

Asymptotic Profiles with Finite Mass in  
One-Dimensional Contaminant Transport  
Through Porous Media: The Fast Reaction Case

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# 1 Introduction

In this paper we consider the large time behaviour of solutions of the convection-dispersion equation with nonlinear capacity

$$\frac{\partial}{\partial t}\{u + u^p\} + \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial x^2} = 0, \text{ with } (x, t) \in R \times R^+, \quad (1.1)$$

subject to the initial condition

$$u(x, 0) = u_0(x) \quad \text{for } x \in R. \quad (1.2)$$

Here  $p$  and  $D$  are positive constants and  $u_0$  is a nonnegative function satisfying the finite mass property

$$u_0 + u_0^p \in L^1(R). \quad (1.3)$$

Problem (1.1)-(1.2) arises as a model for the one-dimensional transport of a solute, with scaled concentration  $u \geq 0$ , through a porous medium. In this model it is assumed that the solute undergoes equilibrium adsorption with the porous matrix. In equation (1.1) the term  $u^p$  denotes the scaled adsorbed concentration. The integrability condition (1.3) implies that initially the total mass, both in solution and adsorbed, is finite.

This work is organised as follows.

## Contents

1. Introduction
2. The model
3. Some analytical remarks
4. The asymptotic solution
  - 4.1 The outer solutions
    - 4.1.1 The case  $p > 2$
    - 4.1.2 The case  $1 < p \leq 2$
    - 4.1.3 The case  $p < 1$
  - 4.2 The uniformity of the outer solutions: the translation layer solutions
    - 4.2.1 The case  $1 < p < 2$
    - 4.2.2 The case  $0 < p < 1$
  - 4.3 Construction of uniformly valid solutions for  $0 < p < 2$
- 5 The numerical method
- 6 Conclusions
- References

- A.1. Error analysis for  $p = 3$  and  $p = 2$
- A.2. Solutions to a boundary value problem
- A.3. The solution for  $D = 0$  when  $p = 1/2$  and  $p = 3/2$

In Section 2 we discuss the physical background of the problem and derive equation (1.1). In Section 3 some analytical properties of solutions are given, such as the occurrence of free boundaries when  $p \in (0, 1)$  and the large time behaviour when the initial distribution satisfies either  $u_0(-\infty) = 1, u_0(+\infty) = 0$  (convergence towards a traveling wave) or  $u_0(-\infty) = 0, u_0(+\infty) = 1$  (convergence towards a rarefaction wave). We also compare our findings with the analytical results of Escabedo, Vazquez & Zuazua [1]. Next, in Section 4, the asymptotic form for pulse type solutions satisfying

$$(u + u^p)(\cdot, t) \in L^1(R) \quad \text{for all } t > 0 \tag{1.4}$$

is considered. We discuss first the outer solutions in Section 4.1 and thereafter, in Section 4.2, the boundary layer solutions which occur for  $0 < p < 1$  and  $1 < p < 2$ . We also compare the asymptotic profiles with the numerical solution of problem (1.1)-(1.2). The algorithm which is discussed in Section 5, is based on a higher-order Godunov approach, which makes it possible to compute solutions of (1.1) with  $D$  small, or even with  $D = 0$ . Some concluding remarks are given in Section 6.

## 2 The model

In this Section we formulate a model for the one-dimensional transport of a one-species contaminant through a porous medium. To begin with, we consider the flow of an incompressible fluid through a homogeneous and saturated porous medium. We shall assume that the flow is steady, macroscopically one dimensional and directed along what is chosen to be the positive  $x$ -axis. It is characterized by the volumetric flux, also known as specific discharge, which will be denoted by  $q(m/s)$ .

In the fluid a one-species solute is present at tracer level concentration  $C(\text{mol}/\text{m}^3)$ . This means that the flow is independent of the solute distribution. We shall therefore take  $q$  to be a known positive constant.

If no adsorption reactions occur between the solute and the surrounding solid part of the porous medium, then the transport is determined by convection, molecular diffusion and mechanical dispersion, see for instance Bear [2] or Freeze & Cherry [3]. However, if adsorption reactions do take place, this has to be taken into account when describing the transport process. In this reactive case we denote by  $S$  (mol/kg porous material) the adsorbed concentrations. If the boundary and flow conditions are such that both  $C$  and  $S$  can be assumed to be constant in planes perpendicular to the  $x$ -axis, implying  $C = C(x, t)$  and  $S = S(x, t)$ , then mass conservation yields the expression (see for example Bolt [4]).

$$\frac{\partial}{\partial t}\{\theta C + \rho S\} + \frac{\partial}{\partial x}\{qC - D\frac{\partial C}{\partial x}\} = 0, \quad (2.1)$$

where  $t$  and  $x$  denote, respectively, time and space coordinates. Here  $\theta(-)$  is the porosity of the porous material,  $\rho$  ( $\text{kg}/\text{m}^3$ ) its bulk mass density and  $D(\text{m}^2/\text{s})$  the coefficient of hydrodynamic dispersion, which is the sum of molecular diffusion and mechanical dispersion.

All coefficients in (2.1) can be considered as being constant and positive. The term  $\rho\frac{\partial S}{\partial t}$  in (2.1) represents the rate of change of concentration on the porous matrix due to adsorption or desorption.

Now we consider the adsorption process. In general, the relation between the concentration in the fluid and the adsorbed concentration is described by a first order ordinary differential equation of the form

$$\frac{\partial S}{\partial t} = kf(C, S), \quad (2.2)$$

where  $k > 0$  ( $s^{-1}$ ) is the rate parameter and  $f$  (mol/kg. porous material) the reaction rate function. In Van Duijn & Knabner [5] general rate functions are considered some of which are discussed below together with some of their properties.

If we can solve the equation

$$f(C, S) = 0 \quad (2.3a)$$

in the form

$$S = \psi(C) \quad (2.3b)$$

then we call  $\psi$  the adsorption isotherm. In many cases, rate functions and isotherms satisfy the following monotonicity properties

$$f(C, S) > (<) 0 \text{ iff } S < (>) \psi(C), \quad (2.4b)$$

$$\left. \begin{array}{l} \psi(0) = 0, \\ \text{and } \psi \text{ strictly increasing and smooth for } C > 0 \end{array} \right\} \quad (2.4b)$$

The isotherms  $\psi$  are sometimes classified according to their behaviour near  $C = 0$ . We say

1.  $\psi$  is of Langmuir (*L*-type) if  
 $\psi$  is strictly concave near  $C = 0$  and  $\psi'(0+) < \infty$ ;
2.  $\psi$  is of Freundlich (*F*-type) if  
 $\psi$  is strictly concave near  $C = 0$  and  $\psi'(0+) = \infty$ ;
3.  $\psi$  is of convex (*S*-type) if  
 $\psi$  is strictly convex near  $C = 0$ .

The distinction between these classes is of importance, because different isotherms may give different transport behaviours for the solutes. In mathematical terms, the regularity and global behaviour of the solutions may be different if  $\psi$  is taken from these different classes. This is clearly the case when considering the asymptotic profiles in Sections 4 and 5.

Well-known examples of isotherms are

1. the Langmuir isotherm where

$$\psi(C) = \frac{K_1 C}{1 + K_2 C} \quad \text{with } K_1, K_2 > 0 \quad (2.5)$$

and

2. the Freundlich isotherm where

$$\psi(C) = K_3 C^p \quad \text{with } K_3, p > 0 \quad (2.6)$$

In (2.6) the isotherm is of  $F$ -type if  $0 < p < 1$  and of  $S$ -type if  $p > 1$ . The case  $0 < p < 1$  occurs in many practical situations, although values for which  $p > 1$  have been used, (see for example Van Genuchten & Cleary [6]).

In this paper we restrict ourselves to the case of fast reactions or equivalently equilibrium adsorption. Mathematically this is achieved by letting  $k \rightarrow \infty$  in (2.2). As a result we have

$$S = \psi(C) \quad (2.7)$$

see also (2.3). The convergence process in which  $k \rightarrow \infty$ , is discussed in detail by Knabner [7] and Van Duijn & Knabner [8]. From a physical view point this limit implies that the adsorption reactions are very rapid in comparison with the flow velocity, so that the adsorbed concentration instantaneously follows the variations of the solute concentration.

We also restrict ourselves with respect to the choice of the isotherm. To be specific we shall consider the case of Freundlich isotherms only, whence the transport equation (2.1) becomes

$$\frac{\partial}{\partial t} \{ \theta C + \rho K_3 C^p \} + \frac{\partial}{\partial x} \{ qC - D \frac{\partial C}{\partial x} \} = 0. \quad (2.8)$$

We shall consider solutions of this equation in the half space

$$Q = \{ (x, t) : -\infty < x < \infty, t > 0 \},$$

and impose the initial condition

$$C(x, 0) = C_0(x)$$

at  $t = 0$ . To eliminate the constants from (2.8) we apply the following scaling and redefinition:

$$p = 1 : \begin{cases} u := C & t := 2 \frac{q}{\theta} (1 + \frac{\rho K_3}{\theta})^{-1} t \\ x := x & D := \frac{D}{q} \end{cases} \quad (2.9)$$

$$p \neq 1 : \begin{cases} u := (\frac{\rho K_3}{\theta})^{\frac{1}{p-1}} C & t := \frac{q}{\theta} t \\ x := x & D = \frac{D}{q} \end{cases} \quad (2.10)$$

This leads to the initial value problem (for all  $p > 0$ )

$$(IVP) \begin{cases} \frac{\partial}{\partial t} \{ u + u^p \} + \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial x^2} = 0 & \text{for } (x, t) \in Q \\ u(x, 0) = u_0(x) & \text{for } x \in R \end{cases} \quad (2.11)$$

$$(2.12)$$

where

$$u_0(x) = \begin{cases} C_0(x) & \text{for } p = 1, \\ (\frac{\rho K_3}{\theta})^{\frac{1}{p-1}} C_0(x) & \text{for } p \neq 1. \end{cases}$$

We shall consider the large time behaviour of solutions of Problem (IVP) for the case where  $u_0$  is a pulse satisfying

$$u_0(\pm\infty) = 0, u_0 \geq 0 (\neq 0) \text{ on } R.$$

In particular we shall require that

$$u_0 + u_0^p \in L^1(R), \quad (2.13)$$

so that the total mass of adsorbed and dissolved concentration is finite.

We note that we could have chosen a scaling which also eliminates the constant  $D$  from equation (2.11). To be specific

$$x := \frac{x}{D} \quad t := \frac{t}{D} \quad (2.14)$$

to give

$$\frac{\partial(u + u^p)}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}. \quad (2.15)$$

However, keeping  $D$  in front of the second derivative allows us to consider the hyperbolic limit  $D \downarrow 0$ . We come back to this point in the concluding remarks of Section 6.

We finally observe that in many cases of practical interest  $\frac{\rho K_3}{\theta} \gg 1$ . This means that the implication of the scaling is quite different for  $p < 1$ ,  $p = 1$  and  $p > 1$ . One has to bear this in mind when comparing solutions of Problem (IVP) for different values of  $p$ .

### 3 Some analytical remarks

We first set

$$\beta(u) = u + u^p \quad \text{for } u \geq 0, \quad (3.1)$$

and write equation (2.11) as

$$\frac{\partial}{\partial t} \beta(u) + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0. \quad (3.2)$$

We have put  $D = 1$  in (3.2) which, as we note from (2.19), can be done without loss of generality. Equation (3.2) is a nonlinear second order equation of parabolic type. Since  $\beta'(u)$  may tend to infinity when  $u$  tends to zero ( $p < 1$ ), equation (3.2) can degenerate at points where its solution vanishes. Therefore we cannot expect Problem (IVP) to have classical solutions for values of  $p$  belonging to the interval  $(0, 1)$ .

Writing

$$w = \beta(u) \quad \text{and} \quad u = \varphi(w) \quad (3.3)$$

where  $\varphi = \beta^{-1}$  denotes the inverse of the function  $\beta$ , we obtain for  $w$  the transformed problem

$$(IVP') \begin{cases} \frac{\partial w}{\partial t} + \frac{\partial \varphi(w)}{\partial x} - \frac{\partial^2 \varphi(w)}{\partial x^2} = 0 & \text{for } (x, t) \in Q \\ w(x, 0) = w_0(x) := u_0(x) + u_0^p(x) & \text{for } x \in R \end{cases} \quad (3.4)$$

The existence and uniqueness theory for Problem (IVP') is well-known and can be found in Gilding [9]. We can therefore use these results to make some statements about the solvability of Problem (IVP) for  $u$ . If we assume  $u_0$ , and hence  $w_0$ , is such that

$$u_0 \in C(R) : \quad (3.6a)$$

there exist numbers  $-\infty < a_1 < a_2 < \infty$  such that

$$u_0(x) = \begin{cases} 0 & -\infty < x \leq a_1 \\ > 0 & a_1 < x < a_2 \\ 0 & a_2 \leq x < \infty. \end{cases} \quad (3.6b)$$

Then we can make the following statements.

If  $p \geq 1$  then Problem (IVP) has a unique classical solution  $u \in C^\infty(Q) \cap C(\bar{Q})$  which satisfies  $u > 0$  in  $Q$ ;

If  $p \in (0, 1)$  then Problem (IVP) has a unique weak (distributional) solution  $u \in C(\bar{Q})$ . At points where  $u > 0$ , the solution is smooth (i.e.  $u \in C^\infty(\{u > 0\})$ ) and satisfies the equation classically. Moreover, there exist functions (for  $i = 1, 2$ )  $s_i \in C([0, \infty))$ , satisfying  $s_i(0) = a_i$  and  $-\infty < s_1(t) < \infty$  for all  $t > 0$ , which form the support of  $u$  in the  $x, t$  plane; i.e.  $u(x, t) > 0$  if and only if  $x \in (s_1(t), s_2(t))$ , for every  $t \geq 0$ . The function  $s_i$  are called interfaces or free boundaries and they occur only in the degenerate case  $p \in (0, 1)$ .

In this paper we are interested in the large time behaviour of solutions of Problem (IVP). First we make some statements about previous work on equation (3.2) which, to the authors' knowledge has dealt exclusively with initial data satisfying

$$u_0(\pm\infty) = u_\pm \quad (3.7)$$

where either

$$0 \leq u_- < u_+ < \infty \quad (3.8a)$$

or

$$\infty > u_- > u_+ \geq 0. \quad (3.9b)$$

We note that (3.2) has travelling waves solutions  $u(x, t) = f(x - ct)$ , which satisfy  $f(\pm\infty) = u_\pm$ , where  $u_+$  and  $u_-$  satisfy (3.8a) if  $p > 1$  and (3.8b) if  $p \in (0, 1)$  and where the wave speed  $c$  is given by

$$c = \frac{u_+ - u_-}{\beta(u_+) - \beta(u_-)}. \quad (3.10)$$

The stability of these travelling waves follows from a result of Osher & Ralston [10]. They employ a contraction property of the semigroup associated with the transformed problem (IVP') to prove convergence, as  $t \rightarrow \infty$  in  $L^1(R)$ , towards a suitably shifted travelling wave.

The large time behaviour for the cases where no travelling waves exist, i.e.

$$p > 1 \quad \text{and} \quad (3.8b) \quad (3.11a)$$

or

$$p \in (0, 1) \quad \text{and} \quad (3.8a) \quad (3.11b)$$

was considered by Van Duijn & De Graaf [11]. For this parameter choice, the contaminant profiles become flatter as time increases. In fact it was shown that  $\beta(u(\cdot, t))$  converges to the transformed solution  $u^*$  of the reduced hyperbolic problem

$$\frac{\partial}{\partial t} \beta(u) + \frac{\partial u}{\partial x} = 0 \quad \text{for } (x, t) \in Q \quad (3.12a)$$

$$u(x, 0) = \begin{cases} u_- & x < 0 \\ u_+ & x > 0 \end{cases} \quad (3.12b)$$

The function  $u^*$  is a rarefaction wave with  $u^* = u^*(x/t)$ . Since  $\beta$  is given by (3.1), it is easy to obtain explicit expressions for  $u^*$ . The convergence analysis gives an estimate for decay rate of  $\|\beta(u(\cdot, t)) - \beta(u^*(\cdot, t))\|_{L^\infty(\mathbb{R})}$  as  $t \rightarrow \infty$ .

We now make some remarks about the solution to Problem (IVP) and the procedures used to find the asymptotic solutions. We first note that the solutions have mass conservation, that is for all  $t > 0$

$$\int_{\mathbb{R}} \{u + u^p\}(x, t) dx = \int_{\mathbb{R}} (u_0 + u_0^p)(x) dx =: M.$$

This property, together with scaling arguments, plays a crucial role in establishing the asymptotic solutions. The existence of a second integral invariant

$$L = \int_{-\infty}^{\infty} e^{-x} (u + u^p) dx \quad (3.13)$$

was pointed out by us by Dr. J.R. King. We refer to this later in the paper.

In constructing the asymptotic solutions we use the following intuitive ideas. In the degenerate case,  $p \in (0, 1)$ , for  $t \rightarrow \infty$  we may write

$$u + u^p \sim u^p$$

and consider the simplified equation

$$\frac{\partial(u^p)}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (3.14)$$

to obtain the asymptotic limit. This procedure is formalized in detail in Section (4.1.3).

When  $p = 1$ , we have the linear equation

$$2 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (3.15)$$

which has the asymptotic profile

$$u(x \cdot t) \rightarrow \frac{M}{\sqrt{2\pi t}} \exp\left\{-\frac{1}{2t} \left(x - \frac{t}{2}\right)^2\right\} \text{ as } t \rightarrow \infty. \quad (3.16)$$

When  $p > 1$  the situation becomes more subtle. Here we first transform to the moving coordinates

$$t = t \quad \text{and} \quad \xi = x - t \quad (3.17)$$

which gives the equation, now with nonlinear convection,

$$\frac{\partial}{\partial t}(u + u^p) - \frac{\partial u^p}{\partial \xi} - \frac{\partial^2 u}{\partial \xi^2} = 0. \quad (3.18)$$

Next we write

$$u + u^p \sim u$$

as  $t \rightarrow \infty$  and obtain the nonlinear convection equation

$$\frac{\partial u}{\partial t} - \frac{\partial u^p}{\partial \xi} - \frac{\partial^2 u}{\partial \xi^2} = 0. \quad (3.19)$$

whence we distinguish the following cases.

$1 < p < 2$ . Here convection dominates with respect to diffusion and the asymptotic profile will result in half an  $N$ -wave, see Section 4.1.2.

$p = 2$ . Here convection and diffusion balance and (3.19) is in fact Burgers' equation for which a limit profile exists in the form of a self-similar solution. It is also given in Section 4.1.2.

$p > 2$ . Here the diffusion term dominates convection. This fact is reflected in the asymptotic profiles which now are symmetrical self-similar solutions of the heat equation, see Section 4.1.1.

The asymptotic forms discussed above are called outer solutions. We note here that convergence (in the  $L^1$  sense) of solutions of (3.19) towards these outer solutions for  $p > 1$  was proved by Escabedo, Vasquez & Zuazua [1] and Escabedo & Zuazua [12]. For  $1 < p < 2$  there are two defects associated with the outer solutions. Firstly they are not continuous and secondly do not have unbounded support. It is necessary therefore to supplement the outer solutions by boundary layer solutions which are valid in thin regions near points of discontinuous behaviour of the outer solution. Except for one case, we can solve the boundary layer equations explicitly. These solutions can be used to do two things: first to render the the outer solution continuous and secondly to give the solution unbounded support. Boundary layers have also to be inserted when  $0 < p < 1$  to smooth out the outer solution and locate the position of the free boundaries which occur since the outer solution predicts, erroneously, a stationary interface at one end of its support.

## 4 The asymptotic solution

### 4.1 The outer solutions

In this Section we construct the large time solution of the scaled equation

$$\frac{\partial(u + u^p)}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}, \quad p > 0, (x, t) \in Q \quad (4.1)$$

with pulse type initial conditions satisfying (2.13). Our approach is to a large extent intuitive in the sense that we postulate an analytic form for the solution and then deduce for what range of values of  $p$  this type of solution is expected to occur. This idea has been successfully used in a number of papers devoted to large time asymptotics for nonlinear diffusion and related equations, particularly for pulse type initial data with either bounded or unbounded support. See for example Grundy [13,14,15].

The nature of the limiting solution depends on the value of the parameter  $p$  and reflects the relative importance of the various terms in (4.1) as  $t$  becomes large. As will become apparent later on it is natural to take the cases  $0 < p < 1$ ,  $1 < p \leq 2$  and  $p > 2$  separately. Let us take  $p > 2$  first.

#### 4.1.1 The case $p > 2$

We start off by transforming equation (4.1), using the moving coordinate system

$$(t, \xi = x - t). \quad (4.2)$$

with  $u = u(\xi, t)$ , giving

$$\frac{\partial}{\partial t}(u + u^p) = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial(u^p)}{\partial \xi} \quad (4.3)$$

We would now expect the spread of the solution to be incorporated by using the similarity variable

$$\eta = \frac{\xi}{t^\delta} = \frac{x - t}{t^\delta} \quad (4.4)$$

where  $\delta > 0$ , together with the change of dependent variable

$$u(\xi, t) = t^\alpha v(\eta, t) \quad (4.5)$$

with  $\alpha < 0$  to simulate temporal decay. In the new variables (4.3) now becomes

$$\underbrace{\left\{ t \frac{\partial v}{\partial t} + \alpha v - \delta \eta \frac{\partial v}{\partial \eta} \right\}}_{(1)} + t^{\alpha(p-1)} \underbrace{\left\{ t \frac{\partial(v^p)}{\partial t} + \alpha p v^p - \delta \eta \frac{\partial(v^p)}{\partial \eta} \right\}}_{(2)}$$

$$= t^{1-2\delta} \overset{\textcircled{3}}{\frac{\partial^2 v}{\partial \eta^2}} + t^{\alpha(p-1)+(1-\delta)} \overset{\textcircled{4}}{\frac{\partial(v^p)}{\partial \eta}} \quad (4.6)$$

The immediate aim of the analysis is to determine the unknown indices  $\alpha$  and  $\delta$ . To assist us in this task we turn to the time invariant, (3.13) namely

$$M = \int_{-\infty}^{\infty} \{u + u^p\} dx \quad (4.7)$$

which, in terms of  $\eta$  and  $v$ , can be written as

$$M = \int_{-\infty}^{\infty} \{t^{\alpha+\delta} v + t^{\alpha p+\delta} v^p\} d\eta \quad (4.8)$$

Thus, for  $p > 1$

$$M \sim t^{\alpha+\delta}, \text{ as } t \rightarrow \infty$$

and hence for  $M$  to be invariant in time

$$\delta = -\alpha \quad (4.9)$$

Remembering that we are seeking solutions in the limit  $t \rightarrow \infty, \eta = 0(1)$ , we now assume that  $v$  and its derivatives with respect to  $\eta$  together with  $t \frac{\partial v}{\partial t}$  are bounded in this limit. With this assumption it is clear that for  $p > 1$  and  $\alpha < 0$  the term (1) dominates the left hand side of (4.6) as  $t \rightarrow \infty$ . Two possibilities now emerge; either (3) dominates (4) on the right hand side and balances with (1), or (4) dominates (3) and balances with (1). By the term balancing we mean asymptotically equivalent as  $t \rightarrow \infty$ .

The first possibility requires that

$$\delta = \frac{1}{2} \quad (4.10)$$

and

$$\alpha(p-1) + 1 - \delta < 0 \quad (4.11)$$

With  $\alpha = -\frac{1}{2}$  from (4.9), the condition (4.11) is simply

$$p > 2 \quad (4.12)$$

whence (4.6) becomes

$$\begin{aligned} & t \frac{\partial v}{\partial t} - \frac{1}{2} \left( v + \eta \frac{\partial v}{\partial \eta} \right) + t^{-(p-1)/2} \left\{ t \frac{\partial(v^p)}{\partial t} - \frac{p}{2} v^p - \frac{\eta}{2} \frac{\partial(v^p)}{\partial \eta} \right\} \\ & = \frac{\partial^2 v}{\partial \eta^2} + t^{-(p-2)/2} \frac{\partial(v^p)}{\partial \eta} \end{aligned} \quad (4.13)$$

We now expand

$$v(\eta, t) = v_0(\eta) + o(1), \quad (4.14)$$

such that  $t \frac{\partial v}{\partial t} = o(1)$ , in the limit  $t \rightarrow \infty, \eta = 0(1)$  and substitute into (4.13). Collecting leading order terms gives

$$v_0'' + \frac{1}{2}(\eta v_0' + v_0) = 0$$

where primes denote differentiation with respect to  $\eta$ . The general solution of this equation is

$$v_0 = Ae^{-\eta^2/4} - Be^{-\eta^2/4} \int_{\eta}^{\infty} e^{s^2/4} ds = Ae^{-\eta^2/4} + o(\eta^{-1}) \text{ as } \eta \rightarrow \infty$$

where A and B are arbitrary constants. Bearing in mind that mass invariance, for  $p > 2$ , in the form (4.8) requires  $v_0(\eta)$  be integrable on  $(-\infty, \infty)$  then we must put  $B = 0$  to give

$$v_0 = Ae^{-\eta^2/4} \quad (4.15)$$

Substituting (4.14) into (4.8), with (4.15), the leading order result gives

$$M = A \int_{-\infty}^{\infty} e^{-\eta^2/4} d\eta$$

or

$$A = \frac{M}{2\sqrt{\pi}}.$$

Thus we have shown that for  $p > 2$ ,

$$u(x, t) = \frac{M}{2\sqrt{\pi}} t^{-\frac{1}{2}} e^{-(x-t)^2/4t} \{1 + o(1)\} \quad (4.16)$$

as  $t \rightarrow \infty, (x-t)/2\sqrt{t} = 0(1)$ . This result is uniform in  $x$ .

It is instructive at this stage to compare the leading order behaviour of (4.16) with the numerical computations. In Figure 1 we plot  $v(\eta, t) \equiv t^{1/2}u(x, t)$  as a function of  $\eta$  for  $p = 3$  and the initial data

$$u_0(x) = H(x+1) - H(x-1) \quad (4.17)$$

for various values of  $t$ . Here  $H(x)$  is the Heaviside function. So with  $M = 4$  from (4.7) the results show a slow but evident convergence to the asymptotic profile

$$\frac{2}{\sqrt{\pi}} e^{-\eta^2/4}$$

the slowness due presumably to the neglect of the dominant convective error term  $t^{-(p-2)/2} \frac{\partial(v^p)}{\partial\eta}$  in (4.13). If we include this in a first order error analysis then we show in Appendix 1 that for  $p = 3$  and  $t \rightarrow \infty$

$$v(\eta, t) = v_0(\eta) + \frac{4t^{-1/2} \log t}{\pi\sqrt{3}} v_0'(\eta) + 0(t^{-1/2}) \quad (4.18)$$

or, equivalently, that

$$v(\eta, t) = v_0(\eta_1) + 0(t^{-1/2}) \quad (4.19)$$

where

$$\eta_1 = \eta + \frac{4t^{-1/2} \log t}{\pi\sqrt{3}}. \quad (4.20)$$

The utility of this device is shown in Figure 2 where the numerical solution is represented as a function of  $\eta_1$ . The accelerated convergence to

$$\frac{2}{\sqrt{\pi}} e^{-\eta_1^2/4}$$

is clearly evident.

We recall here the results of Escobedo et al [1] and Escobedo & Zuazua[12] who proved that for  $p > 2$  the finite mass solutions of (4.3), after neglecting the time derivative of  $u^p$ , converge in the  $L^1$  sense to the solution of the heat equation, obtained by disregarding the derivative  $\frac{\partial(u^p)}{\partial\xi}$  in (3.18). Our analysis includes this term as  $t \rightarrow \infty$  and enables us to improve the convergence to the asymptotic profile.

**4.1.2 The case  $1 < p \leq 2$**  We now turn to the second possibility in (4.6) namely that (4) dominates (3) and balances with (1). By a similar token as before this implies that

$$\alpha(p-1) + 1 - \delta = 0 \quad (4.21)$$

and

$$1 - 2\delta < 0. \quad (4.22)$$

Hence, using (4.9), (4.21) yields

$$\alpha = -\frac{1}{p}, \quad \delta = \frac{1}{p} \quad (4.23)$$

and the inequality (4.22) becomes

$$p < 2 \quad (4.24)$$

With these values of  $\alpha$  and  $\delta$  (4.6) can be written as

$$\begin{aligned} t \frac{\partial v}{\partial t} - \frac{1}{p}(v + \eta \frac{\partial v}{\partial \eta}) + t^{-(p-1)/p} \left\{ t \frac{\partial(v^p)}{\partial t} - v^p - \frac{\eta}{p} \frac{\partial(v^p)}{\partial \eta} \right\} \\ = t^{-(2-p)/p} \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial(v^p)}{\partial \eta} \end{aligned} \quad (4.25)$$

We now expand

$$v = v_0(\eta) + o(1) \quad (4.26)$$

as  $t \rightarrow \infty, \eta = 0(1)$ , with  $t \frac{\partial v}{\partial t} = o(1)$ , which we call the outer expansion. Leading order terms in (4.25) now give

$$(v_0 + \eta v_0') + p(v_0^p)' = 0$$

where again primes denote differentiation with respect to  $\eta$ . The general solution of this equation is

$$v_0^p + \frac{\eta}{p} v_0 = C$$

where  $C$  is an arbitrary constant. In order to fix the value of  $C$  in this solution we use the following argument. For  $C < 0$   $v_0(\eta)$  is double valued with  $\eta \leq -p^2(-C)^{\frac{1}{p}}/(p-1)^{(p-1)/p}$ ; a solution which we reject. On the other hand for  $C > 0$   $v_0(\eta)$  is single valued on  $-\infty < \eta < \infty$  but as  $\eta \rightarrow +\infty$

$$v_0 \sim \frac{pC}{\eta}$$

Since mass invariance requires  $v_0(\eta)$  be integrable on  $(-\infty, \infty)$ , then  $C$  cannot be positive. Thus we must take  $C = 0$  and solution for  $v_0$  is simply

$$v_0 = \left( -\frac{\eta}{p} \right)_+^{1/p-1} \quad (4.27)$$

Now (4.27) is defined for all  $\eta \in R$ . The non trivial part of the asymptotic solution however is confined to the finite interval  $\eta_1 \leq \eta < 0$  by appealing to the mass invariance condition (4.8). Substituting (4.26) into (4.8), with (4.27), gives to leading order

$$M = \int_{\eta_1}^0 \left( -\frac{\eta}{p} \right)^{\frac{1}{p-1}} d\eta$$

and hence

$$\eta_1 = -p \left( \frac{M}{p-1} \right)^{\frac{p-1}{p}} < 0 \quad (4.28)$$

In terms of  $x$  and  $t$  therefore we have shown that for  $1 < p < 2$

$$u(x, t) = t^{-1/p} \left( \frac{t-x}{pt^{1/p}} \right)^{\frac{1}{p-1}} \{1 + o(1)\} \quad (4.29)$$

as  $t \rightarrow \infty$ ,  $\frac{x-t}{t^{1/p}} = o(1)$ . The condition  $\eta_1 \leq \eta \leq 0$  requiring in (4.29) that

$$\eta_1 t^{1/p} \leq x - t < 0.$$

It is convenient now to consider the borderline case  $p = 2$ . In this event the terms (1) (3) and (4) are asymptotically equivalent as  $t \rightarrow \infty$ ,  $\eta = o(1)$  whence

$$\alpha = -\delta = -\frac{1}{2}.$$

Expanding

$$v = v_0(\eta) + o(1)$$

with  $t \frac{\partial v}{\partial t} = o(1)$  as before, yields a second order ordinary differential equation for  $v_0$  which has the solution

$$v_0(\eta) = \frac{e^{-\eta^2/4}}{A + \sqrt{\pi} \operatorname{erf}(\eta/2)} \quad (4.30)$$

where  $A$  is an arbitrary constant given by the asymptotic mass invariance condition

$$M = \int_{-\infty}^{\infty} v_0(\eta) d\eta$$

This condition gives

$$A = \sqrt{\pi}(e^M + 1)/(e^M - 1)$$

Again we can compare this asymptotic result with the full numerical solution. With  $u_0$  given by (4.17),  $v(\eta, t) = t^{1/p}u(x, t)$  converges rather slowly to (4.26) due to the neglected term of  $o(t^{-1/2})$  in (4.13). To include this term we can go through a procedure similar to the one for  $p > 2$ . In Appendix 1 we expand

$$v(\eta, t) = v_0(\eta) - 0.0639t^{-1/2} \log tv'_0(\eta) + o(t^{1/2}) \quad (4.31)$$

suggesting that we may write

$$v(\eta, t) = v_0(\eta_1) + o(t^{1/2}) \quad (4.32)$$

where  $\eta_1 = \eta - 0.0639t^{-1/2} \log t$ . The numerical solution is represented as a function of  $\eta_1$  in Figure 3 where a somewhat faster convergence to the asymptotic profile is evident.

Finally we note that (4.26) agrees with the convergence result of Escabedo et al [1] for these cases. This can be understood if one disregards the time derivative of  $u^p$  in equation (4.3). Also note that the function  $v_0$  in (4.26) is a self similar solution of Burgers' equation ((3.19) with  $p = 2$ ).

#### 4.1.3 The case $p < 1$

Leaving aside the linear case  $p = 1$ , we now come on to discuss the parameter range  $0 < p < 1$ . Here we adopt a different strategy by considering a similarity variable with no translation. So this time we put

$$\eta = \frac{x}{t^\nu} \quad , \quad \nu > 0 \quad (4.33)$$

with

$$u(x, t) = t^\beta v(\eta, t), \quad \beta < 0 \quad (4.34)$$

Making the change of variable in (1.1) gives

$$\begin{aligned} t^{\beta(1-p)} \left\{ t \frac{\partial v}{\partial t} + \beta v - \nu \eta \frac{\partial v}{\partial \eta} \right\} + \left\{ t \frac{\partial(v^p)}{\partial t} + \beta p v^p - \nu \eta \frac{\partial(v^p)}{\partial \eta} \right\} \\ = t^{\beta(1-p)+1-2\nu} \frac{\partial^2 v}{\partial \eta^2} - t^{\beta(1-p)+1-\nu} \frac{\partial v}{\partial \eta} \end{aligned} \quad (4.35)$$

Clearly since  $\beta < 0$  and  $p < 1$ , (2) dominates the left hand side of (4.35) as  $t \rightarrow \infty$ . The question is with which term on the right hand does (2) balance. It turns out that the only consistent possibility is that (4) dominates (3) and takes up the asymptotic balance with (2). This requires

$$\beta(1-p) + 1 - \nu = 0 \quad (4.36)$$

and

$$\beta(1-p) + 1 - 2\nu < 0 \quad (4.37)$$

The second equation relating  $\beta$  and  $\nu$  in addition to (4.36) is obtained from the mass invariance result (4.7). Writing this in terms of  $\eta$  and  $v$  from (4.33) and (4.34) we have

$$M = \int_{-\infty}^{\infty} \left\{ t^{\beta+\nu} v + t^{\beta p+\nu} v^p \right\} d\eta \quad (4.38)$$

and, for  $p < 1$  with  $\beta < 0$ ,

$$M \sim t^{\beta p+\nu} \text{ as } t \rightarrow \infty$$

Hence the invariance of  $M$  requires that

$$\nu = -\beta p \quad (4.39)$$

which, along with (4.36) gives,

$$\beta = -1 \text{ and } \nu = p$$

With these values of  $\beta$  and  $\nu$ , (4.35) becomes

$$t \frac{\partial(v^p)}{\partial t} - p \left\{ v^p + \eta \frac{\partial(v^p)}{\partial \eta} \right\} + t^{-(1-p)} \left\{ t \frac{\partial v}{\partial t} - v - p\eta \frac{\partial v}{\partial \eta} \right\} = t^{-p} \frac{\partial^2 v}{\partial \eta^2} - \frac{\partial v}{\partial \eta} \quad (4.40)$$

We now expand

$$v(\eta, t) = v_0(\eta) + o(1) \quad (4.41)$$

such that  $t \frac{\partial v}{\partial t} = o(1)$ , as  $t \rightarrow \infty, \eta = 0(1)$ . Substitution into (4.40) gives to leading order

$$p \{ v_0^p + \eta (v_0^p)' \} = v_0'$$

where again primes denote differentiation with respect to  $\eta$ . The general solution is given by

$$p\eta v_0^p - v_0 = C$$

with  $C$  constant. A similar argument to that of Section (4.2.1) leads to  $C = 0$  and the solution for  $v_0$  is then

$$v_0 = (p\eta)_+^{1/1-p} \quad (4.42)$$

The nontrivial part of the asymptotic solution is confined to the interval  $0 < \eta < \eta_2$  by the mass invariance condition (4.38), which using (4.41) with (4.42), gives to leading order

$$M = \int_0^{\eta_2} (p\eta)^{p/1-p} d\eta$$

Thus

$$\eta_2 = \left\{ \frac{M}{1-p} \right\}^{1-p} p^{-p} \quad (4.43)$$

In terms of  $x$  and  $t$  and for  $0 < p < 1$  we have now shown that

$$u(x, t) = t^{-1} \left( \frac{px}{t^p} \right)^{1/1-p} \{ 1 + o(1) \} \quad (4.44)$$

as  $t \rightarrow \infty, \frac{x}{t^p} = 0(1)$ . The condition  $0 < \eta < \eta_2$  requires

$$0 < x < \eta_2 t^p.$$

We call (4.38) the outer solution.

## 4.2 The uniformity of the outer solutions:the boundary layer solutions

We now consider the question of uniformity in  $x$  of the outer solutions for  $u(x, t)$  in the limit  $t \rightarrow \infty$ . In the case  $p \geq 2$  it is clear that the asymptotic representations (4.16) and (4.30) are uniformly valid for all  $x$ . For other values of  $p$  the situation is not so straightforward. For instance when  $1 < p < 2$  we must first ask how (4.25) represent the zero order asymptotic solution on  $-\infty < x < \infty$ . Since this has to satisfy  $u = 0$  at  $x = \pm\infty$  then we would expect the structure

$$u(x, t) = \begin{cases} 0, & -\infty < x < t + \eta_1 t^{1/p} \\ t^{-1/p} \left( \frac{t-x}{\eta_1 t^{1/p}} \right)^{1/p-1}, & t + \eta_1 t^{1/p} \leq x < t \\ 0, & x > t \end{cases} \quad (4.45)$$

There are however two objections to this representation. Firstly  $u(x, t)$  is not continuous while in the second place, contrary to what we know, the support is bounded. Both these objections to (4.45) can be met by including boundary layers at the trailing and leading edges of the pulse where  $\eta = \eta_1$  and  $\eta = 0$  respectively. As  $t \rightarrow \infty$  these will be thin on the scale of  $\eta$  but since we include the diffusion term within them, they have the effect of smoothing out the solution and at the same time rendering the support unbounded. We note that this approach has been successfully used to uniformise asymptotic solutions to diffusion-convection equations by Grundy [13,14,15].

A similar situation presents itself in the case  $0 < p < 1$  where, in principle, the uniformisation can be carried out in the same way. In this case the zero order outer representation is

$$u(x, t) = \begin{cases} 0 & x < 0 \\ t^{-1} \left( \frac{px}{t^p} \right)^{1/1-p} & 0 < x < \eta_2 t^p \\ 0 & x > \eta_2 t^p \end{cases} \quad (4.46)$$

Once again this situation is at variance with what we expect since the solution is discontinuous at  $x = \eta_2 t^p$  and there is no moving interface to the left. As in the case  $p > 1$ , these difficulties can be removed by including boundary layers.

Having set out our reasons for seeking boundary layer solutions we devote the remainder of this Section to constructing them.

### 4.2.1 The case $1 < p < 2$

#### (a) The trailing edge layer at $\eta = \eta_1$

Near  $\eta = \eta_1$  we make the change of variable

$$\eta = \eta_1 + t^{-\mu} \zeta, \quad \mu > 0 \quad (4.47)$$

where  $\zeta = 0(1), t \rightarrow \infty$ . This defines a thin layer, on the  $\eta$  scale, of thickness  $t^{-\mu}$  where  $\mu$  has to be found. In this layer we look for a solution which varies on the  $\zeta$  scale so we put

$$v(\eta, t) = w(\zeta, t) \quad (4.48)$$

in (4.25) which now becomes

$$t^{-\mu} \left\{ t \frac{\partial w}{\partial t} - \frac{w}{p} + \left( \mu - \frac{1}{p} \right) \zeta \frac{\partial w}{\partial \zeta} \right\} + t^{-1/p} \left\{ t \frac{\partial(w^p)}{\partial t} + w^p + \mu \zeta \frac{\partial(w^p)}{\partial \zeta} \right\} \\ - \frac{\eta_1}{p} \frac{\partial w}{\partial \zeta} = t^{\mu - \frac{(2-p)}{p}} \frac{\partial^2 w}{\partial \zeta^2} + \frac{\partial(w^p)}{\partial \zeta} \quad (4.49)$$

An essential feature of the boundary layer is that the diffusion term becomes important there. Assuming all  $\zeta$  derivatives are bounded within the boundary layer we therefore put

$$\mu = \frac{2-p}{p} > 0 \quad (4.50)$$

and expand

$$w(\xi, t) = w_0(\xi) + o(1) \quad (4.51)$$

with  $t \frac{\partial w}{\partial t} = o(1)$ , as  $t \rightarrow \infty, \zeta = 0(1)$  in (4.49). Collecting leading order terms gives

$$w_0'' + (w_0^p)' + \frac{\eta_1}{p} w_0' = 0 \quad (4.52)$$

with primes denoting differentiation with respect to  $\zeta$ . The matching condition requires

$$w_0 = \left( \frac{-\eta_1}{p} \right)^{1/p-1} \text{ as } \zeta \rightarrow \infty \quad (4.53)$$

while the boundary conditions are

$$w_0 = w_0' = 0 \text{ as } \zeta \rightarrow -\infty \quad (4.54)$$

Equation (4.46) admits the solution

$$w_0 = \left\{ \frac{\eta_1 e^{-(p-1)\eta_1(\zeta-\zeta_0)/p}}{p[1 - e^{-(p-1)\eta_1(\zeta-\zeta_0)/p}]} \right\}^{1/p-1} \quad (4.55)$$

which satisfies the conditions (4.53) and (4.54) but is only unique to within the arbitrary translational shift  $\zeta_0$ . Unfortunately there appears to be no way of finding  $\zeta_0$  to this order of approximation.

(b) **The leading edge layer near  $\eta = 0$**

Near  $\eta = 0$  we make the change of variable

$$\eta = \chi t^{-\varepsilon}, \quad (4.56)$$

where  $\varepsilon > 0$  will be chosen so that the diffusion term in (4.25) becomes important in the region where the variable  $\chi$  is  $0(1)$ . To see how to scale the independent variable  $v$  within the trailing edge layer we have the outer expansion

$$\begin{aligned} v(\eta, t) &= v_0(\eta) + o(1) \\ &= \left( \frac{-\eta}{p} \right)^{1/p-1} + o(1) \\ &\sim t^{-\varepsilon/p-1} \left( \frac{-\chi}{p} \right)^{1/p-1} \end{aligned}$$

in terms of  $\chi$ . This suggests we put

$$v(\eta, t) = t^{-\varepsilon/p-1} W(\chi, t) \quad (4.57)$$

and expand

$$W(\chi, t) = W_0(\chi) + o(1) \quad (4.58)$$

with  $t \frac{\partial W}{\partial t} = o(1)$  as  $t \rightarrow \infty, \chi = 0(1)$ , using

$$W_0(\chi) \sim (-\chi/p)^{1/p-1} \text{ as } \chi \rightarrow -\infty \quad (4.59)$$

as a matching condition. Making the above changes of variables in (4.25) gives

$$\begin{aligned} &t \frac{\partial W}{\partial t} - \left( \frac{\varepsilon}{p-1} + \frac{1}{p} \right) W + \left( \varepsilon - \frac{1}{p} \right) \chi \frac{\partial W}{\partial \chi} \\ &+ t^{-1/2} \left\{ t \frac{\partial(W^p)}{\partial t} + \left( 1 - \frac{\varepsilon p}{p-1} \right) W^p + \left( \varepsilon - \frac{1}{p} \right) \chi \frac{\partial(W^p)}{\partial \chi} \right\} \\ &= t^{2\varepsilon-(2-p)/p} \frac{\partial^2 W}{\partial \chi^2} + \frac{\partial(W^p)}{\partial \chi}. \end{aligned}$$

We now invoke the condition that the second derivative becomes important in the limit  $t \rightarrow \infty, \chi = 0(1)$ . This demands that

$$\varepsilon = \frac{(2-p)}{2p}$$

and the equation for  $W(\chi, t)$  then becomes

$$\begin{aligned} t \frac{\partial W}{\partial t} - \frac{W}{2(p-1)} - \frac{\chi}{2} \frac{\partial W}{\partial \chi} + t^{-1/2} \left\{ t \frac{\partial(W^p)}{\partial t} + 2W^p - \frac{\chi}{2} \frac{\partial(W^p)}{\partial \chi} \right\} \\ = \frac{\partial^2 W}{\partial \chi^2} + \frac{\partial(W^p)}{\partial \chi} \end{aligned} \quad (4.60)$$

Substituting (4.58) into (4.60) and equating leading order terms gives

$$W_0'' + (W_0^p)' + \frac{1}{2}\chi W_0' + \frac{W_0}{2(p-1)} = 0 \quad (4.61)$$

with primes indicating  $\chi$ -derivatives, which has to be solved subject to (4.59) and the boundary conditions

$$W_0 = 0 \quad , \chi \rightarrow \infty. \quad (4.62)$$

This boundary value problem could not be solved by the authors and its resolution is left as an open question. However some partial results are known. If  $W_0$  is a solution then one can show that  $W_0'(\chi) < 0$  for all  $-\infty < \chi < \infty$ . Hence the function

$$\omega = W_0^{p-1}$$

is strictly monotone. This allows us to consider the inverse

$$\chi = \chi(\omega) \quad \text{for} \quad 0 < \omega < \infty$$

and the positive function

$$y(\omega) = -(W_0^{p-1})'(\chi(\omega)), \omega > 0.$$

This results in the following problem for  $y$  on  $\omega > 0$

$$\begin{cases} y' - p\omega + \frac{2-p}{p-1} \frac{y}{\omega} + \frac{\omega}{2y} \Big\}' = -\frac{1}{2y} \\ y(0) = 0, y(\infty) = \frac{1}{p} \end{cases}, \quad (4.63b) \quad (4)$$

Whenever a solution  $y$  exists it satisfies  $y'(\omega) > 0$  and consequently  $0 < y(\omega) < 1/p$  for all  $\omega > 0$ . Further we can establish that a solution  $y$  approaches the origin according to either

$$y(\omega) \sim \sqrt{p-1} \omega \left\{ \log \frac{1}{\omega} \right\}^{1/2} \quad (4.64a)$$

or

$$y(\omega) \sim A\omega^2 \quad (4.64b)$$

where  $A > 0$  is an unknown constant. A decay as in (4.64a) implies an exponential behaviour as  $\chi \rightarrow \infty$  for  $W_0(\chi)$ , while (4.64b) would imply algebraic behaviour. Based on what we know of the original partial differential equation, we expect and conjecture that the solution  $y$  of (4.63) behaves as in (4.64a). Finally we note that every numerical approach we have tried has been thwarted by some pathological property of the equations or lack of analytic knowledge of the problem.

#### 4.2.2 The case $0 < p < 1$

##### (a) The leading edge layer near $\eta = \eta_2$

We now go through a similar procedure for the case  $0 < 1 < p$  where we expect interfaces to appear within the support of the solution. We put

$$\eta = \eta_2 + t^{-\rho}\zeta \quad , \rho > 0 \quad (4.65)$$

where  $\zeta = 0(1)$  as  $t \rightarrow \infty$ , defining a thin layer on the scale of  $\eta$  of thickness  $t^{-\rho}$  where  $\rho$  is to be found. In this layer we look for solutions of (4.35) where

$$v(x, t) = z(\zeta, t)$$

which, in the new variables, becomes

$$\begin{aligned} & t^{-(\mu+1-p)} \left\{ t \frac{\partial z}{\partial t} - z + (\rho - p)\zeta \frac{\partial z}{\partial \zeta} \right\} - p\eta_2 t^{-(1-p)} \frac{\partial z}{\partial \zeta} \\ & + t^{-\rho} \left\{ t \frac{\partial(z^p)}{\partial t} - pz^p - (\rho + p)\xi \frac{\partial(z^p)}{\partial \zeta} \right\} - p\eta_2 \frac{\partial(z^p)}{\partial \zeta} \\ & = t^{\rho-p} \frac{\partial^2 z}{\partial \zeta^2} - \frac{\partial z}{\partial \zeta} \end{aligned}$$

Choosing  $\rho = p$  has the effect of making the diffusion term  $0(1)$  so the equation for  $z(\xi, t)$  becomes

$$\begin{aligned} & t^{-1} \left\{ t \frac{\partial z}{\partial t} - z \right\} - p\eta_2 t^{-(1-p)} \frac{\partial z}{\partial \zeta} \\ & + t^{-p} \left\{ t \frac{\partial(z^p)}{\partial t} - pz^p - 2p\zeta \frac{\partial(z^p)}{\partial \zeta} \right\} \\ & = \frac{\partial^2 z}{\partial \zeta^2} - \frac{\partial z}{\partial \zeta} + p\eta_2 \frac{\partial(z^p)}{\partial \zeta} \end{aligned} \quad (4.66)$$

Expanding

$$z(\zeta, t) = z_0(\zeta) + o(1)$$

such that  $t \frac{\partial z}{\partial t} = o(1)$  as  $t \rightarrow \infty$ ,  $\zeta = 0(1)$ , yields the equation

$$z_0' - z_0' + p\eta_2(z_0^p)' = 0 \quad (4.67)$$

for  $z_0$  with prime denoting differentiation with respect to  $\zeta$ . This has to be solved subject to the matching condition

$$z_0 \rightarrow (p\eta_2)^{1/1-p}, \quad \zeta \rightarrow -\infty \quad (4.68)$$

and the boundary condition

$$z_0 = z_0' = 0, \quad \zeta \rightarrow \infty. \quad (4.69)$$

The problem (4.67)-(4.69) admits the solution

$$z_0 = \begin{cases} [p\eta_2 - e^{(1-p)(\zeta-\zeta_0)}]^{1/1-p}, & \zeta < \zeta_0 + \frac{\log(p\eta_2)}{1-p} \\ 0, & \zeta > \zeta_0 + \frac{\log(p\eta_2)}{1-p} \end{cases} \quad (4.70)$$

which is unique to within the arbitrary shift  $\zeta_0$  which is not determined to this order of approximation. Clearly (4.64) has an interface at

$$\xi = \zeta_0 + \frac{\log(p\eta_2)}{(1-p)} \quad (4.71)$$

(b) **The trailing edge layer near  $\eta = 0$**

In order to work out the structure of the asymptotic solution near  $\eta = 0$  we have to look in a little more detail at the outer expansion (4.41). To be specific we write

$$v(\eta, t) = (p\eta)^{1/1-p} + t^{-p} \log t v_1(\eta) + t^{-p} v_2(\eta) + t^{-(1-p)} v_3(\eta) + \dots \quad (4.72)$$

Substituting this into (4.40) we find that

$$v_1 = K(p\eta)^{p/1-p} \quad (4.73)$$

$$v_2 = \frac{(p\eta)^{p/1-p}}{(1-p)} \log \eta \left\{ \frac{p^2}{(1-p)^2} - K \right\} + C_2(\eta p)^{p/1-p} \quad (4.74)$$

and

$$v_3 = \frac{(\eta p)^{\frac{2-p}{1-p}}}{p(1-p)} + C_3 \eta p \quad (4.75)$$

where  $K, C_2$  and  $C_3$  are arbitrary constants. Comparing the first two terms in (4.72), with (4.73), reveals that the outer expansion is nonuniformly valid when  $x = 0(\log t)$ , suggesting the inner independent variable

$$\xi = \frac{x}{\log t} = \frac{\eta t^p}{\log t} \quad (4.76)$$

together with the dependent variable

$$u = \left( \frac{\log t}{t} \right)^{1/1-p} Z(\xi, t) \quad (4.77)$$

where we have reverted to the original independent variable  $u$ . Making these changes of variables in (4.1) and expanding

$$Z(\xi, t) = Z_0(\zeta) + o(1) \quad (4.78)$$

yields the equation

$$Z_0' = \frac{p}{(1-p)} Z_0$$

with solution

$$Z_0 = \{p(\zeta - \zeta_0)\}^{1/1-p} \quad (4.79)$$

giving an interface at  $\zeta = \zeta_0$  which is, at the moment, unknown.

To match the inner and outer expansions we first recast (4.72) with (4.73)-(4.75) in terms of  $\zeta$  and  $u$  to give

$$u = \left( \frac{\log t}{t} \right)^{1/1-p} \left\{ (p\zeta)^{1/1-p} + \frac{(p\zeta)^{p/1-p}}{(1-p)} \left[ K - \frac{p^2}{(1-p)^2} \right] + o \left( \frac{\log(\log t)}{\log t} \right) \right\}. \quad (4.80)$$

This must match with (4.71) and (4.73) expanded as  $\zeta \rightarrow \infty$ , namely

$$u = \left( \frac{\log t}{t} \right)^{1/1-p} \left\{ (p\zeta)^{1/1-p} - \frac{p\zeta_0}{(1-p)} (p\zeta)^{p/1-p} + - \right\}.$$

Thus for a successful match we require

$$p\zeta_0 = \frac{p^3}{(1-p)^2} - K. \quad (4.81)$$

To proceed further we note that the diffusion term has yet to be taken into account in the inner region. In order to do this we make a further scaling near  $\zeta = \zeta_0$ , namely

$$\zeta = \zeta_0 + \frac{\xi}{\log t} \quad (4.82)$$

together with

$$u = t^{-1/1-p} U(\xi, t). \quad (4.83)$$

We observe in passing that in terms of  $x$  the scaling (4.82) can be written as

$$x = \zeta_0 \log t + \xi. \quad (4.84)$$

Making these changes of variable the equation for  $U(\xi, t)$  becomes

$$t^{-1} \left\{ t \frac{\partial U}{\partial t} - \frac{U}{(1-p)} - \zeta_0 \frac{\partial U}{\partial \zeta} \right\} + t \frac{\partial(U^p)}{\partial t} - \frac{p}{(1-p)} U^p + \zeta_0 \frac{\partial(U^p)}{\partial \xi} = \frac{\partial^2 U}{\partial \xi^2} - \frac{\partial U}{\partial \xi}. \quad (4.85)$$

Expanding

$$U(\xi, t) = U_0(\xi) + o(1)$$

such that  $t \frac{\partial U}{\partial t} = o(1)$  as  $t \rightarrow \infty$ ,  $\xi = O(1)$  gives the ordinary differential equation

$$U_0'' - U_0' + \frac{p}{(1-p)} U_0^p + \zeta_0 (U_0^p)' = 0 \quad (4.86)$$

for  $U_0$  with primes denoting differentiation with respect to  $\xi$ . The matching condition with the inner solution requires that

$$U_0 \sim (p\xi)^{1/1-p}, \xi \rightarrow \infty \quad (4.87)$$

while the boundary condition demands that

$$U_0 = 0, \quad \xi \rightarrow -\infty \quad (4.88)$$

The problem is to find  $\zeta_0$  and  $U_0$  subject to (4.86)-(4.88).

We show in Appendix 2 that a unique solution to this problem exists if and only if

$$\zeta_0 = -p/1-p \quad (4.89)$$

and in that event

$$U_0 = \begin{cases} [p(\xi - \xi_1)]^{1/1-p}, & \xi \geq \xi_1 \\ 0, & \xi < \xi_1 \end{cases} \quad (4.90)$$

for some  $\xi_1$  which can actually be found using the second integral invariant  $L$  defined by (3.13). So, returning to (4.81) we find that

$$K = \frac{p^2}{(1-p)^2}$$

We note that (4.90) is actually uniformly valid throughout the inner region and can be matched to zero order with the outer expansion. We use this observation in the next Section where we construct uniformly valid solutions for all  $x$ .

Finally from (4.90) we observe that the trailing edge interface is located at  $\xi = \xi_1$  or in view of (4.89) and (4.84) at

$$x = \frac{-p}{(1-p)} \log t + \xi_1.$$

### 4.3 Construction of uniformly valid solution for $0 < p < 2$

In this section we make some remarks about the construction of uniformly valid large time solutions for the parameter range  $0 < p < 2$ . In contrast to the situation for  $p \geq 2$ , for  $0 < p < 2$  the asymptotic analysis does not yield a single formula which approximates  $u(x, t)$  uniformly in  $x$  as  $t \rightarrow \infty$ . To see how such a formula may be constructed we note that the asymptotic solution for both  $0 < p < 1$  and  $1 < p < 2$  can be regarded as an outer expansion linked by two boundary layers to the conditions at infinity. Although this picture is not entirely straightforward for  $0 < p < 1$  the interpretation can still be made since, as noted above, equation (4.90) is uniformly valid in the trailing edge layer.

If we denote the zero order outer solution by  $U_0$ , the zero order leading edge boundary layer solution by  $U_1$  and the zero order trailing edge boundary layer solution by  $U_2$  then for  $0 < p < 1$  we have from (4.46)

$$U_0 = \left\{ \begin{array}{ll} 0 & , \quad x < 0 \\ \left(\frac{px}{t}\right)^{1/1-p} & , \quad 0 < x \leq \eta_2 t^p \\ 0 & , \quad x > \eta_2 t^p \end{array} \right\}$$

and from (4.90)

$$U_1 = \left\{ \begin{array}{ll} t^{-1} \left[ p\eta_2 - e^{(1-p)(x-\eta_2 t^p - \zeta_0)} \right]^{1/1-p} & , \quad x < \eta_2 t^p + \zeta_0 + \frac{\log(p\eta_2)}{(1-p)} \\ 0 & , \quad x > \eta_2 t^p + \zeta_0 + \frac{\log(p\eta_2)}{1-p} \end{array} \right\}$$

while from (4.90) with (4.84)

$$U_2 = \left\{ \begin{array}{ll} 0 & , \quad x < \xi_1 - \frac{p}{1-p} \log t \\ t^{-1/1-p} \left\{ p \left( x + \frac{p}{1-p} \log t - \xi_1 \right) \right\}^{1/1-p} & , \quad x \geq \xi_1 - \beta \log t \end{array} \right\}$$

For  $1 < p < 2$ , the corresponding expressions and domains are from (4.45)

$$U_0 = \left\{ \begin{array}{ll} 0 & , \quad x < t + \eta_1 t^{1/p} \\ t^{-1/p-1} \left( \frac{t-x}{p} \right)^{1/p-1} & , \quad t + \eta_1 t^{1/p} \leq x < t \\ 0 & , \quad x > t \end{array} \right\}$$

from (4.55)

$$U_1 = t^{-1/p} \left\{ \frac{\eta_1}{p \left\{ e^{(p-1)\eta_1(x-t-\eta_1 t^{1/p} - \zeta_0 t^{(p-1)/p})/p} - 1 \right\}} \right\}^{1/p-1}, \quad -\infty < x < \infty$$

and from (4.58)

$$U_2 = t^{-1/2(p-1)} W_0 \left( \frac{x-t}{t^{1/2}} \right), \quad -\infty < x < \infty.$$

The next step in the process is to construct the inner limits of the outer solution, expressions which have to reflect the failure of analyticity of  $U_0$  at the leading and trailing edges. For  $0 < p < 1$  we write the leading edge inner limit of  $U_0$  as

$$(U_0)_1 = \begin{cases} t^{-1}(p\eta_2)^{1/1-p}, & x \leq \eta_2 t^p \\ 0 & , \quad x > \eta_2 t^p \end{cases}$$

and the trailing edge inner limit of  $U_0$  as

$$(U_0)_2 = \begin{cases} t^{-1/1-p}(px)^{1/1-p}, & x > 0 \\ 0 & , \quad x < 0 \end{cases}$$

For  $1 < p < 2$ , corresponding expressions are

$$(U_0)_1 = \begin{cases} 0 & , \quad x < t + \eta_1 t^{1/p} \\ t^{-1/p}(-\eta_1/p) & , \quad x \geq t + \eta_1 t^{1/p} \end{cases}$$

and

$$(U_0)_2 = \begin{cases} t^{-1/p-1} \left( \frac{t-x}{p} \right)^{1/p-1}, & x < t. \\ 0 & , \quad x \geq t \end{cases}$$

The idea now is to construct what, in the language of matched expansions, are called composite solutions and which are uniformly valid for all  $x$ . In this case we use multiplicative composition; for a number of reasons the more familiar additive composition is not applicable here. In a more straightforward situation a leading order composite approximation would be of the form

$$u_c(x, t) = \frac{U_0 U_1 U_2}{(U_0)_1 (U_0)_2} \quad (4.91)$$

(See for example Van Dyke [16]). However this rule cannot be applied directly since  $U_0, (U_0)_1$  and  $(U_0)_2$  are zero over certain domains of  $x$ . Nevertheless the multiplicative nature of  $u_c$  makes it a straightforward matter to modify (4.91) to take this into account. In the case  $0 < p < 1$  we write

$$u_c(x, t) = \begin{cases} 0 & , \quad x < \xi_1 - \log t \\ \frac{U_1 U_2}{(U_0)_1} & , \quad \xi_1 - \log t < x < 0 \\ \frac{U_1 U_2}{(U_0)_1} & , \quad 0 < x < \eta_2 t^p \\ \frac{U_1 U_2}{(U_0)_2} & , \quad \eta_1 t^p < x < \eta_2 t^p + \xi_0 + \frac{\log(p\eta_2)}{1-p} \\ 0 & , \quad x > \eta_2 t^p + \xi_0 + \frac{\log(p\eta_2)}{1-p} \end{cases}$$

This is adapted from (4.91) by omitting  $U_0 = (U_0)_2 = 0$  from both numerator and denominator in the second domain. In the third domain  $U_0 = (U_0)_2$  which cancel while in the fourth domain  $U_0 = (U_0)_1 = 0$  which can again be omitted. The corresponding result for  $1 < p < 2$  is

$$u_c(x, t) = \begin{cases} \frac{U_1 U_2}{(U_0)_2} & x < t + \eta_2 t^{1/p} \\ \frac{U_1 U_2}{(U_0)_1} & t + \eta_1 t^{1/p} < x < t \\ \frac{U_1 U_2}{(U_0)_1} & x > t \end{cases}$$

Although these uniformly valid composite expansions are continuous, they accept discontinuities in their  $x$ -derivatives; namely of  $0(t^{-(p+1)})$  at  $x = \eta_2 t^p$  for  $0 < p < 1$  and of  $0(t^{-2/p})$  at  $x = t + \eta_1 t^{1/p}$  for  $1 < p < 2$ . However this is not a high price to pay since in each case the formula for  $u_c$  is simple and in the case  $0 < p < 1$  very easy to compute. For  $1 < p < 2$  the procedure is complicated by the need to evaluate  $W_0\left(\frac{x-t}{t^{1/2}}\right)$  at prescribed values of its argument.

The main problem with using the composite solutions is that they are indeterminate to within the two arbitrary constants occurring in each of the boundary layer solutions. In principal these cannot be determined from the zero order analysis, although for  $0 < p < 1$   $\xi_1$  can be found from the second integral invariant. However it is possible to simulate the convergence to the asymptotic solution by adjusting these constants so that the composite solution fits, in some adhoc way, the numerical solution at a suitable value of time. From a practical computational point of view the use of such a formula would reduce the excessive computing time required to calculate the slowly converging large time solution using the scheme of Section 5.

## 5 The numerical method

The method used to generate the numerical solutions was developed and analysed by Dawson [17] for nonlinear parabolic equations in one space dimension. Here, we discuss the application of the method to Problem (IVP).

Again we make transformation (3.3) and use the transformed equation as a starting point for the discussion. However here we want to keep the diffusion coefficient  $D$  in the analysis, because this will allow us to do computations for problems in which convection dominates diffusion. Therefore we consider, with  $w = \beta(u)$  and  $u = \varphi(w)$ ,

$$\frac{\partial w}{\partial t} + \frac{\partial \varphi(w)}{\partial x} - D \frac{\partial^2 \varphi(w)}{\partial x^2} = 0 \quad (x, t) \in Q \quad (5.1)$$

Let  $-\infty < \dots < x_{-J-1/2} < x_{-J+1/2} < \infty$  be a partition of  $R$  into grid blocks  $B_j = [x_{j-1/2}, x_{j+1/2}]$ , and let  $x_j$  be the midpoint of  $B_j$ . For simplicity, assume the partition is uniform with mesh spacing  $h > 0$ . Let  $\Delta t > 0$  denote a time-stepping parameter, and let  $t^n = n \Delta t$  and  $t^{n+1/2} = (t^n + t^{n+1})/2$ . For functions  $g(x, t)$ , let  $g_j^n = g(x_j, t^n)$ ,  $g_{j+1/2}^{n+1/2} = g(x_{j+1/2}, t^{n+1/2})$ , etc.

On each grid block  $B_j$ , approximate  $w_j^n$  and  $u_j^n$  by constants  $\omega_j^n$  and  $U_j^n$ , respectively, where

$$\omega_j^n \equiv U_j^n + (U_j^n)^p. \quad (5.2)$$

Discretizing (5.1) by finite differences in space and time, we find

$$\begin{aligned} \frac{w_j^{n+1} - w_j^n}{\Delta t} + \frac{\varphi(w_{j+1/2}^{n+1/2}) - \varphi(w_{j-1/2}^{n+1/2})}{h} \\ - D \frac{\varphi(w_{j+1}^{n+1}) - 2\varphi(w_j^{n+1}) + \varphi(w_{j-1}^{n+1})}{h^2} \approx 0. \end{aligned} \quad (5.3)$$

The term  $\varphi(w_{j+1/2}^{n+1/2})$  is approximated using a higher-order Godunov approach, see van Leer [18]. Assume the time step satisfies the CFL constraint

$$\sup_w \varphi'(w) \Delta t \leq h. \quad (5.4)$$

Expanding in a Taylor series about the point  $(x_j, t^n)$ ,

$$w_{j+1/2}^{n+1/2} = w + \frac{h}{2} w_x + \frac{\Delta t}{2} w_t + O(h^2 + \Delta t h + \Delta t^2), \quad (5.5)$$

where the right side of (5.5) is evaluated at  $(x_j, t^n)$ . Using the differential equation (5.1), we find

$$w_{j+1/2}^{n+1/2} = w + \left(\frac{h}{2} - \varphi'(w) \frac{\Delta t}{2}\right) w_x + O(h^2 + \Delta t). \quad (5.6)$$

Emulating (5.6), define

$$\bar{\omega}_{j+1/2}^{n+1/2} = \omega_j^n + \left(\frac{h}{2} - \varphi'(\omega_j^n)\right) \frac{\Delta t}{2} \delta_x \omega_j^n, \quad (5.7)$$

where  $\delta_x \omega_j^n$  is calculated by a slope-limiting procedure. In particular

$$\delta_x \omega_j^n = \delta_{\lim} \omega_j^n \cdot \text{sign}(\omega_{j+1}^n - \omega_{j-1}^n), \quad (5.8)$$

where

$$\delta_{\lim} \omega_j^n = \begin{cases} \min(|\Delta_+ \omega_j^n|, |\Delta_- \omega_j^n|) & , \text{ if } \Delta_+ \omega_j^n \cdot \Delta_- \omega_j^n > 0, \\ 0 & , \text{ otherwise.} \end{cases} \quad (5.9)$$

Here  $\Delta_+ \omega_j$  is the forward difference  $(\omega_{j+1} - \omega_j)/h$ , and  $\Delta_- \omega_j = (\omega_j - \omega_{j-1})/h$ . Approximating  $\omega_{j+1/2}^{n-1/2}$  by  $\bar{\omega}_{j+1/2}^{n+1/2}$ , we obtain the nonlinear system of equations

$$\begin{aligned} & \frac{\omega_j^{n+1} - \omega_j^n}{\Delta t} + \frac{\varphi(\bar{\omega}_{j+1/2}^{n+1/2}) - \varphi(\bar{\omega}_{j-1/2}^{n+1/2})}{h} \\ & - D \frac{\varphi(\omega_{j+1}^{n+1}) - 2\varphi(\omega_j^{n+1}) + \varphi(\omega_{j-1}^{n+1})}{h^2} = 0. \end{aligned} \quad (5.10)$$

Initially, set  $U_j^0 = u^0(x_j)$  and  $\omega^0 = U_j^0 + (U_j^0)^p$ . Note that, given  $\omega_j^n$  at some time level  $t^n$ , the term  $\varphi(\bar{\omega}_{j+1/2}^{n+1/2})$  is calculated explicitly. Thus, we are left with a symmetric system of nonlinear equations to determine  $\omega_j^{n+1}$ , which we solve by a fixed point iteration.

Let

$$r_j^n = -\frac{\varphi(\bar{\omega}_{j-1/2}^{n+1/2}) - \varphi(\bar{\omega}_{j-1/2}^{n+1/2})}{h}.$$

Then, substituting the definition of  $\omega_j^{n+1}$  into (5.10) we find

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{(U_j^{n+1})^p - (U_j^n)^p}{\Delta t} - D \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} = r_j^n. \quad (5.11)$$

Let  $U_j^{n+1,0}$  be an initial guess for  $U_j^{n+1}$ . Then, given  $U_j^{n+1,k}$ , define

$$\beta_j^{n,k} = \begin{cases} \frac{(U_j^{n+1,k})^p - (U_j^n)^p}{U_j^{n+1,k} - U_j^n} & , \text{ if } U_j^{n+1,k} \neq U_j^n, \\ 0 & , \text{ otherwise,} \end{cases} \quad (5.12)$$

and determine the  $k+1$ -st iterate  $U_j^{n+1,k+1}$  by solving

$$[1 + \beta_j^{n,k}] \frac{U_j^{n+1,k+1} - U_j^n}{\Delta t} - D \frac{U_{j+1}^{n+1,k+1} - 2U_j^{n+1,k+1} + U_{j-1}^{n+1,k+1}}{h^2} = r_j^n. \quad (5.13)$$

This procedure give iterates which satisfy a maximum principle; that is

$$\min_j U_j^n \leq U_j^{n+1,k} \leq \max_j U_j^n,$$

therefore,  $\beta_j^{n,k}$  well-defined.

## 6 Conclusions

When considering the large time behaviour of nonnegative solutions of the nonlinear transport equation (1.1), the value of  $p$  in the nonlinearity plays a crucial role in relation to the large time development of the initial value problem. For pulse type solutions satisfying

$$(u + u^p)(\cdot, t) \in L'(R) \text{ for all } t \geq 0. \quad (6.1)$$

the results of the paper distinguish the following cases.

$p > 2$ : The limit profile is given by the leading order term in (4.16) Note that it is symmetric with respect to the moving coordinate  $x = t$  and that it decays as  $t^{-1/2}$ . In fact the limiting profile is a solution of equation (1.1), with  $D = 1$ , if we ignore the nonlinearity  $u^p$ . Numerical solutions of the initial value problem for  $p = 3$  are displayed in Figures 1 and 2. Figure 1 shows  $t^{1/2}u$  as a function of  $\eta = (x - t)/t^{1/2}$  for increasing values of  $t$ . Because of the slow convergence, a first order analysis was undertaken from which a new independent variable  $\eta_1$  was constructed given by (4.21). This is used in Figure 2, which clearly reveals the accelerated convergence to the final profile.

$p = 2$ : The limit profile now becomes asymmetric with respect to  $x = t$ . It is given by the expression (4.26), which is in fact a similarity solution of Burgers' equation obtained by disregarding  $u^2$  with respect to  $u$  in the time derivative in (4.3). The numerical convergence to the limit profile is illustrated in Figure 3 where again a modified independent variable is used to accelerate the convergence to the limiting form.

$1 < p < 2$ : The limit profile is left-asymmetric with respect to  $x = t$  and with a discontinuity along the curve  $(x - t)/t^{1/p} = \eta_1$ , with  $\eta_1$  given by (4.29). The outer expansion, valid as  $t \rightarrow \infty$ ,  $(x - t)/t^{1/p} = O(1)$ , is given by (4.25) with the leading order term as the limiting solution. In Section 4.2 boundary layer solutions are constructed near the leading and trailing edges of the limit profile. These resolve the non-analytic behaviour of the outer solution near  $\eta = 0$  and  $\eta = \eta_1$  respectively. The boundary layer solution near  $\eta = 0$  poses an interesting but unsolved boundary value problem (4.55) with (4.53) and (4.56). Figure 4 presents sample numerical solutions for the case  $p = 3/2$ . In order to minimise the effect of the diffusion boundary layers on the large time solution we carry out the computations for  $D = 10^{-2}$ . To check the convergence properties of this numerical solution and, since it is dominated by convection in this parameter regime, the large time solution itself, we compare the results with the solution of the initial value problem for  $D = 0$ . This solution is written down for  $p = 3/2$  in Appendix 3 and presented in Figure 5. The slow convergence to the outer limit, shared by both sets of results, is clearly evident.

$0 < p < 1$ : In this case the relevant outer variable is  $\eta = x/t^p$  and the limit profile becomes right-asymmetric with respect to  $x = 0$  with a discontinuity along the curve  $x = \eta_2 t^p$  where the constant  $\eta_2$  is determined by (4.43). Again we have to construct boundary layer solutions near  $\eta = 0$ , where the profile vanishes, and near  $\eta = \eta_2$ . In both cases this leads to solvable boundary layer problems. Figure 6 shows results of the computations for the initial value problem (2.11) with  $D = 10^{-2}$  which, as a check on the convergence, can be compared with the exact solution for  $D = 0$  given in Appendix 3 and displayed in Figure 7. In both cases the slow convergence to the zero order outer solution is clearly apparent.

## References

- [1] Escabedo, J.L. Vazquez & Zuazua. Source type solutions and asymptotic behaviour for a diffusion convection equation.
- [2] J. Bear, Dynamics of Fluids in Porous Media, Amerivan Elsevier 2nd edition, New York, 1972.
- [3] R.A. Freeze & J.A. Cherry, Groundwater, Prentice-Hall, Englewood Cliffs, NJ, 1979.
- [4] G.H. Bolt, Soil Chemistry B. Physico-Chemical Models. Elsevier, Amsterdam, 1979.
- [5] C.J. van Duijn & P. Knabner, Travelling waves in the transport of reactive solutes through porous media: adsorption and binary on exchange - Part I and II, Transport in Porous Media, to appear in 1992.
- [6] M.Th. van Genuchten & R.W. Cleary, Movement of solutes in soil: Computer simulated and laboratory results, in Soil Chemistry B, Physico-Chemical Models, edited by G.H. Bolt, pp. 349-386, Elsevier, Amsterdam, 1982.
- [7] P. Knabner, Mathematische Modelle für Transport und Sorption gelöster Stoffe in porösen Medien, Meth. und Verf. der math. Phys. 36, Frankfurt/M, 1991.
- [8] C.J. van Duijn & P. Knabner, Solute transport in porous media with equilibrium and non-equilibrium multiple-site adsorption: travelling waves, J. Reine Angew. Math. 415, pp. 1-49, 1991. 18. pp. 728-743, 1987.
- [9] B.H. Gilding, Improved theory for a nonlinear degenerate parabolic equation, Ann. Scuola, Norm. Sup. Pisa Cl. Sci. 4 16, pp. 165-224, 1989.

- [10] S. Osher & J. Ralston,  $L^1$ -stability of travelling waves with applications to convective porous media flow, *Comm. Pure Appl. Math.* **35**, pp. 737-749, 1982.
- [11] C.J. van Duijn & J.M. de Graaf, Limiting profiles in contaminant transport through porous media, *SIAM J. Math. Anal.*
- [12] Escabedo & Zuazua. Large time behaviour for solutions of a convection diffusion equation in  $R^N$ .
- [13] R.E. Grundy, Asymptotic solution of a model nonlinear convective diffusion equation. *IMA Journal of App. Maths.*, **31**, pp. 121-137, 1983.
- [14] R.E.Grundy, The Cauchy problem for a nonlinear diffusion equation with absorption and convection. *IMA Journal of App. Maths.*, **40**, pp. 183-204, 1988.
- [15] R.E.Grundy, Asymptotic solutions of a model diffusion reaction equation, *IMA Journal of App. Maths.*, **40**, pp 53-72, 1988.
- [16] M.Van Dyke. *Perturbation methods in fluid mechanics.* (New York, Academic Press. (1964).
- [17] C.N. Dawson, Godunov-mixed methods for advective flow problems in one space dimension, to appear in *SIAM J. Numer. Anal.*
- [18] B. van Leer, Towards the ultimate conservative difference scheme via a second-order sequel to Godunov's method, *J. Comput. Phys.* **32**, pp. 101-136, 1979.

## A.1. Error Analysis for $p = 3$ and $p = 2$

In the first part of this Appendix we compute the leading error term in the expansion (4.14) for  $v(\eta, t)$  when  $p = 3$ . It turns out that we must consider an expansion of the form.

$$v(\eta, t) = v_0(\eta) + t^{-1/2}(\log t)v_1(\eta) + t^{-1/2}v_2(\eta) + o(t^{-1/2}) \quad (A.1.1)$$

where  $v_0(\eta) = \frac{M}{2\sqrt{\pi}}e^{-\eta^2/4}$ . With a few modification the analysis can be extended to all values of  $p > 2$ . Substituting (A.1.1) into (4.13) and equating terms of  $0(t^{-1/2} \log t)$  gives

$$v_1'' + \frac{\eta}{2}v_1' + v_1 = 0 \quad , v_1(\pm\infty) = 0. \quad (A.1.2)$$

If we require  $v_1$  to have exponential decay as  $\eta \rightarrow \pm\infty$ , then (A.1.2) has the solution

$$v_1 = Av'_0(\eta) \quad (\text{A.1.3})$$

where  $A$  is an arbitrary constant. The equation for  $v_2$  can now be written as

$$v_2'' + \frac{\eta}{2}v_2' + v_2 = (A - 3v_0^2)v_0'.$$

with solution

$$v_2 = v_0'(\eta)\omega_2(\eta)$$

where

$$\begin{aligned} \omega_2' = & \frac{Be^{\eta^2/4}}{\eta^2} + \frac{3M^2 e^{\eta^2/4}}{4\pi \eta^2} \int_{\eta}^{\infty} s^2 e^{-3s^2/4} ds \\ & - \frac{Ae^{\eta^2/4}}{\eta^2} \int_{\eta}^{\infty} s^2 e^{-s^2/4} ds \end{aligned}$$

and  $B$  is an arbitrary constant. We now require that  $v_2$  decays exponentially to zero as  $\eta \rightarrow \pm\infty$ . This demands that the exponentially growing terms in  $\omega_2'$  be suppressed, which, as  $\eta \rightarrow +\infty$ , requires  $B$  to be zero. The same condition as  $\eta \rightarrow -\infty$  gives

$$A = \frac{3M^2}{4\pi} \int_{-\infty}^{\infty} s^2 e^{-3s^2/4} ds / \int_{-\infty}^{\infty} s^2 e^{-s^2/4} ds = \frac{M^2}{4\pi\sqrt{3}}$$

and hence (A.1.1) can be written as

$$v(\eta, t) = v_0(\eta) + \frac{M^2}{4\pi\sqrt{3}} t^{-1/2} (\log t) v_0'(\eta) + 0(t^{-1/2})$$

or, equivalently

$$v(\eta, t) = v_0 \left\{ \eta + \frac{M^2}{4\pi\sqrt{3}} t^{-1/2} \log t \right\} + 0(t^{-1/2}) \quad (\text{A.1.4})$$

which gives the representation (4.18) with (4.20).

The case  $p = 2$  can be dealt with in a similar way. With this value of  $p$ , (4.13) becomes

$$\begin{aligned} t \frac{\partial v}{\partial t} - \frac{1}{2} (v + \eta \frac{\partial v}{\partial \eta}) + t^{-1/2} \left\{ t \frac{\partial(v^2)}{\partial t} - \frac{p}{2} v^2 - \frac{\eta}{2} \frac{\partial(v^2)}{\partial \eta} \right\} \\ = \frac{\partial^2 v}{\partial \eta^2} + \frac{\partial(v^2)}{\partial \eta} \end{aligned} \quad (\text{A.1.5})$$

Again we have to expand

$$v(\eta, t) = v_0(\eta) + t^{-1/2} (\log t) v_1(\eta) + t^{-1/2} v_2(\eta) + o(t^{-1/2}) \quad (\text{A.1.6})$$

where this time  $v_0(\eta)$  is given by (4.30). Substituting (A.1.6) into (A.1.5), equating terms  $0(t^{-1/2} \log t)$  and requiring exponential decay on  $v_1$  as  $\eta \rightarrow \pm\infty$  gives

$$v_1 = Bv_0'(\eta).$$

A similar procedure for terms of  $0(t^{-1/2})$  gives the condition on  $B$  for exponential decay of  $v_2$  as  $\eta \rightarrow \pm\infty$  as

$$B = \int_{-\infty}^{\infty} \frac{(v_0^2 + sv_0v_0')v_0'}{v_0^2} e^{-s^2/4} ds / \int_{-\infty}^{\infty} \left(\frac{v_0'}{v_0}\right)^2 e^{-s^2/4} ds.$$

Numerical evaluation gives  $B = -0.0630$  to four decimal places.

## A.2. Solution to a boundary value problem

In this appendix we construct the solution to the boundary value problem represented by (4.86)-(4.88) in Section 4.2.2.

We first show that  $U_0(\xi)$  is monotonically increasing for  $-\infty < \xi < \infty$ . To see this we suppose  $\xi_0$  is a stationary point of  $U_0(\xi)$  so that from (4.86) we have, with  $U_0(\xi_0) > 0$ ,

$$U_0'' = -\frac{p}{(1-p)} U_0^p(\xi_0) < 0$$

so  $\xi_0$  is a maximum. This implies that  $U_0(\xi) < U_0(\xi_0)$  for all  $\xi$  so the boundary condition (4.87) cannot be satisfied, hence  $U_0$  cannot have a stationary point and thus the monotonicity condition must hold. This result now enables us to change the independent variable from  $\xi$  to  $U_0$ . Putting  $U_0 = u$  we write

$$u'(\xi) = y(u) \tag{A.2.1}$$

regarding  $u$  as the independent variable with  $0 < u < \infty$ . This reduces (4.86) to the first order equation

$$\frac{dy}{du} = 1 + \zeta_0 p u^{p-1} - \frac{p}{(1-p)} \frac{u^p}{y} \tag{A.2.2}$$

Since we are assuming that  $u = U_0(\xi)$  has a continuous derivative for all  $\xi$  then we must have  $u'(\xi) \rightarrow 0$  as  $\xi \rightarrow \infty$  and we must solve (A.2.2) subject to the initial condition  $y(0) = 0$ . Now near  $u = 0$

$$y(u) \sim \zeta_0 u^p + \dots$$

and hence from (A.2.1) we see that the support of  $u(\xi)$  must be finite since with  $0 < p < 1$

$$\int_0^u \frac{ds}{y(s)}$$

is bounded. Thus  $u(\xi) \geq 0$  for  $\xi \geq \xi_1$  and  $u(\xi) = 0$  for  $-\infty < \xi \leq \xi_1$  where we may take  $\xi_1 = 0$  since (4.86) is translationally invariant. To see what conditions hold at  $\xi = 0$  we integrate (4.86) between zero and  $\xi > 0$  with  $u(0) = u'(0) = 0$  to give

$$u'(\xi) = \zeta_0 u^p(\xi) + u(\xi) - \frac{p}{(1-p)} \int_0^\xi u^p(s) ds$$

whence dividing by  $u^p(\xi) > 0$  we have

$$\frac{u'(\xi)}{u^p(\xi)} = \zeta_0 + u^{1-p}(\xi) - \frac{p}{(1-p)u^p(\xi)} \int_0^\xi u^p(s) ds$$

Since  $u^p(\xi)$  is monotonically increasing then

$$\int_0^\xi u^p(s) ds < \xi u^p(\xi)$$

so

$$\lim_{\xi \rightarrow 0} \left\{ \frac{u'(\xi)}{u^p(\xi)} \right\} = \zeta_0$$

or

$$\left\{ u^{1-p}(\xi) \right\}' = \zeta_0(1-p) \tag{A.2.3}$$

at  $\xi = 0$ .

We are now in a position to construct the solution. First we put

$$v = u^{1-p} \text{ and } v'(\xi) = Y(v)$$

in (A.2.2) to give

$$\frac{dY}{dv} = \frac{Y-p}{Y} + \frac{p}{(1-p)v} \{ \zeta_0(1-p) - Y \} \tag{A.2.4}$$

where from (A.2.3)

$$Y(0) = (1-p)\zeta_0 \tag{A.2.5}$$

and from (4.87)

$$Y(\infty) = p \tag{A.2.6}$$

To determine  $\zeta_0$  and  $Y(v)$  we proceed as follows. First if we suppose that  $\zeta_0 > p/(1-p)$  then  $Y(0) > p$  and, from (A.2.4),  $Y'(0) > 0$ . Now in order for the condition (A.2.6) to be satisfied there must be a value of  $v > 0$  for which  $Y = \zeta_0(1-p)$  and  $Y' < 0$ .

Clearly from (A.2.4) such a condition cannot hold for  $\zeta_0 > p/(1-p)$ . A similar argument holds if  $\zeta_0 < p/(1-p)$  and so we conclude that

$$\zeta_0 = p/(1-p) \quad (\text{A.2.7})$$

whence the problem becomes

$$\frac{dY}{dv} = \frac{(Y-p)\{(1-p)v - pY\}}{vY(1-p)}$$

$$Y(0) = p$$

Now there are no solutions which increase from  $Y(0) = p$  since  $\frac{dY}{dv} < 0$  for  $Y > p$  and  $Y > \frac{(1-p)v}{p}$ . Similarly there are no solutions which decrease from  $Y(0) = p$ . Thus  $Y = p$  is the only solution satisfying  $Y(0) = p$  and hence with (A.2.7) constitutes the unique solution. In the original variables this becomes

$$U_0 = \begin{cases} \{p(\xi - \xi_1)\}^{1/1-p} & , \quad \xi \geq \xi_1 \\ 0 & , \quad \xi < \xi_1 \end{cases} \quad (\text{A.2.8})$$

which together with (A.2.7) is the unique solution, to within the arbitrary translation  $\xi_1$ , of the boundary value problem (4.86) - (4.88).

### A.3. The solution of the initial value problem for $D = 0$ .

In this Appendix we write down the solution of the equation (2.11) with  $D = 0$  and initial data (4.17) for  $p = 1/2$  and  $p = 3/2$ .

1.  $p = 1/2$  In this case the point  $x = -1$ , where  $u = 0$ , does not move so there is an expansion wave which connects  $u = 0$  to  $u = 1$ . In addition a shock initiated at  $x = 1$  moves to the right with a speed determined by the mass invariance condition. So we have

$$u(x, t) = \begin{cases} 0 & , \quad x < -1 \\ \frac{(1+x)^2}{4(t-x-1)^2} & , \quad -1 \leq (2t/3) - 1 \\ 1 & , \quad (2t/3) - 1 \leq (t/2) + 1 \\ 0 & , \quad x > (t/2) + 1 \end{cases} \quad (\text{A.3.1})$$

This persists until the head of the expansion wave coalesces with the shock at  $t = 12$  and  $x = 7$ . For  $t > 12$  we then have

$$u(x, t) = \begin{cases} 0 & , \quad x < -1 \\ \frac{(1+x)^2}{4(t-x-1)^2} & , \quad -1 \leq x \leq S(t) \\ 0 & , \quad x > S(t) \end{cases} \quad (\text{A.3.2})$$

where

$$S(t) = 4(t+4)^{1/2} - 9, t \geq 12 \quad (\text{A.3.3})$$

In terms of the outer variable of Section 4.1.3. , namely

$$\eta = x/t^{1/2}$$

$$v(\eta, t) = \begin{cases} 0 & , \quad \eta < -t^{-1/2} \\ \frac{(\eta+t^{-1/2})^2}{4(1-\eta t^{-1/2}-t^{-1})^2} & , \quad -t^{-1/2} \leq \eta \leq \eta_2(t) \\ 0 & , \quad \eta \geq \eta_2(t) \end{cases} \quad (\text{A.3.4})$$

with

$$\eta_2(t) = 4(1+4t^{-1/2})^{1/2} - 9t^{-1/2} \quad (\text{A.3.5})$$

2.  $p = 3/2$  Here the point  $x = 1$  translates to the right with unit speed so we put

$$X = x - t$$

so the leading edge of the wave, where  $u = 0$ , is located at  $X = 1$ . There is an expansion wave which links  $u = 0$  and  $u = 1$ , together with a shock which moves to the left in the  $(X, t)$  plane from  $X = -1$  according to mass conservation in the wave. Hence the solution can be written

$$u(X, t) = \begin{cases} 0 & , \quad X < -1 - (t/5) \\ 1 & , \quad -1 - (t/5) \leq X < 1 - (3t)/5 \\ \frac{4(1-X)^2}{9(t-1+X)^2} & , \quad 1 - (3t)/5 < X \leq 1 \\ 0 & , \quad X \geq 1 \end{cases} \quad (\text{A.3.6})$$

This solution persists until  $t = 20$  at which time the shock and expansion wave coalesce. For  $t \geq 20$  we then have

$$u(X, t) = \begin{cases} 0 & , \quad X < X_1(t) \\ \frac{4(1-X)^2}{9(X+t-1)^2} & , \quad X_1(t) \leq X \\ 0 & , \quad X > 1 \end{cases} \quad (\text{A.3.7})$$

where  $X_1(t)$  is given by the solution of a cubic. In terms of the outer variable of Section 4.1.2., namely

$$\eta = X/t^{2/3}$$

we have

$$v(\eta, t) = \begin{cases} 0 & , \quad \eta < \eta_1(t) \\ \frac{4(t^{-2/3}-\eta)^2}{9(1+\eta t^{-1/3}-t^{-1})^2} & , \quad \eta_1(t) \leq \eta \leq t^{-2/3} \\ 0 & , \quad \eta > t^{-2/3} \end{cases} \quad (A.3.8)$$

where

$$\eta_1(t) = t^{-2/3}(12(u+v) - 8)$$

$$u = (-q + \sqrt{q^2 + p^3})^{1/3}$$

$$v = -p/u$$

$$p = t/8 - 9/16$$

and

$$q = t^2/128 - (9t)/64 + 27/64.$$

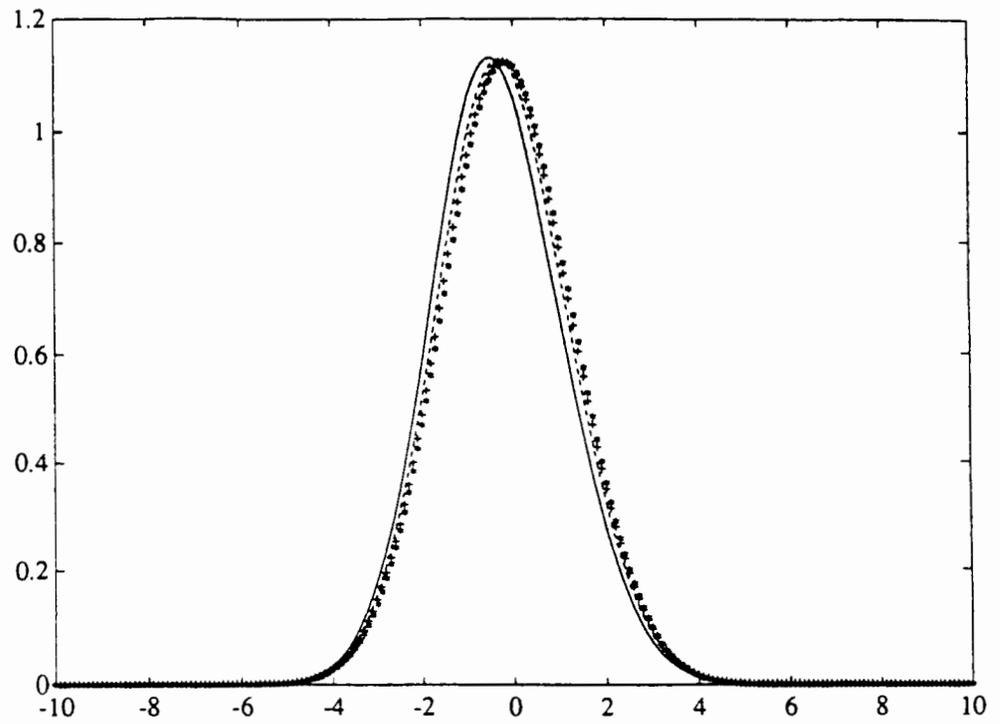


Figure 1: Convergence of the numerical solution to the limiting profile for the case  $p = 3$ .  $ut^{1/2}$  plotted as a function of  $\eta$  for —  $t = 80$ , - - -  $t = 320$ , + + +  $t = 1000$ , \* \* \*  $t = 2000$ .

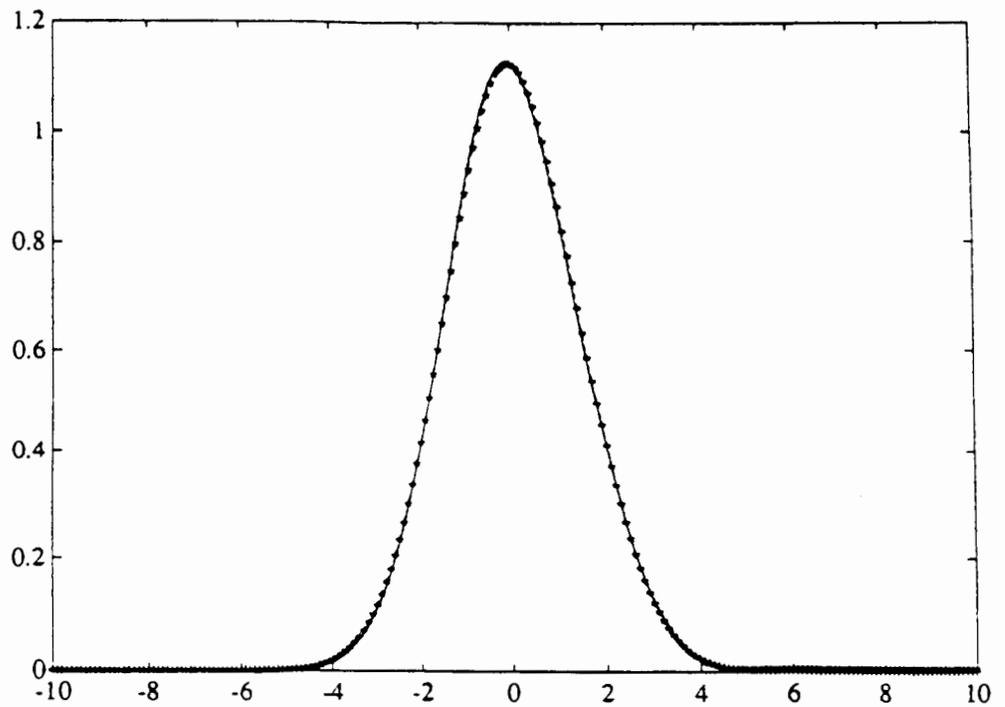


Figure 2: Convergence of the numerical solution to the limiting profile for the case  $p = 3$ .  $ut^{1/2}$  plotted as a function of the modified variable  $\eta_1$  for —  $t = 200$ , - - -  $t = 500$ , + + +  $t = 1000$ , \* \* \*  $t = 1500$ .

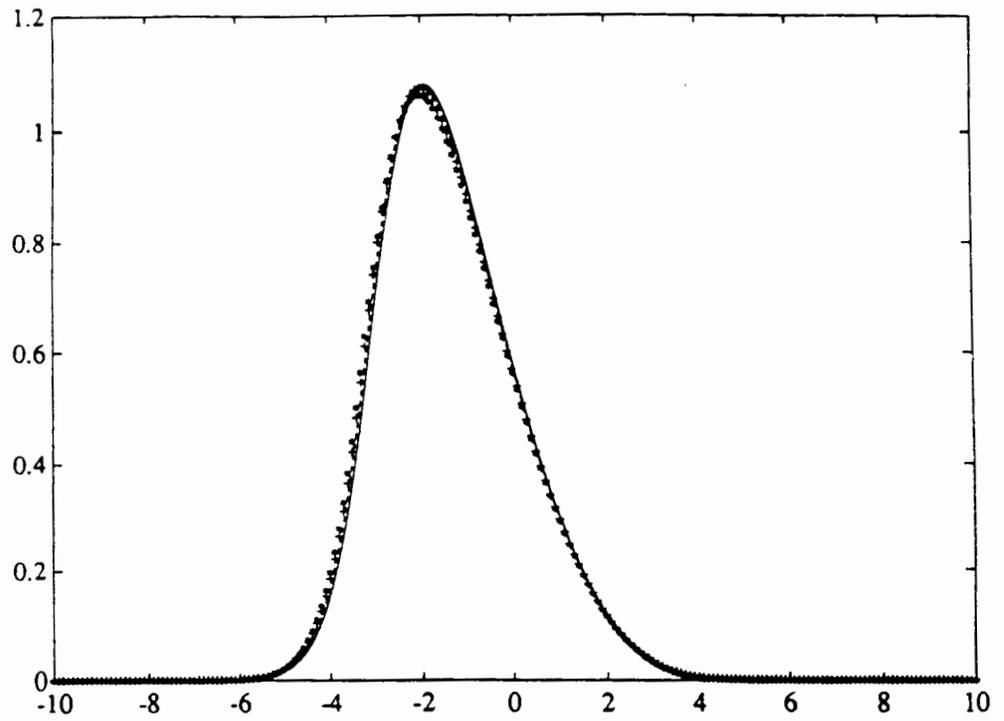


Figure 3: Convergence of the numerical solution to the limiting profile for the case  $p = 2$ .  $ut^{1/2}$  plotted as a function of the modified variable  $\eta_1$  for —  $t = 500$ , - -  $t = 1000$ , + + +  $t = 1500$ , \* \* \*  $t = 3000$ .

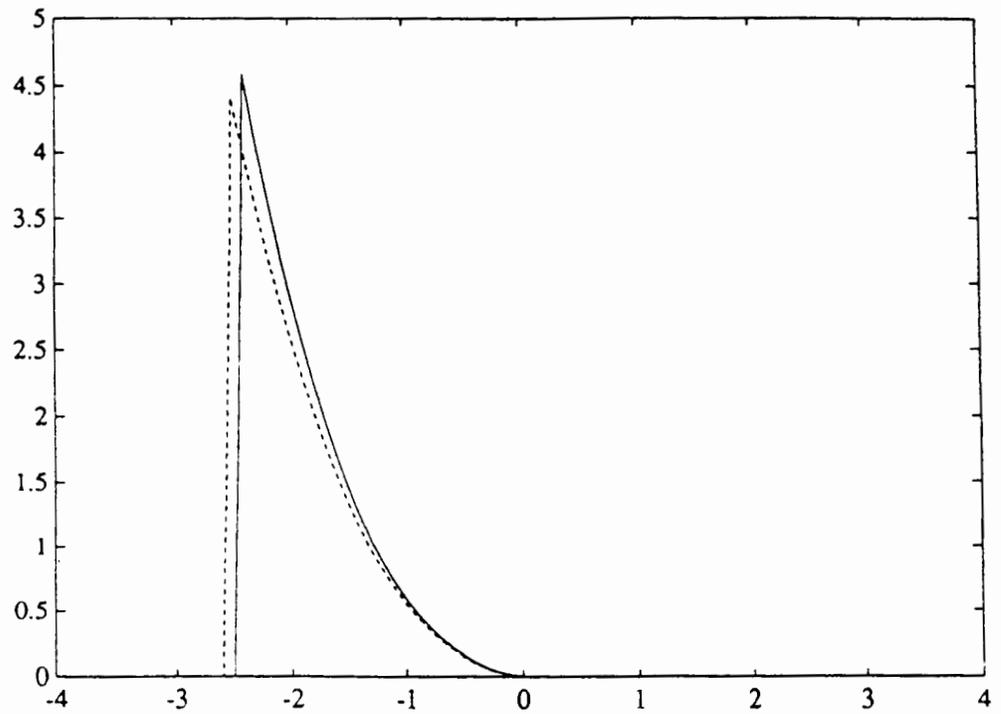


Figure 4: Convergence of the numerical solution to the zero order outer solution for  $p = 1.5$ .  $ut^{2/3}$  plotted as a function of  $\eta$  for —  $t = 1000$ , - - -  $t = 2000$  with  $D = 10^{-2}$ .

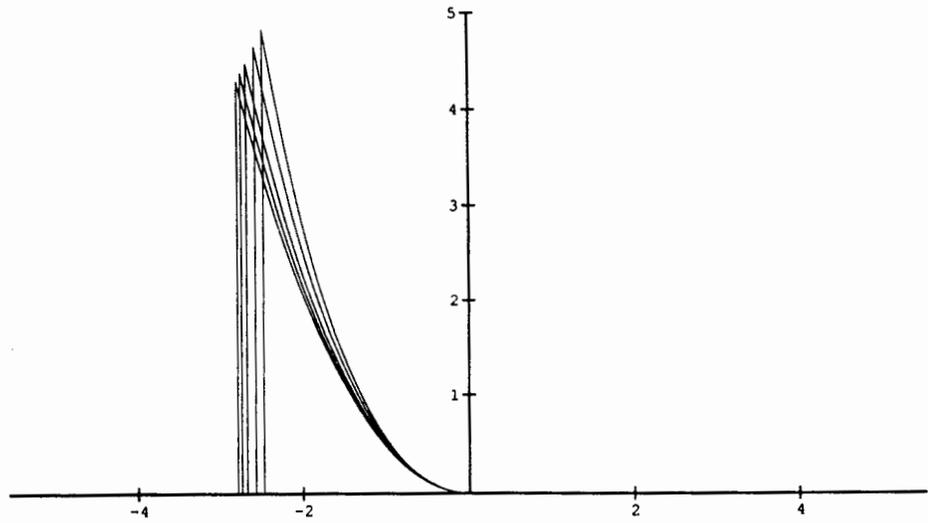


Figure 5: Convergence of the exact solution of the initial value problem to the zero order outer solution for  $D = 0$  and  $p = 1.5$ . The values of time are 1000, 2000, 5000, 10,000 and 20,000.

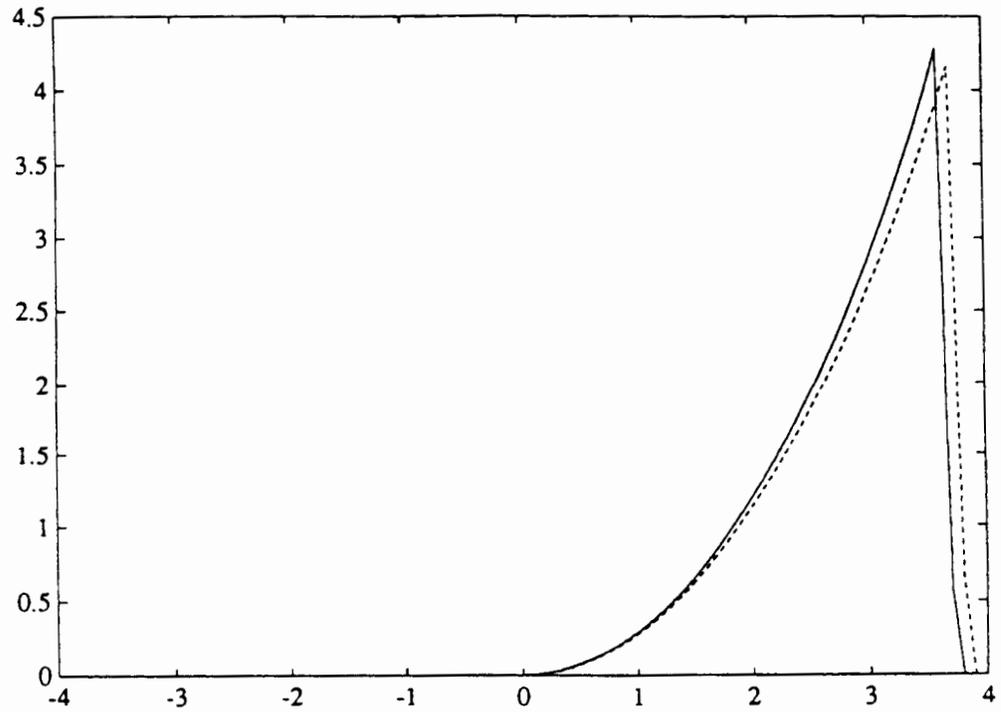


Figure 6: Convergence of the numerical solution to the zero order outer solution for  $p = .5$ .  $ut$  plotted as a function of  $\eta$  for  $\text{---} t = 1000$ ,  $\text{- - -} t = 2000$  with  $D = 10^{-2}$ .

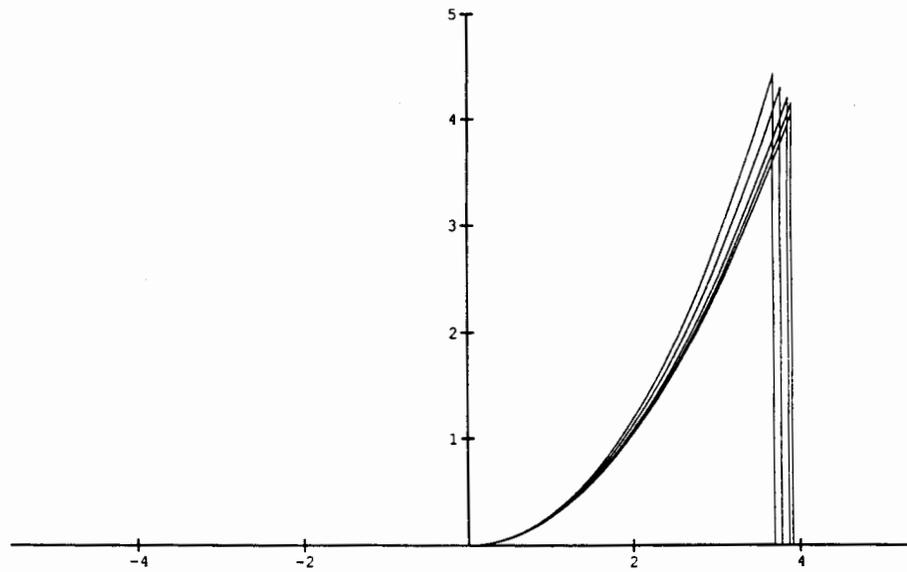


Figure 7: Convergence of the exact solution to the initial value problem for  $D = 0$  and  $p = .5$ . The values of time are 1000, 2000, 5000, 10,000 and 20,000.

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