RICE UNIVERSITY

A STUDY OF CONTINUOUS ELASTIC BEAMS ON DISCRETE NON-LINEAR ELASTIC SUPPORTS

by

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Thesis Director's signature:

Murphy H. Thibodeaux

Houston, Texas

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ABSTRACT

This study discusses an iterative approach to the problem of a continuous beam on discrete non-linear elastic supports and its application. The literature studied revealed that practically no work has been done on engineering approaches to this problem.

The basic equations are developed from flexural theory and are solved by iteration utilizing Newton's linear approximation method. The solutions obtained were checked by energy methods.

Numerical results are presented for a marine fendering system, a typical example of a beam on non-linear elastic supports. For various loading conditions, a comparison is made between an approximate linear solution and the solution obtained using the non-linear approach developed in this study.

ACKNOWLEDGMENT

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I
INTRODUCTION

Beams ·. hich are continuous over discrete supports are very commonly used as structural elements. In the majority of design studies rigid foundations are assumed but quite frequently the effect of support flexibility is of primary interest in the design. There are many cases where flexibility is purposely built into a continuous structure in order to reduce the forces transmitted to the supporting structures or to prevent damage to moving structures generating these forces. For example, marine fendering devices purposely employ beam supports which are selected on the basis of their elastic characteristics.

The case of a linear elastic beam on non-linear elastic supports is of some interest analytically and will be considered in this thesis in detail. The special case of linear elastic supports is also of interest for purposes of comparison since this approximation is frequently used by design engineers. The popularity of this approximation is due to the fact that the solution to the system with linear elastic springs leads to a linear formulation, while the system with non-linear elastic springs leads to a rather involved non-linear formulation. However, a linear approximation may give rise to inconsistency in actual design.

An iterative technique has been developed in this study which can be applied to both linear and non-linear systems. All computations were performed on an IBM 1620 computor. Hand computation is possible on a desk calculator, but may be too time consuming for a beam which is continuous over more than six spans. Considerable time may be saved by utilizing an approximate linear solution as the initial guess.

II

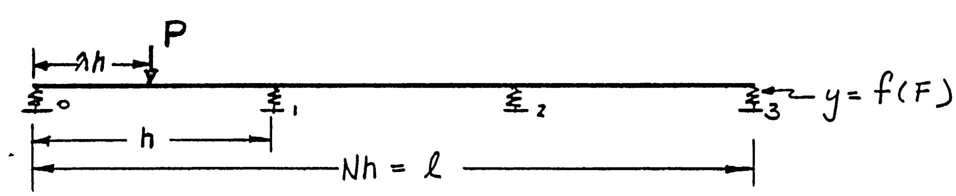
EVALUATION OF SOME ENGINEERING APPROACHES TO THE PROBLEM OF ELASTIC SUPPORTS

In order to select an approach best suited for the task of determining numerical results, a study of available approaches to the problem was conducted. A survey of the literature produced no engineering approaches to the problem of non-linear supports. However, some applications concerned with linear systems were of interest and consequently used as a basis for initial study of the non-linear case.

(a) Maxwell's Method[1]

For the case of a linear system, Maxwell's general method is an excellent one because it leads to a symmetric system of linear algebraic equations which can either be solved by a desk calculator or high speed computor without much difficulty.

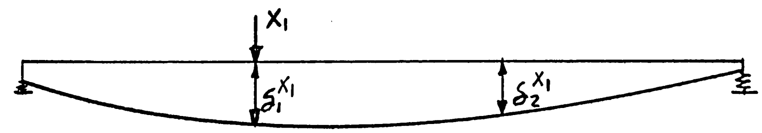
Maxwell's method may be applied as follows: Consider the non-linear spring system shown below.



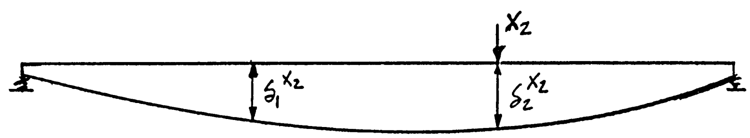
Procedure:

(1) Remove the redundant supports, (1, 2) and find the deflections at all supports due to the external load, P.

(2) Remove the external load, apply a redundant force X_1 and find the deflections at all supports,



(3) Apply a redundant force X_2 and find the deflections at the supports.



(4) Write the equations of geometric compatibility for each redundant support.

$$\Delta_1^P + \delta_1^{x_1} + \delta_1^{x_2} = y_1 = f(x_1)$$

$$\Delta_2^P + \delta_2^{x_1} + \delta_2^{x_2} = y_2 = f(x_2)$$

(5) Solve this system of non-linear algebraic equations.

An important advantage of this method is the fact that each step can be carried out with clear physical meaning. However, the practicability of the method is questionable because of the following reasons:

1. Many non-linear spring systems encountered in practice are characterized by symmetrical load deformation curves, that is, similar response to tension and compression forces. An attempt to analytically describe actual spring behaviors indicated that such symmetrical response can be expressed only by certain odd powered algebraic functions. To obtain a reasonable curve fit for some spring deflection cases requires high order polynomial functions which makes the solution to the compatibility equations extremely difficult.

2. Direct application of the Maxwell approach requires that the load-deformation curve be a continuous function, whereas many non-linear spring systems have discontinuous characteristics.

3. Those springs which have continuous unsymmetrical response are extremely difficult to describe with a single algebraic expression over the working range.

A numerical solution using Maxwell's method was achieved for a beam continuous over four supports with symmetrical loading. The spring load-deformation function assumed was very simple, ($F = KY^2$), but of little practical importance since the beam stiffness had to be selected so that no spring had tensile loading. Formulation of this simple case and relaxation of the resulting non-linear algebraic equations indicated that the approach may be of value when the load-deformation curve can be expressed by such a simple function.

(b) Timoshenko Method[2]

This approach involves the closed form solution to the fourth order linear differential beam equation for the special case of a continuous linear elastic support. Application of the resulting solution has been extended to the case of discrete linear supports by Timoshenko[2] and by Seely and Smith[3]. These applications give only approximate results for the case of discrete support system. The limitations of this approach to discrete supports are discussed in reference (3). Since this method applied to the case of a continuous non-linear support system yields a fourth order non-linear differential equation which seemed too involved for a practical solution, no attempt was made to use this approach for the discrete non-linear system.

(c) Method of Three Moments

This method has been applied to the linear discrete system by Firmage and Chiu[4] who have provided a valuable design aid in publishing influence lines for continuous beams of this type. However, the Three-Moment approach is inherently restricted to linear systems and not applicable to the non-linear supports of primary concern in this study.

(d) Newmark-Austin Method[5]

This method has been advanced as a numerical approach to the problem of discrete linear elastic supports and involves successive corrections to an assumed configuration. The method is primarily intended for desk calculator solutions and becomes tedious for multi-span beams. It does not lend itself readily to high speed computor solutions and is not easily extended to a non-linear system.

(e) Koiter Method[6, 7]

This method is essentially the same as the Newmark-Austin technique except that the integrations performed do not involve numerical approximations. Application prior to this study has been limited to the linear system. Since this method lends itself well to high speed computor programing and is also practical for desk calculator solution, a modification of the Koiter Method as applied by Biezno and Grammel for the linear case was selected as the best available approach to the problem under study.

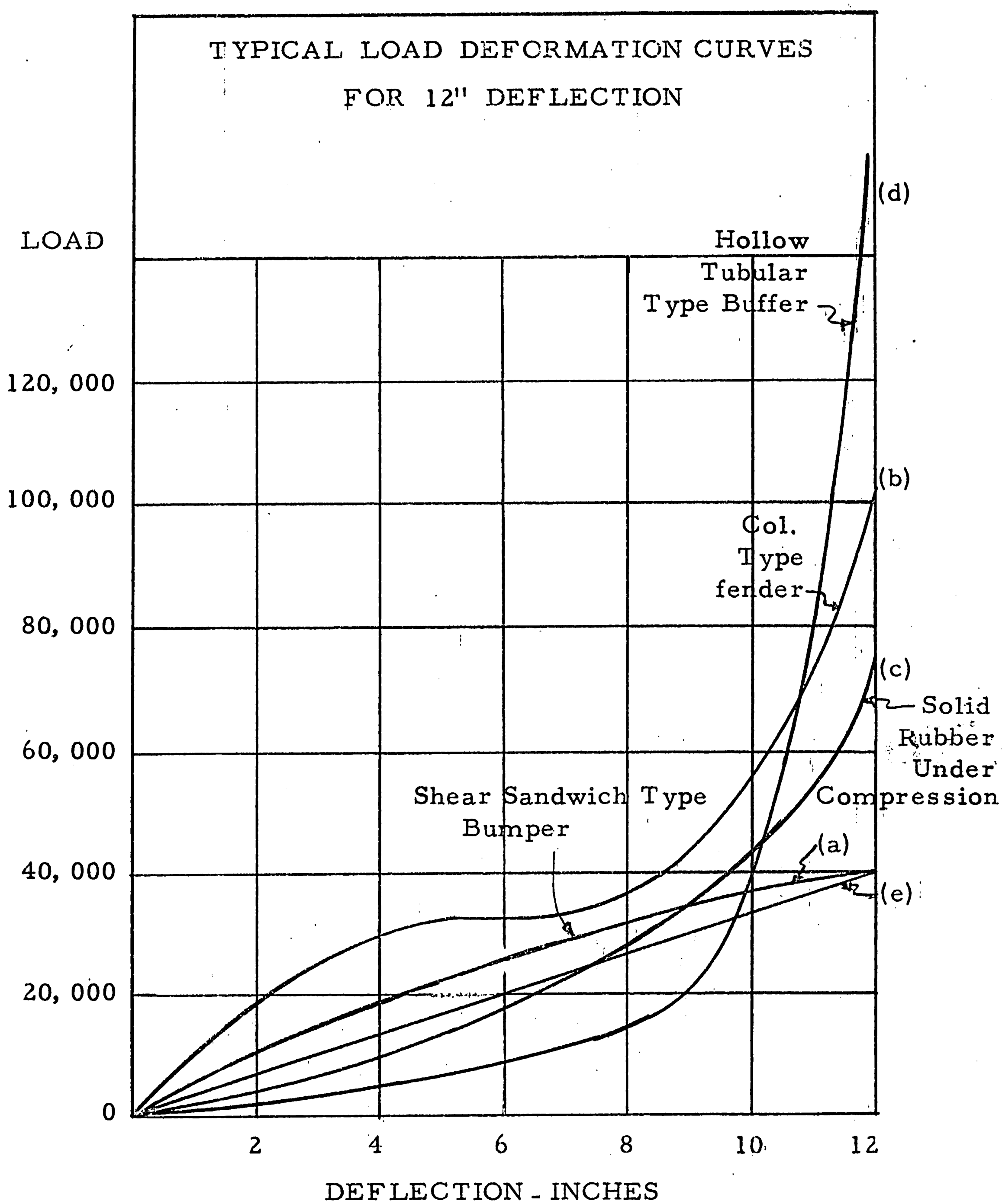
III

TYPICAL LOAD-DEFORMATION RELATIONSHIPS AND THEIR MATHEMATICAL REPRESENTATION

Figure (1) shows some of the load-deformation curves which are commonly used in marine fendering systems. Curves marked (a), (b), (c), and (d) show the load-deformation relations for the non-linear system, while (e) shows the load-deformation curve for a linear system.

A very popular type of spring used in many bumper designs is the Shear Sandwich type whose load-deformation curve is shown in figure (1a). To demonstrate the procedure developed, a curve of this type will be considered. The curve can be expressed in the form of a polynomial, i. e.,

$$F = f\,(ay + by^2 + cy^3 + \cdots\cdots + py^n)$$



TYPICAL LOAD DEFORMATION CURVES
FOR 12" DEFLECTION
LOAD
120, 000
100, 000
80, 000
60, 000
40, 000
20, 000
0
2
4
6
8
10
12
(d)
Hollow
Tubular
Type Buffer
(b)
Col.
Type
fender
(c)
Solid
Rubber
Under
Compression
Shear Sandwich Type
Bumper
(a)
(e)
DEFLECTION - INCHES

FIG. 1

IV

DEVELOPMENT OF ITERATIVE EQUATION

Consider the jth span of a continuous beam on elastic non-linear supports subjected to any given loading as shown in Figure 2. Let V_{jl} and M_{jl} denote the shear and moment at the left end, and V_{jr} and M_{jr} the shear and moment at the right end. Let P_{jn} denote concentrated loads at distances X_{jn} from the left end, and let $\omega_j(x)$ denote the distributed loading.

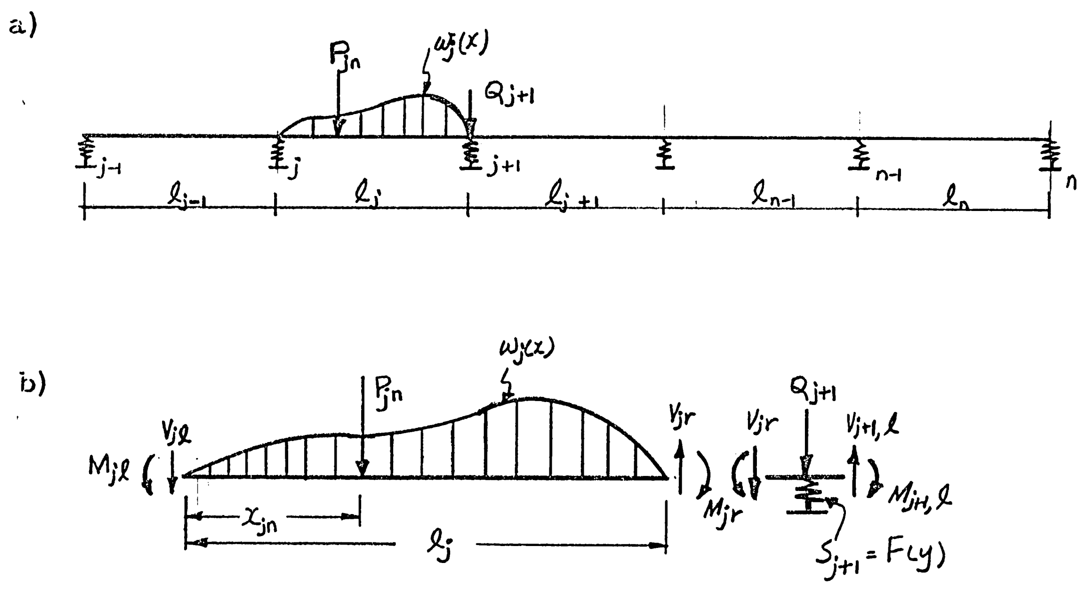


FIG. 2

Then from the statics, the condition $\Sigma V = 0$ gives

$$V_{jr} = V_{jl} + \sum_n P_{jn} + \int_o^{l_j} \omega_j \cdot (x)\, dx \tag{1}$$

$$M_{jr} = M_{jl} + V_{jl} l_j + \sum_n P_{jn} (l_j - x_{jn}) + \int_o^{l_j} (l_j - x)\, \omega_j(x) dx \tag{2}$$

The deflection and slope at the (j+1)th support can be obtained in terms of forces V_{jl}, M_{jl}.

From the elementary beam theory the deflection of the elastic curve is approximated by the following relationship:

$$\frac{d^2 y}{dx^2} = \frac{M}{(EI)_j} \tag{3}$$

Thus,

$$\frac{d^2 y}{dx^2} = \frac{M_{jl}}{(EI)_j} + \frac{V_{jl}}{(EI)_j} (l_j - x) + \frac{M_j^P(x)}{(EI)_j} \tag{4}$$

Where, $M_j^P(x)$ denotes the contribution of moment at a point in the jth span due to any arbitrary loading. For example, the expression $M_j^P(x)$, for a concentrated and uniform load becomes

$$M_j^P(x) = \sum_n P_{nj} \frac{(x_{jn} - x)}{(EI)_j} + \int_o^{\beta} \frac{\omega(x)}{(EI)_j}\, x\, dx \qquad x_{jn} < x \tag{5}$$

Integrating the equation (4) once with respect to x

$$\frac{dy}{dx}(\xi) = \left(\frac{dy}{dx}\right)_{jl} + M_{jl} \int_o^{\xi} \frac{dx}{(EI)_j} + V_{jl} \int_o^{\xi} \frac{x}{(EI)_j}\, dx + \int_o^{\xi} \frac{M_j^P(x)}{(EI)_j}\, dx$$

This is the slope at any point $x = \xi$ in the span considered. Since only the slope at the right end of the beam is of interest at this time, the integration must be performed up to $\xi = l_j$. Then, the slope of the right end is obtained in the following form.

$$\frac{dy}{dx} = \left(\frac{dy}{dx}\right)_{j1} + M_{j1}\int_{o}^{l_j}\frac{dx}{(EI)_{j1}} + V_{j1}\int_{o}^{l_j}\frac{1}{(EI)_j}\,x\,dx + \int_{o}^{l_j}\frac{M_j^P(x)}{(EI)_j}\,dx \qquad (6)$$

In like manner, in order to obtain the deflection at any point $x = \eta$ in the span considered, the equation (6) be integrated once again.

Thus,

$$y(\eta) = y_{j1} + \int_{o}^{\eta}\left(\frac{dy}{dx}\right)_{j1} dx + M_{j1}\int_{o}^{\eta}\int_{o}^{\xi}\frac{dx}{(EI)_j}\,d\xi + V_{j1}\int_{o}^{\eta}\int_{o}^{\xi}\frac{1}{(EI)_j}\,x\,dx\,d\xi$$

$$+ \int_{o}\int_{o}\frac{M_j^P(x)}{(EI)_j}\,dx\,d$$

For the deflection at $= l_j$, the following equation is obtained.

$$y_{jr} = y_{j1} + \left(\frac{dy}{dx}\right)_{j1} l_j + M_{j1}\int_{o}^{l_j}\int_{o}^{\xi}\frac{dx}{(EI)_j}\,d\xi + V_{j1}\int_{o}^{l_j}\int_{o}^{\xi}\frac{1}{(EI)_j}\,dx\,d\xi$$

$$+ \int_{o}^{l_j}\int_{o}^{\xi}\frac{M_j^P(x)}{(EI)_j}\,dx\,d\xi \qquad (7)$$

Considering the free body of the portion of the beam in the immediate vicinity of a support as shown in Figure (2b), the following relationship is established from statics.

$$V_{j+1,1} = V_{jr} + Q_{j+1} - F_{j+1}(y) \qquad (8)$$

where, $F(y)$ is the force introduced by the spring deflection and where Q_{j+1} is a concentrated force directly above the support.

Equations (9) through (13) presented again below for convenience, completely describes the mechanics of the beam problem under consideration and are the basic equations used in this approcah.

$$V_{jr} = V_{j1} + \sum_{n} P_{jn} + \int_{o}^{l_j}\omega_j(x)\,dx \qquad (9)$$

$$M_{jr} = M_{jr} + V_{j1} l_j + \sum_n P_{jn} (l_j - x_{jn}) + \int_0^{l_j} (l_j - x)\omega_j(x)\,dx \tag{10}$$

$$y'_{jr} = y'_{j1} + A_m M_{j1} + A_v V_{j1} + A_p \tag{11}$$

$$y_{jr} = y_{j1} + y'_{j1} l_j + B_m M_{j1} + B_v V_{j1} + B_p \tag{12}$$

$$V_{j+1,\,1} + V_{j1} + Q_{j+1} - F(y) \tag{13}$$

The primes indicate the first derivative with respect to x and the constants A_m, A_v, A_p, B_m, B_v, and B_p, for uniform and concentrated load, are given by the following expressions:

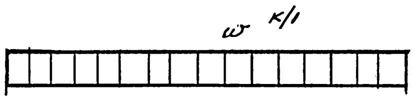
$$A_m = \int_0^{l_j} \frac{dx}{(EI)_j} \qquad A_v = \int_0^{l_j} \frac{1}{(EI)_j}\, x\, dx$$

$$A_p = \sum_n P_{jn} \int_0^{l_j} \frac{(x-x_{jn})}{(EI)_j}\, dx + \int_0^{l_j} \int_0^{\beta} \frac{\omega(x)}{(EI)_j}\, x\, dx\, d\beta$$

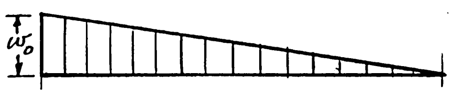
$$B_m = \int_0^{l_j} \int_0^{\xi} \frac{dx}{(EI)_j}\, d\xi \qquad B_r = \int_0^{l_j} \int_0^{\xi} \frac{1}{(EI)_j}\, x\, dx\, d\xi$$

$$B_p = \sum_n P_{jn} \int_{x_{jn}}^{l_j} \int_{x_{jn}}^{\xi} \frac{(x-x_{jn}}{(EI)_j}\, dx\, d\xi + \int_0^{l_j} \int_0^{\xi} \int_0^{\beta} \frac{\omega(x)}{(EI)_j}\, dx\, d\beta\, d\xi \tag{14}$$

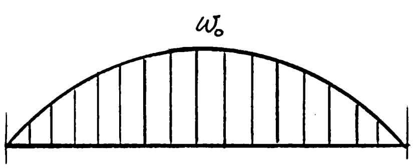
The subscripts designate the terms to which constants apply, i. e., the constants A_m, B_m applied to moment terms, M_{j1}, in the expressions for slope and deflection respectively. The constants are not hard to evaluate once a given loading condition and beam constants are specified. Notice that A_m, A_v, B_m, and B_v are determined by the properties of the cross section. A_p and B_p depend on the properties of the section as well as on the type of loading. Expressions for A_p and B_p are given below for some frequently encountered cases of loading on prismatic beams.



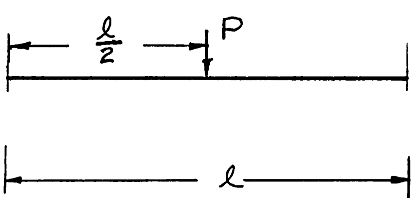
$$A_p = \frac{\omega l^3}{6EI} \qquad B_p = \frac{\omega l^4}{24EI}$$



$$A_p = \frac{\omega_o l^3}{8EI} \qquad B_p = \frac{\omega_o l^4}{30EI}$$



$$A_p = \frac{\omega_o l^3}{10EI} \qquad B_p = \frac{\omega_o l^4}{4sEI}$$



$$A_p = \frac{PL^2}{8EI} \qquad B_p = \frac{PL^3}{48EI}$$

The numerical technique for applying equations (9) through (13) to accomplish a solution will be considered next. First, arbitrary initial values of y_0'' and y_0' at support 1 are assumed. Then the support reaction may be computed. The shear to the right of support (1) may be obtained from equation (13). Equations (9) through (12) may then be used to proceed to support 2. This process may be repeated to obtain a value of V_n and M_n to the right of the last support. The values of V_n and M_n obtained are then compared to known boundary values. Since V and M to the right of the last support are completely dependent upon the initial assumption for y_0 and y_0', they may be related to these initial values. These may be written in functional form as follows:

$$V_n = V_n\,(y_0,\ y_0') \qquad (15)$$

$$M_n = M_n\,(y_0,\ y_0') \qquad (16)$$

Where, V_n and M_n are the shear and moment to the right of the nth support, that is the last support. For simple end supports the condition which must be satisfied to the right of the last support is that $M_n = V_n = 0$. Equations (15) and (16) may be considered residual expressions equivalent to non-linear simultaneous algebraic equations and serve as a guide in establishing the next trial values of y_0, and y_0'.

There are a few suggested methods for solving certain types of non-linear algebraic equations in the literature. Newton's Method is a very convenient one provided initial assumptions are reasonably accurate. It gives faster convergency than any other method studied.

Newton's Method may be applied using the following procedure:

1. Guess initial values, i.e., y_0 and y_0'.
2. Calculate V_n and M_n using equations (9) through (13).
3. Replace V_n and M_n with approximate linear functions $\overline{V}_n$ and $\overline{M}_n$.

$$V_n=\overline{V}_n(y_0, y_0') = V_n(y_0, y_0') + \left(\frac{\partial V_n}{\partial y}\right)_{(y_0, y_0')}(y-y_0) + \left(\frac{\partial V_n}{\partial y'}\right)_{(y_0, y_0')}(y'-y_0') \tag{17}$$

$$M_n=\overline{M}_n(y_0, y_0') = M_n(y_0, y_0') + \left(\frac{\partial M_n}{\partial y}\right)_{(y_0, y_0')}(y-y_0) + \left(\frac{\partial M_n}{\partial y}\right)_{(y_0, y_0')}(y'-y_0') \tag{18}$$

4. Solve the resulting linear equations,

$$\overline{V}_n(y_0, y_0') = 0 \tag{19}$$

$$\overline{M}_n(y_0, y_0') = 0 \tag{20}$$

to obtain new approximate values for y_{01} and y_{01}'.

In order to solve equations (19, 20) numerically, they must be expressed in the difference form shown below.

$$0=V_n(y_0, y_0') + \left[\frac{V_n(y_0+\Delta y_0, y_0') - V_n(y_0, y_0')}{\Delta y_0}\right]\Delta\overline{y}_0$$
$$+\left[\frac{V_n(y_0, y_0'+\Delta y_0') - V_n(y_0, y_0')}{\Delta y_0'}\right]\Delta\overline{y}_0' \tag{21}$$

$$0=M_n(y_0, y_0') + \left[\frac{M_n(y_0+\Delta y_0, y_0') - M_n(y_0, y_0')}{\Delta y_0}\right]\Delta\overline{y}_0$$
$$+\left[\frac{M_n(y_0, y_0' + \Delta y'_0) - M_n(y_0', y_0')}{\Delta y_0'}\right]\Delta\overline{y}'_0 \tag{22}$$

The new approximate values of y_0 and y_0' are obtained simply by adding the values $\Delta\overline{y}_0$ and $\Delta\overline{y}_0'$ to the initial values y_0 and y'_0. Thus,

$$y_{01} = y_0 + \Delta\overline{y}_0$$

$$y'_{01} = y'_0 + \Delta\overline{y}'_0$$

5. Repeat steps 2 through 4 of the above procedure until the boundary conditions at the last support are satisfied.

V

APPLICATIONS

The iterative technique developed is applied to a typical marine fender system. From the standpoint of application, a nine-span continuous beam on ten discrete non-linear elastic supports was considered sufficient. Since the effects transmitted to points remote from the point of application normally decay rapidly after traversing a few continuous spans, most multi-span structures of say 20 or 30 span could be satisfactorily studied by considering only 10 or less supports.

a. Properties of spring

All springs are considered identical having the same load-deformation curve as shown in Figure 3. A mathematical expression for this force function in terms of deflection was found to be

$$F = 44.3y - 14.698y^2 + 2.449y^3.$$

where y is in ft. and F is in kips

The expression is shown as a dotted line in Figure 3. A linear approximation commonly used in design is also shown in the figure. This linear function can be expressed mathematically by

$$F = 24.4y.$$

where y is in ft. and F is in kips

b. Loading

Three separate types of loading are considered as shown in the Fugures 4a and 4b; concentrated, parabolic and uniform loading. Computations were made for the following two studies:

1. A comparison was made between the non-linear solution and the linear approximation for gradually increasing loads on the end span of the beam and on the center span. Both concentrated and uniform loads were considered. The loads were increased until either the elastic limit on the beam or the deflection limits of the springs were reached.

2. A study was made of the effect on the beam and the springs of the shape of the load diagram. Equivalent concentrated, uniform and parabolically distributed loads were studied for both linear and non-linear spring supports. As in case 1, the loads were placed on the end span of the beam and on the center span.

In all cases, it was assumed that the loads were applied symmetrically within a span.

c. Evaluation of constants for beam in Figure 4 and results.

Case 1

(a) Concentrated load at mid-point of a span.

$$A_m = \frac{L}{EI} = \frac{32}{1514708} = 0.2113 \times 10^{-9} \quad k^{-1}\,Ft^{-1}$$

$$A_v = \frac{L^2}{2EI} = \frac{32^2}{2 \times 1514708} = 0.3380 \times 10^{-3} \quad k^{-1}$$

$$A_p = \frac{PL^2}{8EI} = \frac{P \times 32^2}{8 \times 1514708} = 0.8450 \times 10^{-4} \quad P \quad k^{-1}$$

P (kips)	40	80	120	160
A_p	0.338×10^{-2}	0.676×10^{-2}	0.1014×10^{-1}	0.1352×10^{-1}

$$B_m = \frac{L^2}{2EI} = A_v = 0.3380 \times 10^{-3} \quad k^{-1}$$

$$B_v = \frac{L^3}{6EI} = \frac{32}{6 \times 1514708} = 0.3605 \times 10^{-2} \quad k^{-1}\,Ft$$

$$B_p = \frac{PL^3}{48EI} = \frac{P \times 32^3}{48 \times 1514708} = 0.4507 \times 10^{-3} \; P \;\; Ft\,k^{-1}$$

P(kips)	40	80	120	160
B_p(Ft)	0.1803×10^{-1}	0.3606×10^{-1}	0.5408×10^{-1}	0.7211×10^{-1}

(b) Uniform Load

All constants have the same value as in the previous case except the constants A_p and B_p which are functions of the loads.

$$A_p = \frac{\omega l^3}{6EI} = \frac{32^2\,\omega}{6 \times 1514708} = 0.3606 \times 10^{-2}\,\omega \qquad k^{-1}$$

$$B_p = \frac{\omega l^4}{24EI} = \frac{32^4\omega}{24 \times 154708} = 0.2884 \times 10^{-1}\,\omega \qquad Ft^2 \;\; k^{-1}$$

$\omega(k/_1)$	2	4	6	8
A_p	0.7211×10^{-2}	0.1442×10^{-1}	0.2164×10^{-1}	0.2885×10^{-1}
B_p	0.5768×10^{-1}	0.1154	0.1730	0.2307

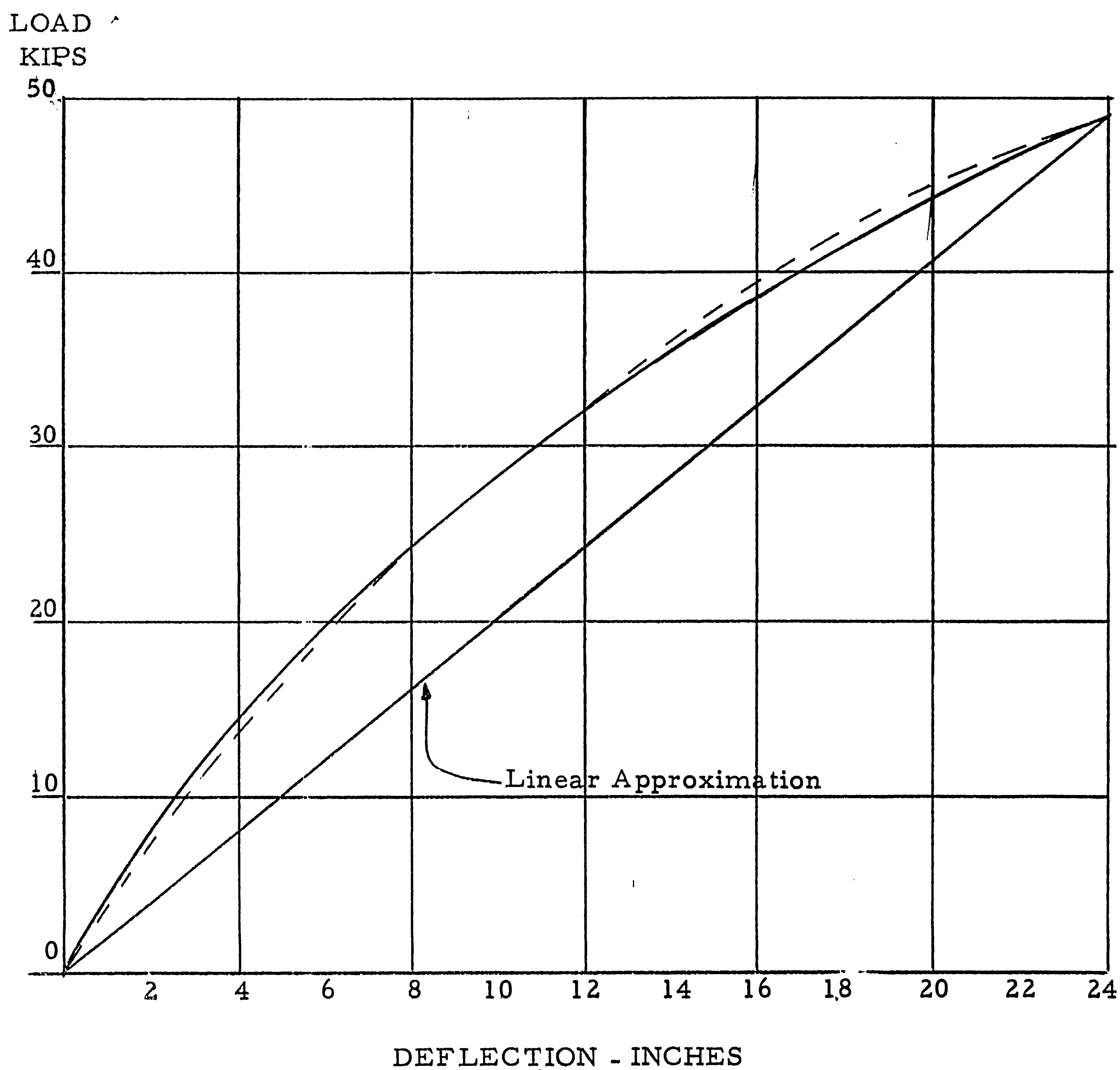

LOAD
KIPS
50
40
30
20
10
0
2
4
6
8
10
12
14
16
18
20
22
24
Linear Approximation
DEFLECTION - INCHES

FIG. 3

BEAM 2-24W120, $I=3635 IN^4$, $EI=1,514,708\ FT^2\ K$

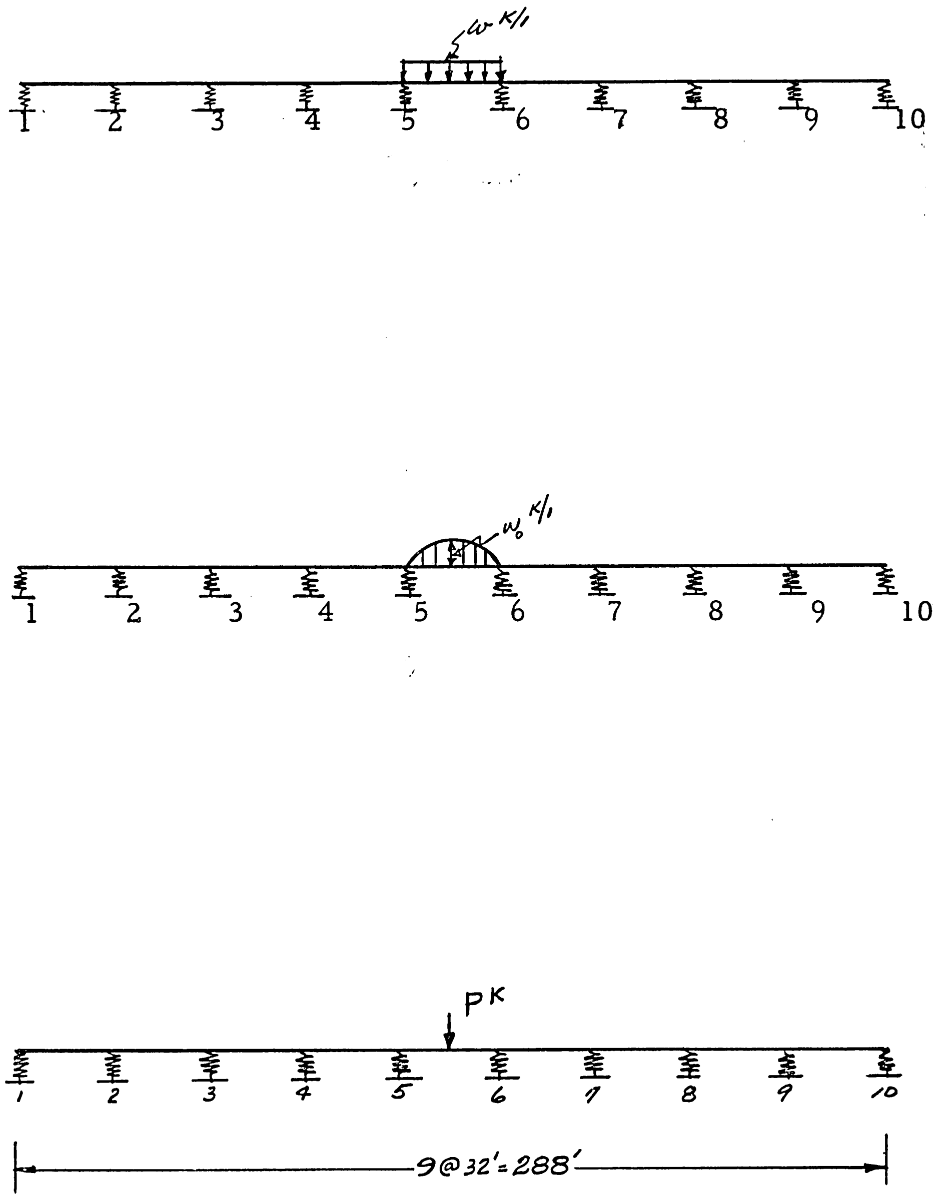


FIG. 4a

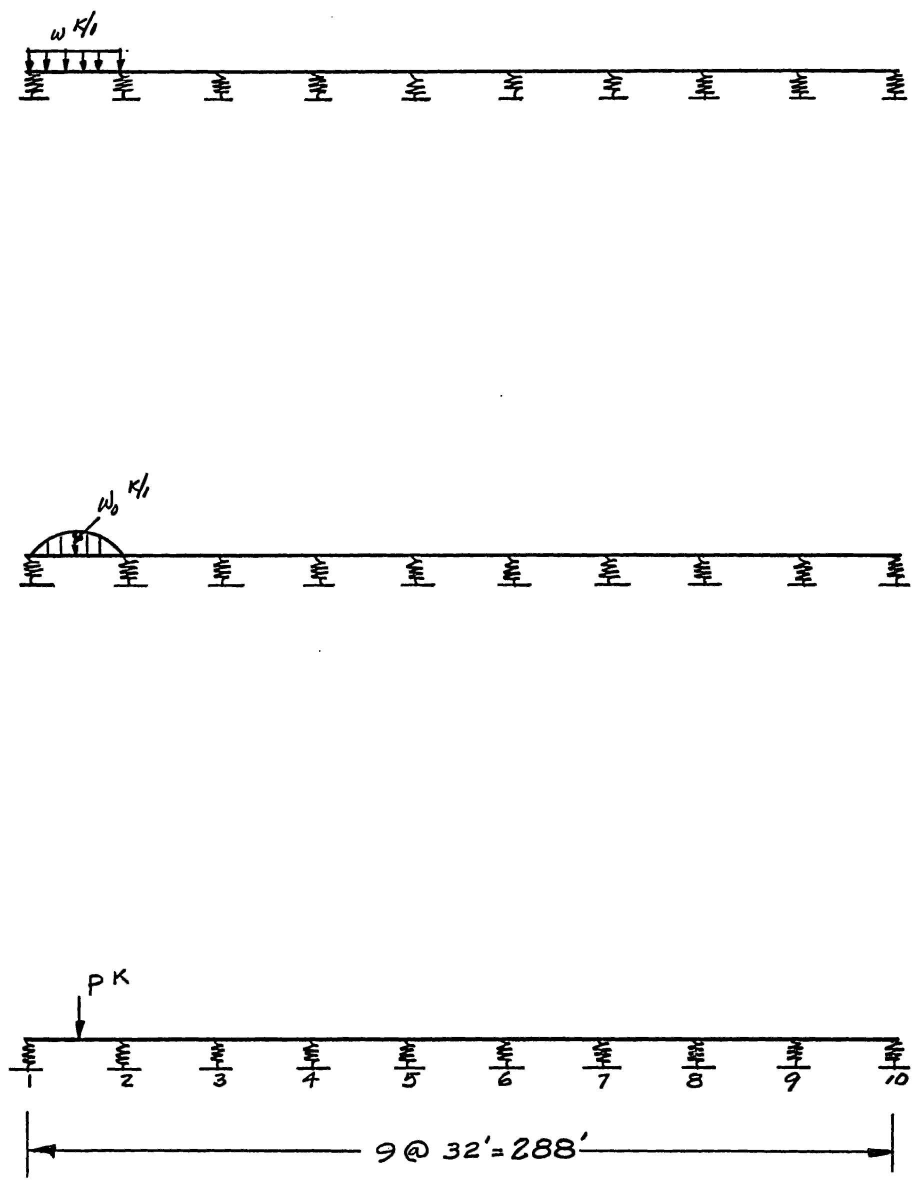


FIG. 4b

Case 2

The constants A_m, A_v, B_m and B_v are the same as Case 1.

(a) Concentrated load of 96 kips

$$A_p = \frac{pl^2}{8EI} = \frac{96 \times 32^2}{8 \times 1514708} = 0.8112 \times 10^{-2}$$

$$B_p = \frac{pl^3}{48EI} = \frac{96 \times 32^3}{48 \times 1514708} = 0.4327 \times 10^{-1} \text{ Ft.}$$

(b) Equivalent uniform load (96^k total load)

$$A_p = \frac{\omega l^3}{6EI} = \frac{3 \times 32^3}{6 \times 1514708} = 0.1082 \times 10^{-1}$$

$$B_p = \frac{\omega l^4}{24EI} = \frac{3 \times 32^4}{24 \times 1514708} = 0.8653 \times 10^{-1} \text{ Ft.}$$

(c) Equivalent parabolic load (96^k total load)

$$A_p = \frac{\omega_0 l^4}{10EI} = \frac{4.5 \times 32^3}{10 \times 1514708} = 0.9735 \times 10^{-2}$$

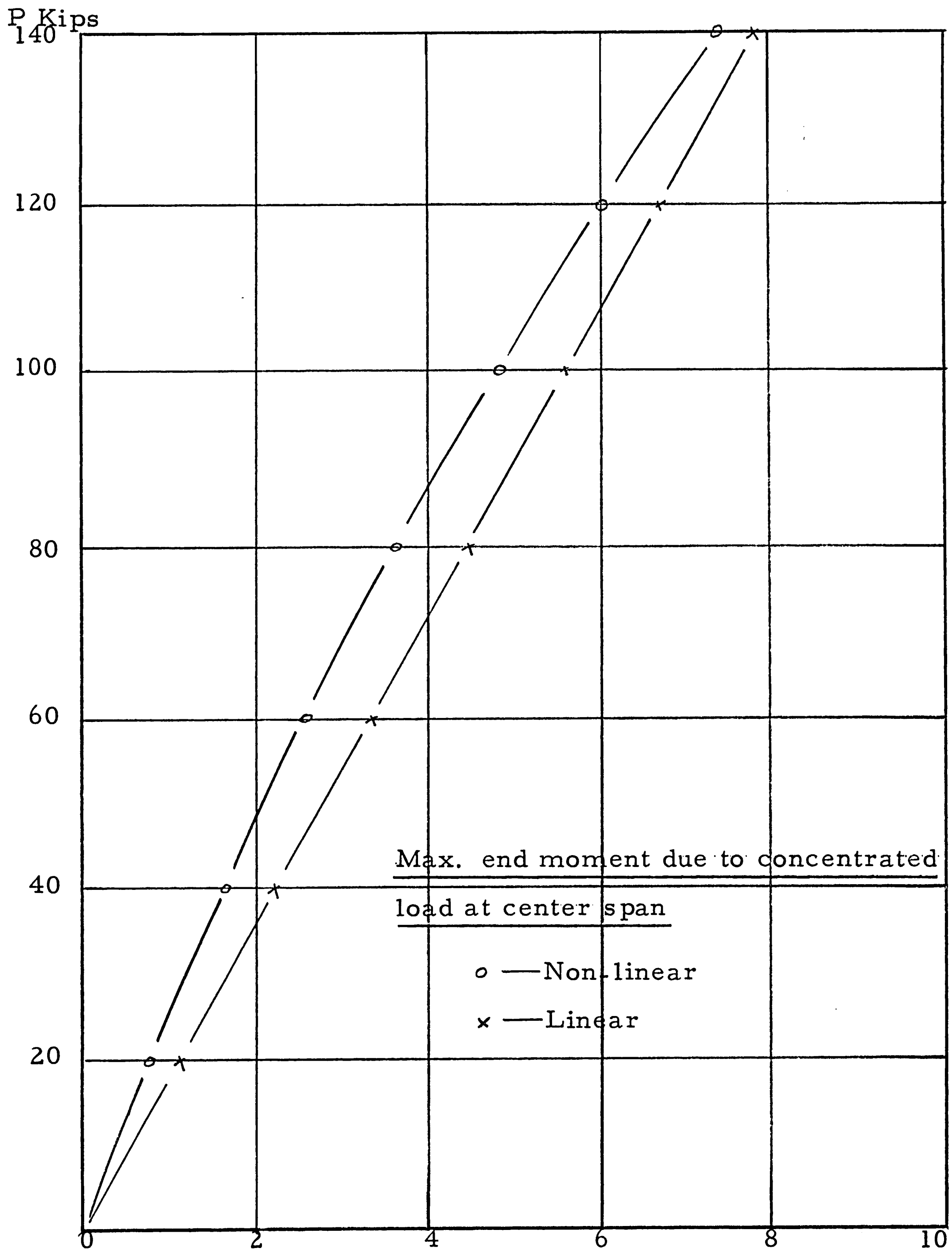
$$B_p = \frac{\omega_0 l^4}{45EI} = \frac{4.5 \times 32^4}{45 \times 1514708} = 0.6924 \times 10^{-1} \text{ Ft.}$$

(1) Results of Case 1

(a) The moments and spring deflections at the supports due to concentrated loads at the center span.

Spt.	P(kips)	M_n(ft-k)	M_l(ft-k)	y_n(ft)	y_l(ft)	$\frac{M_l}{M_n}$	$\frac{y_l}{y_n}$
1	40	0	0	-0.0318	-0.0671		2.1100
	80	0	0	-0.0698	-0.1341		1.9212
	120	0	0	-0.1166	-0.2012		1.7255
	160	0	0	-0.1746	-0.2683		1.5366
2	40	44.57	52.79	-0.0052	+0.0091	1.184	-1.750
	80	96.75	105.59	-0.0082	+0.0182	1.091	-2.219
	120	159.02	158.39	-0.0074	0.0272	0.996	-3.75
	160	233.59	211.18	-0.0004	0.0363	0.904	-90.75
3	40	96.50	98.45	0.0523	0.1201	1.02	2.296
	80	205.09	196.89	0.1201	0.2402	0.96	2.00
	120	328.56	295.34	0.2104	0.3604	0.898	1.712
	160	467.74	393.78	0.3318	0.4805	0.841	1.448
4	40	75.52	49.54	0.1670	0.2872	0.655	1.719
	80	149.78	99.07	0.3691	0.5746	0.661	1.556
	120	219.91	148.60	0.6197	0.8619	0.675	1.390
	160	280.49	198.14	0.9337	1.1491	0.706	1.222
5	40	-169.47	-225.54	0.3081	0.4630	1.330	1.502
	80	-368.60	-451.07	0.6682	0.9260	1.223	1.385
	120	-605.22	-676.60	1.0985	1.3890	1.117	1.264
	160	-884.10	-900.03	1.6174	1.8520	1.018	1.145

Subscripts n and l indicate the forces of the non-linear and linear and linear system respectively.



MOMENT x 10^3 (5th SUPPORT)

FIG. 5

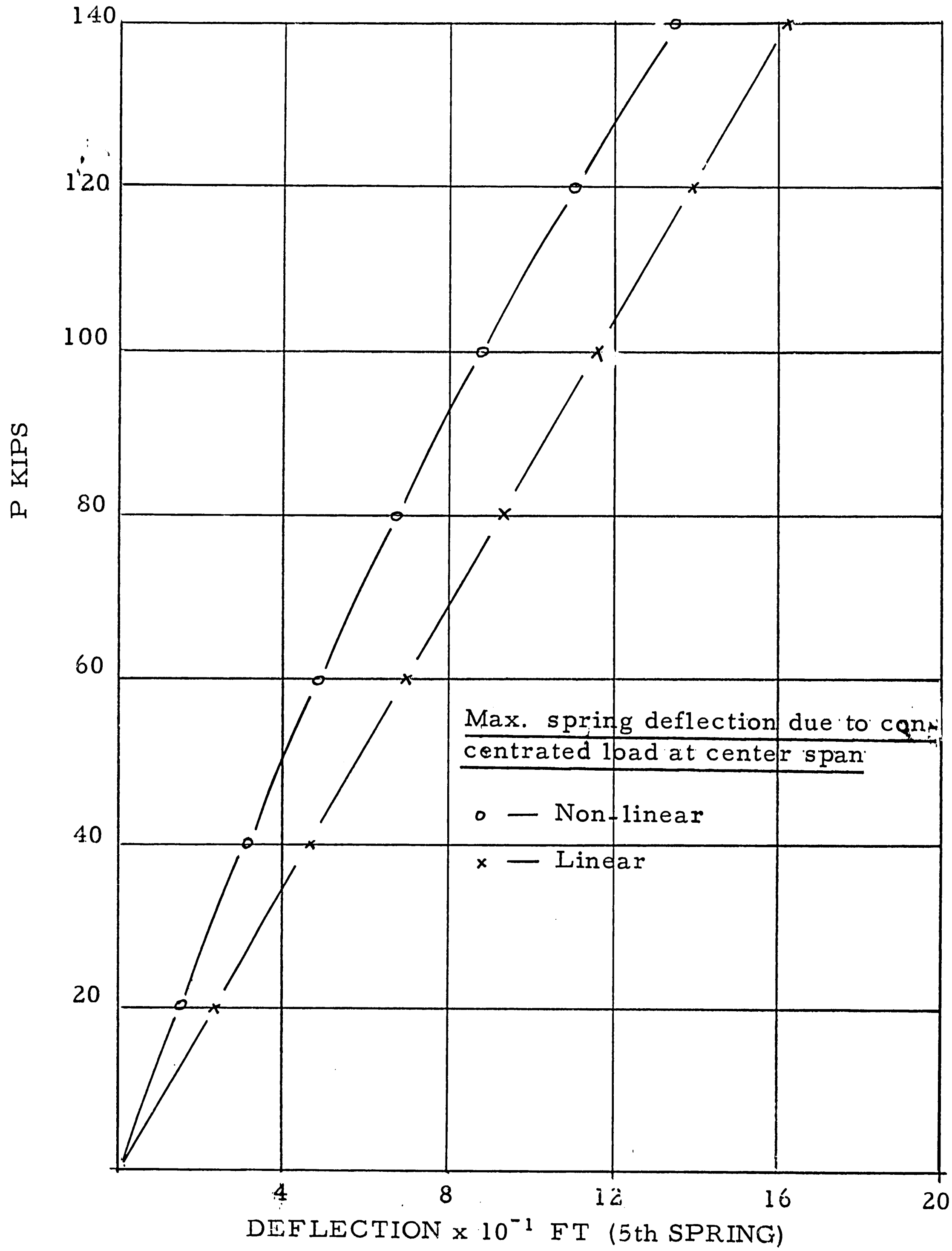

140
120
100
80
60
40
20
P KIPS
4
8
12
16
20
DEFLECTION x 10^{-1} FT (5th SPRING)
Max. spring deflection due to concentrated load at center span
o — Non-linear
x — Linear

FIG. 6

(b) The moments and spring deflections at the supports due to concentrated loads at the end span.

Spt	P(k)	M_n(ft-k)	M_l(ft-k)	y_n(ft)	y_l(ft)	$\frac{M_l}{M_n}$	$\frac{y_l}{y_n}$
1	40	0	0	0.5450	0.8207		1.505
	80	0	0	1.3529	1.6414		1.213
	120	0	0	2.5072	2.4622		0.982
2	40	-5.91	-6.07	0.3643	0.5617	1.027	1.5209
	80	28.94	12.15	0.8852	1.1235	0.419	1.269
	120	87.25	18.22	1.6129	1.6853	0.208	1.044
3	40	164.88	185.63	0.153	0.2649	1.1258	1.7313
	80	397.20	371.26	0.3633	0.530	0.934	1.458
	120	702.8	556.88	0.6690	0.7948	0.792	1.188
4	40	129.38	168.78	0.0256	0.0705	1.3045	2.7539
	80	308.73	337.56	0.0596	0.1411	1.093	2.368
	120	557.00	506.34	0.1163	0.2117	0.909	1.820
5	40	57.82	96.384	-0.0183	-0.0158	1.669	0.8633
	80	137.46	192.77	-0.0446	-0.0316	1.402	0.708
	120	252.59	289.15	-0.0774	-0.0474	1.144	0.613
6	40	12.07	36.44	-0.0204	-0.0356	3.0316	1.745
	80	28.49	72.88	-0.0490	-0.0713	2.5580	1.4551
	120	55.13	109.32	-0.0885	-0.1070	1.9829	1.209
7	40	-5.01	4.58	-0.0111	-0.0278	-0.9141	2.5045
	80	-12.16	9.16	-0.0266	-0.0556	-0.7532	2.0902
	120	-20.46	13.74	-0.0490	-0.0834	-0.6715	1.702

Spt	P(k)	M_n(ft-k)	M_1(ft-k)	y_n(ft)	y_1(ft)	$\frac{M_1}{M_n}$	$\frac{y_1}{y_n}$
	40	-6.44	-5.40	-0.0034	-0.0144	0.8385	4.2352
8	80	-15.49	-10.80	-0.0082	-0.0288	0.6972	3.5121
	120	-27.77	-16.20	-0.0157	-0.0432	0.5833	2.7515
	40	-2.97	-4.04	+0.0003	-0.0034	1.3602	-11.3333
9	80	-7.16	-8.089	0.0009	-0.0068	1.1297	-7.5555
	120	-12.98	-12.12	0.0013	-0.0102	0.9337	-7.8461
	40	0	0	0.0021	+0.0051		2.4285
10	80	0	0	0.005	+0.0103		2.060
	120	0	0	0.0092	0.0154		1.6739

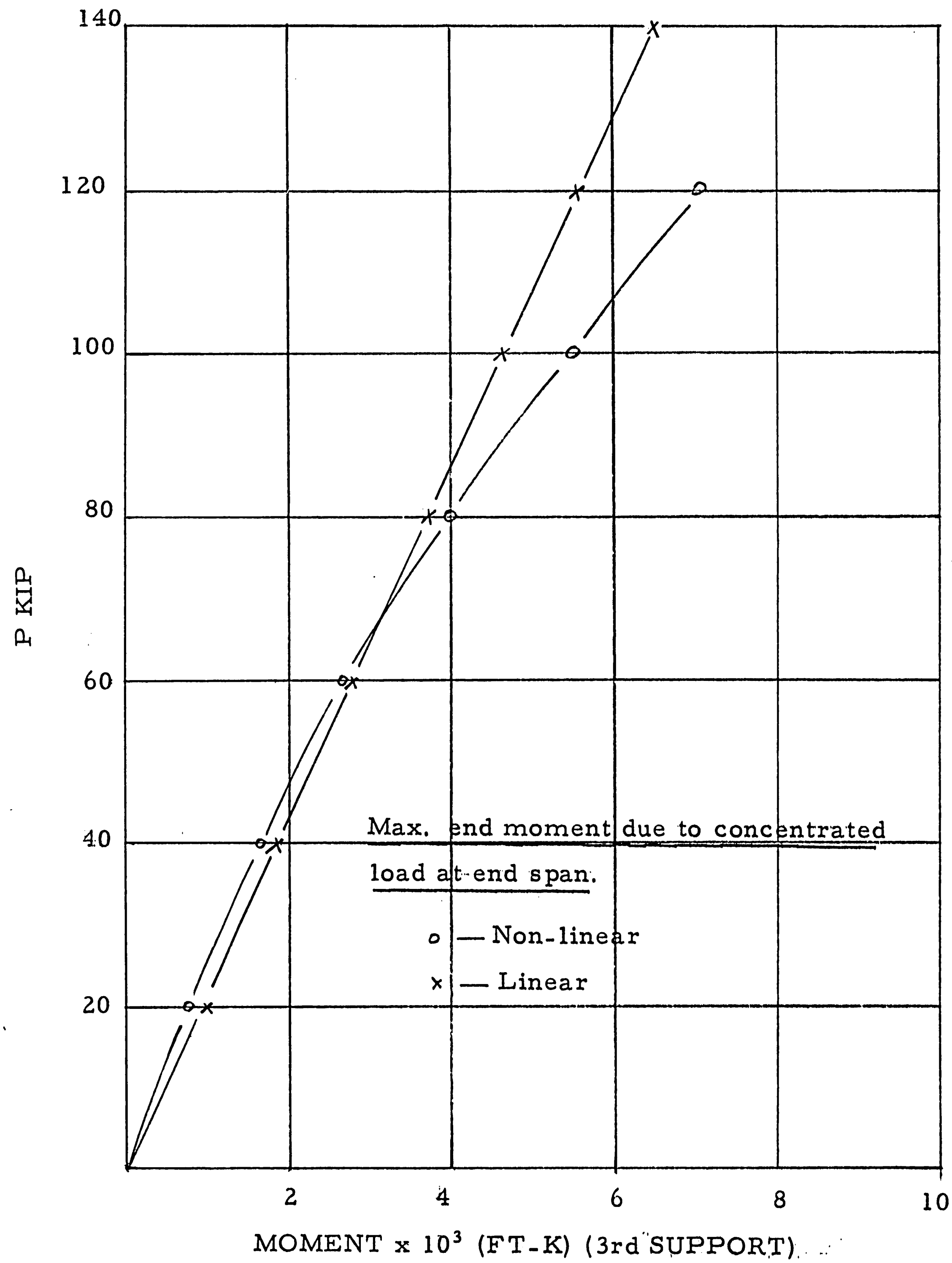
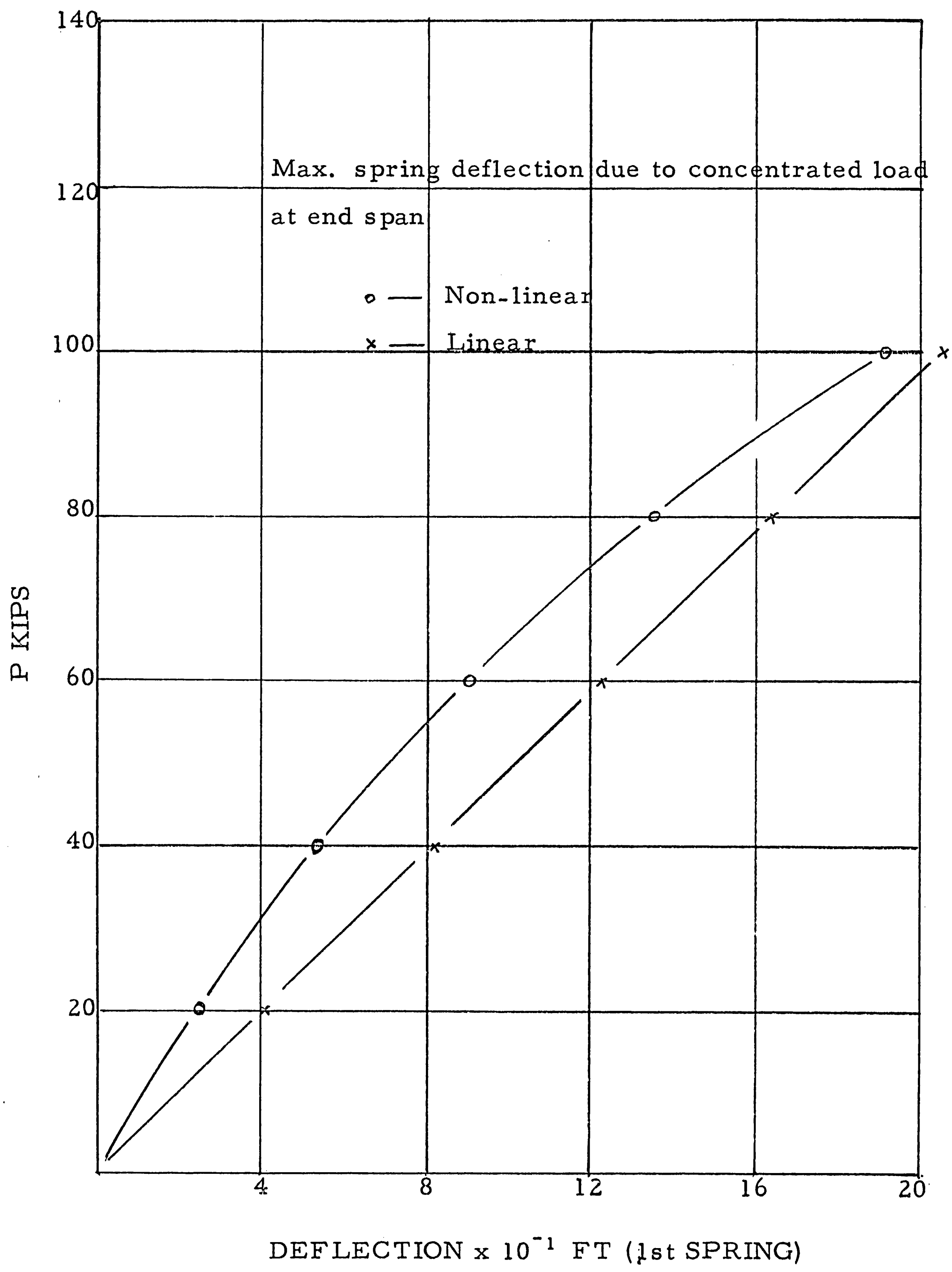

140
120
100
80
60
40
20
P KIP
2
4
6
8
10
Max. end moment due to concentrated load at end span.
o — Non-linear
x — Linear
MOMENT x 10^3 (FT-K) (3rd SUPPORT)

FIG. 7



140
120
100
80
60
40
20
P KIPS
Max. spring deflection due to concentrated load
at end span
o — Non-linear
x — Linear
4
8
12
16
20
DEFLECTION x 10⁻¹ FT (1st SPRING)

FIG. 8

(c) The moments and spring deflections at the supports due to uniform loads at the center span

Spt.	$\omega(k/_1)$	M^n(k-ft)	M^1(k-ft)	y_n(ft)	y_1(ft)	$\frac{M_1}{M_n}$	$\frac{y_1}{y_n}$
	2	0	0	-0.0533	-0.1055		1.979
1	4	0	0	-0.1256	-0.2110		1.679
	6	0	0	-0.2256	-0.3165		1.402
	8	0	0	-0.3359	-0.4220		1.256
	2	74.19	83.04	-0.0047	0.0186	1.186	-4.957
2	4	170.83	166.09	-0.0012	0.0371	0.972	-30.916
	6	296.78	249.14	0.0196	0.0557	0.839	2.841
	8	426.06	332.18	0.0583	0.0743	0.779	1.274
	2	155.06	151.47	0.0947	0.1972	0.978	2.082
3	4	343.41	302.94	0.2388	0.3943	0.882	1.651
	6	565.91	454.41	0.4624	0.5915	0.802	1.274
	8	771.00	605.88	0.7310	0.7887	0.785	1.078
	2	105.78	64.69	0.2846	0.4610	0.615	1.619
4	4	203.15	129.38	0.6766	0.9221	0.636	1.362
	6	272.31	194.03	1.2258	1.383	0.712	1.128
	8	299.82	258.72	1.8362	1.844	0.862	1.004
	2	-310.72	-385.02	0.5056	0.730	1.239	1.443
5	4	-705.20	-770.03	1.1669	1.4572	1.091	1.248
	6	-1196.60	-1155.15	2.0435	2.186	0.965	1.069
	8	-1673.70	-1540.16	2.9776	2.914	0.921	0.978

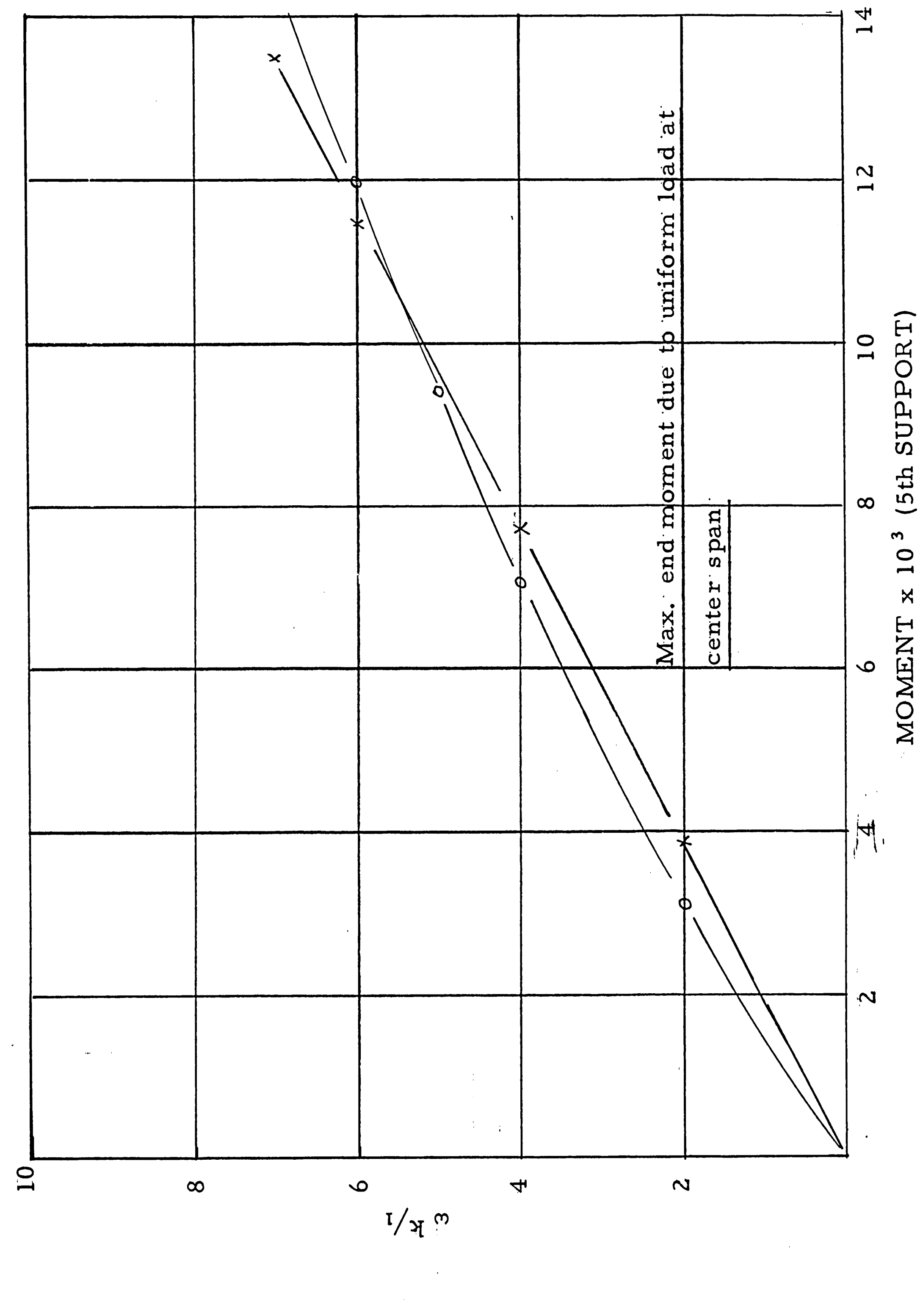

10
8
6
4
2
2
4
6
8
10
12
14
Max. end moment due to uniform load at center span
MOMENT x 10^3 (5th SUPPORT)

FIG. 9

(d) The moments and spring deflections at the supports due to uniform loads at the end span.

Spt	$\omega(k/_1)$	M_n(ft-k)	M_1(ft-k)	y_n(ft)	y_1(ft)	$\frac{M_1}{M_n}$	$\frac{y_1}{y_n}$
	1	0	0	0.4250	0.6609		1.558
1	2	0	0	0.4956	1.3218		1.327
	3	0	0	1.8068	1.9827		1.097
	4	0	0	2.7501	2.6435		0.895
	1	-11.51	-8.25	0.2824	0.4447	0.739	1.574
2	2	1.482	-16.50	0. 6454	0.8894	-11.133	1.378
	3	48.32	-24.78	1.1444	1.3340	-0.512	1.165
	4	76.69	-32.98	1.7414	1.779	-0.430	1.021
	1	124.35	145.44	0.1183	0.2107	1.169	1.781
3	2	286.89	290. 90	0.2674	0.4214	1.013	1.575
	3	508.85	436.32	0.4710	0.6321	0.857	1.342
	4	745.21	581.81	0.7296	0.8428	0.780	1.155
	1	98.91	133.29	0.0205	0.0567	1.347	2.765
4	2	225.49	266.58	0.0450	0.1135	1.005	2.522
	3	397.79	399.86	0.0786	0.1702	1.005	2.165
	4	599.40	533.17	0.1317	0.2270	0.889	1.723
	1	44. 60	76.46	-0.0136	-0.0120	1.714	0.882
5	2	100.98	152.92	-0.0318	-0.0240	1.514	0.754
	3	178.13	229.37	-0.0568	-0.0360	1.287	0.633
	4	274.80	305.84	-0.0807	-0.0481	1.112	0.596

Spt	ω(k/ɩ)	M_n(ft-k)	M_1(ft-k)	y_n(ft)	y_1(ft)	$\frac{M_1}{M_n}$	$\frac{y_1}{y_n}$
6	1	9.53	29.09	-0.0155	-0.0280	3.0524	1.8064
	2	21.26	58.17	-0.0356	-0.0561	2.7361	1.5758
	3	37.53	87.26	-0.0632	-0.0841	2.3725	1.3306
	4	61.73	116.34	-0.0950	-0.1122	1.8846	1.1810
7	1	-3.66	3.79	-0.0085	-0.0220	-1.0355	2.5882
	2	-8.65	7.59	-0.0194	-0.8774	-0.8774	2.2680
	3	-15.35	11.38	-0.0345	-0.0659	-0.7413	1.9101
	4	-20.97	15.17	-0.0531	-0.0879	-0.7234	1.6553
8	1	-4.8	-4.20	-0.0027	-0.0114	0.8750	4.2222
	2	-11.23	-8.39	-0.0061	-0.0229	0.7471	3.7540
	3	-19.92	-12.59	-0.011	-0.0343	0.6320	3.1181
	4	-29.68	-16.78	-0.0273	-0.0458	0.5653	1.6776
9	1	-2.15	-3.18	0.0003	-0.0027	1.4790	-9.0000
	2	-5.22	-6.36	0.0006	-0.0055	1.2183	-9.1666
	3	-9.23	-9.54	0.0011	-0.0082	1.0335	-7.4545
	4	-14.03	-12.68	0.0012	-0.0110	0.9002	-9.1666
10	1	0	0	0.0018	0.0040		2.2222
	2	0	0	0.0037	0.0081		2.1891
	3	0	0	0.0065	0.0121		1.8615
	4	0	0	0.0100	0.0162		1.6200

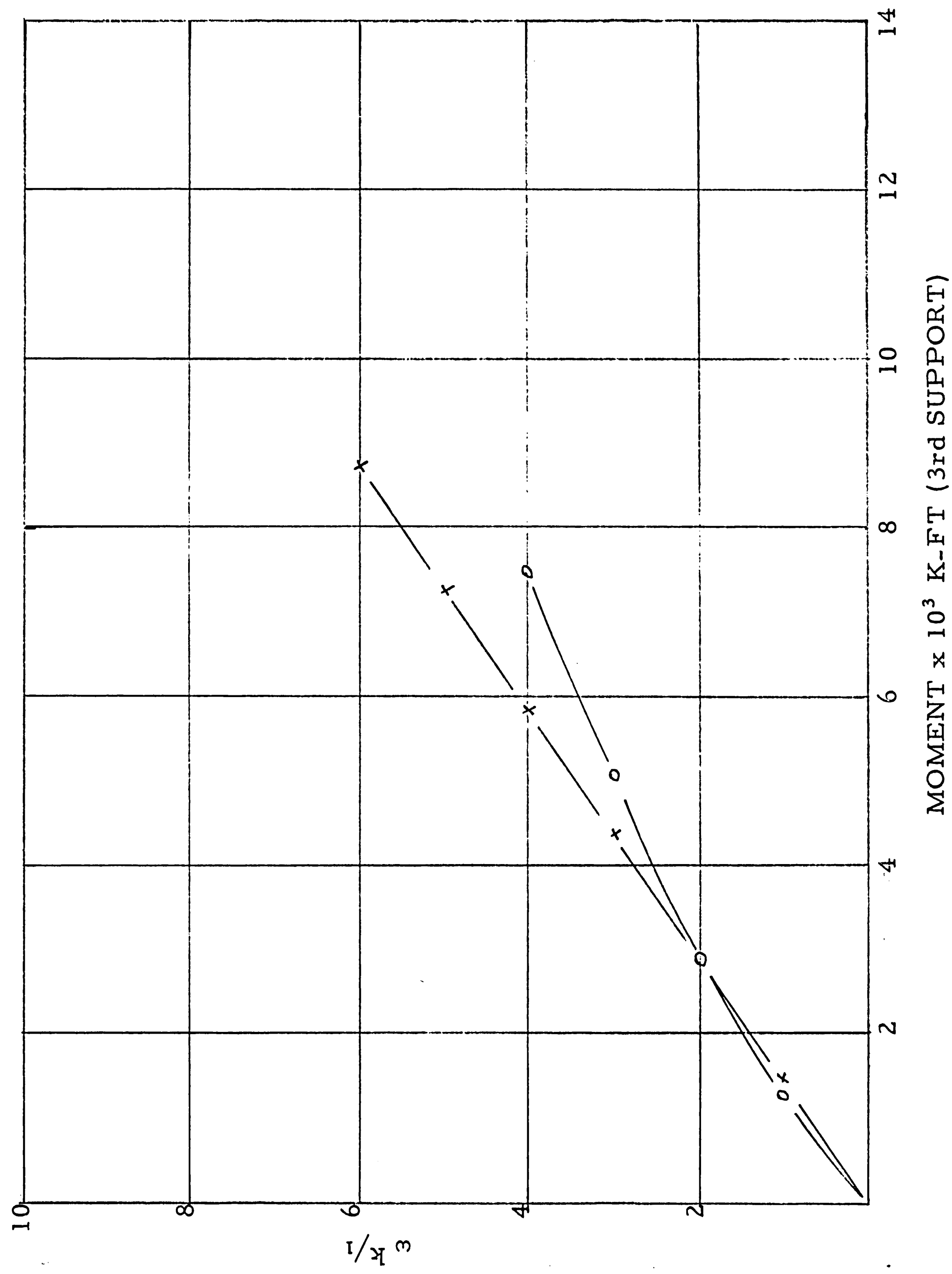


FIG. 10

(2) Results of Case 2

(a) Equivalent load of 96 kips at center span

Types of load	Non-linear system		Linear System		Point of
	M_{max}(ft-k)	y_{max}(ft)	M_{max}(ft-k)	y_{max}(ft)	M_{max}
Concentrated	1225	0.831	1308	1.111	Center of the span
Parabolic	960	0.820	1024	1.10	"
Uniform	879	0.811	961	1.093	"

Maximum spring deflection occurs at the 5th support in all types of loading.

(b) Equivalent load of 96 kips at end span

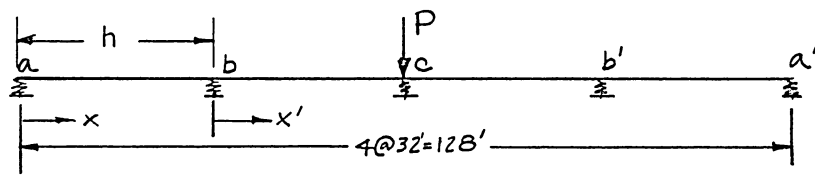
Types of load	Non-linear system		Linear System		Point of
	Mmax(ft-k)	y_{max}(ft)	M_{max}(ft-k)	y_{max}(ft)	M_{max}
Concentrated	739	1.789	775	1.970	Under load
Parabolic	512	1.799	440	1.977	3rd support
Uniform	508	1.805	436	-.983	"

Maximum spring deflection occurs at the 1st support in all types of loading.

VI
ENERGY CHECK

In order to verify the numerical results obtained from this approach an energy check was performed on some particular cases during the early phases of this study. Since the actual problem encountered in marine fendering design is one of energy absorption, the check has some practical significance and will be presented here. For simplicity, a four span continuous non-linear system will be considered. The same size of beam and type of spring which have been considered in the previous example is also considered in this check.

Fig. 11



During the loading process, the work done by the external loads, We, must be equal to the internal energy stored in the system, W_i, i.e.,

$$W_e = W_i$$

or $$W_e \simeq U_b + U_s$$

where, Ub and Us are strain energy stored in the beam and springs, respectively. Due to the non-linear nature of the system, the loads were applied starting from 10 kips up to 100 kips with increment of 10 kips to get the deflection configuration at the load point. The reactions and deflections due to application of this series of loads are shown in tables A and B. With this information energy can be computed as follows:

(a) Bending strain energy in the beam is

$$U_b = \frac{1}{2EI}\int_o^l M^2\,dx$$

Taking the coordinates as in Fig. 11

$$M = R_a\,x + [\,R_a\,(\,h + x'\,) + R_b\,x'\,]$$

Thus, considering symmetry

$$U_b = \frac{1}{EI}\int_o^h (R_a x)^2\,dx + \int_o^h [R_a\,(h+x')]^2\,dx'$$

Performing integration and simplifying

$$U_b = \frac{h^3}{EI}\left[\frac{1}{3}\,(\,8R_a^2 + 5R_aR_b + R_b^2)\right] \tag{25}$$

Substituting the proper values from table (B) into equation (25), the following value for U_b is obtained.

$$EIU_b = 32^3\left[\frac{1}{3}(8 \times 53.7084 + 5 \times 189.8422 + 671.0328)\right]$$

$$= 32^3\left[\frac{1}{3}(429.6672 + 949.2110 + 671.0328)\right]$$

$$= 32^3\left[\frac{1}{3} \times 2049.9110\right] = 22390492.0328$$

$$U_b = \frac{22390492.0328}{1514\,708} = 14.7820 \text{ k - Ft.}$$

(b) Strain energy in the springs

$$U_s = \int_o^k F\,dy = \int_o^k [ay + by^2 + cy^3]dy = \left[\frac{a}{2}y^2 + \frac{b}{3}y^3 + \frac{c}{4}y^4\right]_o^k$$

$$= \frac{a}{2}\,k^2 + \frac{b}{3}k^3 + \frac{c}{4}k^4 \tag{26}$$

Substituting the values for a, b and c from equation 23, the following expression for the strain energy in a spring is obtained.

$$U_s = 22.15\,k^2 - 4.8994k^3 + 0.6122k^4 \tag{27}$$

where, k is maximum deformation of a particular spring and corresponds to y_a, y_b and y_c in the table B.

The total strain energy in the springs is from symmetry

$U_s = 2(U_{s,a} + U_{s,b}) + U_{s,c}$.

Where, $U_{s,a}$, $U_{s,b}$ and $U_{s,c}$ are the strain energy in the springs a, b and c respectively. The computations are carried out as follows:

$y_a^2 = 0.0307 \qquad y_a^3 = 0.0054 \qquad y_a^4 = 0.0009$

$y_b^2 = 0.5571 \qquad y_b^3 = 0.4158 \qquad y_b^4 = 0.3104$

$y_c^2 = 1.1410 \qquad y_c^3 = 1.2188 \qquad y_c^4 = 1.3018$

$U_{s,a} = 0.6616 - 0.0264 + 0.0006 = 0.6376$

$U_{s,b} = 12.3398 - 2.0372 + 0.1900 = 10.4926$

$U_{s,c} = 25{,}2732 - 5.9714 + 0.7970 = 20.2788$

$U_s = 2(U_{s,a} + U_{s,b}) + U_{s,c} = 2(10.4926 + 0.6376) + 20.2788$

$= 22.2604 + 20.2788 = 42.4852$ ft-kip

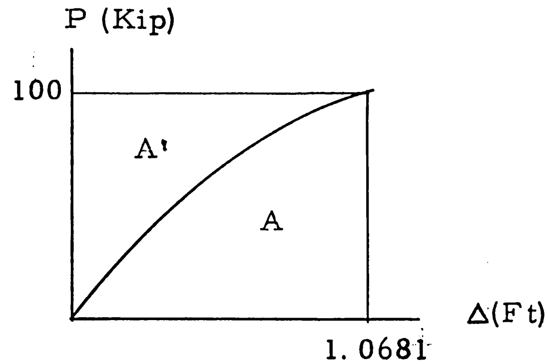
The total internal work is then

$W_i' = U_b + U_s = 14.7820 + 42.4852 = 57.2672$ ft-kips

(c) Work done by the external load

The external work is the area under the curve in Fig. 12 and 13. Simpson's rule has been used to compute this area.

Fig. 13



$$A' = \frac{h}{3}[(y_o + y_n) + 4\,(y_i + \ldots\ldots + y_{n-1}) + 2(y_2 + \ldots\ldots y_n)]$$

$$= \frac{10}{3}[1.0682 + 4(0.0880 + 0.2742 + 0.4761 + 0.6946 + 0.9382)$$

$$+ 2(0.1793 + 0.3730 + 0.5837 + 0.8144)]$$

$$= \frac{10}{3}[1.0682 + 4 \times 2.4729 + 2 \times 1.9504]$$

$$= \frac{10}{3} \times 14.8606 = 49.535 \quad \text{k-ft}$$

and

$$W_e = A = 100 \times 1.0682 - A' = 57.285 \quad \text{kips - ft,}$$

which checks within the limits of computational accuracy the value obtained for internal energy.

(d) Comparison of strain energy between non-linear and approximated linear system.

The strain energy in the approximated linear system has been computed using the information in Table B, and its value is found to be

$$W_i^2 = U_s^2 + U_b^2 = 40.2873 + 17.0815 = 66.3716 \text{ kips - ft.}$$

and

$$\frac{W^l_i}{W^n_i} = \frac{66.3716}{57.2672} = 1.1589,$$

which demonstrates that in this example the linear approximation indicates that about 15% more energy would be stored in the system that is actually the case.

TABLE A and B

A. Loads and Deflections at the Load Point (3rd support)

Load (kip)	10	20	30	40	50	60
Defl. at c. (ft)	0.0880	0.1793	0.2742	0.3730	0.4761	0.5837

	70	80	90	100
	0.6964	0.8144	0.9382	1.0682

B. Deflections and Reactions at the Supports a, b and c Due to 100 kips

Supports	a	b	
y_n (ft)	0.1753	0.7647	1.0681
R_n (kips)	7.3286	25.9044	33.5341
y_1 (ft)	0.369	0.995	1.337
R_1 (kips)	9.0812	24.4691	32.8993

where the subscript l and n indicate the linear and non-linear systems.

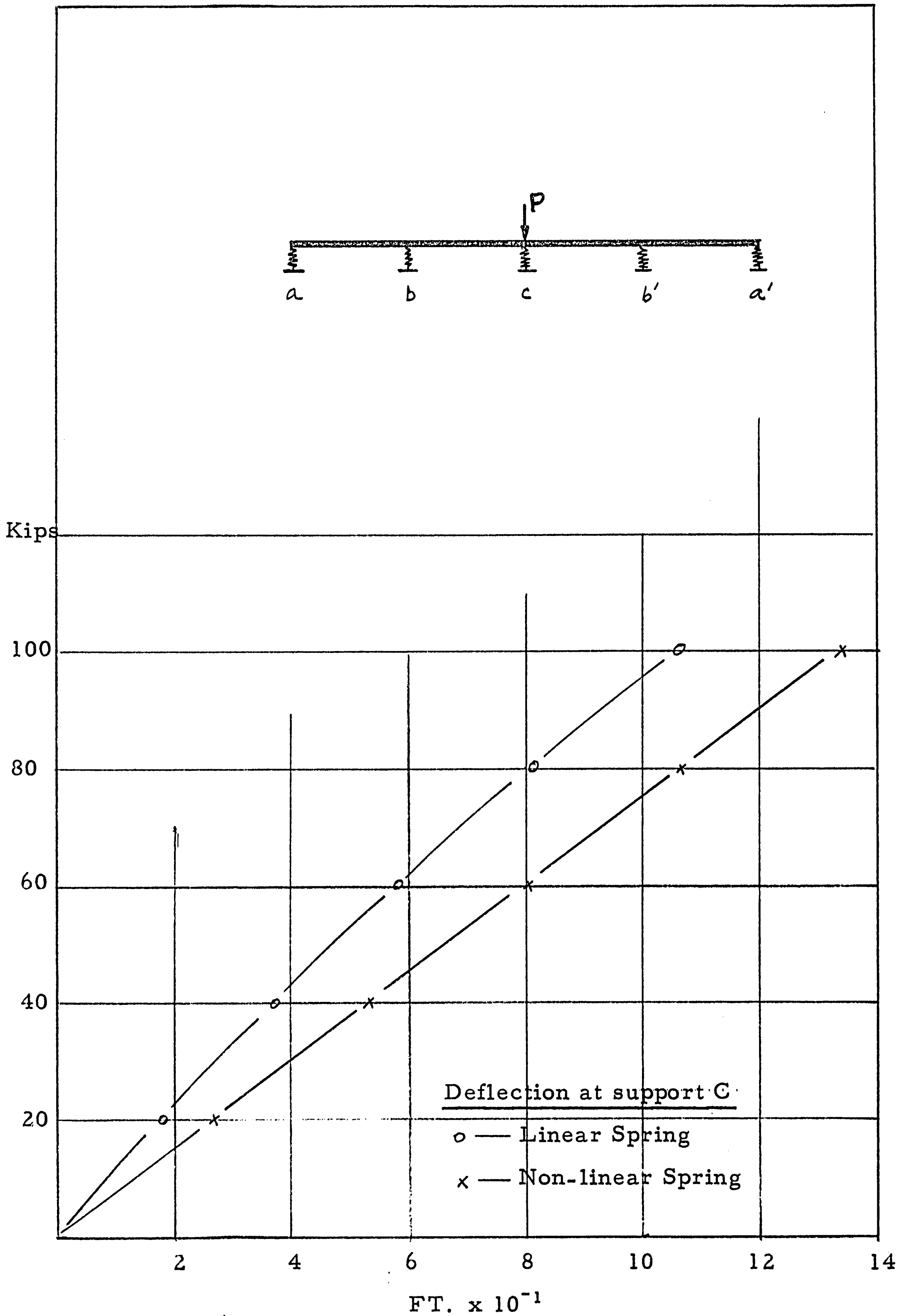

P
a
b
c
b'
a'
Kips
100
80
60
40
20
2
4
6
8
10
12
14
Deflection at support C
o — Linear Spring
x — Non-linear Spring
FT. x 10^{-1}

FIG. 12

VII

CONCLUSION

1. The two criteria for designing beams on discrete elastic supports are generally the beam working at a specified limiting flexural stress, while the springs are operating at their specified limit deflection. Analytical methods based on linear approximations are not able to predict achievement of either of these criteria, and a different method of analysis based on non-linearity is recommended. The technique developed in this study will furnish this alternate approach.

2. Although it was assumed that the mechanical behavior of the spring is the same in both tension and compression in the example considered, the method is still applicable in the case of springs whose mechanical behavior is not the same in tension and compression or when some mathematical discontinuities occur in the force function.

3. No attempt has been made in this study to rigorously resolve the important question of convergence which merits consideration for further study. However, the behavior of many particular examples studied was observed and is worthy of comment here.

For all the numerical examples considered in this study, no difficulty with slow convergence was experienced. In all attempts to force divergence by purposely assuming unrealistic initial displacement and slope values, divergence was very rapid and easily recognizable. No difficulty with an elusive oscillating type divergence was found to occur. No solutions obtained appeared unreasonable from the standpoint of physical behavior of the beam.

4 Solution by desk calculator using the method proposed is practical for beams of not more than 5 or 6 spans.

APPENDIX

References

(1) Maxwell, J. C., On the Calculations of the Equilibrium and Stiffness of Frames, Philosophical Mag. (4), 27, 294, 1864

(2) Timoshenko, S., Strength of Materials Part II, D. Van Nostrand Company, Inc., 1930, pp. 1 - 25

(3) Seely, F. B., Smith, J. O., Advanced Mechanics of Materials, Second Edition, John Wiley and Sons, New York, 1957, pp. 188-219

(4) Firmage, D. A., Chiu, R. H., Influence Lines for Continuous Beams on Elastic Supports, Brigham Young University Press, Provo, Utha, 1961

(5) Newmark, N. M., Austin, W. J., Step by Step Method of Beam on Linear Elastic Supports, Unpublished Class note, Rice University, Houston, Texas, 1962

(6) Koiter, W. T., Bereckening van veerend gesteun de balken, De Ingenieur 55 (1940), No. 35, P. 296, W. 57

(7) Biezno, C. B., Grammel, R., Engineering Dynamics, Volume II, Blackie and Son Limited, 1956, pp. 14-68

General References

Kinney, J. S., Indeterminate Structural Analysis, Addison-Wesley Publishing Co., Inc., 1957

Shanley, F. R., Strength of Materials, McGraw-Hill Book Company, Inc., 1957

Zaguskin, V. L., Hand Book of Numerical Methods for the Solution of Algebraic and Transcendental Equations, Pergamon Press, 1961

Thadani, B. N., Continuous Beams on Elastic Supports, Civil Engineering and Public Work Review, Dec., 1958

Brown, E. H., The Energy Theorems of Structural Analysis, Engineering, Vol. 179, March, 1955