RICE UNIVERSITY

FLUID TEMPERATURE DYNAMICS IN INCOMPRESSIBLE FLUID HEAT EXCHANGER SYSTEMS

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Science

Thesis Director's Signature

Alan J Chapman

Houston, Texas

May, 1971

ACKNOWLEDGEMENTS

The writer wishes to express his appreciation to Dr. A. J. Chapman who has followed the "ins and outs" of the writer's graduate academic career and has provided guidance in the development of this thesis.

The writer also wishes to recognize the assistance of the National Aeronautics and Space Administration under its graduate study program and the Southwest Research Institute for its support of the research for this thesis and for the use of its computer facilities in solution of the problems formulated.

Finally, the writer wishes to make special acknowledgement of the assistance of his wife Virginia in typing the text and of her encouragement throughout his graduate studies.

ABSTRACT

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Linearized partial differential equations for incompressible fluid temperature dynamics in pipes and single-pass heat exchangers are derived. Laplace transform methods are employed to obtain temperature transfer functions (for pipes) and transfer matrices (for heat exchangers). Using a power series approximation for the individual transfer functions of a heat exchanger transfer matrix, Fourier transforms (in Euler form) are obtained for evaluation of frequency response.

Using these models, analysis of multiple heat exchanger systems is described in terms of multiplication of a sequence of suitable transfer matrices (either geometric or causal). The effect of piping on temperatures in heat exchanger systems is shown to be negligible in the steady state and dependent on the static effectiveness of the individual heat exchangers of the system in the transient state.

For analysis of load changes in process design, it is suggested that dynamics of heat exchanger systems be characterized by the overall steady state gain and a single time constant (for each transfer function) determined by evaluation of the phase frequency response of the heat exchanger system.

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NOMENCLATURE

C_1	Total heat capacity of tube wall
C_2	Total heat capacity of shell wall
C_{p_t}	Specific heat capacity of tube fluid
C_{p_s}	Specific heat capacity of shell fluid
θ	Temperature of tube fluid
φ	Temperature of shell fluid
T_t	Temperature of tube material
T_s	Temperature of shell material
h_t	Film coefficient per unit length for the tube fluid to the tube wall
h_s	Film coefficient per unit length for the shell fluid to the shell wall
h_w	Film coefficient per unit length for the shell fluid to the tube wall
H_t, H_s, H_w	Total film coefficients ($H = L \cdot h$)
L	Total length of pipe or heat exchanger
A_t, A_s, A_w	Total contact areas corresponding to the film coefficients, h
m_s, m_t	Total mass of fluid in the shell, tube
w_s, w_t	Mass flow-rate of shell, tube fluids
x	Length coordinate for description of heat exchanger temperature distributions
t	Time
$\bar{\theta}_0, \bar{\varphi}_0$	Laplace transforms of heat exchanger outlet temperatures

$\bar{\Theta}_i, \bar{\varphi}_i$	Laplace transforms of heat exchanger inlet temperatures
τ_s	Time constant of the shell wall
τ_t	Time constant of the tube wall
$\sigma_t, \sigma_s, \sigma_\omega$	Ratios of convective parameters in the partial differential equations for temperature distributions in heat exchangers
r	Ratio of convective heat transfer terms influencing the tube wall
UA	The overall heat transfer coefficient of the heat exchanger
β	Nondimensional parameter which is a measure of the size of a heat exchanger
ε	Nondimensional parameter which is a measure of the size of a heat exchanger
Γ	Nondimensional parameter which measures the static imbalance of the flow streams of a heat exchanger
$\underset{\sim}{y}$	Temperature vector used in solution of the partial differential equations for temperature distributions in heat exchangers
$[I]$	Identity matrix
p	Laplace transform variable for spatial temperature distributions
s	Laplace transform variable for time variations of temperatures
$\mathcal{L}\{F\}, \bar{F}, F(s)$	Laplace transform of F
$\mathcal{L}^{-1}\{F(s)\}$	Inverse Laplace transform of $F(s), \bar{F}$
$[M]$	Matrix of heat exchanger characterization parameters in the partial differential equations
τ_1	Compensated transport time for the tube fluid

τ_2	Compensated transport time for the shell fluid
λ_1, λ_2	Eigenvalues of $[M]$
$[G]$	Heat exchanger transfer matrix for transformation of inlet temperatures to outlet temperatures
τ_p	Compensated time constant for an adiabatic pipe
$F(s)$	Arbitrary transfer function of a heat exchanger transfer matrix
ω	Circular frequency
j	The imaginary unit = $\sqrt{-1}$
a_K	K^{th} moment of a heat exchanger transfer function
c_K	K^{th} cummulant of a heat exchanger transfer function
$[A_K]$	Matrix of "k^{th}" moments for a heat exchanger transfer matrix
f_1, f_2	Coefficients of $[M]$ and $[I]$, respectively, in the expansion of $e^{[M]}$
$[A]$	Arbitrary matrix
$P([A])$	Arbitrary matrix polynomial of $[A]$
$Z_K([A])$	Matric product function in Sylvester's formula
$[G_c]$	Geometric solution transfer matrix for a counter-flow heat exchanger
X_{ω_K}	Modulus of the eigenvectors of $[A]$
ψ_{ω_K}	Argument of the eigenvectors of $[A]$
L_T	Transfer function of a pipe connecting the tube sides of two heat exchangers
τ_{Pt}	Time constant of L_T
L_S	Transfer function of a pipe connecting the shell sides of two heat exchangers

τ_{PS}	Time constant of L_S
G_1	Tube self transfer function
G_2	Tube cross transfer function
G_3	Shell cross transfer function
G_4	Shell self transfer function
$[D]_{(P)}$	Overall parallel-flow heat exchanger system transfer matrix for a system with connecting pipes
$[D]_{(O)}$	Overall parallel-flow heat exchanger system transfer matrix for a system without connecting pipes
$[R]$	Heat exchanger system ratio matrix or column vector
r_{ij}, r_i	Elements of $[R]$
$\mathfrak{z}_1, \mathfrak{z}_2$	Arbitrary complex numbers
α_1, α_2	Modulus of $\mathfrak{z}_1, \mathfrak{z}_2$
γ_1, γ_2	Argument of $\mathfrak{z}_1, \mathfrak{z}_2$
$[P]$	Causal transfer matrix for piping in a counter-flow heat exchanger system
$[Q]$	Overall geometric transfer matrix for a counter-flow heat exchanger system
$[H]_{(P)}$	Overall causal transfer matrix for a counter-flow heat exchanger system with pipes
$[H]_{(O)}$	Overall causal transfer matrix for a counter-flow heat exchanger system without pipes
$\varphi_c^{(i)}$	Common inlet temperature for the shell sides of a heat exchanger system in which the tube sides only are in series
$\theta_c^{(i)}$	Common inlet temperature for the tube sides of a heat exchanger system in which the shell sides only are in series

$U_a(t)$	Unit step function
$[B]$	A matrix component of $[M]$
$[C]$	A matrix component of $[M]$

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I. Introduction

The demands of greater output and efficiency in the process industries have resulted in plant designs with high process flow-rates and small permissable changes in process fluid temperatures at specific plant locations. In nuclear power plants, fluid temperature control is critical and a high degree of reliability is necessary at the design stage of heat exchanger systems for plant load control. Both of these fields have been popular for studies of the dynamics of individual heat exchange components, i.e., either heat exchangers or piping. A comprehensive review of literature applicable to the dynamics of these components is available in References 1 and 2. In virtually all of the articles cited in these references, lumped parameter models of the heat exchange component are developed from the partial differential equations describing the process; subsequently, analog models are suggested and verified for these simplified models. Invariably the values of analog circuit elements (resistances, capacitances, etc.) are set to match the experimental data for a specific heat exchange element and the mathematical model derived merely to lend insight into the required analog circuit structure. This type of representation is desirable from a control engineer's viewpoint as the heat exchange component is easily simulated and various controlling elements can be integrated with the component for evaluation of temperature control.

The process designer, however, cannot afford the luxury of analog representation since he must deal with familiar terms such as heat capacity rates, overall heat transfer coefficients, effectiveness, etc. Futhermore, the system elements which are to be connected have specified physical sizes and cannot be broken into small "lumps" for accurate analog representation. Analytical approximations, then, of the dynamics of heat exchange components are required to facilitate investigation of these components or of fluid systems containing them.

This work investigates the dynamics of heat exchanger systems in which several parallel or counter-flow heat exchangers are coupled to provide an overall steady state heat exchange effectiveness. Approximations to the partial differential equations describing the system components are made, then consideration is given to the ideal dynamics of heat exchange systems (i. e., without connecting piping) and of the modification of these dynamics when piping effects are included.

Overall, this work is directed toward an intermediate ground for system representations, providing the process engineer with an evaluation procedure for the dynamics as well as the steady state performance of heat exchange components.

Equipment selection, comparisons, and preliminary control system evaluations can be made on a rational basis with system dynamics interpreted in terms of component physical characteristics.

II. Analysis of Heat Exchange Components

Two basic heat exchange components are considered in this work, heat exchangers (specifically, single-pass heat exchangers) and piping. The most general consideration of these elements would include variable flowrates and the coupling of thermal transients with the fluid dynamic processes of mass and momentum storage. Various authors have demonstrated (References 3 and 4) that the fluid transient (outlet temperature response to a flowrate change) is at least an order of magnitude faster than the thermal load transient. Since neglect of flowrate variations reduces the complexity of equations describing heat exchanger and pipe temperature dynamics, this restriction is readily justified. Consideration of small pertubations in flowrate about an equilibrium point can be made (Reference 3), where the technique employed is essentially that presented here for temperature forced transients.

The following additional assumptions, which have proven adequate (References 4, 5, and 6) for analysis of thermal exchange in incompressible fluid heat exchangers, are employed:

1. The liquids are incompressible and have constant specific heat and density.
2. Longitudinal heat flow is negligible and radial conduction is infinite in the heat exchanger walls.
3. The velocity and temperature are uniform at a cross

section normal to the flow, i. e., a one dimensional representation is adequate.

4. The outside surface of the shell wall is perfectly insulated.
5. There are no energy sources in the fluids or heat exchanger materials.
6. Steady state heat transfer coefficients can be employed which are uniform throughout the heat exchanger and are constant for a given flowrate.

These assumptions are sufficient to linearize the partial differential equations for energy exchange. For analysis of dynamics of a linear system, several approaches are available. The differential equations describing a process can be solved in the time domain for specific forcing temperature inputs to obtain the transient response. Under the restriction of vanishing initial conditions, transfer functions may be written for the process which are independent of the nature of these forcing temperatures. This second approach will be employed in this work since it incompasses time domain responses by an appropriate transform inversion. A system characterized by means of transfer functions provides generality which a time domain solution does not. Control engineers have developed both system analysis and synthesis techniques by investigation of the dynamics of a system through its transfer functions only (References 7 and 8). By confining analysis to the transform domain, this work will provide both the potential for a specific solution inversion to the time domain and the generality of system data for control engineers.

From this viewpoint, the method of representation of system or component dynamic performance is obvious, i.e., phase and amplitude frequency response of the transfer functions. Conversion of a Laplace transform to a Fourier transform (by the substitution $S=j\omega$) produces a transform which yields the variation of steady state phase and amplitude of a linear system when excited by a sinusoidal input. Frequency response diagrams (Bode plots) can be used to synthesize steady state solutions for periodic forcing functions (decomposable into a Fourier series) or can be used for evaluation of transient response by procedures as given in References 7 and 9.

In the following sections, the governing equations of the dynamics of both heat exchangers and connecting piping are derived. Further simplifying techniques are employed and transfer matrices (for heat exchangers) and transfer functions (for pipes) are obtained through application of the Laplace transformation.

III. Single-Pass Heat Exchanger Dynamics

This type of heat exchanger consists of one tube path and one shell path and is commonly known as a double-pipe heat exchanger, although the governing equations are applicable to a heat exchanger with several parallel tube passes within the shell. Consider a length dx of the parallel or counter-flow heat exchanger of Figure 1. A heat balance on the tube fluid yields:

$$\frac{\partial}{\partial t}\left(m_t c_{pt} \frac{dx}{L} \theta\right) + w_t c_{pt} \frac{\partial \theta}{\partial x} dx + A_t h_t (\theta - T_t) dx = 0 \tag{1}$$

A similar balance for the shell fluid yields:

$$\frac{\partial}{\partial t}\left(m_s c_{ps} \frac{dx}{L} \varphi\right) \pm w_s c_{ps} \frac{\partial \varphi}{\partial x} dx + A_s h_s (\varphi - T_s) dx + A_w h_w (\varphi - T_t) dx = 0 \tag{2}$$

where the plus sign is for parallel-flow and the minus sign is for counter-flow. Heat balances for the metallic regions of the heat exchanger give:

For the tube,

$$A_w H_w (\varphi - T_t) + A_t H_t (\theta - T_t) = C_1 \frac{\partial T_t}{\partial t} \tag{3}$$

for the shell,

$$A_s H_s (\varphi - T_s) = C_2 \frac{\partial T_s}{\partial t} \tag{4}$$

Introduce time constants:

$$\tau_s = \frac{C_2}{A_s H_s} \tag{5}$$

$$\tau_t = \frac{C_1}{A_t H_t} \tag{6}$$

The ratio of convective heat transfer terms for both sides of the tube wall is:

$$r = \frac{A_\omega H_\omega}{A_t H_t} \tag{7}$$

and

$$\sigma_t = \frac{A_t H_t}{\omega_t C_{p_t}} \quad (8); \qquad \sigma_s = \frac{A_s H_s}{\omega_s C_{p_s}} \quad (9); \qquad \sigma_\omega = \frac{A_\omega H_\omega}{\omega_s C_{p_s}} \quad (10)$$

With these definitions, equations (1) through (4) reduce to:

$$\left(\frac{m_t}{\omega_t}\right)\frac{\partial \theta}{\partial t} + L\frac{\partial \theta}{\partial x} + \sigma_t(\theta - T_t) = 0 \tag{11}$$

$$\left(\frac{m_s}{\omega_s}\right)\frac{\partial \varphi}{\partial t} \pm L\frac{\partial \varphi}{\partial x} + \sigma_s(\varphi - T_s) + \sigma_\omega(\varphi - T_t) = 0 \tag{12}$$

$$r(\varphi - T_t) + (\theta - T_t) = \tau_t \frac{\partial T_t}{\partial t} \tag{13}$$

$$(\varphi - T_s) = \tau_s \frac{\partial T_s}{\partial t} \tag{14}$$

The steady state solution to these equations is obtained when

$$\frac{\partial T_s}{\partial t} = \frac{\partial T_t}{\partial t} = \frac{\partial \varphi}{\partial t} = \frac{\partial \theta}{\partial t} = 0$$

Equation (11) becomes,

$$L\frac{\partial \theta}{\partial x} + \sigma_t(\theta - T_t) = 0$$

Since (14) yields, $\varphi - T_s = 0$

equation (12) becomes,

$$\pm L\frac{\partial \varphi}{\partial x} + \sigma_\omega(\varphi - T_t) = 0$$

and equation (13) becomes,

$$r(\varphi - T_t) = -(\theta - T_t)$$

Solving this last equation for T_t and substituting into the above equations, we obtain,

$$L\frac{\partial \theta}{\partial x} + \frac{r\sigma_t}{1+r}\left\{\theta - \varphi\right\} = 0 \tag{15}$$

$$\pm L\frac{\partial \varphi}{\partial x} + \frac{\sigma_\omega}{1+r}\left\{\varphi - \theta\right\} = 0 \tag{16}$$

From the definition of the parameters in the above equations,

$$\frac{\sigma_\omega}{1+r} = \frac{A_\omega H_\omega}{C_{ps}\,\omega_s} \cdot \frac{A_t H_t}{A_t H_t + A_\omega H_\omega}$$

Now,

$$\frac{(A_\omega H_\omega)(A_t H_t)}{A_t H_t + A_\omega H_\omega} = \frac{1}{A_\omega H_\omega} + \frac{1}{A_t H_t} = \frac{1}{UA} \tag{17}$$

where $UA =$ the overall heat transfer coefficient of the heat exchanger.

$$\frac{\sigma_\omega}{1+r} = \frac{UA}{\omega_s C_{ps}}$$

Similarly,

$$\frac{r\sigma_t}{1+r} = \frac{UA}{\omega_t C_{pt}}$$

Define the following additional parameters:

$$\beta = \frac{UA}{2}\left(\frac{1}{\omega_t C_{pt}} + \frac{1}{\omega_s C_{ps}}\right) \tag{18}$$

$$\varepsilon = \frac{UA}{2}\left(\frac{1}{\omega_t C_{pt}} - \frac{1}{\omega_s C_{ps}}\right) \tag{19}$$

where β is a measure of the size of the heat exchanger and $\Gamma = \varepsilon/\beta$ (20) $(-1 < \Gamma < 1)$ is a measure of the static imbalance of the flow streams. Then,

$$\frac{\sigma_\omega}{1+r} = \beta - \varepsilon \tag{21}$$

and

$$\frac{r\sigma_t}{1+r} = \beta + \varepsilon \tag{22}$$

With the above definitions, equations (15) and (16) can be written in matrix notation as:

$$\frac{\partial}{\partial x}\begin{Bmatrix}\theta\\ \varphi\end{Bmatrix} = \frac{1}{L}\begin{Bmatrix}-(\beta+\varepsilon) & (\beta+\varepsilon)\\ \pm(\beta-\varepsilon) & \mp(\beta-\varepsilon)\end{Bmatrix}\begin{Bmatrix}\theta\\ \varphi\end{Bmatrix} \tag{23}$$

where again the top sign in the second row is for parallel-flow and the lower sign is for counter-flow.

The matrix equation (23) is a set of ordinary differential equations of first order and first degree with constant coefficients. The simplest solution is obtained by noting,

$$\frac{d}{dx}\underset{\sim}{y} = \frac{1}{L}[M]\underset{\sim}{y}$$

where $\underset{\sim}{y}$ is the temperature vector and,

$$[M] = \begin{Bmatrix}-(\beta+\varepsilon) & +(\beta+\varepsilon)\\ \pm(\beta-\varepsilon) & \mp(\beta-\varepsilon)\end{Bmatrix} \tag{24}$$

Laplace transforming this equation, $\mathcal{L}(\underset{\sim}{y}) = \underset{\sim}{\bar{y}}$

$$p\underset{\sim}{\bar{y}} - \underset{\sim}{y}(0) = \frac{1}{L}[M]\underset{\sim}{\bar{y}}$$

$$\left\{p[I] - \frac{1}{L}[M]\right\}\underset{\sim}{\bar{y}} = \underset{\sim}{y}(0)$$

$$\underset{\sim}{\bar{y}} = \left\{p[I] - [N]\right\}^{-1}\underset{\sim}{y}(0), \qquad [N] = \frac{1}{L}[M]$$

$$\left\{p[I] - [N]\right\}^{-1} = \frac{[I]}{p} + \frac{[N]}{p^2} + \frac{[N]^2}{p^3} + \cdots\cdots$$

$$\mathcal{L}^{-1}\left\{p[I] - [N]\right\}^{-1} = [I] + [N]x + \frac{[N]^2 x^2}{2!} + \ldots = e^{[N]x}$$

$$\underset{\sim}{y} = e^{[M]\frac{x}{L}}\underset{\sim}{y}(0) \tag{25}$$

Thus, the relation between the temperatures on both ends of the heat exchanger is:

$$\begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}_{X=L} = e^{[M]} \begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}_{X=0}$$

Note that in equation (25), for the parallel-flow heat exchanger,

$$\begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}_{X=L} = \begin{Bmatrix} \theta_o \\ \varphi_o \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}_{X=0} = \begin{Bmatrix} \theta_i \\ \varphi_i \end{Bmatrix}$$

For the counter-flow heat exchanger, however,

$$\begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}_{X=L} = \begin{Bmatrix} \theta_o \\ \varphi_i \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}_{X=0} = \begin{Bmatrix} \theta_i \\ \varphi_o \end{Bmatrix}$$

The solution of the equations for statics of these heat exchangers relates fluid temperatures on a "geometric" basis, i.e., the physical geometry of the fluid flow directions in the coordinates describing the problem dictates the specific temperature elements in the temperature vectors. The formulation for the parallel-flow heat exchanger can be used directly for investigation of statics, but the counter-flow heat exchanger formulation must be modified so that, respectively, temperature vectors related by a transformation matrix contain outlet temperatures or inlet temperatures only. To do this, the "causal" form (References 3 and 4) or "partial inversion" (Reference 10) described in Appendix A is applied to the transformation matrix. Further applications of the causal form will be employed in system integration of the counter-flow heat exchanger (Section X).

For simplicity of the Laplace transform solution to the dynamic equations (11) through (14), variations about the steady state defined by equation (25) are considered. Laplace transforming equations (11) through (14) by the methods of Reference (11), and changing variables so that the temperatures now denote only the time varying component, we obtain:

$$\left(\frac{m_t}{\omega_t}\right) s\bar{\theta} + L\frac{\partial\bar{\theta}}{\partial X} + \sigma_t(\bar{\theta} - \bar{T}_t) = 0 \tag{26}$$

$$\left(\frac{m_s}{\omega_s}\right) s\bar{\varphi} \pm L\frac{\partial\bar{\varphi}}{\partial X} + \sigma_s(\bar{\varphi} - T_s) + \sigma_\omega(\bar{\varphi} - \bar{T}_t) = 0 \tag{27}$$

$$r(\bar{\varphi} - \bar{T}_t) + (\bar{\theta} - \bar{T}_t) = \tau_t s \bar{T}_t \tag{28}$$

$$(\bar{\varphi} - \bar{T}_s) = \tau_s s \bar{T}_s \tag{29}$$

Solving (28) and (29) respectively for $\bar{T}_t$ and $\bar{T}_s$ we have,

$$\bar{T}_s = \bar{\varphi}/1 + \tau_s s \quad \text{and} \quad \bar{T}_t = (\bar{\theta} + r\bar{\varphi})/(1 + r + \tau_t s)$$

With these, equations (26) and (27) become,

$$s\left\{\left(\frac{m_t}{\omega_t}\right)+\frac{\sigma_t \tau_t}{(1+r+\tau_t s)}\right\}\bar{\theta}+\frac{\sigma_t r}{(1+r+\tau_t s)}\bar{\theta}+L\frac{\partial\bar{\theta}}{\partial X}$$
$$-\frac{r\sigma_t}{(1+r+\tau_t s)}\bar{\varphi}=0 \qquad (30)$$

$$s\left\{\left(\frac{m_s}{\omega_s}\right)+\frac{\sigma_s \tau_s}{1+\tau_s s}+\frac{\sigma_\omega \tau_t}{(1+r+\tau_t s)}\right\}\bar{\varphi}+\frac{\sigma_\omega}{(1+r+\tau_t s)}\bar{\varphi}$$
$$\pm L\frac{\partial\bar{\varphi}}{\partial X}-\frac{\sigma_\omega}{(1+r+\tau_t s)}=0 \qquad (31)$$

As shown by Fux (Reference 3), the wall temperature transients or resulting "wall charging" time in a liquid to liquid heat exchanger is significantly faster than the transport process of the fluid through the exchanger. Thus, it is allowable to consider τ_t, τ_s as considerably smaller than $\frac{m_s}{\omega_s}$ or $\frac{m_t}{\omega_t}$

With this, a first order approximation of the process becomes:

$$\tau_1 s\bar{\theta}+L\frac{\partial\bar{\theta}}{\partial X}+\frac{\sigma_t r}{1+r}\bar{\theta}-\frac{\sigma_t r}{1+r}\bar{\varphi}=0 \qquad (32)$$

$$\tau_2 s\bar{\varphi}\pm L\frac{\partial\bar{\varphi}}{\partial X}+\frac{\sigma_\omega}{1+r}\bar{\varphi}-\frac{\sigma_\omega}{1+r}\bar{\theta}=0 \qquad (33)$$

where,

$$\tau_1 \approx \frac{m_t}{\omega_t}+\sigma_t\tau_t \qquad (34)$$

$$\tau_2 \approx \frac{m_s}{\omega_s}+\sigma_s\tau_s+\sigma_\omega\tau_t \qquad (35)$$

τ_1 and τ_2 are termed "compensated" transport times, since they represent an effective lengthening of the fluid transport times in the heat exchanger. This approximation has been shown to be good for the low frequency behavior of liquid to liquid heat exchangers by Fux (Reference 3) and Hansen (Reference 4).

Rearranging the terms of equations (32) and (33), we have:

$$\left\{\frac{r\sigma_t}{1+r} + \tau_1 s\right\}\bar{\theta} + \frac{r\sigma_t}{1+r}\bar{\varphi} + L\frac{\partial\bar{\theta}}{\partial X} = 0 \tag{36}$$

$$\left\{\frac{\sigma\omega}{1+r} + \tau_2 s\right\}\bar{\varphi} - \frac{\sigma\omega}{1+r}\bar{\theta} \pm L\frac{\partial\bar{\varphi}}{\partial X} = 0 \tag{37}$$

Employing the relations for the statics of the heat exchanger, i. e.,

$$\frac{\sigma\omega}{1+r} = \beta - \varepsilon \quad \text{and} \quad \frac{r\sigma_t}{1+r} = \beta + \varepsilon$$

Equations (36) and (37) can be written in matrix notation for comparison with equation (25) as:

$$\frac{\partial}{\partial X}\begin{Bmatrix}\bar{\theta}\\ \bar{\varphi}\end{Bmatrix} = \frac{1}{L}\begin{Bmatrix}-(\beta+\varepsilon)-\tau_1 s & (\beta+\varepsilon)\\ \pm(\beta-\varepsilon) & \mp(\beta-\varepsilon)\mp\tau_2 s\end{Bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\varphi}\end{Bmatrix} \tag{38}$$

The same solution method is valid for (38) as for (25), thus,

$$\begin{Bmatrix}\bar{\theta}\\ \bar{\varphi}\end{Bmatrix}_{X=L} = e^{[M]}\begin{Bmatrix}\bar{\theta}\\ \bar{\varphi}\end{Bmatrix}_{X=0} \tag{39}$$

where,

$$[M] = \begin{Bmatrix} -(\beta+\varepsilon)-\tau_1 s & (\beta+\varepsilon) \\ \pm(\beta-\varepsilon) & \mp(\beta-\varepsilon)\mp\tau_2 s \end{Bmatrix} \tag{40}$$

$e^{[M]}$ can be evaluated by means of Sylvester's Theorem (Reference 12),

$$e^{[M]} = \sum_{i=1}^{2} e^{\lambda_i} \frac{\prod_{\ell \neq i} \left(\lambda_\ell [I] - [M]\right)}{\prod_{\ell \neq i} \left(\lambda_i - \lambda_\ell\right)} \tag{41}$$

Expanding this,

$$e^{[M]} = \frac{1}{\lambda_1-\lambda_2}\left\{\left(e^{\lambda_1}-e^{\lambda_2}\right)[M] - \left(\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}\right)[I]\right\} \tag{42}$$

where $\lambda_{1,2}$ are the eigenvalues of $[M]$. For the parallel-flow heat exchanger, the characteristic equation of $[M]$ is:

$$\begin{vmatrix} -(\beta+\varepsilon)-\tau_1 s-\lambda & (\beta+\varepsilon) \\ (\beta-\varepsilon) & -(\beta-\varepsilon)-\tau_2 s-\lambda \end{vmatrix} = 0$$

yielding,

$$\lambda = -\frac{\left[(\tau_1+\tau_2)s+2\beta\right]}{2} \pm \frac{1}{2}\left[(\tau_1-\tau_2)^2 s^2 + 4\varepsilon(\tau_1-\tau_2)s + 4\beta^2\right]^{1/2} \tag{43}$$

for the counter-flow heat exchanger, the characteristic equation is:

$$\begin{vmatrix} -(\beta+\varepsilon)-\tau_1 s-\lambda & \beta+\varepsilon \\ -(\beta-\varepsilon) & (\beta-\varepsilon)+\tau_2 s-\lambda \end{vmatrix} = 0$$

and,

$$\lambda = \frac{[(\tau_2-\tau_1)s-2\varepsilon]}{2} \pm \frac{1}{2}\left[(\tau_2+\tau_1)^2 s^2 + 4\beta(\tau_1+\tau_2)s + 4\varepsilon^2\right]^{1/2} \tag{44}$$

The desired transfer matrix is obtained by substitution of the appropriate $[M]$ and λ into equation (19). Note that we have obtained geometric relations for the dynamic temperature variations of the fluid streams, as in the case for statics of these heat exchangers. Again, the geometric form is adequate for the parallel-flow heat exchanger, but the causal transformation is required for investigation of the dynamics of counter-flow heat exchangers.

Resulting transfer matrix relations for these heat exchangers have the form:

$$\begin{Bmatrix} \bar{\Theta}_o \\ \bar{\varphi}_o \end{Bmatrix} = [G] \begin{Bmatrix} \bar{\Theta}_i \\ \bar{\varphi}_i \end{Bmatrix} \tag{45}$$

where,

$$[G] = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{46}$$

G_1 and G_4 are termed "self" transfer functions (terms) as they represent the influence of the stream inlet temperature on its own outlet. G_2 and G_3 are "cross" transfer functions (terms) since they represent the influence of the adjacent stream's inlet temperature on the outlet temperature of the first stream.

At this point, it is worthy to note that by conversion of the individual transfer functions to Fourier transforms, these transfer functions can, in principle, be separated into real and imaginary components. Once these components are available, conversion to the Euler form of a complex number will yield amplitude and phase relations for the transform.

The form of the matrix (42) with either set of substitutions for $[M]$ and λ is very complex when fully expanded (References 13, 14, 15, and 16 without the restriction of compensated transport times) and contains several hyperbolic functions. For even a simple choice of parameters, the algebraic reduction of the transfer functions to Euler form is tedious, if not, in many cases, impossible. A method for further simplifications of these equations is required and is developed in Section V.

The following section treats the dynamics of a single adiabatic pipe. The resulting transfer function of this component lends further insight into workable forms for heat exchanger transfer functions.

IV. Pipe Dynamics

The geometry used for analysis of a pipe with an adiabatic exterior wall is shown in Figure 1. The equations describing the dynamics of this pipe with an inlet temperature variation are a special case of the equations for the tube fluid temperature response for the single-pass heat exchanger.

Equation (11) applies as does equation (13) with $r = 0$.

$$\left(\frac{m_t}{\omega_t}\right)\frac{\partial \Theta}{\partial t} + L\frac{\partial \Theta}{\partial X} + \sigma_t(\Theta - T_t) = 0 \tag{11}$$

$$(\Theta - T_t) = \tau_t \frac{\partial T_t}{\partial t} \tag{47}$$

The steady state solution of equations (11) and (47) is Θ =Constant, thus steady state associations of heat exchange components can ignore the presence of piping.

Considering transient variations about the steady state operating point, and denoting the time varying component of the temperature by the original variable, we have

$$\left(\frac{m_t}{\omega_t}\right)s\bar{\Theta} + L\frac{\partial \bar{\Theta}}{\partial X} + \sigma_t(\bar{\Theta} - \bar{T}_t) = 0. \tag{26}$$

$$(\bar{\Theta} - \bar{T}_t) = \tau_t s \bar{T}_t \tag{48}$$

By equation (48),

$$\bar{T}_t = \bar{\Theta}/1+\tau_t s$$

Equation (26) becomes,

$$\left\{\left(\frac{m_t}{\omega_t}\right) + \frac{\sigma_t \tau_t}{1+\tau_t s}\right\} s\bar{\Theta} + L\frac{\partial\bar{\Theta}}{\partial X} = 0 \tag{49}$$

Compensation, as done for heat exchangers, is employed for a first order approximation of the process, yielding:

$$\frac{\partial\bar{\Theta}}{\partial X} = -\frac{1}{L}\tau_p s\bar{\Theta}$$

where,

$$\tau_p \approx \left(\frac{m_t}{\omega_t}\right) + \sigma_t \tau_t \tag{50}$$

Solving the compensated equation, we obtain:

$$\bar{\Theta}(x) = e^{-\tau_p s\, x/L}\,\Theta(0)$$

For $x = L$

$$\bar{\Theta}_{out} = e^{-\tau_p s}\,\bar{\Theta}_{in} \tag{51}$$

From this final form of the transfer function it is readily seen that the pipe produces only a phase lag in the fluid temperature response with no attenuation. In addition, the phase lag for any τ_p is a linear function of frequency.

V. Approximation of Heat Exchanger Transfer Functions

The simplicity of having a transfer function already in Euler form, as is available for fluid conveying pipes, prompts the desire for a similar conversion for the heat exchanger transfer functions.

The following method was evolved by Professor H. M. Paynter (Reference 17) and was utilized for the analysis of heat exchange in subsequent work under his direction (References 3 and 4). This method is utilized to obtain more manageable forms for the heat exchanger transfer functions.

This method apparently has wide applications in applied control theory since application of many control processes can be interpreted in terms of relations and functions derived from probability theory.

The more important aspects of Paynter's method to this work are the realization of two power series for any transfer function of a heat exchanger. These series are:

$$F(s) = \sum_{k=0}^{\infty} a_K \frac{(-s)^K}{K!} \tag{52}$$

where, $$a_K = (-1)^K \lim_{s \to 0} \left[\frac{d^K}{ds^K} \left(F(s) \right) \right] \tag{53}$$

the k^{th} "moment" of $F(s)$.

and $$\ln F(s) = \sum_{K=0}^{\infty} \frac{c_K (-s)^K}{K!} \tag{54}$$

where, C_K is the k^{th} "cummulant" of

a_K and C_K are related by:

$$a_K = c_K + \sum_{m=1}^{K-1} \binom{K-1}{m-1} C_m \, a_{K-m} \tag{55}$$

The following useful relations are derivable from equation (55)

<u>Cummulants</u> <u>from</u> <u>moments</u>

$$C_0 = \ln a_0$$
$$C_1 = a_1$$
$$C_2 = a_2 - a_1^2$$
$$C_3 = a_3 - 3a_2 a_1 + 2a_1^3$$

etc.

<u>Moments</u> <u>from</u> <u>cummulants</u>

$$a_0 = e^{C_0}$$
$$a_1 = C_1$$
$$a_2 = c_2 + c_1^2$$
$$a_3 = c_3 + 3c_2 c_1 + c_1^3$$

etc.

For an individual heat exchanger, the cummulants provide the desired transfer function form. Each transfer function of the transfer matrix is given by:

$$F(s) = e^{C_0 - C_1 s + \frac{c_2}{2!} s^2 - \frac{c_3}{3!} s^3 + \dots} \tag{56}$$

The first four cummulants have been found to adequately represent the low frequency behavior of heat exchangers for control system studies. (References 3, 4, and 13). This exponential relationship is valuable in evaluation of the frequency response of heat exchanger transfer functions, since by conversion to a Fourier transform, the phase and amplitude variations are readily seen from the Euler form.

$$\left.\begin{aligned} F(j\omega) &= e^{(c_0 - c_2/2!\,\omega^2 + \dots)}\, e^{j(-c_1\omega + c_3/3!\,\omega^3 - \dots)} \\ \text{Amp } F &= e^{c_0 - c_2/2!\,\omega^2 + \dots} \\ \text{Phase } F &= -c_1\omega + c_3/6\,\omega^3 - \dots \end{aligned}\right\} \quad (57)$$

Consider the special case of zero frequency. This amounts to a consideration of stable inlet temperatures. This zero frequency case then, reduces to the solution for statics of these heat exchangers.

$$\text{Amp } F = e^{c_0} \quad \text{and} \quad \text{Phase } F = 0.$$

The "zeroth" cummulant matrix completely characterizes the statics of heat exchangers. The zero frequency "gain" applies whether we are considering a steady state solution (using absolute temperatures) or dynamic pertubations from this steady state.

VI. Evaluation of Power Series for Heat Exchanger Transfer Functions

Applying the results of the previous section, we can obtain the moments of the transfer functions in matrix form.

Recall,

$$e^{[M]} = \frac{1}{\lambda_1 - \lambda_2}\left\{(e^{\lambda_1} - e^{\lambda_2})[M] - (\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2})[I]\right\} \qquad (42)$$

where the λ are defined by equation (43) or (44) and $[M]$ is defined by equation (40) with the appropriate choice of sign.

The expansion of $e^{[M]}$ for the parallel-flow heat exchanger or of the causal form of $e^{[M]}$ for the counter-flow heat exchanger produces a transformation matrix $[G]$ (Equation (46)).

The moment matrix can now be noted as:

$$[A_K] = \begin{pmatrix} a_K^{(1)} & a_K^{(2)} \\ a_K^{(3)} & a_K^{(4)} \end{pmatrix} = (-1)^K \lim_{s \to 0} \frac{d^K}{ds^K}[G] \qquad (58)$$

where the superscript refers to the appropriate transfer function.

For the parallel-flow heat exchanger, define:

$$f_1(s) = \frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} \qquad (59) \qquad \text{and} \qquad f_2(s) = \frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_1 - \lambda_2} \qquad (60)$$

Since, at most, only the first four moments of the transfer function power series are required, only three derivatives of these f require evaluation. The evaluation of these derivatives is laborious and is included in Appendix B.

Note that, $[M] = [B] + s[C]$ (61)

where,

$$[B] = \begin{Bmatrix} -(\beta+\varepsilon) & (\beta+\varepsilon) \\ (\beta-\varepsilon) & -(\beta-\varepsilon) \end{Bmatrix} \tag{62}$$

and

$$[C] = \begin{Bmatrix} -\tau_1 & 0 \\ 0 & -\tau_2 \end{Bmatrix} \tag{63}$$

Also note $[M]' = [C]$, where the prime denotes the derivative with respect to the transform variable S .

$$\frac{d}{ds} e^{[M]} = f_1 [C] + f_1' [M] - f_2' [I] \tag{64}$$

$$\frac{d^2}{ds^2} e^{[M]} = 2f_1' [C] + f_1'' [M] - f_2'' [I] \tag{65}$$

$$\frac{d^3}{ds^3} e^{[M]} = 3f_1'' [C] + f_1''' [M] - f_2''' [I] \tag{66}$$

Applying equation (58) and changing notation so that the f and their derivatives are now interpreted in terms of their limiting values as $S \to 0$, we obtain:

$$[A_0] = f_1 [B] - f_2 [I] \tag{67}$$

$$[A_1] = -f_1 [C] - f_1' [B] + f_2' [I] \tag{68}$$

$$[A_2] = 2f_1'[C] + f_1''[B] - f_2''[I] \tag{69}$$

$$[A_3] = -3f_1''[C] - f_1'''[B] + f_2'''[I] \tag{70}$$

By applying the results of Appendix B, we get

$$[A_0] = \begin{Bmatrix} 1 - \frac{1-e^{-2\beta}}{2}(1+\Gamma) & (1+\Gamma)\left(\frac{1-e^{-2\beta}}{2}\right) \\ (1-\Gamma)\left(\frac{1-e^{-2\beta}}{2}\right) & 1-(1-\Gamma)\left(\frac{1-e^{-2\beta}}{2}\right) \end{Bmatrix} \tag{71}$$

$$\begin{aligned}[A_1] = & \left\{ \frac{\tau_1+\tau_2}{2}\left(1 + \frac{e^{-2\beta}-1}{2\beta}\right) - \frac{\varepsilon(\tau_1-\tau_2)}{2\beta}\left(\frac{e^{-2\beta}+1}{2\beta} + \frac{\beta-1}{\beta}\right)\right\}[I] \\ & + \left\{ \frac{e^{-2\beta}-1}{4}(\tau_1+\tau_2) + \frac{\varepsilon(\tau_1-\tau_2)}{4\beta}\left(\frac{e^{-2\beta}-1}{\beta} + e^{-2\beta}+1\right)\right\}\begin{bmatrix} -(1+\Gamma) & (1+\Gamma) \\ (1-\Gamma) & -(1-\Gamma) \end{bmatrix} \\ & + \frac{1-e^{-2\beta}}{2\beta}\begin{bmatrix} -\tau_1 & 0 \\ 0 & -\tau_2 \end{bmatrix} \end{aligned} \tag{72}$$

Due to the complexity of algebraic reduction of the equations for higher order moments, computation of these moments was left for digital computer reduction.

For the counter-flow heat exchanger, Fux (Reference 3) has derived, obviously with much perserverence, expressions for the first cummulant (or moment) for the transfer functions contained in the causal form of equation (42).

Since the λ_i (Equations (43) or (44)) have the form:

$$\lambda_i = d(s) \pm b(s)$$

$$e^{[M]} = e^{d}\left\{\left(\frac{[M]-d[I]}{b}\right)\sinh(b) + [I]\cosh(b)\right\} \qquad (73)$$

a form valid for parallel-flow or counter-flow heat exchangers.

The causal form of $e^{[M]}$ is:

$$[G] = \begin{Bmatrix} \dfrac{e^{d}\left[\dfrac{(|M|-d^2)\sinh^2(b)+b\cosh^2(b)}{b}\right]}{(M_{22}-d)\sinh(b)+b\cosh(b)} & \dfrac{M_{12}\sinh(b)}{(M_{22}-d)\sinh(b)+b\cosh(b)} \\ \\ \dfrac{-M_{21}\sinh(b)}{(M_{22}-d)\sinh(b)+b\cosh(b)} & \dfrac{b}{(M_{22}-d)\sinh(b)+b\cosh(b)} \end{Bmatrix} \qquad (74)$$

Now, $$[A_K] = \begin{pmatrix} a_K^{(1)} & a_K^{(2)} \\ a_K^{(3)} & a_K^{(4)} \end{pmatrix} = (-1)^K \lim_{s\to 0} \frac{d^K}{ds^K}[G] \qquad (58)$$

Fortunately, Fux has completed the algebra for $C_1^{(j)}$

In terms of the formulation used in this work,

$$C_1^{(1)} = \left[\frac{(\beta/\varepsilon)^2\left(1-\frac{\tanh\varepsilon}{\varepsilon}\right)+(1+\beta)\frac{\tanh\varepsilon}{\varepsilon}}{1+\beta\,\frac{\tanh\varepsilon}{\varepsilon}}\right]\frac{(\tau_1+\tau_2)}{2} - \frac{(\tau_2-\tau_1)}{2} \qquad (75)$$

$$C_1^{(2)} = \left[\frac{(1+\beta)\dfrac{\tanh\varepsilon}{\varepsilon} - \beta/\varepsilon^2\left(\dfrac{\varepsilon}{\tanh\varepsilon} - 1\right)}{1+\beta\dfrac{\tanh\varepsilon}{\varepsilon}}\right]\left(\frac{\tau_1+\tau_2}{2}\right) \quad (76)$$

$$C_1^{(3)} = C_1^{(2)} \quad (77)$$

$$C_1^{(4)} = \left[\frac{(\beta/\varepsilon)^2\left(1-\dfrac{\tanh\varepsilon}{\varepsilon}\right) + (1+\beta)\dfrac{\tanh\varepsilon}{\varepsilon}}{1+\beta\dfrac{\tanh\varepsilon}{\varepsilon}}\right]\left(\frac{\tau_1+\tau_2}{2}\right) - \frac{(\tau_2-\tau_1)}{2} \quad (78)$$

The "zeroth" moment can also be found by application of equation (58).

$$[A_0] = \begin{Bmatrix} \dfrac{2\Gamma e^{-2\varepsilon}}{2\Gamma+(1-\Gamma)(1-e^{-2\varepsilon})} & \dfrac{(1+\Gamma)(1-e^{-2\varepsilon})}{2\Gamma+(1-\Gamma)(1-e^{-2\varepsilon})} \\ \dfrac{(1-\Gamma)(1-e^{-2\varepsilon})}{2\Gamma+(1-\Gamma)(1-e^{-2\varepsilon})} & \dfrac{2\Gamma}{2\Gamma+(1-\Gamma)(1-e^{-2\varepsilon})} \end{Bmatrix} \quad (79)$$

Fux and Hansen have shown that the phase frequency response of the counterflow heat exchanger transfer functions is largely determined by the cummulant C_1. Hansen also indicates that since a fluid system in which a counter-flow heat exchanger will be employed will contain additional thermal lags due to connecting piping, approximation by means of C_1 alone is an adequate representation for the phase frequency response of these transfer functions.

Since these considerations have not been made for the parallel-flow heat exchangers, the effect of higher order cummulants was considered. The following section treats this evaluation. Computation of the moments and cummulants was facilitated by the use of a time-sharing computer system.

VII. Numerical evaluation of Heat Exchanger Transfer Functions

The dynamics of the heat exchangers studied in this work are governed by four parameters, $\beta, \varepsilon, \tau_1$ and τ_2. Attempts by both Hansen and Fux to provide methods of concise representation for the first and higher order cummulants were met with only limited success. For limited cases such as $\varepsilon = 0$ or $\tau_1 = \tau_2$ for the counter-flow heat exchanger, some data is available for these cummulants (Reference 4). No particular emphasis was placed in this work on generalized methods for presentation of the cummulants, rather a spectrum of cases of frequency response were examined. In general, very good approximations of the phase lag of parallel-flow heat exchangers was obtained with the use of the first cummulant only.

Before presenting some typical results of frequency response, it is worthwhile to examine a simplified viewpoint for the relationship of the governing parameters. By the definitions of β and ε, we can obtain:

$$\frac{\omega_s c_{ps}}{\omega_t c_{pt}} = \frac{\beta + \varepsilon}{\beta - \varepsilon} \qquad (80)$$

This ratio or its inverse is the "heat capacity rate ratio" of Kays and London (Reference 18). Knowing the minimum heat capacity rate, a choice of either $\beta - \varepsilon$ or $\beta + \varepsilon$ can be made to obtain the number of transfer units (NTU) for the heat exchanger. The statics of the heat exchanger are now interpretable in terms of perhaps a more familiar notation. With the graphical information available in Reference 18, the significance of β and $\mathcal{E}$ on statics is easily interpreted.

For dynamics, if we consider the fluids to have approximately the same specific heats,

$$\frac{\omega_s}{\omega_t} \approx \frac{\beta+\mathcal{E}}{\beta-\mathcal{E}} \tag{81}$$

If we neglect the heat capacity of the shell and tube materials, we obtain,

$$\tau_1 = m_t/\omega_t \tag{82}$$

$$\tau_2 = m_s/\omega_s \tag{83}$$

Then,

$$\frac{\tau_1}{\tau_2} = \frac{m_t}{m_s} \cdot \frac{\omega_s}{\omega_t}$$

and,

$$\frac{m_t}{m_s} = \frac{\tau_1}{\tau_2} \cdot \frac{\omega_t}{\omega_s} \tag{84}$$

Substituting (81) into (84), the relative size of the heat exchanger flow passages is seen. The specific time value of τ_1 or τ_2 has no particular significance other than for comparative purposes between two heat exchangers with the same heat capacity rates and static effectiveness. With the same static effectiveness, a comparatively "larger" value of τ_1 or τ_2 would indicate a physically "bigger" heat exchanger.

As an example, the following values of the governing parameters were chosen:

β =(2, 3) and $\mathcal{E}$ =1.

τ_1 =10 seconds, τ_2 =15 seconds.

(Time lags for heat exchangers in the literature (References 1 and 2) were in general less than one minute.) This heat exchanger has a heat capacity rate ratio of:

$$1/3 \text{ for } \beta=2 \text{ and } 1/2 \text{ for } \beta=3$$

The NTU are given by $\beta+\mathcal{E}$ for both of these heat exchangers. The effectiveness is:

$$\epsilon = .74 \text{ for } \beta = 2$$

$$\epsilon = .67 \text{ for } \beta = 3$$

Interpreting the dynamic nature of the heat exchanger by the simplified viewpoint proposed,

$$\frac{m_t}{m_s} = 2/9 \text{ for } \beta=2 \qquad \text{and } \frac{m_t}{m_s} = 1/3 \text{ for } \beta=3.$$

In either case, a much larger proportion of the total fluid mass in the heat exchanger is contained in the shell for this example.

In figures 2 and 3, the computed phase lag of the self transfer functions are shown. Phase frequency response is plotted as a function of the first and third cummulants and as a function of the first cummulant only. All transfer functions were examined in this fashion and in all cases approximation by means of the first cummulant only appears adequate. Should some set of parameters be found for which this approximation does not hold, higher order terms should be retained.

Figures 4 and 5, respectively, show the results of investigation of the amplitude frequency response for the self transfer functions of the parallel-flow heat exchanger previously examined in Figures 2 and 3. From Figures 4 and 5, it is noted that the normalized amplitude is within 20% of the zero frequency amplitude. While this is a fairly significant percentage, it is obtained only at the highest frequency. The "worst" case for examination of dynamics during process design is to consider the amplitude of an inlet disturbance undiminished as a function of frequency. The constant gain for each transfer function, e^{C_0}, will provide the only output attenuation. This final approximation places the dynamics of a heat exchanger on a similar level of complexity to the pipe considered in Section IV. The low frequency response is modeled as a constant gain plus a single phase lag which is a linear function of frequency.

This conclusion, as previously noted, has already been qualitatively made for counter-flow heat exchangers. Figures 6 and 7 are plots of the phase frequency response of the analogous self terms for a counter-flow heat exchanger with the same governing parameters (β, ε, τ_1 and τ_2) as the parallel-flow heat exchanger discussed above. Typically, a counter-flow heat exchanger appears to have a lower phase lag for each transfer function then does a comparative parallel-flow heat exchanger.

VIII. Heat Exchanger Systems

The connection of heat exchangers and piping is mathematically equivalent to performing successive transformations (or operations)on the fluid temperatures. Mechanically, this is performed by multiplication of a sequence of suitable transfer matrices, each matrix representing the effects of a heat exchange component on the fluid temperatures. In order to visualize the manipulations required to obtain an overall system response to a given disturbance, the system should first be schematically laid out and determinations made of the requisite transformations required to construct an overall system transfer matrix. Once this is completed, the only remaining task is to "grind" through the mechanics of matrix multiplication. The frequency response of the resulting system transfer functions then allows computation of the response to the given disturbance.

For steady state associations of identical heat exchangers, Domingos (Reference 10) has derived matrix relations for systems of series interconnected heat exchangers. The analysis he presents is a logical approach, but the application of Sylvester's Theorem (Reference 12) is more concise for integration of these heat exchangers. Sylvester's Theorem in its most general form is:

$$P([A]) = \sum_{K=1}^{n} P(\lambda_K)\, Z_K([A]) \tag{85}$$

where,
$$Z_K([A]) = \frac{\prod_{r \neq K} ([A] - \lambda_r [I])}{\prod_{r \neq K} (\lambda_K - \lambda_r)} \qquad K = 1, \ldots, n \tag{86}$$

λ_K = eigenvalues of $[A]$

It is easily shown that for "n" heat exchangers in overall series flow, that the system steady state (or dynamic) transfer matrix is given by $[G]^n$ for parallel-flow heat exchangers or for counter-flow heat exchangers by the causal form of a matrix $[Q]$, where $[Q] = [G_c]^n$, $[G_c]$ being the geometric steady state solution for a counter-flow heat exchanger. Then,

$$P([A]) = [A]^n = \lambda_1^{n-1} \left\{ \left[\frac{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^n}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)} \right] [A] - \lambda_2 \left[\frac{1 - \left(\frac{\lambda_2}{\lambda_1}\right)^{n-1}}{1 - \left(\frac{\lambda_2}{\lambda_1}\right)} \right] [I] \right\} \tag{87}$$

Since the steady state transfer matrix merely contains constants and the matrices are only 2 X 2, this manipulation is easily performed. Domingos' results were verified by this approach.

For dynamics, however, each transfer function of each transfer matrix has a frequency dependent term. In addition, for dynamics of heat exchanger systems the effects of connecting piping (in real systems) cannot be neglected. Also, in real systems, it is doubtful that connecting piping between any two heat exchangers will be the same as the connecting piping between any other two heat exchangers. Generalized dynamic relations, then, for a system of heat exchangers would be of academic interest only. Nevertheless, it is instructive to describe the procedure for accomplishing these manipulations. In principle, at each ω, the complex numbers representing the Fourier transforms of each element of the transfer matrix can be manipulated to yield, for eigenvalues:

$$\lambda_K = e^{X\omega_K - j\,\psi\omega_K} \tag{88}$$

Now,

$$\lambda_K^{\,n} = e^{n(X\omega_K - j\,\psi\omega_K)} \tag{89}$$

With this, substitution can be made into equation (87) and an overall transfer matrix determined for the particular frequency.

With a single heat exchanger, inlet disturbances can be ideally considered to be encountered in only one fluid stream. Outlet temperature dynamics then, are influenced only by the "self" transfer function of the disturbed stream and the "cross" transfer function of the adjacent stream. When several heat exchangers are connected, however, excitation produces responses from all characteristic transfer functions, thus producing a response different (and more complicated to compute) than the response of a single heat exchanger.

Many systems or interconnections of heat exchangers could be considered, such as "large" heat exchangers connected to "small" heat exchangers, or various heat exchangers sizes interconnected with pipes. Only knowledge of specific process design objectives would prompt investigation of these systems. Usually, process heat exchangers are "off-the-shelf" items which are employed singly or in multiples to acheive a desired steady state heat exchange effectiveness. These heat exchangers must be connected with piping for fluid transport. It is to this latter case that the balance of this work is directed. Specifically, the application of the previously outlined methodology to various interconnections of two identical heat exchangers is treated. Consideration of a larger number of heat exchangers could be made, but it is suggested that investigations in this direction add additional heat exchangers and piping as individual elements or multiples of the "two coupled heat exchangers" systems described here.

Approaching dynamics on an incremental system basis allows flexibility, such as different lengths of pipe between heat exchangers and can lend greater insight into the dynamics of the chosen system.

In the following two sections numerical examples of heat exchanger systems will be given. For convenience, as well as for continuity, the heat exchanger examples discussed in Section VII will be utilized for system investigations.

IX. Two Identical Parallel-flow Heat Exchangers in Series

The input-output relationship for the heat exchangers of Figure 8 is:

$$\begin{Bmatrix} \theta_o^{(j)} \\ \varphi_o^{(j)} \end{Bmatrix} = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{Bmatrix} \theta_i^{(j)} \\ \varphi_i^{(j)} \end{Bmatrix} \tag{90}$$

The connecting piping couples the heat exchangers by:

$$\begin{aligned} \theta_i^{(2)} &= L_T\, \theta_o^{(1)} \\ \varphi_i^{(2)} &= L_S\, \varphi_o^{(1)} \end{aligned} \tag{91}$$

The overall relationship for the system is:

$$\begin{Bmatrix} \theta_o^{(2)} \\ \varphi_o^{(2)} \end{Bmatrix} = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} L_T & 0 \\ 0 & L_S \end{pmatrix} \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{Bmatrix} \theta_i^{(1)} \\ \varphi_i^{(1)} \end{Bmatrix} \tag{92}$$

The overall transfer matrix is:

$$[D]_{(P)} = \begin{Bmatrix} L_T G_1^2 + L_S G_2 G_3 & L_T G_1 G_2 + L_S G_2 G_4 \\ L_T G_1 G_3 + L_S G_3 G_4 & L_T G_2 G_3 + L_S G_4^2 \end{Bmatrix} \tag{93}$$

where the subscript denotes the transfer matrix with pipes included.

If there were no connecting pipes,

$$[D]_{(0)} = \begin{Bmatrix} G_1^2 + G_2 G_3 & G_1 G_2 + G_2 G_4 \\ G_1 G_3 + G_3 G_4 & G_2 G_3 + G_4^2 \end{Bmatrix} \tag{94}$$

where the subscript denotes the transfer matrix of the association without connecting pipes.

Define now, a matrix $[R]$ whose elements are the ratios of the corresponding elements of $[D]_{(P)}$ and $[D]_{(0)}$.

$$r_{11} = \frac{L_T + L_S\left(\frac{G_2 G_3}{G_1^2}\right)}{1 + \frac{G_2 G_3}{G_1^2}} \tag{95}$$

$$r_{12} = \frac{L_T + L_S\left(\frac{G_4}{G_1}\right)}{1 + \left(\frac{G_4}{G_1}\right)} \tag{96}$$

$$r_{21} = \frac{L_T + L_S\left(\frac{G_4}{G_1}\right)}{1 + \left(\frac{G_4}{G_1}\right)} = r_{12} \tag{97}$$

$$r_{22} = \frac{L_T + L_S\left(\frac{G_4^2}{G_2 G_3}\right)}{1 + \frac{G_4^2}{G_2 G_3}} \tag{98}$$

The phase variation of the transfer functions of this overall system can be examined only through numerical evaluation of the frequency response. It was hoped at the outset of this work that generalization could be made for the effects of connecting pipes on system dynamics. This desire prompted the ratio matrix whose elements are given by equations (95) through (98). It is readily seen, however, that the real components of complex numbers influence the phase angle of a sum of two complex numbers. For example, if z_1 and z_2 are to be summed,

$$z_1 = e^{\alpha_1 + j\gamma_1} \quad ; \quad z_2 = e^{\alpha_2 + j\gamma_2}$$

$$z_1 + z_2 = (e^{\alpha_1}\cos\gamma_1 + e^{\alpha_2}\cos\gamma_2) + j(e^{\alpha_1}\sin\gamma_1 + e^{\alpha_2}\sin\gamma_2)$$

$$\text{PHASE}(z_1 + z_2) = \tan^{-1}\left[\frac{e^{\alpha_1}\sin\gamma_1 + e^{\alpha_2}\sin\gamma_2}{e^{\alpha_1}\cos\gamma_1 + e^{\alpha_2}\cos\gamma_2}\right]$$

Thus even the ratio matrix does not afford an aid to generality. For some fluid systems piping may dominate the system dynamics, i.e., when the time constants of the pipes are much larger than the time constants of the heat exchanger transfer functions. In other systems, the inverse of these conditions may hold. It is worthy to note, however, that regardless of system configuration, piping will produce the same phase lag for the cross transfer terms.

Figures 9 through 12 display phase frequency response for the system dynamics of two parallel-flow heat exchangers in series. All transfer functions of the overall system were examined for cases of the system with piping included and without piping. Heat exchanger parameters selected for this example were those used in Section VII (For β =3 only). The pipe time constants were selected as multiples of the heat exchanger time constants (τ_t and τ_s). For the tube stream connecting pipe, time constants of 5 and 10 seconds were used. For the shell stream connecting pipe, time constants of 7.5 and 15 seconds were used. Also shown in figures (9) through (12), for comparison purposes, is the phase frequency response for a single heat exchanger. In all cases, phase frequency response was found to be linear. Amplitude effects of the piping were found to be negligible.

One item of major importance can be noted from the frequency response for one of the heat exchanger systems studied. For the system with piping time constants of τ_{P_t}=10 and τ_{P_S}=15seconds, both the shell stream self term (Figure 12) and the tube stream cross term (Figure 10) have 180 degree phase lags at approximately .1 radian/second. If a load control system were to be employed which compares (by feedback) the inlet and outlet temperature transients of the heat exchanger, then a disturbance at this critical frequency would result in instability. The control system design must account for the possibility of such an occurrence and provide safegaurds. The process designer can evaluate the need for special precautions by a knowledge of the system inputs and output responses.

X. Two Identical Counter-flow Heat Exchangers in Series

The input-output relationship of the heat exchangers in Figure 13 is:

$$\begin{Bmatrix} \theta_o^{(j)} \\ \varphi_o^{(j)} \end{Bmatrix} = [G] \begin{Bmatrix} \theta_i^{(j)} \\ \varphi_i^{(j)} \end{Bmatrix} \tag{99}$$

$[G]$ must be manipulated to a causal form so that the temperature vectors related can be physically connected in the heat exchanger system. We obtain,

$$\begin{Bmatrix} \theta_o^{(j)} \\ \varphi_i^{(j)} \end{Bmatrix} = [G_c] \begin{Bmatrix} \theta_i^{(j)} \\ \varphi_o^{(j)} \end{Bmatrix} \tag{100}$$

where,

$$[G_c] = \begin{pmatrix} \dfrac{(G_1G_4 - G_2G_3)}{G_4} & \dfrac{G_2}{G_4} \\ -\dfrac{G_3}{G_4} & \dfrac{1}{G_4} \end{pmatrix} \tag{101}$$

Now, the piping produces,

$$\begin{Bmatrix} \theta_i^{(2)} \\ \varphi_i^{(1)} \end{Bmatrix} = \begin{pmatrix} L_T & 0 \\ 0 & L_S \end{pmatrix} \begin{Bmatrix} \theta_o^{(1)} \\ \varphi_o^{(2)} \end{Bmatrix} \tag{102}$$

Now,

$$\begin{Bmatrix} \theta_o^{(2)} \\ \varphi_i^{(2)} \end{Bmatrix} = [G_c][P][G_c] \begin{Bmatrix} \theta_i^{(1)} \\ \varphi_o^{(1)} \end{Bmatrix} \tag{103}$$

where $[P]$ is the causal form for the piping transfer matrix,

$$[P] = \begin{pmatrix} L_T & 0 \\ 0 & \frac{1}{L_S} \end{pmatrix} \tag{104}$$

$$[Q] = [G_c]\,[P]\,[G_c]$$

$$[Q] = \begin{Bmatrix} \dfrac{(G_1G_4 - G_2G_3)^2 L_T - \dfrac{G_2G_3}{L_S}}{G_4^2} & \dfrac{G_2L_T(G_1G_4 - G_2G_3) + \dfrac{G_2}{L_S}}{G_4^2} \\[2ex] \dfrac{-(G_1G_4 - G_2G_3)L_TG_3 - G_3/L_S}{G_4^2} & \dfrac{-L_TG_2G_3 + 1/L_S}{G_4^2} \end{Bmatrix} \tag{105}$$

$$|Q| = |G_c|\,|P|\,|G_c| = \left(\frac{G_1}{G_4}\right)^2 \frac{L_T}{L_S} \tag{106}$$

Since $[Q]$ does not provide the desired connection between input and output temperature vectors, a new causal form must be obtained. That is,

$$\begin{Bmatrix} \Theta_o^{(2)} \\ \varphi_o^{(1)} \end{Bmatrix} = [H] \begin{Bmatrix} \Theta_i^{(1)} \\ \varphi_i^{(2)} \end{Bmatrix} \tag{107}$$

$$[H]_{(P)} = \begin{Bmatrix} \dfrac{G_1^2 L_T}{1 - L_SL_TG_2G_3} & \dfrac{G_2 + G_2L_SL_T(G_1G_4 - G_2G_3)}{1 - L_SL_TG_2G_3} \\[2ex] \dfrac{G_3 + G_3L_SL_T(G_1G_4 - G_2G_3)}{1 - L_SL_TG_2G_3} & \dfrac{L_SG_4^2}{1 - L_SL_TG_2G_3} \end{Bmatrix} \tag{108}$$

where the subscript denotes the transfer matrix with pipes included. If there are no connecting pipes,

$$[H]_{(0)} = \begin{Bmatrix} \dfrac{G_1^{\,2}}{1-G_2G_3} & \dfrac{G_2[(G_1G_4-G_2G_3)+1]}{1-G_2G_3} \\ \dfrac{G_3[(G_1G_4-G_2G_3)+1]}{1-G_2G_3} & \dfrac{G_4^{\,2}}{1-G_2G_3} \end{Bmatrix} \qquad (109)$$

Define, as was done for the parallel-flow heat exchangers in series, a matrix $[R]$, where,

$$r_{11} = L_T\left[\frac{1-G_2G_3}{1-L_SL_TG_2G_3}\right] \qquad (110)$$

$$r_{12} = \left[\left(\frac{L_SL_TG_1G_4}{1-L_SL_TG_2G_3}\right)+1\right] \Bigg/ \left[\frac{G_1G_4}{1-G_2G_3}+1\right] \qquad (111)$$

$$r_{21} = r_{12} \qquad (112)$$

$$r_{22} = L_S\left[\frac{1-G_2G_3}{1-L_SL_TG_2G_3}\right] \qquad (113)$$

For this system of heat exchangers, as for parallel-flow, no generalization can be made on the effects of piping. It can be noted, however, that the cross terms are subjected to an identical phase lag. Furthermore, since the counter-flow heat exchanger has the cross transfer function phase lags identical, the intercoupling of two heat exchangers still produces a system with cross terms of equal phase lag.

The heat exchanger parameters used for the examples in the previous section were also utilized here for example computations of phase frequency response for a counter-flow heat exchanger system. For the heat exchanger parameters chosen, the system static effectiveness is very high, giving for the association:

$$\begin{pmatrix} \theta_o \\ \varphi_o \end{pmatrix}_{S.S.} = \begin{pmatrix} .009 & .99 \\ .49 & .50 \end{pmatrix} \begin{pmatrix} \theta_i \\ \varphi_i \end{pmatrix}_{S.S.}$$

For this example, the outlet tube temperature is virtually equal to the shell input temperature, and the outlet shell temperature is the mixed average of the input temperatures. Since the gain of the tube self term is so low, phase lags for this transfer function are of little importance. This fact also influences the cross terms. Since $Amp(G_1) \approx 0$, the cross terms (in addition to being identical) become:

$$\approx G_2(1 - L_S L_T G_2 G_3)/(1 - L_S L_T G_2 G_3) = G_2$$

The cross term dynamics of this heat exchanger system, then are not influenced by piping or by the association itself. The dynamics of the cross term of the association are the same as the dynamics of a single heat exchanger. The shell self term is noticeably affected by the addition of piping to the heat exchanger system, although not to as large a degree as was noted for the parallel-flow heat exchanger system of the previous section. The frequency response of the shell self transfer function of the association is treated in Figure 14. Pipe time constants utilized for this example were τ_{p_t}=10 seconds for the tube stream and τ_{p_s} =15 seconds for the shell stream.

It is apparent from this example, that the efficiency of fluid "feedback" heat transfer is effective in the dynamic as well as the static case for a counter-flow heat exchanger system. The overall system effect of piping is much smaller than for a comparable parallel-flow heat exchanger system.

One final observation can be made from study of this system. Control instabilities can be experienced in both the system with piping and without (Figure 14) if a control system is not properly matched with the heat exchanger at frequencies (.05-.07 radians/second) which produce -180° phase lag.

XI. Parallel-flow or Counter-flow Heat Exchangers with One Stream in Parallel and the Other in Series

Consider an input—output relation for the heat exchanger of the form of equation (96),

$$\begin{Bmatrix} \theta_o \\ \varphi_o \end{Bmatrix} = [G] \begin{Bmatrix} \theta_i \\ \varphi_i \end{Bmatrix} \tag{114}$$

Refer now to Figure 15. If the shell sides of the heat exchangers are connected, in series,

$$\varphi_i^{(2)} = L_S \, \varphi_o^{(1)} \tag{115}$$

For the tube sides connected in series, refer to Figure 16.

$$\theta_i^{(2)} = L_T \, \theta_o^{(1)} \tag{116}$$

In each of the above cases, the other stream is connected in parallel so that,

$$\theta_i^{(2)} = \theta_i^{(1)} = \theta_c^{(i)} \tag{117}$$

and

$$\varphi_i^{(2)} = \varphi_i^{(1)} = \varphi_c^{(i)} \tag{118}$$

Referring now to Figure 15,

$$\begin{pmatrix} \theta_o^{(1)} \\ \varphi_o^{(1)} \end{pmatrix} = [G] \begin{pmatrix} \theta_c^{(i)} \\ \varphi_1^{(i)} \end{pmatrix} \quad ; \quad \begin{pmatrix} \theta_o^{(2)} \\ \varphi_o^{(2)} \end{pmatrix} = [G] \begin{pmatrix} \theta_c^{(i)} \\ \varphi_i^{(2)} \end{pmatrix}$$

$$\begin{pmatrix} \Theta_o^{(2)} \\ \varphi_o^{(2)} \end{pmatrix} = [G] \begin{pmatrix} \bar{\Theta}_c^{(i)} \\ L_s \varphi_o^{(1)} \end{pmatrix} \tag{119}$$

$$\varphi_o^{(1)} = G_3 \Theta_c^{(i)} + G_4 \varphi_i^{(1)}$$

Substituting,

$$\begin{pmatrix} \Theta_o^{(2)} \\ \varphi_o^{(2)} \end{pmatrix} = \begin{Bmatrix} G_1 + L_s G_2 G_3 & L_s G_2 G_4 \\ G_3 + L_s G_3 G_4 & L_s G_4^2 \end{Bmatrix} \begin{pmatrix} \Theta_c^{(i)} \\ \varphi_i^{(1)} \end{pmatrix} \tag{120}$$

Investigating $\varphi_o^{(2)}$, we conclude that the dynamics of the shell fluid are only influenced by the transfer functions peculiar to that stream. This is a considerable simplification as the overall counter-flow or overall parallel-flow associations incorporate effects of all characteristic transfer functions.

$$\varphi_o^{(2)} = \left\{ G_3 (1 + L_s G_4),\ L_s G_4^2 \right\} \begin{pmatrix} \Theta_c^{(i)} \\ \varphi_i^{(1)} \end{pmatrix} \tag{121}$$

If no pipe was present,

$$\varphi_o^{(2)} = \left\{ G_3 (1 + G_4),\ G_4^2 \right\} \begin{pmatrix} \Theta_c^{(i)} \\ \varphi_i^{(1)} \end{pmatrix} \tag{122}$$

Now a column vector of ratios of transfer functions of the assembly with a pipe and of the assembly without a pipe can be formed:

$$r_1 = \frac{1 + L_S G_4}{1 + G_4} \quad , \quad r_2 = L_S \tag{123}$$

From the value of r_2 , it is readily seen that the only influence of the pipe on the outlet shell temperature is to delay the temperature response, i.e., if the tube forcing temperature is steady. If the tube forcing temperature is varying, the behavior of r_1 must be investigated. Referring now to Figure 16,

$$\begin{pmatrix} \theta_o^{(1)} \\ \varphi_o^{(1)} \end{pmatrix} = [G] \begin{pmatrix} \theta_i^{(1)} \\ \varphi_c^{(i)} \end{pmatrix} \quad ; \quad \theta_i^{(2)} = L_T \, \theta_o^{(1)}$$

$$\begin{pmatrix} \theta_o^{(2)} \\ \varphi_o^{(2)} \end{pmatrix} = [G] \begin{pmatrix} L_T \theta_o^{(1)} \\ \varphi_c^{(i)} \end{pmatrix} \tag{124}$$

$$\begin{pmatrix} \theta_o^{(2)} \\ \varphi_o^{(2)} \end{pmatrix} = \begin{pmatrix} L_T G_1^2 & G_2 (1 + L_T G_1) \\ L_T G_1 G_3 & G_4 + L_T G_2 G_3 \end{pmatrix} \begin{pmatrix} \theta_i^{(1)} \\ \varphi_c^{(i)} \end{pmatrix} \tag{125}$$

Similar to the conclusions for shell sides in series, the dynamics of the tube fluid are only influenced by the transfer functions peculiar to that stream.

$$\theta_o^{(2)} = \left\{ L_T G_1^2 \, , \, G_2 (1 + L_T G_1) \right\} \begin{pmatrix} \theta_i^{(1)} \\ \varphi_c^{(i)} \end{pmatrix} \tag{126}$$

If no pipe was present,

$$\varphi_0^{(2)} = \left\{G_1^{\,2}, \; G_2(1+G_1)\right\}\begin{pmatrix}\theta_i^{(1)}\\ \varphi_c^{(i)}\end{pmatrix} \tag{127}$$

Now, $\bar{R} = (r_1, r_2)$.

$$r_1 = L_T \quad , \; r_2 = \frac{1 + L_T G_1}{1 + G_1} \tag{128}$$

The pipe, therefore, only adds a delay time to the transfer function of the tube stream to itself. If the shell forcing temperature is varying, the behavior of r_2 must be investigated.

XII. Transient Response Approximations

Having reviewed the frequency response of the transfer functions for several heat exchangers and heat exchanger systems, it would appear logical to address the problem of transient response as well. Unfortunately, the linearity of the phase frequency response decays badly beyond .1 radian/second (or approximately 1 cycle/minute) and systems excited by these higher frequencies require additional cummulants in the approximations. Obtaining these higher cummulants is laborious, and in the author's opinion, unwarranted due to the nature of the systems investigated, i.e., single-pass heat exchangers only.

As a very rough approximation of the transient response of a heat exchange system to a load change, it is suggested that the low frequency phase linearity be computed and examined to obtain an equivalent system time constant. The statics of the system will provide an equivalent constant gain. With these, the transient response to any arbitrary input may be compute by the time-axis shifting theorm of the Laplace transformation(Reference 19), i.e.,

$$e^{c_0 - c_1 s} F(s) = \mathcal{L}\left\{e^{c_0} f(t-c_1)\, U_a(t)\right\} \tag{129}$$

where $U_a(t)$ is the unit step function.

With each transfer function of a heat exchanger system modeled in the manner noted, rough approximations of system transient temperatures may be made. The equation (129) will be adequate for arbitrary transient inputs if the frequency content of the input disturbance lies within the low frequency response range or if a harmonic analysis indicates that the input disturbance has a low amplitude for frequencies outside the low frequency spectrum.

XIII. Conclusions

1. Phase dynamics of parallel and counter-flow heat exchangers can be characterized by a linear frequency response term for each element of the transfer matrix. In addition, the amplitude of the transfer function can be approximated by the steady state gain.

2. Pipes used as heat exchanger system connecting elements can be considered to produce only a linear frequency response with no amplitude attenuation.

3. Due to the form of the transfer functions, investigations of the low frequency dynamics of heat exchange systems is reduced to "straight-forward" complex number manipulations.

4. As a consequence of the characterization of the heat exchanger as in (1), the zero frequency transfer matrix of a system of heat exchangers produces the steady state solution matrix as well.

5. Due to the dependence of the frequency response of a system of heat exchangers containing piping on the frequency response of the individual transfer functions of the heat exchanger, no generalized conclusions can be drawn for the effects of piping. Specific heat exchanger and pipe systems have been examined and observations drawn for these systems.

6. Employing simple methods of matrix manipulation, various combinations of heat exchangers can be analyzed. These analyses can point out the need for more sophisticated control investigations for the heat exchanger systems, or provide data for

approximation of temperature transients delivered to subsequent process components.

7. Overall system time constants can be computed, which when employed with the time axis shifting theorem of the Laplace transformation, will allow computation of heat exchanger system response to arbitrary temperature inputs.

XIV. Recommendations for Further Work

The generalized methods presented in this work produce fairly concise means for evaluation of heat exchanger dynamics, however, the first cummulant is a function of four parameters. A more convenient method of evaluating heat exchanger dynamics or a method providing greater physical insight is desired. Hougen (Reference 20) has noted that experimentally obtained transfer functions of heat exchangers can be expressed by a form:

$$F(s) = Ke^{-\tau s} \Big/ \left[\left(s^2/\omega_n^2 \right) + \left(2\zeta s/\omega_n \right) + 1 \right]$$

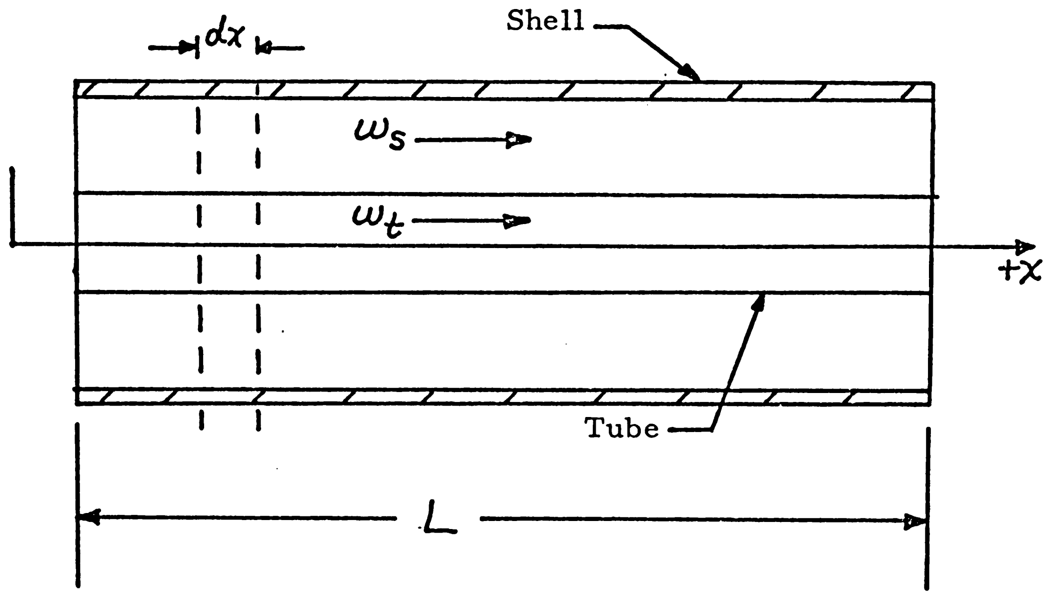
τ is the fluid transport delay time $= \frac{m}{\omega}$ and ζ is a "damping ratio" for a second order system (found to be unity). K is the zero frequency gain of the transfer function. With this approach, the need for compensation is eliminated. In addition, for all heat exchangers examined by Hougen, ω_n was shown to be closely approximated as a linear function of UA . Oldenburger and Goodson (Reference 21) have simplified the transfer function for hydraulic dynamics of pipes (forms which are very similar to equation (73) by expansion of this transfer function in a series of infinite products:

$$G(s) = \prod_{n=0}^{\infty} \left[1 + \left(\frac{2\zeta_n s}{\omega_n} \right) + \left(\frac{s^2}{\omega_n^2} \right) \right]$$

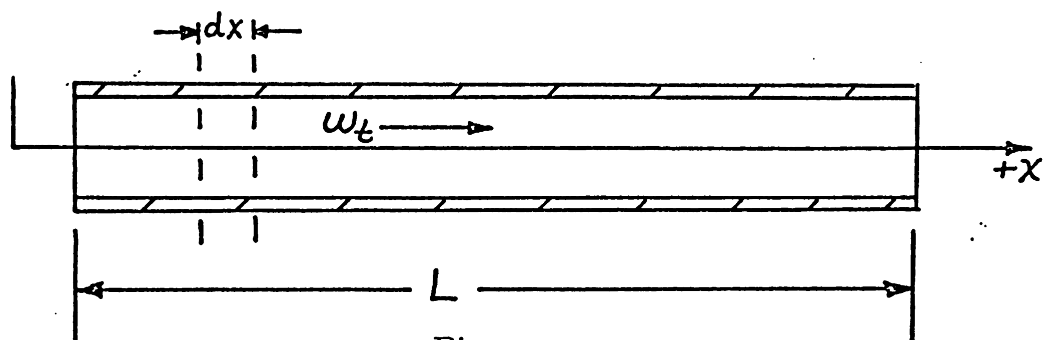
where, only the first term was required to provide an adequate approximation of hydraulic dynamics.

The similarity of these transfer function forms and the potential that a physical interpretation may be more readily obtained, should warrant further investigation into dynamics of heat exchangers.

Figure 1. Geometry for Mathematical Models



Single-Pass Heat Exchanger



Pipe

Figure 2. Phase Frequency Response for Shell Stream Self Term Parallel-flow Heat Exchanger
$\mathcal{E}=1,\ \tau_1=10,\ \tau_2=15$

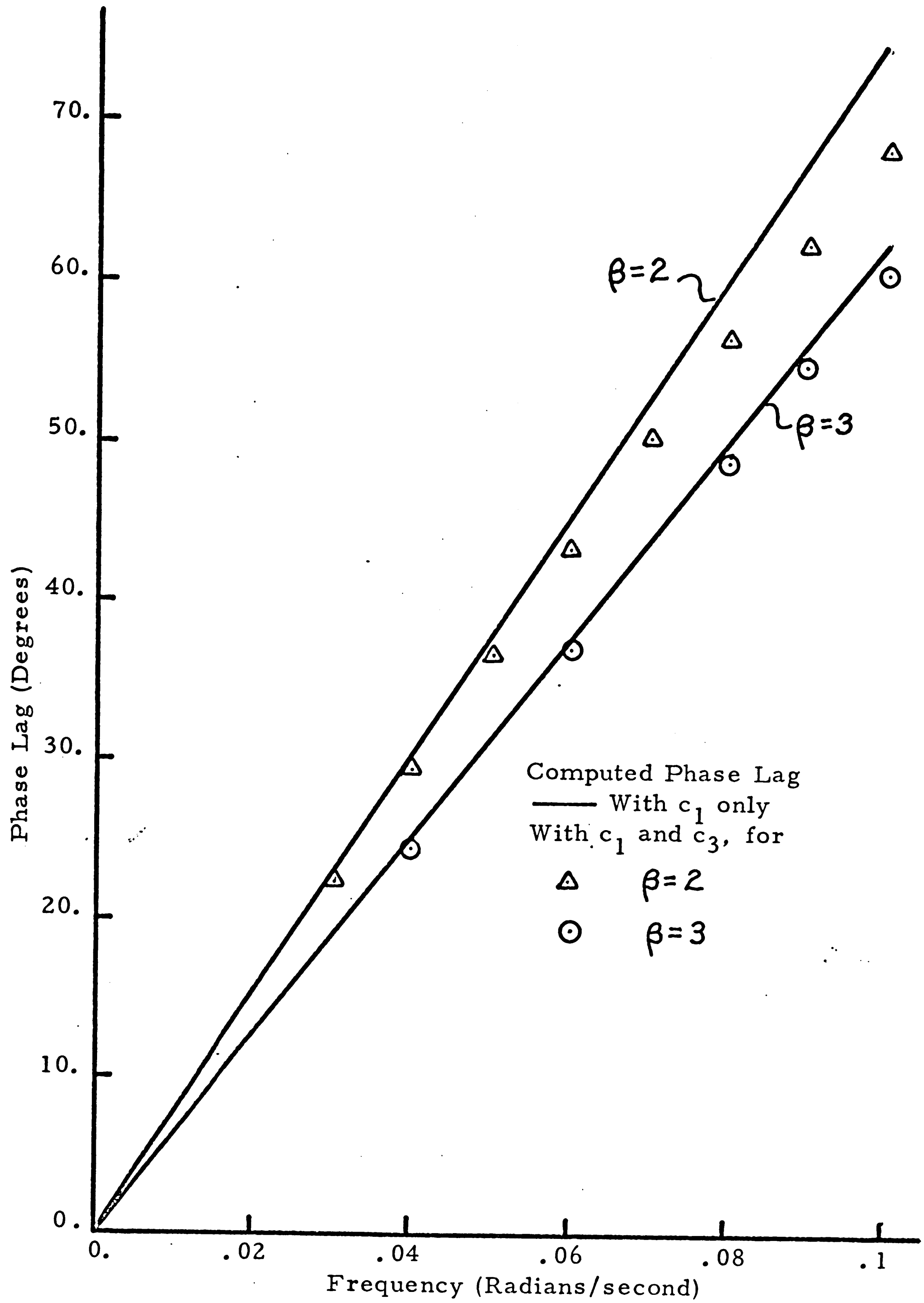


Figure 3. Phase Frequency Response for Tube Stream Self Term Parallel-flow Heat Exchanger
$\mathcal{E}=1,\ \tau_1=10,\ \tau_2=15$

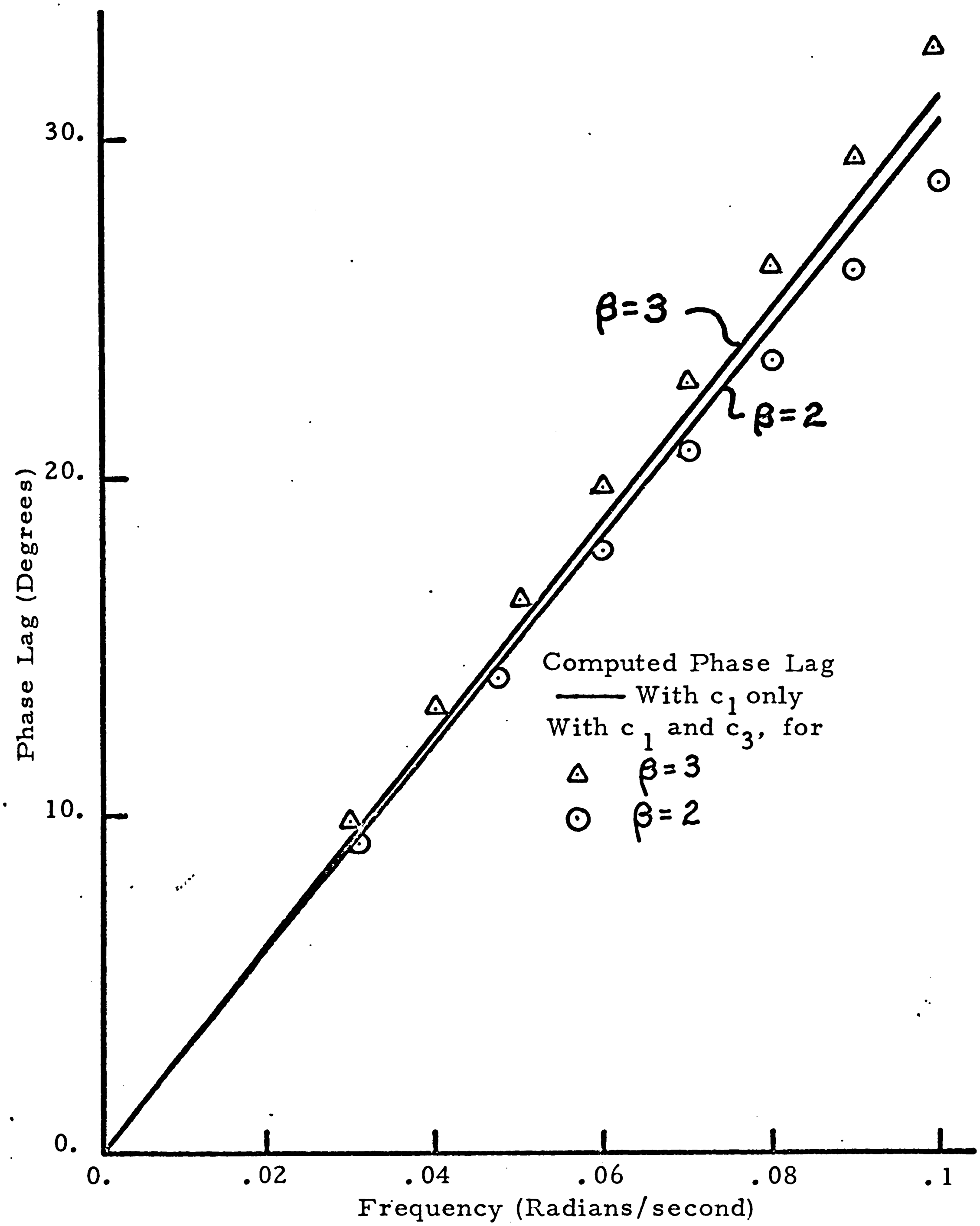


Figure 4. Normalized Amplitude Frequency Response. Tube Stream Self Term Parallel-flow Heat Exchanger
$\varepsilon = 1, \tau_1 = 10, \tau_2 = 15$

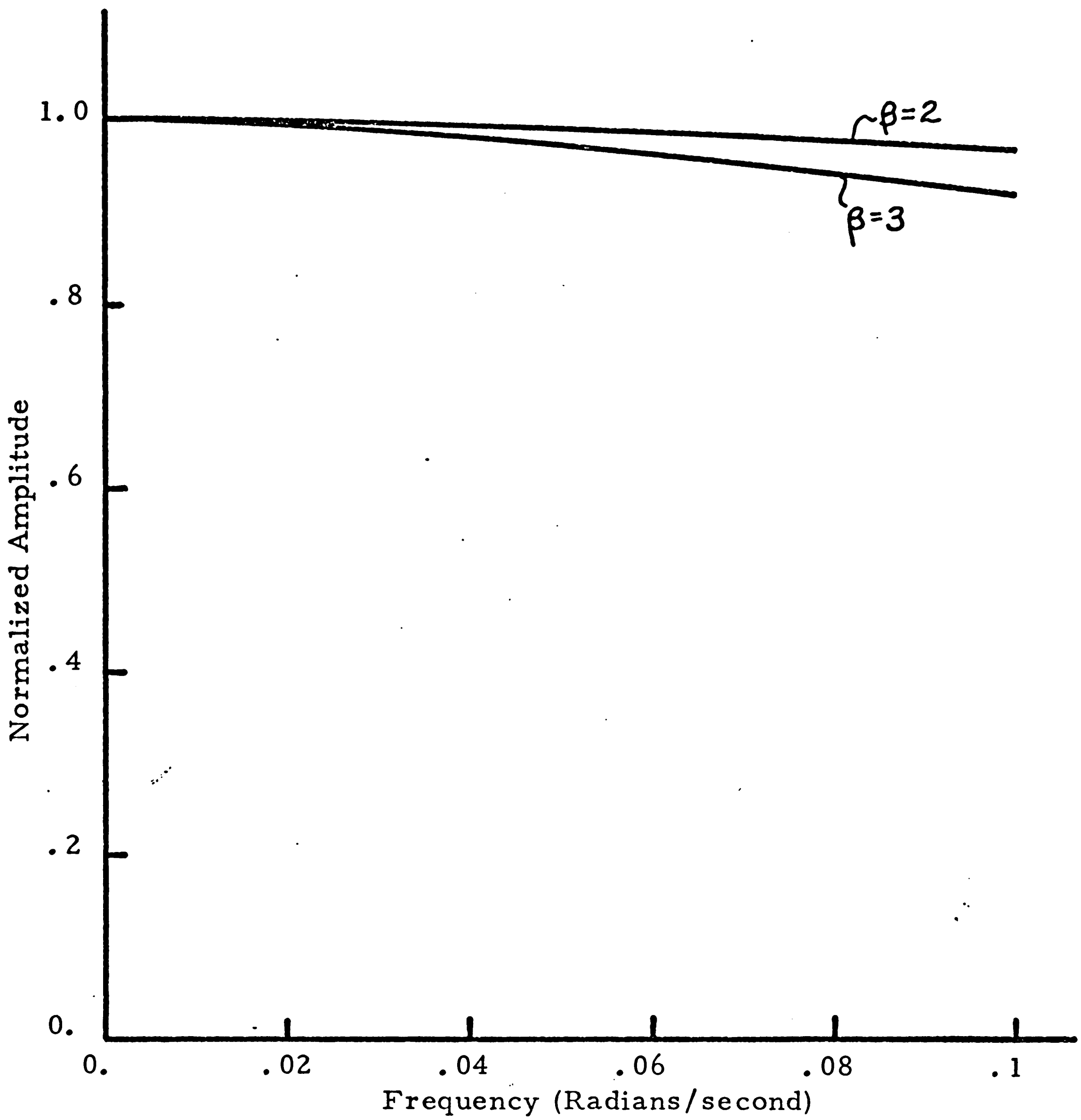


Figure 5. Normalized Amplitude Frequency Response. Tube Stream Cross Term Parallel-flow Heat Exchanger
$\mathcal{E} = 1,\ \tau_1 = 10,\ \tau_2 = 15$

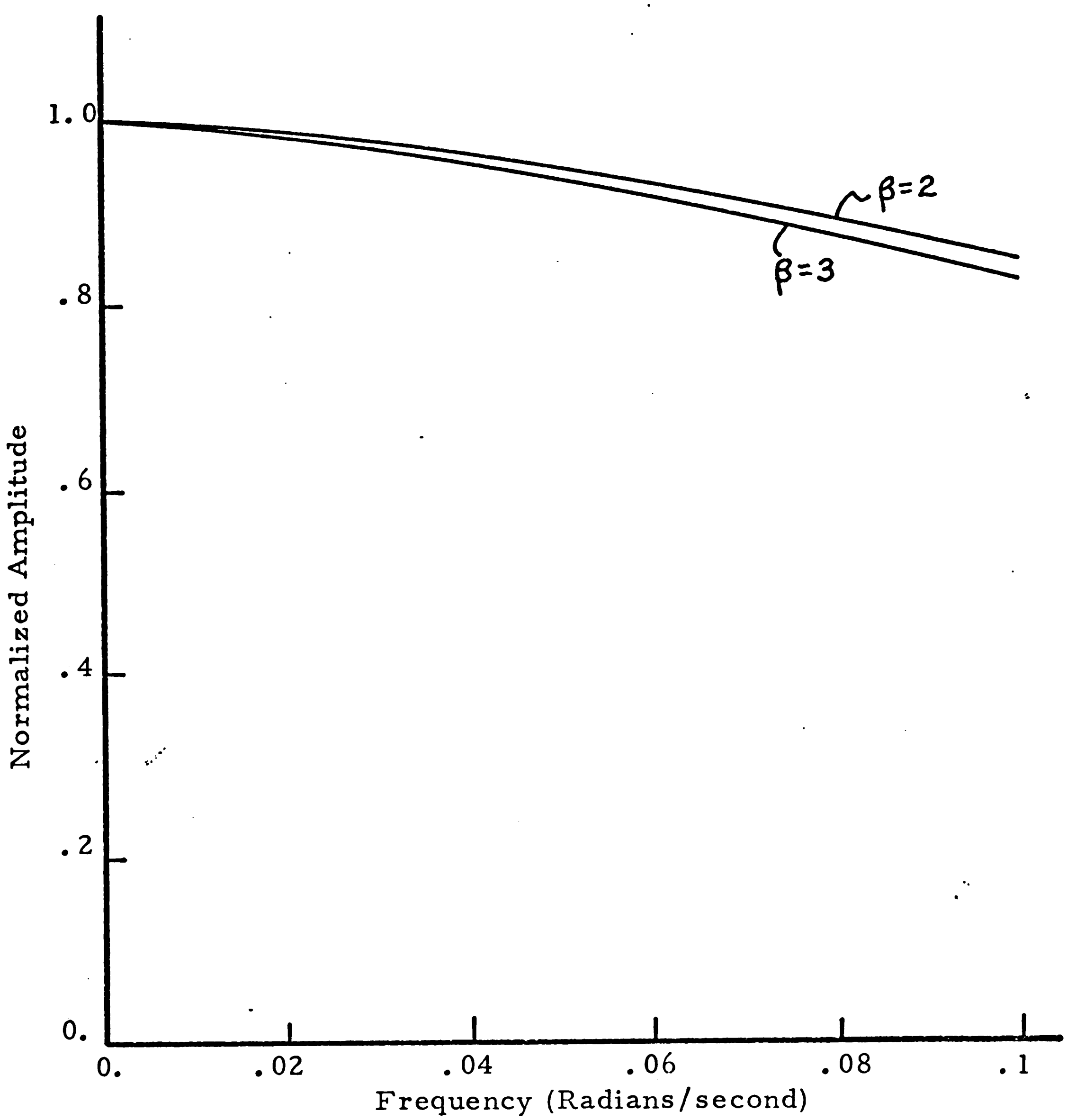


Figure 6. Phase Frequency Response
Shell Stream Self Term
Counter-flow Heat Exchanger
$\varepsilon = 1, \tau_1 = 10, \tau_2 = 15$

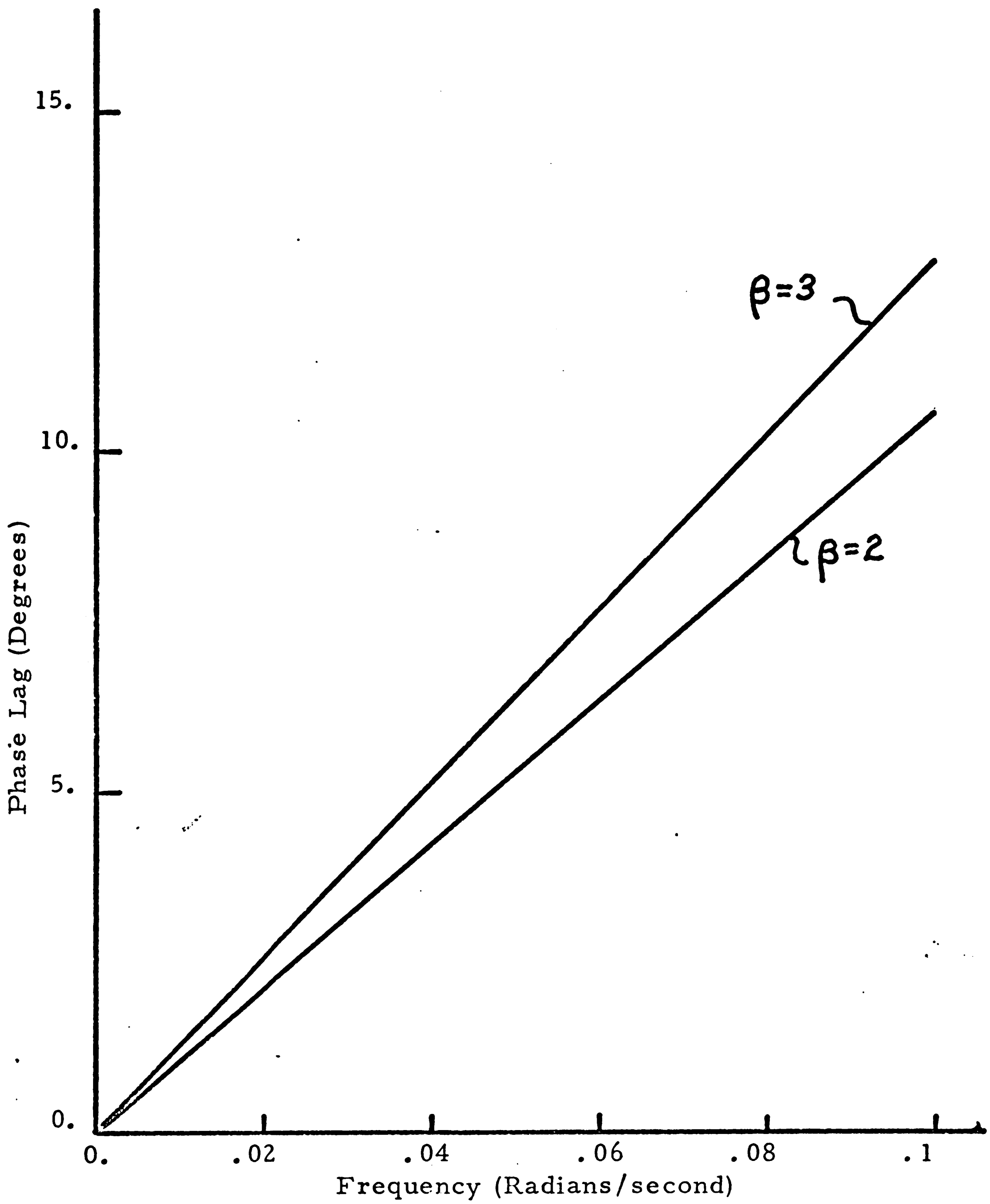


Figure 7. Phase Frequency Response
Tube Stream Self Term
Counter-flow Heat Exchanger
$\mathcal{E}=1,\ \tau_1=10,\ \tau_2=15$

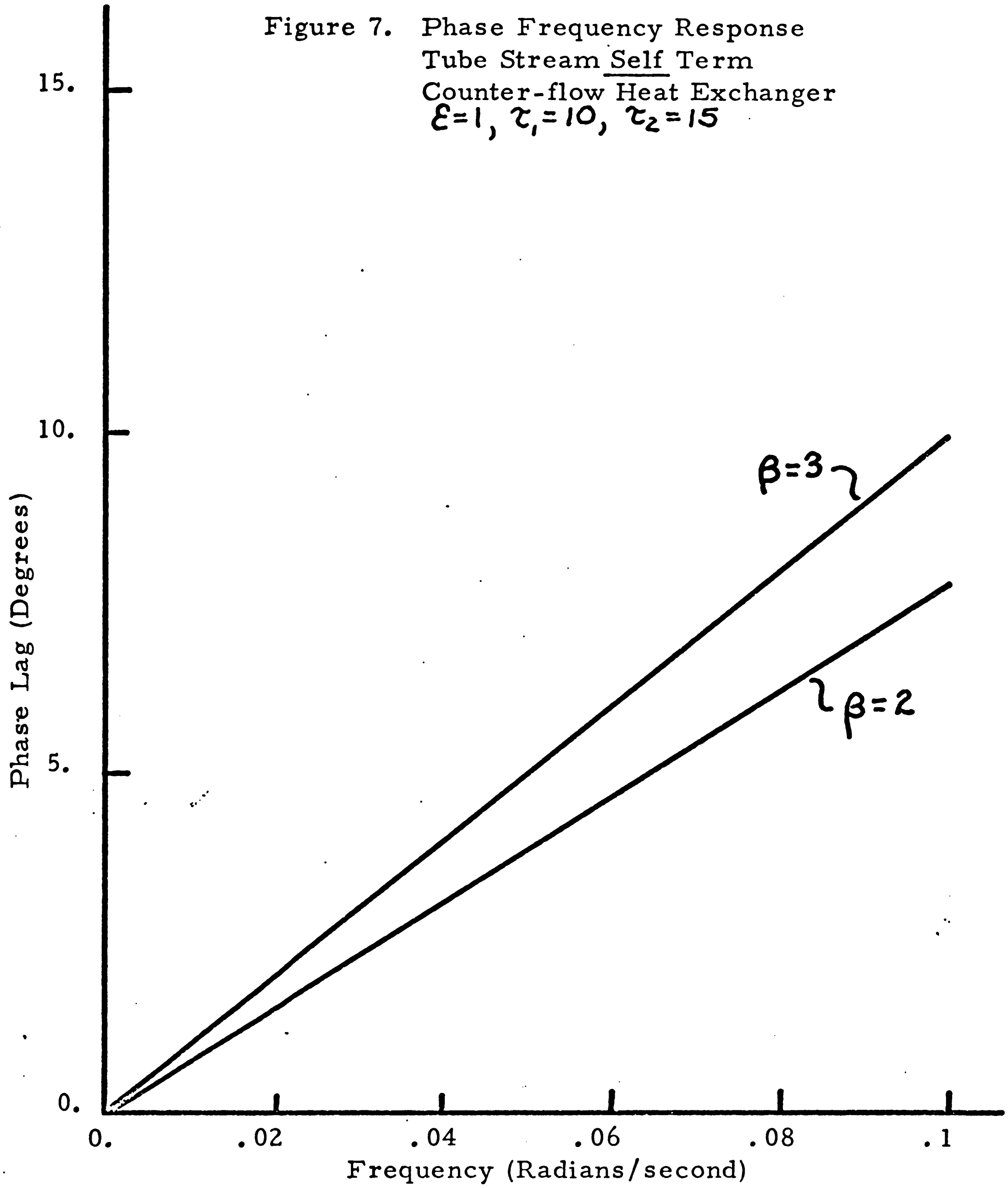
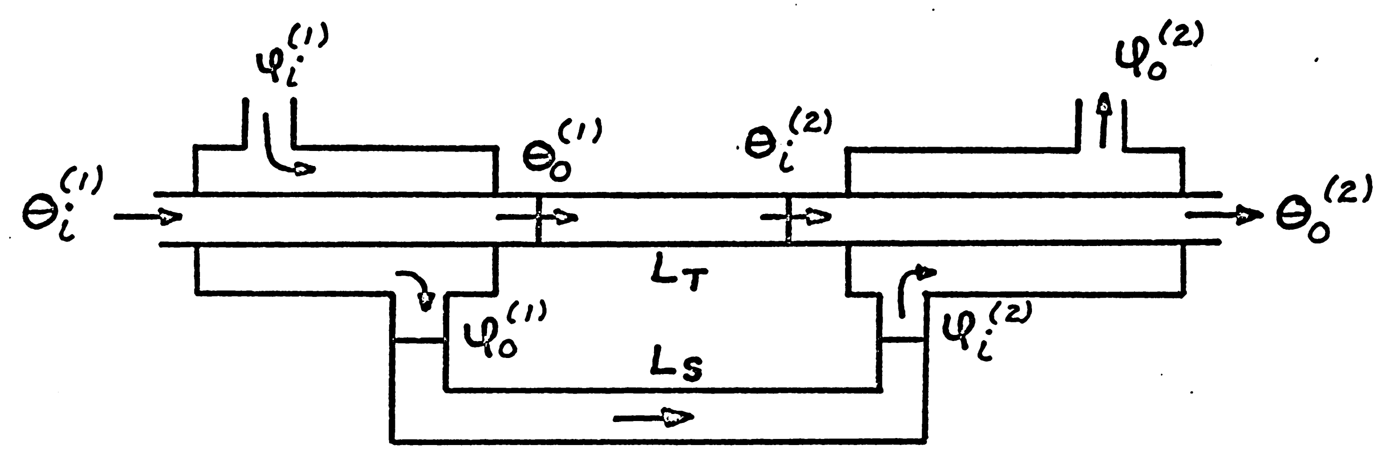


Figure 8. Two Parallel-flow Heat Exchangers In Series



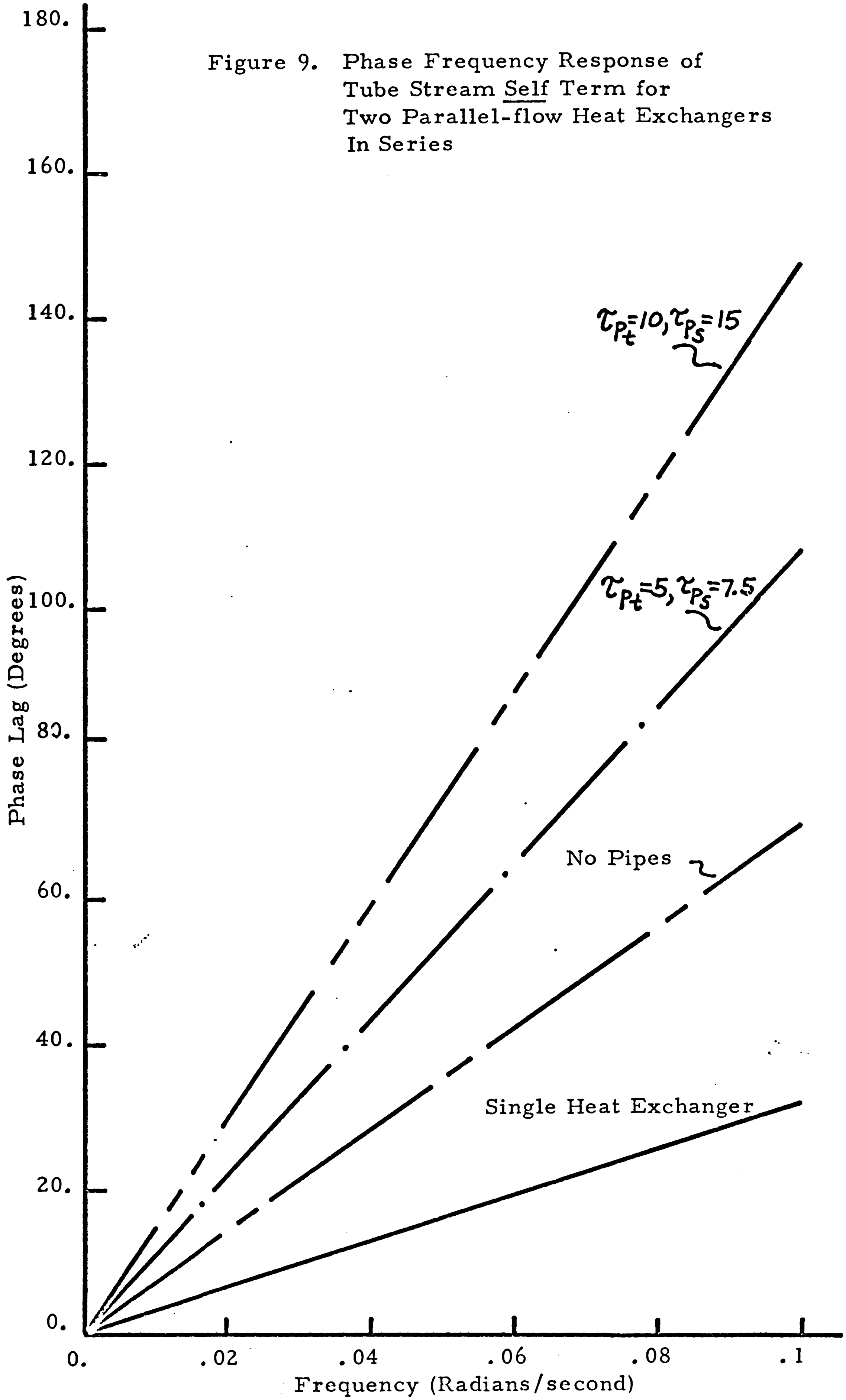


Figure 9. Phase Frequency Response of Tube Stream Self Term for Two Parallel-flow Heat Exchangers In Series

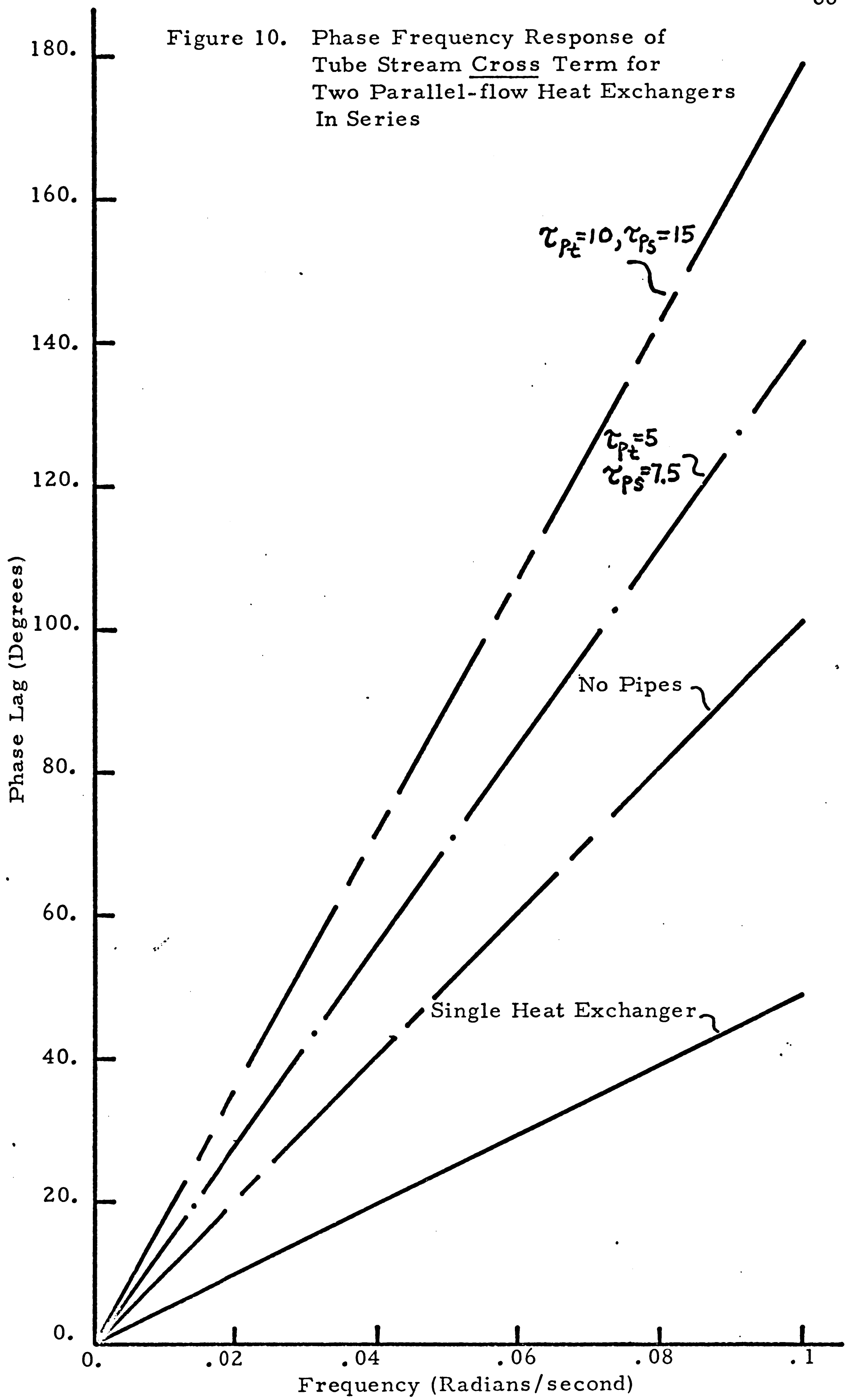


Figure 10. Phase Frequency Response of Tube Stream Cross Term for Two Parallel-flow Heat Exchangers In Series
$\tau_{pt}=10, \tau_{ps}=15$
$\tau_{pt}=5$
$\tau_{ps}=7.5$
No Pipes
Single Heat Exchanger
Phase Lag (Degrees)
180.
160.
140.
120.
100.
80.
60.
40.
20.
0.
0.
.02
.04
.06
.08
.1
Frequency (Radians/second)

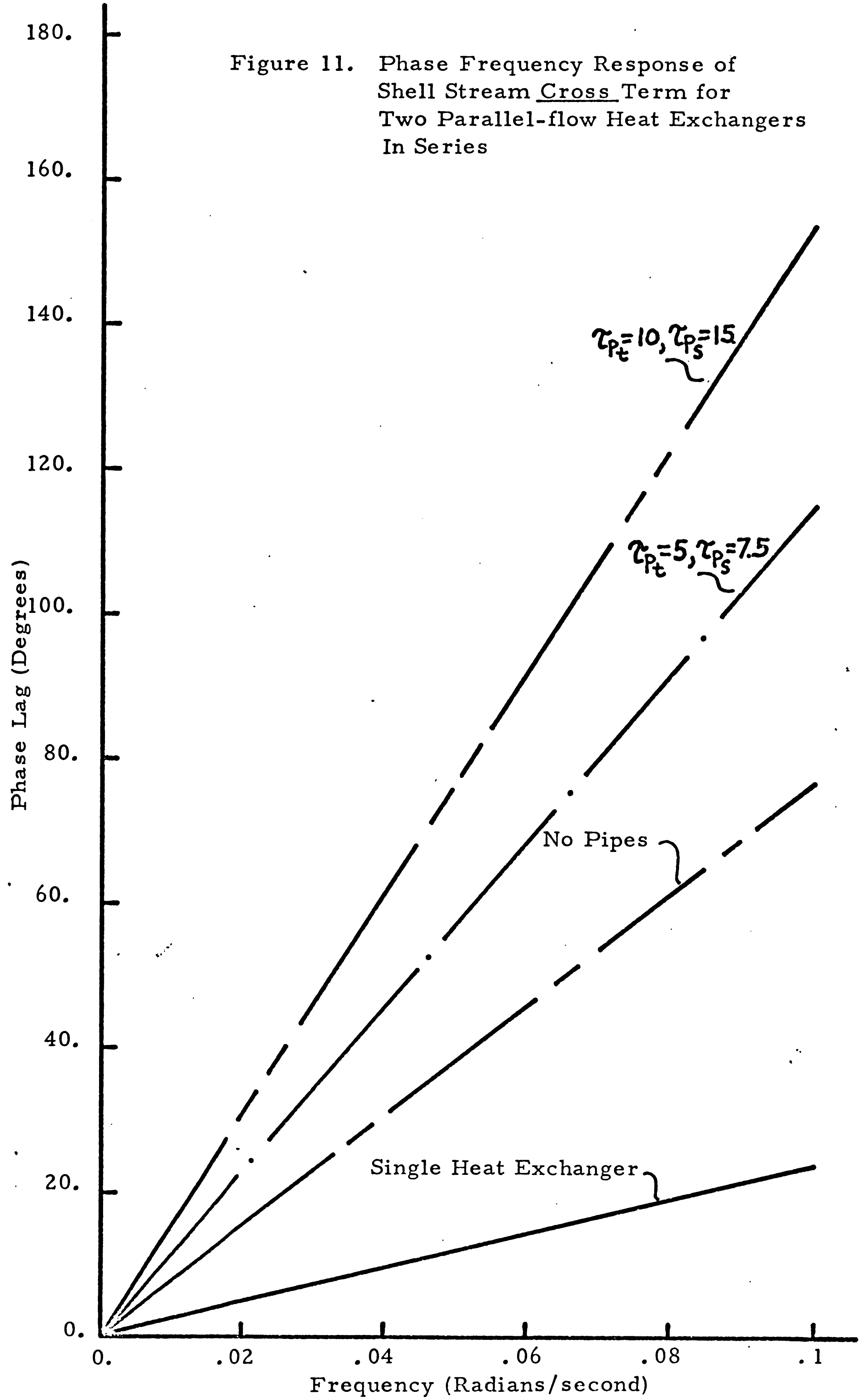


Figure 11. Phase Frequency Response of Shell Stream Cross Term for Two Parallel-flow Heat Exchangers In Series

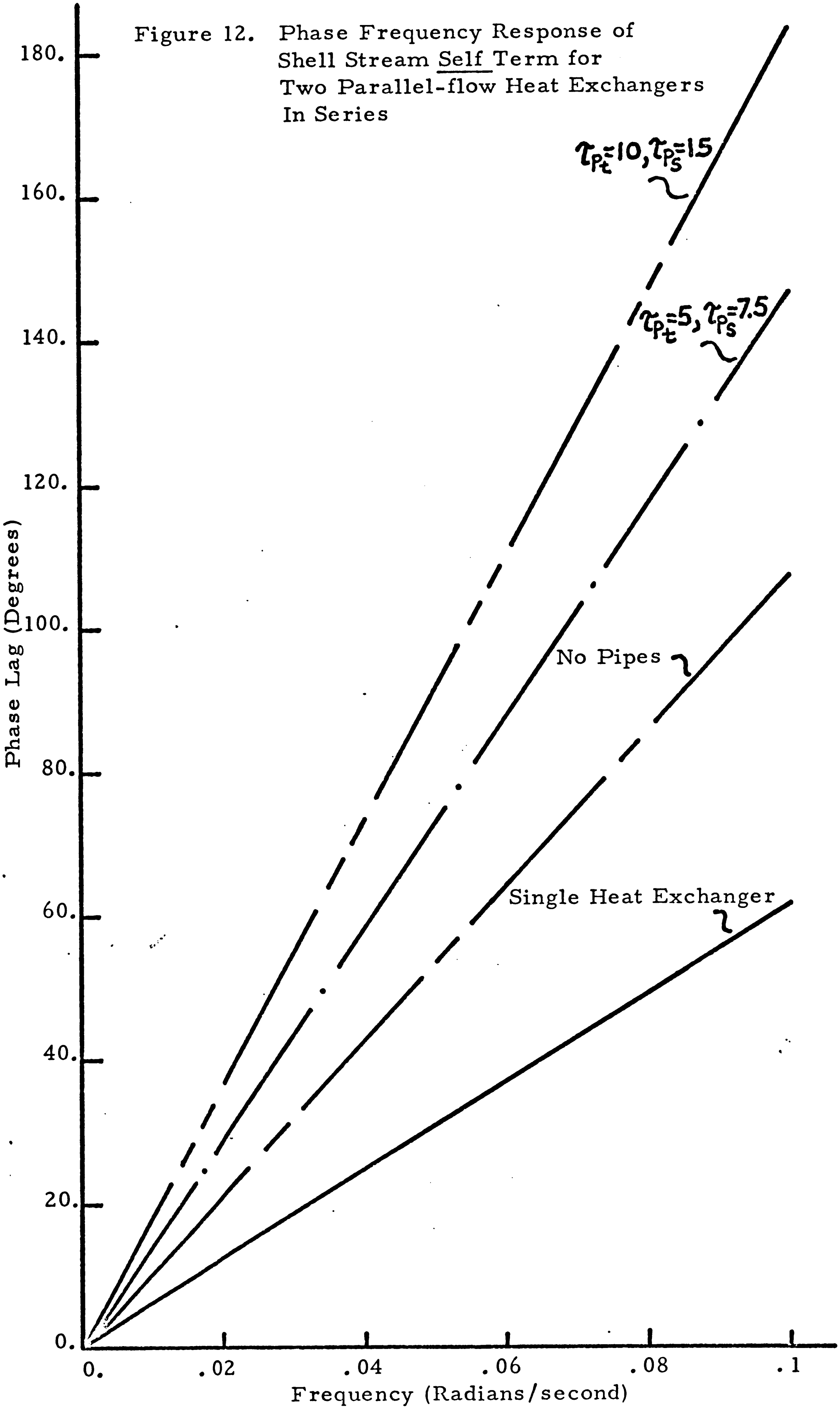
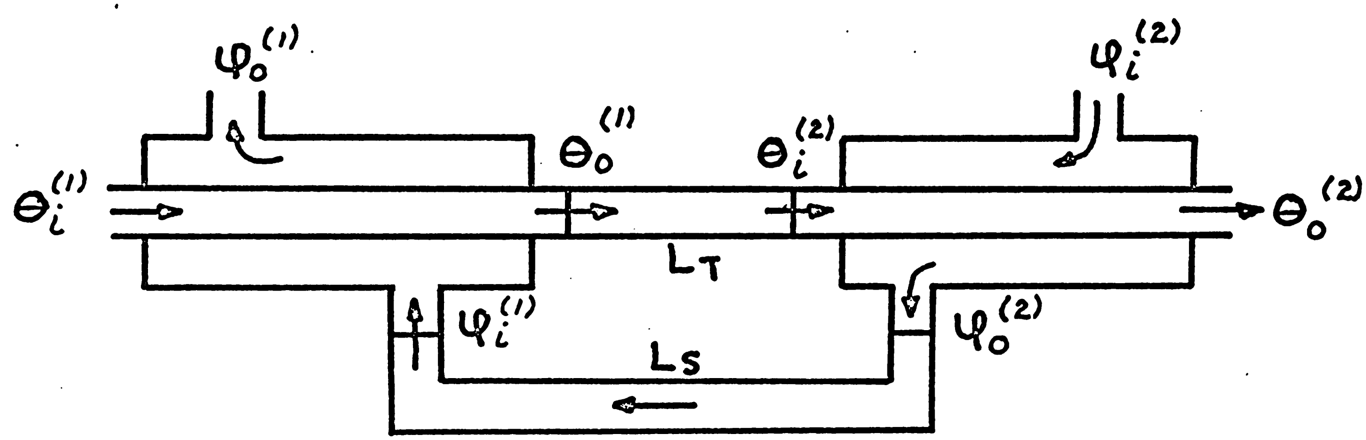


Figure 12. Phase Frequency Response of Shell Stream Self Term for Two Parallel-flow Heat Exchangers In Series
$\tau_{P_t}=10, \tau_{P_s}=15$
$\tau_{P_t}=5, \tau_{P_s}=7.5$
No Pipes
Single Heat Exchanger
Phase Lag (Degrees)
180.
160.
140.
120.
100.
80.
60.
40.
20.
0.
0.
.02
.04
.06
.08
.1
Frequency (Radians/second)

Figure 13. Two Counter-flow Heat Exchangers In Series



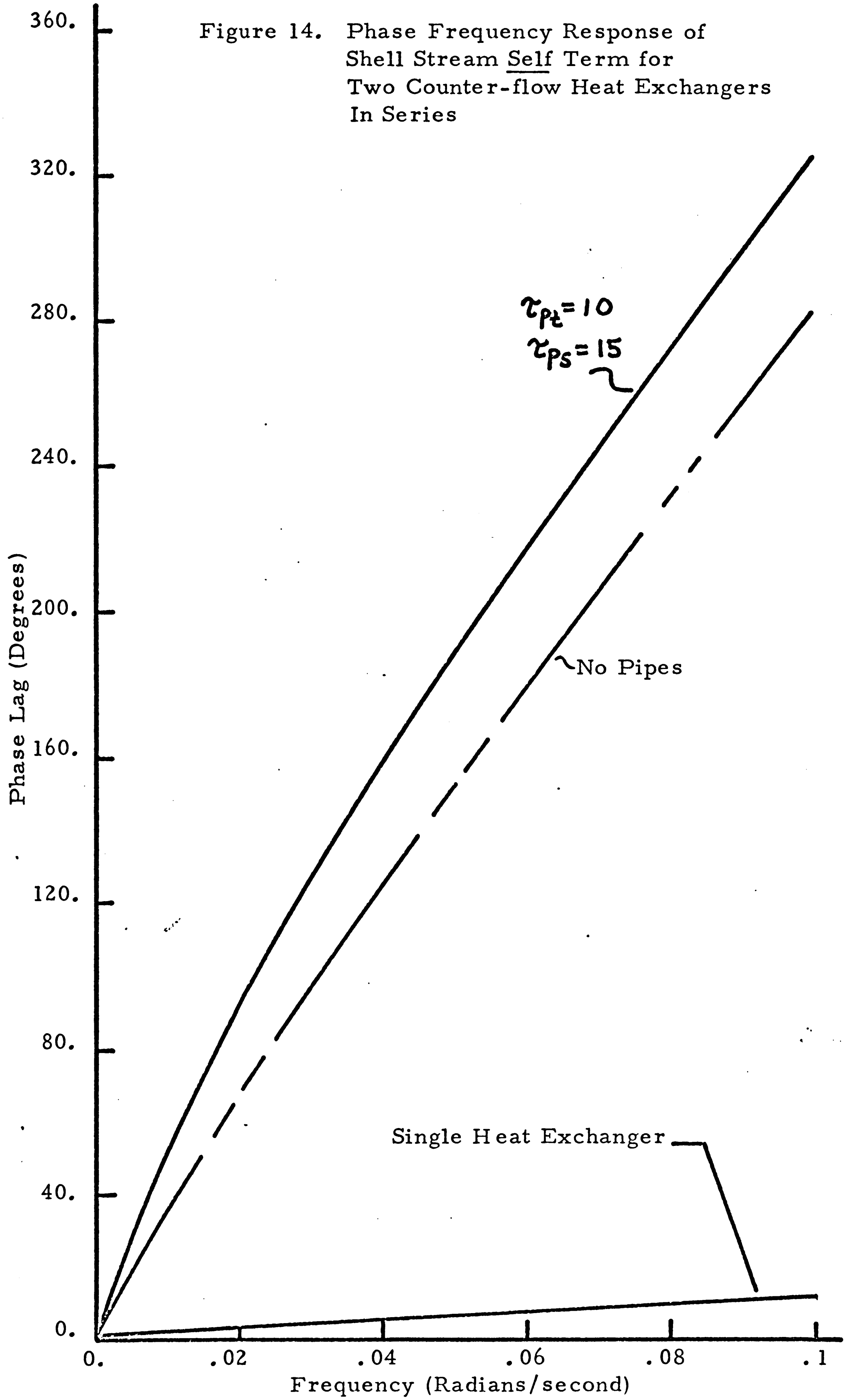
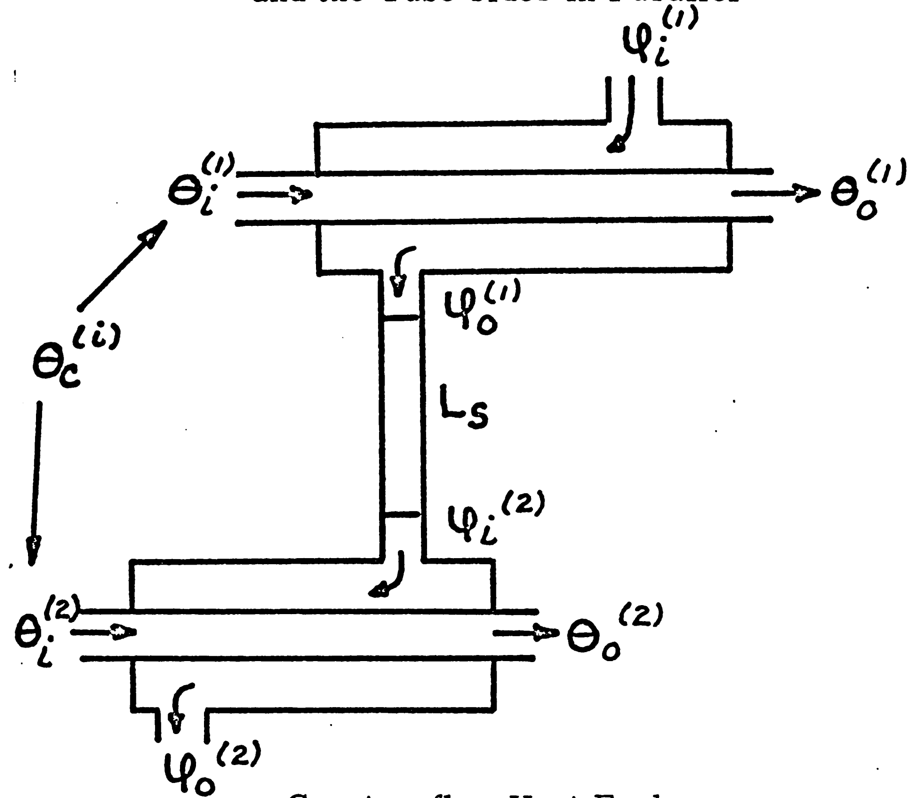
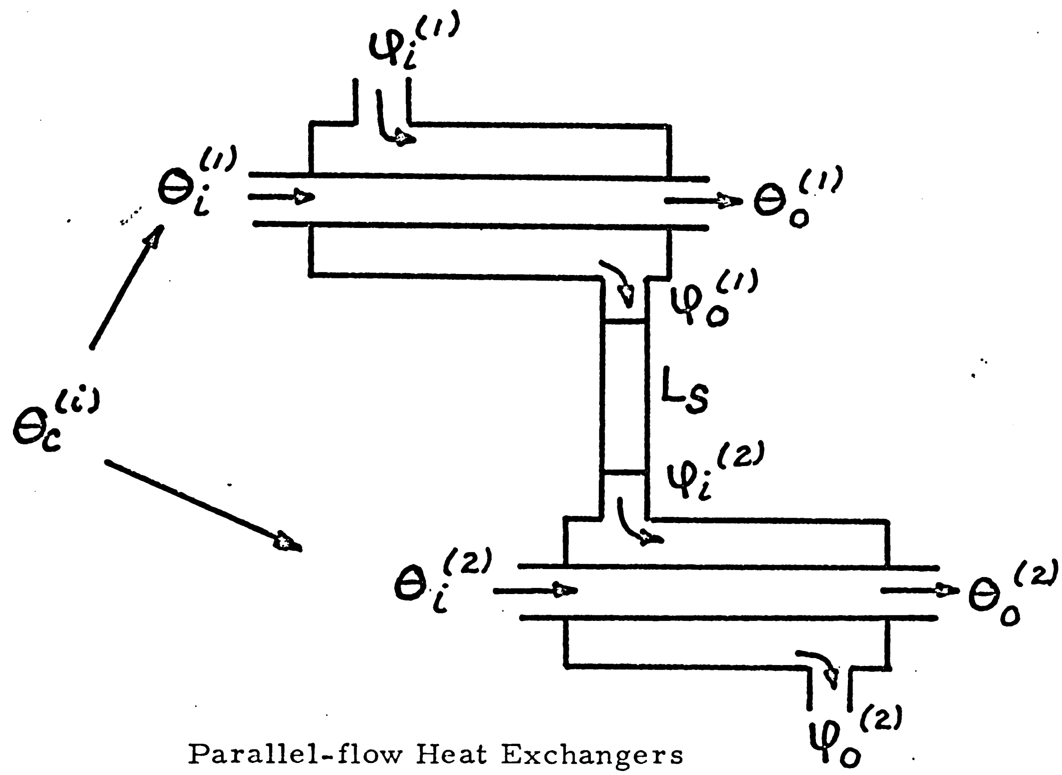


Figure 14. Phase Frequency Response of Shell Stream Self Term for Two Counter-flow Heat Exchangers In Series

Figure 15. Two Heat Exchangers with Shell sides in Series and the Tube sides in Parallel

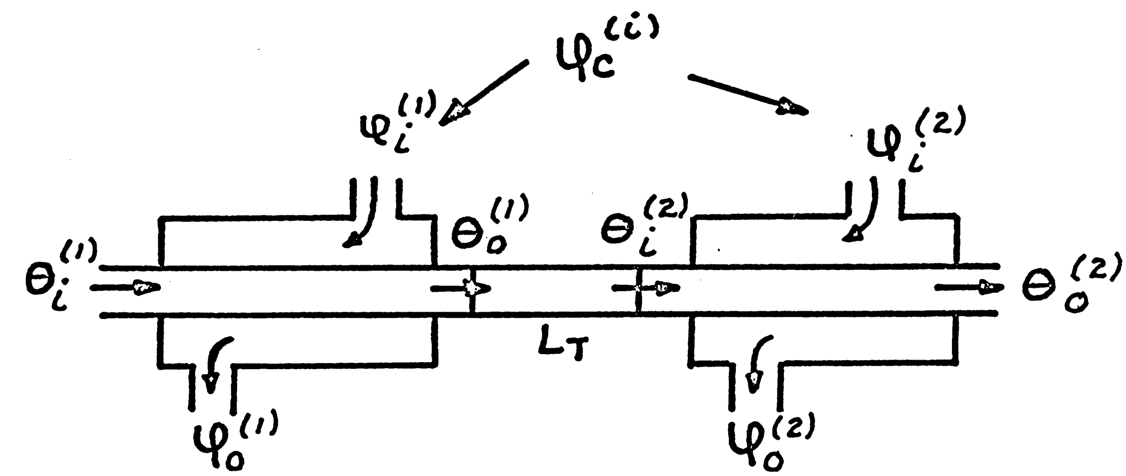


Counter-flow Heat Exchangers

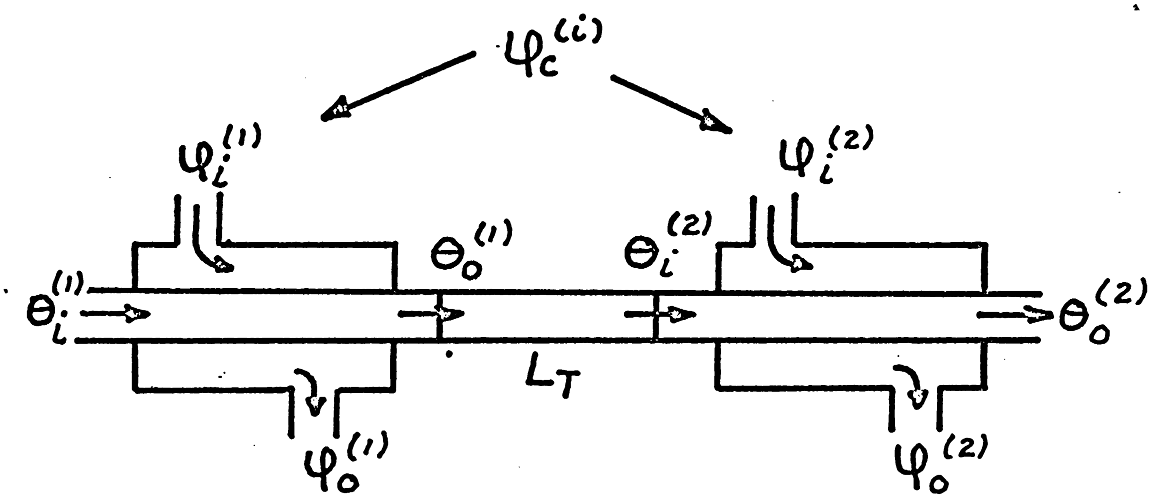


Parallel-flow Heat Exchangers

Figure 16. Two Heat Exchangers with Tube sides in Series and Shell sides in Parallel



Counter-flow Heat Exchangers



Parallel-flow Heat Exchangers

REFERENCES

1. Williams, Theodore J. and Henry J. Morris. "A Survey of the Literature on Heat Exchanger Dynamics and Control," Chemical Engineering Progress Symposium Series, AIChE, No. 36, Vol. 57, pp. 20-33.

2. Finlay, I. C. "Determining the Dynamic Response of Heat Exchangers," (Heat Transfer Survey), 1966, pp. 142-150.

3. Fux, Aimé. "Dynamic Models for Practical Heat Exchangers," M. S. Thesis, Mechanical Engineering Department, M. I. T., 1959.

4. Hansen, Peter D. "The Dynamics of Heat Exchange Processes," Sc. D. Thesis, Mechanical Engineering Department, M. I. T., 1960.

5. Campbell, Donald P. Process Dynamics, New York, John Wiley and Sons, Inc., 1958.

6. Gould, Leonard A. "Approximation of the Dynamic Behavior of Heat Exchangers," Ingenioren, December, 1957, pp. 825-830.

7. Ogata, Katsuhiko. Modern Control Engineering, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970.

8. Bowen, J. L. and P. M. Schultheiss. Introduction to the Design of Servomechanisms, New York, John Wiley and Sons, Inc., 1958.

9. Leonhard, A. "Determination of Transient Response from Frequency Response," Transactions of the ASME, November, 1954, pp. 1215-1236.

10. Domingos, J. D. "Analysis of Complex Assemblies of Heat Exchangers," International Journal of Heat and Mass Transfer, Vol. 12, Pergamon Press, 1969, pp. 537-548.

11. Carslaw, H. S. and J. C. Jaeger. Conduction of Heat in Solids, 2nd Ed., Oxford University Press, 1959.

12. Hildebrand, Francis B. Methods of Applied Mathematics, 2nd Ed., Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1965.

13. Takahashi, Yasundo. "Transfer Function Analysis of Heat Exchange Processes," Automatic and Manual Control, Ed. A. Tustin, Butterworth Scientific Publication, 1952, pp. 235-245.

14. Paynter, H. M. and Yasundo Takahashi. "A New Method of Evaluating Dynamic Response of Counter-flow and Parallel-flow Heat Exchangers," Transactions of the ASME, May, 1956, pp. 749-758.

15. Hsu, James P. and Nathan Gilbert. "Transfer Functions of Heat Exchangers," AIChE Journal, Vol. 8, November, 1962, pp. 593-598.

16. Gaurdabassi, G. and S. Rinaldi. "Heat Exchanger Transfer Functions," ASME paper, 67-HT-5, August, 1967.

17. Paynter, H. M. "On an Analogy between Stochastic Processes and Monotone Dynamic Systems," Regelungstechnik, Oldenbaurg Verlag, Munich, Germany, 1957, pp. 243-250.

18. Kays, W. M. and A. L. London. Compact Heat Exchangers, New York, McGraw-Hill, 1958.

19. Kreyszig, Erwin. Advanced Engineering Mathematics, 2nd Ed., New York, John Wiley and Sons, Inc., 1967.

20. Hougen, J. O. "Simplified Correlations of Process Dynamics," Chemical Engineering Progress, Vol. 59, No. 4, April, 1963, pp. 49-53.

21. Oldenburger, Rufus and R. E. Goodson. "Simplification of Hydraulic Line Dynamics by Use of Infinite Products," Transactions of the ASME, Journal of Basic Engineering, March, 1964, pp. 1-10.

BIBLIOGRAPHY

1. Chapman, Alan J. Heat Transfer, New York, The Macmillan Company, 1960.

2. Ed. by Grabbe, Eugene M., Simon Ramo, and Dean E. Wooldridge. Handbook of Automation, Computation, and Control, Vol. 1, John Wiley and Sons, Inc., 1958.

3. Kern, Donald Q. Process Heat Transfer, New York, McGraw-Hill Book Co., Inc., 1950.

4. Noble, Ben. Applied Linear Algebra, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

5. Pipes, Louis A. Matrix Methods for Engineering, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1963.

6. Thal-Larsen, Herman. "Dynamics of Heat Exchangers and Their Models," Transactions of the ASME, Journal of Basic Engineering, June, 1960, pp. 489-504.

7. Thomson, William T. Vibration Theory and Applications, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1965.

Appendix A Partial Inversion

From the solution of the differential equations for temperature dynamics of a counter-flow heat exchanger, we get:

$$\begin{Bmatrix} \bar{\theta}_o \\ \bar{\varphi}_i \end{Bmatrix} = e^{[M]} \begin{Bmatrix} \bar{\theta}_i \\ \bar{\varphi}_o \end{Bmatrix} \tag{39}$$

To obtain $[G]$ in equation (46) we first write

$$e^{[M]} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = [A] \tag{A1}$$

Now, the second equation (second row) is solved for $\bar{\varphi}_o$ in terms of $\bar{\varphi}_i$ and $\bar{\theta}_i$, yielding:

$$\bar{\varphi}_i = \frac{-A_{21}}{A_{22}} \bar{\theta}_i + \frac{1}{A_{22}} \bar{\varphi}_o \tag{A2}$$

Substituting this result into the first equation,

$$\bar{\theta}_o = \frac{(A_{11} A_{22} - A_{12} A_{21})}{A_{22}} \bar{\theta}_i + \frac{A_{12}}{A_{22}} \bar{\varphi}_o \tag{A3}$$

The required transformation matrix becomes (in terms of the elements of $e^{[M]}$):

$$(G) = \begin{pmatrix} \dfrac{|A|}{A_{22}} & \dfrac{A_{12}}{A_{22}} \\ & \\ -\dfrac{A_{21}}{A_{22}} & \dfrac{1}{A_{22}} \end{pmatrix} \tag{A4}$$

Appendix B   Evaluation of Matrix Coefficients for Parallel-flow Heat Exchanger Moments

This appendix summarizes the first three derivatives of:

$$f_1(s) = \frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} \tag{59}$$

and

$$f_2(s) = \frac{\lambda_2 e^{\lambda_1} - \lambda_1 e^{\lambda_2}}{\lambda_1 - \lambda_2} \tag{60}$$

and the limiting values of parameters in these equations as the laplace transform variable approaches zero.

$$f_1' = \frac{\lambda_1' e^{\lambda_1} - \lambda_2' e^{\lambda_2}}{\lambda_1 - \lambda_2} - \frac{\lambda_1' - \lambda_2}{\lambda_1 - \lambda_2} f_1 \tag{B1}$$

$$f_1'' = \frac{\{e^{\lambda_1}(\lambda_1'^2 + \lambda_1'') - e^{\lambda_2}(\lambda_2'^2 + \lambda_2'')\}}{\lambda_1 - \lambda_2} - 2\left(\frac{\lambda_1' - \lambda_2'}{\lambda_1 - \lambda_2}\right) f_1' - \left(\frac{\lambda_1'' - \lambda_2''}{\lambda_1 - \lambda_2}\right) f_1 \tag{B2}$$

$$f_1''' = \frac{\{e^{\lambda_1}(\lambda_1'^3 + 3\lambda_1'\lambda_1'' + \lambda_1''') - e^{\lambda_2}(\lambda_2'^3 + 3\lambda_2'\lambda_2'' + \lambda_2''')\}}{\lambda_1 - \lambda_2}$$

$$- 3\left(\frac{\lambda_1' - \lambda_2'}{\lambda_1 - \lambda_2}\right) f_1'' - 3\left(\frac{\lambda_1'' - \lambda_2''}{\lambda_1 - \lambda_2}\right) f_1' - \left(\frac{\lambda_1''' - \lambda_2'''}{\lambda_1 - \lambda_2}\right) f_1 \tag{B3}$$

$$f_2' = \frac{\{e^{\lambda_1}(\lambda_2\lambda_1' + \lambda_2') - e^{\lambda_2}(\lambda_1\lambda_2' + \lambda_1')\}}{\lambda_1 - \lambda_2} - \frac{\lambda_1' - \lambda_2'}{\lambda_1 - \lambda_2} f_2 \tag{B4}$$

$$f_2'' = \frac{\left\{e^{\lambda_1}\left(\lambda_2\lambda_1'' + 2\lambda_1'\lambda_2' + \lambda_2\lambda_1'^{2} + \lambda_2''\right)\right\}}{\lambda_1 - \lambda_2}$$

$$- \frac{\left\{e^{\lambda_2}\left(\lambda_1\lambda_2'' + 2\lambda_1'\lambda_2' + \lambda_1\lambda_2'^{2} + \lambda_1''\right)\right\}}{\lambda_1 - \lambda_2}$$

$$-2\left(\frac{\lambda_1' - \lambda_2'}{\lambda_1 - \lambda_2}\right) f_2' - \left(\frac{\lambda_1'' - \lambda_2''}{\lambda_1 - \lambda_2}\right) f_2 \tag{B5}$$

$$f_2''' = \frac{\left\{e^{\lambda_1}\left(\lambda_2\lambda_1''' + \lambda_2'\lambda_1'' + 3\lambda_2\lambda_1'\lambda_1'' + 2\lambda_2'\lambda_1'' + 2\lambda_1'\lambda_2'' + 3\lambda_2'\lambda_1'^{2} + \lambda_2\lambda_1'^{3} + \lambda_2''\lambda_1' + \lambda_2'''\right)\right\}}{\lambda_1 - \lambda_2}$$

$$- \frac{\left\{e^{\lambda_2}\left(\lambda_1\lambda_2''' + \lambda_1'\lambda_2'' + 3\lambda_1\lambda_2'\lambda_2'' + 2\lambda_2''\lambda_1' + 2\lambda_2'\lambda_1'' + 3\lambda_1'\lambda_2'^{2} + \lambda_1\lambda_2'^{3} + \lambda_2'\lambda_1'' + \lambda_1'''\right)\right\}}{\lambda_1 - \lambda_2}$$

$$-3\left(\frac{\lambda_1' - \lambda_2'}{\lambda_1 - \lambda_2}\right) f_2'' - 3\left(\frac{\lambda_1'' - \lambda_2''}{\lambda_1 - \lambda_2}\right) f_2' - \left(\frac{\lambda_1''' - \lambda_2'''}{\lambda_1 - \lambda_2}\right) f_2 \tag{B6}$$

Various derivatives and combinations of the λ are required in the evaluation of the limits of these derivatives. These parameters are given by:

$$\lim_{s \to 0} \lambda_1 = 0 \tag{B7}$$

$$\lim_{s\to 0} \lambda_2 = -2\beta \tag{B8}$$

$$\lim_{s\to 0} \lambda_1' = -\frac{\left[(\tau_1+\tau_2) - \varepsilon/\beta\,(\tau_1-\tau_2)\right]}{2} \tag{B9}$$

$$\lim_{s\to 0} \lambda_2' = -\frac{\left[(\tau_1+\tau_2) + \varepsilon/\beta\,(\tau_1-\tau_2)\right]}{2} \tag{B10}$$

$$\lim_{s\to 0} (\lambda_1-\lambda_2) = 2\beta \tag{B11}$$

$$\lim_{s\to 0} (\lambda_1'-\lambda_2') = \varepsilon/\beta\,(\tau_1-\tau_2) \tag{B12}$$

$$\lim_{s\to 0} \lambda_1'' = \frac{1}{4\beta}(\tau_1-\tau_2)^2\left(1-(\varepsilon/\beta)^2\right) \tag{B13}$$

$$\lim_{s\to 0} \lambda_2'' = -\lim_{s\to 0} \lambda_1'' \tag{B14}$$

$$\lim_{s\to 0} \lambda_1''' = -3/8\;\varepsilon/\beta^3\,(\tau_1-\tau_2)^3\left(1-(\varepsilon/\beta)^2\right) \tag{B15}$$

$$\lim_{s\to 0} \lambda_2''' = -\lim_{s\to 0} \lambda_1''' \tag{B16}$$