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THE COMPUTATION OF OPTIMAL CONTROLS WITH A STATE VARIABLE INEQUALITY CONSTRAINT

by

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ABSTRACT

The Computation of Optimal Controls with
a State Variable Inequality Constraint

by

John L. Tietze

An algorithm for the numerical solution of optimal control problems involving a functional I subject to differential constraints, terminal constraints, and a scalar state variable inequality constraint is developed. First the inequality constraint is converted to an equality constraint by a Valentine-type transformation; then this new problem is scaled so that the subarcs which comprise the total trajectory are normalized to unit length. The resultant normalizing parameters are then included in the overall optimal control problem, and this final problem is solved by a modification of the sequential gradient-restoration algorithm.

The new algorithm has three major advantages:

(i) time normalization transforms variable time corner conditions and variable time terminal conditions into fixed time corner conditions and fixed time terminal conditions, with subsequent ease of implementation.

(ii) inequality satisfaction is enforced at all stages of the algorithm.

(iii) a clear-cut criterion for choosing gradient stepsize is employed.

Four numerical examples illustrating the theory are presented.

1. Introduction

Early work on the constrained† variational problem dates to Valentine in 1937, reference 1. Later work, which included necessary conditions for optimality of constrained problems, was done by Gamkrelidze in reference 2, Dreyfus in reference 3, Berkovitz in references 4 and 5, and Bryson et al. in reference 6. Therefore all the relevant theory for the optimal control problem with state variable inequality constraints was known by the end of 1963. The only problem remaining was the development of computational algorithms to solve the problem stated by the necessary conditions. This problem has occupied the attention of many investigators over the last ten years.

In the construction of an algorithm, two approaches have been followed - direct and indirect. In the direct approach, when a portion of the trajectory lies on the inequality boundary, the control vector is determined so as to keep the trajectory on the boundary. The entrance time onto the boundary and the exit time from the boundary are determined from the necessary conditions of the problem. In the indirect approach, additional state variables are introduced and/or the functional to be minimized is modified suitably to account for the constraint.

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† In the context of this thesis, constrained refers to a problem involving a state variable inequality constraint.

In the area of direct approaches, some of the earlier work was done in references 7-8, while more recent work has been done in references 9-12. The common element in references 7-12 is that the number and sequence of subarcs comprising the extremal arc must be known a priori. On the inequality boundary, the inequality constraint is employed with the equality sign. The equality constraint is then differentiated as many times as required (K times), until the control variable appears explicitly. Then, the equality constraint and its first K-1 derivatives are employed as corner conditions at the entrance to the inequality boundary, while the final derivative is used as a control variable equality constraint to be satisfied everywhere on the boundary.

In the area of indirect approaches, the most widely used are the penalty function and mathematical programming techniques. In the penalty function technique, the original constrained optimization problem is transformed into an equivalent unconstrained optimization problem through suitable modification of the functional to be minimized. In this connection, penalty function techniques were developed in reference 13 in conjunction with a conjugate gradient algorithm and in reference 14 in conjunction with a generalized Newton-Raphson algorithm. In the mathematical programming technique, the system of differential equations is discretized over the interval of integration and the resulting difference equations are solved by Mathematical Programming algorithms. References 15, 16, and 17 discuss several algorithms in detail. The major advantage of these approaches is that the number and sequence of subarcs composing the extremal arc need not be known a priori. Naturally there are some disadvantages. In the penalty function approach, the choice of penalty constant is arbitrary. Generally the problem is solved for increasing values of the penalty constant, and in the limit as the constant becomes infinite the computed solution converges to the actual solution. In reference 18 it is proven that this technique generates behavior which seriously degrades the accuracy of numerical integration techniques. The disadvantage of the mathematical programming approach revolves around the size of the discretization step. For good accuracy it should be small, but to keep the size of the problem within manageable proportions it should be large. There is no clear criterion for choosing a stepsize to yield a desired accuracy.

Another indirect approach, developed in reference 18, uses a Valentine-type transformation, or slack variable, to convert the inequality constraint to an equality constraint. The resulting equality constraint is then differentiated K times until the control variable appears explicitly. Defining the Kth derivative of the slack variable as a new control and defining the other slack variable derivatives as additional state variables, the Kth equation is then solved for the original control in terms of all of the state variables and the new control. The original control is then replaced and the optimization takes place in the new and larger (by K) state space with respect to the new control. Several disadvantages are apparent. Accurate results depend on the slack variable and its derivatives being zero when the trajectory lies on the constraint boundary. Because of the type of numerical algorithm used to solve the problem, the slack variable is never identically zero, even at convergence. Therefore the trajectory will never lie on the constraint boundary, but only approach it. Second, the transformed problem is singular when the trajectory is on the constraint boundary. This means that the new control may have discontinuities at the entrance and exit points of the boundary, a severe problem for numerical integration methods. Finally, it may not be possible to solve explicitly for the old control. This will be generally true if a nonlinear system is being optimized.

This last disadvantage has been removed in references 19 and 20. Instead of solving for the old control, the equation is adjoined to the functional as a non-differential equality constraint via a Lagrange multiplier. This permits solution of the problem without having to take special precautions when solving nonlinear or transcendental equations.

This thesis develops a hybrid algorithm in an attempt to combine the best features of both the direct and indirect approaches. Philosophically speaking, the new algorithm is a combination of the direct approach as outlined in reference 11 with the indirect approach as outlined in reference 19. As in reference 19 the state variable inequality constraint is transformed into an equality constraint by means of a Valentine-type transformation; i.e. a slack variable. This new equality constraint is then differentiated K times until the control variable appears explicitly. The slack variable and its first K-1 derivatives are defined as new state variables, while the Kth derivative is defined as a new control. Finally the Kth derivative of the state variable constraint interrelates the original control with all the state variables and the new control.

While on the state constraint boundary, the new state variables and the new control must be zero. The new algorithm is designed to enforce this zero condition at all times. Therefore one of the disadvantages of the approach of references 18 and 19 is eliminated. To determine the boundary entrance and exit times, the algorithm divides the original problem into separate subarcs, as in reference 11. Then each subarc is normalized to unit length by a parameter, and the parameter vector is included as part of the optimization problem. At convergence the values of the parameter vector will tell exactly when the boundary entrance and exit occurred.

The new algorithm belongs to the sequential gradient-restoration class as defined by Miele, et al. in reference 21. A major property of the algorithm is: at the end of each gradient-restoration cycle the state trajectories satisfy the constraints to a given accuracy. This means that at convergence, the algorithm has produced a sequence of feasible but suboptimal solutions, except for the final cycle.

2. Development of the Algorithm

The basic problem to be solved numerically is the following.

$$\text{minimize} \quad I = g(x)_\tau + \int_0^\tau f(x,u,\theta)\,d\theta \qquad (1)$$

subject to the following constraints

$$dx/d\theta = \Phi(x,u,\theta) \qquad x(0) = \text{given} \qquad (2)$$

$$\Psi(x)_\tau = 0 \qquad (3)$$

$$\text{and} \quad L(x,\theta) \le 0 \qquad (4)$$

where x is a vector of state variables (n long), u is a vector of control variables (m long), Φ is a vector of length n, Ψ is a vector of terminal constraints (q long), and L is a scalar state variable inequality constraint. Now, using a technique due to Valentine, rewrite Equation (4) as an equality constraint by adding a so-called slack variable:

$$L(x,\theta) + \beta^2 = 0 \qquad (5)$$

Equation (5) is now differentiated K times with respect to θ, yielding the following equations:

$$\begin{aligned} L_1(x,\theta) + 2\beta\beta_1 &= 0 \qquad (6)\\ L_2(x,\theta) + 2\beta_1^2 + 2\beta\beta_2 &= 0\\ \cdot \quad \cdot \quad \cdot \quad \cdot &\\ \cdot \quad \cdot \quad \cdot \quad \cdot &\\ L_k(x,u,\theta) + [\text{terms in } \beta_1,,\beta_{k-1}] + \beta\beta_k &= 0 \end{aligned}$$

where the subscripts on L and β denote derivative with respect to θ. Using the Kth equation, one may solve for the control u to obtain:

$$u = G(x, \beta_{k-1}, \ldots\ldots, \beta_1, \beta\beta_k, \theta) \tag{7}$$

Using Equation (7), and treating $\beta, \ldots, \beta_{k-1}$ as additional state variables, the following unconstrained control problem, with β_k as the new control, is obtained:

$$I = g(x)_\tau + \int_0^\tau f(x, G(\ldots.), \theta)\, d\theta \tag{8}$$

$$dx/d\theta = \Phi(x, G(\ldots.), \theta) \qquad x(0) = \text{given} \tag{9-1}$$

$$\Psi(x)_\tau = 0 \tag{9-2}$$

$$\left.\begin{array}{lll} d\beta/d\theta & = \beta_1 & \beta(0) \\ d\beta_1/d\theta & = \beta_2 & \beta_1(0) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ d\beta_{k-1}/d\theta & = \beta_k & \beta_{k-1}(0) \end{array}\right\} \text{determined from } x(0) \text{ and Equations (5) and (6)} \tag{10}$$

Although the original problem has been converted to an unconstrained form, the constrained subarc of the original problem has also been transformed - into a singular subarc. Thus the possibility that β_k will be discontinuous at the junction of the unconstrained and constrained subarcs exists. Since numerical integration techniques operate by approximating the function and its derivative over a grid of points, a discontinuity in the derivative will affect integration accuracy unless the method being used is designed specifically for this case. To bypass this condition, the problem formulated in Equations (8-10) is restated in a slightly different form and then is split into three separate subarcs with the central subarc lying on the constraint boundary. Splitting the problem into three subarcs allows the integration to proceed along each subarc without the problem of integrating across the control discontinuity.†

First, consider a restatement of the problem in the following form. Here the state vector x has been expanded to n+k to include the additional state variables, and the control vector has been expanded to m+1 to include the new control β_k. Further, the final equation in Equation (6) is not explicitly solved for u, but is kept as an equality constraint over the entire interval of integration. Therefore:

$$\text{minimize } I = \tilde{g}(x)_\tau + \int_0^\tau \tilde{f}(\tilde{x},\tilde{u},\theta)d\theta \qquad (11)$$

subject to

$$d\tilde{x}/d\theta = \tilde{\Phi}(\tilde{x},\tilde{u},\theta) \qquad \tilde{x}(0) = \text{given} \qquad (12)$$

$$\tilde{S}(\tilde{x},\tilde{u},\theta) = 0 \qquad (13)$$

$$\tilde{u}_{m+1}(\theta) = \beta_k(\theta) = 0 \qquad \theta_1 \le \theta \le \theta_2 \qquad (14)$$

$$\tilde{x}_{n+i}(\theta_1) = 0 \quad i = 1,\ldots,k \qquad (15)$$

$$\tilde{\Psi}(\tilde{x})_\tau = 0 \qquad (16)$$

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† For the rest of this thesis, the main problem is considered to have three subarcs with the central subarc lying on the constraint boundary. If the constraint boundary is encountered more than once, it is a simple modification of the algorithm to include this case.

Equation (13) is the final equation from the set in Equation (6). Equations (14-15) result from the fact that, on the constraint boundary, β and all of its derivatives must be zero. The trajectories are on the boundary from θ_1 to θ_2, θ_1 being the entry point and θ_2 being the exit point.

The problem outlined above is now transformed into one which has three subarcs of unit length. The entry, exit, and possibly the final time are considered to be parameters which are optimized with the rest of the problem. This device is attributed to Long, reference 22. With this transformation, the problem is changed from one with variable entry and exit points to one with fixed entry and exit points.

The following transformation between t and θ is used:

$$\theta = \begin{cases} \theta_1 t & 0 \le t \le 1 \\ \theta_1 + (\theta_2 - \theta_1)(t-1) & 1 \le t \le 2 \\ \theta_2 + (\theta_3 - \theta_2)(t-2) & 2 \le t \le 3 \end{cases} \qquad (17)$$

where θ_1 is the boundary entrance point, θ_2 is the boundary exit point, and $\theta_3 = \tau$ is the final time, which may be variable. The parameter vector π is defined as (θ_1, θ_2) if the final time is fixed, or $(\theta_1, \theta_2, \theta_3)$ if the final time is free.

The transformed problem now becomes:

$$I = g(x)_3 + \int_0^3 f(x,u,\pi,t)dt \tag{18}$$

$$\dot{x} = \phi(x,u,\pi,t) \qquad x(0)= \text{given} \tag{19}$$

$$S(x,u,\pi,t) = 0 \tag{20}$$

$$N(u) = u_{m+1}(t) = 0 \qquad 1 \le t \le 2 \tag{21}$$

$$M(x)_1 = x_{n+1}(1),\ldots,x_{n+k}(1) = 0 \tag{22}$$

$$\Psi(x,\pi)_3 = 0 \tag{23}$$

The necessary conditions for the above problem are derived in the next section.

2.1 Necessary Conditions

To keep the notation uncluttered, explicit dependence on x, u, π, and t will be eliminated and only indicated where necessary. Further, evaluation of a function at a given time t_1 will not be indicated by a subscript but will be parenthesized like an argument. Thus Equation 23 would become $\Psi(3)$.

Define the Hamiltonian of the problem as

$$H = f - \lambda^T\phi + \rho_s S + \rho_n N \tag{24}$$

where $\lambda(t)$ is an n+k vector of Lagrange multipliers, $\rho_s(t)$ and $\rho_n(t)$ are scalar multipliers, with $\rho_n(t)$ defined as:

$$\rho_n(t) = 0 \qquad 0 \le t \le 1 \;,\; 2 \le t \le 3 \qquad (25)$$
$$\qquad \ne 0 \qquad 1 \le t \le 2$$

The augmented performance index is then

$$J = \int_0^3 (H + \lambda^T \dot{x})dt + \sigma^T M(1) + \mu^T \Psi(3) + g(3) \qquad (26)$$

where σ is a k-vector multiplier and μ is a q-vector multiplier. Integrating by parts over each subarc yields:

$$J = \int_0^3 (H - \dot{\lambda}^T x)dt + \sigma^T M(1) + \mu^T \Psi(3) + g(3) \qquad (27)$$
$$+ (\lambda^T x)\Big|_0^1 + (\lambda^T x)\Big|_1^2 + (\lambda^T x)\Big|_2^3$$

Taking the first variation of J we have:

$$\begin{aligned}\delta J = &\int_0^3 (H_x - \dot{\lambda})^T \Delta x \, dt + \int_0^3 H_u{}^T \Delta u \, dt \qquad (28)\\ &+ [\int_0^3 H_\pi dt + \Psi_\pi(3)\mu + g_\pi(3)]^T \Delta\pi \\ &+ [\lambda(1-) - \lambda(1+) + M_x(1)\sigma]^T \Delta x(1) \\ &+ [\lambda(2-) - \lambda(2+)]^T \Delta x(2) \\ &+ [\lambda(3) + g_x(3) + \Psi_x(3)\mu]^T \Delta x(3) - \lambda(0)^T \Delta x(0)\end{aligned}$$

where continuity of the Δx across the junctions has been utilized.

Since the variations $\Delta x(t)$, $\Delta u(t)$, and $\Delta\pi$ are free and since $\Delta x(0) = 0$, the following conditions must be satisfied if the first variation of J is to vanish.

$$\dot{\lambda} = H_x \tag{29}$$

$$H_u = 0 \tag{30}$$

$$\int_0^3 H_\pi \, dt + g_\pi(3) + \Psi_\pi(3)\mu = 0 \tag{31}$$

$$\lambda(1-) - \lambda(1+) + M_x(1)\sigma = 0 \tag{32}$$

$$\lambda(2-) - \lambda(2+) = 0 \tag{33}$$

$$\lambda(3) + g_x(3) + \Psi_x(3)\mu = 0 \tag{34}$$

subject to:

$$\dot{x} = \phi \qquad x(0) = \text{given} \tag{35}$$

$$S = 0 \tag{36}$$

$$N(u) = 0 \qquad 1 \le t \le 2 \tag{37}$$

$$\Psi(3) = 0 \tag{38}$$

$$M(1) = 0 \tag{39}$$

Summarizing, we seek the functions $x(t)$, $u(t)$, π and the multipliers $\lambda(t)$, $\rho_s(t)$, $\rho_n(t)$, σ , μ which satisfy the constraints (35) - (39) and the optimality conditions (29) - (34).

2.2 Approximate Methods

Since the equations to be solved will be nonlinear in general, some form of iterative approximation must be used in their solution. Therefore define the scalar functionals P and Q which denote the error in the constraint and the optimum conditions respectively. In the context of this thesis the norm of a vector is defined as:

$$Z(a) = a^T a \tag{40}$$

Define:

$$P = \int_0^3 \{Z(\dot{x} - \phi) + Z(S)\}\,dt + \int_1^2 Z(N)\,dt + Z(M) + Z(\psi) \tag{41}$$

$$\begin{aligned} Q = {} & \int_0^3 \{Z(\dot{\lambda} - H_x) + Z(H_u)\}\,dt + \\ & Z\left(\int_0^3 H_\pi\,dt + g_\pi(3) + \psi_\pi(3)\mu\right) + \\ & Z(\lambda(1-) - \lambda(1+) + M_x(1)\sigma) \\ & Z(\lambda(2-) - \lambda(2+)) \\ & Z(\lambda(3) + g_x(3) + \psi_x(3)\mu) \end{aligned} \tag{42}$$

For the exact variational solution $P = 0$ and $Q = 0$, while any approximate solution yields $P > 0$ and/or $Q > 0$. In practice it is difficult to numerically obtain zero, so that a convergence criterion might become:

$$P < \epsilon_1 \quad Q < \epsilon_2 \tag{43}$$

where ϵ_1 and ϵ_2 are small preselected numbers.

The algorithm proposed for the solution of the preceding problem is a variation of the sequential gradient-restoration algorithm of reference 21. The algorithm consists of an alternate succession of gradient phases and restoration phases. In the gradient phase, the value of the functional I is decreased, while in the restoration phase the value of P is reduced to its preselected accuracy. In both phases, the variations in the functions are obtained by placing a quadratic constraint on the control and parameter variations.

2.3 Sequential Gradient-Restoration Algorithm

A generalized algorithm which applies for either the gradient or restoration phase can be derived in the following manner. Let $x(t)$, $u(t)$, and π denote the nominal functions and let $\tilde{x}(t)$, $\tilde{u}(t)$, and $\tilde{\pi}$ denote the varied functions. Further let α denote the stepsize and let $A(t)$, $B(t)$, and C denote the displacements per unit stepsize. These quantities satisfy the definitions:

$$\tilde{x}(t) = x(t) + \Delta x(t) \qquad \tilde{u}(t) = u(t) + \Delta u(t) \tag{44}$$

$$\tilde{\pi} = \pi + \Delta\pi$$

and

$$\Delta x(t) = \alpha A(t) \qquad \Delta u(t) = \alpha B(t) \qquad \Delta\pi = \alpha C \tag{45}$$

The variations $\Delta x(t)$, $\Delta u(t)$, and $\Delta\pi$ are generally selected to have some desirable properties for the problem of interest. For this problem, a desirable property in the gradient phase would be the reduction of J while keeping the increase in P relatively small. For the restoration phase a desirable property would be the decrease of P with only a small change in the value of J. To obtain either property it is only necessary to select the variations so that the first variation of J or P is negative. This means that for a sufficiently small stepsize α, the functional J or P will decrease.

To aid in the selection of the correct variations, the first variations of J and P are given below:

$$\delta J = \int_0^3 (H_x - \dot{\lambda})^T \Delta x dt + \int_0^3 H_u{}^T \Delta u dt \tag{46}$$

$$+ [\int_0^3 H_\pi dt + \Psi_\pi(3)\mu + g_\pi(3)]^T \Delta\pi$$

$$+ [\lambda(1-) - \lambda(1+) + M_x(1)\sigma]^T \Delta x(1)$$

$$+ [\lambda(2-) - \lambda(2+)]^T \Delta x(2)$$

$$+ [\lambda(3) + g_x(3) + \Psi_x(3)\mu]^T \Delta x(3)$$

$$\delta P = 2\{\int_0^3 [(\dot{x} - \phi)^T(\Delta\dot{x} - \phi_x^T \Delta x - \phi_u{}^T \Delta u - \phi_\pi{}^T \Delta\pi) \tag{47}$$

$$+ S^T(S_x{}^T \Delta x + S_u{}^T \Delta u + S_\pi{}^T \Delta\pi)$$

$$+ N^T(N_u{}^T \Delta u)]dt + M(1)^T(M_x{}^T(1)\Delta x(1))$$

$$+ \Psi(3)^T(\Psi_x{}^T(3)\Delta x(3) + \Psi_\pi{}^T(3)\Delta\pi)\}$$

The variations to be used can now be selected. In the following R_1 and R_2 are constants which depend on the phase, i.e. for a gradient phase $R_1 = 0$ and $R_2 = 1$, for a restoration phase $R_1 = 1$ and $R_2 = 0$. With this in mind, the variations $\Delta x(t)$, $\Delta u(t)$, and $\Delta\pi$ must satisfy the following equations.

$$\Delta\dot{x} - \Phi_x{}^T\Delta x - \Phi_u{}^T\Delta u - \Phi_\pi{}^T\Delta\pi + \alpha R_1(\dot{x} - \Phi) = 0 \tag{48}$$

$$\Delta x(0) = 0 \tag{49}$$

$$\Psi_x{}^T(3)\Delta x(3) + \Psi_\pi{}^T(3)\Delta\pi + \alpha R_1\Psi(3) = 0 \tag{50}$$

$$S_x{}^T\Delta x + S_u{}^T\Delta u + S_\pi{}^T\Delta\pi + \alpha R_1 S = 0 \tag{51}$$

$$N_u{}^T\Delta u + \alpha R_1 N = 0 \qquad 1\le t\le 2 \tag{52}$$

$$\dot{\lambda} - R_2 f_x + \Phi_x\lambda - \rho_s S_x = 0 \tag{53}$$

$$\Delta u + \alpha(R_2 f_u - \Phi_u\lambda + \rho_s S_u + \rho_n N_u) = 0 \tag{54}$$

$$\Delta\pi + \alpha\Big[\int_0^3 (R_2 f_\pi - \Phi_\pi\lambda + \rho\ S_\pi)dt + R_2 g_\pi(3) \tag{55}$$

$$+ \Psi_\pi(3)\mu\Big] = 0$$

$$\lambda(1+) - \lambda(1-) - M_x(1)\sigma = 0 \tag{56}$$

$$\lambda(2+) - \lambda(2-) = 0 \tag{57}$$

$$\lambda(3) + R_2 g_x(3) + \Psi_x(3)\mu = 0 \tag{58}$$

$$M_x{}^T(1)\Delta x(1) + \alpha R_1 M(1) = 0 \tag{59}$$

Using the above equations and the definitions of P and Q when subject to these equations , we have:

$$J = - \alpha Q \tag{60}$$

$$P = -2\alpha P \tag{61}$$

which yields the desired descent property in each case provided that $Q > 0$ and/or $P > 0$. On account of Equations (45), Equations (48-59) can be rewritten as:

$$\dot{A} - \Phi_x{}^T A - \Phi_u{}^T B - \Phi_\pi{}^T C + R_1(\dot{x} - \Phi) = 0 \tag{62}$$

$$A(0) = 0 \tag{63}$$

$$\Psi_x{}^T(3)A(3) + \Psi_\pi{}^T(3)C + R_1\Psi(3) = 0 \tag{64}$$

$$S_x{}^T A + S_u{}^T B + S_\pi{}^T C + R_1 S = 0 \tag{65}$$

$$N_u{}^T B + R_1 N = 0 \qquad 1 \le t \le 2 \tag{66}$$

$$\dot{\lambda} - R_2 f_x + \Phi_x \lambda - \rho_s S_x = 0 \tag{67}$$

$$B + R_2 f_u - \Phi_u \lambda + \rho_s S_u + \rho_n N_u = 0 \tag{68}$$

$$C + \int_0^3 (R_2 f_\pi - \Phi_\pi \lambda + \rho_s S_\pi)dt + R_2 g_\pi(3) + \Psi_\pi(3)\mu = 0 \tag{69}$$

$$\lambda(1+) - \lambda(1-) - M_x(1)\sigma = 0 \tag{70}$$

$$\lambda(2+) - \lambda(2-) = 0 \tag{71}$$

$$\lambda(3) + R_2 g_x(3) + \Psi_x(3)\mu = 0 \tag{72}$$

$$M_x{}^T(1)A(1) + R_1 M(1) = 0 \tag{73}$$

For given nominal functions $x(t)$, $u(t)$, π and the constants R_1 and R_2, Equations (44-45) and Equations (62-73) constitute a complete iteration leading to the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ providing one specifies the stepsize α. To repeat, for a gradient step $R_1 = 0$ and $R_2 = 1$; for a restoration step $R_1 = 1$ and $R_2 = 0$. In addition the multipliers ρ_n and ρ_s are defined via Equation (68) so that Equations (65-66) are satisfied exactly.

2.4 The Linear Multipoint Boundary Value Problem

Equations (62-73) represent a linear time-varying boundary value problem. The initial conditions on A are specified by Equation (63), the final conditions on λ are specified by Equation (72), and midpoint conditions on A are specified by Equation (73). The following technique, which is a modification of Miele's Method of Particular Solutions (ref. 23), is proposed for the solution.

In what follows the order of the differential systems 62 and 67 is n, the order of the constraint is k, the number of parameters is p, and the number of terminal constraints is q. First integrate the differential systems 62 and 67 forward in time n+p+k+1 times, and in each integration (subscript i) assign the following values to the n-vector $\lambda(0)$, the p-vector C, and the k-vector σ:

$$\lambda_i(0) = [\delta_{i1}, \delta_{i2}, \ldots\ldots, \delta_{in}]^T \tag{74}$$

$$C_i = [\delta_{i,n+1}, \ldots\ldots, \delta_{i,n+p}]^T \tag{75}$$

$$\sigma_i = [\delta_{i,n+p+1}, \ldots\ldots, \delta_{i,n+p+k+1}]^T \tag{76}$$

for $i = 1,\ldots,n+p+k+1$ and where δ_{ij} is the Kronecker delta. In this way one obtains the functions

$$A_i(t), B_i(t), \lambda_i(t), \rho_{si}(t), \rho_{ni}(t), \sigma_i, C_i \tag{77}$$

Next introduce the $n+p+k+1$ undetermined scalar constants K_i and form the linear combinations

$$A(t) = \sum K_i A_i(t) \qquad B(t) = \sum K_i B_i(t) \qquad C = \sum K_i C_i \tag{78}$$
$$\rho_s(t) = \sum K_i \rho_{si}(t) \qquad \rho_n(t) = \sum K_i \rho_{ni}(t)$$
$$\lambda(t) = \sum K_i \lambda_i(t) \qquad \sigma = \sum K_i \sigma_i$$

By an appropriate choice of the constants K_i and the components of the multiplier μ, these linear combinations can satisfy all the differential equations and the boundary conditions. This choice is determined by the following set of linear equations.

$$\sum K_i = 1 \tag{79}$$
$$\sum K_i (\Psi_x{}^T(3) A_i(3) + \Psi_\pi{}^T(3) C_i) + R_1 \Psi(3) = 0 \tag{80}$$
$$\sum K_i [\int_0^3 (R_2 f_\pi - \Phi_\pi \lambda + \rho_s S_\pi) dt + C_i] + R_2 g_\pi(3) + \Psi_\pi(3)\mu = 0 \tag{81}$$
$$\sum K_i \lambda_i(3) + R_2 g_x(3) + \Psi_x(3)\mu = 0 \tag{82}$$
$$\sum K_i M_x{}^T(1) A_i(1) + R_1 M(1) = 0 \tag{83}$$

The linear equations (79-83) represent $n+p+k+1+q$ equations in the n+p+k+1 unknown multipliers K_i and the q components of μ.

After determining the constants K_i, two methods of combination are possible. If all the quantities in Equation (77) have been saved, the composite solution of Equations (78) can be formed directly. An alternative is to use the K_i's to define $\lambda(0)$, C, and σ and integrate the differential system forward one final time. This final technique is the one used in the examples of this thesis.

2.5 Stepsize Determination

From the solution of the linear multipoint boundary value problem the functions A(t), B(t), C and the multipliers $\lambda(t)$, $\rho_n(t)$, $\rho_s(t)$, σ, μ are available. With these functions, one can form a one-parameter family of solutions:

$$\tilde{x}(t)= x(t) + \alpha A(t) \qquad \tilde{u}(t)= u(t) + \alpha B(t) \tag{84}$$

$$\tilde{\pi}= \pi + \alpha C$$

for which the augmented functional J and the constraint error P take the following form:

$$\tilde{J} = \tilde{J}(\alpha) \qquad \tilde{P} = \tilde{P}(\alpha) \tag{85}$$

In the restoration phase of the algorithm it is desired to have

$$\tilde{P}(\alpha) < \tilde{P}(0) \tag{86}$$

Satisfaction of inequation (86) can be ensured by using a bisection process, starting from the reference stepsize

$$\alpha = 1 \tag{87}$$

This value reduces the constraint error P to zero if all of the constraints are linear in x(t), u(t), and π. If the constraints are nonlinear, the Newton class restoration phase should exhibit quadratic convergence after stabilizing at $\alpha = 1$.

The reference stepsize for the gradient phase is determined by finding the minimum of $\tilde{J}(\alpha)$ along the direction. This stepsize is computed by assuming that $\tilde{J}(\alpha)$ has a quadratic representation, and then finding the minimum of this representation.

Let $\tilde{J}(\alpha)$ be represented by the quadratic form:

$$\tilde{J}(\alpha) = K_0 + K_1\alpha + K_2\alpha^2 \tag{88}$$

and let the coefficients of the quadratic be determined from the values of the ordinate and the slope at $\alpha = 0$ and the value of the ordinate at $\alpha = 1$. This yields the relations:

$$\tilde{J}(0) = K_0 \qquad \tilde{J}_\alpha(0) = K_1 \qquad \tilde{J}(1) = K_0 + K_1 + K_2 \tag{89}$$

From Equations (42-46) we have $\tilde{J}_\alpha(0) = -Q$, therefore

$$K_0 = \tilde{J}(0) \quad K_1 = -Q \quad K_2 = \tilde{J}(1) - \tilde{J}(0) + Q \tag{90}$$

With the coefficients known two possibilities exist, $K_2>0$ or $K_2<0$. If $K_2>0$, the quadratic function (88) has a minimum for the gradient stepsize value

$$\alpha = -K_1/2K_2 \tag{91}$$

If $K_2<0$, the quadratic function does not possess a minimum. This suggests the use of the following values of α in the gradient phase:

$$\alpha = -K_1/2K_2 \qquad \text{if} \quad K_2>0 \tag{92}$$

$$\alpha = 1 \qquad \text{if} \quad K_2<0 \tag{93}$$

Finally, for any given stepsize α, the new parameters must be such that the following inequality is satisfied:

$$\theta_1 < \theta_2 < \theta_3 \tag{94}$$

If it is not, then some systematic reduction of α should be made until satisfaction occurs. Inequation (94) is used to ensure that all subarc lengths will be finite and positive.

3.0 Numerical Computation

Experimental Conditions

In order to evaluate the previous algorithm, four numerical examples were considered. Computations were performed using the Rice University IBM 370/155 digital computer. Double Precision arithmetic was used, and the algorithm was programmed in FORTRAN IV. Each problem was broken into three subintervals with 20 integration steps per subinterval. All differential equations were integrated using Hamming's modified predictor-corrector method with a special Runge-Kutta starting procedure. All definite integrals were computed with a modified Simpson's rule.

Search Conditions

The determination of the gradient stepsize or the restoration stepsize was performed in accordance with Section 2.5. For the gradient phase, the stepsize was subject to the inequalities:

$$\tilde{J}(\alpha) < \tilde{J}(0) \qquad \tilde{P}(\alpha) \leq \tilde{P}(0) + 1 \tag{95}$$

For the restoration phase, the stepsize was subject to the inequality:

$$\tilde{P}(\alpha) < \tilde{P}(0) \tag{96}$$

Before checking either of these inequalities, the following inequality on the parameters must be satisfied:

$$\theta_1 < \theta_2 < \theta_3 \tag{97}$$

this ensures a finite and positive length for each subarc.

<u>Convergence</u>

Convergence of the algorithm was defined through the inequalities:

$$P \le \epsilon_1 \qquad Q \le \epsilon_2 \tag{98}$$

where

$$\epsilon_1 = 10^{-8} \qquad \epsilon_2 = 10^{-4} \tag{99}$$

<u>Nonconvergence</u>

Conversely, nonconvergence was defined as:

$$\text{(a)} \quad N > 50 \tag{100-1}$$

$$\text{(b)} \quad N_s > 10 \tag{100-2}$$

Here N is the number of the iteration and N_s is the number of bisections of the stepsize α required to satisfy ineqn. (95) or (96). Inequation (100) is primarily to stop algorithm execution should nonstandard conditions occur.

Nominal Selection

Experimentation with the algorithm has demonstrated a sensitivity to the selection of nominals. This takes the form of failing [nonconvergence (b)] during the initial restoration because the boundary subarc is reduced to zero length while the parameter perturbations remain finite and such as to reduce the boundary subarc below zero. This effect seems to be due to the fact the S function of Equation (13) is nonzero initially. In an effort to eliminate this behavior, the following rationale is proposed for selecting nominals.

In general the values for the state variables at boundary entrance and exit are known, especially for the newly added slack variables. Therefore choose the nominals so that straight lines join the initial or final conditions to the boundary entrance or exit points respectively. The new control must be zero on the boundary subarc. Therefore choose the nominal to be zero on this subarc and to be 1 on the other subarcs. The sign for the control off the boundary is determined by the sign of the nominal slack variable derivative. Choose the old control so the S function is zero on the central subarc, and otherwise use a constant control. The value of the constant is determined from continuity across the boundaries. The initial value of the parameters does not seem to be critical, and can be estimated from the unconstrained solution if available. Otherwise values of .1

and .9 times the maximum time seem reasonable.

Example 3.1

Consider the problem of minimizing the functional

$$I = \int_0^1 (x^2 + y^2 + .005\, u^2)\, d\theta \tag{101}$$

with respect to the states $x(\theta)$, $y(\theta)$ and the control $u(\theta)$ which satisfy the differential constraints

$$\begin{aligned} dx/d\theta &= y \\ dy/d\theta &= -y + u \end{aligned} \tag{102}$$

and the state variable inequality constraint

$$y - 8(\theta - .5)^2 + .5 \le 0 \tag{103}$$

and the boundary conditions

$$x(0) = 0 \qquad y(0) = -1 \tag{104}$$

In order to ensure satisfaction of the state inequality (103) over the interval of definition, introduce the additional state-slack variable $z(\theta)$ through the following relation:

$$y - 8(\theta - .5)^2 + .5 + .5\, z^2 = 0 \tag{105}$$

which upon differentiation with respect to θ becomes

$$dy/d\theta \;-\; 16(\theta - .5) \;+\; z\,dz/d\theta \;=\; 0 \qquad (106)$$

Now introduce the new control variable $w(\theta)$ defined as:

$$dz/d\theta \;=\; w(\theta) \qquad (107)$$

In light of Eqns. (102), (105), (106), and (107), the following relations among the states and the controls exist:

$$u \;-\; y \;+\; 16(\theta - .5) \;+\; zw = 0 \qquad 0 \le \theta \le 1 \qquad (108)$$

$$w = 0 \qquad \theta_1 \le \theta \le \theta_2$$

where the state is on the boundary from θ_1 to θ_2.

The boundary conditions on the slack variable $z(\theta)$ must be consistent with the definition (105) and the boundary conditions Eqn. (104). This generates an initial condition:

$$z(0) \;=\; -\sqrt{5} \qquad (109)$$

In addition, the following midpoint condition is imposed:

$$z(\theta_1) = 0 \qquad (110)$$

To convert the problem to the model of this thesis,

introduce the normalized time t, as defined by Eqn. (17). When normalized time is employed, the problem is reformulated as that of minimizing the functional

$$I = \int_0^1 \theta_1(x^2 + y^2 + .005\,u^2)dt + \int_1^2 (\theta_2 - \theta_1)(x^2 + y^2 + .005\,u^2)dt + \int_2^3 (1 - \theta_2)(x^2 + y^2 + .005\,u^2)dt \tag{111}$$

with respect to the state x(t),y(t),z(t), the control u(t), w(t), and the parameter components θ_1, θ_2 which satisfy the differential constraints

$$\begin{aligned} \dot{x} &= \theta_1 y \\ \dot{y} &= \theta_1(u - y) \\ \dot{z} &= \theta_1 w \end{aligned} \qquad 0 \le t \le 1 \tag{112}$$

$$\begin{aligned} \dot{x} &= (\theta_2 - \theta_1)y \\ \dot{y} &= (\theta_2 - \theta_1)(u - y) \\ \dot{z} &= (\theta_2 - \theta_1)w \end{aligned} \qquad 1 \le t \le 2$$

$$\begin{aligned} \dot{x} &= (1 - \theta_2)y \\ \dot{y} &= (1 - \theta_2)(u - y) \\ \dot{z} &= (1 - \theta_2)w \end{aligned} \qquad 2 \le t \le 3$$

the control variable constraints

$$u - y + 16(\theta - .5) + zw = 0 \qquad 0 \le t \le 3 \qquad (113)$$

$$w = 0 \qquad 1 \le t \le 2$$

[where θ is defined by Eqn. (17)], and the multipoint conditions

$$x(0) = 0 \qquad x(3) = \text{free} \qquad (114)$$

$$y(0) = -1 \qquad y(3) = \text{free}$$

$$z(0) = -\sqrt{5} \quad z(1) = 0 \quad z(3) = \text{free}$$

In this problem,

$$n = 3 \quad m = 2 \quad p = 2 \quad k = 1 \quad q = 0 \qquad (115)$$

Since $n+p+k+1 = 7$, seven particular solutions are needed for each gradient or restoration iteration.

Assume the nominal state

$$x = -t/3 \quad y = -1 \quad z = \sqrt{5}\,(t-1) \quad 0 \le t \le 1 \qquad (116)$$

$$x = -t/3 \quad y = -1 \quad z = 0 \quad 1 \le t \le 2$$

$$x = -t/3 \quad y = -1 \quad z = \sqrt{5}\,(t-2) \quad 2 \le t \le 3$$

the nominal control

$$u = -5 \qquad w = +1 \qquad 0 \le t \le 1 \qquad (117)$$

$$u = 8t - 13 \qquad w = 0 \qquad 1 \le t \le 2$$

$$u = 3 \qquad w = -1 \qquad 2 \leq t \leq 3$$

and the nominal parameter components

$$\theta_1 = .25 \qquad \theta_2 = .75 \tag{118}$$

These nominals are consistent with the selection criteria outlined in Section 3.0, but they violate Eqns. (112). This being the case the algorithm starts with a restoration phase. After a total of 12 iterations the algorithm has converged to the required accuracy. Tables 1-6 show convergence history, and the optimal trajectories and multipliers for the converged solution.

Table 1 Convergence History for Example 3.1

Iteration	P	Q	I	Mode
0	.1333E+02		.1373E+01	REST
1	.3120E+01		.1180E+01	REST
2	.3640E+00		.7352E+00	REST
3	.4195E-03		.6765E+00	REST
4	.3953E-09	.3455E-01	.6760E+00	GRAD
5	.1782E+00		.2974E+00	REST
6	.9521E-04		.3331E+00	REST
7	.1618E-11	.4975E-02	.3325E+00	GRAD
8	.1571E+00		.1773E+00	REST
9	.1802E-04		.1883E+00	REST
10	.6815E-13	.2444E-03	.1882E+00	GRAD
11	.4167E-02		.1724E+00	REST
12	.6976E-08	.4659E-04	.1728E+00	GRAD

Table 2 Parameter Convergence History for Example 3.1

Iteration	θ_1	θ_2
0	.2500	.7500
1	.3710	.6889
2	.4799	.5942
3	.4755	.5820
4	.4754	.5823
5	.3456	.6019
6	.3707	.5978
7	.3704	.5978
8	.2898	.5789
9	.3034	.5839
10	.3034	.5839
11	.2811	.6237
12	.2826	.6226

Table 3 Converged State Trajectories for Example 3.1

Time	X	Y	Z
0.0	.0000E+00	-.1000E+01	-.2236E+01
0.2	-.3897E-01	-.4339E+00	-.1736E+01
0.4	-.5410E-01	-.1375E+00	-.1292E+01
0.6	-.5781E-01	-.1685E-01	-.8835E+00
0.8	-.5813E-01	-.1105E-01	-.4717E+00
1.0	-.6117E-01	-.1219E+00	-.1442E-12
1.0	-.6117E-01	-.1219E+00	-.1437E-12
1.2	-.7666E-01	-.3214E+00	-.1437E-12
1.4	-.1032E+00	-.4469E+00	-.1437E-12
1.6	-.1357E+00	-.4985E+00	-.1437E-12
1.8	-.1693E+00	-.4761E+00	-.1437E-12
2.0	-.1988E+00	-.3797E+00	-.1437E-12
2.0	-.1988E+00	-.3797E+00	-.1521E-12
2.2	-.2213E+00	-.2116E+00	.2260E+00
2.4	-.2308E+00	-.4576E-01	.5376E+00
2.6	-.2303E+00	.4367E-01	.9284E+00
2.8	-.2258E+00	.7015E-01	.1320E+01
3.0	-.2187E+00	.1369E+00	.1651E+01

Table 4 Converged Control Trajectories for Example 3.1

Time	U	W
0.0	.1213E+02	.9450E+01
0.2	.6863E+01	.8289E+01
0.4	.3323E+01	.7467E+01
0.6	.9787E+00	.7110E+01
0.8	-.7788E+00	.7662E+01
1.0	-.3600E+01	.8980E+01
1.0	-.3600E+01	.0000E+00
1.2	-.2711E+01	.0000E+00
1.4	-.1749E+01	.0000E+00
1.6	-.7127E+00	.0000E+00
1.8	.3976E+00	.0000E+00
2.0	.1582E+01	.0000E+00
2.0	.1582E+01	.2890E+01
2.2	.2185E+01	.3414E+01
2.4	.1747E+01	.4805E+01
2.6	.6571E+00	.5354E+01
2.8	.3952E+00	.4898E+01
3.0	.1932E+01	.3757E+01

Table 5 Lambda Multipliers for Converged Solution of Example 3.1

Time	λ_1	λ_2	λ_3
0.0	.2801E+00	.6178E+00	-.1028E+01
0.2	.2776E+00	.5301E+00	-.7973E+00
0.4	.2722E+00	.4869E+00	-.5937E+00
0.6	.2658E+00	.4653E+00	-.4064E+00
0.8	.2592E+00	.4497E+00	-.2176E+00
1.0	.2525E+00	.4280E+00	-.2025E-03
1.0	.2525E+00	.4280E+00	-.1658E-02
1.2	.2433E+00	.3779E+00	-.1658E-02
1.4	.2312E+00	.3070E+00	-.1658E-02
1.6	.2150E+00	.2256E+00	-.1658E-02
1.8	.1942E+00	.1440E+00	-.1658E-02
2.0	.1691E+00	.7277E-01	-.1658E-02
2.0	.1691E+00	.7277E-01	-.1658E-02
2.2	.1372E+00	.1696E-01	.5202E-02
2.4	.1029E+00	-.1033E-01	.2859E-02
2.6	.6807E-01	-.1636E-01	.9408E-04
2.8	.3362E-01	-.1242E-01	-.8603E-04
3.0	-.1688E-14	.1772E-11	-.6814E-11

Table 6 Remaining Multipliers for Example 3.1

Time	ρ_s	ρ_n
0.0	.1317E+00	
0.2	.1301E+00	
0.4	.1294E+00	
0.6	.1294E+00	
0.8	.1296E+00	
1.0	.1312E+00	
1.0	.1578E+00	-.5639E-03
1.2	.1382E+00	-.5639E-03
1.4	.1113E+00	-.5639E-03
1.6	.8077E-01	-.5639E-03
1.8	.4992E-01	-.5639E-03
2.0	.2240E-01	-.5639E-03
2.0	.2453E-01	
2.2	.1539E-02	
2.4	-.4141E-02	
2.6	-.1078E-02	
2.8	.2883E-02	
3.0	-.2972E-03	

σ = -.1456E-02

Example 3.2

Consider the problem of minimizing the functional

$$I = \tau \tag{119}$$

with respect to the states $x(\theta)$, $y(\theta)$ and the control $u(\theta)$ which satisfy the differential constraints

$$dx/d\theta = u \tag{120}$$
$$dy/d\theta = u^2 - x^2$$

and the state variable inequality constraint

$$.4 - y \geq 0 \tag{121}$$

and the boundary conditions

$$x(0) = 0 \qquad x(\tau) = 1 \tag{122}$$
$$y(0) = 0 \qquad y(\tau) = 0$$

In order to ensure satisfaction of the state inequality (121) over the interval of definition, introduce the additional state-slack variable $z(\theta)$ through the following relation:

$$.4 - y = z^2 \tag{123}$$

which upon differentiation with respect to θ becomes

$$-dy/d\theta \quad = \quad 2z\ dz/d\theta \qquad (124)$$

Now introduce the new control variable $w(\theta)$ defined as:

$$dz/d\theta \quad = w(\theta) \qquad (125)$$

In light of Eqns. (120), (123), (124), and (125), the following relations among the states and the controls exist:

$$x^2 - u^2 - 2zw = 0 \qquad 0 \le \theta \le \tau \qquad (126)$$

$$w = 0 \qquad \theta_1 \le \theta \le \theta_2$$

where the state is on the boundary from θ_1 to θ_2.

The boundary conditions on the slack variable $z(\theta)$ must be consistent with the definition (123) and the boundary conditions Eqn. (122). This generates an initial condition:

$$z(0) = \sqrt{.4} \qquad (127)$$

In addition, the following midpoint condition is imposed:

$$z(\theta_1) = 0 \qquad (128)$$

To convert the problem to the model of this thesis,

introduce the normalized time t, as defined by Eqn. (17). When normalized time is employed, the problem is reformulated as that of minimizing the functional

$$I = \theta_3 \tag{129}$$

with respect to the state x(t),y(t),z(t), the control u(t), w(t), and the parameter components θ_1, θ_2, θ_3 which satisfy the differential constraints

$$\begin{aligned} \dot{x} &= \theta_1 u & 0 \le t \le 1 \\ \dot{y} &= \theta_1 (u^2 - x^2) \\ \dot{z} &= \theta_1 w \end{aligned} \tag{130}$$

$$\begin{aligned} \dot{x} &= (\theta_2 - \theta_1) u & 1 \le t \le 2 \\ \dot{y} &= (\theta_2 - \theta_1)(u^2 - x^2) \\ \dot{z} &= (\theta_2 - \theta_1) w \end{aligned}$$

$$\begin{aligned} \dot{x} &= (\theta_3 - \theta_2) u & 2 \le t \le 3 \\ \dot{y} &= (\theta_3 - \theta_2)(u^2 - x^2) \\ \dot{z} &= (\theta_3 - \theta_2) w \end{aligned}$$

the control variable constraints

$$x^2 - u^2 - 2zw = 0 \qquad 0 \le t \le 3 \tag{131}$$

$$w = 0 \qquad 1 \le t \le 2$$

and the multipoint conditions

$$x(0) = 0 \qquad\qquad x(3) = 1 \tag{132}$$
$$y(0) = -1 \qquad\qquad y(3) = 0$$
$$z(0) = \sqrt{.4} \quad z(1) = 0 \quad z(3) = \text{free}$$

In this problem,

$$n = 3 \quad m = 2 \quad p = 3 \quad k = 1 \quad q = 2 \tag{133}$$

Since $n+p+k+1 = 8$, eight particular solutions are needed for each gradient or restoration iteration.

Assume the nominal state

$$x = -t/3 \quad y = .4t \quad z = \sqrt{.4}(1-t) \quad 0 \le t \le 1 \tag{134}$$
$$x = -t/3 \quad y = .4 \quad z = 0 \quad 1 \le t \le 2$$
$$x = -t/3 \quad y = .4(3-t) \quad z = \sqrt{.4}(t-2) \quad 2 \le t \le 3$$

the nominal control

$$u = 1/3 \quad w = -1 \quad 0 \le t \le 1 \tag{135}$$
$$u = t/3 \quad w = 0 \quad 1 \le t \le 2$$
$$u = 2/3 \quad w = +1 \quad 2 \le t \le 3$$

and the nominal parameter components

$$\theta_1 = \pi/6 \quad \theta_2 = \pi/3 \quad \theta_3 = \pi/2 \tag{136}$$

These nominals are consistent with the selection criteria outlined in Section 3.0, but they violate Eqns. (130). This being the case the algorithm starts with a restoration phase. After a total of 6 iterations the algorithm has converged to the required accuracy. Tables 7-12 show convergence history, and the optimal trajectories and multipliers for the converged solution.

Table 7 Convergence History for Example 3.2

Iteration	P	Q	I	Mode
0	.8685E+00		.1570E+01	REST
1	.2208E+00		.1557E+01	REST
2	.6361E-02		.1565E+01	REST
3	.1030E-05		.1585E+01	REST
4	.5446E-13	.2996E-02	.1585E+01	GRAD
5	.2818E-04		.1580E+01	REST
6	.3233E-10	.9933E-04	.1582E+01	GRAD

Table 8 Parameter History for Example 3.2

Iteration	θ_1	θ_2	θ_3
0	.5235	1.047	1.570
1	.8607	1.074	1.557
2	.8148	.8916	1.565
3	.8172	.9184	1.585
4	.8172	.9182	1.585
5	.7827	.9110	1.580
6	.7832	.9127	1.582

Table 9 Converged State Trajectories for Example 3.2

Time	X	Y	Z
0.0	.0000E+00	.0000E+00	.6324E+00
0.2	.1394E+00	.1231E+00	.5261E+00
0.4	.2733E+00	.2307E+00	.4114E+00
0.6	.4013E+00	.3171E+00	.2879E+00
0.8	.5224E+00	.3771E+00	.1513E+00
1.0	.6310E+00	.4000E+00	-.2878E-12
1.0	.6310E+00	.4000E+00	-.2879E-12
1.2	.6475E+00	.4000E+00	-.2879E-12
1.4	.6645E+00	.4000E+00	-.2879E-12
1.6	.6820E+00	.4000E+00	-.2879E-12
1.8	.6999E+00	.4000E+00	-.2879E-12
2.0	.7182E+00	.4000E+00	-.2879E-12
2.0	.7182E+00	.4000E+00	-.2903E-12
2.2	.8071E+00	.3808E+00	.1385E+00
2.4	.8801E+00	.3249E+00	.2739E+00
2.6	.9367E+00	.2380E+00	.4024E+00
2.8	.9771E+00	.1272E+00	.5222E+00
3.0	.1000E+01	.1686E-12	.6324E+00

Table 10 Converged Control Trajectories for Example 3.2

Time	U	W
0.0	.9031E+00	-.6447E+00
0.2	.8741E+00	-.7076E+00
0.4	.8355E+00	-.7575E+00
0.6	.7975E+00	-.8248E+00
0.8	.7430E+00	-.9224E+00
1.0	.6310E+00	-.9980E+00
1.0	.6310E+00	.0000E+00
1.2	.6475E+00	.0000E+00
1.4	.6645E+00	.0000E+00
1.6	.6820E+00	.0000E+00
1.8	.6999E+00	.0000E+00
2.0	.7182E+00	.0000E+00
2.0	.7182E+00	.1036E+01
2.2	.6058E+00	.1026E+01
2.4	.4827E+00	.9887E+00
2.6	.3622E+00	.9271E+00
2.8	.2390E+00	.8595E+00
3.0	.9842E-01	.7829E+00

Table 11 Lambda Multipliers for Converged Solution of Example 3.2

Time	λ_1	λ_2	λ_3
0.0	.2226E+01	-.9890E+00	.3242E+00
0.2	.2199E+01	-.9890E+00	.2697E+00
0.4	.2117E+01	-.9890E+00	.2074E+00
0.6	.1984E+01	-.9890E+00	.1415E+00
0.8	.1806E+01	-.9890E+00	.7583E-01
1.0	.1583E+01	-.9890E+00	.3248E-02
1.0	.1583E+01	-.9890E+00	.1102E-02
1.2	.1542E+01	-.9890E+00	.1102E-02
1.4	.1503E+01	-.9890E+00	.1102E-02
1.6	.1464E+01	-.9890E+00	.1102E-02
1.8	.1426E+01	-.9890E+00	.1102E-02
2.0	.1390E+01	-.9890E+00	.1102E-02
2.0	.1390E+01	-.9890E+00	.1102E-02
2.2	.1190E+01	-.9890E+00	-.2685E-02
2.4	.9656E+00	-.9890E+00	-.1939E-02
2.6	.7226E+00	-.9890E+00	.2556E-04
2.8	.4686E+00	-.9890E+00	-.4092E-04
3.0	.2060E+00	-.9890E+00	-.1642E-13

Table 12 Remaining Multipliers for Example 3.2

Time	ρ_s	ρ_n
0.0	.1941E+00	
0.2	.2078E+00	
0.4	.2142E+00	
0.6	.2002E+00	
0.8	.1796E+00	
1.0	.2098E+00	
1.0	.3598E-01	.1427E-03
1.2	.2752E-01	.1427E-03
1.4	.1948E-01	.1427E-03
1.6	.1184E-01	.1427E-03
1.8	.4585E-02	.1427E-03
2.0	-.2313E-02	.1427E-03
2.0	-.1387E-01	
2.2	-.3624E-02	
2.4	.5954E-02	
2.6	.2261E-02	
2.8	-.1015E-02	
3.0	.9171E-03	

μ_1 = -.2060E+00
μ_2 = .9890E+00

σ = -.2146E-02

Example 3.3

Consider the problem of minimizing the functional

$$I = \tau \tag{137}$$

with respect to the states $x(\theta), y(\theta)$ and the control $u(\theta)$ which satisfy the differential constraints

$$dx/d\theta = \cos u \tag{138}$$
$$dy/d\theta = \sin u$$

and the state variable inequality constraint

$$1 - (x - 2)^2 - y^2 \leq 0 \tag{139}$$

and the boundary conditions

$$x(0) = 0 \qquad x(\tau) = 4 \tag{140}$$
$$y(0) = 0 \qquad y(\tau) = 0$$

In order to ensure satisfaction of the state inequality (139) over the interval of definition, introduce the additional state-slack variable $z(\theta)$ through the following relation:

$$1 - (x - 2)^2 - y^2 + z^2 = 0 \tag{141}$$

which upon differentiation with respect to θ becomes

$$-2(x - 2)\, dx/d\theta \; - 2y\, dy/d\theta + 2z\, dz/d\theta \; = 0 \qquad (142)$$

Now introduce the new control variable $w(\theta)$ defined as:

$$dz/d\theta \; = w(\theta) \qquad (143)$$

In light of Eqns. (138), (141), (142), and (143), the following relations among the states and the controls exist:

$$2(\, zw - (x - 2)\cos u \; - y \sin u\,) = 0 \quad 0 \leq \theta \leq \tau \qquad (144)$$

$$w = 0 \qquad \theta_1 \leq \theta \leq \theta_2$$

where the state is on the boundary from θ_1 to θ_2.

The boundary conditions on the slack variable $z(\theta)$ must be consistent with the definition (141) and the boundary conditions Eqn. (140). This generates an initial condition:

$$z(0) = \sqrt{3} \qquad (145)$$

In addition, the following midpoint condition is imposed:

$$z(\theta_1) = 0 \qquad (146)$$

To convert the problem to the model of this thesis,

introduce the normalized time t, as defined by Eqn. (17). When normalized time is employed, the problem is reformulated as that of minimizing the functional

$$I = \theta_3 \tag{147}$$

with respect to the state x(t),y(t),z(t), the control u(t), w(t), and the parameter components θ_1, θ_2, θ_3 which satisfy the differential constraints

$$\begin{aligned} \dot{x} &= \theta_1 \cos u & 0 \le t \le 1 \\ \dot{y} &= \theta_1 \sin u \\ \dot{z} &= \theta_1 w \end{aligned} \tag{148}$$

$$\begin{aligned} \dot{x} &= (\theta_2 - \theta_1)\cos u & 1 \le t \le 2 \\ \dot{y} &= (\theta_2 - \theta_1)\sin u \\ \dot{z} &= (\theta_2 - \theta_1)w \end{aligned}$$

$$\begin{aligned} \dot{x} &= (\theta_3 - \theta_2)\cos u & 2 \le t \le 3 \\ \dot{y} &= (\theta_3 - \theta_2)\sin u \\ \dot{z} &= (\theta_3 - \theta_2)w \end{aligned}$$

the control variable constraints

$$2(zw - (x - 2)\cos u - y \sin u) = 0 \qquad 0 \le t \le 3 \tag{149}$$

$$w = 0 \qquad 1 \le t \le 2$$

and the multipoint conditions

$$x(0) = 0 \qquad\qquad x(3) = 4 \tag{150}$$
$$y(0) = 0 \qquad\qquad y(3) = 0$$
$$z(0) = \sqrt{3} \qquad z(1) = 0 \qquad z(3) = \text{free}$$

In this problem,

$$n = 3 \quad m = 2 \quad p = 3 \quad k = 1 \quad q = 2 \tag{151}$$

Since $n+p+k+1 = 8$, eight particular solutions are needed for each gradient or restoration iteration.

Assume the nominal state

$$x = 4t/3 \quad y = 4t(1-t/3)/3 \quad z = \sqrt{3}(1-t) \quad 0 \le t \le 1 \tag{152}$$
$$x = 4t/3 \quad y = 4t(1-t/3)/3 \quad z = 0 \quad 1 \le t \le 2$$
$$x = 4t/3 \quad y = 4t(1-t/3)/3 \quad z = \sqrt{3}(t-2) \quad 2 \le t \le 3$$

the nominal control

$$u = \arctan(.75) \qquad w = -1 \qquad 0 \le t \le 1 \tag{153}$$
$$u = \arctan\,(2-x)/y \qquad w = 0 \qquad 1 \le t \le 2$$
$$u = -\arctan(.75) \qquad w = +1 \qquad 2 \le t \le 3$$

and the nominal parameter components

$$\theta_1 = 1 \qquad \theta_2 = 3 \qquad \theta_3 = 4 \tag{154}$$

These nominals are consistent with the selection criteria outlined in Section 3.0, but they violate Eqns. (148). This being the case the algorithm starts with a restoration phase. After a total of 19 iterations the algorithm has converged to the required accuracy. Tables 13-18 show convergence history, and the optimal trajectories and multipliers for the converged solution.

Table 13 Convergence History for Example 3.3

Iteration	P	Q	I	Mode
0	.2561E+01		.4000E+01	REST
1	.7225E-01		.4552E+01	REST
2	.4425E-04		.4536E+01	REST
3	.1155E-07		.4543E+01	REST
4	.2386E-17	.3872E-01	.4543E+01	GRAD
5	.4162E-02		.4499E+01	REST
6	.2630E-06		.4519E+01	REST
7	.2181E-14	.4638E-02	.4519E+01	GRAD
8	.1896E-03		.4509E+01	REST
9	.1275E-08	.1482E-02	.4513E+01	GRAD
10	.1766E-04		.4511E+01	REST
11	.3565E-11	.5949E-03	.4512E+01	GRAD
12	.5290E-05		.4511E+01	REST
13	.2412E-12	.4940E-03	.4512E+01	GRAD
14	.2356E-05		.4511E+01	REST
15	.6887E-13	.1801E-03	.4512E+01	GRAD
16	.5376E-06		.4511E+01	REST
17	.3863E-14	.1735E-03	.4511E+01	GRAD
18	.2375E-06		.4511E+01	REST
19	.9128E-15	.7036E-04	.4511E+01	GRAD

Table 14 Parameter History for Example 3.3

Iteration	θ_1	θ_2	θ_3
0	.1000E+01	.3000E+01	.4000E+01
1	.1436E+01	.2924E+01	.4552E+01
2	.1413E+01	.2929E+01	.4536E+01
3	.1414E+01	.2933E+01	.4543E+01
4	.1414E+01	.2933E+01	.4543E+01
5	.1487E+01	.2876E+01	.4499E+01
6	.1497E+01	.2886E+01	.4519E+01
7	.1497E+01	.2886E+01	.4519E+01
8	.1563E+01	.2851E+01	.4509E+01
9	.1565E+01	.2854E+01	.4513E+01
10	.1585E+01	.2845E+01	.4511E+01
11	.1585E+01	.2845E+01	.4512E+01
12	.1601E+01	.2838E+01	.4511E+01
13	.1601E+01	.2838E+01	.4512E+01
14	.1611E+01	.2833E+01	.4511E+01
15	.1611E+01	.2833E+01	.4512E+01
16	.1622E+01	.2828E+01	.4511E+01
17	.1622E+01	.2828E+01	.4511E+01
18	.1628E+01	.2826E+01	.4511E+01
19	.1628E+01	.2826E+01	.4511E+01

Table 15 Converged State Trajectories for Example 3.3

Time	x	y	z
0.0	.0000E+00	.0000E+00	.1732E+01
0.2	.2822E+00	.1627E+00	.1406E+01
0.4	.5651E+00	.3243E+00	.1078E+01
0.6	.8480E+00	.4859E+00	.7504E+00
0.8	.1131E+01	.6469E+00	.4163E+00
1.0	.1413E+01	.8097E+00	-.5285E-12
1.0	.1413E+01	.8097E+00	-.5288E-12
1.2	.1622E+01	.9257E+00	-.5288E-12
1.4	.1852E+01	.9890E+00	-.5288E-12
1.6	.2091E+01	.9957E+00	-.5288E-12
1.8	.2324E+01	.9457E+00	-.5288E-12
2.0	.2539E+01	.8417E+00	-.5288E-12
2.0	.2539E+01	.8417E+00	-.5354E-12
2.2	.2831E+01	.6724E+00	.3784E+00
2.4	.3123E+01	.5047E+00	.7193E+00
2.6	.3415E+01	.3367E+00	.1057E+01
2.8	.3708E+01	.1687E+00	.1395E+01
3.0	.4000E+01	-.4380E-12	.1732E+01

Table 16 Converged Control Trajectories for Example 3.3

Time	u	w
0.0	.5309E+00	-.9957E+00
0.2	.5189E+00	-.1003E+01
0.4	.5189E+00	-.1005E+01
0.6	.5189E+00	-.1011E+01
0.8	.5105E+00	-.1061E+01
1.0	.6270E+00	-.1490E+01
1.0	.6270E+00	-.3425E-20
1.2	.3876E+00	-.1590E-20
1.4	.1481E+00	-.5467E-20
1.6	-.9128E-01	-.2201E-20
1.8	-.3307E+00	-.1851E-20
2.0	-.5701E+00	-.2293E-20
2.0	-.5701E+00	.1202E+01
2.2	-.5170E+00	.1030E+01
2.4	-.5218E+00	.1004E+01
2.6	-.5218E+00	.1002E+01
2.8	-.5216E+00	.1001E+01
3.0	-.5297E+00	.9963E+00

Table 17 Lambda Multipliers for Converged Solution, Example 3.3

Time	λ_1	λ_2	λ_3
0.0	.2951E+01	.4986E+00	.1802E+01
0.2	.2657E+01	.3291E+00	.1463E+01
0.4	.2362E+01	.1611E+00	.1122E+01
0.6	.2068E+01	-.6905E-02	.7811E+00
0.8	.1774E+01	-.1740E+00	.4344E+00
1.0	.1480E+01	-.3441E+00	-.5511E-03
1.0	.1480E+01	-.3441E+00	.8705E-03
1.2	.1267E+01	-.4641E+00	.8705E-03
1.4	.1086E+01	-.5150E+00	.8705E-03
1.6	.9551E+00	-.5198E+00	.8705E-03
1.8	.8822E+00	-.5054E+00	.8705E-03
2.0	.8661E+00	-.4988E+00	.8705E-03
2.0	.8661E+00	-.4988E+00	.8705E-03
2.2	.8670E+00	-.4994E+00	-.6039E-03
2.4	.8665E+00	-.4992E+00	-.9071E-04
2.6	.8664E+00	-.4991E+00	.2413E-04
2.8	.8663E+00	-.4991E+00	.1481E-03
3.0	.8665E+00	-.4991E+00	-.5046E-11

Table 18 Remaining Multipliers for Example 3.3

Time	ρ_s	ρ_n
0.0	.8497E+00	
0.2	.8472E+00	
0.4	.8470E+00	
0.6	.8468E+00	
0.8	.8405E+00	
1.0	.9327E+00	
1.0	.6849E+00	.1042E-02
1.2	.5425E+00	.1042E-02
1.4	.4000E+00	.1042E-02
1.6	.2574E+00	.1042E-02
1.8	.1149E+00	.1042E-02
2.0	-.2766E-01	.1042E-02
2.0	-.3928E-01	
2.2	.4115E-02	
2.4	.3563E-03	
2.6	.2748E-03	
2.8	.2735E-03	
3.0	-.1512E-02	

μ_1 = -.8665E+00

μ_2 = .4991E+00

σ = .1421E-02

Example 3.4

Consider the problem of minimizing the functional

$$I = \int_0^1 u^2 \, d\theta \tag{155}$$

with respect to the states $x(\theta), y(\theta)$ and the control $u(\theta)$ which satisfy the differential constraints

$$\begin{aligned} dx/d\theta &= y \\ dy/d\theta &= u \end{aligned} \tag{156}$$

and the state variable inequality constraint

$$.15 - x \geq 0 \tag{157}$$

and the boundary conditions

$$\begin{aligned} x(0) &= 0 \qquad & x(1) &= 0 \\ y(0) &= 1 \qquad & y(1) &= -1 \end{aligned} \tag{158}$$

In order to ensure satisfaction of the state inequality (157) over the interval of definition, introduce the additional state-slack variable $z(\theta)$ through the following relation:

$$.15 - x = z^2 \tag{159}$$

which upon differentiation with respect to θ becomes

$$-dx/d\theta = 2z\ dz/d\theta \tag{160}$$

Since $dx/d\theta$ from Eqn. (156) is only a function of the state, the constraint (157) is of higher order. Therefore introduce the new state variable $v(\theta)$ defined as:

$$dz/d\theta = v(\theta) \tag{161}$$

Equation (160) now takes the following form

$$2zv + y = 0 \tag{162}$$

Differentiating Eqn. (162) with respect to θ yields:

$$2v\ dz/d\theta + 2z\ dv/d\theta + dy/d\theta = 0 \tag{163}$$

Now introduce the new control variable $w(\theta)$ defined as:

$$dz/d\theta = w(\theta) \tag{164}$$

In light of Eqns. (156), (159), (162), and (164), the following relations among the states and the controls exist:

$$2zw + 2v^2 + u = 0 \qquad 0 \le \theta \le 1 \tag{165}$$

$$w = 0 \qquad \theta_1 \le \theta \le \theta_2$$

where the state is on the boundary from θ_1 to θ_2.

The boundary conditions on the slack variables $v(\theta)$, $z(\theta)$ must be consistent with the definitions (159) and (162) and the boundary conditions Eqn. (158). This generates an initial condition:

$$z(0) = \sqrt{.15} \qquad v(0) = -.5/z(0) \tag{166}$$

In addition, the following midpoint condition is imposed:

$$z(\theta_1) = 0 \qquad v(\theta_1) = 0 \tag{167}$$

To convert the problem to the model of this thesis, introduce the normalized time t, as defined by Eqn. (17). When normalized time is employed, the problem is reformulated as that of minimizing the functional

$$I = \int_0^1 \theta_1 u^2 dt + \int_1^2 (\theta_2 - \theta_1) u^2 \, dt + \int_2^3 (1 - \theta_2) u^2 \, dt \tag{168}$$

with respect to the state $x(t), y(t), z(t)$, the control $u(t)$, $w(t)$, and the parameter components θ_1, θ_2 which satisfy the differential constraints

$$\dot{x} = \theta_1 y \qquad 0 \le t \le 1 \qquad (169)$$
$$\dot{y} = \theta_1 u$$
$$\dot{z} = \theta_1 v$$
$$\dot{v} = \theta_1 w$$

$$\dot{x} = (\theta_2 - \theta_1) y \qquad 1 \le t \le 2$$
$$\dot{y} = (\theta_2 - \theta_1) u$$
$$\dot{z} = (\theta_2 - \theta_1) v$$
$$\dot{v} = (\theta_2 - \theta_1) w$$

$$\dot{x} = (1 - \theta_2) y \qquad 2 \le t \le 3$$
$$\dot{y} = (1 - \theta_2) u$$
$$\dot{z} = (1 - \theta_2) v$$
$$\dot{v} = (1 - \theta_2) w$$

the control variable constraints

$$2zw + 2v^2 + u = 0 \qquad 0 \le t \le 3 \qquad (170)$$

$$w = 0 \qquad 1 \le t \le 2$$

and the multipoint conditions

$$x(0) = 0 \qquad x(3) = 0 \qquad (171)$$
$$y(0) = 1 \qquad y(3) = -1$$
$$z(0) = \sqrt{.15} \qquad z(1) = 0 \qquad z(3) = \text{free}$$

$$v(0) = -.5/\sqrt{.15} \quad v(1) = 0 \quad v(3) = \text{free}$$

In this problem,

$$n = 4 \quad m = 2 \quad p = 2 \quad k = 2 \quad q = 2 \tag{172}$$

Since n+p+k+1 = 9, nine particular solutions are needed for each gradient or restoration iteration.

Assume the nominal state

$$\begin{array}{lll} x = .15t & y = 1-t & 0 \le t \le 1 \\ x = .15 & y = 0 & 1 \le t \le 2 \\ x = .15(3-t) & y = 2-t & 2 \le t \le 3 \end{array} \tag{173}$$

$$\begin{array}{lll} z = \sqrt{.15}(1-t) & v = -.5(1-t)/\sqrt{.15} & 0 \le t \le 1 \\ z = 0 & v = 0 & 1 \le t \le 2 \\ z = \sqrt{.15}(t-2) & v = -.5(t-2)/\sqrt{.15} & 2 \le t \le 3 \end{array}$$

the nominal control

$$\begin{array}{lll} u = 2(t-1) & w = 1 & 0 \le t \le 1 \\ u = 0 & w = 0 & 1 \le t \le 2 \\ u = 2(2-t) & w = -1 & 2 \le t \le 3 \end{array} \tag{174}$$

and the nominal parameter components

$$\theta_1 = .10 \qquad \theta_2 = .90 \tag{175}$$

These nominals are slightly inconsistent with the selection criteria outlined in Section 3.0. Due to the sensitivity of the problem, the control u was selected to start and terminate at the unconstrained values and to be joined to the zero value of the central subarc by a straight line segment. This in effect reduces the initial size of the constraint (170-1). The selected nominals violate Eqns. (169) so the algorithm starts with a restoration phase. After a total of 22 iterations the algorithm has converged to the required accuracy. Table 19-24 show convergence history, and the optimal trajectories and multipliers for the converged solution.

Table 19 Convergence History for Example 3.4

Iteration	P	Q	I	Mode
0	.5640E+01		.2666E+00	REST
1	.3561E+01		.2040E+01	REST
2	.1553E+01		.2829E+01	REST
3	.3562E+00		.5846E+01	REST
4	.1443E-03		.6265E+01	REST
5	.1076E-07		.6308E+01	REST
6	.1090E-17	.1483E+00	.6308E+01	GRAD
7	.4008E-02		.5973E+01	REST
8	.3899E-07		.5969E+01	REST
9	.2220E-17	.1449E-01	.5969E+01	GRAD
10	.1412E-04		.5938E+01	REST
11	.1663E-12	.2679E-02	.5938E+01	GRAD
12	.2200E-04		.5932E+01	REST
13	.2416E-11	.2579E-02	.5932E+01	GRAD
14	.1873E-08	.4426E-03	.5928E+01	GRAD
15	.2160E-06		.5928E+01	REST
16	.4247E-16	.6713E-03	.5928E+01	GRAD
17	.1534E-06		.5927E+01	REST
18	.1146E-16	.6336E-03	.5927E+01	GRAD
19	.2110E-11	.3174E-03	.5927E+01	GRAD
20	.5218E-11	.1786E-03	.5926E+01	GRAD
21	.8795E-11	.1083E-03	.5926E+01	GRAD
22	.1321E-10	.6872E-04	.5926E+01	GRAD

Table 20 Parameter History for Example 3.4

ITERATION	θ_1	θ_2
0	.1000	.9000
1	.2199	.6952
2	.3622	.4427
3	.4580	.5712
4	.4516	.5890
5	.4515	.5906
6	.4515	.5906
7	.4666	.5589
8	.4675	.5540
9	.4675	.5540
10	.4699	.5478
11	.4699	.5476
12	.4746	.5360
13	.4747	.5355
14	.4748	.5353
15	.4757	.5331
16	.4757	.5331
17	.4767	.5305
18	.4767	.5305
19	.4767	.5304
20	.4767	.5304
21	.4767	.5303
22	.4767	.5303

Table 21 Converged State Trajectories for Example 3.4

Time	x	y	z	v
0.0	.0000E+00	.1000E+01	.3872E+00	-.1290E+01
0.2	.7659E-01	.6211E+00	.2709E+00	-.1146E+01
0.4	.1213E+00	.3323E+00	.1693E+00	-.9812E+00
0.6	.1428E+00	.1329E+00	.8478E-01	-.7839E+00
0.8	.1494E+00	.2221E-01	.2354E-01	-.4717E+00
1.0	.1499E+00	.8625E-06	.4900E-13	.2209E-12
1.0	.1499E+00	.8625E-06	.4892E-13	.2211E-12
1.2	.1499E+00	.8625E-06	.5130E-13	.2211E-12
1.4	.1499E+00	.8625E-06	.5367E-13	.2211E-12
1.6	.1499E+00	.8625E-06	.5605E-13	.2211E-12
1.8	.1499E+00	.8625E-06	.5843E-13	.2211E-12
2.0	.1499E+00	.8625E-06	.6080E-13	.2211E-12
2.0	.1499E+00	.8625E-06	.5934E-13	.2260E-12
2.2	.1494E+00	-.2432E-01	-.2448E-01	-.4967E+00
2.4	.1423E+00	-.1408E+00	-.8730E-01	-.8065E+00
2.6	.1204E+00	-.3405E+00	-.1720E+00	-.9898E+00
2.8	.7567E-01	-.6265E+00	-.2726E+00	-.1149E+01
3.0	.1941E-12	-.1000E+01	-.3872E+00	-.1290E+01

Table 22 Converged Control Trajectories for Example 3.4

Time	u	w
0.0	-.4466E+01	.1463E+01
0.2	-.3498E+01	.1608E+01
0.4	-.2556E+01	.1862E+01
0.6	-.1642E+01	.2437E+01
0.8	-.6427E+00	.4197E+01
1.0	-.5493E-12	.5604E+01
1.0	.2394E-16	.0000E+00
1.2	.3497E-16	.0000E+00
1.4	.2893E-17	.0000E+00
1.6	.2893E-17	.0000E+00
1.8	.2107E-17	.0000E+00
2.0	.2893E-17	.0000E+00
2.0	.7117E-12	-.5996E+01
2.2	-.7095E+00	-.4410E+01
2.4	-.1704E+01	-.2311E+01
2.6	-.2578E+01	-.1799E+01
2.8	-.3505E+01	-.1586E+01
3.0	-.4515E+01	-.1526E+01

Table 23 Lambda Multipliers for Converged Solution, Example 3.4

Time	λ_1	λ_2	λ_3	λ_4
0.0	.1969E+02	.1082E+02	-.2033E+02	.1526E+02
0.2	.1969E+02	.8941E+01	-.1517E+02	.8642E+01
0.4	.1969E+02	.7063E+01	-.1055E+02	.4129E+01
0.6	.1969E+02	.5185E+01	-.6521E+01	.1430E+01
0.8	.1969E+02	.3306E+01	-.2533E+01	.2140E+00
1.0	.1969E+02	.1428E+01	.6916E-01	-.1317E-02
1.0	.1969E+02	.1428E+01	.1134E-01	.5245E-02
1.2	.1969E+02	.1217E+01	.1134E-01	.5123E-02
1.4	.1969E+02	.1006E+01	.1134E-01	.5002E-02
1.6	.1969E+02	.7955E+00	.1134E-01	.4880E-02
1.8	.1969E+02	.5846E+00	.1134E-01	.4759E-02
2.0	.1969E+02	.3738E+00	.1134E-01	.4638E-02
2.0	.1969E+02	.3738E+00	.1134E-01	.4638E-02
2.2	.1969E+02	-.1476E+01	.3636E-01	.1483E-01
2.4	.1969E+02	-.3327E+01	-.4416E-02	-.7387E-02
2.6	.1969E+02	-.5177E+01	-.4177E-02	-.5143E-02
2.8	.1969E+02	-.7028E+01	-.5097E-03	-.3495E-03
3.0	.1969E+02	-.8878E+01	.1266E-11	.2971E-11

Table 24 Remaining Multipliers for Example 3.4

Time	ρ_s	ρ_n
0.0	.9410E+01	
0.2	.7600E+01	
0.4	.5805E+01	
0.6	.4037E+01	
0.8	.2189E+01	
1.0	.6809E+00	
1.0	.7645E-01	.2807E-03
1.2	.6516E-01	.2742E-03
1.4	.5387E-01	.2677E-03
1.6	.4258E-01	.2612E-03
1.8	.3129E-01	.2547E-03
2.0	.2000E-01	.2482E-03
2.0	.1755E+00	
2.2	-.2764E-01	
2.4	.3520E-01	
2.6	-.8918E-02	
2.8	-.5873E-02	
3.0	.4499E-01	

μ_1 = -.1969E+02
μ_2 = .8878E+01

σ_1 = -.5782E-01
σ_2 = .6562E-02

4.0 Summary and Conclusions

An algorithm for solving the optimal control problem in the presence of a state variable inequality constraint has been developed. Three major features of the algorithm are: (a) simple control of boundary entrance and exit time through the introduction of normalized time; (b) satisfaction of the state inequality constraint at all times, especially when the state trajectory is on the boundary; (c) a well-defined criterion for selecting the gradient stepsize. Several numerical examples have been solved to test the algorithm. In all cases these results are reasonably close to previous results. The inaccuracy is partly due to the new formulation which automatically generates a singular control problem. Experience with the algorithm suggests the following possibilities for improvement:

(i) since the k added state variables are simple integrators, taking advantage of this fact may dramatically improve efficiency.

(ii) the problem associated with the selection of nominals may be alleviated by using the combined gradient-restoration algorithm

(iii) accuracy may be improved by using a sequential conjugate gradient-restoration algorithm

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