Compound Poisson cascades

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Modelling complexity through scaling. Scale invariance and related phenomena have received considerable attention in the past from the point of view of both, analysis and estimation as well as modelling and synthesis. Various kinds of scaling form an undisputable component of empirical data observed in a wide variety of applications ranging from natural phenomena such as hydrodynamic turbulence [9], biology and body rhythms [20] to purely human phenomena created by mankind's activities such as computer networks [13, 16] and financial markets [14]. Often, the existence of scaling has its toll, e.g., leading to high volatility in markets and to large waiting queues in networking. Being of major importance scaling phenomena call for both, appropriate tools of analysis with known accuracy as well as novel models with controllable parameters leading to deeper understanding and allowing for physical interpretations.

Most prominently, self-similar processes have been favored as models for scale invariance for their simplicity. Indeed, any self-similar process $X(t)$ with stationary increments spots an appealing and simple scale invariance. Indeed, let $\delta_\tau X(t) = X(t+\tau) - X(t)$ denote its increments over a lag τ, then

$$\mathbb{E}|\delta_\tau X(t)|^q = c_q \cdot |\tau|^{qH}, \tag{1}$$

where H is the Hurst parameter. This relation is independent of t and holds irrespective of τ which is best described as the absence of a characteristic scale. However, one has to acknowledge the restrictive character of relation (1) that imposes severe constraints for the empirical analysis of actual data. To provide processes with more realistic scaling which are able to match real world data, multiplicative cascades and the framework of multifractal analysis were introduced, allowing a non-linear dependence on the order q of exponents $\zeta(q) \neq qH$ in (1) so that:

$$\mathbb{E}|\delta_\tau X(t)|^q = c_q \cdot |\tau|^{\zeta(q)}. \tag{2}$$

Lamenting the restrictive nature of both, the statistical self-similar scaling of (1) and the multifractal scaling of (2) this paper sets off to construct processes with more flexible and natural scaling properties with the following properties on the progression of moments: (i) it may depend in an arbitrary way on the order, (ii) it can be determined between arbitrary

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scales, and (iii) it may depend in an arbitrary way on scale, not necessarily in form of a power law. To this purpose, one can place multifractal analysis in the more general framework of log infinitely divisible cascades (LIDC) [2, 5, 6, 21] characterized by:

$$\mathbb{E}|\delta_\tau X(t)|^q = c_q \exp[-H(q) \cdot n(\tau)]. \tag{3}$$

Note how the framework of LIDC encompasses scaling in the form of power laws by setting $n(\tau) = -\log(\tau)$. The extra degree of freedom in scale dependence was found highly useful for the analysis and modelling of empirical data in turbulence [7] and computer network traffic [21].

While a collection of theoretical as well as practical synthesis procedures is known and used for the fractional Brownian motion, the synthesis of processes that possess a priori prescribed multiscaling properties as well as other casual characteristics such as second-order stationarity of the increments or a continuous rather than discrete scaling region proved extremely difficult. The celebrated martingale of Mandelbrot studied in [12] for instance as well as the wavelet based cascades more recently introduced in [1] do have multifractal properties that can be prescribed but present neither second-order stationary increments nor continuous scale invariance. Very recently, a small number of attempts were made to try to improve this situation, see [3, 4, 15, 19].

We intend to contribute to the pavement of this difficult path by proposing the definition of new processes that not only match *prescribed multiscaling exponents*, have *stationary increments* and *continuous scale invariance*, but will on top of it *not assume a priori power law behaviors of the moments*. We explain how to tune their correlation structure, as well as their scaling properties, and hint at how to go beyond scaling in form of pure power laws towards more general infinitely divisible scaling. Further, we point out that these cascades represent but the most simple and most intuitive case out of an entire array of infinitely divisible cascades allowing to construct general infinitely divisible processes with interesting scaling properties (see [8] for a detailed presentation). Relying on the idea of infinitely divisible processes the construction of a wide class of LIDC cascades becomes feasible, allowing for instance to introduce log-normal as well as log-stable LIDC cascades. While such generality is beyond the scope of this paper and has to be postponed to a forthcoming paper [18] we shortly present below a more intuitive solution.

Poisson cascading noise. The construction is based on a Poissonian geometry to allow for continuous multiplication. As they possess compound Poissonian statistics we term the resulting processes *compound Poisson cascades.* The basic building blocks for the construction of infinitely divisible compound Poisson cascades are the following functions P_i :

$$P_i(t) = \begin{cases} 1 & \text{if } |t - t_i| \geq \dfrac{r_i}{2}, \\ W_i & \text{if } |t - t_i| < \dfrac{r_i}{2}. \end{cases} \tag{4}$$

where $(t_i, r_i)_{i \in I}$ and W_i are defined as follows (see Figure 1). $(t_i, r_i)_{i \in I}$ is a planar Poisson point process with local arrival rate described by a *control measure* $dm(t, r)$ which we assume to be supported on the half-strip $\mathbb{R} \times]0, 1]$ and which we require to be time-invariant, meaning that the control measure necessarily takes the form,

$$dm(t, r) = g(r) dr dt, \tag{5}$$

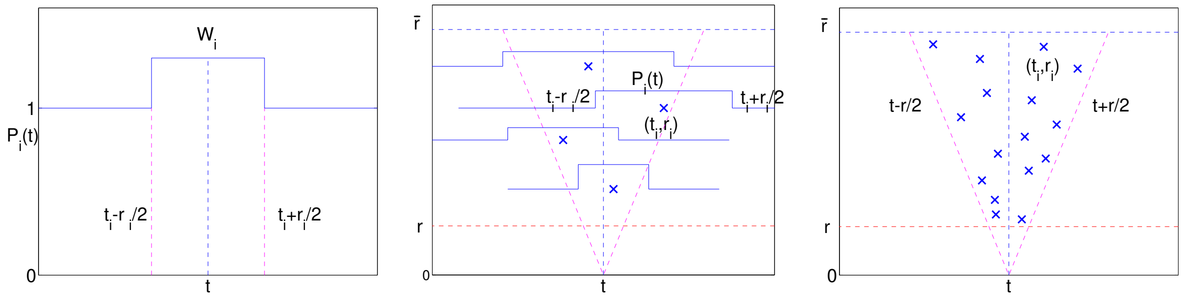

Figure 1: **Compound Poisson cascading.** **(left)** The basic multiplier of the Poisson cascade takes the simple form of a random rectangular functions $P_i(t)$; **(middle)** the cascade is defined as $Q_r(t) \propto \prod_i P_i(t)$. Only those pulse processes $P_i(\cdot)$ will contribute to $Q_r(t)$ for which $|t_i - t| < r_i/2$;**(right)** consequently, $Q_r(t)$ can be viewed as the product of a random number of the random multipliers W_i located in the trapezoid $\mathcal{C}_r(t)$.

for a proper density function g. The multipliers $\{W_i\}_{i\in I}$ are independent identically distributed positive random variables, independent of the point process $(t_i, r_i)_{i\in I}$. The set $(t_i, r_i, W_i)_{i\in I}$ is usually called a marked Poisson process.

Infinitely divisible cascades will be built on the product of these processes P_i, an idea that goes back to Mandelbrot who coined them "cylindrical pulses" [4]. In preparation for defining the cascade consider a fixed point in time t. It follows immediately from (4) that only those pulse processes P_i will contribute to the product for which $|t_i - t| < r_i/2$ (see Figure 1):

$$(t_i, r_i) \in \mathcal{C}_r(t) = \left\{(t', r') \ : \ r' > r \text{ and } t' - \frac{r'}{2} < t < t' + \frac{r'}{2}\right\}. \tag{6}$$

The number of points falling into $\mathcal{C}_r(t)$ is a Poisson random variable of mean $m(\mathcal{C}_r(t))$. We define *Poisson cascading noise* as

$$Q_r(t) = \exp\left[(1 - \mathbb{E}W)m(\mathcal{C}_r(t))\right] \prod_{1>r_i>r} P_i(t) = \exp\left[(1 - \mathbb{E}W)m(\mathcal{C}_r(t))\right] \prod_{\mathcal{C}_r(t)} W_i \tag{7}$$

where the *normalization factor* in front of the product renders $Q_r(t)$ a mean 1 process. A sample of Q_r is shown in Figure 2 (left). Note that Q_r is *stationary.* A further simple property of Q_r is that if forms a T-martingale [11].

Poisson cascading motion. Where there is noise there is motion. We introduce the *Poisson cascading motion* via

$$A_r(t) = \int_0^t Q_r(s)\, ds. \tag{8}$$

Indeed, the increments $\delta_\tau A(t) = A(t+\tau) - A(t) = \int_t^{t+\tau} Q_r(s)\, ds$ of A inherit *stationarity* directly from Q_r. Furthermore, $\mathbb{E}[A_r(t)] = \int_0^t \mathbb{E}[Q_r(s)]\, ds = t$ for all t and all r. At least under certain conditions, $A_r(t)$ is a positive martingale and must converge almost surely. We denote this limit by

$$A(t) = \lim_{r\to 0} A_r(t). \tag{9}$$

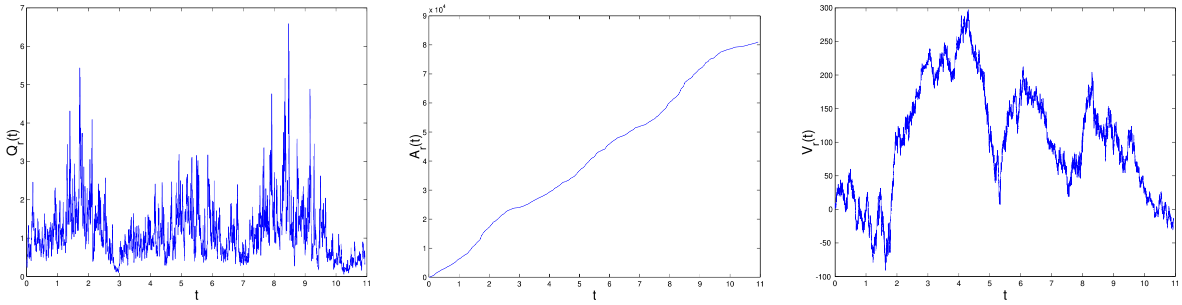

Figure 2: Sample of a realization of $Q_r(t)$ (left), $A_r(t)$ (middle) and $V_r(t)$ (right).

We establish conditions for convergence of A in L^2. A special case of the present cascade A has been studied with great care in [4]. For a sample of a process $A_r(t)$ see Figure 2 (middle).

Poisson cascading Brownian motion In contrast with $A_r(t)$ obtained from a deterministic integral, we introduce the *Poisson cascading Brownian motion* $V_r(t)$ that appears as a stochastic integral of $Q_r(t)$:

$$V(t) = \lim_{r \to 0} V_r(t), \tag{10}$$

where

$$V_r(t) = \int_0^t \sqrt{Q_r(s)} dB(s), \tag{11}$$

whenever it exists, with $\sqrt{Q_r(s)}dB(s)$ the corresponding *Poisson cascading Gaussian noise*. For a sample of a process V_r, see Figure 2 (right). Let us point out that the increments of V_r are (second order) *un-correlated* while *not independent* since they inherit higher order correlations from $Q_r(s)$. Mandelbrot calls this the "blind spot of spectral analysis". Motivated by [10, 17], let us consider the process $\mathcal{B}(t) = B(A_r(t))$ where B denotes the ordinary Brownian motion. It constitutes a form of "multifractal random walk" similar to [3]. They are called *warped Gaussian noise* in [17]. We remark that[1]

$$V_r(\tau) \stackrel{fdd}{=} B(A_r(\tau)). \tag{12}$$

Statistical properties in the scale invariant case. In search for the choice of $dm(r,t)$ providing the most classical form of scaling, i.e., power laws, we simply set

$$dm(t,r) = \frac{c}{r^2} dr dt \qquad \text{for } 0 < r \leq 1, \tag{13}$$

and zero elsewhere. This special case of a Poisson cascading noise was studied in [4], in particular the issue of convergence, establishing the so-called multifractal formalism which relates the function $H(q)$ to local scaling properties of the paths of the Poisson cascading motion. The moments of Q_r, $A(t)$ and $V(t)$ follow power law behaviors given by:

$$\begin{array}{lcl} \mathbb{E}[Q_r^q(t)] & = & r^{c(H(q)-qH(1))} \\ \mathbb{E}[A(t)^q] & \sim & t^{q+c(H(q)-qH(1))}, \\ \mathbb{E}[V_r(t)^q] & \sim & t^{q/2+c(H(q/2)-q/2\,H(1))} \quad \forall t \in [0,1], \end{array} \tag{14}$$

[1]The concept is easily extended to Levy motion or fractional Brownian motion.

where $H(q) = 1 - \mathbb{E}W^q$. We emphasize the fact that these scaling behaviors are continuously valid through the scales, not only for a particular set of discrete scale ratios. Thus, A and V follow an LIDC at least approximatively. Similar scaling laws hold as well for the wavelet transform $T_A(a,t) = \int \psi_{a,t}(s)A(s)ds$ of A.

Conclusion. We have defined here a process $A(t)$ that is characterized by (positive) stationary increments as well as rich *scaling* properties which hold for a continuous range of scales and which can be easily prescribed from the definitions. We also introduced the companion processes $Q_r(t)$ and $V(t)$ that share the specification of $A(t)$. We believe that the combined advantages of fast numerical synthesis together with well-controlled scaling properties make these processes ideal for modeling and studying real world signals with complicated scaling phenomena which are observed, e.g., in computer networks and in hydrodynamic turbulence. The corresponding MATLAB synthesis procedure is available upon request or from our WEB pages. We refer the reader to [8] for a more detailed presentation. We have paid special attention here to the particular case of compound Poisson cascades which exhibits exact scale invariance, i.e., multifractal scaling in the classical form of power laws. Our main goal, remains to synthesize processes obeying to an arbitrary, prescribed LIDC (recall (3)), possibly departing from the reference situation of exact scale invariance or power laws. Furthermore, compound Poisson cascades form only a specific case of infinitely divisible cascades of particular practical interest. It takes only little, however, to extend the definitions of the processes $Q_r(t), A(t)$ and $V(t)$ given in this paper using general stochastic infinitely divisible measure, providing us, e.g., with log-normal cascades of continuous scales. We will elaborate on both extensions, beyond power laws and beyond compound Poisson cascades, in a forthcoming paper [18].

References

[1] A. Arneodo, E. Bacry, and J.F. Muzy. Random cascade on wavelet dyadic trees. *J. Math. Phys.*, 39(8):4142–4164, 1998.

[2] A. Arneodo, J.F. Muzy, and S. Roux. Experimental analysis of self-similarity and random cascade processes: application to fully developed turbulence data. *Journal de Physique II France*, 7:363–370, 1997.

[3] E. Bacry, J. Delour, and J.F. Muzy. Multifractal random walk. *Phys. Rev. E*, 64:026103, 2001.

[4] J. Barral and B. Mandelbrot. Multiplicative products of cylindrical pulses. discussion paper 1287, Cowles Foundation, 1999. submitted to *Probability Theory and Related Fields.*

[5] B. Castaing, Y. Gagne, and E. Hopfinger. Velocity probability density functions of high Reynolds number turbulence. *Physica D*, 46:177–200, 1990.

[6] P. Chainais. *Cascades log-infiniment divisibles et analyse multirésolution. Application à l'étude des intermittences en turbulence.* PhD thesis, E.N.S. Lyon, 2001.

[7] P. Chainais, P. Abry, and J.F. Pinton. Intermittency and coherent structures in a turbulent flow: a wavelet analysis of joint pressure and velocity measurements. *Phys. Fluids*, 11(11):3524–3539, 1999.

[8] P. Chainais, R.H. Riedi, and P. Abry. Compound Poisson cascades. *In Preparation.*

[9] U. Frisch. *Turbulence. The legacy of A. Kolmogorov.* Cambridge University Press, Cambridge, UK, 1995.

[10] P. Gonçalvès and R. Riedi. Wavelet analysis of fractional brownian motion in multifractal time. In *Proceedings of the 17th Colloquium GRETSI, Vannes, France*, Sept. 1999.

[11] J.-P. Kahane. Positives martingales and random measures. *Chinese Ann. Math.*, 8B:1–12, 1987.

[12] J.-P. Kahane and J. Peyrière. Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.*, 22:131–145, 1976.

[13] W. Leland, M. Taqqu, W. Willinger, and D. Wilson. On the self-similar nature of Ethernet traffic (extended version). *IEEE/ACM Trans. Networking*, pages 1–15, 1994.

[14] B. B. Mandelbrot. *Fractals and Scaling in Finance.* Springer New York, 1997.

[15] P. Mannersalo, R. Riedi, and I. Norros. Multifractal products of stochastic processes: construction and some basic properties. *Applied Probability*, submitted August 2001.

[16] K. Park and W. Willinger, editors. *Self-similar network traffic and performance evaluation.* Wiley (Interscience division), 2000.

[17] R. H. Riedi. Multifractal processes. *in: "Long range dependence : theory and applications", eds. Doukhan, Oppenheim and Taqqu*, to appear 2002.

[18] R.H. Riedi, P. Chainais, and P. Abry. Infinitely divisible processes. *In Preparation.*

[19] F. Schmitt and D. Marsan. Stochastic equations generating continuous multiplicative cascades. *Eur. Phys. J. B*, 20:3–6, 2001.

[20] M. Teich, S. Lowen, B. Jost, and K. Vibe-Rheymer. Heart rate variability: Measures end models. *Preprint*, 2000.

[21] D. Veitch, P. Abry, P. Flandrin, and P. Chainais. Infinitely divisible cascade analysis of network traffic data. *Proc. ICASSP 2000 conference*, 2000.