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SUSPENDED IN A BINGHAM MATERIAL.

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INCIPIENT MOTION OF A SPHERICAL BODY
SUSPENDED IN A BINGHAM MATERIAL

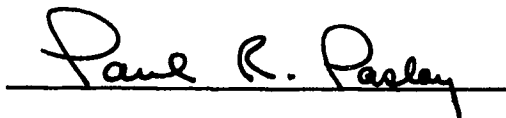
by

Donald Bayne Wood

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's Signature:

A handwritten signature in cursive script, reading "Paul R. Pasley", is written over a horizontal line.

Houston, Texas

May, 1965

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Nomenclature

D	rate of energy dissipation
d_{ij}	deformation rate tensor
F	force acting on the rigid sphere
F_i	i th component of the body force
g	acceleration of gravity
I_2	second invariant of the deformation rate tensor
J_2	second invariant of the stress deviation tensor
n_i	i th component of the unit normal to the surface
p_i	i th component of the unitary surface traction
q_i	i th component of the unitary velocity
R	radius of the sphere
r	radial component of spherical coordinates
S_h	surface bounding a transition layer
S_k	surface bounding a transition layer
S_m	median surface of a transition layer
S_T	stress boundary
S_V	velocity boundary
S_{ij}	stress deviation tensor
t	time, thickness of transition layer
U_0	speed of falling sphere
v_i	i th component of the velocity
x_i	i th Cartesian coordinate

δ	small positive constant
δ_{ij}	Kronecker delta
ϵ_{ij}	unitary deformation rate tensor
θ	latitude angle of spherical coordinates
λ	yield parameter
μ	viscosity
ρ_S	density of the sphere
ρ_B	density of the Bingham material
σ_{ij}	stress tensor
τ_0	critical value of the shear stress
$\tau_{r\theta}$	shear stress in polar coordinates
φ	azimuth angle of spherical coordinates

Introduction

The research leading to this thesis has been directed toward a study of the use of limit analysis theorems for bounding the yield load of a body composed of a Bingham material, and toward applying these theorems to the problem of the incipient motion of a sphere suspended in a Bingham material.

The Bingham material is the incompressible viscoplastic material, which is also often referred to as a rigid viscous material. This material can be best understood by considering its behavior in a simple experiment. If a layer of Bingham material of thickness h is placed between two rigid plates, one plate being rigidly fixed and the other having a force F applied to it as illustrated in Figure 1a, then the upper plate will remain stationary until the value of the shear stress in the Bingham material exceeds the critical value τ_0 . After the value of the shear stress has exceeded τ_0 , then the velocity of the upper plate v increases linearly with the shear stress τ as shown in Figure 1b. The slope of the line in Figure 1b, μ , is the viscosity of the Bingham material. Figure 1c shows the result of performing this same experiment on a rigid-ideally plastic material or Mises material.

In general the physical behavior of the Bingham material is such that it remains rigid under loading until the yield value of the stress is exceeded. After yielding, it flows in the following way. Let a differential volume of Newtonian fluid, a differential volume of Mises plastic solid, and a differential volume of Bingham material be subjected to the same deformation rate. Then the stress in the Bingham material is the sum of the stress in the Newtonian fluid and the Mises solid.

Limit analysis theorems are widely discussed in the literature of plasticity. These theorems provide a very useful means for determining upper and lower bounds on the yield load of a plastic body. The attractiveness of limit analysis in the theory of plasticity stems from the fact that boundary value problems of the mathematical theory of plasticity are very difficult to solve, and determining yield loads by solving the boundary value problems is prohibitively difficult, if not impossible. The boundary value problems for Bingham materials are even more difficult to solve than those for plastic materials; thus, it was considered to be of interest to determine how the limit analysis theorems of plasticity might be applied to Bingham materials.

General interest in the yeilding of Bingham materials arose from interest in the following problem in oil well drilling technology. The drilling fluid used in oil well drilling may be characterized to a first approximation as being a Bingham material. One function of this drilling fluid is to hold the rock chips or cuttings in suspension when circulation of the fluid is stopped. Since these cuttings are more dense than the fluid in which they are suspended, there is a force, due to gravity, acting to pull them downward. Due to the nature of the Bingham material, the particles will remain in suspension unless the gravity force is sufficient to cause the Bingham material to yield, then they will fall toward the bottom of the hole.

Having this practical problem in mind, it is of interest to determine the value of the gravity force, acting on a rigid sphere suspended in a Bingham material, which will cause yielding of the Bingham material; resulting in incipient motion of the sphere.

The following is a brief review of publications pertaining to the subject of this thesis. Bingham^{1*} developed

* Super-scrip numbers refer to the references at the end of the thesis

the one dimensional stress-deformation rate relations for the visco-plastic material. Prager and Hohenemser² formulated the three dimensional constitutive equations for the Bingham material. Oldroyd^{3, 4, 5, 6} solved several specific boundary value problems of the rectilinear flows of a Bingham material and developed boundary layer equations for the Bingham material. Limit analysis theorems for ideal plastics were developed by Drucker, Prager, and Greenburg⁷. Prager⁸ established extremum principles for the boundary value problem of the slow flow of a Bingham material. Paslay and Slibar^{9, 10, 11, 12} have solved several boundary problems for the visco-plastic and made practical applications of the results. Also Paslay and Slibar^{13, 14, 15} have developed constitutive equations for the gelling Bingham material. In the Russian Literature, Tybin¹⁶ and Andres¹⁷ have given approximate solutions to the problem of the sphere falling in a Bingham material.

In performing limit analysis on plastics, it is a common and useful practice to employ discontinuous velocity fields. It has been stated by Paslay and Slibar¹³ that in limit analysis of Bingham materials, the use of discontinuous velocity fields is not permissible. This thesis will show

that it is not necessary to exclude discontinuous velocity fields in limit analysis of Bingham materials. It will be shown here also that the yield force for the sphere in the Bingham material obtained by both Tybin¹⁶ and Andres¹⁷ is lower than the actual value of the yield load.

I. Incipient Flow of a Bingham Material

I.1. The Boundary Value Problem

In a system of rectangular Cartesian coordinates x_i an incompressible Bingham body occupying the three dimensional region V is bounded by the surface S . The body force F_i is specified throughout V . The surface traction T_i is prescribed on the portion S_T of S and the velocity is prescribed to vanish on the portion S_V of S .

The current values of the surface tractions are considered to have been reached in proportional loading. That is, as the values of T_i have increased, they have done so such that

$$T_i(x_1, x_2, x_3, t) = \lambda(t) \cdot p_i(x_1, x_2, x_3) \quad (I.1)$$

where $\lambda(t)$ is a scalar function of time, and $p_i(x_1, x_2, x_3)$ is the component of a vector.

I.2. Basic Relations

The deformation rate tensor d_{ij} is defined in Cartesian coordinates as

$$d_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (I.2)$$

The incompressibility of the Bingham material requires that

$$d_{ii} = \frac{\partial v_i}{\partial x_i} = 0 \quad (I.3)$$

(The summation convention that repeated indices are to be summed is used here.)

Since the inertia forces will be zero in all of the problems considered here, the stresses σ_{ij} must satisfy the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \quad (\text{I.4})$$

The stress deviation tensor is given by

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \quad (\text{I.5})$$

where δ_{ij} is the Kronecker Delta.

$$\sqrt{I_2} = \sqrt{2d_{ij}d_{ij}} \quad (\text{I.6})$$

$$\text{and } \sqrt{J_2} = \sqrt{\frac{1}{2} s_{ij} s_{ij}} \quad (\text{I.7})$$

are the positive square roots of the second invariants of the deformation rate and stress deviation tensors respectively.

The constitutive equations for the Bingham material are

$$2\mu d_{ij} = 0 \quad \text{if } \sqrt{J_2} < \tau_0 \quad (\text{I.8})$$

$$2\mu d_{ij} = \frac{\sqrt{J_2} - \tau_0}{\sqrt{J_2}} s_{ij} \quad \text{if } \sqrt{J_2} \geq \tau_0 \quad (\text{I.9})$$

μ and τ_0 are the viscosity and critical shear stress respectively. (See Figure 1.)

In order to invert Equation (I.9), multiply each side of this equation by itself, and sum on i and j obtaining

$$4\mu^2 d_{ij} d_{ij} = \frac{(\sqrt{J_2} - \tau_0)^2}{J_2} s_{ij} s_{ij} \quad \text{if } \sqrt{J_2} \geq \tau_0 \quad (\text{I.10})$$

which according to Equations (I.6) and (I.7) is

$$\mu^2 I_2 = (\sqrt{J_2} - \tau_0)^2 \quad \text{if } \sqrt{J_2} \geq \tau_0 \quad (\text{I.11})$$

or

$$\mu \sqrt{I_2} = \sqrt{J_2} - \tau_0 \quad \text{if } \sqrt{J_2} \geq \tau_0 \quad (\text{I.12})$$

From the substitution of Equation (I.12) into Equation (I.9), there results

$$s_{ij} \left(\frac{\mu \sqrt{I_2}}{\mu \sqrt{I_2} + \tau_0} \right) = 2 \mu d_{ij} \quad \sqrt{J_2} \geq \tau_0 \quad (\text{I.13})$$

or

$$s_{ij} = 2 \left(\mu + \frac{\tau_0}{\sqrt{I_2}} \right) d_{ij} \quad \text{if } \sqrt{J_2} \geq \tau_0 \quad (\text{I.14})$$

I.3. The State of Incipient Flow

Incipient flow is the state that exists in the body when it first yields. In the case of incipient flow of an

ideal plastic due to proportional loading, it is easy to see what the proper analytical description of the state of incipient flow should be since it is precisely the same state which persists in the body for all time unless the loads are reduced below the yield value. However, due to the presence of the viscous effect in the Bingham material the velocity and stress yields will change with any increase of the loads, the stresses and velocities, in general, increasing. This behavior makes the proper description of the state of incipient flow for a Bingham material less obvious. The analytical description of incipient flow is developed as follows:

As the surface tractions T_i applied to the body are increased in proportional loading, a state of impending visco-plastic flow will be reached such that any further increase of the surface tractions will cause the body to yield, producing regions of the body where the deformation rate is non-zero.

The state of incipient motion may be considered to be steady state visco-plastic flow at vanishingly small velocity. If δ is a small positive number, the steady state velocity field may be expressed as

$$v_i(x_1, x_2, x_3) = \delta q_i(x_1, x_2, x_3) \quad (I.15)$$

and the resulting deformation rate field may be expressed as

$$d_{ij}(x_1, x_2, x_3) = \delta \epsilon_{ij}(x_1, x_2, x_3)$$

Then by Equation (I.14)

$$S_{ij} = 2\mu\delta\epsilon_{ij} + \frac{2\tau_0 \delta \epsilon_{ij}}{\delta \sqrt{2\epsilon_{ij}\epsilon_{ij}}} \quad (I.16)$$

and

$$\lim_{\delta \rightarrow 0} S_{ij} = \frac{2\tau_0 d_{ij}}{\sqrt{2d_{ij}d_{ij}}} \quad (I.17)$$

Equation (I.12) may be written:

$$\mu\delta \sqrt{2\epsilon_{ij}\epsilon_{ij}} = \sqrt{J_2} - \tau_0 \quad (I.18)$$

then it is seen that as $\delta \rightarrow 0$, $\sqrt{J_2} \rightarrow \tau_0$.

Then at incipient flow

$$S_{ij} = \frac{2\tau_0}{\sqrt{I_2}} d_{ij} \quad (I.19)$$

and

$$\sqrt{J_2} = \tau_0 \quad (I.20)$$

in the regions of the body where the impending flow is occurring. In the remainder of the body

$$\sqrt{J_2} < \tau_0, \quad (I.21)$$

$$d_{ij} = 0, \quad (I.22)$$

and the stress deviation tensor is not expressible in terms of the deformation rate tensor.

Equations (I.19), (I.20), (I.21), and (I.22) show that for incipient flow, the viscous effect vanishes and that flow of a Bingham body at vanishingly small velocities is governed by the same relations as the flow of a Mises body, which is discussed extensively by Prager and Hodge in their book, Theory of Perfectly Plastic Solids¹⁸.

I.4 Limit Analysis Theorems

A velocity field which satisfies the equation of incompressibility, the velocity boundary condition, and which yields a stress field from the constitutive equations which satisfies equilibrium and the stress boundary condition, constitutes an exact solution to the boundary value problem of Section I.1. From the exact solution for vanishingly small velocities, the yield load could be determined. Obtaining an exact solution to a boundary problem for a Bingham body is, in general, a formidable task. However, with the understanding of incipient flow developed above, the limit analysis theorems of Drucker, Prager, and Greenburg⁷, may be applied to the boundary value problem of the Bingham material, giving a means of calculating upper and lower bounds on the yield loads, which is reasonably simple to carry out. These limit analysis

theorems which are proved in Appendix A may be stated as follows:

Statistically Admissible Stress Field:

A continuous stress field with piecewise continuous first partial derivatives is statically admissible if it satisfies the boundary condition

$$\sigma^*_{ij} n_j = T^*_i = \lambda^* p_i \quad (I.23)$$

on S_T , where n_j is the unit normal to the surface, the equilibrium Equations (I.17) throughout V , and the yield inequality

$$\sqrt{J_2^*} \leq \tau_0. \quad (I.24)$$

Lower Bound Theorem:

The load parameter λ^* associated with any statically admissible stress field is a lower bound on the yield load parameter.

The upper bound theorem will be stated here for continuous velocity fields, and then section I.5 will be devoted to extending this theorem to discontinuous velocity fields.

Kinematically Admissible Velocity Field

A continuous velocity field \mathbf{v}^*_1 having piecewise

continuous first partial derivatives is kinematically admissible if it vanishes on S_v , satisfies the equation of incompressibility.

$$d_{ii} = \frac{\partial v_i}{\partial x_i} = 0 \quad (I.25)$$

and is such that

$$\int T_i v_i^* dS + \int F_i v_i^* dv \geq 0 \quad (I.26)$$

Equation (I.26) requires that the velocity field be such that the external forces do positive work.

Upper Bound Theorem

A load parameter λ^* may be associated with any kinematically admissible velocity field through the relation

$$\lambda^* \int p_i v_i^* dS + \int F_i v_i^* dv = \tau_o \int \sqrt{I_2^*} dv \quad (I.27)$$

The load parameter λ^* associated with any kinematically admissible velocity field gives an upper bound on the yield load parameter.

I.5 Discontinuous Velocity Fields

In performing limit analysis with plastic materials, it is found to be very convenient to employ discontinuous velocity fields. Due to the fact that velocity discontinuities

cannot occur in a Bingham material during actual flow, it has been thought (see Paslay and Slibar¹³, page 108) that discontinuous velocity fields are not permissible for limit analysis of Bingham materials. The possibility of velocity discontinuities occurring in either of these materials can best be understood by considering their constitutive relations. During actual flow an ideally plastic or Mises material has the constitutive relation

$$s_{ij} = \frac{2\tau_o}{\sqrt{I_2}} \quad d_{ij} = \frac{2\tau_o}{\sqrt{2d_{ij}d_{ij}}} d_{ij} \quad (I.28)$$

At a velocity discontinuity, one of the shear components of the deformation rate tensor becomes infinite while all other components become zero. By inspection of equation (I.28), it is seen that at such a discontinuity the stress will remain finite in an ideal plastic. In fact, in a Mises material the shear stress transmitted across the discontinuity is τ_o (see Prager and Hodge¹⁸, page 163). Inspection of Equation (I.14) reveals that in a Bingham material an infinite shear stress will occur across a velocity discontinuity, should one exist during actual flow. Since an infinite shear stress is obviously not possible, then the conclusion is that the velocity fields

of Bingham materials are never discontinuous during actual flow. However, it will be shown in the following that because the velocities are vanishingly small, the stress can be considered to remain finite across a velocity discontinuity during incipient flow, and that discontinuous velocity fields may be used in performing limit analysis for Bingham materials.

Since the Bingham material is not elastic, the stress power $(S_{ij}d_{ij})$ is entirely dissipative, and the rate at which energy is being dissipated in a differential volume of Bingham material, during deformation, is given by

$$dD = S_{ij}d_{ij}dv \quad (I.29)$$

During incipient flow S_{ij} is given by Equation (I.19) and the energy dissipation rate is

$$dD = \frac{2\tau_0 d_{ij}d_{ij}dv}{\sqrt{2d_{ij}d_{ij}}} = \tau_0 \sqrt{I_2}dv \quad (I.30)$$

Then from Equation (I.30) it is recognized that the term on the right hand side of Equation (I.27) is the rate at which energy is being dissipated by the fictitious velocity field, v_i^* , during incipient flow. The upper bound theorem simply equates the rate at which work is done by the external forces to the rate at which energy is dissipated

for the fictitious velocity field v_i^* . Thus it is the rate of energy dissipation that needs to be examined at discontinuities occurring in a velocity field during incipient flow.

The type of discontinuity to be considered in the following is simply an idealization of a continuous velocity distribution in which the tangential component of the velocity changes very rapidly across a thin transition layer. Let the transition layer be bounded by the surfaces S_h and S_k (see Figure 2). Let S_m be the median surfaces of S_h and S_k . At a generic point P on the median surface S_m , let there be a system of Cartesian coordinates such that x_1 is tangent to the median surface and in the direction of the tangential velocity component at P , x_2 is normal to the median surface, and x_3 has the customary right hand relation to x_1 and x_2 .

Across the transition layer the tangential velocity component changes quite rapidly. That is to say

$$\frac{\partial v_1}{\partial x_2} \gg \frac{\partial v_i}{\partial x_j} \quad i, j \neq 1, 2 \quad (I.31)$$

Accordingly

$$d_{12} \gg d_{ij} \quad i, j \neq 1, 2 \quad (I.32)$$

The energy dissipation rate in the differential volume dV at the generic point P is

$$dD = S_{ij} d_{ij} t dS \quad (I.33)$$

where t is the thickness of the transition layer at P and dS is the differential area of the median surface S_m at P . Considering the constitutive Equation (I.14), Equation (I.29) becomes

$$dD = 2d_{ij} d_{ij} \left(\mu + \frac{\tau_o}{\sqrt{I_2}} \right) t dS \quad (I.34)$$

As t becomes small, d_{12} remains finite and is given by

$$d_{12} = \frac{1}{2} \left(\frac{v_1^h - v_1^k}{t} \right) \quad (I.35)$$

v_1^h and v_1^k being the values of v_1 at the surfaces S_h and S_k respectively. Also as t becomes small, the squares of the deformation rates $(d_{ij})^2$ for $i, j \neq 1, 2$ becomes negligibly small compared to $(d_{12})^2$,

$$d_{ij} d_{ij} \rightarrow 2(d_{12})^2 \quad (I.36)$$

and

$$I_2 \rightarrow 4 (d_{12})^2 \quad (I.37)$$

such that Equation (I.34) becomes

$$dD = \frac{|(\mathbf{v}_1^h - \mathbf{v}_1^k)|}{t} \left[\mu \frac{|(\mathbf{v}_1^h - \mathbf{v}_1^k)|}{t} + \tau_o \right] t dS \quad (I.38)$$

As was done in Section I.3, let $\mathbf{v}_i = \delta \mathbf{q}_i$ then

$$dD = \delta |(q_1^h - q_1^k)| \left[\frac{\mu \delta |(q_1^h - q_1^k)|}{t} + \tau_o \right] dS \quad (I.39)$$

At incipient flow δ is vanishingly small, and the expres-

$$\text{sion} \left[\frac{\mu \delta |(q_1^h - q_1^k)|}{t} + \tau_o \right] \rightarrow \tau_o \quad (I.40)$$

Thus the energy dissipation rate in the very thin but finite transition layer is given by

$$D = \tau_o \int_{S_m} |\Delta \mathbf{v}_m| dS \quad \text{where } \Delta \mathbf{v}_m = (\mathbf{v}_1^h - \mathbf{v}_1^k) \quad (I.41)$$

$\Delta \mathbf{v}_m$ represents the velocity "jump" occurring across the thin transition layer. The thin transition layer may then be represented by a velocity discontinuity $\Delta \mathbf{v}_m$ and the dissipation in the transition layer is determined by integrating $\Delta \mathbf{v}_m$ over the surface of discontinuity.

With this understanding of a velocity discontinuity, the upper bound theorem may be restated as follows:

Upper Bound Theorem for Discontinuous
Velocity Fields

A piecewise continuous velocity field v_i^* having piecewise continuous first partial derivatives is kinematically admissible if it vanishes on S_v , satisfies the equation of incompressibility

$$d_{ii} = \frac{\partial v_i}{\partial x_i} = 0 \quad (I.42)$$

and is such that

$$\int T_i v_i^* dS + \int F_i v_i^* dV \geq 0 \quad (I.43)$$

A load parameter λ^* may be associated with any kinematically admissible velocity field through the relation

$$\lambda^* \int p_i v_i^* dS + \int F_i v_i^* dV = \tau_0 \int \sqrt{I_2^*} dV + \sum_n \tau_0 \int_{S_n} |\Delta v_n^*| dS_n \quad (I.44)$$

where the sum in the last term of Equation (I.44) is extended over all surfaces of discontinuity. The load parameter λ^* associated with any kinematically admissible velocity field gives an upper bound yield load parameter.

II. Incipient Motion of a Spherical Body Suspended in a Bingham Material

II.1. The Problem

An infinite region of Bingham material having a density ρ_B and a critical shear stress τ_o has suspended in it a rigid, spherical body whose density is $\rho_S > \rho_B$, and whose radius is R . Due to the presence of a gravity field and the density difference $(\rho_S - \rho_B)$ there is a downward force F ,

$$F = \frac{4}{3} \pi R^3 (\rho_S - \rho_B) g \quad (\text{II.1})$$

acting on the sphere. (See Figure 3). Up to the point of yielding, the Bingham material resists this force and the sphere remains suspended. The force, F_Y , which causes the Bingham material to yield is proportional to $\tau_o R^2$, thus it is the dimensionless ratio

$$\frac{3F}{4\pi \tau_o R^2} = \frac{R(\rho_S - \rho_B)g}{\tau_o} \quad (\text{II.2})$$

which determines yielding. When the ratio

$$\frac{R(\rho_S - \rho_B)g}{\tau_o} \quad (\text{II.3})$$

reaches the critical value, the sphere will be in a state

of incipient motion. That is, for given values of the material constant there is some maximum size sphere which can be suspended, or for a given size sphere there is some maximum density difference $(\rho_S - \rho_B)$ for which the sphere will not fall. During incipient falling, the Bingham material is considered to adhere to the surface of the sphere.

In the following, upper and lower bounds will be placed on

$$\frac{R (\rho_S - \rho_B) g}{\tau_0} \quad (\text{II.4})$$

by using the limit analysis theorems discussed in Section I.

II.2. Lower Bound

In the present problem, the portion of the boundary S_T is the surface of the sphere, and the stress boundary condition is

$$F = \int_{S_T} T_3 \, dS \quad (\text{II.5})$$

$$\int_{S_T} T_1 \, dS = \int_{S_T} T_2 \, dS = 0 \quad (\text{II.6})$$

where T_3 is the vertical component of the surface traction and T_1 and T_2 are horizontal components. (See Figure 3).

Since gravity and bouyancy forces have been accounted for through the density difference ($P_S - P_B$) the body forces are zero and the equilibrium Equations I.4 become

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad (\text{II.7})$$

According to the lower bound theorem, any continuous stress field which satisfies the equilibrium Equations II.7, the boundary conditions II.5 and II.6, and the yield inequality I.19 is statically admissible and will give a lower bound on F .

Of those stress fields investigated, the greatest lower bound is given by

$$\sigma_r = \sigma_\theta = \sigma_\phi = \frac{24}{25} \left[10 \left(\frac{R}{r} \right) - 6 \left(\frac{R}{r} \right)^2 \right] \tau_o \cos \theta \quad (\text{II.8})$$

$$\tau_{r\theta} = \frac{24}{25} \left[5 \left(\frac{R}{r} \right) - 6 \left(\frac{R}{r} \right)^2 \right] \tau_o \sin \theta \quad (\text{II.9})$$

$$\tau_{r\phi} = \tau_{\theta\phi} = 0 \quad (\text{II.10})$$

where the stresses given above are the physical components of the stress tensor referred to a system of spherical coordinates. (See Figure 4).

In Appendix B it is shown that the stress field given above is statically admissible and that it yields the lower bound

$$\frac{3F}{4\pi\tau_0 R^2} = \frac{R(\rho_S - \rho_B)g}{\tau_0} \geq 5.75 \quad (\text{II.11})$$

II.3 Upper Bound

For the sphere falling at speed U_0

$$FU_0 = \int_{S_T} T_i v_i^* dS \quad (\text{II.12})$$

and $F_i = 0$, thus inequality I.43 is satisfied by any velocity field which is consistent with the adherence of the Bingham material to the falling sphere.

Then according to the upper bound theorem, any velocity field which satisfies the equation of incompressibility I.42 and satisfies the adherence boundary condition on the surface of the sphere will give an upper bound on the ratio

$$\frac{3F}{4\pi\tau_0 R^2} = \frac{R(\rho_S - \rho_B)g}{\tau_0} \quad \text{through}$$

Equation I.44 which in this application becomes

$$FU_0 = \tau_0 \int \sqrt{I_2}^* dv + \sum_n \tau_0 \int_{S_n} |\Delta v_n^*| ds_n \quad (\text{II.13})$$

Among the velocity fields investigated, the one giving the least upper bound is

$$\left. \begin{aligned} v_r^* &= -U_0 \cos \theta \left(\frac{2R}{r} - 1 \right) \\ v_\theta^* &= U_0 \sin \theta \left(\frac{R}{r} - 1 \right) \end{aligned} \right\} R \leq r \leq 2R \quad (\text{II.14})$$

where the velocities given above are the physical components of the velocity vector referred to spherical coordinates.

(See Figure 4). Since the Bingham material adheres to the sphere, the boundary condition at $r = R$ is

$$v_r^* (R) = -U_0 \cos \theta \quad (\text{II.15})$$

$$v_\theta^* (R) = U_0 \sin \theta \quad (\text{II.16})$$

Both velocity components must vanish at the outer flow boundary $r = 2R$, thus the boundary condition there is

$$v_r^* (2R) = v_\theta^* (2R) = 0 \quad (\text{II.17})$$

Then Equations II.14 represent a velocity field which has two finite discontinuities in the component v_θ , i.e.

$$\text{at } r = R \left\{ \begin{aligned} v_\theta^* (R-0) &= U_0 \sin \theta \\ v_\theta^* (R+0) &= 0 \end{aligned} \right. \quad (\text{II.18})$$

$$\text{at } r = 2R \begin{cases} v_{\theta}^* (2R - 0) = \frac{U_o}{2} \sin \theta \\ v_{\theta}^* (2R + 0) = 0 \end{cases} \quad (\text{II.19})$$

It is shown in Appendix C that the velocity field above is kinematically admissible, and that it yields the upper bound

$$\frac{3F}{4\pi \tau_o R^2} = \frac{R(\rho_S - \rho_B) g}{\tau_o} \leq 12.25 \quad (\text{II.20})$$

II.4 Comparison with other Results

It has been shown in the two preceding sections that

$$5.75 \leq \frac{R(\rho_S - \rho_B) g}{\tau_o} \leq 12.25 \quad (\text{II.21})$$

Tybin¹⁶ gives an approximate solution of the boundary value problem of the slow viscoplastic flow of a Bingham material about a sphere in which he obtains the value

$$\frac{R(\rho_S - \rho_B) g}{\tau_o} = 3 \quad (\text{II.22})$$

for the yield ratio.

Andres¹⁷ applies dimensional analysis to the problem of a sphere falling in a Bingham material and obtains

$$\frac{R(\rho_S - \rho_B) g}{\tau_o} = 2.36 \quad (\text{II.23})$$

Since limit analysis places rigorous bounds on the yield load of a body, the results obtained here show that both these values are lower than the actual value of the yield ratio.

Conclusions

From this research the following conclusions may be drawn:

1. The limit analysis theorems of plasticity may be usefully applied to Bingham materials.
2. Discontinuous velocity fields may be employed in performing limit analysis of Bingham materials.
3. Since the state of incipient flow of a Bingham material is identical to the state of steady flow of a rigid-ideally plastic or Mises material, and since 2. above is true, then limit analysis for a body composed of Bingham material is exactly the same as limit analysis for a body composed of a Mises plastic.
4. The value of the dimensionless ratio $\frac{R(\rho_S - \rho_B)g}{\tau_0}$ for which incipient motion of a sphere suspended in a Bingham material will occur is

$$5.75 \leq \frac{R(\rho_S - \rho_B)g}{\tau_0} \leq 12.25.$$

Appendix A

Proofs of the Limit Analysis Theorems

The proofs given here for the Bingham material are essentially the same as the proofs given by Prager and Hodge¹⁸, pages 247 to 251, for the elastic-plastic or Prandtl-Reuss material.

A.1 Lower Bound Theorem

Statically Admissible Stress Field:

A continuous stress field with piecewise continuous first partial derivatives is statically admissible if it satisfies the boundary condition

$$\sigma_{ij} n_j = T_i = \lambda p_i \quad (\text{A.1.1})$$

on S_T , the equilibrium equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0, \quad (\text{A.1.2})$$

throughout V , and the yield inequality $\sqrt{J_2} \leq \tau_0$. (A.1.3)

throughout V .

Lower Bound Theorem:

The load parameter λ^* associated with any statically

admissible stress field is a lower bound on the yield load parameter.

In order to prove the lower bound theorem, it is necessary to first prove two auxiliary theorems, theorem I and the theorem of virtual work.

Theorem I: Let σ_{ij}^* be a statically admissible stress field for the surface tractions $T_i^* = \lambda^* p_i$ and let σ_{ij} and d_{ij} be the actual stress and deformation rate fields that exist during incipient flow, then

$$(\sigma_{ij} - \sigma_{ij}^*) d_{ij} \geq 0 \quad (A.1.4)$$

Proof I: From the definition of the stress deviation tensor

$$s_{ij} = \sigma_{ij} - \delta_{ij} \frac{1}{3} \sigma_{kk} \quad (A.1.5)$$

then

$$(\sigma_{ij} - \sigma_{ij}^*) d_{ij} = (s_{ij} - s_{ij}^*) d_{ij} + \delta_{ij} \frac{1}{3} (\sigma_{kk} - \sigma_{kk}^*) d_{ij} \quad (A.1.6)$$

$$\delta_{ij} d_{ij} (\sigma_{kk} - \sigma_{kk}^*) = d_{ii} (\sigma_{kk} - \sigma_{kk}^*) \quad (A.1.7)$$

but $d_{ii} = 0$ due to incompressibility so that

$$(\sigma_{ij} - \sigma_{ij}^*) d_{ij} = (s_{ij} - s_{ij}^*) d_{ij} \quad (A.1.8)$$

At incipient flow the Bingham body consists of two types of regions, those where $\sqrt{J_2} < \tau_0$ and those where $\sqrt{J_2} = \tau_0$. In Section I.3 it was shown that where $\sqrt{J_2} < \tau_0$

$d_{ij} = 0$, and where $\sqrt{J_2} = \tau_0$

$$d_{ij} = \frac{\sqrt{I_2}}{2\tau_0} s_{ij} \quad (\text{A.1.9})$$

Then in the regions where $\sqrt{J_2} < \tau_0$

$$(s_{ij} - s_{ij}^*) d_{ij} = 0, \quad (\text{A.1.10})$$

and where $\sqrt{J_2} = \tau_0$

$$(s_{ij} - s_{ij}^*) d_{ij} = (s_{ij} - s_{ij}^*) \frac{\sqrt{I_2}}{2\tau_0} s_{ij} \quad (\text{A.1.11})$$

$$= \frac{\sqrt{I_2}}{\tau_0} \left(\frac{1}{2} s_{ij} s_{ij} - \frac{1}{2} s_{ij} s_{ij}^* \right) = \frac{\sqrt{I_2}}{\tau_0} \left(\tau_0^2 - \frac{1}{2} s_{ij} s_{ij}^* \right) \quad (\text{A.1.12})$$

$\frac{\sqrt{I_2}}{\tau_0}$ is a positive quantity, and by the Schwartzian inequality

$$s_{ij}^* s_{ij} \leq \sqrt{s_{ij} s_{ij}} \sqrt{s_{ij}^* s_{ij}^*} \quad (\text{A.1.13})$$

Since s_{ij}^* is statically admissible

$$\sqrt{\frac{1}{2} s_{ij}^* s_{ij}^*} \leq \tau_0 \quad (\text{A.1.14})$$

and since σ_{ij} is the actual stress field existing in the body during incipient flow

$$\sqrt{\frac{1}{2} s_{ij} s_{ij}} = \tau_0 \quad (\text{A.1.15})$$

Then

$$s_{ij} s_{ij}^* \leq 2 \tau_0^2 \quad (\text{A.1.16})$$

showing that the right hand side of (A.1.12) is positive definite and Theorem I is proved.

The Theorem of Virtual Work

Let σ_{ij} denote a stress field which satisfies the equilibrium Equations (I.4) and T_i denote the surface tractions in equilibrium with these stresses. Let v_i be a velocity field which is entirely independent of the stress field, i.e., the stress and velocities are not related through any particular constitutive equation, then the following equation holds

$$\int \sigma_{ij} d_{ij} dV = \int T_i v_i dS + \int F_i v_i dV \quad (\text{A.1.17})$$

Proof of the Theorem of Virtual Work

By Equation (I.2)

$$\sigma_{ij} d_{ij} = \frac{1}{2} \sigma_{ij} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (\text{A.1.18})$$

and since i and j are both summed

$$\sigma_{ij} d_{ij} = \sigma_{ij} \frac{\partial v_i}{\partial x_j} \quad (\text{A.1.19})$$

$$\frac{\partial}{\partial x_j} (\sigma_{ij} v_i) = \frac{\partial \sigma_{ij}}{\partial x_j} v_i + \sigma_{ij} \frac{\partial v_i}{\partial x_j} \quad (\text{A.1.20})$$

Since σ_{ij} satisfies the equilibrium Equations (I.4)

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -F_i \quad (\text{A.1.21})$$

so that

$$\int \sigma_{ij} d_{ij} dv = \int \frac{\partial}{\partial x_j} (\sigma_{ij} v_i) dv + \int F_i v_i dv \quad (\text{A.1.22})$$

By Green's Theorem

$$\int \frac{\partial}{\partial x_j} (\sigma_{ij} v_i) dv = \int (\sigma_{ij} v_i) n_j dS \quad (\text{A.1.23})$$

$$\text{but } \sigma_{ij} n_j = T_i .$$

Then

$$\int \sigma_{ij} d_{ij} dv = \int T_i v_i dS + \int F_i v_i dv \quad (\text{A.1.24})$$

and the theorem of virtual work is proved.

Proof of the Lower Bound Theorem

According to Theorem I

$$\int (\sigma_{ij} - \sigma_{ij}^*) d_{ij} dv \geq 0 \quad (\text{A.1.25})$$

The principle of virtual work (A.1.24) transforms (A.1.25) into

$$\int T_i v_i dS + \int F_i v_i dV - \int T_i^* v_i dS - \int F_i v_i dV \geq 0 \quad (\text{A.1.26})$$

the same body force F_i being associated with either of the two equilibrium states of stress. Since $T_i = \lambda p_i$ and $T_i^* = \lambda^* p_i$ then

$$(\lambda - \lambda^*) \int p_i v_i dV \geq 0 \quad (\text{A.1.27})$$

The integral $\lambda \int p_i v_i dV$ represents rate which the applied loads do work during incipient flow and is therefore positive. Since λ is a positive constant $\int p_i v_i dV$ is positive and therefore

$$\lambda \geq \lambda^* \quad (\text{A.1.28})$$

concluding the proof of the Lower Bound Theorem.

A.2 Upper Bound Theorem

Since it is shown in Section I.5 that a discontinuous velocity field is only a limiting case of a continuous velocity field, the Upper Bound Theorem is proved here for continuous velocity fields.

Kinematically Admissible Velocity Field

A continuous velocity field v_i^* having piece-wise continuous first partial derivatives is kinematically admissible if it vanishes on S_v , satisfies the equation of incompressibility

$$d_{ii} = \frac{\partial v_i}{\partial x_i} = 0 \quad (A.2.1)$$

and is such that

$$\int T_i v_i^* dS + \int F_i v_i^* dV \geq 0 \quad (A.2.2)$$

A load parameter λ^* may be associated with any kinematically admissible velocity field through the relation

$$\lambda^* \int p_i v_i^* dS + \int F_i v_i^* dV = \tau_0 \int \sqrt{I_2^*} dV \quad (A.2.3)$$

Upper Bound Theorem

The load parameter λ^* associated with any kinematically admissible velocity field gives an upper bound on the yield load parameter.

In order to prove the Upper Bound Theorem, it is first necessary to prove an auxiliary theorem, Theorem II.

Theorem II: Let d_{ij}^* be the deformation rate field derived from the kinematically admissible velocity field v_i^* and let

$$\sigma_{ij} d_{ij}^* \leq \tau_0 \sqrt{I_2^*} \quad (A.2.4)$$

Proof II: As was shown in the preceding section of this appendix

$$\sigma_{ij} d_{ij}^* = s_{ij} d_{ij}^* \quad \text{due to incompressibility.}$$

According to Schwarz' inequality

$$s_{ij} d_{ij}^* \leq \sqrt{s_{ij} s_{ij}} \sqrt{d_{ij}^* d_{ij}^*} \quad (\text{A.2.5})$$

$$\text{or } s_{ij} d_{ij}^* \leq \sqrt{\frac{1}{2} s_{ij} s_{ij}} \sqrt{2 d_{ij}^* d_{ij}^*} \quad (\text{A.2.6})$$

$$\text{By definition } \sqrt{2 d_{ij}^* d_{ij}^*} = \sqrt{I_2^*} \quad \text{and}$$

$$\text{during incipient flow } \sqrt{\frac{1}{2} s_{ij} s_{ij}} = \sqrt{J_2} \leq \tau_0$$

so that

$$s_{ij} d_{ij}^* \leq \tau_0 \sqrt{I_2^*} \quad (\text{A.2.7})$$

which concludes the proof of Theorem II.

Proof of the Upper Bound Theorem

According to (A.2.4)

$$\int \sigma_{ij} d_{ij}^* dv \leq \tau_0 \int \sqrt{I_2^*} dv \quad (\text{A.2.8})$$

The principle of virtual work (A.1.24) transforms (A.1.25)

into

$$\int T_i v_i^* dS + \int F_i v_i^* dv \leq \tau_0 \int \sqrt{I_2^*} dv \quad (\text{A.2.9})$$

Since $T_i = \lambda p_i$ then

$$\lambda \int p_i v_i^* dS + \int F_i v_i^* dv \leq \tau_0 \int \sqrt{I_2^*} dv \quad (\text{A.2.10})$$

where λ is the load parameter associated with the actual yield load. From (A.2.10) it follows that for any load parameter λ^* which is associated with a kinematically admissible velocity field v_i^* through Equation (A.2.3)

$$\lambda^* \geq \lambda \quad (\text{A.2.11})$$

and the upper bound theorem is proved.

Appendix B

Obtaining the Lower Bound

The stress field given in Section II.2 is

$$\sigma_r = \sigma_\theta = \sigma_\phi = \frac{24}{25} \tau_0 \left(10 \frac{R}{r} - 6 \frac{R^2}{r^2} \right) \cos\theta \quad (B.1)$$

$$\tau_{r\theta} = \frac{24}{25} \tau_0 \left(5 \frac{R}{r} - 6 \frac{R^2}{r^2} \right) \sin\theta \quad (B.2)$$

$$\tau_{r\phi} = \tau_{\theta\phi} = 0 \quad (B.3)$$

In spherical coordinates the equilibrium equations are
(see Hill 19, page 344.)

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{1}{r} (2\sigma_r - \sigma_\theta - \sigma_\phi + \tau_{r\theta} \cot\theta) = 0 \quad (B.4)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{1}{r} [(\sigma_\theta - \sigma_\phi) \cot\theta + \tau_{r\theta}] = 0 \quad (B.5)$$

$$\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \sigma_\phi}{\partial \phi} + \frac{1}{r} (3\tau_{r\phi} + 2\tau_{\theta\phi} \cot\theta) = 0 \quad (B.6)$$

In the present case Equation (B.6) is identically zero and

(B.4) and (B.5) reduce to

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r} \tau_{r\theta} \cot\theta = 0 \quad (B.7)$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{3}{r} \tau_{r\theta} = 0 \quad (B.8)$$

The necessary derivatives of the stresses are

$$\frac{\partial \sigma_r}{\partial r} = \frac{24}{25} \tau_o \left[-10 \frac{R}{r^2} + 12 \frac{R^2}{r^3} \right] \cos \theta \quad (\text{B.9})$$

$$\frac{\partial \sigma_\theta}{\partial \theta} = \frac{-24}{25} \tau_o \left[10 \frac{R}{r} - 6 \frac{R^2}{r^2} \right] \sin \theta \quad (\text{B.10})$$

$$\frac{\partial \tau_{r\theta}}{\partial r} = \frac{24}{25} \tau_o \left[-5 \frac{R}{r^2} + 12 \frac{R^2}{r^3} \right] \sin \theta \quad (\text{B.11})$$

$$\frac{\partial \tau_{r\theta}}{\partial \theta} = \frac{24}{25} \tau_o \left[5 \frac{R}{r} - 6 \frac{R^2}{r^2} \right] \cos \theta \quad (\text{B.12})$$

and it is seen that the equilibrium equations are satisfied.

Since

$$\sigma_r = \sigma_\theta = \sigma_\phi \quad (\text{B.13})$$

the normal components of the stress deviation tensor

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \quad (\text{B.14})$$

vanish

$$s_{rr} = s_{\theta\theta} = s_{\phi\phi} = 0 \quad (\text{B.15})$$

and the only non-vanishing component is

$$s_{r\theta} = \tau_{r\theta} \quad (\text{B.16})$$

In spherical coordinates the second invariant of the stress deviation tensor,

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \quad (\text{B.17})$$

$$\text{is } J_2 = \frac{1}{2} (s_{rr}^2 + s_{\theta\theta}^2 + s_{\phi\phi}^2) + s_{r\theta}^2 + s_{\theta\phi}^2 + s_{r\phi}^2 \quad (\text{B.18})$$

For this case

$$J_2 = \tau_{r\theta}^2 \quad (\text{B.19})$$

At the surface of the sphere $r = R$ and

$$\sqrt{J_2} = |\tau_{r\theta}| = \frac{24}{25} \tau_0 \sin\theta < \tau_0 \quad (\text{B.20})$$

$$\frac{d\tau_{r\theta}}{dr} = \frac{24}{25} \tau_0 \left[-\frac{5R}{r^2} + 12 \frac{R^2}{r^3} \right] \sin\theta \quad (\text{B.21})$$

Then $\tau_{r\theta}$ has a maximum at

$$r = \frac{12R}{5} \quad \theta = \pi/2 \quad \text{where}$$

$$\sqrt{J_2} = \frac{24}{25} \tau_0 \left[\frac{24}{12} - \frac{25}{24} \right] = \tau_0 \quad (\text{B.22})$$

$$\text{For } r > \frac{12R}{5} \quad \sqrt{J_2} < \tau_0.$$

Thus it is seen that the yield inequality is satisfied, and that the stress represented by Equations (B.1), (B.2), and (B.3) is statically admissible.

The force F associated with the stress field is obtained by integrating the vertical components of the surface tractions over the surface of the sphere. From Figure 4 it is seen that

$$dF = \left(\sigma_r \Big|_{r=R} \cos\theta - \tau_{r\theta} \Big|_{r=R} \sin\theta \right) dA \quad (B.23)$$

The differential area dA is

$$dA = R^2 \sin\theta \, d\theta \, d\phi \quad (B.24)$$

Then the force F is

$$F = \int_0^{2\pi} \int_0^\pi \left(\sigma_r \Big|_{r=R} \cos\theta - \tau_{r\theta} \Big|_{r=R} \sin\theta \right) \sin\theta \, R^2 \, d\theta \, d\phi \quad (B.25)$$

Integrating (B.25) with respect to ϕ yields

$$F = 2\pi R^2 \int_0^\pi \left(\sigma_r \Big|_{r=R} \cos\theta - \tau_{r\theta} \Big|_{r=R} \sin\theta \right) \sin\theta \, d\theta \quad (B.26)$$

Then for the stress field given by Equations (B.1), (B.2), and (B.3)

$$\frac{F}{\tau_o R^2} = \frac{48\pi}{25} \int_0^\pi (4\cos^2\theta + \sin^2\theta) \sin\theta \, d\theta \quad (B.27)$$

Upon performing the integration in Equation (B.27) there results

$$\frac{F}{\tau_o R^2} = 24.2 \quad (B.28)$$

which gives

$$\frac{3F}{4\pi \tau_o R} = \frac{R(\rho_S - \rho_B) g}{\tau_o} = 5.75 \quad (B.29)$$

Appendix C
Obtaining the Upper Bound

The velocity field given in Section II.3 is

$$\left. \begin{aligned} v_r &= -U_0 \cos\theta \left(\frac{2R}{r} - 1 \right) \\ v_\theta &= U_0 \sin\theta \left(\frac{R}{r} - 1 \right) \\ v_\phi &= 0 \end{aligned} \right\} R \leq r \leq 2R \quad (C.1)$$

v_r , v_θ , and v_ϕ being the components of the velocity vector. (See Figure 4.)

$$\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta + v_\phi \vec{e}_\phi$$

In spherical coordinates the elements of the deformation rate tensor are (See Hill¹⁹.)

$$d_{rr} = \frac{\partial v_r}{\partial r} \quad (C.2)$$

$$d_{\theta\theta} = \frac{1}{r} \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) \quad (C.3)$$

$$d_{\phi\phi} = \frac{1}{r \sin\theta} \left(\frac{\partial v_\phi}{\partial \phi} + v_r \sin\theta + v_\theta \cos\theta \right) \quad (C.4)$$

$$d_{r\theta} = \frac{1}{2} \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \quad (C.5)$$

$$d_{r\phi} = \frac{1}{2} \left(\frac{1}{r \sin\theta} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right) \quad (C.6)$$

$$d_{\theta\phi} = \frac{1}{2r \sin\theta} \left(\sin\theta \frac{\partial v_\phi}{\partial \theta} - v_\phi \cos\theta + \frac{\partial v_\theta}{\partial \phi} \right) \quad (C.7)$$

For the velocity field given by Equations (C.1).

$$\frac{\partial v_r}{\partial r} = -U_o \cos\theta \left(-\frac{2R}{r^2} \right) \quad (C.8)$$

$$\frac{\partial v_r}{\partial \theta} = U_o \sin\theta \left(\frac{2R}{r} - 1 \right) \quad (C.9)$$

$$\frac{\partial v_\theta}{\partial r} = U_o \sin\theta \left(\frac{-R}{r^2} \right) \quad (C.10)$$

$$\frac{\partial v_\theta}{\partial \theta} = U_o \cos\theta \left(\frac{R}{r} - 1 \right) \quad (C.11)$$

Then the non-zero elements of the deformation rate tensor are

$$d_{rr} = \frac{2U_o}{R} \cos\theta \left(\frac{R}{r} \right)^2 \quad (C.12)$$

$$d_{\theta\theta} = d_{\phi\phi} = -\frac{U_o}{R} \left(\frac{R}{r} \right)^2 \cos\theta \quad (C.13)$$

In order for the velocity field to be kinematically admissible, the deformation rate tensor must satisfy the equation of incompressibility.

$$d_{ii} = d_{rr} + d_{\theta\theta} + d_{\phi\phi} = 0 \quad (C.14)$$

It is easy to determine by inspection that this is satisfied.

Since the Bingham material is considered to adhere to the surface of the sphere, the boundary condition at $r = R$ is

$$v_r \Big|_{r=R} = -U_o \cos\theta \quad (C.15)$$

$$v_\theta \Big|_{r=R} = U_o \sin\theta \quad (C.16)$$

But from the second of Equations (C.1)

$$v_\theta \Big|_{r=R} = 0 \quad (C.17)$$

thus there is a discontinuity at $r = R$, i.e.,

$$v_\theta (R-0) = U_o \sin\theta \quad (C.18)$$

$$v_\theta (R+0) = 0$$

producing a velocity jump

$$\Delta v_1 = U_o \sin\theta \quad \text{at } r = R.$$

At the outer boundary of the region of flow $r = 2R$, both velocity components should vanish, but

$$v_\theta \Big|_{r=2R} = -\frac{1}{2} U_o \sin\theta \quad (C.19)$$

producing a velocity jump

$$\Delta v_2 = \frac{1}{2} U_o \sin\theta \quad \text{at } r = 2R.$$

After including these discontinuities in the velocity field, it is seen that this velocity field satisfies the boundary conditions. The condition given by inequality (I.40) may be satisfied simply by choosing the proper

direction for the force acting on the sphere. Then since the velocity field being considered satisfies incompressibility, the boundary conditions, and inequality (I.40), it is kinematically admissible.

The second invariant of the deformation rate tensor is

$$I_2 = 2 d_{ij} d_{ij} \quad (C.20)$$

for this case I_2 is

$$I_2 = 2 (d_{rr}^2 + d_{\theta\theta}^2 + d_{\phi\phi}^2) \quad (C.21)$$

$$I_2 = \frac{12 U_o^2}{R^2} \cos^2 \theta \left(\frac{R}{r}\right)^4 \quad (C.22)$$

As is shown in Section II.3, that for this problem the force associated with a velocity field is given by Equation (II.13) which is

$$F U_o = \tau_o \int \sqrt{I_2} dV + \sum_n \tau_o \int_{S_n} |\Delta v_n| dS_n \quad (C.23)$$

Then for the velocity field being considered

$$\begin{aligned} F U_o &= \tau_o \int_0^{2\pi} \int_0^\pi \int_R^{2R} \frac{U_o}{R} \sqrt{12} \cos^2 \theta \left(\frac{R}{r}\right)^2 r^2 \sin \theta dr d\theta d\phi \\ &+ \tau_o \int_0^{2\pi} \int_0^\pi (U_o \sin \theta) R^2 \sin \theta d\theta d\phi \end{aligned}$$

$$+ \tau_o \int_0^{2\pi} \int_0^\pi \left(\frac{U_o}{2} \sin\theta \right) (2R)^2 \sin\theta \, d\theta d\phi \quad (C.24)$$

After integrating with respect ϕ and dividing through by $U_o \tau_o R^2$ Equation (C.24) becomes

$$\begin{aligned} \frac{F}{\tau_o R^2} &= \frac{2\sqrt{12}\pi}{R} \int_0^\pi \int_R^{2R} |\cos\theta| \sin\theta \, dr d\theta \\ &+ 6\pi \int_0^\pi \sin^2\theta \, d\theta \end{aligned} \quad (C.25)$$

which yields

$$\frac{F}{\tau_o R^2} = 2\sqrt{12}\pi + 3\pi^2 = 51.4 \quad (C.26)$$

giving an upper bound of

$$\frac{3F}{4\pi \tau_o R^2} = \frac{R (\rho_S - \rho_B) g}{\tau_o} = 12.25$$

Appendix D

Other Velocity Fields and Stress Fields Investigated

Stress Fields

A stress field of the general form

$$\sigma_r = f(r) \cos\theta \quad (D.1)$$

$$\sigma_\theta = \sigma_\phi = f(r) \cos\theta \quad (D.2)$$

$$\tau_{r\theta} = h(r) \sin\theta \quad (D.3)$$

$$\tau_{r\phi} = \tau_{\theta\phi} = 0 \quad (D.4)$$

will satisfy the equilibrium equation if

$$g(r) = h(r) = \frac{A}{r^2} \quad (D.5)$$

and

$$f(r) = (1 - \sqrt{3}) \frac{A}{r^2} \quad (D.6)$$

Choosing the constant $A = -\tau_o R^2$ gives the largest value of the lower bound for a solution of this form giving

$$\frac{F}{\tau_o R^2} = 11.42 \quad (D.7)$$

The stresses

$$\sigma_r = f(r) \cos\theta \quad (D.8)$$

$$\tau_{r\theta} = g(r) \sin\theta \quad (D.9)$$

$$\sigma_\theta = \sigma_\phi = \tau_{r\phi} = \tau_{\theta\phi} = 0 \quad (D.10)$$

satisfy the equilibrium equations for

$$f(r) = \frac{2A}{r^3} + \frac{B}{r^2} \quad (D.11)$$

$$g(r) = \frac{A}{r^3} \quad (D.12)$$

$$\text{Letting } A = -\tau_o R^3 \quad (D.13)$$

$$\text{and } B = (\sqrt{3} + 2) \tau_o R^2 \quad (D.14)$$

giving the largest value of the yield force for a solution of this form yielding

$$\frac{F}{\tau_o R^2} = 15.7 \quad (D.15)$$

A stress field of the form

$$\sigma_r = \sigma_\theta = \sigma_\phi = f(r) \cos\theta \quad (D.16)$$

$$\tau_{r\theta} = g(r) \sin\theta \quad (D.17)$$

$$\tau_{r\theta} = \tau_{\theta\phi} = 0 \quad (D.18)$$

satisfies the equilibrium equations if

$$f(r) = \frac{A}{r} + \frac{B}{r^2} \quad (D.19)$$

$$g(r) = \frac{1}{2} \frac{A}{r} + \frac{B}{r^2} \quad (D.20)$$

Setting

$$A = \frac{240}{25} \tau_o R \quad (D.21)$$

$$B = \frac{144}{25} \tau_o R^2 \quad (D.22)$$

gives the stress field which is investigated in detail in Appendix B. This choice of constants maximizes the lower bound for a stress field of this form.

Velocity Fields

Considerable time and effort was devoted to obtaining upper bounds from velocity fields which are continuous functions of r and θ in spherical coordinates. The lowest bound ever obtained from a continuous velocity field was

$$\frac{F}{\tau_o R^2} = 56.45 \quad (D.23)$$

After discontinuous velocity fields were shown to be kinematically admissible, the first calculation made for a discontinuous velocity field is that shown in detail in Appendix C which gives a lower upper bound than is given by Equation (D.23).

Other discontinuous velocity fields and the bounds associated with them are:

$$v_r = -U_o \frac{R^4}{r^4} \cos\theta \quad (D.24)$$

$$v_\theta = -U_o \frac{R^4}{r^4} \sin\theta \quad (D.25)$$

which yields

$$\frac{F}{\tau_o R^2} = 59.58 \quad (D.26)$$

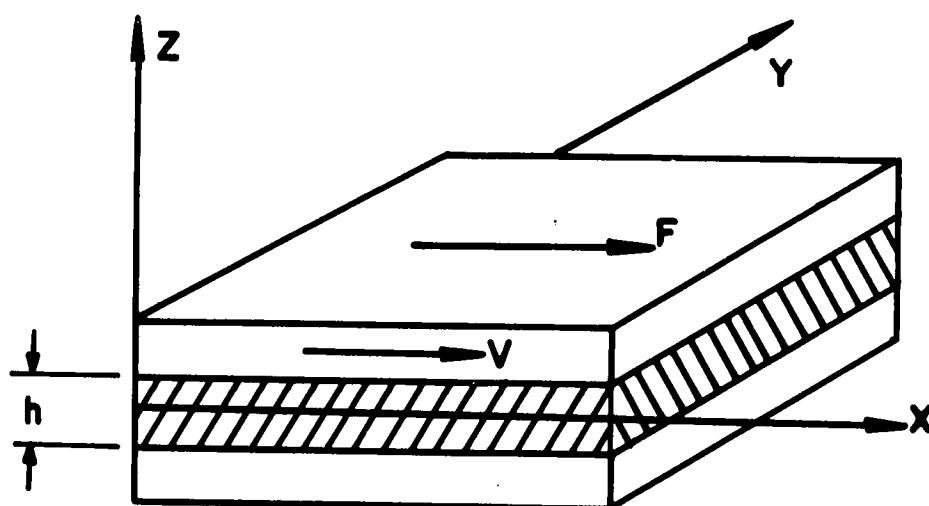
and

$$v_r = -U_o \left(\frac{2R}{r} - 1 \right)^2 \cos \theta \quad (D.27)$$

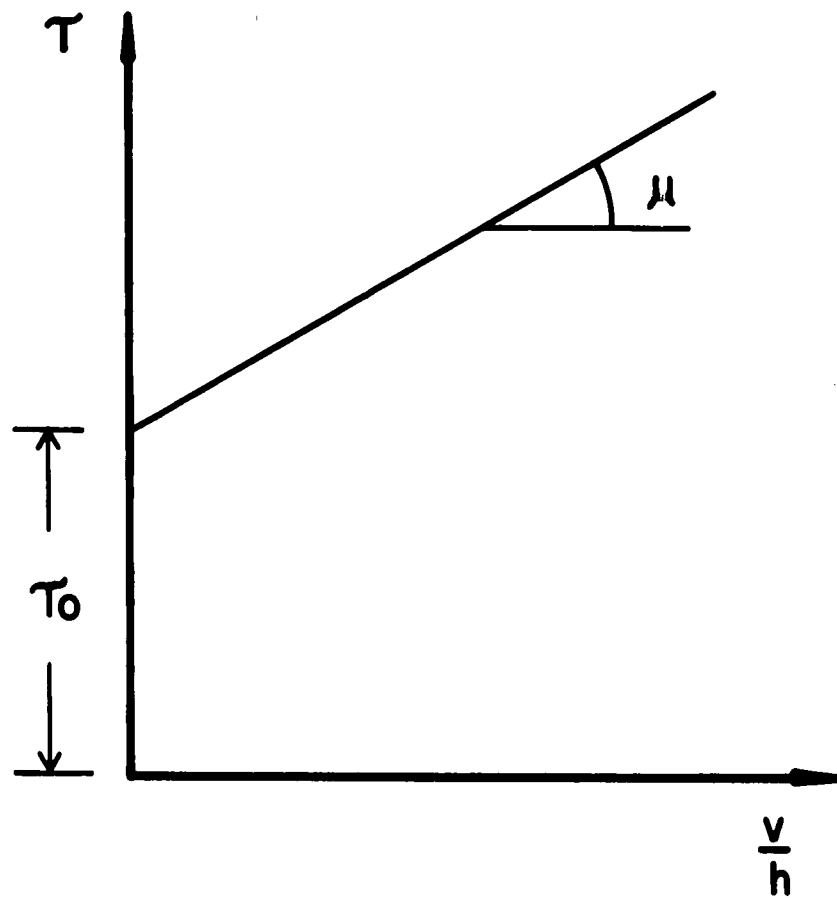
$$v_\theta = -U_o \left(\frac{2R}{r} - 1 \right) \sin \theta \quad (D.28)$$

which yields

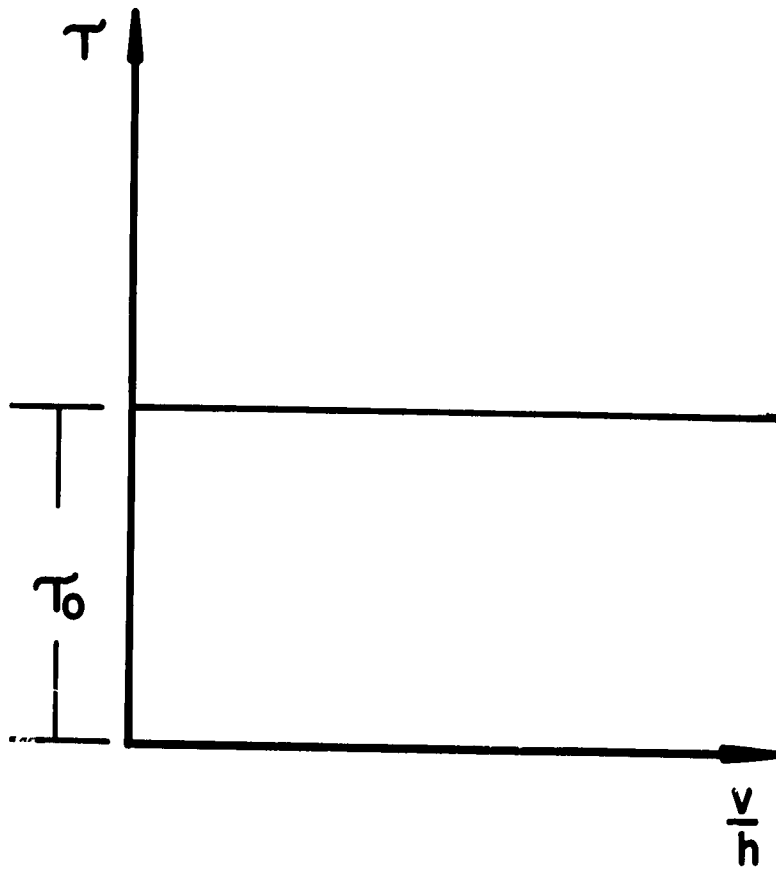
$$\frac{F}{\tau_o R^2} > 65.1 \quad (D.29)$$



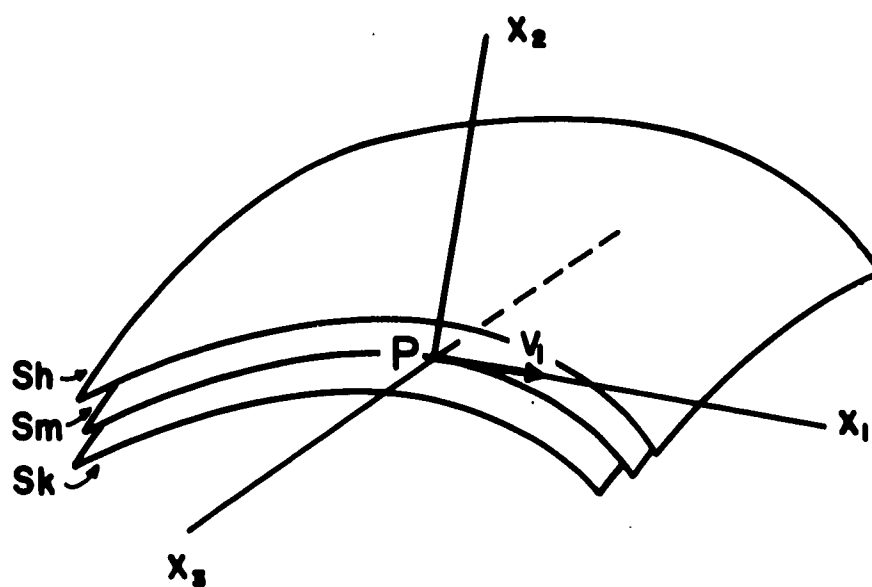
**FIGURE 1 (a)-GEOMETRY OF LAYER OF
MATERIAL BETWEEN FLAT
PLATES**



**FIGURE 1(b) - SHEAR STRESS VERSUS
DEFORMATION RATE FOR
A BINGHAM MATERIAL**



**FIGURE 1(c) - SHEAR STRESS VERSUS
DEFORMATION RATE FOR
A MISES MATERIAL**



**FIGURE 2 - GEOMETRY OF THIN
TRANSITION LAYER**

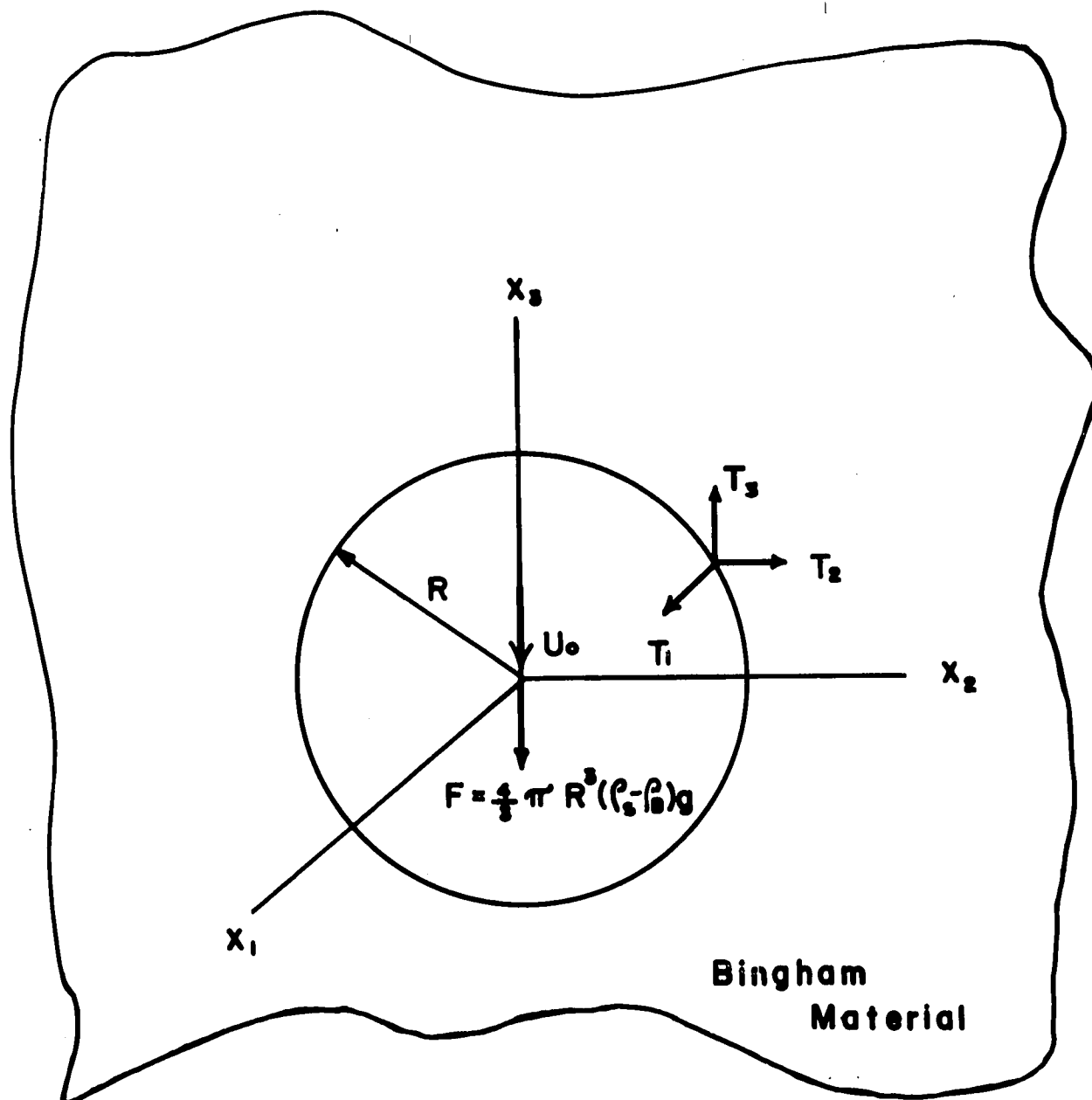


FIGURE 3

GEOMETRY OF SPHERE SUSPENDED IN A
BINGHAM MATERIAL

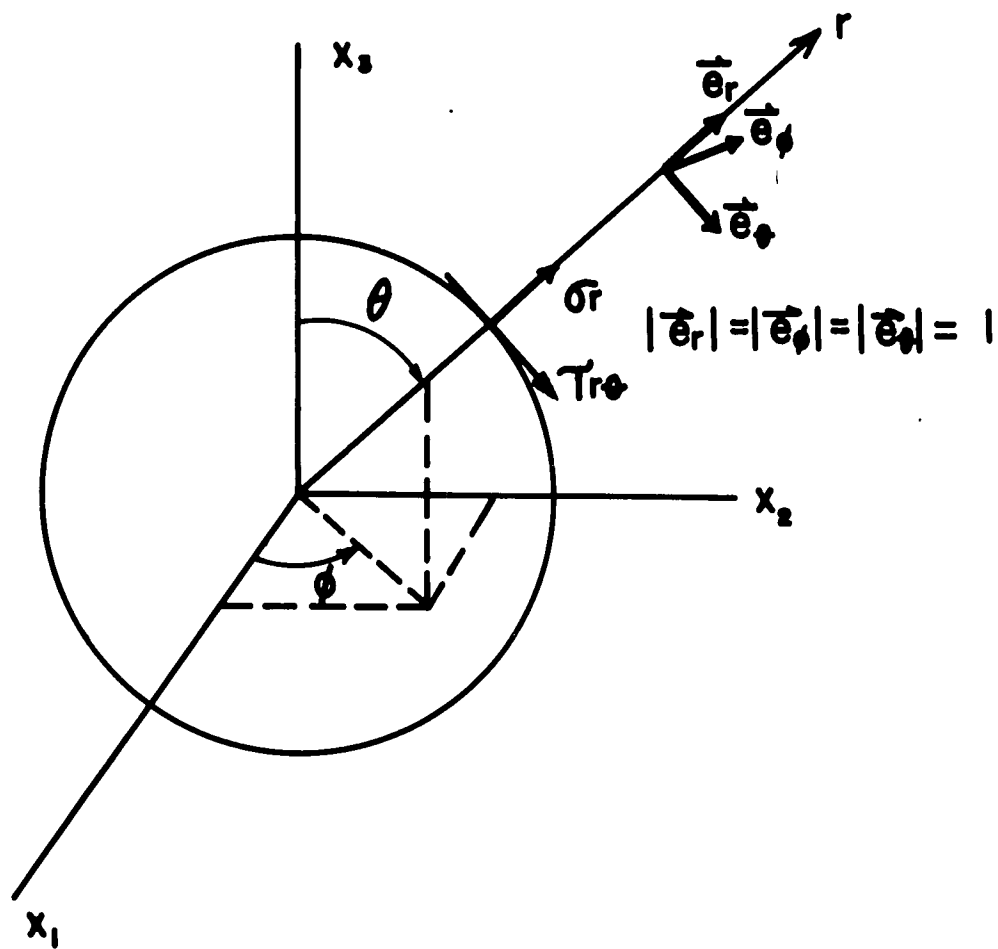


FIGURE 4
SPERICAL COORDINATES

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