{l_{j+1}}$ so that

$$\#\{l \notin \mathcal{L} : \quad \tilde{k}_j < k_l < \tilde{k}_{j+1}\} = M. \tag{6.25}$$

This procedure stops once we would have to choose $\tilde{k}_{j+1} > N-1$. We will call the maximal l so that $\tilde{k}_{l+1}$ is defined $\tilde{L}$. This means that we have now defined

$$0 \leq \tilde{k}_0 < \tilde{k}_1 < \cdots < \tilde{k}_{\tilde{L}} < \tilde{k}_{\tilde{L}+1} \leq N-1.$$

We have the following

Lemma 6.6. *Assume $\sigma \frac{L}{M} \geq 2$, that is* (6.19)*. Then we have that*

$$\tilde{L} \geq (1-2\sigma)\frac{L}{M+1}. \tag{6.26}$$

Proof. By (6.13), we have that

$$\#([1,L]\backslash\mathcal{L}) \geq (1-\sigma)L.$$

We observe now that $l_{j+1} - l_j \geq M+1$, and even

$$l_{j+1} - l_j = M + 1 + \#\{l \in \mathcal{L} : \tilde{k}_j < k_l < \tilde{k}_{j+1}\}.$$

Hence, we may choose

$$\tilde{L} \geq (1-\sigma)\frac{L}{M+1} - 2,$$

and the claim now follows by $2 \leq \sigma \frac{L}{M+1}$. $\square$

We furthermore have the following estimate.

Lemma 6.7. *Assume $\sigma \leq \frac{1}{4}$. Let*

$$\widetilde{\mathcal{L}}_0 = \left\{ l : \quad \tilde{k}_{l+1} - \tilde{k}_{l-1} \geq \frac{16N(M+1)}{\sigma L} \right\}. \tag{6.27}$$

Then we have that

$$\frac{\#\widetilde{\mathcal{L}}_0}{\tilde{L}} \leq \frac{1}{2}\tilde{\sigma}. \tag{6.28}$$

Proof. Since $0 \leq \tilde{k}_0 \leq \tilde{k}_{\tilde{L}+1} \leq N$, we have that

$$\sum_{l=1}^{\tilde{L}} (\tilde{k}_{l+1} - \tilde{k}_{l-1}) = \tilde{k}_{\tilde{L}+1} - \tilde{k}_0 + \tilde{k}_{\tilde{L}} - \tilde{k}_1 \leq 2N.$$

Now, Markov's inequality implies that

$$\#\widetilde{\mathcal{L}}_0 \leq \left(\frac{1}{2} \cdot \frac{\sigma}{2} \right) \cdot \left(\frac{L}{2(M+1)} \right).$$

By (6.26) and $\sigma \leq \frac{1}{4}$, we have that $\frac{1}{\tilde{L}} \leq \frac{2(M+1)}{L}$. Now, the claim follows from $\tilde{\sigma} = \frac{\sigma}{2}$ and the above equation. $\square$

Before coming to the next lemma, we will first introduce the notion of non-resonance.

Definition 6.8. *Given an interval $I \subseteq [0, N-1]$, an energy interval $\mathcal{E}$, and $\varepsilon > 0$, $\{V(n)\}_{n=0}^{N-1}$ is called $(I, \mathcal{E}, \varepsilon)$ non-resonant if, for every $\Lambda \subseteq I$, we have that*

$$\operatorname{dist}(E, \sigma(H_\Lambda)) \geq \varepsilon \tag{6.29}$$

for all $E \in \mathcal{E}$. Otherwise, $\{V(n)\}_{n=0}^{N-1}$ is called $(I, \mathcal{E}, \varepsilon)$ resonant.

Introduce the set $\mathfrak{L}_q$ for $0 \leq q \leq Q$ by

$$\mathfrak{L}_q = \{1 \leq l \leq \tilde{L} : \quad \{V(n)\}_{n=0}^{N-1} \text{ is } ([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2\mathrm{e}^{-\sigma\delta}) \text{ resonant}\}. \tag{6.30}$$

We will now discuss the size of this set.

Lemma 6.9. *There is a set $\mathcal{Q}$ such that*

$$\#\mathcal{Q} \leq \frac{2^{15}}{\tilde{\sigma}} \left(\frac{N(M+1)}{\sigma L} \right)^3 \tag{6.31}$$

and for $q \notin \mathcal{Q}$, we have that

$$\frac{\#\mathfrak{L}_q}{\tilde{L}} \leq \tilde{\sigma}. \tag{6.32}$$

Proof. For l introduce

$$g(l) = \#\{q : \quad \{V(n)\}_{n=0}^{N-1} \text{ is } ([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2\mathrm{e}^{-\sigma\delta}) \text{ resonant}\}.$$

We will now derive an upper bound on $g(l)$. First note that $\sigma(H_\Lambda)$ consists of $\#\Lambda$ elements, so

$$\bigcup_{\Lambda \subseteq [\tilde{k}_{l-1}, \tilde{k}_{l+1}]} \sigma(H_\Lambda)$$

consists of at most $(\tilde{k}_{l+1} - \tilde{k}_{l-1})^3$ elements. For each E in the above set, we have that its $2\mathrm{e}^{-\sigma\delta}$ neighborhood can intersect at most 8 of the $\mathcal{E}_q$ intervals. Thus, we have that

$$g(l) \leq 8(\tilde{k}_{l+1} - \tilde{k}_{l-1})^3.$$

In particular for $l \notin \widetilde{\mathcal{L}}_0$, we have by (6.27) that

$$g(l) \leq 2^{15} \left(\frac{N(M+1)}{\sigma L} \right)^3.$$

Let $h(q) = \#\mathfrak{L}_q$, so that

$$h(q) \leq \#\{l \notin \widetilde{\mathcal{L}}_0 : \quad \{V(n)\}_{n=0}^{N-1} \text{ is } ([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2\mathrm{e}^{-\sigma\delta}) \text{ resonant}\}.$$

We obtain

$$\sum_{q=0}^{Q-1} h(q) \leq \sum_{l \notin \widetilde{\mathcal{L}}_0} g(l) \leq 2^{15} \tilde{L} \left(\frac{N(M+1)}{\sigma L} \right)^3.$$

Let $\mathcal{Q}$ be the set

$$\mathcal{Q} = \{q : \quad h(q) \geq \tilde{\sigma} \tilde{L}\}.$$

Now the claim follows from Markov's inequality. $\square$

We observe that (6.29) implies that

$$\|(H_\Lambda - E)^{-1}\| \leq \frac{1}{2}\mathrm{e}^{\sigma\delta}. \tag{6.33}$$

Lemma 6.10. *Assume for* (l, q) *that* $\{V(n)\}_{n=0}^{N-1}$ *is* $([\tilde{k}_{l-1}, \tilde{k}_{l+1}], \mathcal{E}_q, 2\mathrm{e}^{-\sigma\delta})$ *non-resonant. Then*

$$|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, \tilde{k}_{l\pm 1} \mp 1)| \leq \frac{1}{2}\mathrm{e}^{-\tilde{\delta}} \tag{6.34}$$

for $E \in \mathcal{E}_q$.

Proof. Let $x = \tilde{k}_{l\pm 1}$ (one of the two). Since (6.29), we have that

$$|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, x)| \leq \frac{1}{2}\mathrm{e}^{-\sigma\delta}.$$

By construction of $\tilde{k}_l$, we have sets $\mathcal{J}_\pm$ such that for $j \in \mathcal{J}_\pm$ we have $[k_{j-1}, k_{j+1}] \subseteq [\tilde{k}_l, \tilde{k}_{l\pm 1}] \cup [\tilde{k}_{l\pm 1}, \tilde{k}_l]$. Furthermore, for $j \in \mathcal{J}_+ \cup \mathcal{J}_-$, we have that

$$|G_{[k_{j-1}+1, k_{j+1}-1]}(E, k_j, k_{j\pm 1} \mp 1)| \leq \frac{1}{2}\mathrm{e}^{-\delta}$$

for $E \in \mathcal{E}_q \subseteq \mathcal{E}$.

By the resolvent equation, we find that

$$|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, \tilde{k}_l, x)| \leq \frac{1}{2}\mathrm{e}^{-\sigma\delta}\Big(|G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_-}, x)| \\ + |G_{[\tilde{k}_{l-1}+1, \tilde{k}_{l+1}-1]}(E, k_{j_+}, x)|\Big),$$

where $j_+ = \max(\mathcal{J}_+)$ and $j_- = \min(\mathcal{J}_-)$. Now, by the decay of the Green's function, we know that

$$\begin{aligned}|G_{[\tilde{k}_{l-1}+1,\tilde{k}_{l+1}-1]}(E,\tilde{k}_l,x)| \leq \frac{1}{4}\mathrm{e}^{-(1-\sigma)\delta}\Big(&|G_{[\tilde{k}_{l-1}+1,\tilde{k}_{l+1}-1]}(E,k_{j_--1}+1,x)| \\ &+ |G_{[\tilde{k}_{l-1}+1,\tilde{k}_{l+1}-1]}(E,k_{j_-+1}-1,x)| \\ &+ |G_{[\tilde{k}_{l-1}+1,\tilde{k}_{l+1}-1]}(E,k_{j_+-1}+1,x)| \\ &+ |G_{[\tilde{k}_{l-1}+1,\tilde{k}_{l+1}-1]}(E,k_{j_++1}-1,x)|\Big).\end{aligned}$$

We may iterate this procedure $M = \#\mathcal{J}_+ = \#\mathcal{J}_-$ many times, proving the proposition by our choice of $\tilde{\delta}$. □

Proof of Theorem 6.5. We are essentially done. We observe, that for $q \notin \mathcal{Q}$, we can choose $\mathfrak{L} = \mathfrak{L}_q$, which satisfies

$$\frac{\#\mathfrak{L}}{\tilde{L}} \leq \tilde{\sigma},$$

by (6.32). Furthermore, we then have the estimate on the Green's function on $[\tilde{k}_{l-1}, \tilde{k}_{l+1}]$ by the last lemma for $l \notin \mathfrak{L}$. This finishes the proof that $\{V(n)\}_{n=0}^{N-1}$ is $(\tilde{\delta}, \tilde{\sigma}, \tilde{L}, \mathcal{E}_q)$-critical. □

6.4 Inductive use of the multiscale step

In this section, we develop an inductive way to apply Theorem 6.5. This will lead in the following section to the proof of Theorem 6.2. A major part of this section is taken up by checking inequalities between various numerical quantities, necessary to show that everything converges.

Given numbers $\delta > 0$ and $0 < \sigma \leq \frac{1}{4}$, we will first introduce δ_j, σ_j, and M_j. Introduce $\delta_0 = \delta$ and

$$M_j = 100^{j+1} \tag{6.35}$$

$$\sigma_j = \frac{1}{2^j}\sigma \tag{6.36}$$

$$\delta_{j+1} = (1 - 2\sigma_j)M_j\delta_j. \tag{6.37}$$

This choice is motivated by (6.20) and (6.21). We first observe

Lemma 6.11. *We have that*

$$\prod_{k=0}^{j} M_k = 10^{(j+1)(j+2)} = 10^{j^2} \cdot 1000^j \cdot 100 \tag{6.38}$$

$$\delta_j \geq \mathrm{e}^{-4\sigma} 10^{(j+1)(j+2)}\delta \tag{6.39}$$

$$\sigma_j\delta_j \geq \mathrm{e}^{-4\sigma} 10^{j^2} 500^j 100\sigma\delta. \tag{6.40}$$

Proof. For (6.38), observe that

$$\prod_{k=0}^{j} M_k = 100^{\sum_{k=0}^{j}(k+1)}$$

and $\sum_{k=0}^{j}(k+1) = \frac{(j+1)(j+2)}{2}$.

For (6.39), we have that $\delta_{j+1} = \prod_{k=1}^{j}(1 - \frac{\sigma}{2^k})M_k \cdot \delta$, and since $\prod_{k=1}^{j}(1 -$

$2\frac{\sigma}{2^k}) \geq \prod_{k=1}^{\infty}(1 - 2\frac{\sigma}{2^k})$, we have that

$$\prod_{k=1}^{j}(1 - 2\frac{\sigma}{2^k}) \geq \exp\left(\sum_{k=1}^{\infty}\log(1 - 2\frac{\sigma}{2^j})\right).$$

Now using that $\log(1-x) \geq -2x$ for $0 < x < 1/2$, we have that $\sum_{j=1}^{\infty}\log(1 - 2\frac{\sigma}{2^j}) \geq -4\sigma\sum_{k=1}^{\infty}\frac{1}{2^k} = -4\sigma$ and thus the inequalities follow. $\square$

We let L_j be a sequence of numbers that satisfies

$$(1 - 2\sigma_j)\frac{L_j}{M_j} \leq L_{j+1} \leq \frac{L_j}{M_j}. \tag{6.41}$$

This is motivated by (6.23).

Lemma 6.12. *The L_j satisfy*

$$e^{-4\sigma}e^{-\frac{1}{99}}L10^{-(j+1)(j+2)} \leq L_{j+1} \leq L10^{-(j+1)(j+2)}. \tag{6.42}$$

Proof. Recall from the last lemma that $\prod_{k=1}^{j}(1 - 2\sigma_k) \geq e^{-4\sigma}$. An iteration of (6.41) shows

$$\prod_{k=1}^{j}\frac{1 - 2\sigma_k}{M_k + 1}L_0 \leq L_{j+1} \leq \prod_{k=1}^{j}\frac{1}{M_k + 1}L_0.$$

Since

$$1 \geq \prod_{k=1}^{j}\frac{M_k}{M_k + 1} = \exp\left(-\sum_{k=1}^{j}\log\left(1 + \frac{1}{100^k}\right)\right) \geq \exp(-\frac{1}{99}),$$

we have that (6.38) implies the claim. ☐

We define j_{max} to be the maximal j such that

$$\sigma_{j_{max}} L_{j_{max}} \geq 2M_{j_{max}} \tag{6.43}$$

holds. This is needed in order to satisfy (6.19) in Theorem 6.5. We have that

Lemma 6.13. *If σ stays fixed, then $\delta_{j_{max}} \to \infty$ as $L \to \infty$. Furthermore,*

$$\delta_{j_{max}} L_{j_{max}} \geq \mathrm{e}^{-8\sigma} \mathrm{e}^{-\frac{1}{99}} L\delta \tag{6.44}$$

Proof. We observe that (6.43) only depends on σ and L. Furthermore, if L becomes large, the restriction becomes less and less restrictive.

The second claim follows by (6.39) and (6.42) ☐

We will now start by exploiting the multiscale step stated in Theorem 6.5. We will show

Theorem 6.14. *Assume that*

$$\frac{\sigma L}{M} \geq 2 \tag{6.45}$$

$$|\mathcal{E}| \geq \mathrm{e}^{-\frac{1}{25}\sigma\delta} \tag{6.46}$$

$$\frac{2^{17}\mathrm{e}^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L}\right)^3 \leq \mathrm{e}^{\frac{8}{25}\mathrm{e}^{-4\sigma}\sigma\delta} \tag{6.47}$$

hold and that $\{V(n)\}_{n=0}^{N-1}$ *is* $(\delta, \sigma, L, \mathcal{E})$*-critical, then there is* $\mathcal{E}_0 \subseteq \mathcal{E}$ *satisfying*

$$\frac{|\mathcal{E}_0|}{|\mathcal{E}|} \geq \exp\left(-\frac{25}{4} \frac{\mathrm{e}^{-\frac{8}{25}\sigma\delta}}{\sigma\delta \ln(50)}\right) \tag{6.48}$$

such that $\{V(n)\}_{n=0}^{N-1}$ *is* $(\delta_{j_{max}}, \sigma_{j_{max}}, L_{j_{max}}, \mathcal{E}_0)$*-critical.*

We will now start the proof of this theorem. The proof is based on induction. First, observe that by the assumption that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta, \sigma, L, \mathcal{E})$-critical, we have that $\{V(n)\}_{n=0}^{N-1}$ is $(\delta_0, \sigma_0, L_0, \mathcal{E})$-critical. This means that the base case is taken care of. The main problem with applying induction is that the interval $\mathcal{E}$ will shrink with the induction procedure; that is why we will need to do something slightly more sophisticated. This motivates the following

Definition 6.15. *Given* $\{V(n)\}_{n=0}^{N-1}$. *A collection of intervals* $\{\mathcal{E}_q\}_{q=0}^{Q}$ *is called* (σ, δ, L)*-acceptable if*

(i) For each q, *we have that* $\{V(n)\}_{n=0}^{N-1}$ *is* $(\sigma, \delta, L, \mathcal{E}_q)$*-critical*

(ii) For $q, \tilde{q}$, *we have that* $|\mathcal{E}_q| = |\mathcal{E}_{\tilde{q}}|$.

(iii) We have that

$$|\mathcal{E}_q| \geq \mathrm{e}^{-\frac{1}{25}\sigma\delta} \tag{6.49}$$

for each q.

We first observe that $\{\mathcal{E}\}$ is $(\sigma_0, \delta_0, L_0)$-acceptable, since we assume criticality and (6.46). This implies the following consequence of Theorem 6.5.

Lemma 6.16. *Given* $\{V(n)\}_{n=0}^{N-1}$ *and a collection of* $(\sigma_j, \delta_j, L_j)$*-acceptable intervals* $\{\mathcal{E}_q^j\}_{q=0}^{Q_j}$*, then there exists a collection of intervals* $\{\mathcal{E}_q^{j+1}\}_{q=0}^{Q_{j+1}}$ *that is* $(\sigma_{j+1}, \delta_{j+1}, L_{j+1})$*-acceptable.*

Proof. All but condition (iii) of Definition 6.15 are direct consequences of Theorem 6.5. For (iii) observe that (6.18) implies that

$$|\mathcal{E}_q^{j+1}| \geq \mathrm{e}^{-\sigma_j \delta_j}$$

for any q. Now, observe that since $0 < \sigma_j \leq \frac{1}{4}$ and $M_j \geq 100$, we have that

$$\sigma_{j+1}\delta_{j+1} = \frac{1}{2}\sigma_j(1 - 2\sigma_j)M_j\delta_j \leq 25\sigma_j\delta_j.$$

So the claim follows. ☐

It remains to compare the size of

$$\bigcup_{q=0}^{Q_j} \mathcal{E}_q^j \quad \text{and} \quad \bigcup_{q=0}^{Q_{j+1}} \mathcal{E}_q^{j+1}.$$

For this, we will first need the following lemma.

Lemma 6.17. *Assume* (6.47). *Then we have that*

$$10^{3(j+1)(j+2)} \le e^{\frac{8}{25}\sigma_j\delta_j} \tag{6.50}$$

$$\frac{2^{17}e^{12\sigma}}{\sigma^4}\cdot\left(\frac{N}{L}\right)^3 \le e^{\frac{8}{25}\sigma_j\delta_j}. \tag{6.51}$$

Proof. Since $(j+1)(j+2) \le 50^j$, these inequalities follow from

$$10^3 \le e^{\frac{8}{25}\sigma\delta e^{-4\sigma}} \text{ and } \frac{2^{17}e^{12\sigma}}{\sigma^4}\cdot\left(\frac{N}{L}\right)^3 \le e^{\frac{8}{25}e^{-4\sigma}\sigma\delta}.$$

By $N \ge L$ and $0 < \sigma \le \frac{1}{4}$, we have that

$$10^3 \le 2^{25} \le \frac{2^{17}e^{12\sigma}}{\sigma^4}\cdot\left(\frac{N}{L}\right)^3$$

so both of the above equations follow from (6.47). $\square$

The next lemma will allow us to compare the size of an interval $\mathcal{E}_q^j$ to the size of the intervals $\mathcal{E}_p^{j+1}$ contained in $\mathcal{E}_q^j$.

Lemma 6.18. *We have that*

$$\frac{1}{|\mathcal{E}_q^j|}\cdot\left|\bigcup_{\mathcal{E}_p^{j+1}\subseteq\mathcal{E}_q^j}\mathcal{E}_p^{j+1}\right| \ge 1 - e^{-\frac{8}{25}\sigma_j\delta_j}. \tag{6.52}$$

Proof. By (6.22), we have that

$$\left|\mathcal{E}_q^j\setminus\bigcup_{\mathcal{E}_p^{j+1}\subseteq\mathcal{E}_q^j}\mathcal{E}_p^{j+1}\right| \le \frac{2^{17}100^3}{\sigma_j^4}\frac{N^3}{L_j^3}\cdot|\mathcal{E}_s^{j+1}|.$$

By construction, we have that (6.18) holds, that is, $|\mathcal{E}_s^{j+1}| \le \mathrm{e}^{-\sigma_j \delta_j}$. Hence, we obtain that

$$\left| \mathcal{E}_q^j \backslash \bigcup_{\mathcal{E}_p^{j+1} \subseteq \mathcal{E}_q^j} \mathcal{E}_p^{j+1} \right| \le \frac{2^{17} \mathrm{e}^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L} \right)^3 \cdot 10^{3(j+1)(j+2)} \cdot \mathrm{e}^{-\sigma_j \delta_j}.$$

Since we have that $|\mathcal{E}_q^j| \ge \mathrm{e}^{-\frac{1}{25}\sigma_j \delta_j}$, we obtain that

$$\begin{aligned} \frac{1}{|\mathcal{E}_q^j|} \cdot \left| \bigcup_{\mathcal{E}_p^{j+1} \subseteq \mathcal{E}_q^j} \mathcal{E}_p^{j+1} \right| &\ge 1 - \frac{2^{17} \mathrm{e}^{12\sigma}}{\sigma^4} \cdot \left(\frac{N}{L} \right)^3 \cdot 10^{3(j+1)(j+2)} \cdot \mathrm{e}^{-\frac{24}{25}\sigma_j \delta_j} \\ &\ge 1 - \mathrm{e}^{-\frac{8}{25}\sigma_j \delta_j}, \end{aligned}$$

where we used (6.50) and (6.51). This finishes the proof. $\square$

We now come to

Lemma 6.19. *We have that*

$$\left| \bigcup_{q=0}^{Q_{j+1}} \mathcal{E}_q^{j+1} \right| \ge \left| \bigcup_{q=0}^{Q_j} \mathcal{E}_q^j \right| \cdot (1 - \mathrm{e}^{-\frac{8}{25}\sigma_j \delta_j}). \tag{6.53}$$

Proof. This is a consequence of the last lemma. $\square$

Proof of Theorem 6.14. By the previous discussion, we can choose $\mathcal{E}_0$ such that

$$|\mathcal{E}_0| \ge \prod_{j=1}^{\infty} (1 - \mathrm{e}^{-\frac{8}{25}\sigma_j \delta_j}) |\mathcal{E}|.$$

Using (6.40) and $\log(1-x) \geq -2x$, we find that

$$|\mathcal{E}_0| \geq \exp\left(-2\sum_{j=1}^{\infty} \mathrm{e}^{-\frac{8}{25}\sigma\delta \mathrm{e}^{-4\sigma} 50^j}\right) \geq \exp\left(-2\frac{\mathrm{e}^{-\frac{8}{25}\sigma\delta}}{\frac{8}{25}\sigma\delta\ln(50)}\right),$$

since $\sum_{j=1}^{\infty} \mathrm{e}^{-ta^j} \leq \frac{\mathrm{e}^{-t}}{\ln(a)t}$. □

6.5 Proof of Theorem 6.2

We begin by observing that (6.3) implies that, for L large enough, $\{V(n)\}_{n=0}^{LK-1}$ is $(\delta, \sigma, L, \mathcal{E})$-critical for $\delta = \gamma K$, in the sense of Definition 6.4. To see this, choose $k_j = jK$ and $\mathcal{L}$ as the complement of the set in (6.3). The rest follows. We now use the mechanism of the last two sections to improve the estimate.

Lemma 6.20. *$\{V(n)\}_{n=0}^{LK-1}$ will be $(\hat{\delta}, \hat{\sigma}, \hat{L}, \widehat{\mathcal{E}})$-critical, where $\widehat{\mathcal{E}} \subseteq \mathcal{E}$ satisfies*

$$\frac{|\widehat{\mathcal{E}}|}{|\mathcal{E}|} \geq \exp\left(-\frac{25}{4}\frac{\mathrm{e}^{-\frac{8}{25}\sigma\delta}}{\sigma\delta\ln(50)}\right) \tag{6.54}$$

and by Lemma 6.13, we have that

$$\hat{\delta}\hat{L} \geq \mathrm{e}^{-8\sigma-\frac{1}{99}}\gamma K \cdot L. \tag{6.55}$$

Proof. Since $\{V_\omega(n)\}_{n=0}^{N_t-1}$ is $(\delta, \sigma, L_t, \mathcal{E})$-critical, we now wish to apply Theorem 6.14 to improve this estimate. In order to do this, we still have to ensure that (6.45),(6.46) (6.47) hold. (6.4) implies (6.46). (6.47) is implied by (6.5). For (6.45) observe that it is satisfied if L is large enough. □

We now come to

Lemma 6.21. *We may choose the set $\widehat{\mathcal{E}}$ so that for every $\Lambda \subseteq [0, LK-1]$ and every $E \in \widehat{\mathcal{E}}$, we have*

$$\operatorname{dist}(H, \sigma(H_\Lambda)) \geq \mathrm{e}^{-\hat{\sigma}\hat{\delta}}. \tag{6.56}$$

Proof. This follows by an inspection of the argument of the last section. □

Now repeating the argument to obtain Green's function estimates as done in Lemma 6.10, we obtain the estimates required by Lemma 3.9. Hence, we obtain that

$$\frac{1}{LK} \log \|A(E, LK)\| \geq \mathrm{e}^{-8\sigma} \mathrm{e}^{-\frac{1}{99}} \gamma - \frac{\sqrt{2}}{LK} \tag{6.57}$$

for $E \in \widehat{\mathcal{E}}$. This finishes the proof of Theorem 6.2, using that $\mathrm{e}^{-x} \geq 1 - x$ for $x \geq 1$.

6.6 Proof of Theorem 6.3

We first need the following observation.

Lemma 6.22. *There exists $\omega \in \Omega$, such that the following properties hold:*

(i) We have that

$$L(E) \geq \limsup_{n \to \infty} \frac{1}{n} \log \|A_\omega(E, n)\| \tag{6.58}$$

for all E.

(ii) There are sequences $N_t, L_t \to \infty$ such that $\{V_\omega(n)\}_{n=0}^{N_t-1}$ is $(\gamma K, \sigma, L_t, \mathcal{E})$-critical and

$$\lim_{t\to\infty} \frac{N_t}{L_t} = K. \tag{6.59}$$

Proof. Let Ω_{CS} be the set from Lemma 3.7. This implies that property (i) holds as long as $\omega \in \Omega_{CS}$. Furthermore, we have that $\mu(\Omega_{CS}) = 1$.

Let Ω_g be the complement of the set in (6.8). By Lemma 2.11, we can find a set $\widetilde{\Omega}$ with $\mu(\widetilde{\Omega}) > 0$, and for each $\omega \in \widetilde{\Omega}$ sequences $N_t, L_t \to \infty$ such that property (ii) holds.

So we have that $\Omega_{CS} \cap \widetilde{\Omega}$ is non-empty and by choosing $\omega \in \Omega_{CS} \cap \widetilde{\Omega}$, we are done. □

We now fix ω as in the last lemma, and abbreviate

$$V(n) = V_\omega(n). \tag{6.60}$$

The claim now follows by repeating the arguments of the last section of the proof of Theorem 6.2. Giving more details, we obtain a sequence of sets $\mathcal{E}_t$ satisfying

$$|\mathcal{E}_t| \geq (1 - \mathrm{e}^{-\frac{8}{25}\sigma\gamma K})|\mathcal{E}|$$

and for $E \in \mathcal{E}_t$, we have

$$\frac{1}{N_t} \log \|A(E, N_t)\| \geq \mathrm{e}^{-8\sigma} \mathrm{e}^{-\frac{1}{99}} \gamma + o(1)$$

as $t \to \infty$. Hence, we have that

$$L(E) \geq \mathrm{e}^{-8\sigma} \mathrm{e}^{-\frac{1}{99}} \gamma$$

for

$$E \in \mathfrak{E} = \bigcap_{s \geq 1} \bigcup_{t \geq s} \mathcal{E}_t.$$

We have

Lemma 6.23. *The set* $\mathfrak{E} = \bigcap_{s \geq 1} \bigcup_{t \geq s} \mathcal{E}_t$ *has measure*

$$|\mathfrak{E}| \geq (1 - \mathrm{e}^{-\frac{8}{25}\sigma\gamma K}) |\mathcal{E}|. \tag{6.61}$$

Proof. Let $\mathfrak{E}_s = \bigcup_{t \geq s} \mathcal{E}_t$. We have that $\mathfrak{E}_{s+1} \subseteq \mathfrak{E}_s$ and $|\mathfrak{E}_s| \geq (1 - \mathrm{e}^{-\frac{8}{25}\sigma\gamma K}) |\mathcal{E}|$. This implies the claim, since $\mathfrak{E}_s \subseteq \mathcal{E}$ with $|\mathcal{E}| < \infty$. □

This finishes the proof of Theorem 6.3.

6.7 Applications

We are now ready to prove Theorem 1.5. In fact it follows easily from Proposition 5.7 and Theorem 6.3.

CHAPTER 7

Outline of the second multiscale scheme

The goal of this chapter is to give an outline of the second multiscale scheme, and to derive results from it. In this introduction, I will just state the most basic consequence.

Theorem 7.1. *Let $H_{\underline{\omega}}$ be a family of skew-shift Schrödinger operators with Diophantine frequency. Let $M_1 \geq (K \cdot M_0)^2$ and M_0 be large enough. Assume that*

$$|\{\underline{\omega}: \quad \mathrm{dist}(E, \sigma(H_{\underline{\omega},[1,M_j]})) \geq 1\}| \leq \mathrm{e}^{-(M_j)^{1/2}} \tag{7.1}$$

for $j = 0, 1$. Then for a universal $\gamma_0 > 0$,

$$L(E) \geq \gamma_0. \tag{7.2}$$

We now get down to business.

7.1 Definitions and Notation

In this section, we discuss definitions and notation necessary for the following. The same notation will be used in all the following chapters.

$V(n)$ will always denote a bounded and real-valued sequence, and $H = \Delta + V$ the associated Schrödinger operator on $\ell^2(\mathbb{Z})$. Since we do not make the dependence on the energy E explicit, one might have to replace H by $H - E$. Next, we need the following definition.

Definition 7.2. *Let $[a, b] \subseteq \mathbb{Z}$ be an interval. We call $x, y \in [a, b]$ good if*

$$|x - y| \geq \frac{|b - a|}{10} \tag{7.3}$$

and

$$|x - y| \leq \max(|x - a|, |x - b|, |y - a|, |y - b|). \tag{7.4}$$

The second condition is important to iterate the resolvent equation. We now come to

Definition 7.3. *Let $\gamma > 0$, $1 > \tau > 0$, and $p \geq 0$ be an integer. An interval $[a, b]$ is called (γ, τ, p)-suitable for H if*

$$\|(H_{[a,b]})^{-1}\| \leq \frac{1}{2^p} \mathrm{e}^{(b-a)^\tau} \tag{7.5}$$

and for good $x, y \in [a, b]$, *we have*

$$|\langle e_x, (H_{[a,b]})^{-1} e_y \rangle| \le \frac{1}{2^{p+1}} \mathrm{e}^{-\gamma|x-y|}. \tag{7.6}$$

We recall that $H_{[a,b]}$ denotes the restriction of $H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ to $\ell^2([a, b])$ and that $\langle , \rangle$ denotes the scalar product of $\ell^2(\mathbb{Z})$ and $\{e_x\}_{x\in\mathbb{Z}}$ its standard basis. We will write $\mathcal{B}^N_{\gamma,\tau,p}(H)$ for the set of length N intervals I such that I is not (γ, τ, p)-suitable for H.

We will usually assume the potential $V(n)$ is generated by evaluating a sampling function f along the orbits of the skew-shift $T_\alpha : \mathbb{T}^K \to \mathbb{T}^K$. Here, we will assume that f is a trigonometric polynomial, that is

$$f(\underline{x}) = \sum_{|\underline{\xi}| \le A} \hat{f}(\underline{\xi}) e(\underline{\xi} \cdot \underline{x}), \tag{7.7}$$

where $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$, $|\underline{\xi}| = \max_{1\le k\le K} |\xi_k|$ for $\underline{\xi} \in \mathbb{Z}^K$, and $\underline{\xi} \cdot \underline{x} = \sum_{k=1}^K \xi_k x_k$. Furthemore, we recall that

$$(T_\alpha \underline{\omega})_k = \begin{cases} \omega_1 + \alpha, & k = 1; \\ \omega_k + \omega_{k-1}, & 2 \le k \le K. \end{cases} \tag{7.8}$$

We will also need to impose a Diophantine condition on the frequency α. For $c > 0$, we write $\alpha \in DC(c)$ if for all $a, n \in \mathbb{Z}$ with $n \ge 1$ we have

$$\left|\alpha - \frac{a}{n}\right| \ge \frac{c}{n^3}. \tag{7.9}$$

Furthermore, if $\Omega \subseteq \mathbb{T}^K$, we write $|\Omega|$ for the Lebesgue measure of Ω.

7.2 The result and applications

We will now state the main result, and discuss how Theorem 1.6 follows.

Theorem 7.4. *Let $\frac{1}{K!}\alpha \in DC(c)$, and μ, τ, $\gamma > 0$. Let $M_1 \geq M_0 \geq 1$ with M_0 large enough and assume $(M_1)^\mu \geq 6\gamma L M_0$ and for $j = 0, 1$ that*

$$|\{\underline{\omega}: \quad [1, M_j] \text{ is } (\gamma, \tau, 3)\text{-suitable for } H_{\underline{\omega}}\}| \geq 1 - \mathrm{e}^{-(M_j)^\mu}. \tag{7.10}$$

Introduce for $\beta = \frac{1}{500K^6 \log(K)^2}$

$$N_{min} = \lceil \mathrm{e}^{(M_0)^{\frac{\beta}{2}}} \rceil, \quad N_{max} = \lfloor \mathrm{e}^{(M_0)^\beta} \rfloor. \tag{7.11}$$

Assume $\mu \geq 2\beta$. Then for $N_{min} \leq N \leq N_{max}$ and

$$\hat{\gamma} = \gamma(1 - \frac{1}{M_0}), \quad \hat{\tau} = 1 - \frac{1}{200K^3 \log(K)}, \tag{7.12}$$

we have that

$$|\{\underline{\omega}: \quad [1, N] \text{ is } (\hat{\gamma}, \hat{\tau}, 3)\text{-suitable for } H_{\underline{\omega}}\}| \geq 1 - \mathrm{e}^{-(N)^{2\beta}}. \tag{7.13}$$

Here the largeness condition on M_0 depends on μ, τ, γ, c and K. For M_0 large enough, we can iterate the procedure. Combining this with Lemma 3.9, we obtain the following conclusion.

Corollary 7.5. *Let* $\frac{1}{K!}\alpha \in DC(c)$, *and* μ, τ, $\gamma > 0$. *Let* $M_1 \geq M_0 \geq 1$ *with* M_0 *large enough and assume* $(M_1)^\mu \geq 6\gamma L M_0$ *and, for* $j = 0, 1$, *that*

$$|\{\underline{\omega} : \quad [1, M_j] \textit{ is } (\gamma, \tau, 3)\textit{-suitable for } H_{\underline{\omega}}\}| \geq 1 - \mathrm{e}^{-(M_j)^\mu}. \tag{7.14}$$

Then

$$L(E) \geq \frac{\gamma}{2}. \tag{7.15}$$

Proof. Let $N_0 = M_0$ and introduce

$$N_j = \lceil \mathrm{e}^{(N_{j-1})^{\frac{\beta}{2}}} \rceil.$$

If M_0 is large enough, we obtain that the assumptions of Theorem 7.4 hold again with $M_0 = N_1$ and M_1 choosen in such a way that $(M_1)^\mu \geq 6\gamma L M_0$. Note that now $\mu = 2\beta$. Furthermore, it is easy to see that then the assumptions of Theorem 7.4 hold for $M_0 = N_j$ and a similar choice of M_1. Hence, we obtain

$$|\{\underline{\omega} : \quad [1, N_j] \text{ is } (\gamma_j, \tau, 3)\text{-suitable for } H_{\underline{\omega}}\}| \geq 1 - \mathrm{e}^{-(N_j)^\mu},$$

where $\gamma_j = \gamma_{j-1} \cdot (1 - \frac{1}{N_{j-1}})$, $\gamma_0 = \gamma$. Hence, we obtain by using the Borel–

Cantelli lemma and Lemma 3.9 that

$$\limsup_{N\to\infty} \frac{1}{N} \log \|A_{\underline{\omega}}(E,N)\| \geq \inf_{j\geq 1} \gamma_j$$

for almost every $\underline{\omega} \in \mathbb{T}^K$. By choosing M_0 large enough, we can ensure that $\inf_{j\geq 1} \gamma_j \geq \frac{1}{2}\gamma$, which implies the claim. $\square$

In particular, we obtain Theorem 1.6 using Proposition 5.7. We will now discuss the main ingredients necessary to prove Theorem 7.4, and then prove it.

Let me furthermore point out that our results imply some continuity of the integrated density of states. In fact, we will show the following result, which holds for general ergodic Schrödinger operators. It should be noted that the following result requires the assumption at all energies E.

Theorem 7.6. *Let $\gamma, \tau, \mu, \beta > 0$. Assume that $\mu < \tau$ and for every $E \in \mathbb{R}$ and N sufficiently large*

$$|\{\underline{\omega}: \quad [1,N] \text{ is } (\gamma,\tau,3)\text{-suitable for } H_{\underline{\omega}} - E\}| \geq 1 - \mathrm{e}^{-N^{\mu}}. \tag{7.16}$$

Then for $0 < \tilde{\mu} < \mu$, we have for N sufficiently large

$$\frac{1}{N}\int_{\mathbb{T}^K} \operatorname{tr}\left(P_{[E_0,E_1]}(H_{\underline{\omega}})\right) d\underline{\omega} \leq \mathrm{e}^{-\log\left(\frac{1}{E_1-E_0}\right)^{\tilde{\mu}}}. \tag{7.17}$$

Let $\mathcal{N}(E)$ be the integrated density of states defined as in Section 3.5.

Then this theorem combined with the previous results implies for the skew-shift model that

$$\mathcal{N}(E+\varepsilon) - \mathcal{N}(E-\varepsilon) \leq \mathrm{e}^{-\log(\frac{1}{\varepsilon})^{\beta}}. \tag{7.18}$$

7.3 Ingredients based on the resolvent equation

In this section, we discuss parts of the proof that are essentially based on the resolvent equation, which relates matrix elements of the resolvent on a large scale to the ones on small scales. Given $x \in [a, b] \subseteq \Lambda$ and $y \in \Lambda \setminus [a, b]$, the resolvent equation states that

$$\begin{aligned} \langle e_x, (H_\Lambda)^{-1} e_y \rangle = &- \langle e_x, (H_{[a,b]})^{-1} e_a \rangle \langle e_{a-1}, (H_\Lambda)^{-1} e_y \rangle \\ &- \langle e_x, (H_{[a,b]})^{-1} e_b \rangle \langle e_{b+1}, (H_\Lambda)^{-1} e_y \rangle. \end{aligned} \tag{7.19}$$

Naively, one might guess that nothing is gained by it, since both sides involve $(H_\Lambda)^{-1}$, but we will usually know that the entries involving $(H_{[a,b]})^{-1}$ are exponentially small. Combining this with weak estimates on $(H_\Lambda)^{-1}$, we are able to obtain strong ones for it. The first implementation of this is

Theorem 7.7. *Let $\gamma, \tau > 0$, $1 \leq M \leq L$ and assume*

$$\gamma M^{1-\tau} \geq 6, \quad M^\tau \geq \max(\log(L)), 6), \quad 10(4M^\tau + M + \log(2)) \leq L^\tau. \tag{7.20}$$

Assume

$$(7.21) \qquad |\{\underline{\omega}: \quad [1, M] \text{ is not } (\gamma, \tau, 3)\text{-suitable for } H_{\underline{\omega}}\}|. \leq \mathrm{e}^{-M^{\mu}}$$

Then there exists a set $\Omega_1 \subseteq \mathbb{T}^K$ *with the following properties.*

(i) $|\Omega_1| \geq 1 - L\mathrm{e}^{-M^{\mu}}$.

(ii) For $\underline{\omega} \in \Omega_1$, *we have*

$$(7.22) \qquad \|(H_{\underline{\omega},[1,L]})^{-1}\| \leq \mathrm{e}^{4M^{\tau}},$$

(iii) For $\underline{\omega} \in \Omega_1$, $[1, L]$ *is* $(\hat{\gamma}, \tau, 3)$*-suitable for* $H_{\underline{\omega}}$ *where* $\hat{\gamma} = \gamma - \frac{1}{L^{1-\tau}}$.

We will also show

Theorem 7.8. *Let* $0 < \rho < \tau$, $\mu, \nu > 0$, $0 < q < \left(\frac{1}{120}\right)^2$, *and* $1 \leq L_0 \leq N$. *Assume* $N \gg 1$ *and*

$$(7.23) \qquad N^2 \mathrm{e}^{-(L_0)^{\nu}} \leq \mathrm{e}^{-N^{\mu}}, \quad N^q \geq (L_0)^2.$$

Assume for $L_0 \leq L \leq N$ *that*

$$(7.24) \qquad |\{\underline{\omega}: \quad \|(H_{\underline{\omega},[1,L]})^{-1}\| \leq \mathrm{e}^{L^{\rho}}\}| \geq 1 - \mathrm{e}^{-L^{\nu}}$$

and for $\underline{\omega} \in \mathbb{T}^K$

$$\#\big(\Lambda \subseteq [1, N] : \Lambda \in \mathcal{B}^{L_0}_{\gamma,\tau,0}(H_{\underline{\omega}})\big) \leq N^{1-q}. \tag{7.25}$$

Then for $\hat{\gamma} = \gamma \cdot \left(1 - \log(N)^{-\frac{1}{2}}\right)$

$$|\{\underline{\omega} : \quad [1, N] \in \mathcal{B}^{N}_{\hat{\gamma},\tau,3}(H_{\underline{\omega}})\}| \leq \mathrm{e}^{-N^{\mu}}. \tag{7.26}$$

Here $N \gg 1$ means that $N \geq N_0 = N_0(\rho, K, \gamma_0)$, where $\gamma_0 \leq \tilde{\gamma}$. In fact, this condition can be explicitly read of from Theorem 9.1.

The proofs of Theorem 7.7 and 7.8 are both deterministic. This means that they follow from statements for single operators combined with basic probabilistic estimates. Let me furthermore point out the big difference between the two results. Theorem 7.7 needs that all intervals of length M are suitable, but it does not require resolvent estimates on all scales. Theorem 7.8 allows for intervals that are not suitable, however it requires resolvent estimates on all scales.

This difference is also manifest in the proofs. For Theorem 7.7, most of the work is needed to ensure that the norm of the resolvent is bounded, and then the decay of the off-diagonal terms follows through a simple iteration of (7.19), whereas for Theorem 7.8, we already know that the resolvent is a bounded operator, and we only need to work to obtain the decay of the off-diagonal terms.

It should also be pointed out that Theorem 7.8 is the analog of what Kirsch calls the *analytic estimate* in Chapter 10 of his lecture notes [18].

7.4 Ingredients based on ergodicity

We will prove the following uniform recurrence result. It is an improved version of the answer to the question of how often an iterate of a point $\underline{\omega}$ under the skew-shift lands in a small ball. The proof depends on a bound on this number, which we give in Theorem 11.3. I will discuss further aspects of the proof after its statement.

Theorem 7.9. *Let $\gamma, \tau, \mu, c > 0$ and $K \geq 1$. There exists $L_0 = L_0(K, c) > 0$, $G_2 = G_2(K) > 0$, and $W_3 = W_3(K, c) > 0$. Define*

$$q = \frac{1}{50K^3 \log(K)}. \tag{7.27}$$

Let $L \geq M \geq 1$ and $\frac{1}{K!}\alpha \in DC(c)$. Assume

$$\max(M^{G_2}, L_0(K, c)) \leq L \leq \mathrm{e}^{\frac{1}{2}M^{\mu}}, \tag{7.28}$$

$$|\{\underline{\omega} : \quad [1, M] \in \mathcal{B}^M_{\gamma,\tau,3}(H_{\underline{\omega}})\}| \leq \frac{W_3}{M^{G_2}} \cdot \frac{1}{L^{Kq}}. \tag{7.29}$$

Then for $\underline{\omega} \in \mathbb{T}^K$,

$$\#\big(\Lambda \subseteq [1, L] : \Lambda \in \mathcal{B}^M_{\gamma,\tau,0}(H_{\underline{\omega}})\big) \leq L^{1-q}. \tag{7.30}$$

The first step of the proof is to derive Theorem 11.3, which bounds the number of iterates of the skew-shift, which land in a small ball. The proof relies on estimates on the growth of exponential sums, which I review in Chapter 10. The next step is to extend Theorem 11.3 to semi-algebraic sets, which is done in Theorem 11.7.

The last step is then to show that the set of $\underline{\omega}$ where $[1, L]$ is not suitable for $H_{\underline{\omega}}$ can be contained in a semi-algebraic set. This is done in Section 11.5. In fact, the current proof of this result uses that f is a trigonometric polynomial. However, this is unnecessary, as explained in Remark 11.10.

However, there is also another direction in which Theorem 7.9 could be improved. At the moment (7.30) holds for every $\underline{\omega} \in \mathbb{T}^K$. However, for all our applications it would be sufficient if we had (7.30) up to an exceptional set of $\underline{\omega}$ of measure e^{-L^σ} for some $\sigma > 0$. In particular, we know that the analog of this theorem for random Schrödinger operators has this probabilistic nature. See for example Theorem 10.22 in the lecture notes of Kirsch in [18]. Hence, I expect that if one wanted to make the constant L_0 in this theorem quantitative in K, one should proceed like that. The current dependence is at least like $L_0 \gtrsim K^K$.

7.5 Ingredients based on complex-analysis

The results of this section depend on $\underline{\omega} \mapsto H_{\underline{\omega}}$ being an analytic function. We make the necessary property explicit in the following definition.

Definition 7.10. $\mathbb{R}^K \ni \underline{\omega} \mapsto H_{\underline{\omega}}$ *is nice, if the following hold.*

(i) $H_{\underline{\omega}}$ *extends to an analytic function* $\mathbb{C}^K \ni \underline{z} \mapsto H_{\underline{z}}$.

(ii) There exist constants $C_1, C_2 > 0$ *such that for* $\Lambda \subseteq [1, N]$ *and* $\rho > 0$, *we have for all* $\underline{z} \in \mathbb{C}^K$ *satisfying* $|\mathrm{Im}(z_k)| \leq \rho$ *for* $1 \leq k \leq K$ *that*

$$\|H_{\underline{z},\Lambda}\| \leq C_1 \mathrm{e}^{C_2 \rho N^K}. \tag{7.31}$$

We will show the following theorem, where q could be a parameter, but will be $q = \frac{1}{50K^3 \log(K)}$ in all applications.

Theorem 7.11. *Let* $\gamma, \tau > 0$ *and* $L \geq M \geq 1$. *Assume* $\underline{\omega} \mapsto H_{\underline{\omega}}$ *is nice and*

$$L \geq (8000K \log(K))^{\frac{8}{q}} \tag{7.32}$$

$$\gamma M^{\tau} \geq \max(const, K \log(L)) \tag{7.33}$$

$$\gamma M^{1-\tau} \geq 6 \tag{7.34}$$

$$L^{\frac{q}{3}} \geq 24 M^2. \tag{7.35}$$

Furthermore assume

$$\#\big(\Lambda \subseteq [1, L] : \Lambda \in \mathcal{B}^M_{\gamma,\tau,0}(H_{\underline{\omega}})\big) \leq L^{1-q} \tag{7.36}$$

$$|\{\underline{\omega} : \quad \|(H_{\underline{\omega},[1,L]})^{-1}\| \leq \mathrm{e}^{L^{\frac{q}{3}}}\}| \geq 1 - \mathrm{e}^{-5\gamma K M}. \tag{7.37}$$

Then

$$(7.38) \qquad |\{\underline{\omega}: \quad \|(H_{\underline{\omega},[1,L]})^{-1}\| \leq \mathrm{e}^{L^{1-\frac{q}{3}}}\}| \geq 1 - \mathrm{e}^{-L^{\frac{q}{4}}}.$$

It should be remarked that in (7.37) and (7.38), the left hand sides are the same, but the right hand side in (7.38) is much closer to 1 than in (7.37). This is why I would describe the application of this theorem as *improving a probability*.

The proof of this theorem will be given in Chapter 12. The essential idea is to use a high dimensional version of Cartan's lemma (Lemma 12.1) to improve the probability in (7.37). For this it will be important that, using (7.36), we can reduce the size of $H_{\underline{\omega},[1,L]}$ from an $L \times L$ matrix to an $L^{1-q} \times L^{1-q}$ matrix.

Using Lemma 7.15, one can convert (7.38) to the following Wegner-type estimate

$$(7.39) \qquad \frac{1}{L} \int_{\mathbb{T}^K} \mathrm{tr}(P_{[-\varepsilon,\varepsilon]}(H_{\underline{\omega},[1,L]}))d\underline{\omega} \leq \mathrm{e}^{-L^{\frac{q}{4}}},$$

where $\varepsilon = \mathrm{e}^{-L^{1-\frac{q}{3}}}$.

7.6 Recipe to prove Theorem 7.4

We first need to show

Lemma 7.12. (7.31) *holds.*

Proof. By (7.7) and a computation involving the skew-shift, we have $|V(n)| \leq C_1 \mathrm{e}^{An^K \rho}$ as long as $|\mathrm{Im}(z_k)| \leq \rho$ for $1 \leq k \leq K$. The claim follows. $\square$

We now proceed to give the proof of Theorem 7.4. Recall that $q = \frac{1}{50K^3 \log(K)}$ as in (7.27). Pick $N \in [N_{min}, N_{max}]$ and introduce

$$\hat{L}_0 = \left\lceil \left(\frac{1}{2}\right)^{\frac{4}{q}} \cdot N^{\frac{8\beta}{q}} \right\rceil . \tag{7.40}$$

This is chosen such that (7.23) holds with $\mu = 2\beta$ and $\nu = \frac{q}{4}$. Also note that $\frac{8\beta}{q} = 40 \cdot q$. An easy estimate shows that

$$\hat{L}_0 \geq \left(\frac{1}{4}\right)^{\frac{4}{q}} \cdot \mathrm{e}^{40q \cdot (M_0)^{\beta/2}} \tag{7.41}$$

independent of the choice of $N \in [N_{min}, N_{max}]$.

Let me begin by giving the general outline of the proof, which is shown in Figure 7.1. The general idea of the flowchart is to give an idea of where what happens and what ingredients are used. With *Scale* M_j, I denote in this flowchart (7.10), by *Scale* $\hat{L}_0$ *in scale* N, the conclusion of Theorem 7.9, and by *Wegner estimate at scale* L, the conclusion of Theorem 7.11. Otherwise, I hope that the flowchart is self-explanatory.

Let me now proceed by making the first step precise. Again, the general outline is presented in Figure 7.2. In this figure the clouds indicate which of the previous sections the necessary results to make the conclusion can be found. Furthermore, by *weak probability*, I indicate an estimate of the form

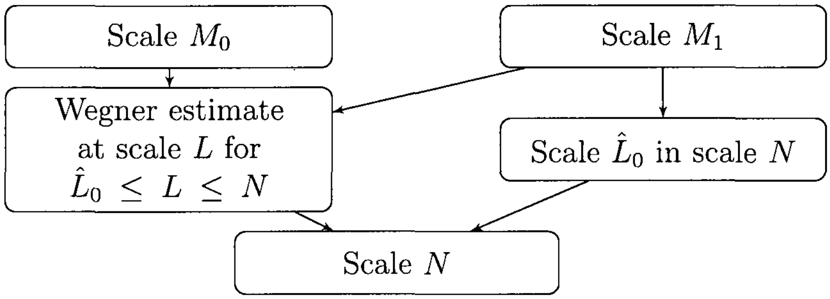

Figure 7.1: General outline of the proof

(7.37), that is, where the probability is still small, but not of size $e^{-L^{\sigma}}$, where L is the length scale.

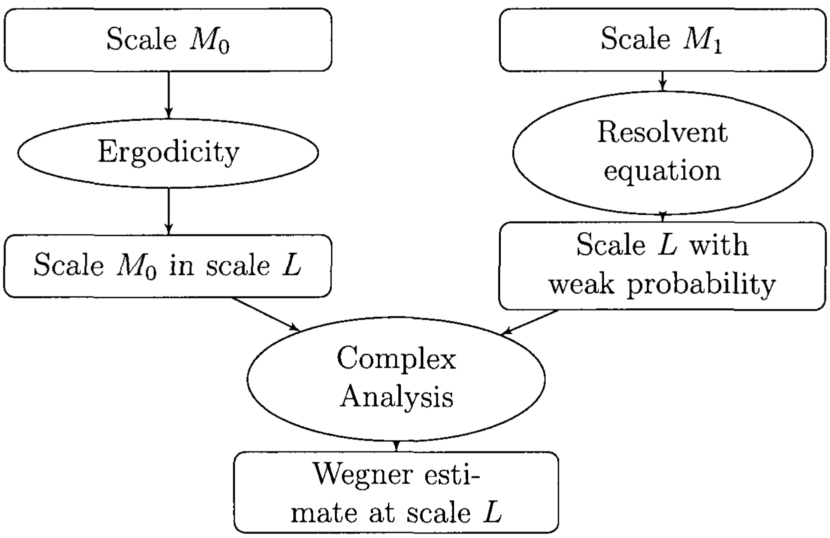

Figure 7.2: First step of the proof. Implemented in Proposition 7.13.

Proposition 7.13. *Assume* (7.10), $(M_1)^{\mu} \geq 6\gamma K M_0$, *and that* $M_0 \geq 1$ *is large enough. Then for* $\hat{L}_0 \leq L \leq N$

$$|\{\underline{\omega} : \quad \|(H_{\underline{\omega},[1,L]})^{-1}\| \leq e^{L^{1-\frac{q}{3}}}\}| \geq 1 - e^{-L^{\frac{q}{4}}}. \tag{7.42}$$

Proof. By (7.41), one obtains the lower bound in (7.28) as long as M_0 is large enough. The upper bound is clear from definition of $N \leq N_{max} = e^{(M_0)^\beta}$, where $\beta \leq \frac{1}{2}\mu$. By Theorem 7.9 with $M = M_0$, we can conclude that

$$\#\{\Lambda \subseteq [1, L] : \quad \Lambda \in \mathcal{B}^{M_0}_{\gamma,\tau,0}(H_{\underline{\omega}})\} \leq L^{1-q}.$$

This is (7.36) with $M = M_0$.

Again by assuming that $M_0 \leq M_1$ is large enough, we are able to conclude that (7.20) holds for $M = M_1$. By Theorem 7.7 applied with $M = M_1$, we can conclude that

$$|\{\underline{\omega} : \quad \|(H_{\underline{\omega},[1,L]})^{-1}\| \leq e^{L^{\frac{q}{3}}}\}| \geq 1 - Le^{-(M_1)^\mu} \geq 1 - Ne^{-(M_1)^\mu}.$$

For M_0 large enough, we have that $\gamma K M_0^{1-\frac{\mu}{2}} \geq 1$. This and the assumption imply (7.37) with $M = M_0$. We see that (7.32) to (7.35) again hold for M_0 large enough. Thus by Theorem 7.11, we obtain the conclusion (here we just use $M_0 \gg 1$). $\square$

We now come to the second part of the proof. The key steps are shown again in the flowchart in Figure 7.3.

Proposition 7.14. *Assume $M_0 \geq 1$ is large enough. Then for every $\underline{\omega} \in \mathbb{T}^K$*

$$\#\{\Lambda \subseteq [1, N] : \quad \Lambda \in \mathcal{B}^{\hat{L}_0}_{\hat{\gamma},\tau,0}(H_{\underline{\omega}})\} \leq N^{1-q}. \tag{7.43}$$

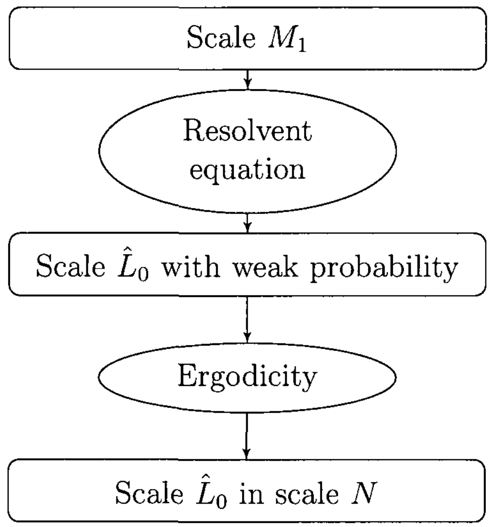

Figure 7.3: Second step of the proof. Implemented in Proposition 7.14.

Proof. (7.20) holds for M_0 large enough, since $M_1 \geq M_0$. Apply Theorem 7.7 with $M = M_1$ and $L = \hat{L}_0$, to conclude that

$$|\{\underline{\omega} : \quad [1, \hat{L}_0] \text{ is } (\hat{\gamma}, \tau, 3)\text{-suitable for } H_{\underline{\omega}}\}| \geq 1 - \hat{L}_0 \mathrm{e}^{-(M_1)^{\mu}}.$$

Hence (7.29) holds with $M = L_0$ and $L = N$, as long as M_0 is large enough. The same is true for (7.28). Now apply Theorem 7.9 to conclude the claim. ☐

We now give again the picture, from which the final step of the proof follows.

Proof of Theorem 7.4. Having (7.42) and (7.43), we are now in a position to apply Theorem 7.8. This finishes the proof of Theorem 7.4. ☐

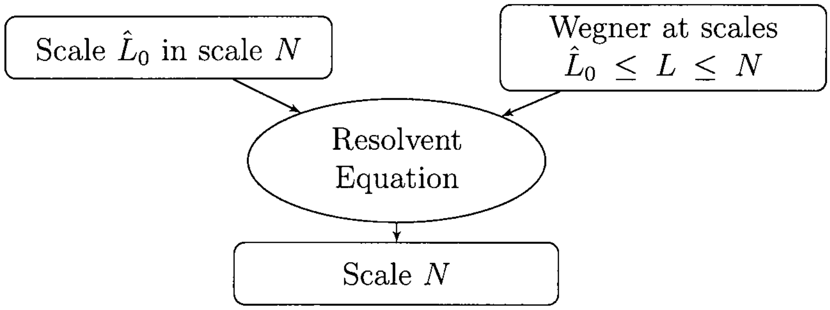

Figure 7.4: Final step of the proof

7.7 Proof of Theorem 7.6

We need the following

Lemma 7.15. *Assume that*

$$|\{\underline{\omega} : \quad \|(H_{[1,N]} - E)^{-1}\| \leq \lambda\}| \geq 1 - \varepsilon. \tag{7.44}$$

Then

$$\frac{1}{N} \int_{\mathbb{T}^K} \operatorname{tr}(P_{[E-\frac{1}{\lambda}, E+\frac{1}{\lambda}]}(H_{\underline{\omega},[1,N]}))d\underline{\omega} \leq \varepsilon. \tag{7.45}$$

Proof. By assumption there exists $\Omega_1 \subseteq \mathbb{T}^K$ such that $|\Omega_1| \geq 1 - \varepsilon$ and for $\underline{\omega} \in \Omega_1$, we have

$$\operatorname{tr}(P_{[E-\frac{1}{\lambda}, E+\frac{1}{\lambda}]}(H_{\underline{\omega},[1,N]})) = 0.$$

The claim now follows from the bound

$$\operatorname{tr}(P_{[E-\frac{1}{\lambda},E+\frac{1}{\lambda}]}(H_{\underline{\omega},[1,N]})) \leq N,$$

which holds for any $\underline{\omega} \in \mathbb{T}^K$. $\square$

By (7.16), we can apply Theorem 7.7 with $M = N$ and $L = \lfloor \mathrm{e}^{\frac{1}{2}N^\mu} \rfloor$ for $N \geq 1$ sufficiently large to conclude that

$$|\{\underline{\omega} : \quad \|(H_{[1,L]} - E)^{-1}\| \leq \mathrm{e}^{4N^\tau}\}| \geq 1 - \mathrm{e}^{-\frac{1}{2}N^\mu}. \tag{7.46}$$

By the previous lemma, this implies by partitioning an energy interval $[E_0, E_1]$ into pieces of length $2\mathrm{e}^{-4N^\tau}$ that

$$\frac{1}{L}\int_{\mathbb{T}^K} \operatorname{tr}(P_{[E_0,E_1]}(H_{\underline{\omega},[1,L]}))d\underline{\omega} \leq \left(\frac{E_1 - E_0}{2}\mathrm{e}^{4N^\tau} + 1\right)\mathrm{e}^{-\frac{1}{2}N^\mu}. \tag{7.47}$$

Proof of Theorem 7.6. By a standard decoupling argument, we also obtain for $P \gg L$ that

$$\frac{1}{P}\int_{\mathbb{T}^K} \operatorname{tr}(P_{[E_0,E_1]}(H_{\underline{\omega},[1,P]}))d\underline{\omega} \leq \left(\frac{E_1 - E_0}{2}\mathrm{e}^{4N^\tau} + 1\right)\mathrm{e}^{-\frac{1}{2}N^\mu} + 3\mathrm{e}^{-\frac{1}{2}N^\mu}.$$

Choose $N = \frac{1}{8}\log\left(\frac{1}{E_1 - E_0}\right)$ to obtain the claim. $\square$

7.8 Some properties of suitability

We begin with the following lemma about perturbations of the inverse, whose proof we leave to the reader.

Lemma 7.16. *Let A be an invertible operator and $\|\tilde{A} - A\| \leq \frac{1}{2\|A^{-1}\|}$. Then*

$$\|\tilde{A}^{-1}\| \leq 2\|A^{-1}\| \tag{7.48}$$

$$\|\tilde{A}^{-1} - A^{-1}\| \leq 2\|A^{-1}\|\|\tilde{A} - A\|. \tag{7.49}$$

Rewriting this lemma using the notion of suitability, we obtain

Lemma 7.17. *Let $[a, b]$ be (γ, τ, p)-suitable for H. Assume $\frac{1}{2}\gamma(b-a) \geq (b-a)^{\tau}$ and*

$$\|\tilde{H} - H\| \leq \frac{1}{2^{p+2}} \mathrm{e}^{-\gamma(b-a)}. \tag{7.50}$$

Then $[a, b]$ is $(\gamma, \tau, p-1)$-suitable for $\tilde{H}$.

CHAPTER 8

First application of the resolvent equation

In this chapter, we develop a mechanism on how to show that a large interval is suitable assuming one knows that all its subintervals of one size are suitable. The essential point here is to show that the resolvent remains a bounded operator. In particular, we will prove Theorem 7.7.

8.1 Deterministic results

In this section, we present the main results of this section.

Theorem 8.1. *Let $N \geq M \geq 1$, $\gamma > 0$, $0 < \tau < 1$ be given. Assume*

$$\gamma M^{1-\tau} \geq 6, \quad M^{\tau} \geq \max(\log(N), 6). \tag{8.1}$$

Let Λ be a union of length M intervals satisfying $|\Lambda| \leq N$. Assume that *every interval* $[n, n+M-1] \subseteq \Lambda$ *is* (γ, τ)*-suitable. Then*

$$\|(H_\Lambda)^{-1}\| \le \mathrm{e}^{4M^\tau}. \tag{8.2}$$

We also have decay of the off-diagonal elements.

Proposition 8.2. *Assume* Λ *is an interval. Let* $x, y \in \Lambda$ *be good in the sense of Definition 7.2. Then we have*

$$|\langle e_x, (H_\Lambda)^{-1} e_y\rangle| \le \frac{1}{2}\mathrm{e}^{-\gamma_1 |x-y|}, \tag{8.3}$$

where $\gamma_1 = \gamma - 10\frac{\log(2)+M+4M^\tau}{N}$.

Proof. We assume without loss of generality that

$$\mathrm{dist}(x, \{a, b\}) \ge \mathrm{dist}(x, y) \ge \frac{|\Lambda|}{10}.$$

Assume for simplicity that M is even and introduce $s = \frac{1}{2}M$. Furthermore, introduce

$$x_j = x + js, \quad J = \lfloor \frac{x-y}{s} \rfloor.$$

We see that $\{x_j\}_{j=-J}^{J} \subseteq \Lambda$. Furthermore, by the resolvent equation applied to the interval $[x_{j-1}+1, x_{j+1}-1]$, we obtain that

$$|\langle e_{x_j}, (H_\Lambda)^{-1} e_y\rangle| \le \mathrm{e}^{-\gamma s} \cdot \max(|\langle e_{x_{j-1}}, (H_\Lambda)^{-1} e_y\rangle|, |\langle e_{x_{j+1}}, (H_\Lambda)^{-1} e_y\rangle|).$$

This implies that

$$|\langle e_x, (H_\Lambda)^{-1} e_y\rangle| \leq \mathrm{e}^{-\gamma s \cdot J} \cdot \max_{-J \leq j \leq J} |\langle e_{x_j}, (H_\Lambda)^{-1} e_y\rangle|.$$

The claim follows. □

As a simple corollary, we obtain

Corollary 8.3. *Assume that* $4M^\tau \leq |\Lambda|^\tau$. *Then* Λ *is* $(\gamma, \tau, 0)$*-suitable for* H.

Proof. This follows by comparing definitions. □

8.2 Results for ergodic Schrödinger operators

Let us now discuss the consequences for ergodic Schrödinger operators. We will assume that (Ω, μ) is a probability space and $T : \Omega \to \Omega$ an invertible and ergodic transformation. For a bounded measurable function $f : \Omega \to \mathbb{R}$, we define the potential by $V_\omega(n) = f(T^n \omega)$ and the Schrödinger operator $H_\omega = \Delta + V_\omega$. A direct consequence of the result of last section is

Corollary 8.4. *Let* $N \geq M \geq 1$, $\gamma > 0$, $0 < \tau < 1$ *be given. Assume* (8.1). *Furthermore, assume*

$$\mu(\{\omega : [1, M] \text{ is } (\gamma, \tau) - \text{suitable for } H_\omega\}) \geq 1 - \varepsilon. \tag{8.4}$$

Then there exists a set Ω_1 *with the following properties.*

(i) $\mu(\Omega_1) \geq 1 - N\varepsilon$.

(ii) *For* $\omega \in \Omega_1$

$$\|(H_{\omega,[1,N]})^{-1}\| \leq \mathrm{e}^{4M^\tau}. \tag{8.5}$$

(iii) *For* $x, y \in [1, N]$ *good*

$$|\langle e_x, (H_{\omega,[1,N]})^{-1} e_y \rangle| \leq \frac{1}{2}\mathrm{e}^{-\gamma_1 |x-y|}, \tag{8.6}$$

where $\gamma_1 = \gamma - 10\frac{\log(2)+M+4M^\tau}{N}$.

Proof. Denote by Ω_0 the set from (8.4). Introduce

$$\Omega_1 = \bigcap_{n=1}^{N} T^{-n}\Omega_0.$$

That $\mu(\Omega_1) \geq 1 - N\varepsilon$ follows, and also that for $\omega \in \Omega_1$, H_ω satisfies the assumptions of Theorem 8.1 with $\Lambda = [1, N]$. □

We are now ready for

Proof of Theorem 7.7. This follows from the previous corollary, that $\mathrm{e}^{4M^\tau} \leq \frac{1}{8}\mathrm{e}^{L^\tau}$, and the choice of $\hat{\gamma}$. □

8.3 Proof of Theorem 8.1

The basic strategy of the proof will be to show that if $H_\Lambda u = Eu$ has a non-trivial solution u, then $|E|$ must be large. In order to do so, we will need to analyze the restrictions of u to intervals of size M. Let us begin by introducing the necessary notation. For $[a,b] \subseteq \Lambda$ and $u \in \ell^2(\Lambda)$ a solution of $H_\Lambda u = Eu$, we have for $n \in [a,b]$ that

$$(8.7)\quad u(n) = -\langle e_n, (H_{[a,b]} - E)^{-1} e_a \rangle u(a-1) - \langle e_n, (H_{[a,b]} - E)^{-1} e_b \rangle u(b+1),$$

where $u(x) = 0$ if $x \notin \Lambda$. We furthermore recall that

$$(8.8)\qquad \frac{1}{\|(H_\Lambda)^{-1}\|} = \inf_{\|v\|=1} \|H_\Lambda v\|.$$

Since H_Λ is self-adjoint, the infimum is attained by an eigenvector v to an eigenvalue E. Let v be such a normalized eigenvector

$$(8.9)\qquad H_\Lambda v = Ev, \quad |E| = \frac{1}{\|(H_\Lambda)^{-1}\|}.$$

Denote by $m \in \Lambda$ a point such that $|v(m)| \geq |v(n)|$ for all $n \in \Lambda$. In particular, we have

$$(8.10)\qquad |v(m)|^2 \geq \frac{1}{N}$$

We denote by $\partial\Lambda$ the boundary of Λ. If we write Λ as the union of disjoint intervals $[a_t, b_t]$ with $b_t \leq a_{t+1} - 2$, then

$$\Lambda = \bigcup_t [a_t, b_t], \quad \partial\Lambda = \bigcup_t \{a_t, b_t\}. \tag{8.11}$$

We have the following lemma

Lemma 8.5. *Assume* $\gamma M \geq 10 \log(4)$ *and*

$$|E| < \frac{1}{8} \mathrm{e}^{-2M^\tau}. \tag{8.12}$$

We have that $\mathrm{dist}(m, \partial\Lambda) < \frac{M}{10}$.

Proof. Let $n \in \Lambda$ be contained in an interval $[a, b]$ of length M, such that

$$|a - n|, |b - n| \geq \frac{M}{10}.$$

Apply Lemma 7.17 with $\tilde{H} = H - E$. We may thus conclude

$$|\langle e_a, (H - E)^{-1} e_n \rangle|, |\langle e_b, (H - E)^{-1} e_n \rangle| \leq \frac{1}{4}.$$

Thus, we obtain by (8.7) that

$$|v(n)| \leq \frac{1}{4}|v(a-1)| + \frac{1}{4}|v(b+1)| \leq \frac{1}{2}(\max(|v(a-1)|, |v(b+1)|),$$

which implies that either $|v(a-1)|$ or $|v(b+1)|$ would be larger than $|v(n)|$.

This shows $m \neq n$, finishing the proof. $\square$

Remark 8.6. *This lemma gives us some information about the structure of v. Denote*

$$\alpha = \mathrm{e}^{-\frac{1}{10}\gamma M} + 4|E|\mathrm{e}^{2M^\tau}.$$

Then, as in the last lemma, we infer for $j \geq 1$ an integer and $\operatorname{dist}(n, \partial\Lambda) \geq \frac{j}{M}10$ *that $|u(n)| \leq \alpha^j$. This means that u is localized close to $\partial\Lambda$.*

We are now ready for

Proof of Theorem 8.1. Choose an interval $m \in [a, b]$ such that either $a - 1 \notin \Lambda$ or $b + 1 \notin \Lambda$. We may assume the first case, and that $|m - a| \leq \frac{M}{10}$. For some c such that $|a - c| \geq \frac{M}{3}$ and $|b - c| \geq \frac{M}{3}$, as in the last lemma, we may conclude using Lemma 7.17 and (8.7) that

$$|v(c)|, |v(c+1)| \leq \frac{1}{2}\mathrm{e}^{-\frac{\gamma M}{3}} + |E|2\mathrm{e}^{2M^\tau}.$$

Define a vector u by

$$u(n) = \begin{cases} v(n), & a \leq n \leq c; \\ 0, & \text{otherwise.} \end{cases}$$

We can compute

$$H_{[a,b]}u(n) = \begin{cases} Eu(n), & a \leq n \leq c-1; \\ -v(c+1), & n = c; \\ v(c), & n = c+1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we conclude that

$$\|H_{[a,b]}u\| \leq |E| + \mathrm{e}^{-\frac{\gamma M}{3}} + 4|E|\mathrm{e}^{2M^\tau}.$$

Futhermore from (8.10) and $[a,b]$ being (γ,τ)-suitable, we have $\|H_{[a,b]}u\| \geq N^{-1/2}\mathrm{e}^{-M^\tau} \geq 2\mathrm{e}^{-2M^\tau}$. Combining these two inequalities, we obtain

$$2\mathrm{e}^{-2M^\tau} - \mathrm{e}^{-\frac{\gamma M}{3}} \geq \mathrm{e}^{-2M^\tau} \geq 5|E|\mathrm{e}^{2M^\tau}.$$

This implies the claim. $\square$

CHAPTER 9

Second application of the resolvent equation

The goal of this chapter will be to prove Theorem 7.8. The main ingredient will be an iteration of the resolvent equation, which we present in Proposition 9.4, and an argument increasing the length of good subsets, presented in Section 9.4.

9.1 The statement for a single operator

The following theorem is a determinstic analog of Theorem 7.8. We will let $H : \ell^2([1,N]) \to \ell^2([1,N])$ be a Schrödinger operator, without further assumptions.

Theorem 9.1. *Let* $0 < \rho < \tau < 1$ *and* $\gamma_0 > 0$. *Assume that for every* *subinterval* $\Lambda \subseteq [1, N]$ *satisfying* $|\Lambda| \geq L_0$, *we have*

$$\|(H_\Lambda)^{-1}\| \leq \mathrm{e}^{|\Lambda|^\rho}. \tag{9.1}$$

Furthermore assume for $0 < q < \frac{1}{120^2}$ *that*

$$\#\{\Lambda \subseteq [1, N] : \quad \Lambda \in \mathcal{B}^{L_0}_{\gamma_0,\tau,0}(H)\} \leq N^{1-q}. \tag{9.2}$$

Assume $N^q \geq (L_0)^2$ *and*

$$\log(N) \geq \max\left(\frac{\log(10) - \log(\gamma_0)}{1 - \rho^{1/2}}, \left(\frac{32}{q}\right)^2 \cdot \frac{1}{\log(\rho^{-1})^4}\right). \tag{9.3}$$

Introduce

$$\gamma_\infty = \left(1 - \frac{1}{\log(N)^{\frac{1}{2}}}\right) \cdot \gamma_0. \tag{9.4}$$

Then $[1, N]$ *is* $(\gamma_\infty, \tau, 3)$*-suitable for* H.

One might wonder why this theorem does not impose a smallness condition on N, with the conclusion improving as $N \to \infty$. The reason is that as $N \to \infty$, the assumption (9.2) improves.

We now proceed to prove Theorem 7.8. First, we will show the following lemma, which will allow us to check (9.1).

Lemma 9.2. *Assume* (7.24) *and that*

$$N^2 \mathrm{e}^{-(L_0)^\nu} \leq \mathrm{e}^{-N^\mu}. \tag{9.5}$$

There exists a set Ω_1 such that

(i) $|\Omega_1| \geq 1 - \mathrm{e}^{-N^\mu}$,

(ii) for every $\underline{\omega} \in \Omega_1$ and subinterval $\Lambda \subseteq [1, N]$ satisfying $|\Lambda| \geq L_0$, we have

$$\|(H_\Lambda(\underline{\omega}))^{-1}\| \leq \mathrm{e}^{|\Lambda|^\rho}. \tag{9.6}$$

Proof. There are less then N^2 of choices of $1 \leq a < b \leq N$ such that $b - a \geq L_0$. Furthermore for each, we have

$$|\{\underline{\omega} : \quad \|(H_{[a,b]})^{-1}\| > \mathrm{e}^{(b-a)^\rho}\}| \leq \mathrm{e}^{-(b-a)^\nu}.$$

Define

$$\Omega_2 = \bigcap_{1 \leq a < b \leq N, b-a \geq L_0} \{\underline{\omega} : \quad \|(H_{[a,b]})^{-1}\| > \mathrm{e}^{(b-a)^\rho}\}.$$

By the previous observations, we have that

$$|\Omega_2| \leq N^2 \mathrm{e}^{-(L_0)^\nu} \leq \mathrm{e}^{-N^\mu}.$$

Now, let $\Omega_1 = \mathbb{T}^K \setminus \Omega_2$ and the claim follows. □

We are now able to proof Theorem 7.8 assuming that Theorem 9.1 holds.

Proof of Theorem 7.8. By (7.25), (9.2) holds for every $\underline{\omega}$. Let Ω_1 be the set from the previous lemma. We have that (9.1) holds for every $\underline{\omega} \in \Omega_1$. Hence, the claim follows. $\square$

9.2 Using the resolvent equation

In this section, we wish to prove the following proposition, which provides an abstract version of the resolvent equation iteration, which we use. We begin with the following definition used to quantify decay of the Green's function.

Definition 9.3. *Let $\gamma > 0$. An interval $I \subseteq [1, N]$ has γ-decay, if for any good $x, y \in I$, we have*

$$|\langle e_x, (H_I)^{-1} e_y\rangle| \leq \frac{1}{2} \mathrm{e}^{-\gamma|x-y|}. \tag{9.7}$$

We remark that if $[a, b]$ has γ-decay and $\|(H_{[a,b]})^{-1}\| \leq \mathrm{e}^{(b-a)^\tau}$ then $[a, b]$ is (γ, τ)-suitable for H. We also recall the resolvent equation. If $x \in [a, b] \subseteq \Lambda$ and $y \in \Lambda \setminus [a, b]$, we have that

$$\begin{aligned} \langle e_x, (H_\Lambda)^{-1} e_y\rangle = &- \langle e_x, (H_{[a,b]})^{-1} e_a\rangle \langle e_{a-1}, (H_\Lambda)^{-1} e_y\rangle \\ &- \langle e_x, (H_{[a,b]})^{-1} e_b\rangle \langle e_{b+1}, (H_\Lambda)^{-1} e_y\rangle. \end{aligned}$$

The following proposition, will allow us to pass from scale to scale as long as, we have information on the size of the resolvents. It's proof is a basic iteration argument of the above equation.

Proposition 9.4. *Let $\gamma > 0$, $0 < \sigma < \frac{1}{30}$ and $M \geq L \geq 1$. Assume for every interval $\Lambda \subseteq [1, M]$ of length $|\Lambda| \geq L$ that*

$$\|(H_\Lambda)^{-1}\| \leq \frac{1}{3} e^{\frac{1}{4}\gamma L}. \tag{9.8}$$

Furthermore assume

$$\#\{\Lambda \subseteq [1, M] : \quad |\Lambda| = L, \quad H_\Lambda \text{ does not have } \gamma \text{ decay}\} \leq \sigma \cdot \frac{M}{L}. \tag{9.9}$$

Then $H_{[1,M]}$ has $\hat{\gamma} = (1 - 30\sigma)\gamma$-decay.

Define Ξ as the $M/2 + 2$ neighborhood of the union over all intervals Λ of length L, which do not have γ-decay. It is easy to see that

$$|\Xi| \leq 3\sigma M. \tag{9.10}$$

Furthermore write

$$\Xi = \bigcup_{t=1}^{T} [a_t, b_t] \tag{9.11}$$

for disjoint intervals $J_t = [a_t, b_t]$. We also have that $T \leq \sigma M / L$.

Lemma 9.5. *For* $\operatorname{dist}(y, \{a_t, b_t\}) > M$ *and* $x \in \{a_t, b_t\}$, *we have*

$$|\langle e_x, (H_{J_t})^{-1} e_y\rangle| \le \frac{1}{3}. \tag{9.12}$$

Proof. By construction of Ξ, we have that the intervals $[a_t, a_t + M - 1]$ and $[b_t - M + 1, b_t]$ have γ-decay. Denote $\tilde{a} = a_t + \lceil M/2 \rceil$, $\tilde{b} = b_t - \lceil M/2 \rceil$. Then a computation yields that

$$\begin{aligned} |\langle e_x, (H_{J_t})^{-1} e_y\rangle| &\le |\langle e_x, (H_{J_t})^{-1} e_{\tilde{a}-1}\rangle| |\langle e_{\tilde{a}}, (H_{[\tilde{a},\tilde{b}]})^{-1} e_y\rangle| \\ &\quad + |\langle e_x, (H_{J_t})^{-1} e_{\tilde{b}+1}\rangle| |\langle e_{\tilde{b}}, (H_{[\tilde{a},\tilde{b}]})^{-1} e_y\rangle| \\ &\le \frac{1}{3} \mathrm{e}^{\frac{1}{4}\gamma L} \cdot \max(|\langle e_x, (H_{J_t})^{-1} e_{\tilde{a}-1}\rangle|, |\langle e_x, (H_{J_t})^{-1} e_{\tilde{b}+1}\rangle|). \end{aligned}$$

We also obtain that

$$\begin{aligned} |\langle e_x, (H_{J_t})^{-1} e_{\tilde{a}-1}\rangle| &\le |\langle e_x, (H_{J_t})^{-1} e_{a_t+M-1}\rangle| |\langle e_x, (H_{[a_t, a_t+M-1]})^{-1} e_{\tilde{a}-1}\rangle| \\ &\le \frac{1}{6} \mathrm{e}^{-\frac{1}{4}\gamma L}. \end{aligned}$$

The claim follows by a similar computation for b_t. □

Proof of Proposition 9.4. By the previous lemma, we can iterate the resolvent equation until the boundary, only picking up decaying terms. The decay even always beats the growth of the number of terms. Hence, we will obtain for $x, y \in \Lambda$ good, that

$$|\langle e_x, (H_{[1,M]})^{-1} e_y \rangle| \leq \frac{1}{2} \mathrm{e}^{-\gamma|x-y|+3\sigma\gamma M},$$

where the last term accounts for the bad bits. The claim now follows. $\square$

9.3 Definitions for the proof of Theorem 9.1

The goal of this section is to define the basic quantities used in the proof of Theorem 9.1. We will also prove various estimates on these quantities. We begin by introducing a sequence of scales

$$L_0 \ll L_1 \ll L_j \ll \dots L_{j_{max}} \ll N.$$

Define

$$\alpha = \frac{1}{\rho^{\frac{1}{2}}}. \tag{9.13}$$

Lemma 9.6. *Assume that*

$$\log(N) \geq \frac{\log(10) - \log(\gamma_0)}{1 - \rho^{\frac{1}{2}}} \tag{9.14}$$

then

$$N^{\alpha \cdot \rho - 1} \leq \frac{\gamma_0}{10}. \tag{9.15}$$

Proof. A computation shows that (9.15) is equivalent to

$$(\rho)^{\frac{1}{2}} - 1 \leq \frac{\log(\gamma_0) - \log(10)}{\log(N)}.$$

The claim follows, since the assumption implies that the right hand side is $\geq \rho^{\frac{1}{2}} - 1$. $\square$

We note that the assumptions of this lemma are satisfied once $\gamma_0 \geq 10$. Introduce a sequence of scales by

$$L_{j+1} = \lfloor (L_j)^{\alpha} \rfloor. \tag{9.16}$$

We observe that we have $L_j \approx (L_0)^{\alpha^j}$. We have that

Lemma 9.7. *Denote by j_{max} the number such that*

$$L_{j_{max}} < N \leq L_{j_{max}+1}. \tag{9.17}$$

We then have that

$$j_{max} = \frac{1}{\log(\alpha)} \cdot \log\left(\frac{\log(N)}{\log(L_0)}\right) \leq 2\frac{\log(\log(N))}{\log(\rho^{-1})}. \tag{9.18}$$

Proof. From $L_{j_{max}} \approx L_0^{\alpha^{j_{max}}}$, we obtain

$$\alpha^{j_{max}} \approx \frac{\log(N)}{\log(L_0)}.$$

The claim follows by taking another logarithm. $\square$

Furthermore define

$$\sigma = \frac{1}{\log(N)}. \tag{9.19}$$

Introduce

$$\gamma_j = (1 - 30\sigma)^j \gamma_0. \tag{9.20}$$

Lemma 9.8. *Assume that*

$$\log(N) \geq \left(\frac{120}{\log(\rho^{-1})} \right)^4. \tag{9.21}$$

Then

$$\gamma_{j_{max}} \geq \gamma_0 \left(1 - \frac{1}{\log(N)^{\frac{1}{2}}} \right). \tag{9.22}$$

Proof. We have $\gamma_{j_{max}} = \gamma_0 \cdot (1 - 30\sigma)^{j_{max}}$. We compute

$$\begin{aligned}(1 - 30\sigma)^{j_{max}} &= \exp\left(\log(1 - 30\sigma) \cdot j_{max}\right) \geq 1 + \log(1 - 30\sigma) \cdot j_{max} \\ &\geq 1 - 60\sigma j_{max} \geq 1 - 120\frac{\sigma}{\log(\rho^{-1})} \cdot \log(\log(N))\end{aligned}$$

where we used (9.18) in the last line. Now use that $\sigma = \frac{1}{\log(N)}$, and that

$\log(\log(N)) \le 4\log(N)^{1/4}$ to conclude

$$(1-30\sigma)^{j_{max}} \ge 1 - \left(\frac{120}{\log(\rho^{-1})}\frac{1}{\log(N)^{1/4}}\right) \cdot \frac{1}{\log(N)^{\frac{1}{2}}}$$

The claim follows, since by assumption $\left(\frac{120}{\log(\rho^{-1})}\frac{1}{\log(N)^{1/4}}\right) \le 1$. □

Introduce a sequence of densities

$$d_0 = \frac{1}{N^{q/2}}, \quad d_j = \frac{1}{\sigma^j} d_0 = \frac{\log(N)^j}{N^{q/2}}. \tag{9.23}$$

Lemma 9.9. *Assume that*

$$\log(N) \ge \max\left(\left(\frac{32}{q\log(\rho^{-1})}\right)^2, 256q^2\right). \tag{9.24}$$

Then

$$d_{j_{max}} \le \sigma. \tag{9.25}$$

Proof. We compute

$$\log(d_{j_{max}}) = \log(d_0) + j_{max}\log(\log(N)) \le -\frac{q}{2}\log(N) + j_{max}\log(\log(N)).$$

By (9.18), we conclude

$$\begin{aligned}
\log(d_{j_{max}}) &\leq -\frac{q}{2}\log(N) + \frac{1}{\log(\rho^{-1})}\log(\log(N))^2 \\
&\leq -\left(\frac{q}{2} - \frac{16}{\log(\rho^{-1})}\cdot\frac{1}{\log(N)^{1/2}}\right)\log(N) \\
&\leq -\frac{q}{4}\log(N),
\end{aligned}$$

where we used the assumption in the last line. Hence, we conclude that $d_{j_{max}} \leq N^{-\frac{q}{4}}$, and the claim holds. □

9.4 Increasing the length

In this section, we will prove Theorem 9.1. The proof will proceed by showing that resolvent estimates hold for larger and larger intervals, with not too small density. It is noteworthy here that the density, where the estimate holds decreases as the scales get large. The following lemma reformulates (9.2).

Lemma 9.10. *Assume* (9.2) *and* $L_0 \leq N^{q/2}$, *then we have that*

$$\#\{I \subseteq [1,N]: \quad |I| = L_0 \quad \textit{does not have } \gamma_0\textit{-decay}\} \leq d_0 \frac{N}{L_0}. \tag{9.26}$$

Proof. This follows by comparing definitions. □

We also bring (9.1) into a more convenient form.

Lemma 9.11. *Let $\Lambda \subseteq [1, N]$ with $L_0 \leq |\Lambda| \leq L_{j+1}$, we then have*

$$\|(H_\Lambda)^{-1}\| \leq \frac{1}{3} e^{\frac{1}{4}\gamma_j L_j}. \tag{9.27}$$

Proof. We have $L_{j+1} \leq (L_j)^\alpha = (L_j)^{\frac{1}{\rho}} \cdot (L_j)^{\alpha - \frac{1}{\rho}}$. Thus

$$(L_{j+1})^\rho \leq L_j \cdot (L_j)^{\alpha\rho - 1} \leq L_j \cdot N^{\alpha\rho-1} \leq \frac{\gamma_0}{10} L_j \leq \frac{\gamma_j}{5} L_j$$

by Lemma 9.6 and (9.22). The claim now follows by (9.1) and that $\gamma_j L_j \geq 20\log(3)$. □

We will also need the following

Definition 9.12. *Let $J \subseteq [1, N]$ be an interval of length L_{j+1}. J is called σ-bad, if*

$$\#\{\textit{disjoint length } L_j \textit{ intervals in } J \textit{ without } \gamma_j \textit{ decay}\} \geq \sigma \frac{L_{j+1}}{L_j}. \tag{9.28}$$

Otherwise J is called σ-good.

This definition is motivated by the following version of Proposition 9.4.

Proposition 9.13. *Let $J \subseteq [1, N]$ be an interval of length $|J| = L_{j+1}$. Assume that J is σ-good, then J has γ_{j+1}-decay.*

Proof. We observe that the previous lemma and definition ensure the conditions of Proposition 9.4 with $M = L_{j+1}$, $L = L_j$, and $\gamma = \gamma_j$. Hence, the claim follows. □

We have the following

Lemma 9.14. *Assume that*

(9.29) $$\#\{\textit{disjoint length } L_j \textit{ intervals in } [1,N] \textit{ without } \gamma_j \textit{ decay}\} \le d_j \frac{N}{L_j}.$$

Then

(9.30)

$$\#\{\textit{disjoint length } L_{j+1} \textit{ intervals in } [1,N] \textit{ without } \gamma_{j+1} \textit{ decay}\} \le d_{j+1} \frac{N}{L_{j+1}}.$$

Proof. By the previous proposition, it suffices to show that

$$\#\{\text{disjoint } \sigma\text{-bad intervals in } [1,N] \text{ of length } L_{j+1}\} \le \frac{d_j}{\sigma} \frac{N}{L_{j+1}}.$$

Assume the converse and conclude that we have more then

$$\frac{d_{j+1}}{\sigma} \frac{N}{L_{j+1}} \cdot \sigma \frac{L_{j+1}}{L_j} = d_j \frac{N}{L_j}$$

intervals of length L_j in $[1, N]$. This is a contradiction finishing the proof. □

We finally note

Proposition 9.15. *We have that* $[1, N]$ *has* $\gamma_{j_{max}}$*-decay.*

Proof. This is again an application of Proposition 9.4, which is possible by (9.25). □

Proof of Theorem 9.1. The decay condition follows from the previous proposition. The condition on the norm of $(H_{[1,N]})^{-1}$ holds by assumption (9.1). $\square$

CHAPTER 10

Distribution of exponential sums

In this chapter, we will discuss some results about exponential sums. These will be used in the next chapter to derive the uniform recurrence results for the skew-shift. For $c > 0$, we write $\alpha \in DC(c)$ if for all integers $a, n \in \mathbb{Z}$, we have

$$\left|\alpha - \frac{a}{n}\right| \geq \frac{c}{n^3}. \tag{10.1}$$

Furthermore, we denote $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$. The main result is

Theorem 10.1. *Let $K \geq 1$ and define*

$$p = \frac{1}{15K^2 \log(K)}. \tag{10.2}$$

Let $P(n)$ be a polynomial of degree $1 \leq k \leq K$ with leading coefficient $\alpha \in$

$DC(c)$, then for $L \geq 1$

$$\left|\sum_{n=1}^{L} e(P(n))\right| \leq \frac{W_0}{c} L^{1-p}, \tag{10.3}$$

where $W_0 = W_0(K) > 0$ is a constant.

The rest of this chapter contains the derivation of this theorem from the literature. It can be skipped for people only interested in understanding the multiscale procedure. Most of the results will be taken from Montgomery's lecture notes [33]. We will provide some details to make the dependence on the number $c > 0$ in the Diophantine condition explicit, since we will need it in the next chapter.

We will comment on the optimality of the results derived here after Theorem 10.10. The essential situation is that, one has that the exponent p in Theorem 10.1 must satisfy $p \gtrsim \frac{1}{K^2}$.

10.1 Diophantine numbers

The goal of this section is to discuss how well a real number α can be approximated by rationals with small denominator. As usual, we denote

$$\|x\| = \operatorname{dist}(x, \mathbb{Z}). \tag{10.4}$$

We begin with a simple observation. Let $\alpha \in \mathbb{R}$ and n be a nonzero integer. Then $\|n\alpha\| \leq \varepsilon$ implies that there exists an integer a such that

$$(10.5) \qquad \left|\alpha - \frac{a}{n}\right| \leq \frac{\varepsilon}{n}.$$

We now come to Dirichlet's theorem.

Theorem 10.2. *Let $\alpha \in \mathbb{R}$ and $N \geq 1$. Then there exists $1 \leq q \leq N$ such that*

$$(10.6) \qquad \|q\alpha\| \leq \frac{1}{N}.$$

This theorem is a consequence of the pigeonhole principle, and can for example be found as Theorem 4.1 in Nathanson's book [34]. Our goal in the following will be to impose a restriction on α so that the above q are not too small. For $c > 0$, we write $\alpha \in DC(c)$, for Diophantine condition, if

$$(10.7) \qquad \|\alpha n\| \geq \frac{c}{n^2}$$

for all integers $n \neq 0$. Since the series $\sum_{n\geq 1} \frac{1}{n^2}$ converges, one can show that $\bigcup_{c>0} DC(c)$ has full measure. Let me remark that the meaning of (10.7) is that there are no integer solutions a and $n \geq 1$ of

$$(10.8) \qquad \left|\alpha - \frac{a}{n}\right| \leq \frac{c}{n^3}.$$

So α is badly approximable by rationals. The first lemma studies the stability of this condition under multiplication of α by an integer.

Lemma 10.3. *Assume $\alpha \in DC(c)$. Then for $1 \leq m \leq M$, we have $m\alpha \in DC(\frac{c}{M^2})$.*

Proof. Compute $\|n \cdot m\alpha\| \geq \frac{c}{(mn)^2} \geq \frac{cM^{-2}}{n^2}$. The claim follows. $\square$

We will need the following variant of Dirichlet's Theorem for Diophantine numbers.

Lemma 10.4. *Let $\alpha \in DC(c)$ and $N \geq 1$. Then there exists $\sqrt{cN} \leq q \leq N$ such that*

$$\|q\alpha\| \leq \frac{1}{N}. \tag{10.9}$$

Proof. The existence of $1 \leq q \leq N$ with (10.9) follows from Theorem 10.2. By $\alpha \in DC(c)$, we have that

$$\frac{c}{q^2} \leq \|q\alpha\| \leq \frac{1}{N}.$$

This implies $q^2 \geq cN$. $\square$

10.2 The cases $K = 1$ and $K = 2$

In this section, we will present the results for polynomials of low degree. We begin with the case of linear polynomials.

Theorem 10.5. *Assume $\alpha \in DC(c)$. Then for $P(n) = \alpha n + \beta$, we have*

$$\left|\sum_{n=1}^{L} e(P(n))\right| \leq \frac{2}{\pi}\frac{1}{c}. \tag{10.10}$$

Proof. Since $e(\alpha n + \beta) = e(\beta) \cdot e(\alpha)^n$, we may compute

$$\sum_{n=1}^{N} e(P(n)) = e(\beta)e(\alpha)\frac{e(N\alpha) - 1}{e(\alpha) - 1} = e(\beta)e(\frac{N+1}{2}\alpha)\frac{\sin(\pi N\alpha)}{\sin(\pi\alpha)}.$$

From this, we obtain the upper bound $\left|\sum_{n=1}^{L} e(P(n))\right| \leq \frac{1}{|\sin(\pi\alpha)|}$. Now, use that $\sin(x) \geq \frac{1}{2}x$ for $0 < x < \pi/2$ to conclude the claim. □

We also state the result for quadratic polynomials.

Theorem 10.6. *There exists a constant V_0 such that for $\alpha \in DC(c)$, we have for a polynomial $P(n) = \alpha n^2 + \beta n + \gamma$ that*

$$\left|\sum_{n=1}^{L} e(P(n))\right| \leq V_0\sqrt{L}\left(\frac{1}{\sqrt{c}} + 2\sqrt{\log(L)}\right). \tag{10.11}$$

Proof. This follows from Theorem 2.2. from [33] combined with Lemma 10.4. □

10.3 Vinogradov's Method

In this section, we will treat polynomials of degree ≥ 3. The method used here was originally developed by Vinogradov. Introduce for $\underline{\alpha} \in [0,1]^K$ and

$L \geq 1$

$$(10.12) \qquad f(\underline{\alpha}, L) = \sum_{n=1}^{L} e(\alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_K n^K).$$

Introduce

$$(10.13) \qquad J_{b,K}(L) = \int_{[0,1]^K} |f(\underline{\alpha}, L)|^{2b} d\underline{\alpha}.$$

We will need Theorem 4.2. from [33]. This result is known as Vinogradov's mean value theorem.

Theorem 10.7. *Let $K, b \geq 1$ be integers. There exists a constant $V_1 = V_1(b, K) > 0$, such that*

$$(10.14) \qquad J_{b,K}(L) \leq V_1 \cdot L^{2b-(1-\delta)\frac{1}{2}k(k+1)}$$

where $\delta = \mathrm{e}^{-b/k^2}$.

And also Theorem 4.4. from [33].

Theorem 10.8. *There is an universal constant $V_2 > 0$. Let $P(n) = \sum_{k=0}^{K} \alpha_k n^k$. Assume that there is q such that*

$$(10.15) \qquad \|\alpha_K q\| \leq \frac{1}{q}$$

then for $b \geq 1$

$$\left|\sum_{n=1}^{L} e(P(n))\right| \leq V_2(b^k k)^{\frac{1}{2b}} \cdot L \cdot \left(\frac{J_{b,K-1}(3L)}{L^{2b-K(K-1)/2}}\right)^{\frac{1}{2b}}$$

$$\cdot \left(\frac{1}{q} + \frac{\log(q)}{L} + \frac{q\log(q)}{L^K}\right)^{\frac{1}{2b}}. \tag{10.16}$$

Next, we show

Lemma 10.9. *Let* $K \geq 3$ *and* $0 < c < \frac{1}{2}$, $\alpha \in DC(c)$, *and* $L \geq 3$. *Then there exists* q *such that* $\|q\alpha\| \leq \frac{1}{q}$ *and*

$$\frac{1}{q} + \frac{\log(q)}{L} + \frac{q\log(q)}{L^K} \leq \frac{5\log(L)}{\sqrt{c}}\frac{1}{L}. \tag{10.17}$$

Proof. By Lemma 10.4, we may choose q such that $\sqrt{c}L \leq q \leq L^2$ and $\|q\alpha\| \leq \frac{1}{q}$ hold. The claim follows through a computation. □

Furthermore, we may compute that

$$\left(\frac{J_{b,K-1}(3L)}{L^{2b-K(K-1)/2}}\right)^{\frac{1}{2b}} \leq 3(V_1)^{\frac{1}{2b}} L^{\frac{\delta}{2b}\frac{k(k-1)}{2}}.$$

For the choice $b = \lfloor 3k^2\log(k)\rfloor$, one may compute for $k \geq 3$ that $\delta k(k-1) < \frac{1}{2}$. Hence, we may conclude that

Theorem 10.10. *There exists a constant* $V_3 = V_3(K) > 0$, *such that*

$$\left|\sum_{n=1}^{L} e(P(n))\right| \leq \frac{V_3}{\sqrt{c}} \cdot L^{1-\frac{1}{11k^2\log(k)}}. \tag{10.18}$$

Further applications of the methods of this thesis will require to obtain better control of the recurrence of the skew-shift. The naive first idea to do this is to improve the above theorem. However, this is not easily possible as discussed in the survey of Ford [20]. In particular, it is shown that the best possible bound in Theorem 10.7 is $\delta = 0$. So, we are quite close for the considered region $b \gtrsim k^2 \log(k)$, which means $\delta \lesssim \frac{1}{k}$. Let me furthermore recommend the survey of Ford here as a source for other application of these bounds for exponential sums.

Some slight improvements in particular by making the dependence on K are possible in Theorem 10.7 by taking the results of Wooley from [38]. However, these results are not major improvements.

CHAPTER 11

Return times to the set of non-suitability

The goal of this chapter is to derive Theorem 7.7. The proof will be done in two steps. First, we will derive recurrence results for the skew-shift using the bounds on exponential sums from the last chapter. This will be achieved in Theorem 11.3.

Then, we will study the structure of the set $\mathcal{B}^M_{\gamma,\tau}(H_{\underline{\omega}})$ using semi-algebraic geometry, and reduce questions about recurrence to them to questions about the recurrence to small balls.

11.1 Selberg Polynomials

In this section, we discuss properties of *Selberg polynomials*, which allow us to majorize characteristic functions of small balls in $\mathbb{T}^K$. We will use this to convert the question about the recurrence to a small ball to estimates on exponential sums. Given $A \subseteq \mathbb{T}^K$, we denote by χ_A the characteristic function of A and $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$.

Lemma 11.1. *Let $a \in \mathbb{R}$ and $\varepsilon > 0$. There are $|\hat{f}_{a,\varepsilon}(l)| \leq 5\varepsilon$ and $d \leq 2/\varepsilon$ such that for*

$$f_{a,\varepsilon}(x) = \sum_{l=-d}^{d} \hat{f}_{a,\varepsilon}(l) e(lx), \tag{11.1}$$

we have $f_{a,\varepsilon}(x) \geq \chi_{[a-\varepsilon,a+\varepsilon]}(x)$ for all $x \in \mathbb{R}$.

Proof. This can be achieved by choosing f to be a Selberg polynomial, see (21+) on page 6 in [33]. For the decay of a_l see (22) on page 8 in [33]. □

Denote by $|\underline{x}|_\infty = \sup_{1 \leq k \leq K} |x_k|$ and

$$B_\varepsilon(\underline{a}) = \{\underline{x} \in \mathbb{T}^K : \quad |\underline{x} - \underline{a}|_\infty < \varepsilon\}. \tag{11.2}$$

The following proposition is a multidimensional version of the last lemma.

Proposition 11.2. *Let $\underline{a} \in \mathbb{T}^K$ and $0 < \varepsilon < \frac{1}{10}$. Then there exists a trigonometric polynomial of degree $d \leq 2/\varepsilon$ given by*

$$f(\underline{x}) = \sum_{|\underline{\xi}| \leq d} \hat{f}(\underline{\xi}) e(\underline{\xi} \cdot \underline{x}) \tag{11.3}$$

such that $\chi_{B_\varepsilon(\underline{a})} \leq f$ and $|\hat{f}(\underline{\xi})| \leq (5\varepsilon)^K$.

Proof. Choose $f = \prod_{k=1}^{K} f_{a_k,\varepsilon}$ with the $f_{a_k,\varepsilon}$ as in Lemma 11.1. $\square$

11.2 Probability to be in a small ball

The goal of this section is to prove the following theorem, which bounds the probability to land in a small ball.

Theorem 11.3. *Let $K \geq 1$ and $c > 0$ and define $p = \frac{1}{15K^2 \log K}$. There is a constant $W_1 = W_1(K, c) > 0$, such that for $L \geq 1$, $\frac{1}{K!}\alpha \in DC(c)$, and $\underline{\omega}, \underline{a} \in \mathbb{T}^K$*

$$\begin{aligned}\#\{n = 1, \dots, L : \quad & \|T_\alpha^n \underline{\omega} - \underline{a}\| \leq \varepsilon\} \\ & \leq (5\varepsilon)^K L + \frac{W_1}{\varepsilon^2} L^{1-p}.\end{aligned} \tag{11.4}$$

As in the previous chapter, $DC(c)$ denotes the set of all real numbers α such that

$$\|n\alpha\| \geq \frac{c}{n^2} \tag{11.5}$$

holds for all integers $n \neq 0$. We furthermore remark, that with W_0 is Theorem 10.1, we have that $W_1 = (20)^K \frac{4W_0}{c}$.

We begin the proof by replacing the claim about a ball, by a claim about exponential sums. Denote by $f(\underline{x}) = \sum_{|\underline{\xi}| \leq \frac{2}{\varepsilon}} \hat{f}(\underline{\xi}) e(\underline{\xi} \cdot \underline{x})$, the function obtained from Proposition 11.2. We compute

$$
\begin{aligned}
\#\{n = 1, \dots, L : \quad \|T_\alpha^n \underline{\omega} - \underline{a}\| \le \varepsilon\} &= \sum_{n=1}^{L} \chi_{B_\varepsilon(\underline{a})}(T_\alpha^n \underline{\omega}) \\
&\le \sum_{n=1}^{L} f(T_\alpha^n \underline{\omega}) \le \sum_{n=1}^{L} \sum_{|\xi| \le \frac{2}{\varepsilon}} \left| \hat{f}(\underline{\xi}) e(\underline{\xi} \cdot T_\alpha^n \underline{\omega}) \right| \\
&\le (5\varepsilon)^K \sum_{|\xi| \le \frac{2}{\varepsilon}} \left| \sum_{n=1}^{L} e(\underline{\xi} \cdot T_\alpha^n \underline{\omega}) \right| \\
&= (5\varepsilon)^K L + (5\varepsilon)^K \sum_{0 < |\xi| \le \frac{2}{\varepsilon}} \left| \sum_{n=1}^{L} e(\underline{\xi} \cdot T_\alpha^n \underline{\omega}) \right|
\end{aligned}
\tag{11.6}
$$

In order to control the terms $\left|\sum_{n=1}^{L} e(\underline{\xi} \cdot T_\alpha^n \underline{\omega})\right|$ for $|\underline{\xi}| \le \frac{2}{\varepsilon}$, we will need the following lemma

Lemma 11.4. *Let $0 < |\xi| \le \frac{2}{\varepsilon}$. Denote by $1 \le k \le K$ the number such that $\xi_k \neq 0$ and $\xi_l = 0$ for $l \ge k+1$. Then for $\underline{\omega} \in \mathbb{T}^K$*

$$
\underline{\xi} \cdot T_\alpha^n \underline{\omega} = \frac{\xi_k \alpha}{k!} n^k + \dots, \tag{11.7}
$$

where $\dots$ denotes a polynomial of degree $k-1$.

Proof. This can be shown using induction. □

We can thus apply Theorem 10.1 with $\alpha \in DC(\frac{c\varepsilon^2}{4})$. Hence, we obtain

$$
\left| \sum_{n=1}^{L} e(\underline{\xi} \cdot T_\alpha^n \underline{\omega}) \right| \le \frac{4W_0}{c\varepsilon^2} L^{1-p}. \tag{11.8}
$$

Proof of Theorem 11.3. There are less then $(4/\varepsilon)^K$ many $\underline{\xi}$ such that $|\underline{\xi}| \leq 2/\varepsilon$. Hence

$$(5\varepsilon)^K \sum_{0<|\xi|\leq\frac{2}{\varepsilon}} \left| \sum_{n=1}^{L} e(\underline{\xi} \cdot T_\alpha^n \underline{\omega}) \right| \leq (20)^K \frac{4W_0}{c\varepsilon^2} L^{1-p}.$$

The claim now follows. □

11.3 Semi-algebraic sets

In this section, we introduce the notion of semi-algebraic set and study its properties. This notion is important, since it will allow us to reduce the question if a point is in a semi-algebraic set $\mathcal{S}$ to the question, if that point is in a small ball. More information on semi-algebraic sets can be found in the book of Bochnak, Coste, and Roy [2] and in Chapter 9 of Bourgain's book [6].

We begin with giving the basic definitions. We denote by $\mathbb{R}[X_1, \dots, X_n]$ the set of all polynomials in the n variables $X_1, \dots, X_n$. That is all functions of the form

$$P(X_1, \dots, X_n) = \sum_{0\leq k_1,\dots,k_n\leq d} P_{k_1,\dots,k_n} X_1^{k_1} \cdots X_n^{k_n}, \tag{11.9}$$

where $d \geq 0$ is some integer and $P_{k_1,\dots,k_n}$ are real numbers. We denote the lowest possible choice of d by $\deg(P)$ and call it the degree of the polynomial.

It will be important in the following, that we have

$$\deg(P \cdot Q) \leq \deg(P) + \deg(Q). \tag{11.10}$$

We now come to the definition of a semi-algebraic set $\mathcal{S}$.

Definition 11.5. *A set $\mathcal{S} \subseteq \mathbb{R}^n$ is a semi-algebraic set of degree at most $s \cdot d$, if there exist polynomials*

$$P_1, \ldots, P_s \in \mathbb{R}[X_1, \ldots, X_n] \tag{11.11}$$

whose degree is bounded by d such that

$$\mathcal{S} = \bigcup_j \bigcap_{l \in \mathcal{L}_j} \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \quad P_l(x_1, \ldots, x_n) s_{jl} 0\}, \tag{11.12}$$

where $\mathcal{L}_j \subseteq \{1, \ldots, s\}$ and $s_{jl} \in \{\leq, \geq, =\}$.

Here the degree of the semi-algebraic set is the minimum over all possible choices for s and d. We will write $\deg(\mathcal{S})$ for the degree of $\mathcal{S}$. One can see that the notion of semi-algebraic sets and their degree give nice properties of the underlying set from the result of Milnor [32], which says that the number of components of $\mathcal{S}$ is $O(\deg(S)^C)$, where $C = C(n)$ is an universal constant.

We will need the following result on the structure of semi-algebraic sets due to Gromov [22] and Yodin [39].

Proposition 11.6. *Let $n \geq 1$. There are constants $G_1 > 0$ and $\varepsilon_0 > 0$. Let*

$\mathcal{S} \subseteq [0,1]^n$ be a semi-algebraic set of degree B, and $0 < \varepsilon < \varepsilon_0$. If

$$|\mathcal{S}| \leq \varepsilon^n, \tag{11.13}$$

then we can cover $\mathcal{S}$ by less then

$$B^{G_1} \left(\frac{1}{\varepsilon}\right)^{n-1} \tag{11.14}$$

many ε balls.

Proof. For the proof use Lemma 3.3. in [22] and then follow Bourgain as in Corollary 9.6. [6]. □

11.4 Return times to semi-algebraic sets

In this section, we will prove the following result, which is the extension of Theorem 11.3 to semi-algebraic sets. We recall that $p = \frac{1}{15K^2 \log(K)}$.

Theorem 11.7. *Let $K \geq 1$, $c > 0$ and define $q = \frac{1}{60K^3 \log(K)}$. There are constants $W_2 = W_2(c, K), W_3 = W_3(c, K) > 0$. Let $\mathcal{S} \subseteq \mathbb{T}^K$ be a semi-algebraic set of degree B. Assume L satisfies*

$$W_2 \cdot B^{(K+2)G_1} \leq L^{p/2} \leq W_3 \cdot \frac{1}{|\mathcal{S}|} \frac{1}{B^{K \cdot G_1}}. \tag{11.15}$$

Then for $\underline{\omega} \in \mathbb{T}^K$ *and* $\frac{1}{K!}\alpha \in DC(c)$

$$\#\{n = 1, \dots, L : \quad T_\alpha^n \underline{\omega} \in \mathcal{S}\} \leq L^{1-q}. \tag{11.16}$$

In the above theorem, we have that

$$W_2 = 4 \cdot (10)^K \cdot 5^{K^2} \cdot W_1, \quad W_3 = \frac{1}{2^K 5^{K^2}}. \tag{11.17}$$

Furthermore G_1 is the constant from Proposition 11.6. We will now prove this result using Theorem 11.3 and Proposition 11.6. In the following, we choose $r > 0$ such that

$$|\mathcal{S}| \leq r^K, \quad \deg(\mathcal{S}) \leq B \tag{11.18}$$

hold. The assumption on the degree holds by assumption, and for the other one, we need $r \geq |\mathcal{S}|^{1/K}$. We will begin with the following lemma

Lemma 11.8. *Assume that*

$$\left(\frac{2W_1 B^{G_1}}{r^{K+1}}\right)^{1/(p-q)} \leq L \leq \left(\frac{1}{2 \cdot 5^K B^{G_1} r}\right)^{1/q}. \tag{11.19}$$

Then for any $\underline{\omega} \in \mathbb{T}^K$, *we have that*

$$\#\{n : \quad 1 \leq n \leq L : \quad T_\alpha^n \underline{\omega} \in \mathcal{S}\} \leq L^{1-q}. \tag{11.20}$$

Proof. By Proposition 11.6, we may cover $\mathcal{S}$ by $T = B^{G_1} r^{K-1}$ many balls of radius r. Denote these by $B_r(\underline{a}_1), \ldots, B_r(\underline{a}_T)$. By Theorem 11.3, we obtain that for $1 \le t \le T$

$$\#\{n : \quad 1 \le n \le L : \quad T_\alpha^n \underline{\omega} \in B_r(\underline{a}_t)\} \le (5r)^K L + \frac{W_1}{r^2} L^{1-p}.$$

Since $\mathcal{S} \subseteq \bigcup_{t=1}^T B_r(\underline{a}_t)$, we obtain

$$\#\{n : \quad 1 \le n \le L : \quad T_\alpha^n \underline{\omega} \in \mathcal{S}\} \le T \cdot \left((5r)^K L + \frac{W_1}{r^2} L^{1-p} \right).$$

Hence, we see that the claim holds as long as

$$B^{G_1} r^{-K+1} (5r)^K L \le \frac{1}{2} L^{1-q}, \quad \frac{W_1}{r^2} L^{1-p} B^{G_1} r^{-K+1} \le \frac{1}{2} L^{1-q}.$$

These inequalities are implied by (11.19) as one checks through a computation. $\square$

Proof of Theorem 11.7. Choose r as

$$r = \frac{1}{2 B^{G_1} 5^K L^q}$$

such that the inequality on the right hand side of (11.19) holds. It thus remains to check that

$$4 \cdot (10)^K \cdot 5^{K^2} W_1 B^{(K+2)G_1} \le L^{p-q(K+2)}.$$

Since $p = \frac{1}{15K^2 \log(K)}$ and $q = \frac{1}{60K^3 \log(K)}$, we have that $L^{p/2} \leq L^{p-(K+2)q}$. This leads to the modified condition

$$4 \cdot (10)^K \cdot 5^{K^2} W_1 B^{(K+2)G_1} \leq L^{p/2}.$$

This is the right hand side of (11.15). We furthermore, recall that we need to require $r \geq |\mathcal{S}|^{1/K}$. Since $L^{p/2} \leq L^{K \cdot K}$, this leads to the left hand side of (11.15). $\square$

11.5 Semialgebraic structure of suitability

In this section, we will show how the set of suitable operators can be contained in a semi-algebraic set. We will begin by stating the necessary assumptions on the potential $V(n) = f(T_\alpha^n \underline{\omega})$. For $\underline{\xi} \in \mathbb{Z}^K$ introduce

(11.21) $$|\underline{\xi}| = \max_{1 \leq k \leq K} |\xi_k|.$$

We will assume that f is a trigonometric polynomial of degree A, that is

(11.22) $$f(\underline{\omega}) = \sum_{|\underline{\xi}| \leq A} \hat{f}(\underline{\xi}) e(\underline{\xi} \cdot \underline{\omega}).$$

Here $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ and $\underline{\xi} \cdot \underline{\omega} = \sum_{k=1}^K \xi_k \omega_k$ as usual. We now state the main result of this section.

Theorem 11.9. *Let $\gamma > 0$, $0 < \tau < 1$. Denote by $\Omega_1 \subseteq \mathbb{T}^K$ the set where*

$[1, N]$ *is* $(\gamma, \tau, p+2)$*-suitable for* $H(\underline{\omega})$*. For* $N \geq 10$ *large enough. There exists* $\Omega_2 \subseteq \mathbb{T}^K$ *with the following properties*

(i) Ω_2 *is semi-algebraic of degree* $\deg(\Omega_2) \leq N^{K+5}$.

(ii) $\Omega_1 \subseteq \Omega_2$.

(iii) We have for $\underline{\omega} \in \Omega_2$ *that* $[1, N]$ *is* (γ, τ, p)*-suitable for* $H(\underline{\omega})$.

Let me begin by a remark.

Remark 11.10. *The assumption that* f *is a polynomial is not really necessary. It would be sufficient to assume the existence of constants* $C, c, \sigma > 0$ *such that*

$$\sum_{|\underline{\xi}| \geq A} |\hat{f}(\underline{\xi})| \leq C \cdot \mathrm{e}^{-cA^{\sigma}} \tag{11.23}$$

for all $A \geq 1$*. The argument would then carry through, if one approximates* f *by a degree* $N^{\frac{1}{\sigma}+1}$ *degree polynomial* f_N *before truncating the exponential sums. Property (i) of Theorem 11.9 would then be replaced by* $\deg(\Omega_2) \leq N^{K+\frac{1}{\sigma}+6}$.

We now begin with the preparations necessary for the proof. Introduce the map $\tilde{T}_\alpha : \mathbb{R}^K \to \mathbb{R}^K$ by

$$(\tilde{T}_\alpha \underline{\omega})_k = \begin{cases} \omega_1 + \alpha, & k = 1; \\ \omega_k + \omega_{k-1}, & 2 \leq k \leq K. \end{cases} \tag{11.24}$$

We then have $(T_\alpha \underline{\omega})_k = (\tilde{T}_\alpha \underline{\omega})_k \pmod 1$. Furthermore, we have

Lemma 11.11. *For $1 \leq k \leq K$, $(\tilde{T}^n_\alpha \underline{\omega})_k$ is a polynomial of degree k and $|(\tilde{T}^n_\alpha \underline{\omega})_k| \leq \mathrm{e} n^k$.*

Proof. It follows using induction that

$$(\tilde{T}^n_\alpha \underline{\omega})_k = \binom{n}{k}\alpha + \sum_{l=0}^{k-1} \binom{n}{l} \omega_{k-l}.$$

Since $\binom{n}{k} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!} \leq \frac{n^k}{k!}$ and $\sum_{l=0}^{\infty} \frac{1}{l!} \leq \mathrm{e}$, the claim follows. □

For $M \geq 1$, introduce the cut-off potential V_M by

$$V_M(n) = \sum_{|\underline{\xi}| \leq A} \sum_{m=0}^{M} \frac{(2\pi \mathrm{i})^m}{m!} \hat{f}(\underline{\xi}) (\underline{\xi} \cdot \tilde{T}^n_\alpha \underline{\omega})^m. \tag{11.25}$$

We have

Lemma 11.12. *Assume $M \geq 4\pi \mathrm{e}^2 A N^K$ then*

$$|V(n) - V_M(n)| \leq \frac{\|\hat{f}\|_{\ell^1(\mathbb{Z}^K)}}{2^M \cdot \mathrm{e}}. \tag{11.26}$$

Furthermore V_M is a polynomial of degree less then M in $\underline{\omega}$.

Proof. We compute

$$V(n) - V_M(n) = \sum_{|\underline{\xi}| \leq A} \hat{f}(\underline{\xi}) \sum_{m=M+1}^{\infty} \frac{(2\pi \mathrm{i})^m}{m!} (\underline{\xi} \cdot \tilde{T}^n_\alpha \underline{\omega})^m.$$

Using that $\frac{1}{m!} \leq \frac{1}{\mathrm{e}} \left(\frac{\mathrm{e}}{m}\right)^m$ and denoting $C = 2\pi A N^K \mathrm{e}^2$, we obtain that

$$\begin{aligned} |V(n) - V_M(n)| &\leq \frac{\|\hat{f}\|_{\ell^1(\mathbb{Z}^K)}}{\mathrm{e}} \sum_{m=M+1}^{\infty} \left(\frac{C}{m}\right)^m \\ &\leq \frac{\|\hat{f}\|_{\ell^1(\mathbb{Z}^K)}}{\mathrm{e}} \frac{1}{1 - C/(M+1)} \left(\frac{C}{M+1}\right)^{M+1}. \end{aligned}$$

The claim follows. $\square$

Define the operator $\tilde{H}^M_{[1,N]}$ as $H_{[1,N]}$ with the role of V replaced by V_M. By the previous lemma, we then have that

$$\|\tilde{H}^M_{[1,N]}(\underline{\omega}) - H_{[1,N]}(\underline{\omega})\| \leq \frac{\|\hat{f}\|_{\ell^1(\mathbb{Z}^K)}}{2^M \cdot \mathrm{e}}. \tag{11.27}$$

Introduce Ω_2 as the set, where $[1, N]$ is $(\gamma, \tau, p+1)$-suitable for $\tilde{H}^M(\underline{\omega})$. We have the following lemma

Lemma 11.13. *Assume*

$$M \geq \frac{\gamma}{2\log(2)} N \tag{11.28}$$

Then

(i) $\Omega_1 \subseteq \Omega_2$.

(ii) For $\underline{\omega} \in \Omega_2$, *we have that* $[1, N]$ *is* (γ, τ, p)*-suitable for* $H(\underline{\omega})$.

Proof. By Lemma 7.17, we need to ensure that

$$\frac{\|\hat{f}\|_{\ell^1(\mathbb{Z}^K)}}{2^M \cdot \mathrm{e}} \leq \frac{1}{2^{p+4}} \mathrm{e}^{-\gamma N}.$$

This can be achieved as long as (11.28) holds and N is large enough. □

We now come to

Proof of Theorem 11.9. First observe that Definition 7.3 involves less then N^2 polynomial inequalities involving sums of elements $\langle e_x, (\tilde{H}^M_{[1,N]}(\underline{\omega}))^{-1} e_y \rangle$. By Lemma 3.8, we can write these elements as ratios of determinants. These are polynomials of degree $\leq 2N$ in the values $V(n)$ of the potential. Hence, we can conclude that

$$\deg(\Omega_2) \leq 2N^3 M.$$

Because of the restrictions imposed in Lemma 11.12 and (11.28), we can choose $N^K \lesssim M \leq \frac{1}{2} N^{K+1}$ for N large enough. The claim now follows. □

11.6 Proof of Theorem 7.9

In this section, we will prove Theorem 7.9. We begin with the following lemma

Lemma 11.14. *There exists a semi-algebraic set $\mathcal{S}$ of degree $\deg(\mathcal{S}) \leq M^{K+5}$, such that for $\underline{\omega} \notin \mathcal{S}$, we have that $[1, M]$ is (γ, τ)-suitable for $H_{\underline{\omega}}$.*

Proof. Introduce Ω_1 as the set of all $\underline{\omega}$ such that $[1, M]$ is $(\gamma, \tau, 3)$-suitable for $H_{\underline{\omega}}$. By (7.29), we have $|\Omega_1| \geq 1 - \mathrm{e}^{-M^{\mu}}$. By Theorem 11.9, we can find a semi-algebraic set $\Omega_2 \supseteq \Omega_1$ of degree $\leq M^{K+3}$ such that for $\underline{\omega} \in \Omega_2$, $[1, M]$ is (γ, τ)-suitable for $H_{\underline{\omega}}$. We let $\mathcal{S} = \mathbb{T}^K \setminus \Omega_2$ and the claim follows. $\square$

We now come to

Proof of Theorem 7.9. This follows by applying Theorem 11.7 to the set $\mathcal{S}$ from the previous lemma. $\square$

By (11.15), one can interfere the following smallness condition on $\mathcal{S}$:

$$|\mathcal{S}| \leq \frac{W_3}{N^{G_2}} \cdot \frac{1}{L^{p/2}}, \tag{11.29}$$

where $G_2 = G_2(K) = 2 \cdot G_1 \cdot K^2$ (with G_1 as in Proposition 11.6) and $p = \frac{1}{15K^2 \log(K)}$. This is exactly (7.29).

CHAPTER 12

Cartan's Lemma and Consequences

In this chapter, we will provide the mechanism on how to improve probabilities. In particular, we will prove Theorem 7.11. The main ingredient will be Cartan's lemma, which tells us that the set, where an appropriately normalized subharmonic function vanishes, has small measure.

Theorems of the type of the main result of this chapter have first appeared in the work of Bourgain, Goldstein, and Schlag [11] on Anderson localization on the lattice $\mathbb{Z}^2$. They were then improved by Bourgain in [5], [6], [8], and [9].

12.1 Cartan's Lemma

The following result is known as Cartan's lemma. It provides the basic idea of the result, although we will need a multidimensional version, which we discuss below.

Lemma 12.1. *Let* $\varphi : \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$ *be a subharmonic function satisfying*

$$\sup_{|z|<1} \varphi(z) < 1, \quad \varphi(0) > -1. \tag{12.1}$$

Then there exists a constant $C_{\mathrm{Cartan}} > 0$ *such that*

$$|\{y \in [-\frac{1}{2}, \frac{1}{2}] : \quad |\varphi(y)| > \lambda\}| \leq \mathrm{e}^{-C_{\mathrm{Cartan}}\lambda}. \tag{12.2}$$

Proof. The proof is part of Section 1 of [5]. See also Section 11.2 and 11.3 in Levin's book [30]. □

We will now discuss the main result of the paper [35] by Nazarov, Sodin, and Volberg. It proves a dimension independent statement of Cartan's lemma in $\mathbb{C}^K$.

Introduce for $\underline{z} \in \mathbb{C}^K$ the norm

$$|\underline{z}|_2 = \left(\sum_{k=1}^{K} |z_k|^2\right)^{1/2}. \tag{12.3}$$

We will use similar for vectors in $\mathbb{R}^K$, $\mathbb{T}^K$ and so on. Let $B = \{\underline{z} \in \mathbb{C}^K :$

$|z|_2 < 1\}$ be the unit ball and $f : B \to \mathbb{C}$ an analytic function. Define the degree d_f by

$$d_f = \log\left(\frac{1}{|f(0)|}\sup_{|\underline{z}|_2<1}|f(\underline{z})|\right), \tag{12.4}$$

$C = 64$, and $\sigma = 384 \cdot d_f$. Furthermore, define the number $M(f)$ by

$$|\{|\underline{\omega}|_2 \le \frac{1}{2} : \quad |f(\underline{\omega})| \ge M(f)\}| = \frac{1}{\mathrm{e}}. \tag{12.5}$$

From [35], we know that

$$M(f) \le (\mathrm{e}C)^{\sigma}\|f\|_{L^1([-\frac{1}{2},\frac{1}{2}]^K)}. \tag{12.6}$$

Theorem 1 in [35] states that

$$|\{|\underline{\omega}|_2 \le \frac{1}{2} : \quad |f(\underline{\omega})| \le \frac{M(f)}{(C\lambda)^{\sigma}}\}| \le \frac{1}{\lambda}. \tag{12.7}$$

Combining these things, we obtain

Theorem 12.2. *Let $f : B \to \mathbb{C}$ be an analytic function and set $\varphi(\underline{z}) = \log(|f(\underline{z})|)$. Assume that*

$$\varphi(0) \ge -1, \quad \sup_{|\underline{z}|_2<1}\varphi(\underline{z}) < 1. \tag{12.8}$$

We have for $\lambda > 1$

$$(12.9) \qquad |\{|\underline{\omega}|_2 \leq \frac{1}{2} : \quad |\varphi(\underline{\omega})| \geq \lambda\}| \leq \mathrm{e}^{-\frac{1}{768}\lambda}.$$

Proof. Follows from the previous discussion, noting that $d_f \leq 2$ and $\|f\|_{L^1} \leq$ e. $\square$

In particular, we see that a possible choice for C_{Cartan} is $C_{Cartan} = \frac{1}{768}$. Let me furthermore remark that in [7], Bourgain shows how to derive a version of this theorem from Lemma 12.1, where one has to replace $\mathrm{e}^{-C\lambda}$ by $\mathrm{e}^{-C\lambda^{1/K}}$ on the right hand side.

12.2 A matrix valued Cartan Theorem

In this section, we will show a variant of Cartan's lemma for matrix valued functions. In addition to imposing conditions (iii) and (iv), which are similar to the ones of Cartan's lemma, we will also impose a condition on submatrices being nice. This will allow us to obtain better estimates. Similar results can be found in Bourgain's book [6] in Chapter 14. The proof here largely parallels Bourgain's argument.

Theorem 12.3. *Let* $\gamma > 0$, $0 < \tau < 1$. *Let* $0 < \kappa < 1$ *and* $\rho > 0$ *satisfy*

$$(12.10) \qquad \kappa + 2\rho < 1.$$

Let $\underline{\omega}_0 \in \mathbb{T}^K$ and $r > 0$. Assume the inequalities

$$M^\tau \geq \max(6, \log(N)), \quad \gamma M^{1-\tau} \geq 6, \quad N^\rho \geq 24 M^2. \tag{12.11}$$

Furthermore, assume the following conditions.

(i) There exists a set $\Xi \subseteq [1, N]$ containing less than MN^κ elements.

(ii) Let $I \subseteq [1, N]$ be an interval of length M satisfying $I \cap \Xi = \emptyset$. For $|\underline{z} - \underline{\omega}_0|_2 \leq 2r$, we have that I is (γ, τ)-suitable for $H(\underline{z})$.

(iii) For $|\underline{z} - \underline{\omega}_0|_2 \leq 2r$, we have

$$\|H_{[1,N]}(\underline{z})\| \leq \mathrm{e}^{4M^\tau}. \tag{12.12}$$

(iv) We have

$$\|H_{[1,N]}(\underline{\omega}_0)^{-1}\| \leq \mathrm{e}^{\frac{1}{3M} N^\rho}. \tag{12.13}$$

Then

$$|\{|\underline{\omega} - \underline{\omega}_0|_2 \leq r : \quad \|H_{[1,N]}(\underline{\omega})^{-1}\| \geq \mathrm{e}^{N^{\kappa + 2\rho}}\}| \leq r^K \mathrm{e}^{-\frac{1}{2000} N^\rho}. \tag{12.14}$$

The proof of this theorem will take the remainder of this section. It will culminate in us being able to choose a good function φ in (12.22) to apply Cartan's Lemma to. This φ will be the determinant of the Schur complement, to which the inversion can be reduced by our suitability assumption. For this, we will first have to study the set J, where it holds.

We let J be the union of all length M intervals in $[1, N]$ disjoint from Ξ. We note that any interval $I \subseteq J$ of length $|I| = M$ will be (γ, τ)-suitable. We first compute

Lemma 12.4. *We have*

$$|J^c| \leq 3MN^\kappa. \tag{12.15}$$

Proof. Compute $|J| \geq N - 3MN^\kappa$. $\square$

By assumption (ii) and Theorem 8.1, we have

Lemma 12.5. *Assume that $M^\tau \geq \max(6, \log(N))$ and $\gamma M^{1-\tau} \geq 6$. For $\underline{z}$ satisfying $|\underline{z} - \underline{\omega}_0|_2 \leq 2r$, we have that*

$$\|(H_J)^{-1}(\underline{z})\| \leq \mathrm{e}^{4M^\tau}. \tag{12.16}$$

We will need the Schur complement formula

Lemma 12.6 (Schur complement). *Assume A is invertible. Then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{12.17}$$

is invertible, if and only if

$$S = D - CA^{-1}B \tag{12.18}$$

is invertible. Furthermore then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}. \tag{12.19}$$

We apply the Schur complement formula with

$$A = H_J, \quad D = H_{J^c}, \quad B = C^* \tag{12.20}$$

where we have with $J = \bigcup_s [a_s, b_s]$

$$C = \sum_s (\langle e_{a_s}, . \rangle e_{a_s - 1} + \langle e_{b_s}, . \rangle e_{b_s + 1}) \tag{12.21}$$

We remark for further reference that $\|B\| \leq 1$. Define

$$\varphi(\underline{z}) = \log|\det(S(\underline{z}))| \tag{12.22}$$

We record that

Lemma 12.7. *Assume that $8M \leq N^\rho$. Then for $|\underline{\omega} - \underline{\omega}_0|_2 < 2r$*

$$\|H_\Lambda(\underline{\omega})^{-1}\| \leq \mathrm{e}^{|\varphi(\underline{\omega})|} \mathrm{e}^{\frac{1}{2}N^{\kappa+2\rho}}. \tag{12.23}$$

Proof. By the Schur complement formula, we compute

$$\|H_\Lambda(\underline{\omega})^{-1}\| \leq 8(1 + \|S^{-1}\|)(1 + \|H_J^{-1}\|)^2.$$

By Lemma 12.5, we compute

$$8(1 + \|H_J^{-1}\|)^2 \leq 32\mathrm{e}^{8M^\tau} \leq \mathrm{e}^{\frac{1}{4}N^{\kappa+2\rho}}.$$

Next, we observe that we may bound $\|S\| \leq 5\|H_J^{-1}\| \leq 5\mathrm{e}^{4M^\tau}$. Using this and (12.15), we interfer for any minor $\tilde{S}$ of S that

$$\log|\det(\tilde{S})| \leq (\log(5) + 4M^\tau) \cdot 3MN^\kappa \leq 16M^2N^\kappa \leq \frac{1}{2}N^{\kappa+2\rho}.$$

Hence, we may conclude

$$(1 + \|S^{-1}\|) \leq \frac{3MN^\kappa}{|\det(S)|}\mathrm{e}^{\frac{1}{2}N^{\kappa+2\rho}}.$$

The claim now follows. □

Furthermore, we have that

Lemma 12.8. *Assume* $24M^2 \leq N^\rho$. *We have for* $|\underline{z} - \underline{\omega}_0|_2 < 2r$ *that*

$$|\varphi(\underline{z})| \leq N^{\kappa+\rho}. \tag{12.24}$$

Proof. Denote by λ_j the eigenvalues of S. By Lemma 12.5 and (12.12), we have that

$$|\lambda_j| \leq \|H(\underline{z})\| + \|H_J(\underline{z})^{-1}\| \leq 2\mathrm{e}^{4M^\tau}.$$

Hence, we conclude using (12.15) that

$$\varphi(z) = \sum_j |\log |\lambda_j|| \leq 24M^2N^\kappa.$$

This finishes the proof. $\square$

Lemma 12.9. *We have that*

$$|\varphi(\underline{\omega}_0)| \geq -N^{\kappa+\rho} \tag{12.25}$$

Proof. We observe that $\|S(\underline{\omega}_0)^{-1}\| \leq \|H_{[1,N]}(\underline{\omega}_0)^{-1}\| \leq \mathrm{e}^{\frac{1}{3M}N^\rho}$ by (12.13). By (12.15) with λ_j the eigenvalues of $S(\omega_0)$

$$\varphi(\omega_0) = -\sum_j |\log \frac{1}{|\lambda_j|}| \geq -3MN^\kappa \inf_j |\log \frac{1}{|\lambda_j|}|.$$

Now use that $\frac{1}{|\lambda_j|} \leq \|S(\underline{\omega}_0)^{-1}\|$ to conclude the result. $\square$

Proof of Theorem 12.3. Define for $|\underline{z}|_2 < 1$

$$\tilde{\varphi}(\underline{z}) = \frac{1}{N^{\kappa+\rho}}\varphi\left(2r\underline{z} + \underline{\omega}_0\right).$$

The previous two lemmas imply that $\tilde{\varphi}(0) > -1$ and $\sup_{|\underline{z}|_2<1} \tilde{\varphi}(\underline{z}) < 1$. We may thus apply Theorem 12.2 to $\tilde{\varphi}$ with

$$\lambda = \frac{1}{2} N^{\rho}.$$

Hence, we obtain a set $\widetilde{\Omega} \subseteq \{\underline{z} : \quad |\underline{z}|_2 < \frac{1}{2}\}$ of measure $\widetilde{\Omega} \leq \mathrm{e}^{-\frac{1}{1536} N^{\rho}}$ such that $\tilde{\varphi}(\underline{\omega}) \geq -\frac{1}{2} N^{\rho}$ on it. Rewriting this in terms of φ, we obtain

$$|\varphi(\underline{\omega})| \leq \frac{1}{2} N^{\kappa + 2\rho}.$$

The claim now follows by (12.23) and a change of variables. □

12.3 Proof of Theorem 7.11

Our first task to prove Theorem 7.11 will be to extend condition (7.36) from a single $\underline{\omega}$ to a small neighborhood. Introduce

$$\Xi(\underline{\omega}, p) = \{\Lambda \subseteq [1, L] : \quad \Lambda \in \mathcal{B}^{M}_{\gamma,\tau,p}(H_{\underline{\omega}})\}. \tag{12.26}$$

By (7.36), we have $\#\Xi(\underline{\omega}, 1) \leq L^{1-q}$. We will now show that for $\underline{z} \in \mathbb{C}^d$ with $|\underline{z} - \underline{\omega}|_2$ small enough, we have

$$\Xi(\underline{z}, 0) \subseteq \Xi(\underline{\omega}, 1).$$

In the following, we let $|\mathrm{Im}(\underline{z})| = \max_{1 \le k \le K} |\mathrm{Im}(z_k)|$. We begin with the following lemma

Lemma 12.10. *Assume for $\Lambda \subseteq [1, N]$ and $|\mathrm{Im}(\underline{z})| \le \rho$*

$$\|H_\Lambda(\underline{z})\| \le C_1 \mathrm{e}^{C_2 \rho N^K}. \tag{12.27}$$

Let $|\mathrm{Im}(\underline{z})| \le \frac{1}{N^K}$, then for $1 \le k \le K$ we have

$$\|\frac{\partial}{\partial z_k} H_\Lambda(\underline{z})\| \le C_3 N^{2K}, \quad C_3 = C_1 \mathrm{e}^{2C_2}. \tag{12.28}$$

Proof. Abbreviate $f(\underline{z}) = H_\Lambda(\underline{z})$. Let $f(z) = f(z_1, \dots, z_{k-1}, z, z_{k+1}, \dots, z_K)$. Then f is an analytic function, satisfying for $|\mathrm{Im}(z)| \le \frac{2}{N^K}$ that

$$|f(z)| \le C_1 \mathrm{e}^{2C_2}.$$

By Cauchy's integral formula

$$f'(z_k) = -\frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_k| = \frac{1}{N^K}} \frac{f(\zeta)}{(\zeta - z_k)^2} d\zeta.$$

The claim follows. $\square$

Next, we discuss how stable the assumption of being (γ, τ)-suitable is for a length M interval in $[1, N]$. We will later apply the proposition with $p = 1$.

Lemma 12.11. *Let $p \ge 1$, $\gamma, \tau > 0$. Let $\Lambda \subseteq [1, N]$ be a length M interval,* *that is (γ, τ, p)-suitable for $H(\omega_0)$. Assume*

$$\gamma M^{1-\tau} \geq 2, \quad \gamma M \geq \max(K \log(N), (p+1)\log(2) + \log(C_3)) \tag{12.29}$$

and

$$|\underline{\omega}_0 - \underline{z}|_2 \leq \mathrm{e}^{-4\gamma M}. \tag{12.30}$$

Then Λ is $(\gamma, \tau, p-1)$-suitable for $H(\underline{z})$.

Proof. The previous lemma implies that

$$\|H(\underline{\omega}_0) - H(\underline{z})\| \leq C_3 N^{2K} |\underline{z} - \underline{\omega}_0|_2.$$

We note that $N^{2K} \leq \mathrm{e}^{2\gamma M}$ and $2^{p+2} C_3 \mathrm{e}^{\gamma M}$ by assumption. Thus

$$\|H(\underline{\omega}_0) - H(\underline{z})\| \leq \frac{1}{2^{p+2}} \mathrm{e}^{-\gamma M}.$$

The claim now follows by Lemma 7.17. $\square$

We collect the consequences of this lemma, in the next proposition.

Proposition 12.12. *Let $\underline{\omega}_0 \in \mathbb{T}^K$ and $r = \frac{1}{2}\mathrm{e}^{-4\gamma M}$. There exists a set Ξ such that assumptions (i) and (ii) of Theorem 12.3 hold with $\kappa = 1 - q$ and $N = L$.*

Proof. We first observe that the previous lemma implies that for $|\underline{z} - \underline{\omega}_0|_2 < r$, we have $\Xi(\underline{z}, 0) \subseteq \Xi(\underline{\omega}_0, 1)$. Choose

$$\Xi = \bigcup_{\Lambda \in \Xi(\underline{\omega}_0, 1)} \Lambda.$$

We have that $\#\Xi \leq M \cdot \#\Xi(\underline{\omega}_0, 1)$. The claim follows. □

Lemma 12.13. *Assume* (12.27) *and* $\gamma M \geq K \log(L)$, *then condition (iii) of Theorem 12.3 holds for* $r = \frac{1}{2} \mathrm{e}^{-4\gamma M}$, $N = L$, *and* M *large enough.*

Proof. A quick computation shows that $|H_{[1,L]}(\underline{z})| \leq C_1 \mathrm{e}^{\frac{1}{2} C_2}$ as long as $|\mathrm{Im}(\underline{z})| \leq r$. The claim follows as long as $4M^{\tau} \geq \log(C_1) + \frac{1}{2} C_2$. □

It now only remains to ensure condition (iv) of Theorem 12.3. We will need the Vitali covering lemma. It can be found for example as Lemma 4.8. in Falconer's book [19].

Lemma 12.14. *Let* $\mathcal{C}$ *be a family of balls contained in some bounded region of* $\mathbb{R}^n$. *Then there is a (finite or countable) disjoint subcollection* $\{B_i\}$ *such that*

$$\bigcup_{B \in \mathcal{C}} B \subseteq \bigcup_i \tilde{B}_i \tag{12.31}$$

where $\tilde{B}_i$ *is the closed ball concentric with* B_i *and of four times the radius.*

We denote in the following

$$B_r(\underline{\omega}_0) = \{\underline{\omega} \in \mathbb{R}^K : \quad |\underline{\omega} - \underline{\omega}_0|_2 < r\}. \tag{12.32}$$

Furthermore, we recall that in $\mathbb{R}^K$, we have that

$$|\{\underline{x} \in \mathbb{R}^K : \quad \sum_{k=1}^{K} |x_k|^2 \leq 1\}| = \frac{\pi^{K/2}}{\Gamma(\frac{K}{2}+1)}. \tag{12.33}$$

We now come to

Lemma 12.15. *Let $r = \frac{1}{2}\mathrm{e}^{-4\gamma M}$ as in Lemma 12.11 and assume* (7.37). *Let $L \geq M$ and assume $\gamma M \geq 10\log(K)$. There exists a set $\Omega_0 \subseteq \mathbb{T}^K$ with the following properties:*

(i) The cardinality of Ω_0 is bounded

$$\#\Omega_0 \leq \frac{\pi^{K/2} \cdot 4^K}{\Gamma(\frac{K}{2}+1)} \mathrm{e}^{4\gamma K M} \tag{12.34}$$

(ii) We have the following covering of $\mathbb{T}^K$

$$\mathbb{T}^K = \bigcup_{\omega_0 \in \Omega_0} B_r(\underline{\omega}_0) \tag{12.35}$$

(iii) For $\underline{\omega}_0 \in \Omega_0$, we have

$$\|H_{[1,L]}(\underline{\omega}_0)^{-1}\| \leq \mathrm{e}^{L^{\frac{9}{3}}}. \tag{12.36}$$

Proof. By (7.37), there exists Ω_1 satisfying $|\Omega_1| \geq 1 - \mathrm{e}^{-5\gamma K M}$ and for $\underline{\omega} \in \Omega_1$, we have

$$\|H_{[1,L]}(\underline{\omega})^{-1}\| \leq \mathrm{e}^{L^{\frac{9}{3}}}$$

Since

$$|B_{\frac{1}{4}r}(\underline{\omega})| = \frac{\pi^{K/2}}{\Gamma(\frac{K}{2}+1)} \frac{1}{4^K} \frac{1}{r^K} > \mathrm{e}^{-5\gamma K M}.$$

we have that $\mathbb{T}^K \setminus \Omega_1$ cannot contain such a ball of radius $\frac{1}{4}r$. Hence, we have that

$$\mathbb{T}^K = \bigcup_{\underline{\omega}\in\Omega_1} B_{\frac{1}{4}r}(\underline{\omega}).$$

We can now obtain the Vitali covering lemma to the collection of balls $\{B_{\frac{1}{4}r}(\underline{\omega})\}_{\underline{\omega}\in\Omega_1}$. Denote these by $B_{\frac{r}{4}}(\underline{\omega}_1), \dots B_{\frac{r}{4}}(\underline{\omega}_T)$. Using that these are disjoint and $\mathbb{T}^K \supseteq \bigcup_{t=1}^T B_{\frac{r}{4}}(\underline{\omega}_t)$, we obtain

$$1 = |\mathbb{T}^K| \geq \left| \bigcup_{t=1}^T B_{\frac{r}{4}}(\underline{\omega}_t) \right| = T \cdot \frac{\pi^{K/2} \cdot 4^K}{\Gamma(\frac{K}{2}+1)} r^K$$

The claim follows. □

In order to prove Theorem 7.11, we will apply Theorem 12.3 with $\kappa = 1-q$ and $\rho = \frac{q}{3}$. We will need the following lemma

Lemma 12.16. *Assume that*

$$L \geq (8000 K \log(K))^{\frac{12}{q}} \tag{12.37}$$

then

$$\frac{\pi^{K/2} \cdot 4^K}{\Gamma(\frac{K}{2}+1)} \cdot \mathrm{e}^{-\frac{1}{2000} L^{\frac{q}{3}}} \leq \mathrm{e}^{-L^{\frac{q}{4}}}. \tag{12.38}$$

Proof. This is a somewhat lengthy computation, where one uses the bound $\log(\Gamma(\frac{K}{2}+1)) \leq 2K\log(K)$. $\square$

We are now ready for

Proof of Theorem 7.11. The previous discussions show that we can apply Theorem 12.3 to each ball $B_r(\underline{\omega})$ for $\omega \in \Omega_1$. We can thus conclude that the claim holds except in an exceptional set of measure

$$\leq \frac{\pi^{K/2} \cdot 4^K}{\Gamma(\frac{K}{2}+1)} \cdot \mathrm{e}^{-\frac{1}{2000}L^{\frac{q}{3}}} \leq \mathrm{e}^{-L^{\frac{q}{4}}}.$$

This finishes the proof. $\square$

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APPENDIX A

Numerics

The eigenvectors were computed with the following *Mathematica* code. I believe that the code is self-explanatory enough, that I do not need to make further comments.

```
size = 30;
lam = 0.9;
alpha = Sqrt[2.]  - 1;
KK = 2;
H0 = DiagonalMatrix[Table[1, {k, 1, size - 1}], 1]
   + DiagonalMatrix[Table[1, {k, 1, size - 1}], -1];
H1 = H0 + DiagonalMatrix[Table[2 lam Cos[2 Pi alpha n^KK],
   {n, 1, size}]];
evs = Eigenvectors[H1];
```

```
Table[ListPlot[Abs[evs[[k]]], PlotRange -> {-0.1, 1}],
   {k, 1, size}]
```