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SOME IMAGINARY TETRAD TRANSFORMATIONS OF EINSTEIN SPACES

by

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Thesis Director's signature:

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CHAPTER I

INTRODUCTION

A. The Newman-Janis Transformation

Interest in complex transformations of solutions to Einstein's field equation was initiated by the work of Newman and Janis [1]. They were able to "transform," by means of a complex "transformation," the Schwarzschild metric into the Kerr metric, the field of a spinning spherical mass. The "transformation" that they employed was ad hoc and left unanswered the very important question of the general role of complex transformations in connecting solutions of Einstein's equation.

The Schwarzschild metric, written in its usual form is

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad . \qquad \text{(I-1)}$$

This can be expressed in terms of the Eddington coordinates

$$ds^2 = -(1-\frac{2M}{r})du^2 + 2drdu - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad \text{(I-2)}$$

where now

$$u = t - r - 2M \ \ell n(r-2M)$$

is a retarded time for this space. The contravariant form of this metric is

$$g^{00} = 0 \quad , \quad g^{11} = -(1 - \frac{2M}{r}) \quad , \quad g^{01} = 1$$

$$g^{22} = -\frac{1}{r^2} \quad , \quad g^{33} = -\frac{1}{r^2\sin^2\theta} \ ,$$

which can be written in terms of the pseudoorthonormal tetrad (see Appendix B),

$$g^{\mu\nu} = - \ell^{\mu}n^{\nu} - \ell^{\nu}n^{\mu} + m^{\mu}\bar{m}^{\nu} + m^{\nu}\bar{m}^{\mu} .$$

This tetrad is

$$\begin{aligned} \ell^{\mu} &= (0,1,0,0) , \\ n^{\mu} &= (1, -\tfrac{1}{2}(1 - \tfrac{2M}{r}), 0,0) , \\ m^{\mu} &= \frac{1}{\sqrt{2}} (0,0, \tfrac{1}{r}, \tfrac{i}{r\sin\theta}) . \end{aligned} \tag{I-3}$$

Newman and Janis now allow the coordinates u and r to become complex, and then write the tetrad

$$\begin{aligned} \ell^{\mu} &= (0,1,0,0) \\ n^{\mu} &= (1, -\tfrac{1}{2}(1-M(\tfrac{1}{r} + \tfrac{1}{\bar{r}})),0,0) \\ m^{\mu} &= \frac{1}{\sqrt{2}} (0,0, \tfrac{1}{r}, \tfrac{i}{r\sin\theta}) . \end{aligned} \tag{I-4}$$

Next, this tetrad is transformed in the usual way, allowing

$$\begin{aligned} u' &= u - ia\cos\theta \\ r' &= r + ia\cos\theta \end{aligned} \tag{I-5}$$

where now u' and r' are required to be real.

The transformed tetrad is

$$\ell'^{\mu} = (0,1,0,0)$$

$$n'^{\mu} = (1, -\tfrac{1}{2}\left[1 - \frac{2Mr}{(r^2+a^2\cos^2\theta)}\right],0,0)$$

$$m^{\mu'} = [\sqrt{2}(r'+ia\cos\theta)]^{-1}(ia\sin\theta,-ia\sin\theta,1,\frac{i}{\sin\theta})$$

and if the metric

$$g'^{\mu\nu} = \ell'^{\mu}n'^{\nu} - \ell'^{\nu}n^{\mu} + m^{\mu}\bar{m}^{\nu} + \bar{m}^{\mu}m^{\nu} ,$$

is calculated it is found to be the Kerr metric.

This procedure has a completely ad hoc character. There is no reason why n^{μ}, for instance should have the particular form that it has. Above all, this transformation works on the contravariant tetrad but not on the covariant one. Nor can it be made to work on the general tensor space of this manifold. Even the metric itself cannot be transformed into another real metric.

Doubtless, in the face of these very serious objections, this transformation would have been considered a generally unproductive idea except for one thing. The transformation was tried again by Newman et al. on the Reissner Nordström metric [2] (the charged version of the Schwarzschild metric), and the result was the charged version of the Kerr metric (now called the Kerr-Newman metric).

Subsequently, Newman and Demianski [3] used the transformation

$$u' = u - i(a\cos\theta + 2b\ln\sin\theta) + 2ib\ln\tan\frac{\theta}{2}$$
$$n' = r + i(a\cos\theta + b)$$

to produce the charged and uncharged versions of a combined Kerr-NUT metric. When $b = 0$, the solution is the Kerr metric, and when $a = 0$ it is the metric discovered by

Newman, Unti, and Torrence [4] in 1963. Demianski [5], finally, was able to extend this procedure to include yet another parameter in his metric, and discovered that this represented the maximum number of parameters to which this transformation could be extended.

In spite of the ad hoc nature of this procedure, its multiple successes tended to confirm Newman's belief that their results are not fortuitous. The most likely explanation lay in the fact that all of these metrics belong to a class of spaces that are called the Kerr-Schild metrics.

B. The Kerr-Schild Metrics and Newman's Complex Minkowski Space

Kerr and Schild, in a study of the Kerr metric [6] discovered that both the Schwarzschild and the Kerr metrics could be written in the form

$$g_{\mu\nu} = \eta_{\mu\nu} - 2M\ell_\mu\ell_\nu \qquad \text{(I-6)}$$

where

$$\eta_{\mu\nu} = \text{diag.}(-1,1,1,1)$$

and where ℓ_μ is a vector such that

$$\eta_{\mu\nu}\ell^\mu\ell^\nu = g_{\mu\nu}\ell^\mu\ell^\nu = 0 \ .$$

The spaces are also algebraically degenerate (see Appendix B, Section C) and ℓ_μ defines one of the principal null directions, i.e. one of the directions along which the field propagates.

What is significant about this is that the geometry of these spaces is entirely determined by $\eta_{\mu\nu}$, and the vector ℓ_μ and its derivatives. This is not unlike the electromagnetic case, where the field is described by the flat space metric, and a vector and its derivatives.

Newman investigated the complex transformation of E. M. fields [7]. Maxwell's equations can be written,

$$\text{curl } w = i\dot{w}$$

$$\nabla \cdot w = 0$$

where $w = E + iB$. If a translation is made of the coordinate origin, this yields the same solution as before. However if the translation is complex, then $w(x^\mu - ib^\mu)$ is a distinct solution to Maxwell's equations.

For example, consider an imaginary translation along the z axis of the Coulomb field, $E = e/r^3\ (x,y,z)$. This translation gives

$$w' = \frac{e}{(r^T)^3}\ (x,y,z-ia)$$

where $r^T = (x^2+y^2+(z-ia)^2)^{\frac{1}{2}}$. w is still a solution of Maxwell's equation, but it has an electric monopole (e), a magnetic dipole (iea), an electric quadrupole moment (ea^2) etc. In fact this field was already known to Newman as the field of the charged Kerr metric.

These same results were found by Newman to apply to the Weyl tensor of the linearized Schwarzschild field (the mass monopole field). In this case, the transformed Weyl tensor is that of the linearized Kerr metric.

This parallel between the linearized Weyl tensor and the electromagnetic field tensor led him to speculate that the Weyl tensor of these metrics is a field defined upon complex Minkowski space. The Kerr field is realized on various real "slices" of this complex Minkowski space, corresponding to different values of the parameter a.

In order to extend these results to the full Schwarzschild metric, he gave the Kerr-Schild metrics a new form [8]

$$ds^2 = ds^2_{Flat} + \lambda(\ell_\mu dx^\mu)^2 \ , \qquad \text{(I-7)}$$

where ds^2_{Flat} is not necessarily the Minkowski metric, but is a flat space metric intrinsic to the particular metric that is being considered, i.e. it represents a particular slice of complex Minkowski space. The vector ℓ^μ is, as before, a principal null vector.

For Schwarzschild space (a = 0) the ds^2_{Flat} does represent Minkowski space

$$ds^2 = -2du^2 - 2dudr + r^2(d\theta^2+\sin^2\theta d\varphi^2) + \frac{4M}{r}\,du^2 \qquad \text{(I-8)}$$

with the tetrad

$$\ell^\mu = (0,1,0,0) \quad , \quad m^\mu = \frac{1}{r}(0,0,1, \frac{i}{\sin\theta}) \ ,$$

$$n^\mu = (1,-1,0,0) \ . \qquad \text{(I-9)}$$

If the transformation

$$u = u' + ia\cos\theta' \; ,$$

$$r = r' - 2ia\cos\theta' \; ,$$

$$\cos\theta = \frac{r'\cos\theta' - 2ia}{r' - 2ia\cos\theta} \; ,$$

$$\cos 2(\varphi - \varphi') = \frac{r'^2 - 4a^2}{r'^2 + 4a^2} \; ,$$

which is equivalent to a translation along the imaginary z axis

$$z \rightarrow z - ia$$

then ds^2_{Flat} in (I-8) above becomes

$$ds^2 = -2du'^2 - 2du'dr' + 2a^2\sin^2\theta dr d\varphi'$$
$$+ \tfrac{1}{2}(r'^2 + 4a^2\cos^2\theta')(d\theta'^2 + \sin^2\theta' d\varphi'^2)$$
$$+ 2a^2\sin^4\theta' d\varphi'^2 \; . \qquad \text{(I-10)}$$

This is just the Kerr metric with $m = 0$.

Thus far, then, Newman's attempt to extend his results in the linearized Schwarzschild metric to the full metric is successful. But the tetrad does not transform well at all, and, in fact, is not even real. The same is true for λ in (I-9), and Newman is again forced to correct these deficiencies by ad hoc assumptions. He gives a procedure for finding the new ℓ^μ, and λ, and then calculate the Newman-Penrose components of the Weyl tensor.

In spite of the actual failure of this attempt to account for the success of the Newman-Janis transformation, Newman is probably right in looking to the Kerr-Schild form of the metric to provide an explanation. This form of the metric is sufficiently close to the linearized form of the metric to enable it to behave in the same way as does its linearized form under these imaginary translations. These ad hoc procedures are sufficient to make up the difference.

C. The Weyl Unitary Trick

The foregoing approaches to the complex transformations of solutions to Einstein's field equations, even were they successful, would apply to only a small class of spaces. I originally sought to discover more widely applicable transformations.

Professor Howard Resnikoff suggested that the more likely approach to this problem lay in Weyl's "unitary trick." Weyl, in order to classify certain non-compact Lie groups, rotated them onto a compact Lie group that was already classified [9].

The most general way to perform this "trick" is to complexify the tangent space to the manifold (the space of vectors tangent to the curves in the manifold). Suppose that the tangent spaces to two real manifolds are spanned by the two sets of basis vector fields $\{e_1, e_2, \ldots, e_n\}$

and $\{e_1', e_2', \ldots, e_n'\}$ respectively. Every vector field in the first of these is some linear combination of these

$$V = \lambda_1 e_1 + \lambda_2 e_2 + \ldots \lambda_n e_n \ ,$$

where the λ_i's are real numbers. The same is true in the primed space. By the complexification of these spaces is meant that the λ_i's are allowed to take on complex values. When this happens, the original manifold is now to be seen as a real subspace of a 2n dimensional complex space.

Suppose that in this larger manifold, the two basis sets can be related by a complex transformation

$$e_i' = \gamma_i^{\ j} e_j \ .$$

It is possible, by this means, to study the structure of the one manifold in terms of the other. This is Weyl's unitary trick.

Sometimes it is not necessary to do this through the tangent spaces. Sometimes the manifold itself can be rotated directly through its coordinate representation. The example of this cited by Professor Resnikoff was that of the circle,

$$x^2 + y^2 = 1 \ .$$

If x and y are allowed to take on complex values, then this is written,

$$(x + iu)^2 + (y + iw)^2 = 1 \ .$$

Now the circle is just the subspace of this two dimensional complex manifold $u = w = 0$. On the other hand, the manifold $u = y = 0$ is the hyperbola

$$x^2 - w^2 = 1 .$$

Any appropriate function defined upon the circle can be analytically continued to one defined upon the hyperbola. For example, the $\sin\theta$ is continued into the $\sinh\theta$, the $\cos\theta$ into the $\cosh\theta$, etc. Most importantly, relations that hold for these functions in one space, will hold in the other. Thus $\cos^2\theta + \sin^2\theta = 1$, and if $\theta \to i\theta$, then $\cosh^2\theta - \sinh^2\theta = 1$.

It was this example that suggested to me the rotation of the four dimensional sphere into the de Sitter universe. The coordinate representation of this universe has long been known to be the hyper-hyperboloid of one sheet embedded in R^5 [10],

$$x^2 + y^2 + z^2 + w^2 - v^2 = 1 .$$

I realized that $S^4(R)$ could be rotated into this by means of the coordinate transformation $v \to iv$, the same one that takes the circle into the hyperbola in the above example. I next undertook the enumeration of all other four dimensional spaces that could be obtained from $S^4(R)$ by such transformations. There are three more of these besides $S^4(R)$ and the de Sitter universe. These are the four dimensional pseudosphere, the anti-de Sitter universe, and another space of signature zero which is of no interest here.

My next thought was to extend these transformations to other spaces besides these very simple ones. The Schwarzschild space, for instance, does not have such a readily available coordinate representation. If the lessons learned from the de Sitter universe were to be extended to it, the more generally applicable method of rotating the tangent space rather than the manifold itself must be undertaken. To implement this for the rotation of $S^4(R)$, the metric must be put in terms of the intrinsic coordinates of the sphere (the spherical angles ψ, χ, θ, φ), and the tetrad of one-forms extracted. Two basic types of rotation are required to do all of these transformations. The first of these, designated as the type 0 transformation, takes one of the legs of the tetrad of $S^4(R)$ and makes it into the time-like leg of the de Sitter universe. This same transformation makes one of the legs of the pseudosphere into the time-like leg of the anti-de Sitter universe.

The other transformation designated the type I transformation changes the sign of the signature of the metrics. Thus the type I transformation maps the de Sitter universe onto the anti-de Sitter universe, and the sphere onto the pseudosphere.

The type 0 transformation from the four sphere to the de Sitter universe involves the choice of a particular leg of the tetrad of the sphere to be the time-like one in the de Sitter universe. Since the legs of the tetrad of the four sphere are equivalent, there is no reason why that particular leg had to be chosen. There are three more legs, and any one of them could have been made into the time-like leg of the transformed space. Thus there are three other spaces like the de Sitter universe that result from type 0 transformations from the four sphere. By composing the two type 0 transformations one can arrive at another transformation from the de Sitter universe onto each of these alternate universes. These transformations will involve the exchange of one space-like and the time-like leg of the tetrad. These are called the type II transformations.

This can be done upon the anti-de Sitter universe also. If a type I transformation is done upon the de Sitter universe, and a type II is done upon the anti-de Sitter universe which results, then one has a transformation from the de Sitter universe to an alternate version of the anti-de Sitter universe. This composition of a type I transformation with a type II transformation is called a type III transformation.

In working through the example of the de Sitter universe I found that four types of transformations will relate a member of this family of spaces to any other member of the family. The type 0 will relate positive or negative definite metrics to Lorentz metrics. The type I relates positive definite metrics to negative definite, and Lorentz metrics to Lorentz metrics. Type II transformations are useful in swapping the time-like legs of a Lorentz metric with one of its space-like legs. Finally, the type III transformation is simply a composition of a type I and a type II transformation.

Having worked out these transformations for the $S^4(R)$ family of spaces, I investigate if these transformations have validity beyond this particular family of spaces. This work is reported in Chapter III. The mathematical tools that were used were Cartan's structure equation described in Appendix A. For each type of transformation, these equations permit the calculation of the structure function of the transformed space in terms of those of the original space. This in turn permits the components of the connection and the components of the curvature tensor for the transformed space to be related to the same objects in the original space.

For types I, II, and III, it is possible to calculate the Einstein tensor. From this, I prove that a vacuum solution of the field equations ~~is~~ rotated onto another vacuum solution by these transformations.

Although, normally, solutions that have $T_{ij} \neq 0$ will not rotate onto other realistic solutions to the field equation, adjustments can sometimes be made. An example of this is the rotation of the Einstein static solution onto the Gödel universe. The necessary adjustment, in this case, is a stretch in the time-like direction.

The next step in this work was to check the physical significance of these transformations. This is done in Chapter IV for two sets of parameters, the hydrodynamical parameters of the cosmological fluid, and the Newman-Penrose spin coefficients. The first of these tells how the behavior of test particles in the transform space, relate to behavior of similar particles in the original space. The latter tell how null fields (e.g. the photon and graviton fields) propagate in this space.

The significance of these physical parameters are explained in detail in order to make the thesis to some extent self-contained. The way in which these parameters transform in the family of space-times related to $S^4(R)$ is described in some detail.

Finally, I felt that this work would be enhanced if a more physical application could be made than the ones already undertaken. Since the interest in complex transformations began with the Schwarzschild metric, I returned to it, originally in the hopes that the Newman-Janis transformation could be explained. This particular hope was not realized but transformations were discovered that related the Schwarzschild metric to the Kantowski-Sachs metrics.

At the same time, I was anxious to find some significant transformations on radiative metrics. The Bateman metrics, and Robinson-Trautman metrics were natural candidates. These lead nowhere. The next candidates considered were the Einstein-Rosen-Bondi cylindrical waves. These lead me to consider the Gowdy universes, that are related to these waves. Two things stood out about the

Gowdy T_3 universes. First of all, they included the Kantowski-Sachs universes. Secondly, they all possessed two hypersurface orthogonal, [30], mutually orthogonal space-like Killing vector fields. This meant that there should be transformations that related them to the Weyl metrics which possess two hypersurface orthogonal, mutually orthogonal Killing vector fields, one time-like and one space-like. There should be a family of Weyl metrics containing the Schwarzschild metric related to these spaces by a type II transformation.

These transformations were done after a careful review of the properties of these metrics and in fact two families of Weyl metrics were discovered. The first was, of course, the ones related to the Schwarzschild metric obtained by a type II transformation. The second was a larger family of Weyl metrics.

In calculating the Newtonian potential for these metrics, I found that both families contained terms that indicated that they are the fields of the <u>interior</u> of a shell of matter. The potentials of the two families of metrics were distinguished by the presence of a point mass at the origin in the class that contained the Schwarzschild metric.

D. Future Prospects

The last section reviews the accomplishments of this thesis. There are a number of problems yet to be considered before complex transformation can be considered an adequately investigated technique for constructing solutions to Einstein's equation.

First of all, more general transformations need to be investigated. The first line of investigation in this matter would be to pursue the effect of imaginary transformation together with scale changes. An example of this is the transformation of the Einstein static universe to the Gödel universe. Next, the matter of the Newman-Janis transformation needs to be pursued. This has something to do with the complex translations. Obviously, there is something more to it than that, however.

Chapter II - $S^4(R)$

A. Introduction

The idea of the complex rotation of a space is actually quite simple. Before proceeding with the program of this thesis, and setting forth the general results that are to come, it seems best to first illustrate these ideas with a simple example.

Consider the space $S^4(R)$, that is the ordinary 4-sphere. As will be seen, this space can be rotated into the de Sitter and anti-de Sitter universes which are well known, well studied [14,15] cosmologies, and also into the four dimensional pseudosphere.

These rotations will be done in Section A as a simple coordinate rotation, but will later be done in Section B by means of a tetrad rotation. Six other types of spaces can be obtained from these by two different types of rotations. These will serve as a model for the general rotations that will be defined and examined in the next chapter.

B. The Four Sphere

The coordinate representation of the four sphere as imbedded in R^5 is

$$x^2 + y^2 + z^2 + w^2 + u^2 = 1 ,$$

with
$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 + du^2 \,.$$

If the coordinates of $S^4(R)$ are allowed to become complex, then it becomes the four dimensional <u>complex</u> space $S^4(C)$, of which $S^4(R)$ is a real four dimensional subspace. This is called the <u>complexification</u> of $S^4(R)$. A collection of real subspaces can be realized by setting either the real or the imaginary part of each coordinate equal to zero. This collection is

$x^2 + y^2 + z^2 + w^2 + u^2 = 1$	$S^4(R)$
$x^2 + y^2 + z^2 + w^2 - u^2 = 1$	de Sitter universe
$x^2 + y^2 + z^2 - w^2 - u^2 = 1$	space x
$x^2 + y^2 - z^2 - w^2 - u^2 = 1$	anti ~~de~~ Sitter universe
$x^2 - y^2 - z^2 - w^2 - u^2 = 1$	pseudosphere

Of these, two spaces have Lorentz signature (the de Sitter and anti ~~de~~ Sitter universes), two have positive or negative definite signature (the sphere and the pseudosphere) and one (space x) has signature zero. The latter space is of no interest here and will be dropped from consideration. The remaining four will be studied further.

One can transform each space into the others by imaginary rotations. For example, $u \rightarrow iu$ transforms $S^4(R)$ into the de Sitter universe. This actually gives little information about the spaces themselves. Additional information can be gained by going to an intrinsic coordinate system, for example that of the spherical angles

$$\begin{aligned} x &= \cos\psi \sin\chi \sin\theta \cos\varphi \\ y &= \cos\psi \sin\chi \sin\theta \sin\varphi \\ z &= \cos\psi \sin\chi \cos\theta \\ w &= \cos\psi \cos\chi \\ u &= \sin\psi \end{aligned} \qquad \text{(II-1)}$$

The metric for $S^4(R)$ obtained from this is

$$ds^2 = d\psi^2 + \cos^2\psi(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)) \qquad \text{(II-2)}$$

If $u \to iu$, then $\psi \to it$, and the metric becomes

$$ds^2 = -dt^2 + \cosh^2 t(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)) \qquad \text{(II-3)}$$

just the metric for the de Sitter universe.

The anti-de Sitter universe is obtained from the de Sitter by making two more coordinates imaginary. This can be done by letting u become real again ($t \to it$), and then letting $\chi \to i\chi$, making x, y, and z imaginary. The resulting metric is

$$ds^2 = dt^2 - \cos^2 t(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)) \ , \qquad \text{(II-4)}$$

which is the anti-de Sitter universe. This is further transformed into the pseudosphere by the transformation $t \to i\psi$, whose metric is

$$ds^2 = -d\psi^2 - \cos^2\psi(d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2) \qquad \text{(II-5)}$$

Figure II-1 is a diagram of these results.

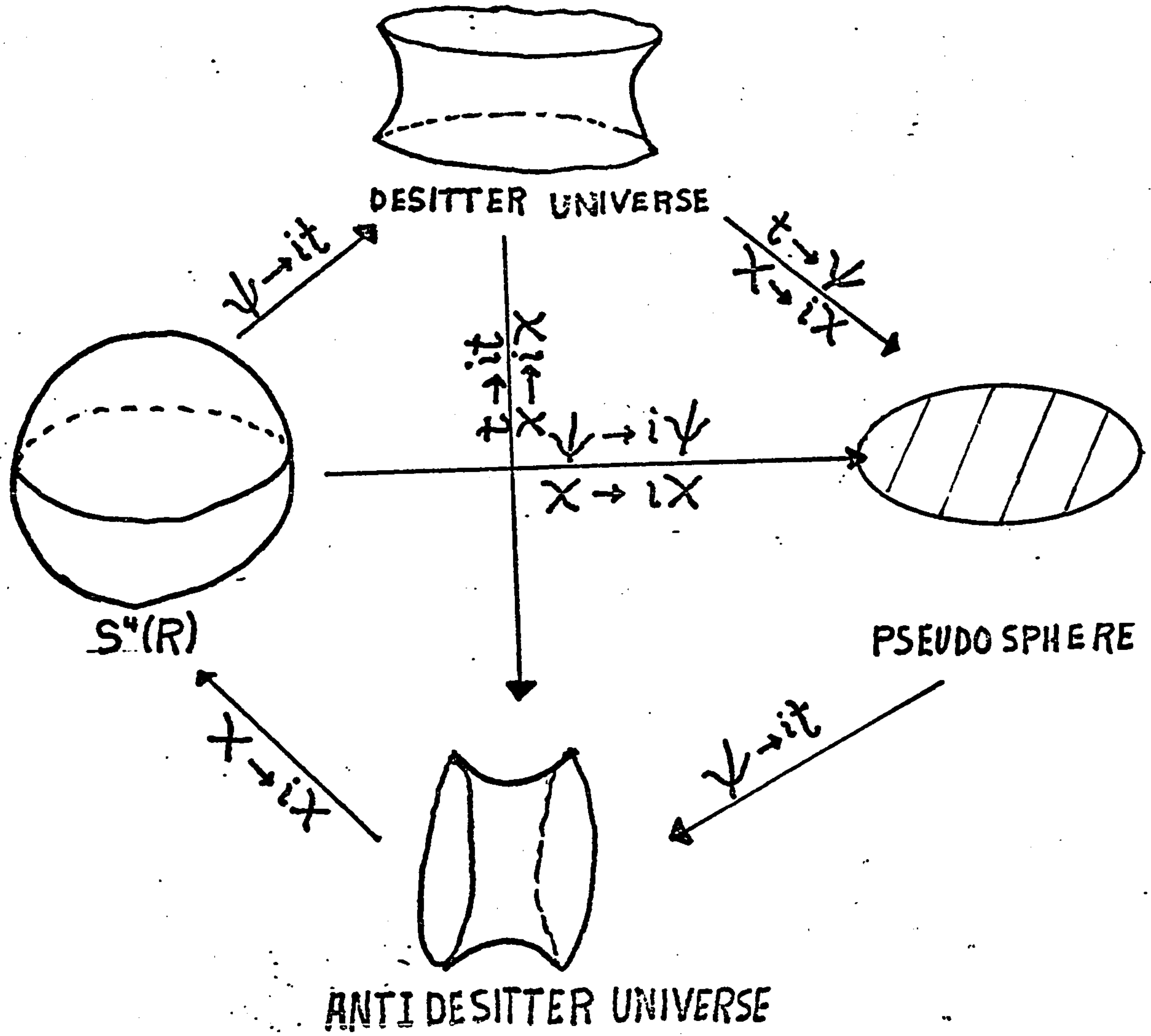


Fig. II-1

C. <u>The Tetrad Approach</u>

The tangent space (space of contravariant vectors) of any four dimensional manifold is spanned by four vectors called a tetrad. These can be written in the form $X_i = X_i{}^{\mu} \frac{\partial}{\partial x^{\mu}}$, from which follow the transformation properties of contravariant vectors. The space of covariant vectors is called the cotangent space, and its elements are called <u>one forms</u> which are written in the form $\omega^i = X^i_{\mu} dx^{\mu}$.

This space is obviously spanned by a tetrad of these one forms. It is always possible to choose this basis so that the metric can be written

$$g_{\mu\nu} = \eta_{ij} X^{i}_{\ \mu} X^{j}_{\ \nu}, \quad (i,j = 0,1,2,3)$$

where

$$\eta_{ij} = \mathrm{diag}(\epsilon,1,1,1)$$

and where $\epsilon = +1$ for signature + 4 spaces, and where $\epsilon = -1$ for signature + 2 spaces. If $X^{0}_{\mu} X^{0\mu} = \epsilon$, and $X^{i}_{\mu} X^{j\mu} = \delta^{ij} (i,j=1,2,3)$, $X^{i}_{\mu} X^{0\mu} = 0$, then the tetrad is called an orthonormal one, and the metric is said to be written in <u>normal form</u>.

With these preliminaries, it is possible to exhibit the tetrad of one-forms for the spaces that have been considered.

$$\begin{array}{lll}
 & \underline{\text{Sphere}} & \underline{\text{de Sitter}} \\
\omega^0 & d\psi & dt \\
\omega^1 & \cos\psi d\chi & \cosh t d\chi \\
\omega^2 & \cos\psi \sin\chi d\theta & \cosh t \sin\chi d\theta \\
\omega^3 & \cos\psi \sin\chi \sin\theta d\varphi & \cosh t \sin\chi \sin\theta d\varphi \\
 & \underline{\text{Anti de Sitter}} & \underline{\text{Pseudosphere}} \\
\omega^0 & dt & d\psi \\
\omega^1 & \cos t d\chi & \cosh\psi d\chi \\
\omega^2 & \cos t \sinh\chi d\theta & \cosh\psi \sinh\chi d\theta \\
\omega^3 & \cos t \sinh\chi \sin\theta d\varphi & \cosh\psi \sinh\chi \sin\theta d\varphi
\end{array} \qquad \text{(II-6)}$$

The transformations that have been made can now be written down in terms of the effect on the tetrad of one forms. In going from the sphere to the de Sitter universe the transformation was

$$\omega^{o\,\prime}(x') = -i\omega^{o}(x')$$
$$\omega^{i\,\prime}(x') = \omega^{i}(x') \qquad \text{(II-7)}$$

where $\omega^{o}(x')$ indicates the original one form, that has been transformed by the coordinate transformation. For example, for the sphere $\omega^{o}(x') = i\,dt$, and $\omega^{o\,\prime}(x') = -i(idt) = dt$. The transformation that was made from the de Sitter to the anti-de Sitter universe is

$$\omega^{i\,\prime}(x') = -i\omega^{i}(x'). \quad (i=0,1,2,3) \qquad \text{(II-8)}$$

Using these two forms of the imaginary rotations, one can transform from $S^4(R)$ to all of the spaces on Fig. II-1. In subsequent chapters these will be called the type 0 and type I transformations respectively.

Returning to the list of four dimensional spaces, one realizes that the coordinates x, y, z, w, u are homogeneous on R^5, and can be interchanged. Thus, the de Sitter space might be represented by

$$x^2 - y^2 + z^2 + u^2 + w^2 = 1 \;.$$

If the transformations to the intrinsic coordinate system were correspondingly changed, the space would be exactly the same as before. But if the two coordinate systems were left in the same relationship as in II-1, the transformation to the de Sitter universe might be effected by letting $\varphi \to \underline{i}t$, $\psi \to \varphi$ which gives the metric

$$ds_I^2 = -\cos^2\varphi\sin^2\chi\sin^2\theta dt^2 + \cos^2\varphi d\chi^2 + \cos^2\varphi\sin^2\chi d\theta^2 + d\varphi^2 . \qquad \text{(a) (II-8)}$$

This is equivalent to choosing ω^3 of $S^4(R)$ rather than ω^o for the time-like direction. The basic manifold here is the same as the de Sitter space, but opposite points have been identified in the space. This gives it a different set of properties [14,16]. In terms of the tetrad formalism, this space can be derived from the de Sitter universe by exchanging the roles of the ω^o and ω^3, i.e. making ω^3 rather than ω^o the time-like leg of the tetrad. This is the transformation

$$\begin{aligned} \omega^{o\prime}(x') &= -i\omega^3(x') \\ \omega^{3\prime}(x') &= -i\omega^o(x') \\ \omega^{a\prime}(x') &= \omega^a(x') . \qquad (a=1,2) \end{aligned} \qquad \text{(II-9)}$$

This suggests that two other spaces could be derived from the de Sitter universe by making ω^1 and ω^2 the time-like legs. Two other transformations similar to II-9 could be written down, except ω^3 would be replaced by ω^1 or ω^2. These transformations are called Type II transformations.

The two additional metrics are

$$ds_{II}^2 = -\cos^2\zeta dt^2+d\zeta^2+\cos^2\zeta\sinh^2 t(d\theta^2+\sinh^2\theta d\varphi^2) \qquad \text{(b) (II-8)}$$

$$ds_{III}^2 = -\cos^2\zeta\sin^2\chi dt^2+d\zeta^2+\cos^2\zeta(d\chi^2+\sin^2\chi\sinh^2\theta d\varphi^2) \qquad \text{(c) (II-8)}$$

Figure II-2 is drawn to illustrate II-8(b), where the θ and φ coordinates have been suppressed. It is shown as embedded in R^5.

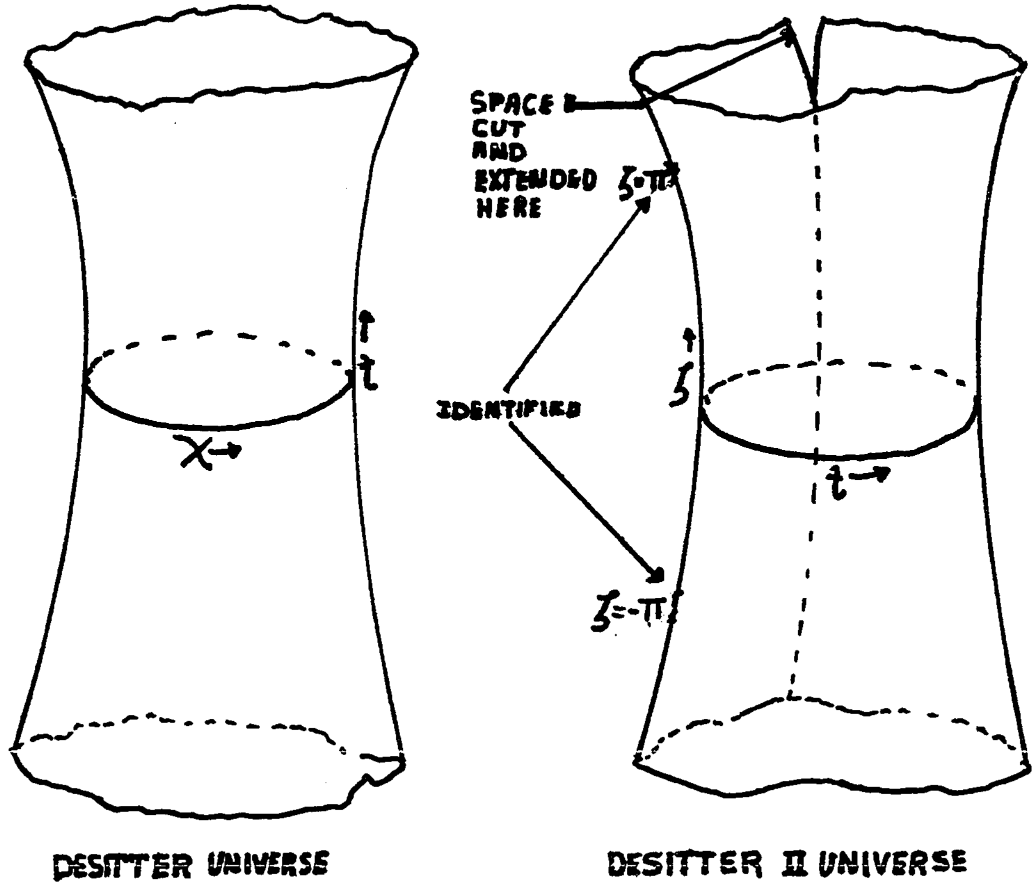


Fig. II-2

This space is readily seen to violate causality, but a covering space can be constructed such that this does not happen. This becomes evident if $\cos^2\zeta$ is factored from the metric

$$ds_{II}^2 = \cos^2\zeta\left(-dt^2 + \frac{d\zeta^2}{\cos^2\zeta} + \sinh^2 t\,(d\theta^2 + \sinh^2\theta\, d\varphi^2)\right).$$

Letting $$r = \int \frac{d\zeta}{\cos\zeta} = \ln \tan\left(\frac{\pi}{4} + \frac{\zeta}{2}\right)$$

one obtains the metric

$$ds_{II}^2 = \operatorname{sech}^2 r(-dt^2 + dr^2 + \sinh^2 t(d\theta^2 + \sinh^2\theta d\varphi^2)) ,$$

which does not violate causality.

The other spaces obtained in this way have for their (r,t) space the two sphere, and so essentially violate causality. If a type II transformation is done on the anti-de Sitter universe, three other universes would be derived from it. Since, however, the anti-de Sitter universe is derived from the de Sitter by a type I transformation, these varieties of the anti-de Sitter universe could be obtained from the de Sitter by this combination of type I and type II, which we call type III.

D. Summary

The four sphere is a manifold that can be embedded in R^5, and in terms of the coordinates of R^5, has the simple expression

$$x^2 + y^2 + z^2 + w^2 + u^2 = 1 .$$

A set of imaginary rotations has been constructed for this space which transforms it into the de Sitter universe, the anti-de Sitter universe, and the pseudosphere. These are

$u \to iu \quad x^2+y^2+z^2+w^2-u^2 = 1$ de Sitter universe

$\left.\begin{array}{l} u \to iu \\ w \to iw \\ z \to iz \end{array}\right\} \quad x^2+y^2-z^2-w^2-u^2 = 1$ anti-de Sitter universe

$\begin{array}{l} u \to iu \\ w \to iw \\ z \to iz \\ y \to iy \end{array} \quad x^2-y^2-z^2-w^2-u^2 = 1$ pseudosphere.

The metrics for these spaces can be calculated in terms of the spherical angles ψ, χ, θ, φ, and they are

$$ds^2 = d\psi^2+\cos^2\psi(d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2) \quad \text{(sphere)},$$

$$ds^2 = -dt^2+\cosh^2 t(d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2) \quad \text{(de Sitter)},$$

$$ds^2 = -dt^2+\cos^2 t(d\chi^2+\sinh^2\chi(d\theta^2+\sin^2\theta d\varphi^2) \quad \text{(anti-de Sitter)},$$

$$ds^2 = +dt^2+\cosh^2 t(d\chi^2+\sinh^2\chi(d\theta^2+\sin^2\theta d\varphi^2) \quad \text{(pseudosphere)}.$$

The transformations given above are now given by

$\psi \to it$ sphere $\to$ de Sitter

$\left.\begin{array}{l} t \to it \\ x \to i\chi \end{array}\right\}$ de Sitter $\to$ anti-de Sitter
sphere $\to$ pseudosphere.

The metrics above are in normal form and can be written

$$ds^2 = \varepsilon(\omega^0)^2+(\omega^1)^2+(\omega^2)^2+(\omega^3)^2$$

where $\varepsilon = +1$ for the sphere and pseudosphere and $\varepsilon = -1$ for the de Sitter and anti-de Sitter universes, and where $\{\omega^i\}$ is an orthonormal tetrad of one forms. The transformation above can be given in terms of this orthonormal tetrad. The first of these, that transform the sphere into the de Sitter universe, is given by

$$\omega^{o\prime} = -i\ \omega^{o} ,$$

$$\omega^{i\prime} = \omega^{i} , \quad (i = 1,2,3)$$

and these are called the type 0 transformation. The second transformation, which transforms the sphere into the pseudosphere and the de Sitter universe into the antide Sitter universe, are given by

$$\omega^{i\prime} = i\ \omega^{i} ,$$

and these are called type I.

In addition to these two rotations, two more are needed for completeness. In making the type 0 transformation from the four-sphere to the de Sitter universe, a particular leg of the orthonormal tetrad of the sphere was chosen to be the time-like leg of the de Sitter universe. Three other possible choices could be made. If the de Sitter universe is rotated back upon the four-sphere, a new leg rotated as the time-like leg of the new space, this would be equivalent to

$$\omega^{o\prime} = -i\ \omega^{a}$$

$$\omega^{a\prime} = -i\ \omega^{o}$$

$$\omega^{j\prime} = \omega^{j} \quad (j = o,a)$$

where ω^{a} is a space-like leg and ω^{o} is the time-like leg in the de Sitter universe. These are called type II transformations.

Finally, a type III transformation is defined to be the composition of a type II transformation, and a type I transformation. If a type II transformation was done upon the de Sitter universe, then a type III transformation would relate that space to the anti-de Sitter universe.

Table of Transformations

Type	Example	Tetrad Transformation
0	4-sphere → de Sitter universe	$\omega^{o\prime}(x') = -i\omega^{o}(x')$ $\omega^{i\prime}(x') = \omega^{i}(x')$ (i=1,2,3)
I	de Sitter universe → anti-de Sitter	$\omega^{i\prime}(x') = -i\ \omega^{i}(x')$ (i=0,1,2,3)
II	exchange of ω^o and ω^a in the de Sitter universe	$\omega^{o\prime}(x') = -i\ \omega^{a}$ $\omega^{a\prime}(x') = -i\ \omega^{o}(x')$ $\omega^{i\prime}(x') = \omega^{i}(x')$ ($i\neq o,a$)
III	space that results from exchange of ω^o and ω^a in the de Sitter universe → anti-de Sitter universe	$\omega^{o\prime}(x') = -\omega^{a}(x')$ $\omega^{a\prime}(x') = -\omega^{o}(x')$ $\omega^{i\prime}(x') = -i\ \omega^{i}(x')$ ($i\neq o,a$)

Chapter III

Tetrad Transformations

A. Introduction

In the previous chapter, the space $S^4(R)$ was rotated into the de Sitter and anti-de Sitter universes, and into the pseudosphere by two tetrad rotations that were called type 0 and type I respectively. Another transformation was given that exchanged the time-like leg of the tetrad with one of the space-like legs in the de Sitter and anti-de Sitter universes. This was called a type II transformation. Finally, the combination of the type II with type I was called a type III transformation. There, the example suggested the transformations, and no rigorous basis was laid for their general application.

In this chapter, each of these transformations will be investigated, and general identities given between the important geometrical object of the original space and the corresponding object in the transformed space. It will be shown that a space derived from a solution to the field equation with $T_{ij} = 0$ by means of a type I, II, or III transformation is itself a solution to those equations. Finally two exceptions are noted and explained.

B. Type 0 Transformations

In the previous section, a type 0 transformation was defined to fit a specific example, the rotation of $S^4(R)$

into de Sitter space. More generally, a complex transformation is

$$x'^i = x'^i(x^j) \qquad (i,j = 0,1,2,3)$$

such that

$$\omega^{0'}(x') = -i\omega^0(x')$$
$$\omega^{i'}(x') = \omega^i(x') \qquad (i = 1,2,3) ,$$

where $\omega^{0'}(x')$, $\omega^{i'}(x')$, and $\omega^i(x')$ are real one forms, and $\omega^0(x')$ is imaginary. This ensures that the transformed space is again a real space, and that the tangent and cotangent spaces are real vector spaces. In the original positive definite space,

$$ds^2 = \omega^{0^2} + \Sigma\, \omega^{i^2} .$$

The transformed space is Lorentz signature because $\omega^0(x')$ is imaginary.

The general relations between the components of the connection of the transformed space and those of the original space can be derived from the structure equations (see Appendix A)

$$d\omega^{i'}(x') = -\omega^{i'}{}_{j'} \wedge \omega^{j'} = -\tfrac{1}{2}\, C^{i'}_{k'j'}\, \omega^{k'} \wedge \omega^{j'} .$$

In terms of the time-like and space-like components, one has $(a = 1,2,3)$

$$d\omega^{0'}(x') = -\tfrac{1}{2}\, C^{0'}_{0'a'}\, \omega^{0'} \wedge \omega^{a'} - \tfrac{1}{2}\, C^{0'}_{a'b'}\, \omega^{a'} \wedge \omega^{b'}$$
$$= \tfrac{1}{2}\, iC^{0'}_{0'a'}\, \omega^0 \wedge \omega^a - \tfrac{1}{2}\, C^{0'}_{a'b'}\, \omega^a \wedge \omega^b = -id\omega^0(x')$$
$$= -i(-\tfrac{1}{2}\, C^0_{0a}\, \omega^0 \wedge \omega^a - \tfrac{1}{2}\, C^0_{ab}\, \omega^0 \wedge \omega^b) .$$

From this,

$$\underline{C^{o'}_{o'a'} = C^{o}_{oa}} \quad , \quad \underline{C^{o'}_{a'b'} = -i\ C^{o}_{ab}} \qquad \text{(III-1a)}$$

and

$$d\omega^{a'}(x') = -\tfrac{1}{2}\ C^{a'}_{o'b'}\omega^{o'}\wedge\omega^{b'} - \tfrac{1}{2}\ C^{a'}_{b'c'}\omega^{b'}\wedge\omega^{c'}$$

$$= \tfrac{1}{2}\ iC^{a'}_{o'b'}\omega^{o}\wedge\omega^{b} - \tfrac{1}{2}\ C^{a'}_{b'c'}\omega^{b}\wedge\omega^{c}$$

$$= d\omega^{a}(x') = -\tfrac{1}{2}\ C^{a}_{ob}\omega^{o}\wedge\omega^{b} - \tfrac{1}{2}\ C^{a}_{bc}\omega^{b}\wedge\omega^{c}\ .$$

Thus,

$$\underline{C^{a'}_{o'b'} = i\ C^{a}_{ob}} \quad , \quad \underline{C^{a'}_{b'c'} = C^{a}_{bc}}\ . \qquad \text{(III-1b)}$$

From (A-12) the coefficients of the connection can be calculated to be

$$\underline{\Gamma^{o'}_{o'a'} = \Gamma^{o}_{oa}} \quad , \quad \underline{\Gamma^{o'}_{a'b'} = -i\ \Gamma^{o}_{ab}} \quad , \quad \underline{\Gamma^{a'}_{o'b'} = i\ \Gamma^{a}_{ob}}$$

and

$$\underline{\Gamma^{a'}_{b'o'} = i\ \Gamma^{a}_{bo}}\ . \qquad \text{(III-2)}$$

Since $\omega^{i'}{}_{j'} = \Gamma^{i}_{kj}\omega^{k}$, one has

$$\omega^{o'}{}_{a'} = \Gamma^{o'}_{b'a'}\omega^{b'} + \Gamma^{o'}_{o'a'}\omega^{o} = -i(\Gamma^{o}_{ba}\omega^{b} + \Gamma^{o}_{oa}\omega^{o}) = -i\omega^{o}{}_{a},$$

$$\omega^{a'}{}_{o'} = i\omega^{a}{}_{o}, \text{ and}$$

$$\omega^{a'}{}_{b'} = \omega^{a}{}_{b}\ .$$

Using the remaining structure equation, the components of the curvature tensor can be calculated as

$$d\omega^{o'}{}_{a'} - \omega^{o'}{}_{e'}\wedge\omega^{e'}{}_{a'} + R^{o'}{}_{a'o'b'}\omega^{o'}\wedge\omega^{b'} + \tfrac{1}{2} R^{o'}{}_{a'b'c'}\omega^{b'}\wedge\omega^{c'}$$

$$= + i\omega^{o}{}_{e}\wedge\omega^{e}{}_{a} - iR^{o'}{}_{a'o'b'}\omega^{o}\wedge\omega^{b} + \tfrac{1}{2} R^{o'}{}_{a'b'c'}\omega^{b}\wedge\omega^{c}$$

$$= -id\omega^{o}{}_{a} = +i\omega^{i}\omega^{o}{}_{e}\wedge\omega^{e}{}_{a} - iR^{o}{}_{aob}\omega^{o}\wedge\omega^{b} - \frac{i}{2} R^{o}{}_{abc}\omega^{b}\wedge\omega^{c} ,$$

and

$$R^{o'}{}_{a'o'b'} = R^{o}{}_{aob} , \quad R^{o'}{}_{a'b'c'} = iR^{o}{}_{abc} . \qquad \text{(III-3a)}$$

In the same way, one gets

$$R^{a'}{}_{b'o'd'} = iR^{a}{}_{bod} , \quad R^{a}{}_{b'c'd'} = R^{a}{}_{bcd} . \qquad \text{(III-3b)}$$

Contracting these, one gets the Ricci tensor and scalar curvature

$$R_{a'b'} = R_{ab} , \quad R_{o'o'} = -R_{oo} , \quad R_{o'a'} = iR_{oa} , \quad R' = R . \qquad \text{(III-3c)}$$

Suppose that the transformed space is a solution of the field equation

$$R_{o'o'} - \tfrac{1}{2}\eta_{o'o'}R' + \Lambda\eta_{o'o'} = \kappa T_{o'o'}$$

$$R_{a'o'} = \kappa T_{o'a'}$$

$$R_{a'b'} - \tfrac{1}{2}\eta_{a'b'}R' + n\eta_{a'b'} = \kappa T_{a'b'}$$

since the transformed space is of Lorentz signature[42] (+2), and the original space is positive definite, the primed and unprimed metrics are

$$\eta_{i'j'} = \mathrm{diag}(-1,1,1,1)$$

and $\eta_{ij} = \text{diag}(1,1,1,1)$, or

$$\eta_{o'o'} = -\eta_{oo},\ \eta_{a'b'} = \eta_{ab}\ .$$

Transforming the field equations

$$- R_{oo} + \tfrac{1}{2}\,\eta_{oo} - \Lambda\,\eta_{oo} = \kappa\, T_{o'o'}$$

$$iR_{ao} = \kappa\, T_{o'a'}$$

$$R_{ab} - \tfrac{1}{2}\,\eta_{ab}R + \Lambda\eta_{ab} = \kappa\, T_{a'b'}\ .$$

This shows that one can define a stress-energy tensor on the positive definite metric such that

$$\begin{aligned} T_{o'o'} &= -\,T_{oo} \\ T_{o'a'} &= -i\,T_{oa} \\ T_{a'b'} &= T_{ab}\ . \end{aligned} \qquad \text{(III-4)}$$

For $T_{i'j'}$ to be that of a perfect fluid requires only that $u'_o = -i\,u_o$, as would be expected.

C. <u>Type I Transformation</u>

If $\bar{x}'^{i} = \bar{x}'^{i}(x^j)$ is a complex transformation that induces (in the sense discussed for the type 0 transformation) a transformation on the tetrad of one forms such that:

$$\omega^{i'}(x') = -i\omega^{i}(x')$$

then the components of the connection can be calculated as before

$$d\omega^{i'}(x') = -\tfrac{1}{2}\, C^{i'}_{j'k'}\,\omega^{j'}\wedge\omega^{k'} = \tfrac{1}{2}\, C^{i'}_{j'k'}\,\omega^{j}\wedge\omega^{k}$$

$$= -id\omega'(x') = i\, C^{i}_{jk}\,\omega^{j}\wedge\omega^{k} \ .$$

Therefore

$$C^{i}_{j'k'} = i\, C^{i}_{jk} \ ,$$

$$\Gamma^{i'}_{j'k'} = i\, \Gamma^{i}_{jk} \ , \qquad \text{(III-5)}$$

$$\omega^{i'}_{j'} = \omega^{i}_{j} \ .$$

This means that

$$d\omega^{i'}_{j'} = -\, \omega^{i}_{k'}\wedge\omega^{k'}_{j'} + \tfrac{1}{2}\, R^{i'}_{j'k'\ell'}\,\omega^{k'}\wedge\omega^{\ell'}$$

$$= -\, \omega^{i}{}_{k}\wedge\omega^{k}{}_{j} - \tfrac{1}{2}\, R^{i'}_{j'k'\ell'}\,\omega^{k}\wedge\omega^{\ell}$$

$$= d\omega^{i}_{j} = -\, \omega^{i}_{k}\wedge\omega^{k}_{j} + \tfrac{1}{2}\, R^{i}_{jk\ell}\,\omega^{k}\wedge\omega^{\ell} \ ,$$

giving

$$R^{i}_{j'k'\ell'} = -\, R^{i}_{jk\ell} \qquad \text{(III-6a)}$$

and

$$R_{ij} = -\, R_{ij} \ . \qquad \text{(III-6b)}$$

But due to the fact that $\eta_{i'j'} = -\,\eta_{ij'}$, $R = R$

and

$$R_{i'j'} - \tfrac{1}{2}\,\eta_{i'j'}R' + \Lambda\eta_{i'j'} = K\, T_{i'j'}$$

$$R_{ij} - \tfrac{1}{2}\,\eta_{ij}\, R + \Lambda\eta_{ij} = -\, K\, T_{ij} \ .$$

Thus

$$T_{i'j'} = -\, T_{ij'} \ . \qquad \text{(III-7)}$$

In general, this transformation rotates a universe with positive energy density and pressure into one with negative energy density and pressure. The sign of the cosmological constant is also changed (the form of the field equations changes with the change of the sign of the signature).

Before going on to discuss the other two types of transformations, it is interesting to note an apparent exception to the above. The metric of the dust filled Robertson-Walker universe, with k = +1, can be written

$$ds^2 = -dt^2 + R^2(d\chi^2+\sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)) ,$$

where

$$R = R_o(1 - \cos\eta)$$

$$t = R_o(\eta - \sin\eta) .$$

A type I transformation on this metric is $t \to it$, and $\chi \to i\chi$ yielding the metric

$$ds^2 = dt^2-R^2(d\chi^2+\sinh^2\chi(d\theta^2+\sin^2\theta d\varphi^2)$$

with

$$R = R_o(1 - \cosh \eta) ,$$

$$t = R_o(\eta - \sinh \eta) .$$

This is the R. W. (k = -1) universe except that R has the opposite of its usual sign. The R. W. (k = -1) universe is known to have positive energy density.

The explanation for this apparent discrepancy from the above results lies in the fact that, for this universe, $\rho \sim R^{-3}$. When the type I transformation is made from the $k = +1$ to the $k = -1$ universe, the energy density is indeed negative, but so is R. The change to the usual sign for R corrects the negative sign of ρ. This example stands as a warning against the simplistic generalizations from the properties of these transformations.

D. Types II and III Transformations

A complex transformation of type II transforms the tetrad of one-forms as

$$\begin{aligned} \omega^{0'}(x') &= -\, i\omega^{1}(x') \\ \omega^{1'}(x') &= -\, i\omega^{0}(x') \\ \omega^{i}(x') &= \omega^{i}(x') \;. \end{aligned} \tag{III-7}$$

The components of the structure tensor and the connection must be calculated. This is done by means of the structure equations, as before. From this, the identities relating the components of the structure tensor and the components of the connection of the original and transformed spaces are

$$\begin{aligned} &C^{0'}_{1'0'} = iC^{1}_{01} \;;\quad C^{1'}_{0'1'} = iC^{0}_{10} \;;\quad C^{a}_{0'1'} = C^{a}_{01} \\ &C^{0'}_{a'b'} = -iC^{1}_{ab} \;;\quad C^{1'}_{a'b'} = -iC^{0}_{ab} \;;\quad C^{0'}_{0'a'} = C^{1}_{1a} \\ &C^{1'}_{1'a'} = C^{0}_{0a} \;;\quad C^{1'}_{0'a'} = C^{0}_{1a} \;;\quad C^{0'}_{1'a'} = C^{1}_{0a} \\ &C^{a'}_{0'b'} = iC^{a}_{1b} \;;\quad C^{a'}_{1b'} = iC^{a}_{0b} \;;\quad C^{a'}_{b'c'} = C^{a'}_{b0'} \end{aligned} \tag{III-8a}$$

$$\Gamma^{0'}{}_{01'} = i\Gamma^{1}_{10} \;;\quad \Gamma^{a'}{}_{0'0'} = -\Gamma^{a}_{11} \;;\quad \Gamma^{a'}{}_{0'1'} = -\Gamma^{a}_{10}$$
$$\Gamma^{a'}{}_{1'0'} = -\Gamma^{a}_{01} \;;\quad \Gamma^{a'}{}_{1'1'} = -\Gamma^{a}_{00} \;;\quad \Gamma^{1'}{}_{a'0'} = \Gamma^{0}_{a1} \qquad \text{(III-8b)}$$
$$\Gamma^{0'}{}_{a'1'} = \Gamma^{1}_{a0} \;;\quad \Gamma^{0'}{}_{a'b'} = -i\Gamma^{1}_{ab} \;;\quad \Gamma^{1}{}_{a'b'} = -i\Gamma^{0}_{ab}$$
$$\Gamma^{a'}{}_{0'b'} = i\Gamma^{a}_{1b} \;;\quad \Gamma^{a'}{}_{1'b'} = i\Gamma^{a}_{0b} \;;\quad \Gamma^{a'}{}_{b'c'} = \Gamma^{a}_{bc}$$

Also, using the second structure equation, the components of the curvature tensor may be calculated:

$$R^{0'}{}_{1'0'1'} = R^{1}_{001} \;;\quad R^{0'}{}_{1'0'a'} = i\, R^{1}_{01a}$$
$$R^{0'}{}_{1'1'a'} = iR^{1}_{00a} \;;\quad R^{0'}{}_{1'b'a'} = R^{1}_{0ba}$$
$$R^{0'}{}_{a'0'b'} = R^{1}_{a1b} \;;\quad R^{0'}{}_{a'1'b'} = R^{1}_{a0b} \qquad \text{(III-9a)}$$
$$R^{0'}{}_{a'b'c'} = -iR^{1}_{abc} \;;\quad R^{1'}{}_{a'1'b'} = R^{0}_{a0b}$$
$$R^{1'}{}_{a'b'c'} = -iR^{0}_{abc} \;;\quad R^{a'}{}_{b'c'd'} = R^{a}_{bcd}$$

From these, the Ricci tensor is obtained:

$$R_{0'0'} = -R_{11} \qquad R_{0'a'} = i\, R_{1a}$$
$$R_{1'1'} = -R_{00} \qquad R_{1'a'} = i\, R_{0a} \qquad \text{(III-9b)}$$
$$R_{a'b'} = R_{ab} \qquad R_{0'1'} = -R_{01}$$

It is to be seen from this that only if $T_{ij} = 0$ will one solution to the field equation be rotated onto another. It is possible, however, that such a transformation could be made, and that ia could be a solution of the field equations if one leg of the tetrad is "stretched," i.e. if the scale is changed in the direction of that particular leg. An example of this will be seen in connection with the type III transformation, which will be discussed next.

If a complex coordinate transformation induces the transformation on the tetrad

$$\begin{aligned}
\omega^{0'}(x') &= -\,\omega^{1}(x') \\
\omega^{1'}(x') &= -\,\omega^{0}(x') \\
\omega^{2'}(x') &= -\,i\omega^{2}(x') \\
\omega^{3'}(x') &= -\,i\omega^{3}(x') \;,
\end{aligned} \tag{III-10}$$

then it is a type III transformation. It is easy to see that this space will be a solution of the homogeneous field equations if the original space was. This follows from the fact the type III transformation is simply a type I transformation followed by type II transformation. Using the same methods previously employed, identities between the components of the structure tensor, the connections, in the original and transformed spaces are

$$\begin{aligned}
&C^{0'}_{1'0'} = -\,C^{1}_{01} \;;\quad C^{a'}_{0'1'} = iC^{a}_{01} \;;\quad C^{1'}_{0'1'} = -\,C^{0}_{10} \\
&C^{0'}_{a'b'} = C^{1}_{ab} \;;\quad C^{1'}_{a'b'} = C^{0}_{ab} \;;\quad C^{0'}_{0'a'} = iC^{1}_{1a} \\
&C^{1'}_{1'a'} = iC^{0}_{0a} \;;\quad C^{0'}_{1'a'} = iC^{1}_{0a} \;;\quad C^{1'}_{0'a'} = iC^{0}_{1a} \\
&C^{a'}_{0'b'} = -\,C^{a}_{1b} \;;\quad C^{a'}_{1'b'} = -\,C^{a}_{0b} \;;\quad C^{a'}_{b'c'} = C^{a'}_{b'c'}
\end{aligned} \tag{III-11}$$

$$\begin{aligned}
&\Gamma^{0'}_{0'1'} = -\,\Gamma^{1}_{10} \;;\quad \Gamma^{1'}_{1'0'} = -\,\Gamma^{0}_{01} \;;\quad \Gamma^{a'}_{0'0'} = -i\Gamma^{a}_{11} \\
&\Gamma^{a'}_{1'0'} = -\,i\Gamma^{a}_{01} \;;\quad \Gamma^{a'}_{0'1'} = -\,i\Gamma^{a}_{10} \;;\quad \Gamma^{a'}_{1'1'} = -\,i\Gamma^{a}_{00} \\
&\Gamma^{0'}_{a'1'} = i\Gamma^{1}_{a0} \;;\quad \Gamma^{0'}_{a'b'} = \Gamma^{1}_{ab} \\
&\Gamma^{1'}_{a'b'} = \Gamma^{0}_{ab} \;;\quad \Gamma^{a'}_{0'b'} = -\,\Gamma^{a}_{1b} \;;\quad \Gamma^{a'}_{1'b'} = -\,\Gamma^{a}_{0b} \\
&\Gamma^{a'}_{b'c'} = i\,\Gamma^{a}_{bc}
\end{aligned} \tag{III-12}$$

This gives the following relations between the components of the curvature tensor and the Ricci tensor in the original and the transformed spaces

$$
\begin{aligned}
&R^{0'}{}_{1'0'1'} = -R^{1}{}_{001} \quad ; \quad R^{0'}{}_{1'1'a'} = -iR^{1}{}_{00a} \\
&R^{0'}{}_{1'0'a'} = -iR^{1}{}_{01a} \quad ; \quad R^{0'}{}_{1'b'a'} = -R^{1}{}_{0ba} \\
&R^{0'}{}_{a'0'b'} = -R^{1}{}_{a1b} \quad ; \quad R^{1'}{}_{a'1'b'} = -R^{0}{}_{a0b} \\
&R^{1'}{}_{a'0'b'} = -R^{0}{}_{a1b} \quad ; \quad R^{0'}{}_{a'b'c'} = iR^{1}{}_{abc} \qquad \text{(III-13)} \\
&R^{1'}{}_{a'b'c'} = iR^{0}{}_{abc} \quad ; \quad R^{3'}{}_{2'3'2'} = -R^{3}{}_{232} \\
&R_{0'0'} = -R_{11} \quad ; \quad R_{1'1'} = -R_{00} \quad ; \quad R_{0'1'} = -R_{10} \\
&R_{0'a'} = -iR_{1a} \quad ; \quad R_{1'a'} = -iR_{0a} \quad ; \quad R_{ab} = -R_{ab}
\end{aligned}
$$

As has been seen, the type II transformation on the de Sitter universe exchanges a space-like leg of the tetrad and the time-like one. Acting on the anti~~de~~ Sitter universe, it does the same thing. Obviously, a type III transformation maps the de Sitter onto one of these versions of the anti~~de~~ Sitter universe, and vice versa.

More generally, by relabeling the space-like legs of the tetrad, a type III transformation can be written

$$
\begin{aligned}
\omega^{0'} &= \omega^{2} \\
\omega^{2'} &= \omega^{0} \\
\omega^{1'} &= -i\omega^{3} \\
\omega^{3'} &= -i\omega^{1}
\end{aligned}
$$

The effect of this transformation ~~is~~ is to interchange ℓ and m, and n and $\bar{m}$, where ℓ, n, m are the usual pseudoorthonormal tetrad of the Newman-Penrose formalism (see Appendix B]. Under some circumstances, this is a very significant transformation.

E. The Transformation of the Einstein-Static Universe into the Gödel Universe

An example of the type III transformation plus a "time-like stretch" is the transformation of the Einstein static universe into the Gödel universe [11,12]. From previous results, a type III transformation from the Einstein static universe would not be expected to yield a solution to the field equations since $T_{\mu\nu} \neq 0$. In this case, however, a "stretch" along the $\frac{\partial}{\partial t}$ direction enables one to obtain another solution to the field equations.

The Einstein static universe is R x S^3, where R is the time-line, and S^3 is the space-like hypersurface. In this transformation the space-like hypersurface becomes a time-like hypersurface, and R is now simply an ignorable space-like coordinate.

The metric for the Einstein static universe is

$$ds^2 = - dt^2 + \frac{1}{4}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)$$

where the σ_i's are the left invariant one-forms of S^3 in euler angle coordinates [13], given by

$$\sigma_z = - (d\psi + \cos\theta d\varphi)$$

$$\sigma_x = \sin\psi d\theta - \cos\psi \sin\theta d\varphi$$

$$\sigma_y = \cos\psi d\theta + \sin\psi \sin\theta d\varphi .$$

The transformation is

$$\theta = -(2ir + \pi) \quad , \qquad d\theta = -2idr \ ,$$

$$\psi = \sqrt{2t} + \varphi \ , \qquad d\psi = \sqrt{2}\, dt + d\varphi \ ,$$

$$t = z \ . \qquad dt = dz \ .$$

In terms of these variables, the one-forms are

$$\sigma_z = -(\sqrt{2}\, dt + d\varphi + \cos(2ir+\pi)\, d\varphi)$$

$$= -(\sqrt{2}\, dt + d\varphi - \cosh 2r\, d\varphi)$$

$$= -\sqrt{2}(dt - \sqrt{2}\, \sinh^2 r\, d\varphi)$$

$$\sigma_x = \sin(\sqrt{2}\, t + \varphi)(-2idr) - \cos(\sqrt{2}\, t+\varphi)\sin[-(2ir+\pi)]\, d\varphi$$

$$= -2i(\sin(\sqrt{2}\, t+\varphi)dr + \cos(\sqrt{2}\, t+\varphi)\sinh r \cosh r\, d\varphi)$$

$$\sigma_y = \cos(\sqrt{2}\, t+\varphi)(-2idr) + \sin(\sqrt{2}\, t+\varphi)\sin[-(2ir+\pi)]\, d\varphi$$

$$= -2i(\cos(\sqrt{2}\, t+\varphi)dr - \sin(\sqrt{2}\, t+\varphi)\sinh r \cosh r\, d\varphi)$$

If one sets

$$\omega^0 = dt \ ,$$

$$\omega^1 = \frac{\sigma_x}{2} \ ,$$

$$\omega^2 = \frac{\sigma_y}{2} \ ,$$

$$\omega^3 = \frac{\sigma_z}{2} \ ,$$

then in the new coordinates

$\omega^o = dz$

$\omega^1 = -i[\sin(\sqrt{2}\ t+\varphi)dr + \cos(\sqrt{2}\ t+\varphi)\sinh r \cosh r\ d\varphi]$

$\omega^2 = -i[\cos(\sqrt{2}\ t+\varphi)dr - \sin(\sqrt{2}\ t+\varphi)\sinh r \cosh r\ d\varphi]$

$\omega^3 = -\frac{1}{\sqrt{2}}(dt - \sqrt{2}\ \sinh^3 r\ d\varphi)$

so that

$$\omega^{o'} = \omega^3 ,$$
$$\omega^{3'} = \omega^o ,$$
$$\omega^{1'} = -i\omega^2 ,$$
$$\omega^{2'} = -i\omega^1 ,$$

and one has a type III transformation. A simple calculation of the Ricci tensor proves that this does not yield a physical solution to the field equations ($T_{ij} = 4\ {}_{i'}\delta_{j'}$). However, if the tetrad of one-forms is chosen so that

$$\omega^{o''} = \sqrt{2}\ \omega^{o'}$$
$$\omega^{i''} = \omega^{i'} ,$$

that is:

$\omega^{o''} = -(dt - \sqrt{2}\ \sinh r\ d\varphi)$

$\omega^{1''} = -[\cos(\sqrt{2}\ t+\varphi)dr - \sin(\sqrt{2}\ t+\varphi)\sinh r \cosh r\ d\varphi]$

$\omega^{2''} = -[\sin(\sqrt{2}\ t+\varphi)dr + \cos(\sqrt{2}\ t+\varphi)\sinh r \cosh r\ d\varphi]$

$\omega^{3''} = dz ,$

then this <u>is</u> a solution to the field equation with the metric

$ds = -(dt-\sqrt{2}\ \sinh^2 r\ d\varphi)^2 + dr^2 + \cosh^2 r \sinh^2 r\ d\varphi^2 + dz^2$.

This is just the second form of the metric for the Gödel universe.

F. Conclusion

The following is a summary of the results of this chapter.

1. Type 0 transformation

$$\omega^{0'}(x') = -i\omega^{0}(x')$$
$$\omega^{i'}(x') = \omega^{i}(x')\ (i = 1,2,3)$$

Components of the connection

$$\Gamma^{0}_{0'a'} = \Gamma^{0}_{0a} \quad ; \quad \Gamma^{0'}_{a'b'} = -i\Gamma^{0}_{ab} \quad ; \quad \Gamma^{a'}_{0'b'} = i\Gamma^{a}_{0b}$$
$$\Gamma^{a'}_{b'0'} = i\Gamma^{a}_{b0}$$

Components of the curvature tensor

$$R^{0'}_{a'0'b'} = R^{0}_{a0b} \quad ; \quad R^{0'}_{a'b'c'} = -iR^{0}_{abc}$$
$$R^{a'}_{b'0'd'} = iR^{a}_{b0d} \quad ; \quad R^{a'}_{b'c'd'} = R^{a}_{bcd}$$

2. Type I transformation

$$\omega^{i'}(x') = -\ i\omega^{i}(x')$$

The connection

$$\Gamma^{i'}_{j'k'} = i\Gamma^{i}_{jk}$$

The curvature tensor

$$R^{i'}_{j'k'\ell'} = -R^{i}_{jk\ell}$$

3. Type II transformation

$$\omega^{0'}(x') = -i\omega^{1}(x')$$

$$\omega^{1'}(x') = -i\omega^{0}(x')$$

$$\omega^{i'}(x') = \omega^{i}(x')$$

The connection

$$\Gamma^{0'}_{0'1'} = i\Gamma^{1}_{10} \quad ; \quad \Gamma^{a'}_{0'0'} = -\Gamma^{a}_{11} \quad ; \quad \Gamma^{a'}_{0'1'} = -\Gamma^{a}_{10}$$

$$\Gamma^{a'}_{1'0'} = -\Gamma^{a}_{01} \quad ; \quad \Gamma^{a'}_{1'1'} = -\Gamma^{a}_{00} \quad ; \quad \Gamma^{1'}_{a'0'} = \Gamma^{0}_{a1}$$

$$\Gamma^{0'}_{a'1'} = \Gamma^{1}_{a0} \quad ; \quad \Gamma^{0'}_{a'b'} = -i\Gamma^{1}_{ab} \quad ; \quad \Gamma^{1}_{a'b'} = -i\Gamma^{0}_{ab}$$

$$\Gamma^{a'}_{0'b'} = i\Gamma^{a}_{1b} \quad ; \quad \Gamma^{a'}_{1'b'} = i\Gamma^{a}_{0b} \quad ; \quad \Gamma^{a}_{b'c'} = \Gamma^{a}_{bc}$$

The curvature tensor

$$R^{0'}_{1'0'1'} = R^{1}_{001} \quad ; \quad R^{0'}_{1'0'a'} = iR^{1}_{01a}$$

$$R^{0'}_{1'1'a'} = iR^{1}_{00a} \quad ; \quad R^{0'}_{1'b'a'} = R^{1}_{0ba}$$

$$R^{0'}_{a'0'b'} = R^{1}_{a1b} \quad ; \quad R^{0'}_{a'1'b'} = R^{1}_{a0b}$$

$$R^{0'}_{a'b'c'} = -iR^{1}_{abc} \quad ; \quad R^{1'}_{a'1'b'} = R^{0}_{a0b}$$

$$R^{1'}_{a'b'c'} = -iR^{0}_{abc} \quad ; \quad R^{a'}_{b'c'd'} = R^{a}_{bcd}$$

4. Type III transformation

$$\omega^{0'}(x') = -\omega^{1}(x')$$

$$\omega^{1'}(x') = -\omega^{0}(x')$$

$$\omega^{2'}(x') = -i\omega^{2}(x')$$

$$\omega^{3'}(x') = -i\omega^{3}(x')$$

The connection

$$\Gamma^{0'}_{0'1'} = -\Gamma^{1}_{10} \quad ; \quad \Gamma^{1'}_{1'0'} = -\Gamma^{0}_{01} \quad ; \quad \Gamma^{a'}_{0'0'} = -i\Gamma^{a}_{11}$$

$$\Gamma^{a'}_{1'0'} = -i\Gamma^{a}_{01} \quad ; \quad \Gamma^{a'}_{0'1'} = -i\Gamma^{a}_{10} \quad ; \quad \Gamma^{a'}_{1'1'} = -i\Gamma^{a}_{00}$$

$$\Gamma^{0'}_{a'1'} = i\Gamma^{1}_{a0} \quad ; \quad \Gamma^{0'}_{a'b'} = \Gamma^{1}_{ab} \quad ; \quad \Gamma^{1'}_{a'b'} = \Gamma^{0}_{ab}$$

$$\Gamma^{a'}_{0'b'} = -\Gamma^{a}_{1b} \quad ; \quad \Gamma^{a'}_{1'b'} = -\Gamma^{a}_{0b} \quad ; \quad \Gamma^{a'}_{b'c'} = i\Gamma^{a}_{bc}$$

The curvature tensor

$$R^{0'}_{1'0'1'} = -R^{1}_{001} \quad ; \quad R^{0'}_{1'1'a'} = -iR^{1}_{00a}$$

$$R^{0'}_{1'0'a'} = \cancel{-}iR^{1}_{01a} \quad ; \quad R^{0'}_{1'b'a'} = -R^{1}_{0ba}$$

$$R^{0'}_{a'0'b'} = -R^{1}_{a1b} \quad ; \quad R^{1'}_{a'1'b'} = -R^{0}_{a0b}$$

$$R^{1'}_{a'0'b'} = -R^{0}_{a1b} \quad ; \quad R^{0'}_{a'b'c'} = iR^{1}_{abc}$$

$$R^{1'}_{a'b'c'} = iR^{0}_{abc} \quad ; \quad R^{3'}_{2'3'2'} = -R^{3}_{232}$$

Chapter IV

Physical Interpretation of These Transformations

A. Introduction

In order to give some physical meaning to the foregoing results, one must relate the physical characteristics of the original space to those of the transformed space. This can be done for two sets of parameters.

All elementary cosmologies are constructed on the premise that the universe is a perfect fluid. The hydrodynamical parameters that characterize such a universe are treated in Section B. These parameters describe the motion of the fluid and represent a decomposition of the properties of the covariant derivative of u, the tangent to the streamlines, onto a hypersurface orthogonal to the streamlines. The first such parameter is θ, the trace of this projection of $u_{\mu;\nu}$. It is a measure of the rate at which neighboring streamlines diverge or converge. When this quantity is subtracted from the projection of $u_{\mu;\nu}$, a traceless matrix remains. The antisymmetric part of this is the vorticity tensor, $\omega_{\mu\nu}$, which measures the rate at which the streamlines of matter twist about each other. The symmetric part is the shear, $\sigma_{\mu\nu}$, which measures the anisotropy of the expansion of the fluid. When θ, $\sigma_{\mu\nu}$, and $\omega_{\mu\nu}$ are expressed with respect to the orthonormal tetrad, they are the trace, the symmetric, and the antisymmetric parts of a 3 x 3 matrix. They can be directly and simply related to the components of the connection, whose transformation properties were studied in the last chapter. Once these parameters are known for one space, they can be calculated for all the spaces that can be obtained from it by a type I, II or III transformation. The de Sitter universe and its transforms are examples of this.

The second set of important parameters that characterize a space are those relating to the null paths of that space. These are the twelve spin coefficients of the Newman-Penrose formalism, and the five components of the complex Weyl tensor which are treated in Section C.

The orthonormal tetrad formalism employs a tetrad of vectors to span the space of the vector fields of the space time, one of which is time-like, and the other three of which are space-like. The Newman-Penrose formalism employs two null vectors ℓ and n, and two space-like vectors which are contained in the real and imaginary parts of a complex vector, m. The spin coefficients are to these vectors what the components of the connection are to the orthonormal tetrad. Two of these tell whether or not the null paths tangent to ℓ and n are geodesics, and two more tell whether or not the parameters used are affine or not. If these two vectors are geodesic, then there are four other spin coefficients that give the expansion, vorticity, and shear of their paths.

The spin coefficients of this formalism are combinations of the components of the connection in the tetrad formalism, and hence their transformation properties are known. These can be generally stated for the case of the type I transformation, but not for the others since the time-like vector can swap with any of the three space-like vectors under these transformations. The examples of the de Sitter universe, and its transforms are worked out. In the de Sitter universe, the two null vectors ℓ and n are geodesics. When ω^o is swapped with ω^1, ℓ and n keep their identity, and are still geodesic. But when ω^o is swapped with ω^2 or ω^3, then ω^1 must be swapped with ω^3 or ω^2 respectively for the new ℓ and n to be geodesic.

This situation is true for all algebraically special metrics, and merits a separate investigation in Section D.

B. Hydrodynamical Parameters

As has been indicated, most cosmologies are constructed on the premise that the universe is a perfect fluid. In terms of finding a solution to Einstein's field equations this means that

$$T_{ij} = (P + \rho)u_i u_j + \eta_{ij}$$

where P is the fluid pressure, ρ its density, and u is the vector tangent to its streamlines.

The tangent to the streamlines of the fluid must be time-like and can be chosen to be the time-like leg of the orthonormal tetrad. A Lorentz transformation will put it into the form $u = \partial/\partial t$ [see Appendix A, Section A]. The motion of the fluid is characterized by the covariant derivative of u (or X_o), which is customarily decomposed into three parts.

The first "piece" of $u_{\mu;\nu}$ that is important is its trace $u^{\mu}{}_{;\mu}$. This quantity measures the isotropic expansion of the fluid, as can be seen from the following elementary considerations.

From the rule on covariant gradients,

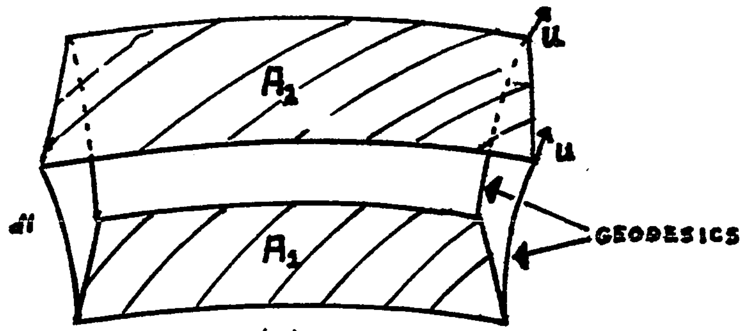
$$u^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g}\; u^{\mu}) \quad .$$

Suppose that u^{μ} is geodesic, and that the coordinate system is cartesian at a point (and hence along a geodesic). An infinitesimal volume ΔV is constructed about the geodesic, bounded on the sides by geodesics, and on the top and bottom by surface increments that are orthogonal to the geodesics, and having areas A_1 and $A_2 = A_1 + \Delta A$. This is illustrated in Fig. [IV-1a].

Integrating over the 4-volume:

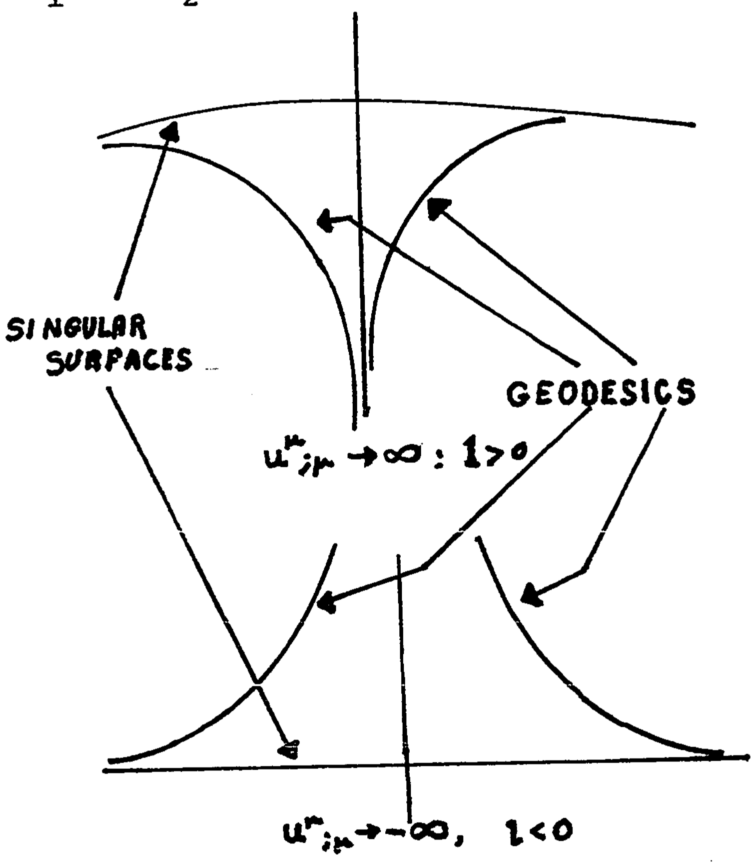
$$\int u^{\mu}{}_{;\mu}\, dv = \int \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} (\sqrt{-g}\; u^{\mu}) \sqrt{-g}\; d^4x = \int \sqrt{-g}\; u^{\mu}\, ds^{\mu}$$

$$= \int_{A_1}^{A_2} ds = A_2 - A_1 = \Delta A$$



(a)

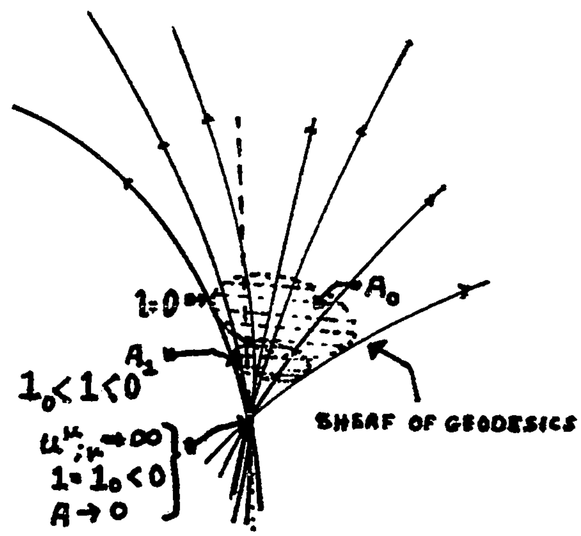
The element of volume Δv bounded by geodesics, and by A_1 and A_2.



(b)

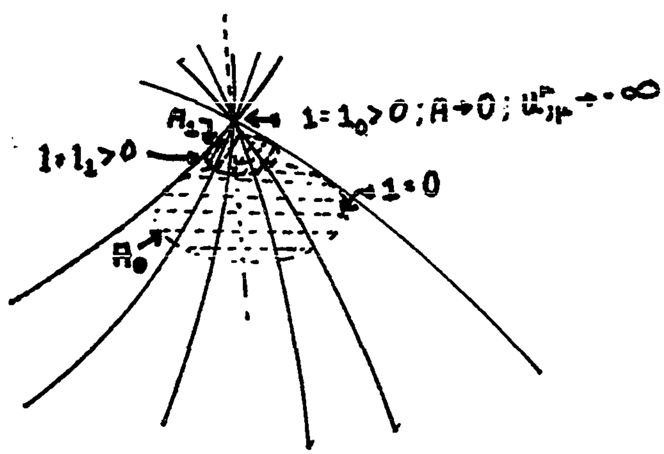
Two cases where the spacetime is singular at $\ell = \ell_o$.

Fig. [IV-1]



(c)

The case where $u^{\mu}{}_{;\mu} \to \infty$, $\ell < 0$, and the sheaf of geodesics has an intersection in the past.



(d)

The case where $u^{\mu}{}_{;\mu} \to -\infty$, $\ell > 0$, and the sheaf of geodesics has an intersection in the future.

where ds_μ is the element of area orthogonal to the x^μ coordinate direction, and dS is the element of area orthogonal to u^μ. In this infinitesimal volume,

$$\Delta A \approx u^\mu{}_{;\mu} A_1 d\ell$$

or

$$\frac{dA}{d\ell} \sim u^\mu{}_{;\mu} A_1$$

If now, $u^\mu{}_{;\mu} > 0$, $dA/d\ell > 0$, and the area orthogonal to u, the cross section of a sheaf of geodesics, will increase as ℓ increases. If $u^\mu{}_{;\mu} = 0$, this area will remain constant, and if $u^\mu{}_{;\mu} < 0$, it will decrease. Thus, $u^\mu{}_{;\mu}$ is a measure of the rate at which neighboring geodesics converge or diverge. It must be noted that this deduction is based upon the behavior of a sheaf of geodesics, and $\frac{dA}{d\ell}$ is a measure of the rate of change of the "cross section" of this sheaf. Nothing is indicated as to the shape of the cross section, or how the geodesics cross it.

Integrating the expression above, the behavior of A over some small increment will be approximately

$$A \sim \exp[\int (\mu^\mu{}_{;\mu}) d\ell]$$

When $u^\mu{}_{;\mu} \to 0$, as has been said before, A remains constant, and the geodesics neither converge or diverge. Another case that needs to be considered is what happens when $u^\mu{}_{;\mu}$ becomes singular, i.e. when $u^\mu{}_{;\mu} \to \infty$ or $u^\mu{}_{;\mu} \to -\infty$. In the cases where $u^\mu{}_{;\mu} \to \infty$, $\ell > 0$ or $u^\mu{}_{;\mu} \to -\infty$, $\ell < 0$, $A \to \infty$. If ℓ remains finite, then this space-time has a singularity, since these geodesics could not be continued beyond that value of ℓ. This is illustrated in Fig. [IV-1b]. If, on the other hand, $u^{\mu}{}_{;\mu} \to \infty$, $\ell < 0$, or $u^{\mu}{}_{;\mu} \to -\infty$, $\ell > 0$, then $A \to 0$. In the first instance, it means that the geodesics are diverging from a point. In the second instance, it means that they are converging toward one. These are illustrated in Fig. [IV-1c and d].

There is another "piece" of $u_{\mu;\nu}$ that has obvious significance. This is the antisymmetric part of it. The quantity $u_{[\mu;\nu]} = \frac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu}) = \frac{1}{2}(u_{\mu,\nu} - u_{\nu,\mu})$ should have some relation to the vorticity of the fluid, as it does in classical hydrodynamics. The difference is that in classical hydrodynamics, the vorticity is related to the three dimensional curl of the velocity vector field, whereas in this case, it is related to a four dimensional "curl." It is necessary to reduce the dimensionality of this quantity. This can be done by projecting $u_{[\mu;\nu]}$ onto the hypersurface orthogonal to u by means of the projection operator $h_{\mu\nu} = (g_{\mu\nu} + u_\mu u_\nu)$ [4]. The resulting vorticity tensor is

$$\omega_{\mu\nu} = h_\mu{}^\rho \, h_\nu{}^\sigma \, u_{[\rho,\sigma]} \ .$$

This, now, is a genuine, three dimensional "curl" that is related to the rate at which the cosmological fluid rotates in its rest frame.

The tensor that describes the way that the cosmological fluid "flows" is $u_{\mu;\nu}$. Two "pieces" of this represent important parameters describing this flow. One of these is the vorticity tensor that measures the rotation of the fluid. The other is the divergence of the fluid. If the original tensor $u_{\mu;\nu}$ is projected upon the hypersurface orthogonal to it and the vorticity subtracted off, there remains

$$\Theta_{\mu\nu} = h_\mu{}^\rho\, h_\nu{}^\sigma\, u_{\rho;\sigma} - \omega_{\mu\nu}\ ,$$

$$= h_\mu{}^\rho\, h_\nu{}^\sigma\, u_{\rho;\sigma} - h_\mu{}^\rho\, h_\nu{}^\sigma\, u_{[\rho;\sigma]}\ ,$$

$$= h_\mu{}^\rho\, h_\nu{}^\sigma\, (u_{\rho;\sigma} - \tfrac{1}{2}(u_{\rho;\sigma} - u_{\sigma;\rho}))\ ,$$

$$= h_\mu{}^\rho\, h_\nu{}^\sigma\, \tfrac{1}{2}(u_{\rho;\sigma} + u_{\sigma;\rho}) = h_\mu{}^\rho\, h_\nu{}^\sigma\, u_{(\mu;\nu)}\ ,$$

where ( ) always denotes symmetrization. The trace of this tensor is

$$g^{\mu\nu}(\delta_\mu{}^\rho + u_\mu u^\rho)(\delta_\nu{}^\sigma + u_\nu u^\sigma)\ \tfrac{1}{2}(u_{\rho;\sigma} + u_{\sigma;\rho})$$

$$= g^{\mu\nu}(\delta_\mu{}^\rho\delta_\mu{}^\sigma + \delta_\mu{}^\rho u_\nu u^\sigma + u_\mu u^\rho \delta_\nu{}^\sigma + u_\mu u_\nu u^\rho u^\sigma)\tfrac{1}{2}(u_{\rho;\sigma} + u_{\sigma;\rho})$$

$$= (g^{\rho\sigma} + u^\rho u^\sigma)\ \tfrac{1}{2}(u_{\rho;\sigma} + u_{\sigma;\rho})$$

$$= g^{\rho\sigma} u_{\rho;\sigma} + u^\rho u^\sigma u_{\sigma;\rho} = u^\sigma{}_{;\sigma}\ .$$

Now, the trace free part of $\Theta_{\mu\nu}$ is what remains after the vorticity and expansion have been subtracted from $u_{\mu;\nu}$, and it is called the shear

$$\sigma_{\mu\nu} = h_\mu{}^\rho\, h_\nu{}^\sigma u_{(\rho;\sigma)} - \frac{h_{\mu\nu}}{3}\,\theta$$

where $\theta = g^{\mu\nu}\Theta_{\mu\nu} = u^{\mu}{}_{;\mu}$. Since the trace of $\Theta_{\mu\nu}$ is , the isotropic expansion, it might be concluded that $\Theta_{\mu\nu}$ is an expansion tensor. This means that the shear, the expansion tensor minus its isotropic part, is a measure of the anisotropy of the expansion.

To reiterate, $u_{\mu;\nu}$ is decomposed into three parts,

$$\text{Expansion:} \quad \theta = u^{\mu}{}_{;\mu} \tag{IV-1a}$$

$$\text{Vorticity:} \quad \omega_{\mu\nu} = h_{\mu}{}^{\rho}\, h_{\nu}{}^{\sigma} u_{[\rho;\sigma]} \tag{IV-1b}$$

$$\text{Shear:} \quad \sigma_{\mu\nu} = h_{\mu}{}^{\rho} h_{\nu}{}^{\sigma} u_{(\rho;\sigma)} - \frac{1}{3} h_{\mu\nu}\theta \tag{IV-1c}$$

These quantities are illustrated in Fig. [IV-2].

The next step is to express these quantities with respect to the tetrad basis

$$\sigma_{ij} = X_i{}^{\mu}\, X_j{}^{\nu}(h_{\mu}{}^{\rho} h_{\nu}{}^{\sigma} u_{(\rho;\sigma)} - \frac{h_{\mu\nu}}{3}\theta)$$

$$\omega_{ij} = X_i{}^{\mu}\, X_j{}^{\nu}(h_{\mu}{}^{\rho} h_{\nu}{}^{\sigma} u_{[\rho;\sigma]})$$

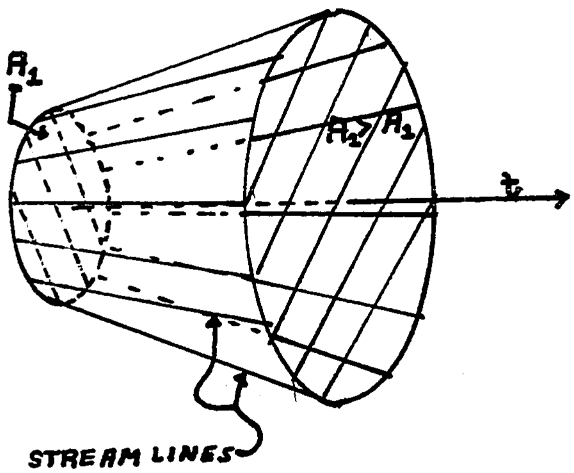
To evaluate these expressions, one needs

$$X_i{}^{\mu} h_{\mu}{}^{\rho} = X_{\mu}{}^{i}(\delta_{\mu}{}^{\rho} + u_{\mu}{}^{\rho}) = X_i{}^{\rho} - \delta_i{}^{o} X_o = \begin{cases} 0 & \text{if } i = 0 \\ X_{\bar{a}}{}^{\rho} & \text{if } i=a\neq 0 \end{cases}$$

Now, these definitions become

$$\begin{aligned}
\sigma_{ij} = \sigma_{ab} &= X_a{}^{\rho} X_b{}^{\sigma} u_{(\rho;\sigma)} - \frac{\delta_{ab}}{3}\theta \\
&= u_{(a;b)} - \frac{\delta_{ab}}{3}\theta \\
&= \tfrac{1}{2}(\Gamma_{ab}{}^{o} + \Gamma_{ba}{}^{o}) - \frac{\delta_{ab}}{3}\theta
\end{aligned} \tag{IV-2a}$$

$$\omega_{ij} = \omega_{ab} = \tfrac{1}{2}(\Gamma_{ab}{}^{o} - \Gamma_{ba}{}^{o}) = \tfrac{1}{2}\, C_{ab}{}^{o} \tag{IV-2b}$$

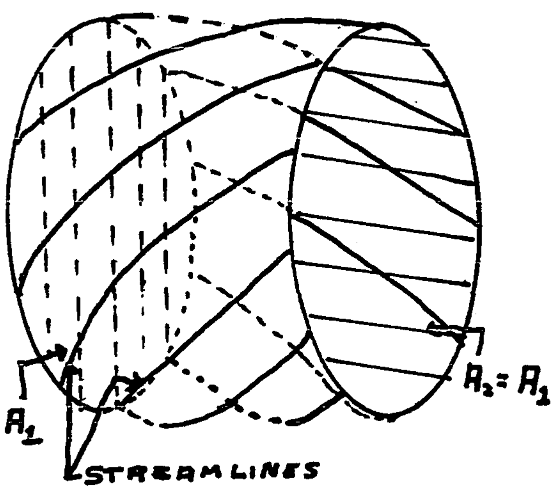


(a)

$$\theta = 0$$

$$\sigma_{ab} = \omega_{ab} = 0$$

Case of simple expansion. Geodesics orthogonal to A_1 and A_2, and shape of A_1 simply and isotropically enlarged.

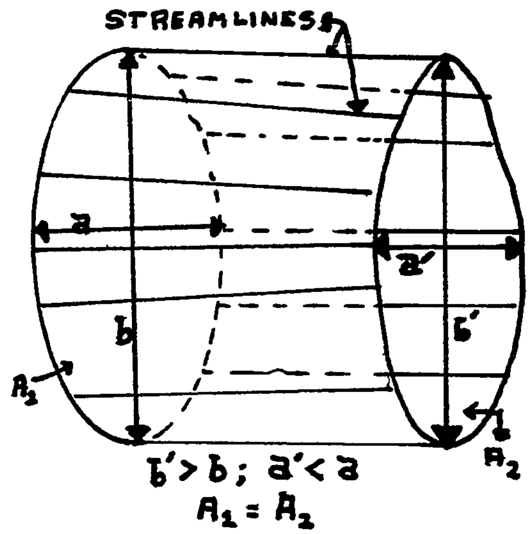


(b)

$$\theta = 0 = \sigma_{ab}$$

$$\omega_{ab} \neq 0$$

Case of simple vorticity: $A_1 = A_2$, and the shape of A_1 is exactly that of A_2, but the congruences twist around each other.



(c)

$$\theta = \omega_{ab} = 0, \ \sigma_{ab} \neq 0$$

Consequently $A_1 = A_2$, but the circular form of A_1 is squeezed into the elongated elipse of A_2.

Figure [IV-2]

Illustration of the physical meaning of the expansion (a), vorticity (b), and shear (c).

From (a) above, one concludes that

$$\theta = u^a{}_{;a} = \Gamma_{ao}{}^a \tag{IV-2c}$$

Before going on to discuss the transformation properties of these parameters, it is necessary at this point to make note of the fact that solutions of the field equations that do transform "well" under the transformations discussed here, are, in fact, empty spaces. They have no fluid and hence no actual velocity field. This, however, does not affect the significance of the hydrodynamical parameters. In empty space solutions, it is customary to introduce "test particles" into the space, and the vector u now becomes tangent to the geodesics which these particles follow.

With the definition of these hydrodynamical parameters, their transformation properties can be studied. The transformation of $\Gamma_{ab}{}^o$ and $C_{ab}{}^o$ under each of the complex tetrad transformations is required. The following table summarizes these results.

Type I: $\Gamma^{o'}_{a'b'} = -i\Gamma^{o}_{ab}$ $C^{o'}_{a'b'} = -iC^{o}_{ab}$ (a=1,2,3)

Type II: $\Gamma^{o'}_{1'a'} = -i\Gamma^{1}_{oa}$ $\Gamma^{o'}_{a'b'} = -i\Gamma^{1}_{ab}$ $\Gamma^{o}_{11} = i\Gamma^{1}_{oo}$ (a=2,3)

$C^{o'}_{1'a'} = -iC^{1}_{oa}$ $C^{o'}_{a'b'} = -iC^{1}_{ab}$

Type III: $\Gamma^{o'}_{1'a} = i\Gamma^{1}_{oa}$; $\Gamma^{o'}_{a'b'} = \Gamma^{1}_{ab}$; $\Gamma^{o'}_{a1} = i\Gamma^{1}_{ao}$

$\Gamma^{o}_{11} = -\Gamma^{1}_{oo}$

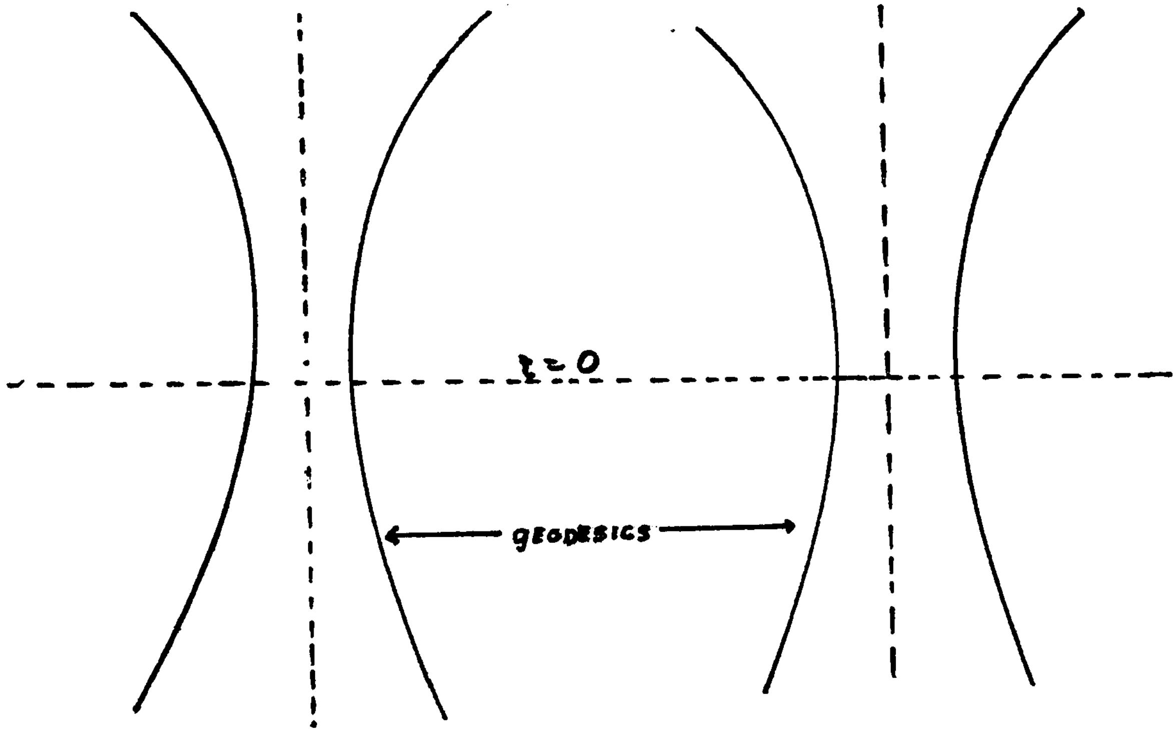
$C^{o'}_{1a} = iC^{1}_{oa}$; $C^{o'}_{a'b'} = C^{1}_{ab}$ $C^{o}_{a1} = iC^{1}_{ao}$ (a,b=2,3)

For a type I transformation, $\theta \rightarrow -i\theta$, $\sigma_{ab} \rightarrow i\sigma_{ab}$, $\omega_{ab} \rightarrow -i\omega_{ab}$. The information that this gives is that rotational universes transform into rotational universes, and irrotational ones into irratational ones. The same applies to the shear. Shear free universes transform into shear free ones, and shearing ones into shearing ones.

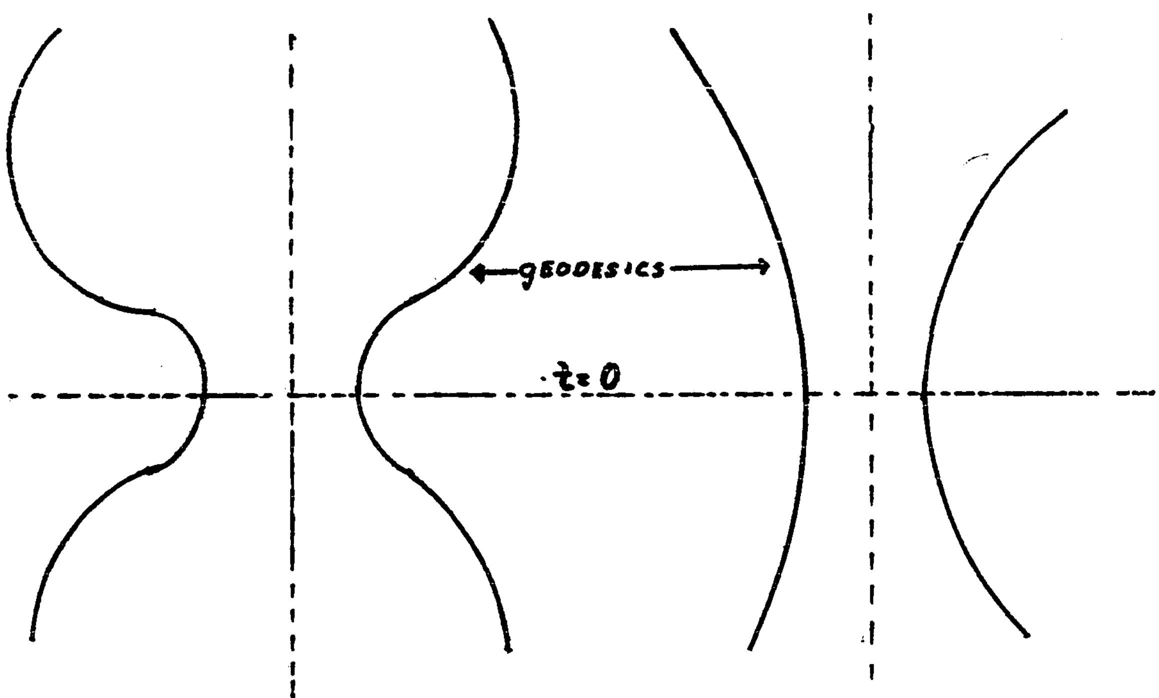
There are many simple instances (e.g. the de Sitter universe) where the expansion $\theta = \theta(t)$ where t is the time, and where one of the coordinate transformation inducing the type I transformations is $t \rightarrow it$. Under the type I transformation, $\theta' = \Gamma^{i'}_{i'o'} = -i\Gamma^{i'}_{io} = -i\theta$. But this means that $\theta\,(it') = \pm\, i\,\theta(t')$, and θ is an odd function of t. Consider seven of these odd functions t, sin t, tan t, cot t, sinh t, tanh t, coth t. In the following table, the zeros and infinities of these functions and their transformed spaces are displayed. Since the transforms of these functions are included among the functions themselves, the table can be read so that either the unprimed quantities (labeled U) represent the original space, and the primed quantities (labeled P) the transformed space, or vice versa.

	θ		Zeros		Neg. Inf.		Pos. Inf.	
	U	P	U	P	U	P	U	P
a)	t	t'	0	0	$-\infty$	$-\infty$	∞	∞
b)	sin	sinh t'	$n\pi$	0	-	-	-	-
c)	tan t	tanh t'	$n\pi$	0	$(2n+1)\frac{\pi}{2}^{-}$	-	$(2n+1)\frac{\pi}{2}^{+}$	-
d)	cot t	coth t'	$(2n+1)\frac{\pi}{2}$	-	$n\pi^{-}$	0^{-}	$n\pi^{+}$	0^{+}

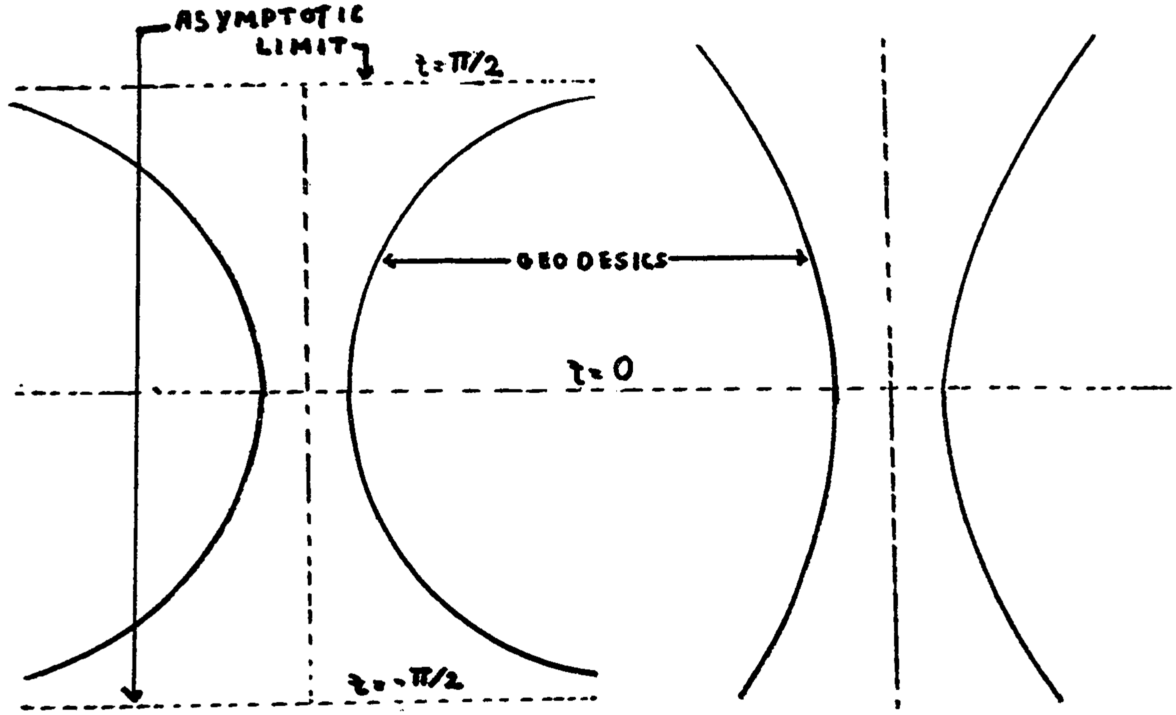
(The small +'s and -'s indicate that this value is approached from the direction of increasing (+) or decreasing (-) t.) The behavior of each of these spaces is indicated in Fig. [IV-3].



(a) $\theta = t$ $\theta' = t'$



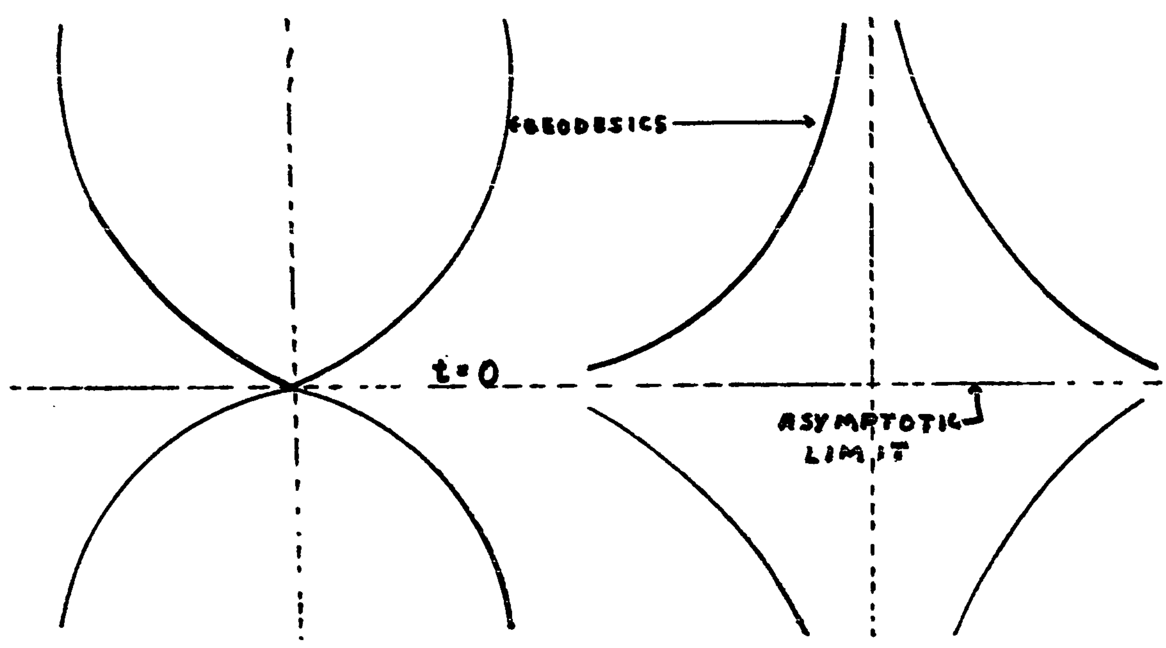
(b) sin t $\theta' = \sinh t'$



(c)

$\theta = \tan t$
geodesics diverge as $t \rightarrow \pi/2$ or
$t \rightarrow -\pi/2$

$\theta' = \tanh t'$



(d)

$\theta = \cot t$
geodesics cross at $t = 0$

$\theta' = -\coth t'$
geodesics diverge
at $t = 0$

Less general information can be deduced from the transformation of the connection in the case of the types II and III transformations since $\Gamma^o_{ij} \rightarrow \Gamma^1_{ij}$, and Γ^1_{ij} is less amenable to interpretation than is Γ^o_{ij}.

The example of the de Sitter universe, that has been used previously, illustrates the properties of these transformed spaces.

First of all, the components of the connection for the de Sitter universe are

$$\begin{aligned} \Gamma^1_{10} &= \Gamma^2_{20} = \Gamma^3_{30} = \tanh t \\ \Gamma^2_{21} &= \Gamma^3_{31} = \frac{\cot x}{\cosh t} \\ \Gamma^3_{32} &= \frac{\cot \theta}{\cosh t \sin x} \end{aligned} \qquad \text{(IV-3)}$$

The parameters for this universe are then

$$\theta = 3 \tanh t$$

$$\sigma_{ab} = 0 = \omega_{ab}$$

If a type I transformation is done upon the de Sitter universe, then, in accord with the previous claims, the new parameters will be

$$\theta' = -3 \tan t$$

$$\sigma'_{ab} = \omega'_{ab} = 0 \ .$$

The picture of the behavior of this universe (the anti-de Sitter universe) is like that in Fig. [IV-C2], where the right side represents the de Sitter universe, and the left side that of the antide Sitter universe. A type II transformation on the de Sitter universe would be given by the metrics II-8. For the metric (II-8b),

$$C^{o'}_{a'b'} = -iC^{1}_{ab} \quad C^{o'}_{o'1'} = iC^{1}_{1o}$$

$$\Gamma^{i'}_{i'o'} = i\Gamma^{i}_{i1} \quad , \quad \Gamma^{o'}_{a'b'} = -i\Gamma^{1}_{ab}$$

$$\Gamma^{o'}_{o'1} = i\Gamma^{1}_{1o}$$

where the primed objects are in the rotated space. This means that

$$\theta' = \Gamma^{i'}_{i'o'} = i\Gamma^{i}_{i1} = \frac{2 \coth x}{\cos t}$$

$$\sigma'_{ab} = \omega'_{ab} = 0 \ ,$$

and the paths of any object whose path is tangent to $X_o{}'$ will still be irrotational and shear free.

In comparing this result with that for the de Sitter universe one important point must be noted. For the de Sitter universe, as can be seen from IV-3, $\Gamma^{o}_{oi} = 0$ for all i. This means that X^o is geodesic, i.e. $X^{o}_{\mu;\nu} X_o{}^{\nu} = 0$. In the case of the transformed spaces, however, $\Gamma^{o'}_{o'1'} = \tan t \neq 0$, and $X^{o'}$ is not geodesic. This means that θ', σ'_{ab}, and ω'_{ab} describe the path of an accelerating particle and is therefore of little physical interest. The same result will hold for the type III transformation when done upon the de Sitter universe. $X^{o'}$ will not be tangent to the path of the test particle there either.

These spaces are best approached, as shall be seen in the next two sections, by means of the Newman-Penrose formalism.

C. The Newman-Penrose Formalism [15]

The Newman-Penrose formalism is perhaps more useful for the description of the classes of universes that are considered here because they are empty, and the null properties have better definition than hydrodynamic ones. In addition, the Newman-Penrose spin coefficient involve more parameters than do the hydrodynamic parameters.

Define the usual pseudoorthonormal tetrad ℓ, n, m, $\bar{m}$ in terms of the orthonormal tetrad

$$\begin{aligned} \ell^* &= \frac{1}{\sqrt{2}} (X_o + X_1) \\ n^* &= \frac{1}{\sqrt{2}} (X_o - X_1) \\ m^* &= \frac{1}{\sqrt{2}} (X_2 + \underline{i}X_3) \ . \end{aligned} \tag{IV-4a}$$

By the same token, one defines the covariant form of these

$$\begin{aligned} \ell_* &= - \frac{1}{\sqrt{2}} (+ \omega^o - \omega^1) \\ n_* &= - \frac{1}{\sqrt{2}} (\omega^o - \omega^1) \\ m_* &= \frac{1}{\sqrt{2}} (\omega^2 + i\omega^3) \end{aligned} \tag{IV-4b}$$

They are defined in such a way that

$$\ell_\mu \ell^\mu = n_\mu n^\mu = m^\mu m_\mu = n^\mu m_\mu = \ell_\mu m^\mu = 0$$

$$-\ell_\mu n^\mu = m^\mu \bar{m}_\mu = 1 \qquad \text{(IV-4c)}$$

and

$$g_{\mu\nu} = -n_\mu \ell_\nu - \ell_\mu n_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu \; . \qquad \text{(IV-4d)}$$

For the pseudoorthonormal tetrad, the components of the connection are called spin coefficients. These are related to the component of the connection of the orthonormal tetrad. Below is a table of these spin coefficients.

$$\begin{aligned}
&\kappa = \ell_{\mu;\nu} m^\mu \ell^\nu \,, \quad \varepsilon = \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu \ell^\nu - m_{\mu;\nu} \bar{m}^\mu \ell^\nu) \\
&\rho = \ell_{\mu;\nu} m^\mu \bar{m}^\nu \,, \quad \sigma = \ell_{\mu;\nu} m^\mu m^\nu \,, \quad \tau = \ell_{\mu;\nu} m^\mu n^\nu \\
&\nu = -n_{\mu;\nu} \bar{m}^\mu n^\nu \,, \quad \gamma = \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu) \\
&\mu = -n_{\mu;\nu} \bar{m}^\mu n^\nu \,, \quad \lambda = -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu \,, \quad \pi = n_{\mu;\nu} \bar{m}^\mu \ell^\nu \\
&\alpha = \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu) \,, \quad \beta = \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu)
\end{aligned} \qquad \text{(IV-5a)}$$

From this it can be seen that

$$\ell_{\mu;\nu} \ell^\nu = \kappa \bar{m}_\mu + \bar{\kappa} m_\mu - (\varepsilon + \bar{\varepsilon}) \ell_\mu$$

and $$n_{\mu;\nu} n^\nu = + \nu m_\mu + \bar{\nu} \bar{m}_\mu + (\gamma + \bar{\gamma}) n_\mu \; .$$

Thus $\kappa = 0$ means that the null paths tangent to ℓ are geodesic, and $\nu = 0$ means that the null paths tangent to n are likewise geodesic. This shows that if $\kappa = 0$, $(\varepsilon+\bar{\varepsilon})$ can be made zero by letting $\ell' = \varphi \hat{\ell}$ for some φ. Likewise $(\gamma+\bar{\gamma})$ can be made zero if $\nu = 0$ by allowing $n' = \psi n$ for an appropriate ψ. Note, however, that only in rare cases can these be done simultaneously.

When these conditions hold ($(\epsilon+\bar{\epsilon}) = 0$, $\kappa = 0$) the parameter $\rho = \frac{1}{2}(-\ell^{\mu}{}_{;\mu} + i \text{ curl } \ell)$, where curl $\ell = (\ell_{[\mu;\nu]}\ell^{\mu;\nu})^{\frac{1}{2}}$. Also $\sigma\bar{\sigma} = \frac{1}{2}(\ell_{(\mu;\nu)}\ell^{\mu;\nu} - \ell^{\mu}{}_{;\mu})$. Briefly, for the null geodesics tangent to ℓ, $\frac{1}{2}(\rho+\bar{\rho})$ corresponds to θ with respect to the streamline of matter, $\frac{1}{2}|\rho-\bar{\rho}|^2$ to $\omega_{ij}\omega^{ij}$, and $\sigma\bar{\sigma}$ to $\sigma_{ij}\sigma^{ij}$. $(\rho+\bar{\rho})$ is the divergence of the neighboring null geodesics, σ measures the shear, and $\rho-\bar{\rho}$ is the vorticity. The quantity τ measures the way that ℓ changes as one moves along η, as can be seen from

$$\ell_{\mu;\nu}n^{\nu} = \tau\bar{m}_{\mu} + \bar{\tau}m_{\mu} - (\gamma+\bar{\gamma})\ell_{\mu} .$$

The parameters ν, $-\mu$, $-\lambda$, π correspond respectively to κ, ρ, σ, τ, except that the congruence to which they pertain is tangent to n rather than ℓ.

The other important quantities needed in the Newman-Penrose formalism are the components of the Weyl tensor. These are

$$\psi_o = -C_{\alpha\beta\gamma\delta}\ell^{\alpha}m^{\beta}\ell^{\gamma}m^{\delta} ,$$

$$\psi_1 = -C_{\alpha\beta\gamma\delta}\ell^{\alpha}n^{\beta}\ell^{\gamma}m^{\delta} ,$$

$$\psi_2 = -\tfrac{1}{2} C_{\alpha\beta\gamma\delta}(\ell^{\alpha}n^{\beta}\ell^{\gamma}n^{\delta} - \ell^{\alpha}n^{\beta}m^{\gamma}\bar{m}^{\delta}) , \qquad \text{(IV-5b)}$$

$$\psi_3 = C_{\alpha\beta\gamma\delta}\ell^{\alpha}n^{\beta}n^{\gamma}\bar{m}^{\delta} ,$$

$$\psi_4 = C_{\alpha\beta\gamma\delta}n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta} ,$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor in the coordinate basis.

The five complex quantities ψ contain all of the independent components of the Weyl tensor (see Appendix B). For a space with $\Lambda = 0$ $C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$, and the ψ_i's can be calculated with $R_{ijk\ell}$ rather than $C_{ijk\ell}$. Even in the case where the cosmological constant $\Lambda \neq 0$, the ψ_i's can be calculated with the $R_{ijk\ell}$'s, except that ψ_2 must be corrected by adding $\Lambda/3$.

The physical significance of the Weyl tensor is to be seen from the Bianchi identities, which can be written [14]

$$C^{\alpha\beta\gamma\delta}{}_{;\delta} = R^{\gamma[\alpha;\beta]} + \frac{1}{6} g^{\gamma[\beta}R^{;\alpha]} \quad .$$

By comparing this with Maxwell's equation

$$F^{\alpha\beta}{}_{;\beta} = J^{\alpha} \ ,$$

one can regard the Bianchi identities as a field equation for $C^{\alpha}{}_{\beta\gamma\delta}$, giving the part of the curvature at a point which results from matter distribution at another point (local matter distributions described entirely in terms of $R_{\mu\nu}$). In other words, the Weyl tensor describes the way in which the gravitational field propagates in a space time.

Specifically, these complex components of the Weyl tensor indicate the Petrov type [15,17] of the space time (see Appendix B). For example, if only $\psi_o \neq 0$, the space is Petrov type N, a pure radiation field, with a propagation vector n_μ. If only $\psi_4 \neq 0$ then the space is a Petrov type N with propagation vector ℓ_μ. If only ψ_1 or ψ_3 is not zero, then the space is type III, with n_μ or ℓ_μ respectively as the propagation vector. If only $\psi_2 \neq 0$, then the space is type D with propagation vectors n_μ and ℓ_μ. If, however, both ψ_1 and $\psi_2 \neq 0$, then the space is type II. From these ψ_i's one can tell whether there is a pure gravitational radiation mode, a gravitational "intermediate" field (type III), a gravitational "near" field or some combination thereof.

Two of the Newman-Penrose field equations merit inspection:

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \phi_{oo}$$

$$D\sigma = (\rho+\bar{\rho})\sigma + (3\epsilon-\bar{\epsilon})\sigma + \psi_o$$

$$\phi_{oo} = \tfrac{1}{2} R_{ij}\ell^i \ell^j \quad ,$$

where $\kappa = \epsilon+\bar{\epsilon} = 0$, and $D\rho = \rho,_\mu \ell^\mu$. Assume that at some space-like hypersurface, $\sigma = \rho = 0$. ϕ_{oo} is the energy density that would be seen by an observer whose path is tangent to ℓ_μ. As one moves off this surface along ℓ^μ, the rate at which ρ increases is proportional to ϕ_{oo}. The rate at which σ increases is proportional to ψ_o. If $\phi_{oo} = 0$ at this surface, then σ will increase along the congruence of ℓ^μ and consequently $\rho+\bar{\rho}$ will increase. The effect of gravitational radiation is to increase the shear along the direction orthogonal to the direction of propagation. The spin coefficients σ and λ couple to the radiation modes of the field ψ_o and ψ_4.

To show the effect of types II and III transformations, it is necessary to relate the spin coefficient and the complex Weyl tensor of the Newman-Penrose formalism, and the components of the connection and the curvature tensor in the orthonormal tetrad formalism. These are given in Appendix C.

Suppose that a type II transformation is made on a metric. If $\ell = -(\omega^0 - \omega^1)$ is geodesic ($\kappa = 0$) then ℓ' will also be a geodesic (one has merely exchanged ω^0 and ω^1). But in case the type II transformation exchanges ω^0 and ω^2, or ω^0 and ω^3, then only rarely will the new ℓ be geodesic. One has to look for a new radial frame, to go with the new time frame to give a new geodesic ℓ. This process will be illustrated by the examples of the three metrics obtained from the de Sitter universe by type II transformations given in II-8 of the last chapter. First (II-8b) is obtained from the de Sitter universe by the exchange of ω^0 and ω^1, since $\ell' = -(\omega^{0\prime} - \omega^{1\prime}) = i(\omega^1 - \omega^0) = i\ell$, ℓ' will be geodesic if ℓ is.

The metric (II-8a) is obtained from the de Sitter metric by a coordinate transformation $t \to i\varphi$, $\varphi \to it$, exchanging ω^0 and ω^3. The components of the connection for the de Sitter universe are

$$\Gamma^1_{10} = \tanh t = \Gamma^2_{20} = \Gamma^3_{30}$$

$$\Gamma^2_{21} = \Gamma^3_{31} = \frac{\cot \chi}{\cosh t}\,, \quad \Gamma^3_{32} = \frac{\cot\theta}{\sin\chi \cosh t}\,.$$

Under the coordinate transformations given above

$$\Gamma^1_{10} = i \tan\varphi = \Gamma^2_{20} = \Gamma^3_{30}$$

$$\Gamma^2_{21} = \Gamma^3_{31} = \frac{\cot\chi}{\cos\varphi}, \quad \Gamma^3_{32} = \frac{\cot\theta}{\sin\chi \cos\varphi} .$$

To make it easy to effect the transformation, the indices one and three are now exchanged ($1 \leftrightarrow 3$), giving

$$\Gamma^3_{30} = \Gamma^2_{20} = \Gamma^1_{10} = i \tan\varphi$$

$$\Gamma^2_{23} = \Gamma^1_{13} = \frac{\cot\chi}{\cos\varphi} \Gamma^1_{12} = \frac{\cot\theta}{\sin\chi \cos\varphi} .$$

The relations (III-8) are now used to give the connection for the transformed space,

$$\Gamma^{0'}_{0'1'} = - \tan\varphi = \Gamma^{2'}_{2'1'} = \Gamma^{3'}_{3'1} \quad , \Gamma^{2'}_{2'3}$$

$$= \Gamma^{0'}_{0'3'} = \frac{\cot\chi}{\cos}$$

$$\Gamma^{0'}_{02} = \frac{\cot\theta}{\sin\chi\cos\varphi}$$

From the calculation described in Appendix C one obtains

$$\kappa = - \frac{1}{2\sqrt{2}} \left(\frac{\cot\theta}{\sin\chi\cos\varphi} + i \frac{\cot\chi}{\cos\varphi}\right) \neq 0.$$

The choice of ℓ with $\omega^o = \cos\varphi \sin\chi \sin\theta \, d\theta$ and $\omega^1 = d\zeta$ is not very physical. As it turns out, the correct choice is $\omega^o = \cos\varphi \sin\chi \sin\theta \, d\theta$ and $\omega^1 = \cos\varphi \sin\chi \, d\theta$. In this case $\kappa = 0 = \sigma = (\rho - \bar{\rho})$. The null congruence represented by this ℓ is geodesic, shear free and irrotational.

Since (II-8c) is given by an $\omega^o \leftrightarrow \omega^1$ exchange on (II-8a), $\kappa = 0$ in the transformed space also.

The next step would logically be the calculation of the ψ_i's (the components of the Weyl tensor). This, however, is unnecessary since they are zero for the de Sitter universe [16]. Since the components of the Weyl tensor transform in the same manner as the components of the curvature tensor they will also be zero for the transformed space.

Type II transformations acting on the de Sitter universe give the three space-times II-8. The space-time (II-8a) comes from an exchange of ω^o and ω^3. However, once ω^3 is chosen as the time-like leg of the tetrad, $-(\omega^{o\prime}-\omega^1)$ is no longer geodesic. If ω^1 and ω^2 are exchanged, then $-(\omega^{o\prime}-\omega^{1\prime})$ is geodesic.

The complete transformation is:

$$\begin{aligned} \omega^{o\prime}(x') &= -i\omega^2(x') \;, \\ \omega^{2\prime}(x') &= -i\omega^o(x') \;, \\ \omega^{1\prime}(x') &= \omega^3(x') \;, \\ \omega^{3\prime}(x') &= \omega^1(x') \;. \end{aligned}$$

D. <u>The ℓ-$\bar{m}$ Exchange</u>

When ω^o and ω^2 were exchanged above, ω^1 and ω^3 were exchanged as well. This means that an exchange is also made between the legs of the pseudoorthonormal tetrad:

$$\begin{aligned} \ell' &= -(\omega^{o\prime}-\omega^{1\prime}) = -(-i\omega^2-\omega^3) = i(\omega^2-i\omega^3) = i\bar{m} \\ n' &= -(\omega^o+\omega^1) = -(-i\omega^2+\omega^3) = i(\omega^2+i\omega^3) = im \\ m' &= (-\omega^2+i\omega^3) = (-i\omega^o+i\omega^1) = -i(\omega^o-\omega^1) = i\ell \\ n' &= i\bar{m} \end{aligned} \qquad \text{(IV-6)}$$

One may call this very special type II transformation an ℓ-$\bar{m}$ exchange.

To see the importance of an ℓ-$\bar{m}$ exchange calculate the values of the spin coefficient of the transformed space in terms of those of the original space:

$$\kappa' = i\bar{\sigma} \quad (\varepsilon-\bar{\varepsilon})' = i(\alpha+\bar{\beta}) \quad (\alpha+\bar{\beta})' = -i(\gamma-\bar{\gamma})$$

$$\pi' = i\bar{\mu} \quad \rho' = i\bar{\tau} \quad (\alpha-\bar{\beta}) = i(\gamma+\bar{\gamma})$$

$$(\varepsilon+\bar{\varepsilon}) = -i(\alpha-\bar{\beta}) \quad \lambda = -i\bar{\nu} \quad \sigma' = i\bar{\kappa}$$

$$\nu' = i\bar{\lambda} \quad (\gamma+\bar{\gamma})' = i(\bar{\alpha}-\beta) \quad \mu' = i\bar{\pi}$$

$$(\gamma-\bar{\gamma}) = i(\bar{\alpha}+\beta) \quad \tau' = i\bar{\rho} .$$

If the original space had a shear free geodesic tangent to ℓ, then $\kappa = 0$, $\sigma = 0$. But this means that $\kappa' = 0$, $\sigma' = 0$. The class of spaces for which this exchange works is very wide indeed, encompassing the entire class of algebraically special spaces (see Appendix B). The result of all this is that a type II transformation can be made on an algebraically specialized space, and the resulting space will always be algebraically specialized. This is confirmed by the fact that $\psi_i \to \psi_i$ in these transformations. Not only will the resulting space be algebraically specialized, but also it will be of the same Petrov type as the original space.

E. Conclusion

The types I, II and III transformations relate the components of the connection and the curvature tensor in one space to those same objects in the transformed space. In order to give a relation between the physical structure in one space to that of the other, it is necessary to interpret these quantities physically. This has been done for two sets of parameters.

The first of these are the hydrodynamical parameters. These are the divergence of the streamlines of matter θ, the vorticity of these streamlines ω_{ab}, and their shear σ_{ab}. The divergence θ of these streamlines gives the rate at which neighboring streamlines diverge. The vorticity gives the rate at which they curl about each other. The shear measures the anisotropy of the expansion.

If the orthonormal tetrad is chosen so that X_o, its time-like leg, is tangent to the streamlines of matter, there are simple relations between these parameters and the components of the connection. These are

$$\theta = \Gamma^{a}_{ao}$$

$$\omega_{ab} = \tfrac{1}{2}\, C^{o}_{ab} = \tfrac{1}{2}(\Gamma^{o}_{ab} - \Gamma^{o}_{ba})$$

$$\sigma_{ab} = \tfrac{1}{2}(\Gamma^{o}_{ab} + \Gamma^{o}_{ba}) - \frac{\delta_{ab}}{3}\,\theta$$

Using the tables in Chapter III that relate the components of the connection in the original space to those of the transformed space for each type of transformation, it is possible to obtain these parameters for the transformed space from the components of the connection in the original one.

The parameters themselves, however, only relate directly in the case of the type I transformation.

The second set of parameters are the spin coefficients of the Newman-Penrose formalism. Instead of having the time-like leg of the orthonormal tetrad tangent to the streamlines of matter, the Newman-Penrose pseudoorthonormal tetrad has two null vectors, ℓ and n and a complex space-like vector m. The spin coefficients are to this pseudoorthonormal tetrad what the connection is to the orthonormal tetrad. If at least one of these null vectors is tangent to a congruence of null geodesics, then these parameters have an easy interpretation. The real part of one of these, $\rho = \ell_{\mu;\nu} m^{\mu} \bar{m}^{\nu}$, measures the divergence of this congruence, while its imaginary part measures their vorticity. Another of these, $\sigma = \ell_{\mu;\nu} m^{\mu} m^{\nu}$, measures the shear of this congruence. There are two other spin coefficients that give these same quantities for the null congruences tangent to n.

There are twelve spin coefficients altogether, and they are sums of the components of the connection for the orthonormal tetrad. For any of the transformations that are considered here, the transformations of the spin coefficients can be written down from the tables in Chapter III.

For type I transformations, each spin coefficient for the transformed space is given directly in terms of the transform of that coefficient in the original space. This same is true for a type II transformation where ω^o and ω^1 are swapped. Otherwise, it is more complicated.

A particularly interesting situation arises in the case of the type II transformation of an algebraically special space. When ω^o and ω^2 are swapped, one must also swap ω^1 and ω^3 in order to keep ℓ geodesic. Or when ω^o is swapped with ω^3, then ω^1 and ω^2 must be swapped. This means that $\ell \rightarrow \bar{m}$, $m \rightarrow \ell$. Under these transformations, algebraically special spaces are mapped onto algebraically special spaces.

Chapter V

Transformation on the Schwarzschild Metric and on the Gowdy T_3 Universes

A. Introduction

This thesis contains two original contributions to the study of General Relativity. The first of these is the development of the types 0, I, II and III transformations. This was developed in Chapters II and III. The second contribution is contained in this chapter and is the discovery of three families of new metrics obtained by the application of these transformations to the Gowdy T_3 universes.

First of all, these transformations are applied to the Schwarzschild metric. This is done in Section A. The types II and III transformations yield the Kantowski-Sachs closed and open universes respectively.

Because the former of these metrics is contained in the Gowdy T_3 universe, these transformations are applied there. The three types of transformations yield the three new families of metrics.

The first of these is the result of the application of the type II transformation to a special subset of these Gowdy T_3 universes. The resulting metrics are Weyl metrics and contain the Schwarzschild metric as a special case.

The second of these new metrics is the result of a type III transformation. These also are Weyl metrics, and do not contain any singularities at $r = 0$.

The third family of metrics is the result of a type I transformation, and reduces to the Kantowski-Sachs open universe as a special case.

B. Schwarzschild Metric

The static gravitational field of a spherically symmetric object is given by the Schwarzschild metric

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\varphi^2)$$

By making a coordinate transformation [18]

$$\frac{r}{M} - 1 = \cosh \eta$$

we have $r = 2m \cosh^2 \frac{\eta}{2}$,

and $$1 - \frac{2m}{r} = \tanh^2 \frac{\eta}{2} .$$

Finally, the metric takes the form

$$ds^2 = -\tanh^2 \frac{\eta}{2} dt^2 + 4m^2\cosh^4 \frac{\eta}{2}(d\eta^2+d\theta^2+\sin^2\theta d\varphi^2) .$$

The one forms are

$$\omega^o = \tanh \frac{\eta}{2} dt$$

$$\omega^1 = 2m \cosh^2 \frac{\eta}{2} d\eta$$

$$\omega^2 = 2m \cosh^2 \frac{\eta}{2} d\theta^2$$

$$\omega^3 = 2m \cosh^2 \frac{\eta}{2} \sin\theta d\varphi .$$

This metric covers the entire exterior region of the Schwarzschild space up to the horizon at $r = 2m$.

If now, $\eta \to it$, $t \to \xi$, these forms transform to

$$\omega^o = i \tan t/2 \, d\zeta$$

$$\omega^1 = i2m \cos^2 t/2 \, dt$$

$$\omega^2 = 2m \cos^2 t/2 \, d\theta$$

$$\omega^3 = 2m \cos^2 t/2 \sin\theta d\varphi \, .$$

A type II transformation yields the result

$$\omega^{o\prime} = 2m \cos^2 t/2 \, dt$$

$$\omega^{1\prime} = 2m \cos^2 t/2 \, d\theta$$

$$\omega^{2\prime} = 2m \cos^2 t/2 \sin\theta d\varphi$$

$$\omega^{3\prime} = \tan t/2 \, d\zeta \, .$$

The metric is

$$ds^2 = 4m^2 \cos^4 t/2[-dt^2 + d\theta^2 + \sin^2\theta d\varphi^2] + \tan^2 t/2 \, d\zeta^2 \, .$$

This is just an example of the Kantowski-Sachs closed universe [19] (see Appendix D for a description of these spaces). This universe is isometric to the Schwarzschild metric within the horizon, as can be seen by making the transformation $\cos t = \frac{r}{m} - 1$.

If the transformation $\theta \to i\theta$, $\eta \to t$, $t \to r$ is made, the orthonormal tetrad is given by

$$\omega^o = \tanh t/2 \, dr$$

$$\omega^1 = 2m \cosh^2 t/2 \, dt$$

$$\omega^2 = i2m \cosh^2 t/2 \, d\theta$$

$$\omega^3 = i2m \cosh^2 t/2 \sinh \theta d\varphi \, .$$

A type III transformation yields the tetrad

$$\omega^0 = 2m \cosh^2 t/2\, dt$$
$$\omega^1 = \tanh t/2\, dr$$
$$\omega^2 = 2m \cosh^2 t/2\, d\theta$$
$$\omega^3 = 2m \cosh^2 t/2 \sinh \theta d\varphi \,,$$

which gives the metric

$$ds^2 = 2m \cosh^4 t/2\, [d\theta^2 - dt^2 + \sinh^2\theta d\varphi^2] + \tanh^2 t/2\, dr^2 \,.$$

This is recognizable as the open vacuum Kantowski-Sachs metric.

C. The Gowdy Universes [20]

The Kantowski-Sachs closed universe is contained in a family of spaces discovered by Gowdy, the Gowdy T_3 universes. The transformation from the Kantowski-Sachs universe to the Schwarzschild metric, when tried on the Gowdy T_3 universes, might yield a whole family of spaces that contains the Schwarzschild metric.

To investigate this possibility, a careful review, needs to be made of these spaces. The Gowdy universes are closed generalizations of the metric for the Einstein-Rosen-Bondi cylindrical plane waves [21], and the Gowdy T_3 universes possess space-like hypersurfaces that are isometric to theethree dimensional torus.

An Einstein-Rosen-Bondi space time is one that possesses two mutually orthogonal, hypersurface orthogonal, space-like killing vector fields. The metric for such a space is

$$ds^2 = L^2\left\{e^{2a}(d\theta^2-dt^2) + R(B^{-1}e^{2W}d\sigma^2 + Be^{-2W}d\delta^2)\right\} \qquad \text{(V-1)}$$

where a, W, B and R are functions of θ and t only. The coordinates θ and t are not fixed completely. If $u = t-\theta$, and $v = t+\theta$, then $u = F(\tilde{u})$, $v = G(\tilde{v})$ are the most general transformation that preserve the form of the metric.

For convenience in writing down the field equations, derivatives with respect to u are denoted by a subscript--, and those with respect to v are denoted by the subscript +. Also, since W and B are redundant functions, it is convenient to define

$$\psi = W - \tfrac{1}{2}\,\ell n\, B.$$

The independent Einstein field equations are,

$$R_+a_+ = R\psi_+^{\ 2} + \tfrac{1}{2}R_{++} - \tfrac{1}{4}R(R_+/R)^2 \qquad \text{(V-2a)}$$

$$R_-a_- = R\psi_-^{\ 2} + \tfrac{1}{2}R_{--} - \tfrac{1}{4}R(R_-/R)^2 \qquad \text{(V-2b)}$$

$$\frac{\partial}{\partial\theta}\left(R\frac{\partial\psi}{\partial\theta}\right) - \frac{\partial}{\partial t}\left(R\frac{\partial\psi}{\partial t}\right) = 0 \qquad \text{(V-2c)}$$

$$\frac{\partial^2 R}{\partial\theta^2} - \frac{\partial^2 R}{\partial t^2} = 0\ . \qquad \text{(V-2d)}$$

For the Einstein-Rosen cylindrical waves, $R = \theta$, or $R = t$. Gowdy chooses $R = \sin\theta\, \sin t$. This will give a universe that is globally distinct from the Einstein-Rosen waves, although not locally distinct.

To see this, note that

$$R = \sin\theta \sin t = -\tfrac{1}{2}(\cos v - \cos u). \qquad \text{(V-3)}$$

But, as has been said previously, one can always choose

$$\tilde{u} = F(u) \ ,$$
$$\tilde{v} = G(v) \ ,$$

where F and G are arbitrary functions of u and v respectively, and still preserve the form of the metric. If one chooses

$$\tilde{u} = F(u) = -\cos u$$
$$\tilde{v} = G(v) = -\cos v$$

then,

$$R = \tilde{v} - \tilde{u} = \tilde{\theta}$$

or if $\qquad \tilde{u} = F'(u) = \cos u$

then $\qquad R = \tilde{v} + \tilde{u} = \tilde{t}.$

The Gowdy universes are not, then, distinct in their local structure from the Einstein-Rosen cylindrical waves.

They are, however, globally distinct. To see this, consider the vector grad R, which is an invariant feature of the spacetime whose magnitude determines the "c energy" of the spacetime [22]. In the two versions of the Einstein-Bondi spacetimes given above, grad R is space-like for $R = \theta$, and time-like for $R = t$. The Gowdy universes, however, contains two regions in which grad R is time-like, and two in which it is space-like, joined together by a hypersurface where R,α is null. These are shown in Fig. V-1 below. Thus, globally, the Gowdy universes consist of four Einstein-Rosen

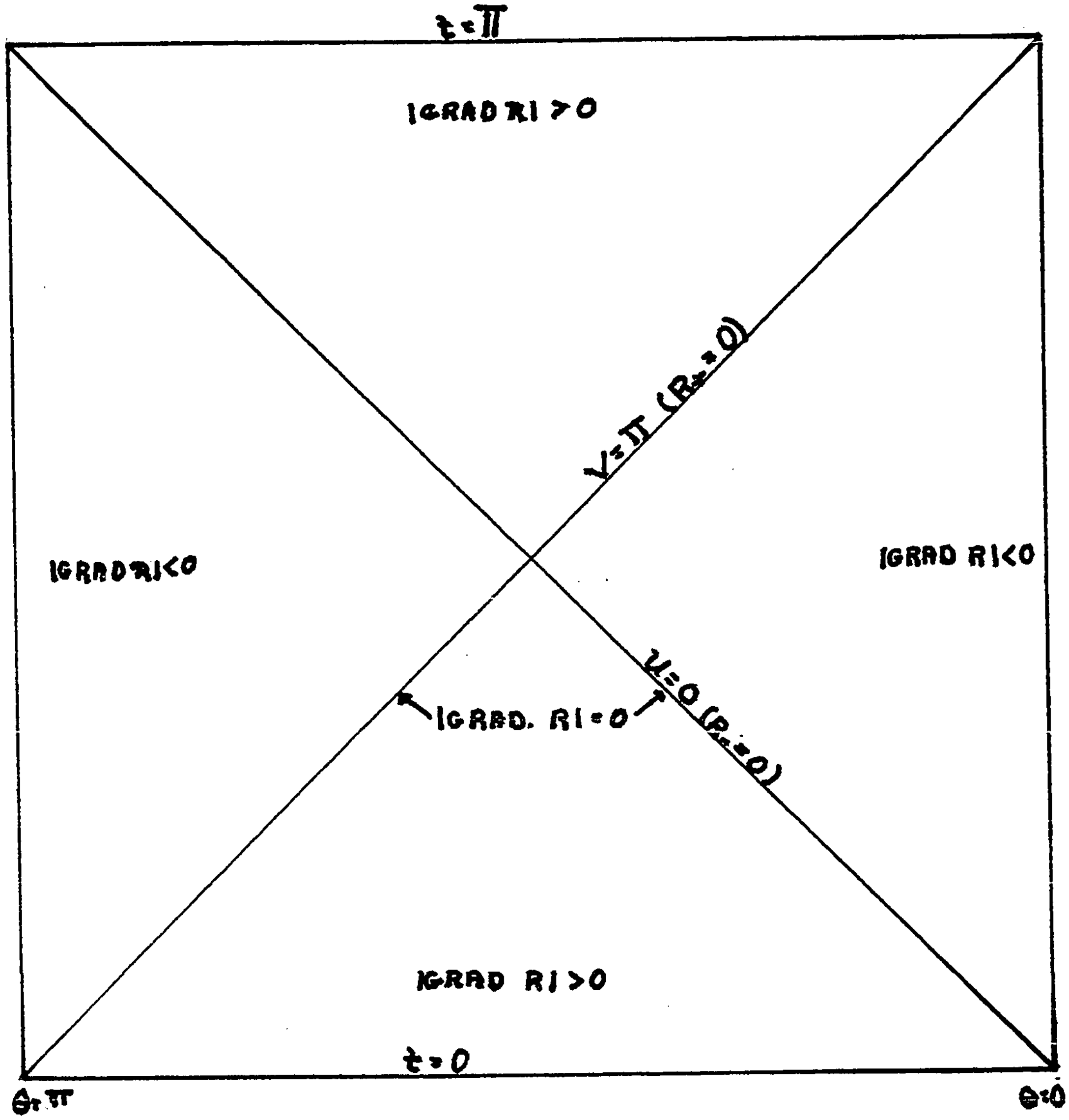


Fig. V-1

"patches," two which are like the R = θ waves, and two of which are like the R = t waves, pasted together to form a closed universe.

Care must be taken in putting together such universes as these. As can be seen from their metric they will be regular only on the coordinate patch as displayed in Fig. V-1, having initial and final collapse singularities at $t = 0$, and $t = \pi$ respectively, and coordinate singularities at $\theta = 0$ and at $\theta = \pi$.

Along the diagonals of this patch, where $u = 0$, and $v = \pi$, $(\text{grad } R) = 0 = [R_+R_-]^{\frac{1}{2}}$. $R_+ = 0$ along $v = \pi$, $R_- = 0$ along $u = 0$. This means that along these hypersurfaces, a_- and a_+ are not defined (cf. [V-2 a&b]). Along the diagonal $v = \pi$, where $R_+ = 0$, [V-2a] becomes

$$\psi_+ = \pm[2\cos u/2]^{-1} \qquad \text{(V-4a)}$$

and along $u = 0$,

$$\psi_- = \pm[2 \sin v/2]^{-1} \qquad \text{(V-4b)}$$

If Eq. [V-2c] (a wave equation) is written in terms of u and v, and evaluated along the diagonal, it becomes a first order differential equation for ψ_+ and ψ_- respectively, and the actual values of these functions given above for each diagonal, are solutions of this wave equation. Thus the constraints (V-4 above) are propagated in time, and need only be imposed at one time. A convenient choice of this time is $t = \pi/2$ where the diagonal cross at $\theta = \pi/2$. One therefore requires that either

$$\partial\psi/\partial\theta = 0 \quad \text{and} \quad \partial\psi/\partial t = \pm 1 \qquad \text{(V-5a)}$$

or

$$\partial\psi/\partial\theta = \pm 1 \quad \text{and} \quad \partial\psi/\partial t = 0 \ . \qquad \text{(V-5b)}$$

Each possible pair of constraints lead to distinct solutions.

Gowdy now chooses

$$W = \Sigma_\ell [A_\ell P_\ell(\cos\ t) + C_\ell Q_\ell(\cos\ t)] P_\ell(\cos\theta)$$

and, for the T_3 universes,

$$B = \sin\theta \sin t \ , \tag{V-6}$$

with the resulting metric

$$ds^2 = L^2 \sin^2 t [e^{2(\gamma - W)}(d\theta^2 - dt^2) + e^{-2W}\sin^2\theta d\delta^2] + L^2 e^{2W} d\sigma^2 \ , \tag{V-7a}$$

where

$$\gamma = a + W - \ell n \sin t \ . \tag{V-7b}$$

Two sets of regularity conditions must be imposed. Those in (V-5) keep the metric regular at $u = 0$, and at $v = \pi$. Further regularity conditions must be chosen to deal with singularities at $\theta = 0$ and $\theta = \pi$. The necessary conditions are $\gamma = \frac{\partial\gamma}{\partial\theta} = 0$ at $\theta = 0$ and $\theta = \pi$. The condition $\frac{\partial\gamma}{\partial\theta} = 0$ is automatically satisfied from equations (V-2 a&b), and so the only constraint necessary is $\gamma(0,t) = \gamma(\pi,t) = 0$.

The unique solution incorporating the constraint $\gamma(0,t) = 0$ is

$$\gamma = \int_0^\theta dy \sin y \left\{ 2\sin t \left(\frac{W_+^{\ 2}}{\sin(t+y)} + \frac{W_-^{\ 2}}{\sin(t-y)} \right) - \cos y (\sin^2 t - \sin^2 y)^{-1} \right\} \tag{V-8}$$

The remaining regularity condition $\gamma(\pi,t) = 0$ can be imposed at $t = \frac{\pi}{2}$, with the resulting integral constraint

$$\int_0^\pi d\theta \, \tan\theta[(\partial W/\partial t)^2 + (\partial W/\partial\theta)^2 - 1]|_{\theta = \pi/2} = 0 \, . \qquad \text{(V-9a)}$$

The other set of constraints (V-5) now becomes

$$\partial W/\partial\theta_{\theta = t = \pi/2} = 0 \text{ and } \frac{\partial W}{\partial t}\Big|_{\theta = t = \pi/2} = \pm 1 \qquad \text{(V-9b)}$$

or

$$\partial W/\partial\theta|_{\theta = t = \pi/2} = \pm 1 \text{ and } \frac{\partial W}{\partial t}\Big|_{\theta = t = \pi/2} = 0 \, . \qquad \text{(V-9c)}$$

These two sets of constraints restrict the admissible values of A_ℓ and C_ℓ in W. If $\partial W/\partial\theta$ and $\partial W/\partial t$ are calculated from (V-6) and substituted into (V-9b) this is equivalent to the constraint

$$\sum_n C_{2n+1} = 0 \quad , \quad \sum_n C_{2n} = \pm 1$$

and if substituted into (V-9c), it is equivalent to

$$\sum_n C_{2n} = 0 \qquad \sum_n C_{2n+1} = \pm 1 \, .$$

Once the C_ℓ's have been chosen to satisfy these constraints, then (V-9a) constrains the values of the A_ℓ's. There are <u>no</u> solutions such that $C_\ell = 0$ for all ℓ.

$$\omega^{0'} = -L \coth \frac{t}{2} \exp[\sum_{\ell} A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta)]dt$$

$$\omega^{1'} = -2L \sinh^2 \frac{t}{2} e^{\gamma'} \exp(\sum_{\ell} - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta)]d\theta$$

$$\omega^{2'} = -2L \sinh^2 \frac{t}{2} \sin\theta \exp(\sum_{\ell} - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta))d\delta$$

$$\omega^{3'} = 2L \sinh^2 \frac{t}{2} e^{\gamma'} \exp(\Sigma - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta))d\eta$$

with the new metric

$$ds^2 = -L^2 \coth^2 \frac{\eta}{2} \exp[2\sum_{\ell} A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta)]dt^2$$
$$+ 4L^2 \sinh^4 \frac{t}{2} \exp[2\sum_{\ell} - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta)] \qquad \text{(V-10)}$$
$$\times \left\{e^{2\gamma}(d\eta^2 + d\theta^2) + \sin^2\theta d\delta^2\right\} .$$

It will be recalled that the original metric had two orthogonal, hypersurface orthogonal space-like killing vectors, $\partial/\partial\delta$, and $\partial/\partial\sigma$. Now $\frac{\partial}{\partial\delta}$ is still as it was, but $\partial/\partial\sigma$ is now $\partial/\partial t$ and is an hypersurface orthogonal killing vector. This means that (V-10) is a static metric, with another space-like, angular hypersurface orthogonal killing vector. These spaces then belong to the Weyl metrics (see Appendix D), and reduce to the Schwarzschild metrics as $A_\ell \to 0$, and as $\eta \to \eta - i\pi$. Making this transformation

$$ds^2 = -\tanh^2 \frac{\eta}{2} \exp[2\sum_{\ell} A_\ell P_\ell(-\cosh \eta) P_\ell(\cos\theta)]dt^2$$
$$+ 4L^2 \cosh^4 \frac{\eta}{2} \exp[-2\sum_{\ell} A_\ell P_\ell(-\cosh \eta) P_\ell(\cos\theta)]$$
$$\times \left\{e^{2\gamma}(d\eta^2 + d\theta^2) + \sin^2\theta\, d\delta^2\right\}$$

If $C_\ell = \pm\delta_{\ell o}$, and if $A_\ell = 0$ for all ℓ, then

$$W = \pm \ln \cot t/2$$

and $\frac{\partial W}{\partial t} = \mp \csc t$, and this clearly satisfies the constraints (V-9a&b]. From this, $\gamma = 0$,

$$e^{\gamma - W} = \frac{1}{\cot t/2} = \tan t/2$$

and

$$ds^2 = L^2 4\sin^2 \frac{t}{2} \cos^2 t/2[\tan^2 t/2(d\theta^3 - dt^2) + \tan^2 \frac{t}{2} \sin^2\theta d\delta^2]$$

$$+ L^2 \cot^2 \frac{t}{2} d\sigma^2$$

$$= 4L^2 \sin^4 \frac{t}{2}[(d\theta^2 - dt^2) + \sin^2\theta d\delta^2] + L^2 \cot^2 \frac{t}{2} d\sigma^2 .$$

If $t \rightarrow t + \pi$, and $\sigma \rightarrow \frac{\sigma}{L}$,

$$ds^2 = 4L^2 \cos^4 \frac{t}{2}[(d\theta^2 - dt^2) + \sin^2\theta d\delta^2] + \tan^2 \frac{t}{2} d\sigma^2 .$$

This is the usual version of the closed Kantowski-Sachs metric given in the previous section. As has been demonstrated, a type II transformation on this space yields the Schwarzschild space.

The presence of the Schwarzschild metric among the transforms of these spaces is a strong inducement to further investigate their properties under the complex transformations developed in this thesis.

From the metric (V-7a), the tetrad of one forms is

$$\omega^o = L \sin t \; e^{(\gamma - W)} dt$$

$$\omega^1 = L \sin t \; e^{(\gamma - W)} d\theta$$

$$\omega^2 = L \sin t \sin\theta \; e^{-W} d\delta$$

$$\omega^3 = L \; e^{W} d\sigma \; ,$$

where W is defined by (V-6). Since it is necessary, under these transformations, that these one forms be either imaginary or real, the effect of imaginary coordinate transformations on W and γ must be tested thoroughly. If $\theta \to i\theta$, W is real since the Legendre polynomials of the first kind become ring functions. If however $t \to i\eta$, W will in general become complex due to the presence of the logarithm part of $Q_\ell(\cos t)$. Thus

$$Q_n(\cos t) = P_n(\cos t) \,\ell n \cot \frac{t}{2} - \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(\cos t) P_{n-m}(\cos t)$$

and when $t \to it$

$$Q_n(\cosh t) = P_n(\cosh t) \,\ell n[-i \coth \frac{t}{2}]$$

$$- \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(\cosh t) P_{n-m}(\cosh t)$$

$$= P_n(\cosh t) \,\ell n[\coth t/2] - i P_n(\cosh t) \frac{\pi}{2}$$

$$- \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(\cosh t) \quad P_{n-m}(\cosh t) \; .$$

In general e^W is a complex number, and the resulting space will be complex, and none of the rotations that have been discussed here will be possible. The single exception is the case where $C_o = 1$, and the resulting space is that of Kantowski-Sachs discussed previously. This does not mean, however, that the Kantowski-Sachs closed universe is the only one of its kind since the A_ℓ's have not been constrained.

The metrics for which $C_o = \pm 1$, form a family with

$$W = \sum_\ell A_\ell P_\ell(\cos\ t) P_\ell(\cos\theta) + \ln\ \cot \frac{t}{2} \ .$$

When $t \to i\eta$

$$W = \sum_\ell A_\ell P_\ell(\cosh\ \eta) P_\ell(\cos\theta) + \ln\ \coth \frac{\eta}{2} - \frac{i\pi}{2} \ ,$$

and

$$e^W = -\ i[\exp(\sum_\ell A_\ell P_\ell(\cosh\ \eta) P_\ell(\cos\theta)]\coth \frac{\eta}{2}$$

$$e^{-W} = i[\exp(-\sum_\ell A_\ell P_\ell(\cosh\eta) P_\ell(\cos\theta)]\tanh \frac{\eta}{2} \ .$$

To see how γ is affected by all of this, V-8 can be written

$$\gamma = \int_0^\theta dy\ \sin y\left\{2\sin\ t\left[\frac{(W_+{}^2+W_-{}^2)\sin\ t\ \cos\theta + (W_+{}^2=W_-{}^2)\cos\ t\ \sin\theta}{\sin^2 t - \sin^2\theta}\right]\right.$$

$$\left. -\ \cos\ y(\sin^2 t - \sin^2 y)^{-1}\right\}$$

The derivatives of W will all be real, and

$$W_+ = \frac{\partial W}{\partial\theta} - i\ \frac{\partial W}{\partial\eta}$$

$$W_- = \frac{\partial W}{\partial\theta} + i\ \frac{\partial W}{\partial\eta}$$

and so $W_+ = \bar{W}_-$, and $W_+{}^2 + W_-{}^2$ is real, and $W_+{}^2 - W_-{}^2$ is imaginary. This is what is required for γ to be real.

The tetrad is now

$$\omega^o = L \sin t\, e^\gamma \exp[-\sum_\ell A_\ell P_\ell(\cos t) P_\ell(\cos\theta)] \tan\frac{t}{2}\, dt$$

$$= 2L \sin^2\frac{t}{2}\, e^\gamma \exp[-\sum_\ell A_\ell P_\ell(\cos t) P_\ell(\cos\theta)]\, dt$$

$$\omega^1 = 2L \sin^2\frac{t}{2}\, e^\gamma \exp[-\sum_\ell A_\ell P_\ell(\cos t) P_\ell(\cos\theta)]\, d\theta$$

$$\omega^2 = 2L \sin^2\frac{t}{2} \sin\theta \exp[-\sum_\ell A_\ell P_\ell(\cos t) P_\ell(\cos\theta)]\, d\delta$$

$$\omega^3 = L \cot\frac{t}{2} \exp[\sum_\ell A_\ell P_\ell(\cos t) P_\ell(\cos\theta)]\, d\sigma.$$

Under the transformation $t \to -i\eta$, $\sigma \to t$

$$\omega^o(x') = 2Li \sinh^2\frac{t}{2}\, e^{\gamma'} \exp(\sum_\ell - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta))\, d\eta$$

$$\omega^1(x') = -2L \sinh^2\frac{t}{2}\, e^{\gamma'} \exp(\sum_\ell - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta))\, d\theta$$

$$\omega^2(x') = -2L \sinh^2\frac{t}{2} \sin\theta \exp(\sum_\ell - A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta))\, d\delta$$

$$\omega^3(x') = -iL \coth\frac{t}{2} \exp[\sum_\ell A_\ell P_\ell(\cosh t) P_\ell(\cos\theta)\, dt\ .$$

Finally, the type II transformation

$$\omega^{o\prime} = -i\omega^3$$
$$\omega^{3\prime} = -i\omega^o$$
$$\omega^{1\prime} = \omega^1$$
$$\omega^{2\prime} = \omega^2$$

gives a new, rotated, tetrad

If $\eta \to \cosh^{-1}(\frac{r}{m} - 1)$, then this will have the same coordinates as the Schwarzschild metric.

Vorhees [23] notes that in the case of a Weyl metric

$$ds^2 = e^{2\varphi}dt^2 + \gamma_{ij}\, dx^i\, dx^j ,$$

φ is the Newtonian potential. In the Schwarzschild coordinates,

$$\varphi = W = \frac{m}{r} - \sum_\ell A_\ell P_\ell(\frac{r}{m} - 1)P_\ell(\cos\theta$$

$$\sim \frac{m}{r} - A_\ell r^\ell P_\ell \cos\theta .$$

Type III transformations can also be made upon these spaces. If $\theta \to ir$, $t \to \theta$, $\sigma \to t$, W will remain real, and so will γ. The tetrad of one forms becomes

$$\omega^0 = L \sin\theta\, e^{(\gamma-W)}d\theta$$

$$\omega^1 = iL \sin\theta\, e^{(\gamma-W)}dr$$

$$\omega^2 = iL \sin\theta \sinh r\, e^{-W}d\delta$$

$$\omega^3 = L\, e^W\, dt$$

and under the type III transformation, the new tetrad is

$$\omega^{0'} = L\, e^{W'}dt$$

$$\omega^{1'} = L \sin\theta \sinh r\, e^{-W'}d\delta$$

$$\omega^2 = L \sin\theta\, e^{(\gamma-W')}d\theta$$

$$\omega^3 = L \sin\theta\, e^{(\gamma-W')}dr .$$

The metric for this space is

$$ds^2 = -L^2 e^{2W} dt^3 + L^2 \sin^2\theta \left\{ e^{2(\gamma - W)} (d\theta^2 + dr^2) + \sinh^2 r \; e^{-2W} d\delta^2 \right\}.$$

This is certainly a standard Weyl metric whose Newtonian potential is

$$\varphi = W = \sum_{\ell} [A_\ell P_\ell(\cos\theta) + C_\ell Q_\ell(\cos\theta)] P_\ell(\cosh r)$$

These spaces lack the spherically symmetric part of the potential, and have replaced it with a part in $Q_\ell(\cos\theta)$ that is singular at $\theta = 0$.

A type I transformation can be done on the Gowdy T_3 universe if $t \to it$, and $\theta \to i\theta$. In this case, it is again necessary for $C_\ell = 0$ for all $\ell \neq 0$, and for $C_o = 1$. This assures that W will not have a variable, imaginary part, and that γ will be real. In these cases,

$$W = \sum_{\ell} A_\ell P_\ell(\cosh t) P_\ell(\cosh \theta) + \ell n \coth \frac{t}{2} - \frac{i\pi}{2},$$

and

$$e^W = -i \coth \frac{t}{2} \exp[\sum_{\ell} A_\ell P_\ell(\cosh t) P_\ell(\cosh\theta)]$$

$$= -i \coth \frac{t}{2} e^{\tilde{W}}.$$

The transformed γ will be designated γ'. The tetrad now becomes

$$\omega^o = iL \sinh t \tanh \frac{t}{2} e^{(\gamma' - \tilde{W})} dt$$

$$= 2iL \sinh^2 \frac{t}{2} e^{(\gamma' - \tilde{W})} dt$$

$$\omega^1 = 2iL \sinh^2 \frac{t}{2} e^{(\gamma' - \tilde{W})} d\theta$$

$$\omega^2 = 2iL \sinh^2 \frac{t}{2} e^{-\tilde{W}} \sinh \theta d\delta$$

$$\omega^3 = i \coth \frac{t}{2} e^{\tilde{W}} d\sigma$$

The type I transformation is given by

$$\omega^{o\prime}(x') = -i\omega^{o}(x') ,$$

giving for the rotated tetrad

$$\omega^{o\prime} = 2L\sinh^2\frac{t}{2}\, e^{(\gamma'-\widetilde{W})}dt$$
$$\omega^{1\prime} = 2L\sinh^2\frac{t}{2}\, e^{(\gamma'-\widetilde{W})}d\theta$$
$$\omega^{2\prime} = 2L\sinh^2\frac{t}{2}\, e^{-\widetilde{W}}\sinh\theta d\delta$$
$$\omega^{3\prime} = \coth\frac{t}{2}\, e^{\widetilde{W}}d\sigma .$$

The metric is now

$$ds^2 = 4L^2\sinh^4\frac{t}{2}\, e^{-2\widetilde{W}}[e^{2\gamma'}(d\theta^2-dt^2)+\sinh^2\theta d\delta^2]$$
$$+\coth^2\frac{t}{2}\, e^{2\widetilde{W}}\, d\sigma^2 ,$$

which, when $A_\ell \to 0$, reduces to the open Kantowski-Sachs metric.

D. Conclusions

The Schwarzschild metric transforms by a type II transformation into the closed Kantowski-Sachs universe, and by a type III transformation into the open Kantowski-Sachs universe.

This leads to an investigation of the Gowdy universes, a series of closed universes locally like the Einstein-Rosen cylindrical waves, among which are the Kantowski-Sachs closed univers. After a thorough review of these spaces it it was shown that a type II transformation induced by the coordinate transformation $t \rightarrow i\eta$ acting upon all of these universes where $C_\ell = 0$ if $\ell \neq 0$, and $C_0 = 1$, transformed them into a new family of Weyl metrics that reduced to the Schwarzschild metric as $A_\ell \rightarrow 0$. The Newtonian potentials of these spaces show that they represent a spherically symmetric source surrounded by a distant, nonspherically symmetric matter distribution.

A second new family of these Weyl metrics resulted from the type III transformation induced by $\theta \rightarrow ir$ upon all of the Gowdy universes. These lacked the point source at $r = 0$, but kept the distant matter distribution, and had, in addition, a piece of the potential that is irregular at $\theta = 0$.

Finally a type I transformation on the first class of Gowdy T_3 universe where $C_\ell = \pm\, \delta_{0\ell}$, transformed them into a third family of new metrics that reduced to the open Kantowski-Sachs universe as $A_\ell \rightarrow 0$.

Chapter VI

Results and Conclusions

A. Results

This work began as an attempt to perform the Weyl "unitary trick" upon solutions to Einstein's field equation. These were first applied to a family of spaces that can be obtained by the imaginary rotation of $S^4(R)$. This led to the first original result of this thesis. Four of these imaginary rotations were discovered. The first of these, the type 0, rotates any solution of the field equations back onto a positive definite metric. The second, the type I, rotates one space that is a vacuum solution to the field equations onto another with the opposite sign of the scalar curvature. The third transformation, the type II, exchanges a space-like leg of the tetrad of the space, and the time-like leg. These transformations also map one vacuum solution onto another vacuum solution. Finally, the type III transformation is a composition of the type I with the type II.

These transformations were then applied to the Schwarzschild metric and to the Gowdy universes. Out of these came the second original contribution of this work. When these transformations were applied to the Gowdy T_3 universes, one family of metrics was discovered that had the Schwarzschild metric as the limit when the parameters

$A_\ell \to 0$. Another had the open Kantowski-Sachs as the limit when $A_\ell \to 0$. A third family was discovered that reduces to no space in particular.

All of these new metrics seem to be interior to a shell of matter.

These two contributions need now to be reviewed in some detail.

B. The Rotations

The type 0 transformation maps a space with signature ± 4 onto a space with Lorentz signature. Examples of these rotations are the rotation of the de Sitter universe into the four-sphere, and the rotation of the pseudosphere into the antide Sitter universe.

All Lorentz spaces have their symmetry reduced by the requirement that one axis be time-like, and that the other three be space-like. This means that it is sometimes easier to deal with the positive definite manifold than with the indefinite one. This is true, for instance, when working with the complex structures of the manifold [24]. In these cases, it is better to work with the positive definite signature manifold and then to transform into the Lorentz signature space. The type 0 transformation makes this possible.

The type I transformation maps a space into another space whose scalar curvature has the opposite sign. If this space is a vacuum solution to Einstein's field equations, then the transform will also be a solution, and the sign of Λ, the cosmological constant, will be the negative of that of the original space. If the one space is a closed space, the other one will be open, and vice versa. The ideal example of this is the transformation between the de Sitter and the anti-de Sitter universe.

The type II transformation exchanges the time-like and one space-like leg of the tetrad. If a space is a vacuum solution to the field equation, then its transform under the type II transformation will be also. For every vacuum metric, there will be three other spaces related by a type II transformation (provided that the necessary coordinate transformation exists). Again, the classic examples of this are the transformations from the de Sitter universe to the universes whose metrics are given in II-8.

A particularly interesting example of the type II transformation is the ℓ-$\bar{m}$ exchange. In choosing the Newman-Penrose pseudoorthonormal tetrad, it is desirable to label the space-like legs of the orthonormal tetrad so that $\ell = -\frac{1}{\sqrt{2}}(\omega^o + \omega^1)$ is geodesic (is tangent to the actual paths of photons). When ω^o is exchanged with ω^2 or ω^3 in a type II transformation, the space-like legs of the tetrad are again relabeled so that $\ell' = -\frac{1}{\sqrt{2}}(\omega^{o\prime} + \omega^{1\prime})$ is geodesic. For the algebraically special metrics, it is found that in order for this to obtain when $\omega^o = -i\omega^2$, then $\omega^{1\prime} = \omega^3$, and when $\omega^{o\prime} = -i\omega^3$, $\omega^{1\prime} = \omega^2$. This means that in the pseudoorthonormal tetrad $\ell \rightarrow \bar{m}$, $n \rightarrow m$. The Petrov type is preserved under these transformations.

The type III transformation is simply a type I transformation followed by a type II transformation. Since both of these transformations, acting on a vacuum solution of Einstein's field equations would yield another such solution, then the type III transformation will do the same. An example of these transformations are those from the metric given in II-8 to the anti-de Sitter universe.

C. The New Metrics

When the types I, II and III transformations were applied to the Gowdy T_3 universes, three new metrics resulted that seem to be of some physical interest. The most significant of these results from the application of the type II transformation to the set of the Gowdy T_3 universe with $C_\ell = \pm \delta_{\ell o}$. The resulting metrics are Weyl metrics whose limit as $A_\ell \rightarrow 0$ for all ℓ, is just the Schwarzschild metric. Otherwise, these spaces are a new set of Weyl metrics, and seem to represent the field with a point source at the origin, and interior to some shell of matter. The metric of this space is

$$ds^2 = -L^2 \coth^2 \frac{t}{2} \exp[2 \sum_\ell A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta)] dt^2$$

$$+ 4L^2 \sinh^4 \frac{t}{2} \exp[\sum_\ell -2A_\ell P_\ell(\cosh \eta) P_\ell(\cos\theta)] \left\{e^{2\gamma}(d\eta^2 + d\theta^2)\right.$$

$$\left. + \sin^2\theta \, d\delta^2\right\}$$

where γ is that of V-8, with $t = i\eta$.

The next most significant of these spaces is the one obtained from the action of a type III transformation upon the Gowdy T_3 universes. The metric of this space is

$$ds^2 = -L^2 e^{2W} dt^2 + L^2 \sin^2\theta \left\{ e^{2(\gamma - W)} (d\theta^2 + dr^2) + \sinh^2 r \, e^{-2W} d\delta^2 \right\}$$

where $W = \sum_{\ell} [A_\ell P_\ell(\cos\theta) + C_\ell Q_\ell(\cos\theta)] P_\ell(\cosh r)$

and where γ is again defined by V-8, but with $\theta \to ir$, $t \to \theta$.

This metric is again a Weyl metric, and seems to be the field of a space interior to a shell of matter.

Finally, a type I transformation, acting on the same set that the type II transformation acted on (those with $C_\ell = \pm \delta_{\sigma\ell}$) gives a set of metrics that reduce to the Kantowski-Sachs open universe as $A_\ell \to 0$. Their metric is

$$ds^2 = 4L^2 \sinh^4 \frac{t}{2} \, e^{-2W} [e^{2\gamma}(d\theta^2 - dt^2) + \sinh^2\theta \, d\delta^2]$$

$$+ \coth^2 \frac{t}{2} \, e^{2W} d\sigma^2$$

where now $W = \sum_{\ell} P_\ell(\cosh t) P_\ell(\cosh\theta)$ and where γ is defined by V-8 with $t \to it$, $\theta \to i\theta$.

The last one of these metrics is of doubtful value as a description of an actual physical system. As has been said, as $A_\ell \to 0$, this universe becomes the vacuum Kantowski-Sachs open universe. When this universe contains matter, it has a negative energy density.

The other two, on the other hand, are certainly worth further investigation. What is needed is, first of all, an explicit expression for the constraint (V-9a) on the A_ℓ. One suspects that these will be related to the mass distribution in the shell of matter surrounding the origin. This should be demonstrated, if possible. Finally, it would be helpful to work out some particular examples.

Appendix A

Geometric Formalism

A. Manifolds, Vectors, and One-Forms [25]

The idea of a differentiable manifold is a generalization of the idea of an Euclidian space. It is, in fact, a space that can be nicely mapped, neighborhood by neighborhood, onto a Euclidian space. To define a manifold exactly, some other definitions are necessary. An m dimensional Euclidian space is denoted R^m. A mapping between two open subsets of R^n is differentiable if the coordinates of one set can be expressed as differentiable functions of the other.

A homoeomorphism between two topological spaces X and Y is a one to one mapping

$$f : X \to Y$$

such that both f and f^{-1} are continuous.

If M is a topological space such that any two points can be each contained in two disjoint open sets, it is called a Hausdorf space. If M is a Hausdorf space, an open chart on M is a pair (U,φ), where U is an open subset of M, and where φ is a homeomorphism of U onto R^m. On U, the subset of R^m, $\varphi(U)$, acts as a coordinate system. It is called a coordinate patch, and the continuity of φ and φ^{-1} guarantee that these coordinates will be well behaved. These are shown in Fig. A-1.

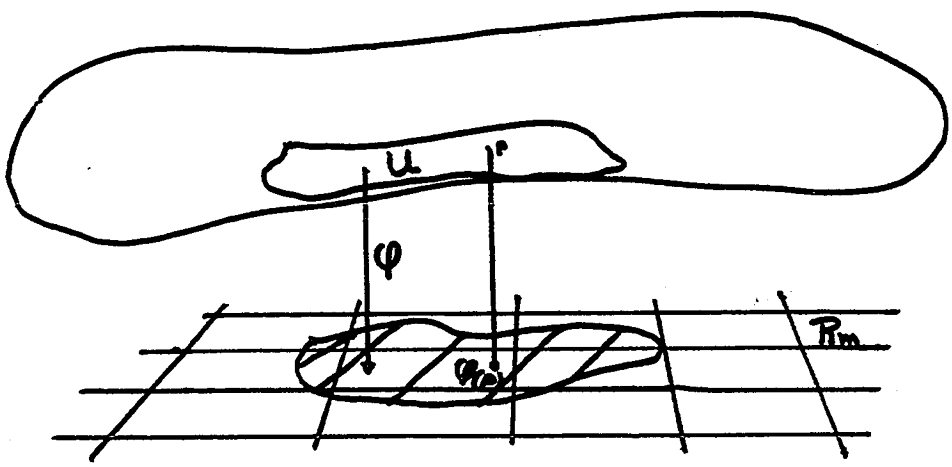


Figure A-1

A manifold is a space that has a collection of such open charts $(U_\alpha, \varphi_\alpha)_{\alpha \varepsilon A}$ that covers the entire manifold, i.e.

$$M = \bigcup_{\alpha \varepsilon A} U_\alpha$$

It is further required that these open charts be mathematically consistent with each other. If U_α and U_β are two intersecting open sets, the homeomorphism φ_α maps U_α into $\varphi_\alpha(U_\alpha)$, an open set of R^m, and φ_β maps U_β into $\varphi_\beta(U_\beta)$, another open set of R^m. Each of these open sets of R^m contains a subset, $\varphi_\alpha(U_\alpha \cap U_\alpha)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ that are images of a common open set in M. Since φ_α is a homeomorphism, φ_α^{-1} is a one to one continuous map from $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $U_\alpha \cap U_\beta$.

$$\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow U_\alpha \cap U_\beta$$

and φ_β takes $U_\alpha \cap U_\beta$ into $\varphi_\beta(U_\alpha \cap U_\beta)$

$$\varphi_\beta : U_\alpha \cap U_\beta \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) .$$

Thus

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

where $\circ$ denotes the composition of the two mappings (the mapping that results from the action, first, of φ_α^{-1} on $\varphi_\alpha(U_\alpha \cap U_\beta)$, followed by the action of φ_β on its image). This is illustrated in Fig. A-2. It is required that $\varphi_\beta \circ \varphi_\alpha^{-1}$ be differentiable.

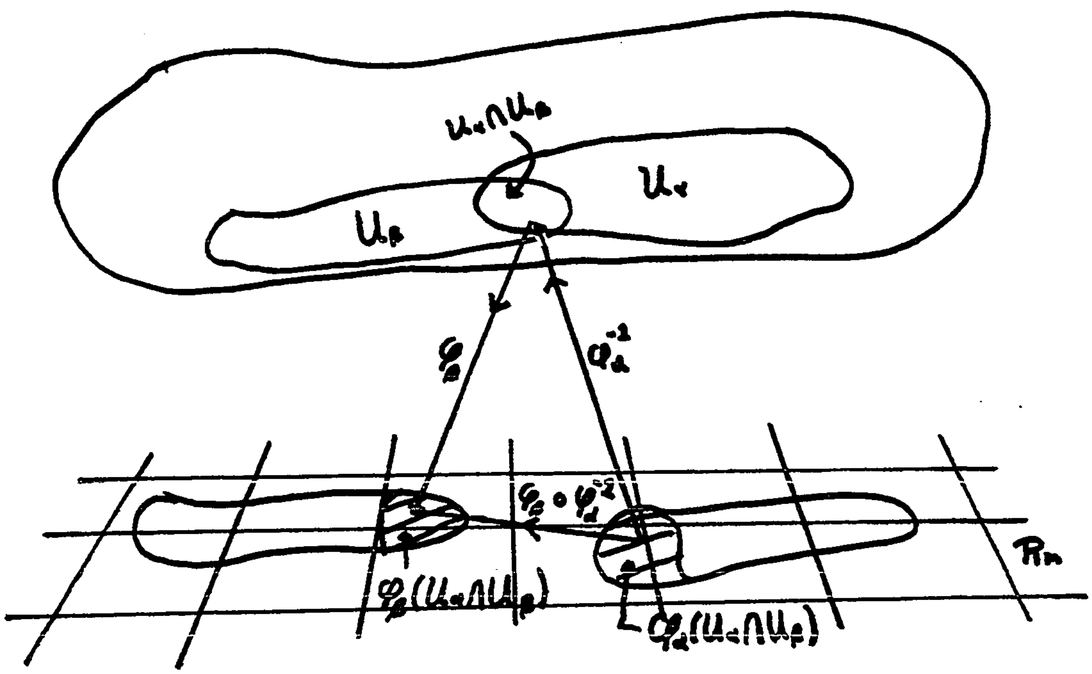


Figure A-2

Suppose that f is a real valued function on M (that is, f assigns a real number to each point of M). This function f is said to be differentiable at some point p of M if there is an open chart $(U_\alpha, \varphi_\alpha)$ containing p such that $f \circ \varphi_\alpha^{-1}$ is differentiable at p. It is said to be differentiable if it is differentiable at each point p of M. The complete collection of such differentiable functions is $C^\infty(M)$.

If $f, g \varepsilon C^\infty(M)$, then a vector field is a mapping

$$X : C^\infty(M) \to C^\infty(M)$$

such that

$$\begin{aligned} &1) \quad X(\alpha f + \beta g) = \alpha Xf + \beta Xg \\ &2) \quad X(fg) = f(Xg) + (Xf)g \ , \end{aligned} \qquad \text{(A-1)}$$

where α and β are real numbers. The collection of vector fields is denoted $D^1(M)$.

It follows from (2) that if f is a constant

$$X(fg) = f(Xg) = f(Xg) + (Xf)g$$

then $Xf = 0$. Suppose that (U,φ) is an open chart on M with p an arbitrary point is U. Let $\varphi(q) = (x_1(q), x_2(q), \ldots, x_m(q)$, where $q\varepsilon U$, and $f^* = f \quad \varphi^{-1}$ for $f\varepsilon C^\infty(M)$. f^* is a function of the coordinates in $\varphi(U)$. Further, let V be an open subset of U such that $\varphi(V)$ is an open ball in $\varphi(U)$ with a center $\varphi(p) = (a^1, a^2, \ldots, a^m)$, and let $(x^1, x^2, \ldots, x^m) \varepsilon \varphi(V)$.

$$f^*(x^1,x^2,\ldots,x^m) = f^*(a^1,a^2,\ldots,a^m) + \sum_{i=1}^{m} \frac{\partial f^*}{\partial x^i}(a^1,a^2,\ldots,a^m)(x^i - a^i) + \tfrac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 f^*}{\partial x^i \partial x^j}(a^1,a^2,\ldots,a^m)(x^i-a^i)^2 + \ldots .$$

Transferring back into M,

$$f(q) = f(p) + \sum_{i=1}^{m} (x_i(q) - x_i(p))\left(\frac{\partial f^*}{\partial x_i}\right)_{\varphi(p)}$$

$$+ \sum_{i,j=1}^{m} (x\ (q) - x\ (p))^2\left(\frac{\partial^2 f^*}{\partial x_i}\right)_{\varphi(p)} + \cdots .$$

If, now, $X \in D^1(U)$

$$Xf(q) = Xf(p) + \sum_{i=1}^{m} X[x\ (q)-x\ (p)]\left(\frac{\partial f^*}{\partial x^i}\right)_{\varphi(p)}$$

$$+ \sum_{i,j=1}^{m} X(x\ (q)-x\ (p))^2\left(\frac{\partial^2 f^*}{\partial x^i \partial x^j}\right)_{\varphi(p)} + \cdots .$$

But $Xf(p) = 0$ since $f(p)$ is just a constant part of f as q ranges over the points of V. Also, for the same reason $X(x_i(q)-x_i(p)) = Xx_i(q)$. Therefore

$$Xf(q) = \sum_{i=1}^{m} Xx_i(q)\left(\frac{\partial f^*}{\partial x^i}\right)_{\varphi(p)} + \sum_{i,j=1}^{m} X(x_i(q)-x_i(p))^2$$

$$\times \left(\frac{\partial^2 f^*}{\partial x^i \partial x^j}\right)_{\varphi(p)}$$

and as $q \to p$,

$$Xf(p) = \sum Xx_i(p)\left(\frac{\partial f^*}{\partial x^i}\right)_{\varphi(p)} \tag{A-2}$$

Thus, $X = \sum_{i=1}^{m} Xx^i(p)\left(\frac{\partial}{\partial x^i}\right) \circ \varphi$ is the local realization of the vector field.

In the ordinary parlance of physics, a vector field is simply a function that assigns a vector to each point of a space. It is far from obvious that the vector fields defined above are, in fact, these vector fields. In order to show that this is so, it is necessary to check the transformation properties of X defined above under a local coordinate transformation.

Let $\{X^i\} = \{Xx^i\}$, where, as before, the x^i are the cooadinate function on U. If $x^{i'} = x^{i'}(x^j)$ is a local coordinate transformation on $\varphi(V)$, then

$$\frac{\partial}{\partial x^i} = \frac{\partial x^{j'}}{\partial x^i} \frac{\partial}{\partial x'^j}$$

and

$$X = X^i \frac{\partial}{\partial x^i} = X^i \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j} = X^{j'} \frac{\partial}{\partial x'^j}$$

where

$$X^{j'} = X^i \frac{\partial x^{j'}}{\partial x^i} .$$

The components of X, then, transform like a contravariant vector. <u>Every element of $D^1(M)$ defines a contravariant vector field</u>.

Suppose, on the contrary, that a contravariant vector field $\{X^i\}$ is given. Then certainly $X = X^i \frac{\partial}{\partial x^i}$ defines a mapping

$$X : C^\infty(M) \to C^\infty(M).$$

$D^1(M)$, then, is just the space of all contravariant vectors over M.

At each point $p \varepsilon M$, $D^1(p)$ is just the space of all vectors at that point. $D^1(p)$ is the <u>tangent space</u> at p. A vector field $X \varepsilon D^1(M)$ defines a vector at each point $p \varepsilon M$, and there is a family of curves in M tangent to X at each point of M. These integral curves are called the <u>congruence</u> of X in M.

If $X,Y \varepsilon D^1(M)$, a product can be defined

$$(X,Y) \rightarrow XY = X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial}{\partial x^j}\right) .$$

This product is not a vector. Another product, however,

$$[X,Y] = XY - YX \qquad \text{(A-3)}$$

is a vector field, and it is called the Lie derivative of Y with respect to X.

$D^1(p)$, the tangent space at p, is just a vector space and is denoted T_p. A <u>linear functional</u> on a vector space T_p is a mapping ω that maps T_p into the real numbers

$$\omega : T_p \rightarrow R .$$

An example of this is the inner product. If ω and θ are linear functional on a vector space V, then $a\ \theta + b\ \omega$ is a linear functional where a and b are real numbers. Thus, the space of linear functionals on a vector space V is itself a vector space $\tilde{V}$, called the dual space. Thus, at each point of M, a dual space exists to $D^1(p)$. This can be denoted $D_1(p)$.

The basis of $D^1(U)$ is $\left\{\frac{\partial}{\partial x^i}\right\}$, as can be seen from (A-2). Likewise, $D_1(U)$ has a basis $\left\{\omega^i\right\}$, such that if $\theta \varepsilon D_1(U)$, $\theta = \theta_i \omega^i$. These ω^i can be chosen so that $\omega^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j$. Later it will be shown that these $\omega^i = dx^i$.

$D^1(M)$ has already been identified as the collection of contravariant vector fields over M. Since $X_\mu Y^\mu$ is a real number, then the collection of covariant vector fields is dual to the collection of contravariant vector fields.

It is possible to define a product on vector spaces such that the product space is itself a vector space. This is called the tensor product. If V and W are vector spaces, and $A,B\epsilon V$, and $C,P\epsilon W$, then $(aA+bB) \otimes (cC+dD) = acA \otimes C + adA \otimes D + BcB \otimes C + bdB \otimes D \epsilon V \quad W$.

From what has been said before, $D^1(M)$ can be producted with itself, or with $D_1(M)$

$$D^1(M) \otimes D^1(M) \otimes \ldots \quad D^1(M) \otimes D_1(M) \otimes \ldots \otimes D_1(M)$$

and the resulting space is a vector space. These are called the tensor fields over M. Since $\left\{\frac{\partial}{\partial x^i}\right\}$ is a basis for $D^1(M)$ and $\{dx^i\}$ is a basis for $D_1(M)$, the basis for the tensor fields of M are

$$\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \ldots \quad \frac{\partial}{\partial x^k} \otimes dx^\ell \otimes \ldots \otimes dx^m \text{ , i.e. for}$$

any tensor T,

$$T = T^{ij\ldots k}{}_{\ell\ldots m} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \ldots \quad \frac{\partial}{\partial x^k} \otimes dx^\ell \otimes \ldots \otimes dx^m \,.$$

It is easy to see that these tensor fields under a coordinate transformation $x^{j'} = x^{j'}(x^i)$, transform

$$T^{i'\ldots j'}{}_{\ell'\ldots m'} = T^{k\ldots n}{}_{x\ldots t} \frac{dx^{i'}}{dx^k} \cdots \frac{dx^{j'}}{dx^n} \frac{dx^s}{dx^{\ell'}} \cdots \frac{dx^k}{dx^n} \,.$$

Having now defined the tensor product of two one-forms, another product can be defined. This is the exterior product (or wedge product). If θ, $\omega \varepsilon D_1(M)$ then

$$\theta \wedge \omega \equiv \theta \otimes \omega - \omega \otimes \theta$$

Since $\theta = \theta_\mu dx^\mu$, $\omega = \omega_\nu dx^\nu$,

$$\theta \wedge \omega = (\theta_\mu \omega_\nu - \theta_\nu \omega_\mu) dx^\mu \otimes dx^\nu .$$

The complete collection of all such forms is called $A_2(M)$. One can go on and define $A_s(M)$ for any number $s > 1$ as the collection of s-forms of the form $\omega_1 \quad \omega_2 \quad \cdots \quad \omega_s$. If $A_o = C^\infty(M)$, and $A_1 = D_1$, A_s is defined for all $s \geq 0$. In these cases, if $f_1 g \varepsilon A_o$, $f \wedge g = fg$, and $f \wedge \theta = f\theta$, where $\theta \varepsilon A_1$.

The exterior derivative is defined so that

1. If $\omega \varepsilon A_n$, $d\omega \varepsilon A_{n+1}$,
2. If $f \varepsilon A_o$, then $df = \dfrac{\partial f}{\partial x^i} dx^i$, and $df(x) = Xf$,
3. If $\omega^2 \varepsilon A(M)$, $\omega^1 \varepsilon A_n(M)$

$$d(\omega^1 \wedge \omega^2) = d\omega^1 \wedge \omega^2 + (-1) \quad \omega^1 \wedge d\omega^2 ,$$

4. and $d^2 = d(d) = 0$.

It was stated before that the basis elements of $D_1(M)$, ω^i, such that $\omega^i(x_j) = \delta^i{}_j$, are $\omega^i = dx^i$. This can now be proved from (2) above

$$df(x) = \frac{\partial f}{\partial x^i} dx^i\left(x^j \frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i} x^j dx^i\left(\frac{\partial}{\partial x^j}\right)$$

$$= Xf = x^i \frac{\partial f}{\partial x^i} .$$

This can be so only if $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i{}_j$.

On any manifold, an <u>affine connection</u> ∇ is the rule by which the covariant derivatives of vectors are calculated. This is defined in terms of the operator ∇ which obeys the two rules

$$\begin{aligned} &1) \quad \nabla_{fX + gY}(Z) = f\nabla_X(Z) + g\nabla_Y(Z) \\ &2) \quad \nabla_X(fZ) = f\nabla_X(Z) + (Xf)Z \end{aligned} \tag{A-4}$$

where $f, g \in C^\infty(M)$ and $X, Z, Y \in D^1(M)$. Let $\{X_i\}$ be the basis for a vector space, and define,

$$\nabla_{X_i} X_j = \Gamma_{ij}{}^k X_k \ .$$

Allow $Y = Y^i X_i$ and $Z = Z^i X_i$, and

$$\nabla_X Z = \nabla_{Y^i X_i}(Z^j X_j) = Y^i \nabla_{X_i}(Z_j X_j)$$

$$= Y^i Z^j \nabla_{X_i} X_j + Y^i X_i{}^\mu \frac{\partial Z^j}{\partial x^\mu} X_j$$

$$= Y^i\left(Z^j \Gamma_{ij}{}^k X_k + X_i{}^\mu \frac{\partial Z^j}{\partial x^\mu} X_j\right)$$

$$= Y^i\left(Z^j \Gamma_{ij}{}^k + X_i{}^\mu \frac{\partial Z^k}{\partial x^\mu}\right) X_k \ .$$

The quantity $Y^i\left(Z^j \Gamma_{ij}{}^k + X_i{}^\mu \frac{\partial Z^k}{\partial x^\mu}\right)$ is the $k^{\underline{th}}$ component of a vector to the basis defined above. Furthermore if $X_i = \partial/\partial X^i$, then this is

$$Y^\mu\left(\Gamma_{\mu\nu}{}^\rho Z^\nu + \frac{\partial Z^\rho}{\partial x^\mu}\right)\frac{\partial}{\partial x^\rho} \ . \tag{A-5}$$

which is $(X^{\mu}Z^{\rho}{}_{;\mu}) \frac{\partial}{\partial x^{\rho}}$.

Before the significance of (A-5) can be discussed, two more quantities must be introduced. These are the torsion and curvature of M. These are respectively

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad \text{(A-6)}$$

for X, Y, $Z \epsilon D^1(M)$.

For the purposes of this thesis, and, indeed, most of the work in general relativity [29], the connection defined in (A-4) is restricted to the pseudo-Riemannian connection. This means that

1 $T(X,Y) = 0$

2 $\nabla_x g = 0$

where g is the metric.

Define $g(X,Y) \equiv g_{\mu\nu}X^{\mu}Y^{\nu} = X_{\mu}Y^{\nu}$, and let $\{X^{\mu}\}$ be a basis of $D^1(M)$. Then condition (1) means

$$\begin{aligned}[X_i,X_j] &= \nabla_{X_i} X_j - \nabla_{X_j} X_i \\ &= (\Gamma_{ij}{}^k - \Gamma_{ji}{}^k)X_k .\end{aligned}$$

Since $[X_i,X_j]$ is a vector field in $D^1(M)$, it can be expressed with respect to the basis $\{X_i\}$. Let, then,

$$[X_i,X_j] \equiv C_{ij}{}^k X_k$$

where the $C_{ij}{}^k$, the components of $[X_i,X_j]$ with respect to $\{X_i\}$, are called the structure function. Condition (1) above implies that

$$\Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k} = C_{ij}{}^{k} \; . \qquad \text{(A-7)}$$

If $\{X_i\} = \{\partial/\partial x^i\}$, then

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \text{ and } \Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k} = 0 \; .$$

Now apply ∇_Z to $g(X,Y)$. Condition (2) above implies that

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \; .$$

Let $\{X_i\}$ be a basis of $D^1(M)$, and define the metric g with respect to that basis,

$$g(X_i, X_j) = g_{ij} \; .$$

$$X_k g(X_i, X_j) = g(\Gamma_{ki}{}^{\ell} X_\ell, X_j) + g(X_i, \Gamma_{kj}{}^{\ell} X_\ell)$$

$$X_k g_{ij} = g_{\ell j}\Gamma_{ki}{}^{\ell} + g_{i\ell}\Gamma_{kj}{}^{\ell} \qquad \text{(A-8)}$$

If $\{X_i\} = \{\partial/\partial x^i\}$, then (A-8) says

$$-\frac{\partial g_{ij}}{\partial x^k} + g_{j\ell}\Gamma_{ki}{}^{\ell} + g_{i\ell}\Gamma_{kj}{}^{\ell} = 0$$

or $$+\frac{\partial g_{ij}}{\partial x^k} = [ki,j] + [kj,i] \; ,$$

which gives the usual expression for the Riemannian connection in terms of the metric tensor.

Returning to (A-5) it can be seen that if the connection is the pseudo-Riemannian one, then $\Gamma_{\mu\nu}{}^{\rho}$ will be the usual, symmetric Christoffel symbols of the second kind, and (A-5) is just the usual covariant derivative of Z with respect to X.

To investigate the curvature tensor, let

$$R(X_i, X_j)X_k = R^k{}_{\ell ij}X_k \qquad \text{(A-9)}$$

Now, again let the basis of the tangent space be $\{\partial/\partial x^k\}$ and recall that $\nabla_{\partial/\partial x^\mu}(\partial/\partial x^\nu) = \Gamma_{\mu\nu}{}^\rho \dfrac{\partial}{\partial x^\rho}$. Then

$$R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)\frac{\partial}{\partial x^\rho} = \nabla_{\frac{\partial}{\partial x^\mu}}\left(\nabla_{\frac{\partial}{\partial x^\nu}}\left(\frac{\partial}{\partial x^\rho}\right)\right) - \nabla_{\frac{\partial}{\partial x^\nu}}\left(\nabla_{\frac{\partial}{\partial x^\nu}}\left(\frac{\partial}{\partial x^\rho}\right)\right)$$

$$- \nabla_{\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right]}\left(\frac{\partial}{\partial x^\rho}\right) = \nabla_{\frac{\partial}{\partial x^\mu}}\left(\Gamma_{\nu\rho}{}^\sigma \frac{\partial}{\partial x^\sigma}\right) - \nabla_{\frac{\partial}{\partial x^\nu}}\left(\Gamma_{\mu\rho}{}^\sigma \frac{\partial}{\partial x^\sigma}\right)$$

$$= \Gamma_{\nu\rho}{}^\sigma \nabla_{\frac{\partial}{\partial x^\mu}}\left(\frac{\partial}{\partial x^\sigma}\right) + \frac{\partial \Gamma_{\nu\rho}{}^\sigma}{\partial x^\mu}\frac{\partial}{\partial x^\sigma} - \Gamma_{\mu\rho}{}^\sigma \nabla_{\frac{\partial}{\partial x^\nu}}\left(\frac{\partial}{\partial x^\sigma}\right) - \frac{\partial \Gamma_{\mu\rho}{}^\sigma}{\partial x^\nu}\frac{\partial}{\partial x^\sigma}$$

$$= \Gamma_{\nu\rho}{}^\sigma\, \Gamma_{\mu\sigma}{}^\tau \frac{\partial}{\partial x^\tau} + \Gamma_{\nu\rho,\mu}{}^\sigma \frac{\partial}{\partial x^\sigma} - \Gamma_{\mu\rho}{}^\sigma\, \Gamma_{\nu\sigma}{}^\tau \frac{\partial}{\partial x^\tau} + \Gamma_{\mu\rho,\nu}{}^\sigma \frac{\partial}{\partial x^\sigma}$$

$$= \left(\Gamma_{\nu\rho,\mu}{}^\tau - \Gamma_{\mu\rho,\nu}{}^\tau + \Gamma_{\nu\rho}{}^\sigma\Gamma_{\mu\sigma}{}^\tau - \Gamma_{\mu\rho}{}^\sigma\Gamma_{\nu\sigma}{}^\tau\right)\frac{\partial}{\partial x^\sigma} = R^\tau{}_{\rho\mu\nu}\frac{\partial}{\partial x^\tau}$$

This means that

$$R^\tau{}_{\rho\mu\nu} = \Gamma_{\nu\rho,\mu}{}^\tau - \Gamma_{\mu\rho,\nu}{}^\tau + \Gamma_{\nu\rho}{}^\sigma\Gamma_{\mu\sigma}{}^\tau - \Gamma_{\mu\rho}{}^\sigma\Gamma_{\nu\sigma}{}^\tau ,$$

which is the usual definition of the curvature tensor. In the next section it shall be calculated with respect to the tetrad frame.

B. Tetrad Formalism [26,27]

All of the manifold being treated in this thesis are of dimension 4. Therefore, generalization beyond this dimensionality is unnecessary.

In a four dimensional manifold, the tangent space is obviously spanned by four vectors. Likewise, the space of one-forms is spanned by four one-forms. The vectors are related to the one-forms by the metric. If X is the vector, ω the one-form dual to it,

$$X = X^{\mu} \frac{\partial}{\partial x^{\mu}}$$

and

$$\omega = X_{\mu} dx^{\mu}$$

$$X^{\mu} = g^{\mu\nu} X_{\nu} \ .$$

The four dimensional spaces relevant in general relativity have Lorentz signature (sig. ±2), and have their tangent space spanned by four linearly independent field vectors, $\{X_i\}$ (i=0,1,2,3), one of which is time-like (i=0) and the rest space-like (an alternative to this will be discussed in Appendix B). These can be chosen to be orthogonal, and normalized so that $g_{\mu\nu} X_i{}^{\mu} X_j{}^{\nu} = \eta_{ij}$, where

$$\eta_{ij} = \mathrm{diag}(-1,1,1,1) \ .$$

In addition, the above relation can be inverted:

$$g_{\mu\nu} = \eta_{ij} X_{\mu}{}^{i} X_{\nu}{}^{j} \ ,$$

where $X_\mu{}^i$ are the covariant forms of $X_i{}^\mu$. In this case

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ij}X_\mu{}^i X_\nu{}^i dx^\mu dx^\nu$$
$$= \eta_{ij}\omega^i\omega^j = \omega^{02} + \omega^{12} + \omega^{22} + \omega^{32} \; , \qquad \text{(A-4)}$$

and ω^i are the one-forms dual to X_i. Such a tetrad of vectors and one-forms is called an orthonormal tetrad, and the line element, when expressed in terms of it, is said to be in normal form. The metric g_{ij} expressed with respect to the orthonormal frame is just η_{ij}. From (A-8), then $\eta_{\ell j}\Gamma_{ki}{}^\ell + \eta_{\ell i}\Gamma_{kj}{}^\ell = 0$. This means that

$$-\Gamma_{ki}{}^0 + \Gamma_{k0}{}^i = 0 \quad \text{or} \quad \Gamma_{ki}{}^0 = \Gamma_{k0}{}^i \qquad \text{(A-10)}$$

and $\Gamma_{ki}{}^j + \Gamma_{ki}{}^j = 0$ or $\Gamma_{ki}{}^j = -\Gamma_{kj}{}^i$, where $i = 1,2,3.$

From (A-4), one would like to have the components of the connection $\Gamma_{ij}{}^k$ in terms of the usual Christoffel symbols of the second kind. To do this, the general basis (specifically the orthonormal tetrad) is expressed with respect to $\{\partial/\partial x^\mu\}$

$$X_i = X_i{}^\mu \frac{\partial}{\partial x^\mu}$$

From (A-4, one computes

$$\nabla_{X_i}X_j = \Gamma_{ij}{}^k X_k = \nabla_{\left(X_i\frac{\partial}{\partial x^\mu}\right)}\left(X_j{}^\mu \frac{\partial}{\partial x^\nu}\right)$$

$$= X_i{}^\mu \nabla_{\frac{\partial}{\partial x^\mu}}\left(X_j{}^\nu \frac{\partial}{\partial x^\nu}\right) = X_i{}^\mu X_j{}^\nu \nabla_{\frac{\partial}{\partial x^\mu}}\left(\frac{\partial}{\partial x^\nu}\right) + X_i{}^\mu \frac{\partial}{\partial x^\mu} X_j{}^\nu \frac{\partial}{\partial x^\nu}$$

$$= X_i{}^{\mu} X_j{}^{\nu} \Gamma_{\mu\nu}{}^{\rho} \frac{\partial}{\partial x^{\rho}} + X_i{}^{\mu} \frac{\partial X_o{}^{\rho}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\rho}}$$

$$= X_i{}^{\mu}\left(\frac{\partial X_j{}^{\rho}}{\partial x^{\mu}} + \Gamma_{\mu\nu}{}^{\rho} X_j{}^{\nu}\right) \frac{\partial}{\partial x^{\rho}}$$

$$= (X_i{}^{\mu} X^{\rho}_{j;\mu}) \frac{\partial}{\partial x^{\rho}} = \Gamma_{ij}{}^{k} X_k = \Gamma_{ij}{}^{k} X_k{}^{\rho} \frac{\partial}{\partial x^{\rho}} .$$

Thus

$$X_k{}^{\rho}\, \Gamma_{ij}{}^{k} = X_i{}^{\mu} X^{\rho}_{j;\mu} \quad \text{or}$$

$$\Gamma_{ij}{}^{k} = X_i{}^{\mu} X^{\rho}_{j;\mu} X_{\rho}{}^{k} = - X^{k}_{\rho;\mu} X_j{}^{\rho} X_i{}^{\mu} \quad . \tag{A-11}$$

From the relations (A-7) and (A-10) it is possible to calculate the $\Gamma_{ij}{}^{k}$ from the $C_{ij}{}^{k}$. First of all, let $i,j,k = 1,2,3$. Then

$$\Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k} = C_{ij}{}^{k} = \Gamma_{ij}{}^{k} - \Gamma_{ji}{}^{k}$$

$$\Gamma_{ki}{}^{j} - \Gamma_{ik}{}^{j} = C_{ki}{}^{j} = + \Gamma_{ij}{}^{k} + \Gamma_{ki}{}^{j}$$

$$-\Gamma_{jk}{}^{i} + \Gamma_{kj}{}^{i} = - C_{jk}{}^{i} = + \Gamma_{ji}{}^{k} + \Gamma_{kj}{}^{i}$$

From this,

$$\Gamma_{ij}{}^{k} = \tfrac{1}{2}(C_{ij}{}^{k} + C_{ki}{}^{j} - C_{jk}{}^{i}) \quad . \tag{A-12}$$

This procedure can be used to calculate $\Gamma_{ij}{}^{o}$, $\Gamma_{oi}{}^{j}$, and $\Gamma_{io}{}^{j}$,

$$\Gamma_{ij}{}^{o} = \tfrac{1}{2}(C_{ij}{}^{o} - C_{oi}{}^{j} + C_{jo}{}^{i}) \tag{A-12b}$$

$$\Gamma_{oj}{}^{i} = \tfrac{1}{2}(C_{oj}{}^{i} + C_{io}{}^{j} + C_{ji}{}^{o}) \tag{A-12c}$$

Now, calculating the curvature tensor in terms of the orthonormal tetrad, one defines

$$R(X_i,X_j)X_k = R^{\ell}{}_{ijk}X_\ell \ .$$

From the definition (A-6),

$$R(X_i,X_j)X_\ell = \nabla_{X_i}\nabla_{X_j}X_\ell - \nabla_{X_j}\nabla_{X_i}X_\ell - \nabla_{[X_i,X_j]}X_\ell$$

$$= \nabla_{X_i}(\Gamma_{j\ell}{}^k X_k) - \nabla_{X_j}(\Gamma_{i\ell}{}^k X_k) - C_{ij}{}^k\nabla_{X_k}X_\ell$$

$$R(X_i,X_j)X_\ell = \Gamma_{j\ell}{}^k\nabla_{X_i}X_k + X_i\Gamma_{j\ell}{}^k X_k - \Gamma_{i\ell}{}^k\nabla_{X_j}X_k$$

$$- X_j\Gamma_{i\ell}{}^k X_k - C_{ij}{}^k\Gamma_{k\ell}{}^s X_s$$

$$= \Gamma_{j\ell}{}^k\Gamma_{ik}{}^s X_s - \Gamma_{i\ell}{}^k\Gamma_{jk}{}^s X_s + X_i\Gamma_{j\ell}{}^s X_s$$

$$- X_j\Gamma_{i\ell}{}^s X_s - C_{ij}{}^k\Gamma_{k\ell}{}^s X_s$$

$$= (\Gamma_{j\ell}{}^k\Gamma_{ik}{}^s - \Gamma_{i\ell}{}^k\Gamma_{jk}{}^s + X_i\Gamma_{j\ell}{}^s - X_j\Gamma_{i\ell}{}^s$$

$$- C_{ij}{}^k\Gamma_{k\ell}{}^s)X_s = R^s{}_{\ell ij}X_s \ .$$

Therefore

$$R^s{}_{\ell ij} = \Gamma_{j\ell}{}^k\Gamma_{ik}{}^s - \Gamma_{i\ell}{}^k\Gamma_{jk}{}^s + X_i\Gamma_{j\ell}{}^s - X_j\Gamma_{i\ell}{}^s - C_{ij}{}^k\Gamma_{k\ell}{}^s \ . \tag{A-13}$$

The following table summarizes the results of this section.

I. The tetrad:

$$\{X_i\} i = 0,1,2,3 \ ; \ g_{\mu\nu}X_i{}^\mu X_j{}^\nu = \eta_{ij}, \text{ where}$$

$$\text{where } \eta_{ij} = \mathrm{diag}(-1,1,1,1)$$

$$g_{\mu\nu} = \eta_{ij}X_\mu{}^i X_\nu{}^j \ ; \ X_i{}^\mu X_\mu{}^j = \delta_i{}^j \ ,$$

$$\text{where } X_\mu{}^i = \eta^{ij}g_{\mu\nu}X_j{}^\nu \ .$$

II. The Connection

$$\Gamma_{ij}{}^k = - X_{\mu;\nu}{}^k X_i{}^\nu X_j{}^\mu$$

$$\Gamma_{ij}{}^k - \Gamma_{ji}{}^k = C_{ij}{}^k \quad \text{where}$$

$$[X_i,X_j] = (X_i{}^\mu X_{j;\mu} - X_j{}^\mu X_{i;\mu}) = (X_i{}^\mu X_{j;\mu} - X_j{}^\mu X_{i;\mu})$$

$$= (X_i X_j - X_j X_i) = C_{ij}{}^k X_k$$

and

$$\Gamma_{ij}{}^k = \tfrac{1}{2}(C_{ij}{}^k + C_{ki}{}^j - C_{jk}{}^i) \quad (i,k,j = 1,2,3)$$

$$\Gamma_{ij}{}^o = \tfrac{1}{2}(C_{ij}{}^o - C_{oi}{}^j + C_{jo}{}^i)$$

$$\Gamma_{oj}{}^i = \tfrac{1}{2}(C_{oj}{}^i + C_{io}{}^j + C_{ji}{}^o)$$

The Curvature Tensor

$$R^s_{\ell ij} = \Gamma_{j\ell}{}^k\Gamma_{ik}{}^s - \Gamma_{i\ell}{}^k\Gamma_{jk}{}^s + X_i\Gamma_{j\ell}{}^s - X_j\Gamma_{i\ell}{}^s - C_{ij}{}^k\Gamma_{k\ell}{}^s$$

D. The Structure Equation [25]

Let $\{X_i\}$ be an orthonormal vector tetrad, and $\{\omega^i\}$ the dual tetrad of one-forms. From previous sections, since $\omega^i = X^i{}_\mu dx^\mu$,

$$d\omega^i = \frac{\partial X^i{}_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu \ .$$

But

$$X^i{}_\mu dx^\mu = \omega^i$$

$$dx^\mu = X^\mu{}_i \omega^i \ ,$$

so

$$d\omega^i = \frac{\partial X^i{}_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu = \left(\frac{\partial X^i{}_\mu}{\partial x^\nu} X_j{}^\nu X_k{}^\mu\right)\omega^j \wedge \omega^k$$

$$= \left(\frac{\partial X^i{}_\mu}{\partial x^\nu} X_j{}^\nu X_k{}^\mu - \frac{\partial X^i{}_\nu}{\partial x^\mu} X_j{}^\nu X_k{}^\mu\right)\omega^j \wedge \omega^k \qquad j < k$$

$$= -\ (\Gamma_{jk}{}^i - \Gamma_{kj}{}^i)\,\omega^j \wedge \omega^k \qquad j < k$$

$$= -\ C_{jk}{}^i \omega^j \wedge \omega^k \qquad j < k$$

$$= -\ \Gamma_{jk}{}^i\ \omega^j \wedge \omega^k$$

Now define a one-form

$$\omega^i{}_k = \Gamma_{jk}{}^i \omega^j \ , \qquad \text{(A-14a)}$$

the first structure equation can be written

$$d\omega^i = -\ \omega^i{}_k \wedge \omega^k \qquad \text{(A-14b)}$$

This structure equation, written in a more general form, is

$$d\omega^i = -\omega^i{}_k \quad \omega^k + \tfrac{1}{2}T_{jk}{}^i\omega^j \wedge \omega^k \quad ,$$

where $T_{jk}{}^i$ are the components of the torsion tensor which are equal to zero for Riemann manifolds. If one were using a non-Riemannian geometry (non-symmetric connection) then the full structure equation would be needed.

To go on, the exterior derivative of $\omega^i{}_k$ is taken:

$$d\omega^i{}_k = \Gamma_{jk,\mu}{}^i \, dx^\mu \wedge \omega^j + \Gamma_{jk}{}^i d\omega^j$$

$$dx^\mu = X_\ell{}^\mu \, \omega^\ell$$

$$d\omega^j = -\omega^j{}_m \wedge \omega^m = -\Gamma_{nm}{}^j\omega^n \wedge \omega^m \; .$$

Then

$$\begin{aligned} d\omega^i{}_k &= \Gamma_{jk,\mu}{}^i X_\ell{}^\mu \omega^\ell \wedge \omega^j - \Gamma_{jk}{}^i\Gamma_{nm}{}^j\omega^n \wedge \omega^m \\ &= \left(X_\ell\Gamma_{jk}{}^i - X_j\Gamma_{\ell k}{}^i - \Gamma_{sk}{}^i C_{\ell j}{}^s\right)\omega^\ell \wedge \omega^j \qquad \ell < j \\ &= \tfrac{1}{2}\, R^i{}_{k\ell j}\omega^\ell \wedge \omega^j - \Gamma_{jk}{}^\tau\Gamma_{\ell\tau}{}^i\omega^\ell \wedge \omega^j \\ &= \tfrac{1}{2}\, R^i{}_{k\ell j}\omega^\ell \wedge \omega^j - \omega^i{}_\tau \wedge \omega^\tau{}_k \end{aligned}$$

This yields the second structure equation

$$d\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \tfrac{1}{2}\, R^i{}_{j\ell m}\omega^\ell \wedge \omega^m \; . \tag{A-14c}$$

The two together are

$$\begin{aligned} d\omega^i &= -\omega^i{}_j \wedge \omega^j = -C_{jk}{}^i\omega^j \wedge \omega^k \quad (j < k) \\ d\omega^i{}_j &= -\omega^i{}_k \wedge \omega^k{}_j + \tfrac{1}{2}\, R^i{}_{j\ell m}\omega^\ell \wedge \omega^m \end{aligned} \tag{A-14}$$

The utility of these equations in doing calculations is enormous. The procedure is simple.

1. Put the metric in normal form:

$$ds^2 = -\omega^{0\,2} + \omega^{1\,2} + \omega^{2\,2} + \omega^{3\,2}$$

2. Use (A-14b) to calculate $C_{jk}{}^{i}$.

3. Use (A-12) to calculate the $\Gamma_{jk}{}^{i}$, and thence $\omega^{i}{}_{j} = \Gamma_{kj}{}^{i}\omega^{k}$.

4. Use (A-14c) to calculate $R^{i}{}_{jk\ell}$, and then contract to calculate R_{ij} and R.

This is the procedure used in Chapter III of this thesis to do the calculations for the Type 0, I, II, and III transformations.

Appendix B

The Newman-Penrose Formalism

A. Introduction

As was said in Appendix A, the tangent space of a manifold is spanned by a tetrad of vectors, $\{X_i\}$ $(i = 0,1,2,3)$ such that X_o is time-like, and $X_a (a = 1,2,3)$ is space-like and

$$g_{\mu\nu} X_i{}^{\mu} X_j{}^{\nu} = \eta_{ij} \tag{B-1a}$$

$$g_{\mu\nu} = \eta_{ij} X_{\mu}{}^{i} X_{\nu}{}^{j} \tag{B-1b}$$

where $$X_{\mu}{}^{i} = \eta^{ij} g_{\mu\nu} X_j{}^{\mu} \tag{B-1c}$$

and $$\eta_{ij} = \text{diag.}\ (-1,1,1,1) \tag{B-1d}$$

In 1963 E. Newman and R. Penrose [15] employed an alternative tetrad as a basis for their formalism.

Instead of one time-like leg, and three space-like legs, one could always substitute two <u>null</u> legs, and two space-like legs into the tetrad. If $\{X_i\}$ is an orthonormal tetrad, then let

$$\ell^* = \frac{1}{\sqrt{2}} (X_o + X_1) \ ,$$

$$n^* = \frac{1}{\sqrt{2}} (X_o - X_1) \ ,$$

and (ℓ, n, X_2, X_3) is a tetrad such as has been described. It is called a pseudoorthonormal tetrad. Now, however,

$$g_{\mu\nu}\ell^{\mu}n^{\mu} = -1$$

and $$g_{\mu\nu} = -\ell_{\mu}n_{\nu} - n_{\mu}\ell_{\nu} + X_{\mu}{}^{2}X_{\nu}{}^{2} + X_{\mu}{}^{3}X_{\nu}{}^{3} \ .$$

Newman and Penrose used this alternative tetrad with one difference. They constructed from X_2 and X_3 a complex vector $m = \frac{1}{\sqrt{2}}(X_2 - iX_3)$. The Newman-Penrose tetrad now consisted of $(\ell, n, m, \bar{m})$. The following relations hold for them

$$\ell^{\mu}\ell_{\mu} = \ell^{\mu}\bar{m}_{\mu} = \ell^{\mu}\bar{m}_{\mu} = n^{\mu}m_{\mu} = n^{\mu}\bar{m}_{\mu} = n^{\mu}n_{\mu}$$

$$= m^{\mu}m_{\mu} = \bar{m}^{\mu}\bar{m}_{\mu} = 0 \qquad \text{(B-2)}$$

$$\ell^{\mu}n_{\mu} = -\, m^{\mu}\bar{m}_{\mu} = -1$$

and

$$g_{\mu\nu} = -\,\ell_{\mu}n_{\nu} - n_{\mu}\ell_{\nu} + m_{\mu}\bar{m}_{\nu} + \bar{m}_{\mu}m_{\mu} \ .$$

If one allows

$$k_o = \ell \ ; \quad k_1 = n \ ; \quad k_2 = m \ ; \quad k_3 = \bar{m} \ ,$$

then (B-2) can be expressed in a way that is parallel to (B-1),

$$g_{\mu\nu}k_i{}^{\mu}k_j{}^{\nu} = \eta_{ij} \qquad \text{(B-3a)}$$

$$g_{\mu\nu} = \eta_{ij}k^{i}k^{j} \qquad \text{(B-3b)}$$

where $$k^{i}{}_{\mu} = \eta^{ij}g_{\mu\nu}k_j{}^{\nu} \qquad \text{(B-3c)}$$

and where $$\eta_{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ . \qquad \text{(B-3d)}$$

As previously

$$ds^2 = \eta_{ij}\omega^i\omega^j \quad ,$$

where the one forms are those of the orthonormal tetrad, and η_{ij} is defined by (B-1d), so

$$ds^2 = \eta'_{ij}\omega^{i'}\,\omega^{j'} \qquad \text{(B-4a)}$$

where

$$\omega^{i'} = k_{\mu}{}^{\dot{k}}\,dx^{\mu} \qquad \text{(B-4b)}$$

and η'_{ij} is defined by (B-3d).

These vectors and one forms are defined by their orthogonality conditions up to the transformations of the six parameter Lorentz group $L_+^{\uparrow}$. These can be written

$$\tilde{\ell}^{\mu} = \ell^{\mu} \qquad \text{(B-5a)}$$

$$\tilde{m}^{\mu} = m^{\mu} + a\ell^{\mu}$$

$$\tilde{n}^{\mu} = n^{\mu} + a\tilde{m}^{\mu} + \bar{a}m^{\mu} + a\bar{a}\ell^{\mu}$$

$$\tilde{\ell}^{\mu} = \lambda\ell^{\mu} \qquad \text{(B-5b)}$$

$$\tilde{n}^{\mu} = \lambda^{-1}n^{\mu}$$

$$\tilde{m}^{\mu} = e^{i\varphi}m^{\mu}$$

$$\tilde{\ell}^{\mu} = \ell^{\mu} + b\bar{m}^{\mu} + \bar{b}m^{\mu}$$

$$\tilde{m}^{\mu} = m^{\mu} + bn^{\mu} \qquad \text{(B-5c)}$$

$$n^{\mu} = n^{\mu}$$

where a and b are complex numbers, and λ and φ are real. (B-5a) and (c) are the two parameter null rotations, (B-5b) the ordinary Lorentz transformation in the ℓ-n plane, and spatial rotations in the m-$\bar{m}$ plane.

B. The Spin Coefficients

Using the definition of (B-4d), the structure equations (A-13a) can be written

$$\begin{aligned} d\omega^i &= -\,\omega^i{}_j \;\; \omega^j \\ &= -\,\gamma^i{}_{kj}\; \omega^k \;\; \omega^j \quad . \end{aligned} \tag{B-6}$$

The $\gamma_{ikj} = \eta_{i\ell}\gamma^{\ell}{}_{kj}$ are called the spin coefficients, and answer to the Γ^k_{kj} of the tetrad formalism. [When Eq. (A-5) is applied to the pseudoorthonormal tetrad, the spin coefficients can be calculated.] In the following table, twelve independent spin coefficients (linear combination of those defined above) are given.

$$\begin{aligned}
\kappa &= \gamma_{020} = \ell_{\mu;\nu} m^{\mu}\ell^{\nu} \;; \quad \pi = -\gamma_{130} = -n_{\mu;\nu}\bar{m}^{\mu}\ell^{\nu} \\
\varepsilon &= \tfrac{1}{2}(\gamma_{010}-\gamma_{230}) = \tfrac{1}{2}(\ell_{\mu;\nu}n^{\mu}\ell^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}\ell^{\nu}) \\
\rho &= \gamma_{023} = \ell_{\mu;\nu} m^{\mu}\bar{m}^{\nu} \;; \quad \lambda = -\gamma_{133} = -n_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu} \\
\alpha &= \tfrac{1}{2}(\gamma_{013}-\gamma_{233}) = \tfrac{1}{2}(\ell_{\mu;\nu}n^{\mu}\bar{m}^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu}) \\
\beta &= \tfrac{1}{2}(\gamma_{012}-\gamma_{232}) = \tfrac{1}{2}(\ell_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}m^{\nu}) \\
\sigma &= \gamma_{022} = \ell_{\mu;\nu} m^{\mu}m^{\nu} \;; \quad \mu = -\gamma_{132} = -n_{\mu;\nu}\bar{m}^{\mu}m^{\nu} \\
\nu &= -\gamma_{131} = -n_{\mu;\nu}\bar{m}^{\mu}n^{\nu} \;; \quad \tau = \gamma_{021} = \ell_{\mu;\nu}m^{\mu}n^{\nu} \\
\gamma &= \tfrac{1}{2}(\gamma_{011} - \gamma_{231}) = \tfrac{1}{2}(\ell_{\mu;\nu}n^{\mu}n^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}m^{\nu})
\end{aligned} \tag{B-7}$$

As before, physical significances can be given to these quantities. From the expressions for the spin coefficients above,

$$\ell_{\mu;\nu}\ell^{\nu} = \kappa\bar{m}_{\mu} + \bar{\kappa}m_{\mu} - (\epsilon+\bar{\epsilon})\,\ell_{\mu}\ , \qquad \text{(B-8a)}$$

$$\ell_{\mu;\nu}n^{\nu} = -(\gamma+\bar{\gamma})\,\ell_{\mu} + \tau\bar{m}_{\mu} + \bar{\tau}m_{\mu}\ , \qquad \text{(B-8b)}$$

$$\ell_{\mu;\nu}m^{\nu} = \bar{\rho}m_{\mu} - (\bar{\alpha}+\beta)\,\ell_{\mu} + \sigma\bar{m}_{\mu}\ , \qquad \text{(B-8c)}$$

$$\ell_{\mu;\nu}\bar{m}^{\nu} = \rho\bar{m}_{\mu} - (\alpha+\bar{\beta})\ \ _{\mu} + \bar{\sigma}m_{\mu}\ . \qquad \text{(B-8d)}$$

From (B-8a) $\kappa = 0$ is the condition that the null path tangent to ℓ be a geodesic. If $\kappa = 0$, then a function φ can be chosen so that $\tilde{\ell}_{\mu} = \varphi\ell_{\mu}$, and $\tilde{\epsilon}+\tilde{\bar{\epsilon}} = 0$.

From (B-8), one can write down the expression for $\ell_{\mu;\nu}$,

$$\ell_{\mu;\nu} = -[\kappa\bar{m}_{\mu}+\bar{\kappa}m_{\mu}-(\epsilon+\bar{\epsilon})\,\ell_{\mu}]n_{\nu} - [-(\gamma+\bar{\gamma})\,\ell_{\mu}+\tau\bar{m}_{\mu}+\bar{\tau}m_{\mu}]\,\ell_{\nu}$$
$$+ [\bar{\rho}m_{\mu}-(\bar{\alpha}+\beta)\,\ell_{\mu} + \sigma\bar{m}_{\mu}]\bar{m}_{\mu} + [\rho\bar{m}_{\mu}-(\alpha+\bar{\beta})\,\ell_{\mu}+\bar{\sigma}m_{\mu}]m_{\nu}\ . \qquad \text{(B-9)}$$

Thus

$$\ell^{\mu}{}_{;\mu} = g^{\mu\nu}\ell_{\mu;\nu} = -(\epsilon+\bar{\epsilon}) + (\rho+\bar{\rho}) \qquad \text{(B-10)}$$

is the measure of the rate of convergence or divergence of the null paths tangent to ℓ [28]. When $\kappa = 0$, $\epsilon+\bar{\epsilon} = 0$, $\ell^{\mu}{}_{;\mu} = (\rho+\bar{\rho})$.

One can project $\ell_{\mu;\nu}$ onto a hypersurface orthogonal to ℓ and n by means of a projection operator $h_{\mu\nu} = (g_{\mu\nu} + 2\ell_{(\mu}n_{\nu)}) = (m_{\mu}\bar{m}_{\nu} + m_{\nu}\bar{m}_{\mu})$. Allow

$$\Theta_{\mu\nu} = h_{\mu}{}^{\rho}h_{\nu}{}^{\sigma}\ell_{\rho;\sigma} - \frac{h_{\mu\nu}}{2}\,(\ell^{\mu}{}_{;\mu})$$
$$= (m_{\mu}\bar{m}^{\rho} + \bar{m}_{\mu}m^{\rho})\,(m_{\nu}\bar{m}^{\sigma} + \bar{m}_{\nu}m^{\sigma})\,\ell_{\rho;\sigma}$$
$$- \frac{m_{\mu}\bar{m}_{\nu} + \bar{m}_{\nu}m_{\mu})\,(\rho+\bar{\rho})}{2}$$
$$= \bar{\sigma}m_{\mu}m_{\nu} + \sigma\bar{m}_{\mu}\bar{m}_{\nu} + (\rho-\bar{\rho})\bar{m}_{[\mu}m_{\nu]}\ .$$

$$\Theta_{[\mu\nu]} = (\rho-\bar{\rho})m_{[\mu}m_{\nu]} = \Omega_{\mu\nu} \quad \text{(B-11a)}$$

$$\Theta_{(\mu\nu)} = \bar{\sigma}m_\mu m_\nu + \sigma\bar{m}_\mu\bar{m}_\nu = \Sigma_{\mu\nu} \quad \text{(B-11b)}$$

The vorticity of the null geodesics tangent to ℓ is given by $\Omega_{\mu\nu}$, and the shear by $\Sigma_{\mu\nu}$. From this,

$$\Omega_{\mu\nu}\Omega^{\mu\nu} = (\rho-\bar{\rho})^2 = \ell_{[\mu;\nu]}\,\ell^{[\mu;\nu]} = \ell_{[\mu;\nu]}\,\ell^{\mu;\nu} \quad \text{(B-12a)}$$

$$\Sigma_{\mu\nu}\Sigma^{\mu\nu} = 2\sigma\bar{\sigma} = \ell_{(\mu;\nu)}\,\ell^{\mu;\nu} - \tfrac{1}{2}(\ell^{\mu}{}_{;\mu})^2 \quad \text{(B-12b)}$$

Thus, the expansion and vorticity of the null congruence tangent to ℓ^μ is given by the real and imaginary parts of ρ, while the shear is given by σ.

Another parameter that is associated with this null congruence is τ, which, as can be seen from (B8-b), is the measure of how ℓ_μ change as it is transported along n_μ. These five spin coefficients, κ, $\epsilon+\bar{\epsilon}$, ρ, σ, τ give a complete description of its properties.

These same calculations could be done for n^μ, and the coefficients, ν, $\gamma+\bar{\gamma}$, μ, λ and π would be found to do the same for the null congruence tangent to it.

C. The Weyl Tensor and the Petrov-Pirani Classification of Space Times

The Weyl tensor is defined in terms of the curvature tensor, $R_{\mu\nu\rho\sigma}$, the Ricci tensor, and the scalar curvature R

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + g_{\mu[\sigma}R_{\rho]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{R}{3}\, g_{\mu[\rho}g_{\sigma]}{}^{\nu} \; . \tag{B-13}$$

Like the curvature tensor

$$C_{[\mu\nu][\rho\sigma]} = C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu} \; , \tag{B-14a}$$

$$C_{\mu[\nu\rho\sigma]} = 0 \quad , \tag{B-14b}$$

$$C_{\mu\nu[\rho\sigma;\tau]} = 0 \; , \tag{B-14c}$$

and in addition it is trace free,

$$C^{\mu}{}_{\nu\mu\sigma} = 0 \; . \tag{B-14d}$$

In general relativity the effects of matter are completely accounted for by the Ricci tensor and its trace R. The Weyl tensor is that part of the curvature tensor that is not determined locally by the matter distribution. It governs the way in which the gravitational field propagates [14]. To see this, contract the equation

$$R_{\mu\nu[\rho\sigma;\tau]} = R_{\mu\nu\rho\sigma;\tau} + R_{\mu\nu\sigma\tau;\rho} + R_{\mu\nu\tau\rho;\sigma} = 0$$

from which

$$R_{\nu\sigma;\tau} - R_{\nu\tau;\sigma} + R^{\rho}{}_{\nu\sigma\tau;\rho} = 0 \; .$$

From (B-13), one has

$$C^{\rho}{}_{\nu\sigma\tau;\rho} = R_{\nu[\tau;\sigma]} - \frac{1}{6}\, g_{\nu[\tau}R_{;\sigma]} \tag{B-15}$$

or

$$C^{\tau\sigma\nu\rho}{}_{;\rho} = R^{\nu[\tau;\sigma]} + \frac{1}{6}\, g^{\nu[\sigma}R^{;\tau]}$$

If this is written,

$$C^{\tau\sigma\nu\rho}{}_{;\rho} = J^{\tau\sigma\nu}$$

where $J^{\tau\sigma\nu} = R^{\nu[\tau;\sigma]} + \frac{1}{6} g^{\nu[\sigma}R^{;\tau]}$, then it can be compared with Maxwell's equation

$$F^{\mu\nu}{}_{;\nu} = J^{\mu} .$$

Thus $J^{\tau\sigma\nu}$ is a kind of current vector, serving as a source for $C^{\tau\sigma\nu\rho}$, and Eq. (B-15) is a field equation for the propagation of this tensor field.

One may press the analogy between $C_{\mu\nu\rho\sigma}$ and $F_{\mu\nu}$ further. It is well known that $F_{\mu\nu}$ can be classified as to its eigenvectors. In the general case, there are two distinct eigenvectors. In the special case, where $E^2-H^2 = E\cdot B = 0$, then there is only one eigenvector whose eigenvalue is zero (i.e. if k^a is the eigenvector, then $F_{ab}k^a = 0$). This is the null field, as distinct from the non-null field. It corresponds to a pure radiation field, whereas the non-null field has sources present.

This same kind of analysis has proved valuable in gravitational theory. The Weyl tensor can also be classified by its eigenvectors.

With respect to an orthonormal tetrad, the Weyl tensor has the following symmetries

$$C^{k}{}_{jk\ell} = \eta^{ik}C_{ijk\ell} = 0 , \qquad \text{(B-16a)}$$

$$C_{ijk\ell} = -C_{jik\ell} = -C_{ij\ell k} = C_{ji\ell k} = C_{k\ell ij} , \qquad \text{(B-16b)}$$

and

$$C_{ijk\ell} + C_{ik\ell j} + C_{i\ell jk} = 0 \; . \qquad \text{(B-16c)}$$

(B-16a) gives the following equalities between the components of the tensor,

$$\begin{array}{ll} C_{0101} = -C_{2323} & C_{1020} = C_{1323} \\ C_{0202} = -C_{1313} & C_{1030} = C_{1232} \\ C_{0303} = -C_{1212} & C_{0323} = -C_{0121} \\ C_{1202} = -C_{1303} & C_{0232} = -C_{0131} \end{array} \qquad \text{(B-17a)}$$

$$C_{0203} = C_{1213} \; ,$$

while (B-16c) gives the relation

$$C_{0123} + C_{0231} + C_{0312} = 0 \; . \qquad \text{(B-17b)}$$

$C_{ijk\ell}$, as can be seen from (B-16b) is antisymmetric in the first and last two indices, and symmetric under the exchange of these pairs. If one labels each pair of indices by an integer,

$$(23) = 1, \; (31) = 2, \; (12) = 3, \; (10) = 4, \; (20) = 5,$$
$$(30) = 6,$$

then $C_{ijk\ell}$ can be written as a symmetric, 6 x 6 tensor, [17] C_{MN}, where A and B are now the labels given above for the pairs. If

$$C_{MN} = \begin{pmatrix} B & A \\ C & E \end{pmatrix} \; ,$$

where A, B, C, E are 3 x 3 tensors, the symmetry of C_{MN} in M and N implies that $C = A^T$. Also, from the relations in

(B-17a) it is clear that $A = A^T$ and $B = B^T$. Further, $C^i_{oio} = 0$ shows that B is trace free, and (B-17b) shows that A is also. Finally, the relations (B-17a) show that $E = -B$. Thus

$$C_{MN} = \begin{pmatrix} B & A \\ A & -B \end{pmatrix} .$$

If another tensor D is defined

$$D \equiv A + iB \tag{B-18}$$

or

$$D = \begin{pmatrix} (C_{0101}+iC_{0123}) & (C_{0102}+iC_{0131}) & (C_{0103}+iC_{0112}) \\ \text{sym.} & (C_{0202}+iC_{0231}) & (C_{0203}+iC_{0212}) \\ & & (C_{0303}+iC_{0312}) \end{pmatrix}$$

then this tensor can be brought into canonical form by means of the three dimensional complex rotation group (which is isomorphic to the Lorentz group).

Three cases can now be distinguished. This tensor can have three, two or one independent eigenvectors, corresponding to the case where it has two non-zero eigenvalues, one non-zero eigenvalue, on no non-zero eigenvalues. These correspond to the Petrov-Pirani classifications I, II and III respectively. In each case the canonical form for D is

$$D_I = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & (-\lambda_1-\lambda_2) \end{pmatrix} \tag{B-19a}$$

$$D_{II} = \begin{pmatrix} \lambda - i\mu & \mu & 0 \\ \mu & \lambda + i\mu & 0 \\ 0 & 0 & -2\lambda \end{pmatrix} \tag{B-19b}$$

$$D_{III} = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & i\mu \\ \mu & i\mu & 0 \end{pmatrix} \tag{B-19c}$$

where λ is complex and μ real.

If $\lambda_1 = \lambda_2$ in (B-19a), and $\mu = 0$ in (B=19b), then $D_I = D_{II}$. This is the Petrov type I D (or simply type D), and both the Schwarzschild and Kerr metrics belong to this classification. If $\lambda = 0$ in (B-19b), then

$$D_{II} = \begin{pmatrix} -i\mu & \mu & 0 \\ \mu & i\mu & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and this is the Petrov type N, a pure radiation field. The relations between these are to be seen in the Penrose diagram

$$\begin{tikzcd} & & \mathrm{I} \arrow[dl] \arrow[d] \\ & \mathrm{II} \arrow[r] \arrow[dl] \arrow[d] & \mathrm{D} \arrow[d] \\ \mathrm{III} \arrow[r] & \mathrm{N} \arrow[r] & 0 \end{tikzcd}$$

where 0 is the type when all of the components of the Weyl tensor are zero (as in the de Sitter universe), and where the arrows point in the direction of increasing algebraic specialization.

When the Weyl tensor is expressed with respect to the Newman-Penrose pseudoorthonormal tetrad, the independent components are

$$
\begin{aligned}
\psi_o &= -C_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta \\
\psi_1 &= -C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta \ell^\gamma m^\delta \\
\psi_2 &= -\tfrac{1}{2}C_{\alpha\beta\gamma\delta}(\ell^\alpha n^\beta \ell^\gamma n^\delta - \ell^\alpha n^\beta m^\gamma \bar{m}^\delta) \\
\psi_3 &= C_{\alpha\beta\gamma\delta}\ell^\alpha n^\beta n^\gamma \bar{m}^\delta \\
\psi_4 &= -C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta
\end{aligned}
\qquad \text{(B-20)}
$$

and where $C_{\alpha\beta\gamma\delta}$ is the coordinate representation of this tensor. The real and imaginary parts of these five complex quantities give the ten independent real components of the Weyl tensor.

The Newman-Penrose form of this tensor is related closely to the components of the complex tensor D on which the Petrov-Pirani classification tensor is based. In Appendix C, the ψ_i's are given directly in terms of the orthonormal tetrad components of the Weyl tensor. Comparing these to (B-18), they can be written in terms of the components of D,

$$
\begin{aligned}
\psi_o &= -\tfrac{1}{2}(\bar{D}_{22}-\bar{D}_{33}+2i\bar{D}_{23}) \\
\psi_1 &= \tfrac{1}{2}(\bar{D}_{12}+i\bar{D}_{13}) \\
\psi_2 &= -\tfrac{1}{2}\, D_{11} \\
\psi_3 &= \tfrac{1}{2}(\bar{D}_{12}-i\bar{D}_{13}) \\
\psi_4 &= -\tfrac{1}{2}(\bar{D}_{22}-\bar{D}_{38}-2i\bar{D}_{23}) \ .
\end{aligned}
$$

Suppose that, of these, only $\psi_2 \neq 0$.

$$\psi_o = \psi_4 = 0 \qquad D_{22} = D_{33} \text{ and } D_{23} = 0$$

$$\psi_1 = \psi_2 = 0 \qquad D_{12} = D_{13} = 0$$

This means that

$$D = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$$

and the space time under consideration is type D. If only $\psi_o \neq 0$, then

$$D_{12} = D_{13} = \theta = D_{11} \quad ,$$

and $\psi_4 = 0$

$$A_{33} = -B_{23} = -A_{22} = -\mu$$

and

$$B_{33} = A_{23} = -B_{22} = \sigma \ .$$

The tensor D is now in the form,

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu - i\sigma & -i(\mu+i\sigma) \\ 0 & -i(\mu+i\sigma) & -(\mu-i\sigma) \end{pmatrix}$$

A null rotation will now make $\mu = 0$, and D is that of a type N field. The same thing can be done if only $\psi_4 \neq 0$, with the same results except for a change of sign on the diagonal, i.e. a permutation of rows 2 and 3. This means, as it turns out, that ℓ^μ is the propagation vector, rather than n^μ. If only $\psi_1 \neq 0$, then

$$D_{11} = 0 \ , \ D_{22} = D_{22} = 0 \quad , \quad D_{23} = 0$$

$$\bar{D}_{12} = i\bar{D}_{13}$$

$$D = \begin{pmatrix} 0 & \mu & -i\mu \\ \mu & 0 & 0 \\ -i\mu & 0 & 0 \end{pmatrix}$$

This is obviously type III. Again, the same will be true if only $\psi_3 \neq 0$ except that ℓ^μ rather than n^μ will be the propagation vector.

Again, suppose that $\psi_o = -\psi_4$, and $\psi_1 = \psi_3 = 0$. Then

$$D = \begin{pmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \mu & \lambda \end{pmatrix}$$

and this is type II.

Appendix C

Equalities Between the Spin Coefficient of the Newman-Penrose Formalism and the Connection of the Orthonormal Tetrad

First of all, the components of the spin coefficients in terms of the components of the connection of the orthonormal tetrad are:

$$\kappa = -\frac{1}{2\sqrt{2}}\left\{\Gamma^0_{02}-\Gamma^1_{12} + \Gamma^0_{12}-\Gamma^1_{02} + i(\Gamma^0_{03}-\Gamma^1_{13}+\Gamma^0_{13}-\Gamma^1_{03})\right\}$$

$$\epsilon+\bar{\epsilon} = \frac{1}{\sqrt{2}}\left\{\Gamma^0_{01} + \Gamma^1_{10}\right\}$$

$$\epsilon-\bar{\epsilon} = \frac{i}{\sqrt{2}}\ (\Gamma^2_{03} + \Gamma^2_{13})$$

$$\rho = -\frac{1}{2\sqrt{2}}\ (\Gamma^i_{i0}+\Gamma^i_{i1} + i(C^0_{23} + C^1_{23})) - (\epsilon+\bar{\epsilon})$$

$$\sigma = -\frac{1}{2\sqrt{2}}\left[(\Gamma^0_{22}-\Gamma^0_{33}) - (\Gamma^1_{22}-\Gamma^1_{33}) + i\left\{(\Gamma^0_{23}+\Gamma^0_{32}) - (\Gamma^1_{23}+\Gamma^1_{32})\right\}\right]$$

$$\tau = -\frac{1}{2\sqrt{2}}\left\{\Gamma^i_{i2}+i\Gamma^i_{i3} - C^2_{10}-iC^3_{10}\right\} - \tfrac{1}{2}(\bar{\alpha}-\beta)$$

$$\nu = \frac{1}{2\sqrt{2}}\ (\Gamma^0_{02}-\Gamma^1_{12}+\Gamma^1_{02}-\Gamma^0_{12}-i[\Gamma^0_{03}-\Gamma^1_{13}+\Gamma^1_{03}-\Gamma^0_{13}])$$

$$\gamma+\bar{\gamma} = \frac{1}{\sqrt{2}}\ (\Gamma^0_{01} - \Gamma^1_{10})$$

$$\gamma-\bar{\gamma} = \frac{i}{\sqrt{2}}\ (\Gamma^2_{03} - \Gamma^2_{13})$$

$$\mu = \frac{1}{2\sqrt{2}}\ (\Gamma^i_{i0} - \Gamma^i_{i1} - i(C^0_{23}+C^1_{23})) + (\gamma+\bar{\gamma})$$

$$\lambda = \frac{1}{2\sqrt{2}}\ (\Gamma^0_{22}-\Gamma^0_{33}+\Gamma^1_{22}-\Gamma^1_{33}-i[\Gamma^0_{23}+\Gamma^0_{32}+\Gamma^1_{23}+\Gamma^1_{32}])$$

$$\pi = \frac{1}{2\sqrt{2}}\,(\Gamma^{i}_{i2} - i\Gamma^{i}_{i3} + C^{2}_{10} - iC^{3}_{10}) + (\alpha-\bar{\beta})$$

$$(\alpha+\bar{\beta}) = \frac{1}{\sqrt{2}}\,(\Gamma^{0}_{21} + i\Gamma^{0}_{31})$$

$$(\alpha-\bar{\beta}) = \frac{1}{\sqrt{2}}\,(i\Gamma^{2}_{23} - \Gamma^{3}_{32})$$

The components of the complex Weyl tensor are related to the components of that tensor with respect to the orthonormal tetrad in the following way:

$$\psi_o = -\tfrac{1}{2}(C_{0202}-C_{0303}+2C_{0212}+i[2C_{0203}+C_{0213}+C_{0312}])$$

$$\psi_1 = \tfrac{1}{2}(C_{0102}-C_{1012}+i(C_{0103}-C_{1013}))$$

$$\psi_2 = -\tfrac{1}{2}(C_{0101}+iC_{0123})$$

$$\psi_3 = \tfrac{1}{2}(C_{0102}+C_{1012}-i(C_{0103}+C_{1013}))$$

$$\psi_4 = -\tfrac{1}{2}(C_{0202}-C_{0303}-2C_{0212}-i(2C_{0203}-C_{0213}-C_{0312}))$$

Now, the components of the connection wrt of the orthonormal tetrad are given in terms of the spin coefficients:

$$\Gamma^{0}_{01} = \frac{1}{\sqrt{2}}\,((\varepsilon+\bar{\varepsilon}) + (\gamma+\bar{\gamma}))$$

$$\Gamma^{0}_{02} = -\frac{1}{2\sqrt{2}}\,(\kappa+\bar{\kappa} + \tau+\bar{\tau} \div \pi-\bar{\pi} - \nu-\bar{\nu})$$

$$\Gamma^{0}_{03} = -\frac{i}{2\sqrt{2}}\,(\kappa-\bar{\kappa} + \tau-\bar{\tau} + \pi-\bar{\pi} + \nu-\bar{\nu})$$

$$\Gamma^{0}_{12} = -\frac{1}{2\sqrt{2}}\,(\kappa+\bar{\kappa} - \tau-\bar{\tau} - \pi-\pi - \nu-\bar{\nu})$$

$$\Gamma^{0}_{13} = -\frac{i}{2\sqrt{2}}\,(\kappa-\bar{\kappa} --\tau\mp\bar{\tau} - \bar{\pi}+\pi - \bar{\nu}+\nu)$$

$$\Gamma^0_{23} = \frac{i}{2\sqrt{2}} (\sigma-\bar{\sigma} + \rho-\bar{\rho} + \lambda-\bar{\lambda} + \mu-\bar{\mu})$$

$$\Gamma^0_{32} = \frac{i}{2\sqrt{2}} (\sigma-\bar{\sigma} - \rho+\bar{\rho} + \lambda-\bar{\lambda} - \mu+\bar{\mu})$$

$$\Gamma^0_{21} = \frac{1}{\sqrt{2}} (\alpha+\bar{\beta}) + (\bar{\alpha}+\beta)$$

$$\Gamma^1_{10} = \frac{1}{\sqrt{2}} \left\{(\epsilon+\bar{\epsilon}) - (\gamma+\bar{\gamma})\right\}$$

$$\Gamma^1_{12} = \frac{1}{2\sqrt{2}} (\kappa+\bar{\kappa} - \tau-\bar{\tau} + \pi+\bar{\pi} - \nu-\bar{\nu})$$

$$\Gamma^1_{13} = -\frac{i}{2\sqrt{2}}(\kappa-\bar{\kappa} - \tau+\bar{\tau} + \pi-\bar{\pi} + \nu-\bar{\nu})$$

$$\Gamma^1_{02} = \frac{1}{2\sqrt{2}} (\kappa+\bar{\kappa} + \tau+\bar{\tau} + \pi+\bar{\pi} + \nu+\bar{\nu})$$

$$\Gamma^1_{03} = \frac{i}{2\sqrt{2}} (\kappa-\bar{\kappa} + \tau-\bar{\tau} - \pi+\bar{\pi} - \nu-\bar{\nu})$$

$$\Gamma^1_{23} = -\frac{i}{2\sqrt{2}} (\sigma-\bar{\sigma} + \rho-\bar{\rho} - \lambda+\bar{\lambda} - \mu+\bar{\mu})$$

$$\Gamma^1_{32} = -\frac{i}{2\sqrt{2}} (\sigma-\bar{\sigma} - \rho+\bar{\rho} - \lambda+\bar{\lambda} + \mu-\bar{\mu})$$

$$\Gamma^2_{20} = -\frac{1}{2\sqrt{2}} (\sigma+\bar{\sigma} + \rho+\bar{\rho} - \mu-\bar{\mu} - \lambda-\bar{\lambda})$$

$$\Gamma^2_{21} = -\frac{1}{2\sqrt{2}} (\sigma+\bar{\sigma} + \rho+\bar{\rho} + \lambda+\bar{\lambda} + \mu+\bar{\mu})$$

$$\Gamma^2_{23} = \frac{i}{\sqrt{2}} \left\{(\bar{\alpha}-\beta) - (\alpha-\bar{\beta})\right\}$$

$$\Gamma^2_{03} = -\frac{i}{\sqrt{2}} ((\epsilon-\bar{\epsilon}) + (\gamma-\bar{\gamma})$$

$$\Gamma^2_{13} = -\frac{i}{\sqrt{2}} ((\epsilon-\bar{\epsilon}) - (\gamma-\bar{\gamma}))$$

$$\Gamma^3_{30} = \frac{1}{2\sqrt{2}} (\sigma+\bar{\sigma} - \rho-\bar{\rho} - \lambda-\bar{\lambda} + \mu+\bar{\mu})$$

$$\Gamma^3_{31} = \frac{1}{2\sqrt{2}} (\sigma+\bar{\sigma} - \rho-\bar{\rho} + \lambda+\bar{\lambda} - \mu-\bar{\mu})$$

$$\Gamma^3_{32} = - \frac{1}{\sqrt{2}} ((\bar{\alpha}-\beta) + (\alpha-\bar{\beta}))$$

Appendix D

Metrics and Spaces

A. The de Sitter and anti-de Sitter Universes [10,14]

The curvature tensor of a space of constant curvature can be written

$$R_{\mu\nu\rho\sigma} = \frac{R}{12} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$
$$= \frac{R}{6} g_{\mu[\rho}g_{\sigma]\nu} \qquad \text{(D-1)}$$

and

$$R_{\mu\rho} = \frac{R}{4} g_{\mu\rho} \ .$$

For Lorentz signature metrics, there are three unique solutions to D-1, depending on the value of R. If $R > 0$, then this is the de Sitter universe. If $R < 0$, then it is the anti-de Sitter universe. If $R = 0$, it is Minkowski space.

There are two possible physical interpretations of these spaces. If one assumes that the cosmological constant $\Lambda = 0$, then $8\pi T_{\mu\nu} = -\frac{1}{4} Rg_{\mu\nu}$. If this universe is a perfect fluid, then

$$8\pi T_{\mu\nu} = 8\pi[(p+\rho)u_\mu u_\nu + pg_{\mu\nu}] = -\frac{R}{4} g_{\mu\nu}$$

and so

$$p = -\rho = -\frac{R}{32} \ .$$

This negative pressure is not very reasonable physically, and so it is usual to adopt the alternative interpretation that $\Lambda = \frac{R}{4}$, and so

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{R}{4} g_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 .$$

The de Sitter universe is a hyper-hyperboloid embedded in R^5

$$x^2 + y^2 + z^2 + w^2 - u^2 = \alpha^2 ,$$

where $\Lambda = R/4 = 1/\alpha$. The scale of these coordinates can always be chosen so that $\Lambda = 1/\alpha = 1$. By letting

$$u = \sinh t \qquad x = \cosh t \sin\chi \sin\theta \cos\varphi$$

$$w = \cosh + \cos\chi \qquad y = \cosh + \sin\chi \sin\theta \sin\varphi$$

$$z = \cosh + \sin\chi \cos\theta$$

this metric is

$$ds^2 = -dt^2 + \cosh^2 t(d\chi^2 + \sin^2\chi(d\theta^2+\sin^2\theta d\varphi^2)) \qquad \text{(D-1)}$$

These coordinates cover the entire hyper-hyperboloid. These are shown in Fig. D-1, with two angular variables (θ and φ) suppressed.

The t = constant hypersurfaces are three spheres, and the geodesic normals are lines that converge to minimum spacial separation, and then diverge. A more familiar form of the metric for the de Sitter space is

$$ds^2 = -d\hat{t}^2 + e^{2\hat{t}}(d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2) .$$

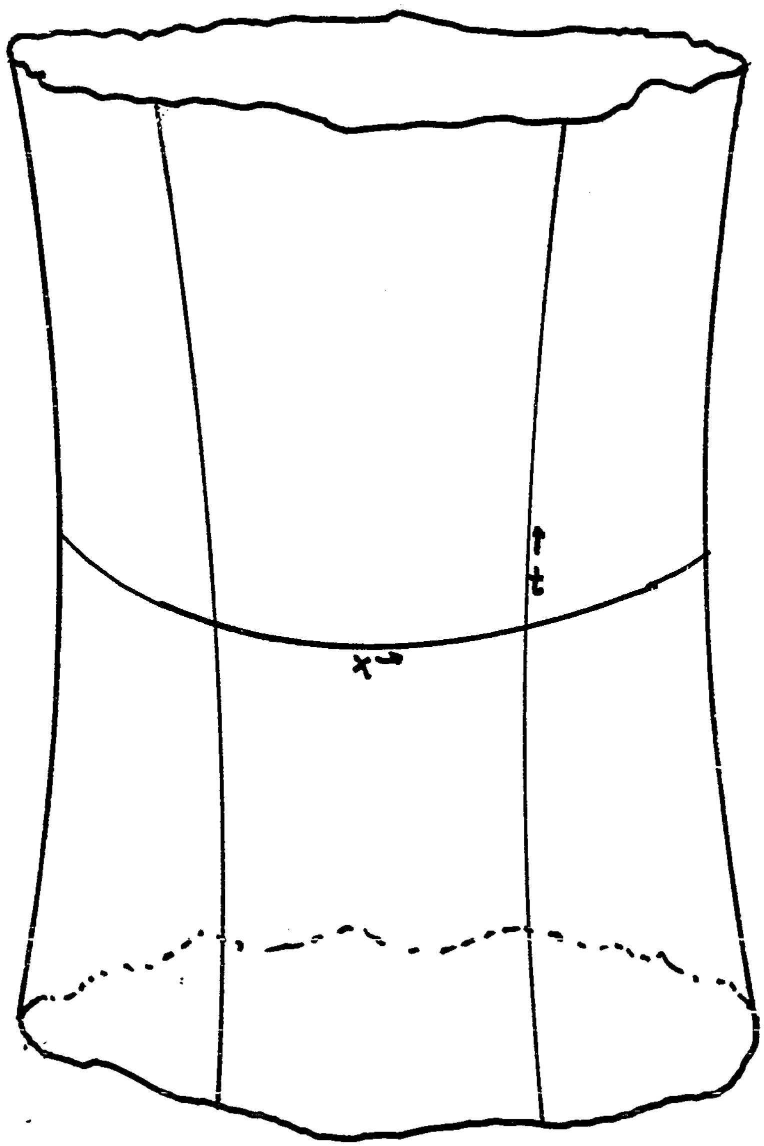


Figure D-1

This metric can be obtained from the metric for R^5 on the hyper-hyperboloid if

$$\hat{t} = \ell n(w+u) \quad , \quad \hat{x} = \frac{x}{w+u} \quad , \quad \hat{y} = \frac{y}{w+u} \quad , \quad \hat{z} = \frac{z}{w+u} \; .$$

It covers only the coordinate patch such that $w+u > 0$.

This piece of de Sitter space is the <u>steady state</u> universe, and represents the limit of the Friedmann universe as the matter density $\rho \to 0$.

The antide Sitter universe is the space of constant curvature such that $R < 0$. It is also an hyper-hyperboloid in R^5

$$-x^2 - y^2 - z^2 + u^2 + w^2 = 1 ,$$

as shown in Fig. D-2.

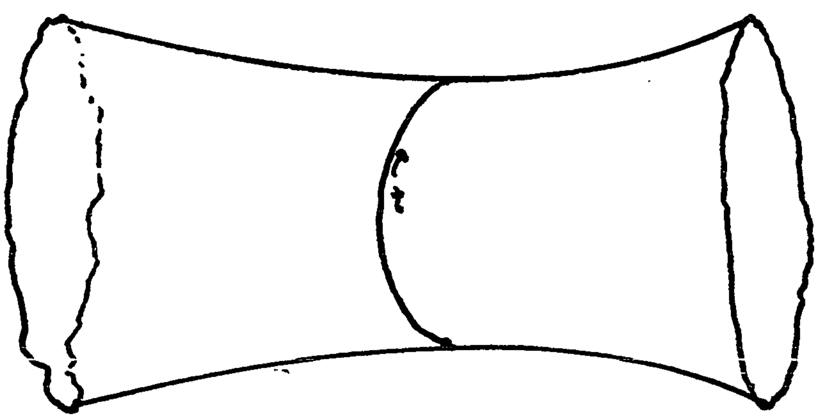


Figure D-2

If $\quad x = \sin t \cosh \chi \sin\theta \cos\varphi \qquad u = \sin t \sinh \chi$

$\quad y = \sin t \cosh \chi \sin\theta \sin\varphi \qquad w = \cos t$

$\quad z = \sin t \cosh \chi \cos\theta$

then the metric for this space is

$$ds^2 = -dt^2 + \cos^2 t(dx^2 + \sinh^2 x(d\theta^2 + \sin^2\theta d\varphi^2)) \ . \qquad \text{(D-2)}$$

The antide Sitter universe contains closed time-like lines, and so violate causality. This violation is not essential, since a covering space exists where this is not so. This can be seen from the static form of the metric

$$ds^2 = -\cosh^2 r\, dt^2 + dr^3 + \sinh^2 r(d\theta^2 + \sin^2\theta d\varphi^2) \quad . \qquad \text{(D-3)}$$

The static form covers all of the covering space of the antide Sitter universe, whereas the metric (D-2) covers only part of it.

B. The Einstein Static Universe

This universe represents the first model of the universe, and in order to obtain it, Einstein introduced the cosmological constant Λ into his field equations.

Einstein noted that without the cosmological constant, homogeneous, isotropic solutions to his field equations either expanded or contracted. Since Hubble had not at that time discovered the expansion of the universe, Einstein naturally used this static model. This space is just

$$u^2 + x^2 + y^2 + z^2 = 1$$

on five dimensional Minkowski space, with $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + du^2$. If

$$x = \cos\chi \sin\theta \cos\varphi \qquad z = \cos\chi \cos\theta$$
$$y = \cos\chi \sin\theta \sin\varphi \qquad u = \sin\chi$$

then the metric is

$$ds^2 = -dt^2 + dx^2 + \sin^2 x(d\theta^2 + \sin^2\theta d\varphi^2) \quad .$$

If, however,

$$u = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}\ , \qquad y = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}\ ,$$

$$x = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}\ , \qquad z = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}\ ,$$

and if

$$\sigma_x = 2(xdu - udx - zdy + ydz) = \sin\psi\, d\theta - \cos\psi \sin\theta d\varphi$$

$$\sigma_y = 2(ydu + zdx - udy - xdz) = \cos\psi\, d\theta + \sin\psi\, \sin\theta d\varphi$$

$$\sigma_z = 2(zdu - ydx + xdy - udz) = -(d\psi + \cos\theta d\varphi)\ ,$$

then this metric is written

$$ds^2 = -dt^2 + \tfrac{1}{4}(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2)\ .$$

The coordinates here are the Euler angles, and the σ_i's are the left invariant one forms of $S^3(R)$.

When Einstein's field equations are computed for this universe, $\Lambda = 1$, and the matter density, $\rho = \frac{1}{4\pi}$.

C. Gödel Universe [11]

The line element of the Gödel universe can be written

$$ds^2 = \tfrac{1}{2}\left\{-(dx_o + e^{x_1}dx_2)^2 + dx_1{}^2 + \frac{e^{2x_1}}{2}dx_2{}^2 + dx_3{}^2\right\} \qquad \text{(D-4a)}$$

or

$$ds^2 = -(dt - \sqrt{2}\sinh^2 r d\varphi)^2 + dr^2 + \sinh^2 r \cosh^2 r d\varphi^2 + dz^2\ . \qquad \text{(D-4b)}$$

As can be seen from Chapter III, the Gödel universe is akin to the Einstein static universe. It has $\Lambda = -1$, and $\rho = \frac{1}{4\pi}$.

The Gödel universe has two interesting properties. First of all, the fluid is rotational. The Gödel universe is the first such cosmological solution of this kind to be discovered. The vector $u^{\mu} = (1,0,0,0)$ is tangent to the streamlines of the matter in this universe. With this velocity field,

$$\omega_{\mu\nu} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{x^1} & 0 \\ 0 & e^{x^1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \omega^j = \frac{\varepsilon^{jik\ell}}{6\sqrt{-g}} u_i \omega_{k\ell}$$

is the angular velocity of distant matter. From the expression for $\omega_{\mu\nu}$, $\omega^3 = 1/\sqrt{2}$ is the only non-vanishing component of this vector.

The fact that the distant matter in this universe rotates relative to an inertial frame means that this universe does not obey Mach's principle.

The other interesting property in this universe is that it violates causality. To see this note that the circle $t = \text{const}$, $z = \text{const}$,

$$\begin{aligned} ds^2 &= (-2 \sinh^4 r + \sinh^2 r \cosh^2 r) d\varphi^2 \\ &= \sinh^2 r (1 - \sinh^2 r) d\varphi^2 . \end{aligned}$$

Then $ds^2 \le 0$ on this circle if $r \ge \sinh^{-1} 1$, and the circle is time-like.

D. The Kantowski-Sachs Universes

R. Kantowski and R. K. Sachs [19] set out to discover certain space times with specified symmetries. The two universes that were discovered have the form

$$ds^2 = -dt^2 + X^2(t)dr^2 + Y^2(t)d\Omega^2 .$$

For the Kantowski-Sachs closed universe

$$\begin{aligned} X &= \epsilon + (\epsilon\eta + b)\tan \eta \\ Y &= a \cos^2\eta \\ t - t_o &= a(\eta + \tfrac{1}{2} \sin^2\eta) \\ d\Omega^2 &= (d\theta^2 + \sin^2\theta d\varphi^2) \\ \rho &= \epsilon \sec^4\eta/a^2[1+(\eta+b)\tan \eta] \end{aligned}$$

and a, b, ϵ are constants with

$$\epsilon = 0,1, \quad -\infty < a < \infty, \quad a \neq 0$$

$$-\tfrac{1}{2}\pi \leq b < 0 .$$

For the Kantowski-Sachs open universe

$$\begin{aligned} X &= \epsilon - (\epsilon\eta + b)\tanh \eta \\ Y &= a \cosh^2\eta \\ t - t_o &= a(\eta + \tfrac{1}{2} \sinh^2\eta) \\ d\Omega^2 &= (d\theta^2 + \sinh^2\theta d\varphi^2) \\ \rho &= -\epsilon \operatorname{sech}^4\eta/a^2[1-(\eta+b)\tanh \eta] \end{aligned}$$

and ϵ and a are as defined previously and $0 \leq b < 0$.

The closed vacuum Kantowski-Sachs universe (when $\epsilon = 0$) is isomorphic to the Schwarzschild metric within the horizon. This is shown to be true in Chapter V.

The space sections of these universes are homogeneous, but not isotropic.

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16. As can be seen from Appendix D, for the de Sitter Universe,

$$R_{abcd} = \frac{2\Lambda}{3} g_{a[c}g_{b]d} ,$$

$$R_{ab} = \Lambda g_{ab} .$$

Therefore

$$\begin{aligned} C_{abcd} &= R_{abcd}+g_{a[d}R_{c]b}+g_{b[c}R_{d]a}+ \frac{4}{3} \Lambda g_{a[c}g_{d]b} \\ &= \frac{2\Lambda}{3} g_{a[c}g_{b]d} + \Lambda g_{a[d}g_{c]b} + \Lambda g_{a[d}g_{c]b} \\ &\quad + \frac{4}{3} \Lambda g_{a[c}g_{d]b} = 0 \end{aligned}$$

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30. If s is an affine parameter of the congruence tangent to a killing vector field, then a family of hypersurfaces are defined by s = constant. If the killing vector is orthogonal to these surfaces, then it is said to be hypersurface orthogonal.