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Asynchronous Bipolar-Equivalent Optical CDMA

by

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Abstract

We analyze the use of the bipolar-equivalent coding method in a incoherent fiber optic time-encoded CDMA communication system in an attempt to emulate the behavior of a coherent bipolar system. Specifically, we consider two possible asynchronous versions of the system which we refer to as appended and interleaved. We then describe a unified theoretical analysis for these systems based on the average interference parameter and the signal-to-noise ratio. We then consider the performance of the appended and interleaved systems and compare them to the performance of similar bipolar systems using length-127 m-sequences and Gold codes. We find that interleaving performs best of the methods considered—performance is very similar to the bipolar system we were trying to emulate, except for a factor of two increase in bandwidth requirement.

Acknowledgments

I would like to thank my advisor Jim Young, as well as the rest of my thesis committee, for their time and insight. I would also like to thank my fellow graduate students Yile Guo, Tasshi Dennis, Lim Nguyen, and Ediz Demirciler for their help and support. I'm sure to have left people out, so I want to thank them at this time and apologize for the omission.

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Introduction

1.1 Motivation

The increasing demand for capacity in communication networks has led to increasingly complicated protocols for efficiently allocating available bandwidth to a large number of users. Code-division multiple access, or CDMA, is a spread spectrum technique which typically allows a very large number of possible subscribers, but takes advantage of the fact that not all subscribers are transmitting all the time [7]. Standard wireless CDMA systems are based on the use of bipolar codes, which are modulated directly on the bipolar data stream. In asynchronous systems these are pseudo-random codes such as maximum-length sequences or Gold codes which have the property that the correlation between any two codes, and therefore any two users' transmission, is very small.

In a CDMA system, a given subscriber is given a unique signature sequence that modulates that user's data. The user correlates the incoming signal with the same sequence, referred to as the reference code, with a receiver type called a matched filter or correlator (look ahead to Figure 3.3). If the correlation is found to be large, it means the data was modulated with the same sequence. The linearity of the correlation operation means this will work regardless of the number of users. Of course, while the correlation of other users' transmission, called the interference, is small in any single case, it adds up linearly with an increasing number of users and can overwhelm the desired transmission. This is why we refer to this system as interference limited, meaning, our primary concern for system performance is the intra-user interference.

To properly perform in a CDMA system, the codes must be transmitted in a way which retains their bipolar nature. The most common method is binary phase-shift keying, or BPSK, where the carrier is transmitted with one of two phases which differ by 180 degrees.

In an optical CDMA system the frequencies are too high for coherent detection. We must therefore use incoherent binary transmission, where the intensity of the carrier is modulated on & off.

If we tried to blindly apply our bipolar codes to this system by replacing BPSK with on-off keying, we would get a intra-user interference which resembled the case in the bipolar system, but with an additional offset equal to the weight of the code which was proportional to the number of users. Bipolar-equivalent coding is a different approach to applying that same bipolar CDMA system to unipolar systems. The bipolar code sequence is separated into two unipolar sequences which are transmitted separately and recombined at the receiver. Obviously, this requires some way of keeping the two unipolar components separate to work correctly. Bipolar-equivalent coding was originally implemented in a spectrally-encoded system where the code chips were spread over two separate groups of wavelength slots and transmitted in parallel. Everything, therefore, was kept synchronous and this system reproduced the bipolar behavior of the codes.

Our interest here, however, is in time-encoded systems that transmit the code chips in series, as in the wireless system. Such systems can only take full advantage of the bandwidth reuse of CDMA when asynchronous. In an asynchronous system there will, in general, be an unknown delay between any two users' transmission. The output of the correlator will not just be the correlation of the two users' codes, but the shifted correlation. We must therefore be concerned with what the correlation between two users' transmission is for all possible shifts. If, as we would like to do, we multiplex the two unipolar components of the bipolar code together somehow, we must consider the shifted cross-correlation between this multiplexed signal and another user's signal multiplexed similarly. In this thesis we consider multiplexing the unipolar components at the bit level, which we call appending them, and at the chip level, which we call interleaving them. There may, of course, be other ways to do this, and it could be the case that synchronicity at some level is necessary in unipolar systems. In this thesis, however, we find that with the requirements that our system be unipolar, time-encoded, and asynchronous, we can still achieve bipolar performance with only a factor of two increase in bandwidth.

This thesis consists of an analysis of all possible correlations between bipolar-equivalent signals using interleaving and appending in a time-encoded asynchronous system, and therefore, computes the interference between two users in such a bipolar-equivalent system. The key to our analysis of the bipolar-equivalent system is to relate its behavior to true bipolar systems. In such systems, the behavior of the codes is well known. We seek to describe the correlation and therefore interference in the bipolar-equivalent system in terms of the bipolar system's correlation and interference. Doing this allows us to compare our bipolar-equivalent system's performance to the true bipolar system's performance to see if we have succeeded in translating the advantages of CDMA to a unipolar system. We describe a unified method of analysis of the performance of the system using average interference parameters which give the system's signal-to-noise ratio which then can be used to approximate the probability of error of the system. We compare the interleaved and appended asynchronous system to the bipolar system both when fully asynchronous, and when partially synchronized.

1.2 Previous Work

1.2.1 Bipolar Analysis

Our analysis of Chapter 3 relies heavily on the methodology derived by Pursley in [4] and other papers for the bipolar system. Starting from the cross-correlation function, Pursley derives the average interference parameter, or AIP, on the way to an expression for the signal-to-noise ratio, or SNR, which he derives as

$$\mathrm{SNR} = \left(\frac{1}{6L^3} \sum_{\substack{k=1 \\ k \neq i}}^{N} r_{x,y} \right)^{-\frac{1}{2}}, \tag{1.1}$$

The AIPs are, for the so-called worst-case,

$$r_{k,i} = 3\mu_{k,i}(0), \tag{1.2}$$

which refers to the case where different users are synchronized at the chip level (but not necessarily at the bit level), and for the so-called average-case where different users are completely asynchronous,

$$r_{k,i} = 2\mu_{k,i}(0) + \mu_{k,i}(1). \tag{1.3}$$

The term $\mu(n)$ is called a correlation parameter and it can be described as the autocorrelation of the cross-correlation function between pairs of users' codes. $\mu(0)$, then, is essentially the variance of the two-user interference. Since different users' signals are assumed to be uncorrelated, we can sum the variances of the 2-user interferences to get the variance of the total N-user interference, which we use to get the SNR. In this thesis, we will deriver our results in terms of the same parameters to compare the unipolar results to the bipolar results.

Now, we will quickly review the work that has been done in incoherent optical CDMA.

1.2.2 Bipolar-Equivalent Coding

All the explicitly bipolar-equivalent coding work done thus far has been based on the use of spectrally-encoded rather than time-encoded systems. Obviously, there are nontrivial differences between the two methods, particularly when it comes to an asynchronous system, as we will see. The first paper on spectrally-encoded bipolar-equivalent coding is the paper by Nguyen, et al [1], where the very basic idea is presented along with some discussion of the system's statistics. In this system, Nguyen makes use of a rather complex system of gratings and masks to achieve the spreading of the code over the spectrum which, with modern technology, does not appear to be particularly cost-effective.

1.2.3 Related Methods

There have been a few papers published describing methods which take a different approach or outlook, but end up quite similar to the time-encoded bipolar-equivalent systems we describe in this thesis. There are two papers in particular which we will discuss here. The paper by Tancevsky, et al [9], entitled "Incoherent asynchronous optical CDMA" provides a rather simplified analysis of a system where the bipolar Gold code is essentially Manchester coded to form a unipolar code of twice the length. Thus far, the system is identical to the interleaved system we discuss later. The receiver is rather similar, in principle, to the interleaved system as well. Tancevski uses the unipolar Manchester coded sequence as reference in the correlator, where in the interleaved system, the reference is the bipolar version of the same code. We will not go into unipolar reference receivers in this thesis, except to say that they produce an additional scaling factor from the bipolar reference and, when combined with a unipolar transmitted signal, a large offset which must be removed somehow. Tancevski, et al., do this using balanced detection.

In "New architecture for incoherent optical CDMA to achieve bipolar capacity," [13] Zaccarin and Kavehrad form composite sequences from the bipolar codes. The type of composite sequences they use are called Kronecker sequences and are formed by taking the Kronecker product, or tensor product, of the original code with some new code. The original code is generally some arbitrary sequence such as a Gold code while the new code is usually a single short sequence such as a Barker sequence. The Barker sequence is the "outer" sequence, meaning the composite sequence is a concatenation of Gold codes, the "inner" sequences, each repeat being modulated by the respective bit of the Barker sequence. The correlative behavior of such sequences has been discussed several times throughout spread-spectrum literature, but probably the most detailed analysis is provided in the 1983 paper by Stark and Sarwate, "Kronecker sequences for spread-spectrum communications" [12], wherein they considered bipolar systems. But as we will show in this thesis, it is relatively easy to relate a code's correlative behavior in a bipolar system to its correlative behavior in a unipolar system. While Zaccarin and Kavehrad use Barker sequences of length four and higher as outer sequences in their work, their method would be very similar to the appended method described in this thesis if they used the Barker sequence of length two, $(0, 1)$, as the outer sequence. Further, though the Kronecker sequence analysis always assumes some single short outer code and arbitrary inner code, the method could be reversed to describe a system with short inner sequences and arbitrary outer sequences. If the length-two Barker sequence was used as the inner sequence, this method would be very similar to the interleaved system we describe.

1.2.4 Other Unipolar Systems

The work by O'Farrel entitled "Code-division multiple-access (CDMA) techniques in optical fiber local area networks" [2] consists entirely of the analysis of the asynchronous time-encoded unipolar CDMA system with bipolar reference. For his codes he simply uses the unipolar version of bipolar codes such as Gold codes and considers a receiver which uses the bipolar version of the same code as reference. He goes on to analyze the system performance through the use of the average interference parameter, or AIP, and gets expressions for the signal-to-noise ratio of the system. We show in the next chapter how our asynchronous bipolar-equivalent system can be analyzed the same way by considering super codes.

One final type of system we should mention is the unipolar system with unipolar reference. This category of system actually includes the method of Tancevsky [9] discussed earlier, but also includes the very different area of so called optical orthogonal codes. We will not go into this last method due to the optical orthogonal codes' poor correlative properties and small code set size. All of these methods, as well as the unipolar system with bipolar reference discussed above, are outlined in the popular article by Parham, et al [3].

1.3 Overview of the Thesis

In Chapter 2, we describe bipolar-equivalent coding and show how it leads to the appending and interleaving methods in a time-encoded system when using a single fiber for transmission. We then show how these are equivalent to normal unipolar systems where unipolar codes of twice the length are transmitted with the bipolar versions of these same codes used as reference at a correlator receiver. These systems are, therefore, unipolar transmission/bipolar reference systems as described in [3].

In Chapter 3, we go on to derive a unified analysis for unipolar transmission/bipolar reference systems based on the analogous analysis of bipolar transmission/bipolar reference systems of [4]. We reach an expression for the signal-to-noise ratio of the systems based on their codes' cross-correlation parameters.

In Chapter 4, we compute the cross-correlations of the appended and interleaved systems, which leads us to their respective correlation parameters and, therefore, signal-to-noise ratios. We find simple expressions for the bipolar-equivalent systems allowing us to write their correlations and correlation parameters in terms of bipolar correlation parameters: A result which allows us to compare the performance of different systems by just computing the correlation parameters for specific codes.

In Chapter 5, we perform the comparison described above for different types of codes. We start by considering perfect random codes which have perfect correlation performance. This gives us a result which we would expect codes with increasingly good randomness properties would approach. We then compare the average interference parameters for the different systems using length-127 m-sequences. We see that the interleaved system performs very similarly to the average-case bipolar system, while the appended system is similar to the worst-case bipolar system. Because of the limited size of the set of m-sequences, we change to Gold codes which have larger code sets with somewhat decreased randomness properties. We see that the Gold codes still yield similar results to the m-sequences in terms of the average interference parameters. We go on to compute the signal-to-noise ratios which we use to approximate the probability of error. We perform an exact analysis of the probability of error due to interference and find our approximation is adequate for comparing system performance. We also perform an analysis of the probability of error due to the Poisson statistics of optical receivers in a hypothetical system. Overall, we find that the interleaved system does a very good job of reproducing the bipolar system behavior, thereby achieving our goal.

Chapter 6 is our conclusion where we discuss the relevance of our results as well as discuss future problems and ideas worth considering.

1.3.1 Statement of the Problem

To summarize what we have said, then, the purpose of this work is to produce a time-encoded unipolar CDMA system based on the bipolar-equivalent coding method that achieves performance comparable to that of bipolar systems using the same codes.

Chapter 2

Bipolar-Equivalent Coding

The primary goal here is to see how we can analyze our bipolar-equivalent coded codes in a system which is not only time-encoded, but also asynchronous. We accomplish this by showing how we can describe the bipolar-equivalent systems as unipolar transmission/bipolar reference systems which use new, so-called "supercodes". Recognizing this fact will then allow us to compare the performance of these systems when asynchronous by looking at the shifted correlations of the supercodes. In particular, we consider two types of time-domain bipolar-equivalent systems, the appended and interleaved systems.

2.1 Basic Bipolar-Equivalence

The key to bipolar-equivalent coding is recognizing that we can describe a bipolar sequence of length L, a_k, with $a_{k,n} \in \{-1,+1\}$, as the difference of two unipolar sequences of length L, $u_k - \overline{u}_k$, with $u_{k,n}, \overline{u}_{k,n} \in \{0,+1\}$. We then transmit u_k and $\overline{u}_k$ in our unipolar system, and subtract them at the receiver. Obviously, if we want to properly rebuild a_k, we must keep u_k and $\overline{u}_k$ separate, and we must keep track of which is which. To start, we consider them transmitted over two separate fibers, essentially spatially-multiplexed, as shown in Figure 2.1. Note that we have taken advantage of the linearity of the correlator-type receiver to move the subtracting operation which rebuilds a_i after the correlation. This can be seen in the following,

$$C_{U_k,A_i} = C_{u_k,a_i} + C_{\overline{u}_k,-a_i} = C_{u_k-\overline{u}_k,a_i} = C_{a_k,a_i}, \tag{2.1}$$

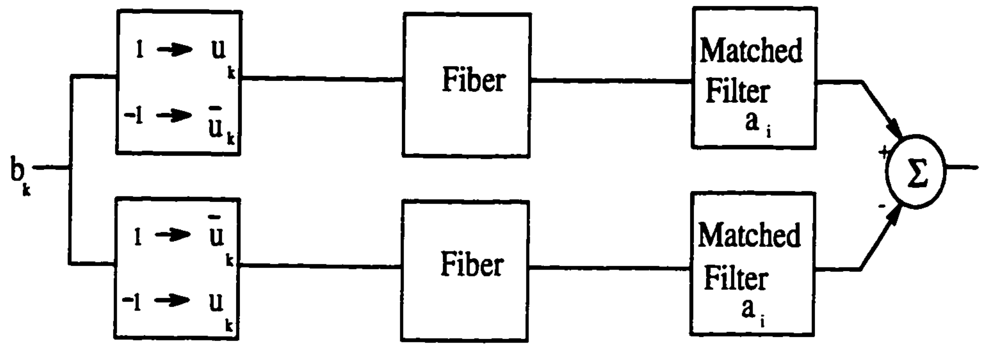


Figure 2.1 Spatially multiplexed bipolar-equivalent system.

where $C_{x,y}$ can be any type of correlation function for vectors x and y; the only property we make use of is its bilinearity.

2.2 Single-Fiber Time-Encoding

In the spectrally-encoded system, as we have said before, the codes are spread over specific wavelength slots. There is no need to consider further complications since one would not expect the wavelengths to change over transmission. It would be trivial to implement a spectrally-encoded system like this using a single fiber. In time-encoding, on the other hand, we cannot put u_k and $\overline{u}_k$ in parallel in any way. We must multiplex them together in series somehow. The most obvious method of doing this is shown in Figure 2.2, where T is the total bit period, the time taken to transmit u_k.

Now we must transmit half as often to fit u_k and $\overline{u}_k$ after each other on the same channel, so we have paid the price of doubling our bandwidth requirement to halve our fiber requirements. If we keep this system synchronized properly, it will obviously give the same output as the bipolar system we are trying to emulate. In fact, we can see that, in general, the output of the transmitter will be either $[u_k, \overline{u}_k]$, or $[\overline{u}_k, u_k]$ depending on what the data bit, b_k, is. That transmitted signal can then be seen to

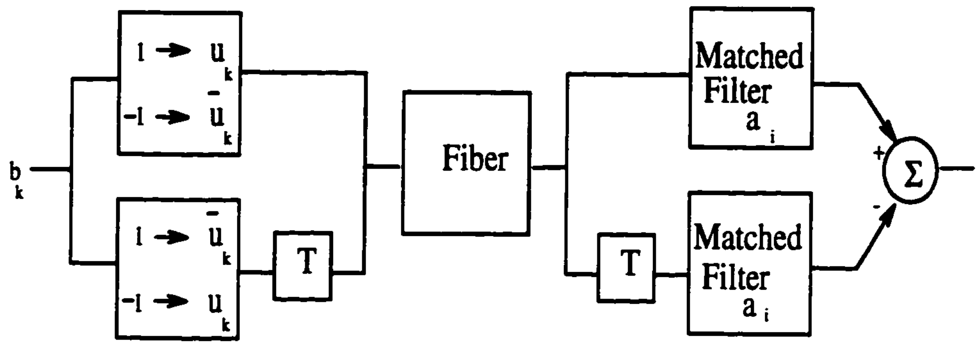


Figure 2.2 Single fiber bipolar-equivalent system.

be correlated at the receiver with $[a_k, -a_k]$. The net effect is as simple as one half of the system of Figure 2.1, and it is depicted in Figure 2.3.

This we call the "appended" system since we have appended u_k and $\overline{u}_k$ together to form a new "supercode" $[u_k, \overline{u}_k]$.

To see why this distinction is useful, imagine the transmitter in Figure 2.3 (or Figure 2.2 for that matter) was $\frac{1}{2}T$ out of synchronization with the i-th receiver (shown in the figure). We would then have the correlation of a_i with part of u_k subtracted from the correlation of a_i with the rest of u_k, and something similar with $\overline{u}_k$. While this is not overly difficult to work out, it is much simpler to think of the problem in terms of shifted correlations between the length $2L$ supercodes $[u_k, \overline{u}_k]$ and $[a_k, -a_k]$.

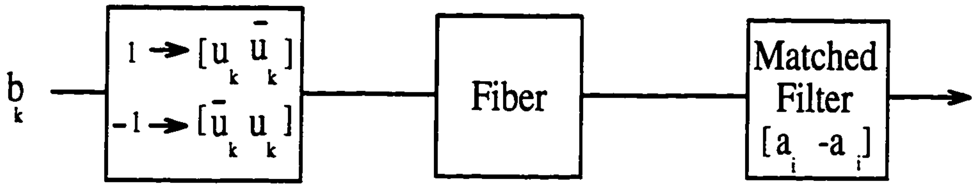
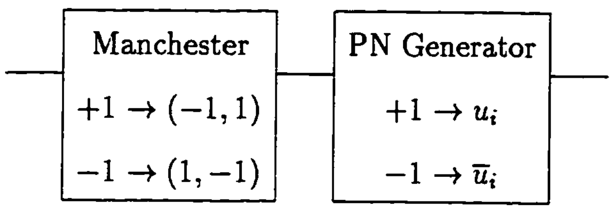


Figure 2.3 The equivalent appended system.

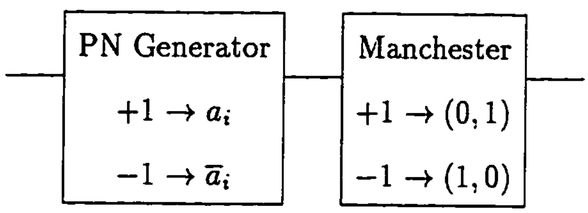
This appended system is one of two apparent ways to combine u_k and $\overline{u}_k$. The other would be to multiplex them at the chip level, a system corresponding to Figure 2.2 with a delay of T_c, the chip period, used to fit the unipolar codes together. This second system we call the "interleaved" system.

2.2.1 Manchester Meanings

Here we show how the appended and interleaved systems can arise from the use of Manchester coding as we mentioned in Chapter 1. Below, we see how the appended system's transmitter can be drawn as the concatenation of a Manchester coding block with a normal code generator to produce an appended output.



Reversing the order gives us a more traditional application of Manchester coding in the following.



We would also expect the receivers for the respective systems would have their reference codes produced in the same way.

2.2.2 Unipolar Transmission/Bipolar Reference

Back to the supercodes now, we write the appended supercode as U_i^{app} where

$$U_i^{app} = (u_{i,1}, u_{i,2}, \ldots, u_{i,L}, \overline{u}_{i,1}, \overline{u}_{i,2}, \ldots, \overline{u}_{i,L}), \tag{2.2}$$

and the reference code for this system is

$$A_i^{app} = (a_{i,1}, a_{i,2}, ..., a_{i,L}, -a_{i,1}, -a_{i,2}, ..., -a_{i,L}). \tag{2.3}$$

It is not too difficult to see that the interleaved system will have the supercodes

$$U_i^{int} = (u_{i,1}, \overline{u}_{i,1}, u_{i,2}, \overline{u}_{i,2}, ..., u_{i,L}, \overline{u}_{i,L}), \tag{2.4}$$

and

$$A_i^{int} = (a_{i,1}, -a_{i,1}, a_{i,2}, -a_{i,2}, ..., a_{i,L}, -a_{i,L}), \tag{2.5}$$

at the transmitter and receiver, respectively. These codes, then, would be used in a system like the one in Figure 2.4.

The final point to be made about this is to stress how, in every way, it is a unipolar transmission/bipolar reference system. The only thing new is our use of specially made codes. Even so, we can see how A_i and U_i are merely the bipolar and unipolar versions of the same code respectively for either system. Further, we can now determine what the output of the correlator will be for arbitrary timing differences between users by knowing the correlation between the supercodes in a given system. In the next chapter we will make use of this fact.

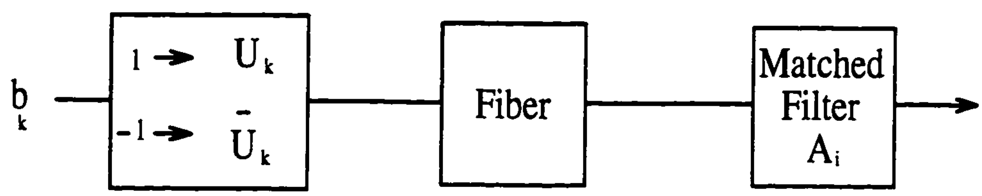


Figure 2.4 The equivalent unipolar transmission/bipolar reference system with supercodes.

Chapter 3

General System Analysis

In this chapter we introduce and analyze the asynchronous unipolar transmitter/bipolar reference CDMA system generally. This will allow us to describe the behavior of the systems of the previous chapter with the supercodes U_i^{app} or U_i^{int}, or for that matter, any unipolar code, super or not. We start off relating the unipolar and bipolar codes in the first—and most important—equation. This allows us to follow the basic route of [4], except that we carry along additional scale and offset terms (and we normalize things differently). We describe the interference in terms of the cross-correlations of whatever codes are used, then we show how the interference will have an extra term which varies with the number of users. Of course, we don't need to address that problem since for the bipolar-equivalent supercodes we use, the extra term will be zero (we will see that in the next chapter). Next, we derive an expression for the signal-to-noise ratio which uses the average interference parameter, a kind of fourth order correlation. The average interference parameter is made up of correlation parameters which are functions of the cross-correlation of the codes. So when, in the next chapter, we find out what the cross-correlations of our supercodes are, we will be able to compute the systems' signal-to-noise ratio, an important performance measure.

3.1 Uni- and Bi-Polarity

In the bipolar system, the i-th user is given a bipolar signature code sequence a_i of length L with $a_{i,n}$ in $\{-1, +1\}$. The unipolar version of this code, u_i with $u_{i,n}$ in

$\{0, +1\}$ can be produced by changing all the -1's in a_i to zeros. We can write a_i as $u_i - \overline{u}_i$, and we can write $\overline{u}_i$ as $\mathbf{1}_i - u_i$. Putting these two ideas together gives

$$u_i = +\frac{1}{2}a_i + \frac{1}{2}\mathbf{1}_i, \tag{3.1}$$

$$\overline{u}_i = -\frac{1}{2}a_i + \frac{1}{2}\mathbf{1}_i. \tag{3.2}$$

We have introduced the sequence $\mathbf{1}_i$ (the subscript is there simply to remind us where the $\mathbf{1}$ term came from), a sequence of ones of length L, which is our way of denoting the offset between bipolarity and unipolarity. An example is given in Figure 3.1.

We will continue to use u's to represent unipolar codes and a's to represent bipolar codes, though we will occasionally use x and y to represent arbitrary codes. Note, also, that the first subscript on the codes always refers to the user while the second subscript is the specific code chip.

3.2 The Unipolar Transmitter

To send the n-th data bit $b_i^{(n)}$ in the usual direct sequence bipolar scheme, the i-th user would transmit

$$S_{a_i}^{(n)}(t) = \sum_{j=0}^{L-1} p(t - jT_c)a_{i,j}b_i^{(n)}, \tag{3.3}$$

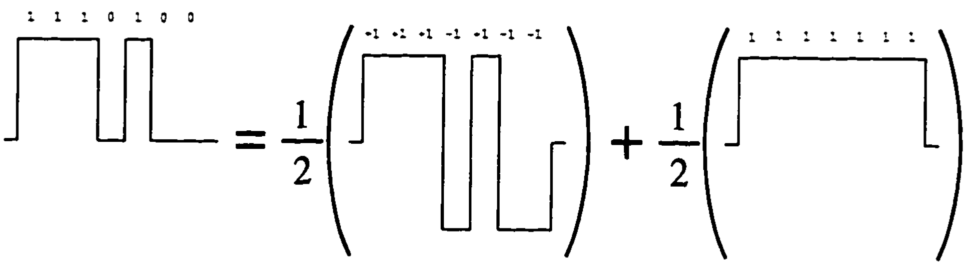


Figure 3.1 Unipolar sequence as sum of bipolar sequence and "ones sequence"

where $p(t)$ is some pulse shape, and T_c is the chip period. The total period of the transmitted signal, then, is $LT_c = T$. b_i usually takes values from $\{-1, +1\}$, and so this method is known as sequence inversion keying (SIK). It is also known as binary phase-shift keyed CDMA.

In the unipolar system, the analogous method is sequence complement keying (SCK) where we send the code or its complement depending on the data as in

$$S_{u_i}^{(n)}(t) = \begin{cases} \sum_{j=0}^{L-1} p(t-nT)u_{i,j} & \text{for } b_i^{(n)} = +1 \\ \sum_{j=0}^{L-1} p(t-nT)\overline{u}_{i,j} & \text{for } b_i^{(n)} = -1. \end{cases} \tag{3.4}$$

Employing Equation 3.2, this gives us

$$S_{u_i}^{(n)}(t) = \sum_{j=0}^{L-1} p(t-kT)\left(\frac{1}{2}a_{i,j}b_i^{(n)} + \frac{1}{2}\right) \tag{3.5}$$

where the data bit $b_i^{(n)}$ is the same bipolar number as in the SIK system. Therefore, we have related the "unipolar transmission" part of unipolar transmission/bipolar reference to bipolar transmission. Basically, all of the differences between the bipolar system and our unipolar system will come from this relation.

3.3 The Channel

The model for the asynchronous optical communication channel as seen by the i-th user is shown in Figure 3.2, and the total received signal is given by

$$s(t) = \sum_{k=1}^{N} S_{u_k}(t-\tau_k) + \eta(t). \tag{3.6}$$

where N is the total number of active users, $\eta(t)$ is zero mean additive white Gaussian noise. The system is asynchronous since we have delays between different users where τ_k is the relative delay of the k-th user with respect to the i-th user.

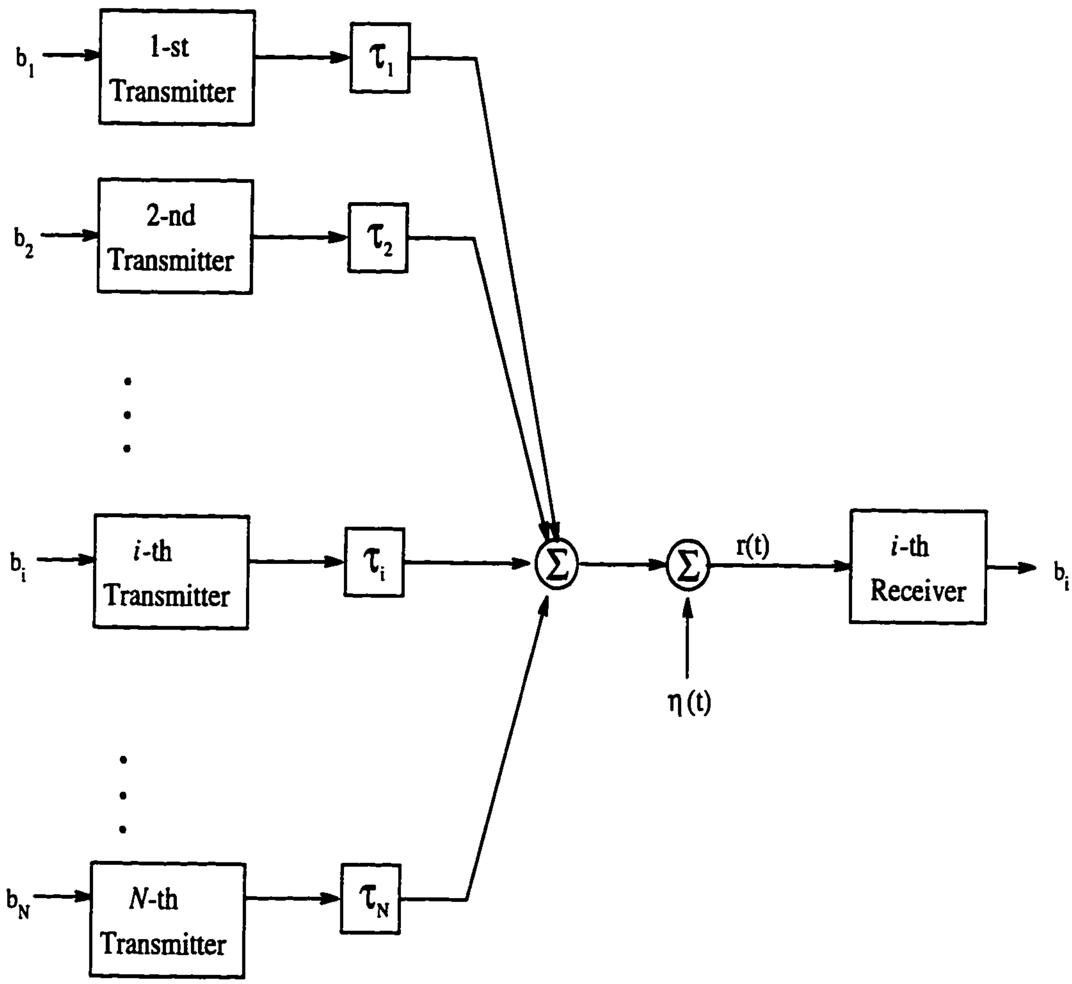


Figure 3.2 The Asynchronous CDMA System Model

3.4 The Matched-Filter Receiver

The simplest and most popular type of receiver used in CDMA systems is the matched-filter, or correlator, shown in Figure 3.3 [2].

The matched-filter receiver has the advantages that it is linear, allowing the machinations of the last chapter, and that it can be implemented almost completely passively using tapped-delay lines (see Section 5.4 for an example). We say that this receiver is matched to the sequence a_i for the i-th user. The output of the receiver is then

$$Z(T) = \int_0^T s(t)a_i(t)dt, \tag{3.7}$$

where we use $a_i(t)$, the impulse response of the i-th matched filter, to refer to the continuous-time version of the code a_i. Putting in our expression for $s(t)$, this gives

$$Z(T) = \int_0^T S_{u_i}(t)a_i(t)dt + \int_0^T \sum_{\substack{k=1 \\ k\neq i}}^{N} S_{u_k}(t)a_i(t) + \int_0^T \eta(t)a_i(t)dt. \tag{3.8}$$

We describe the three parts of this signal as

$$Z(T) = Z_S + Z_I + Z_\eta. \tag{3.9}$$

A signal part (the wanted signal), an interference part (the unwanted signal), and a noise part.

Using Equations (2.1) and (2.3), we write the signal term as

$$Z_S = PT_c \sum_{j=0}^{L-1} \left(b_i^{(n)} \frac{1}{2} a_{i,j} a_{i,j} + \frac{1}{2} a_{i,j} \right) = b_i^{(n)} \left(\frac{PT}{2} \right) + \left(\frac{PT_c}{2} W_i \right), \tag{3.10}$$

where we have used the fact that u_i and a_i are synchronized ($\tau_i = 0$). W_i is the discrete weight of the code, $\sum_{j=0}^{L-1} a_{i,k}$, and PT is the power per bit of the transmitted signal.

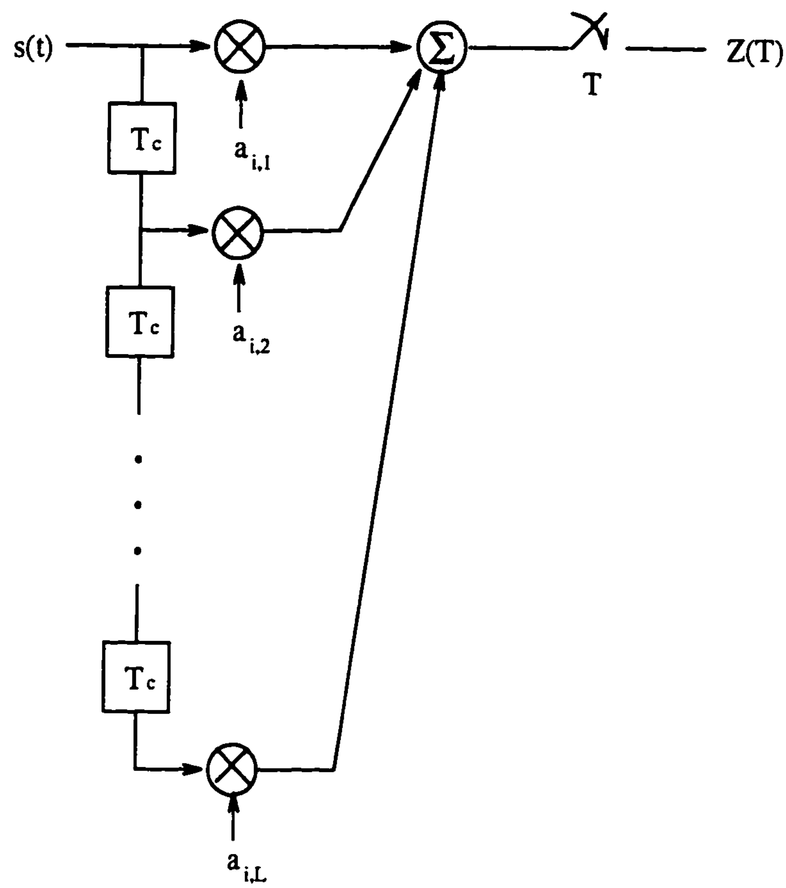


Figure 3.3 The i-th User's Matched Filter Receiver.

The term we are most concerned about is the interference term, which we write as

$$Z_I = PT_c \sum_{\substack{k=1 \\ k\neq i}}^{N} I_{k,i}(\tau_k), \tag{3.11}$$

where we have introduced $I_{i,k}$, the normalized two-channel interference between the i-th and k-th channel. We write it as

$$I_{k,i}(\tau_k) = \frac{1}{2T_c}\{b_k^{(0)} R_{a_k,a_i}(\tau_k) + b_k^{(-1)} \hat{R}_{a_k,a_i}(\tau_k) + T_c W_i\} \tag{3.12}$$

introducing the continuous-time partial cross-correlations,

$$R_{a_k,a_i}(\tau) = \int_0^{\tau} a_k(t-\tau)a_i(t)dt \tag{3.13}$$

$$\hat{R}_{a_k,a_i}(\tau) = \int_{\tau}^{T} a_k(t-\tau)a_i(t)dt. \tag{3.14}$$

The final term, Z_η, the additive channel noise, we will neglect in this treatment since we are considering an interference-limited system.

3.4.1 Correlation Situation

If we assume a rectangular pulse as the chip shape, we can write the partial cross-correlations as

$$R_{a_k,a_i}(\tau) = C_{a_k,a_i}(l-L)T_c + [C_{a_k,a_i}(l+1-L) - C_{a_k,a_i}(l-L)](\tau - lT_c) \tag{3.15}$$

$$\hat{R}_{a_k,a_i}(\tau) = C_{a_k,a_i}(l)T_c + [C_{a_k,a_i}(l+1) - C_{a_k,a_i}(l)](\tau - lT_c) \tag{3.16}$$

where l is the discrete timing delay in number of chips, $lT_c \leq \tau \leq (l+1)T_c$, and we have introduced the aperiodic cross-correlation function (CCF),

$$C_{a_k,a_i}(l) = \begin{cases} \sum_{j=0}^{L-1-k} a_{k,j}a_{i,j+l} & , \quad 0 \leq l \leq L-1 \\ \sum_{j=0}^{L-1+k} a_{k,l-j}a_{i,j} & , \quad 1-L \leq l < 0 \\ 0 & , \quad |l| \geq L \end{cases} \tag{3.17}$$

We will make use of a more simple looking version of Equation 2.18 by requiring that for any code x_k, $x_{k,j}$ equals zero when $j < 0$ or $j > L-1$. We then get

$$C_{a_k,a_i}(l) = \sum_{j=0}^{L-1} a_{k,j} a_{i,j+l}. \tag{3.18}$$

If both codes are bipolar as above, we will write C_{a_k,a_i} as $C_{k,i}$ from now on.

Note that when we discuss the interference from a unipolar code u_k, we can do so in terms of the correlation from the corresponding bipolar code, a_k, since we have separated the difference into the W_i term using Equation 3.12. We assume continuous transmission for all users so we can replace the aperiodic continuous weight with the discrete weight. The corresponding bipolar aperiodic terms, on the other hand, must be considered, as we will see later, due to the randomness of the data.

With the CCF, we can see more clearly the origin of W_i. If we write u_k as $\frac{1}{2}a_k + \frac{1}{2}\mathbf{1}_k$, then the CCF of u_k with some code a_i is

$$C_{u_k,a_i}(l) = C_{\frac{1}{2}a_k+\frac{1}{2}\mathbf{1}_k,a_i}(l) = \frac{1}{2}\{C_{k,i}(l) + C_{\mathbf{1}_k,a_i}(l)\}, \tag{3.19}$$

and we already know this $C_{\mathbf{1}_k,a_i}(l)$ term as W_i. The equation above, however, is of an aperiodic CCF so we should not expect it to result in a constant. For a normal system with continuous transmission, however, the net effect of a series of these will, in fact, produce the constant W_i.

To get a relatively simple expression for the interference, we start by introducing the even and odd periodic bipolar cross-correlation functions, respectively

$$\theta_{k,i}(l) = C_{k,i}(l-L) + C_{k,i}(l), \tag{3.20}$$

$$\hat{\theta}_{k,i}(l) = C_{k,i}(l-L) - C_{k,i}(l). \tag{3.21}$$

Note from Equation 3.12 that the interference depends on the data at any give time as well as the previous data bit. For the four different possible combinations of data at any time, $(b_k^{(-1)}, b_k^{(0)})$ equal to $(+1,+1)$, $(+1,-1)$, $(-1,+1)$, and $(-1,-1)$, we can write the respective two-channel interferences $I_{k,i}^{++}$, $I_{k,i}^{+-}$, $I_{k,i}^{-+}$, and $I_{k,i}^{--}$ as

$$I_{k,i}^{++} = \frac{1}{2}\{+\theta_{k,i}(l) + [\theta_{k,i}(l+1) - \theta_{k,i}(l)](\frac{\tau_i}{T_c} - l) + W_i\}, \tag{3.22}$$

$$I_{k,i}^{--} = \frac{1}{2}\{-\theta_{k,i}(l) - [\theta_{k,i}(l+1) - \theta_{k,i}(l)](\frac{\tau_i}{T_c} - l) + W_i\}, \tag{3.23}$$

$$I_{k,i}^{+-} = \frac{1}{2}\{+\hat{\theta}_{k,i}(l) + [\hat{\theta}_{k,i}(l+1) - \hat{\theta}_{k,i}(l)](\frac{\tau_i}{T_c} - l) + W_i\}, \tag{3.24}$$

$$I_{k,i}^{-+} = \frac{1}{2}\{-\hat{\theta}_{k,i}(l) - [\hat{\theta}_{k,i}(l+1) - \hat{\theta}_{k,i}(l)](\frac{\tau_i}{T_c} - l) + W_i\}. \tag{3.25}$$

3.4.2 Mean of the Interference

We make the standard assumptions that the data is an independently distributed Bernoulli process with $P(b = +1) = P(b = -1) = \frac{1}{2}$, and that the timing delays, τ_k are uniformly distributed between 0 and T.

First we compute the mean of $I_{i,j}$, $E(I_{i,j})$, the expectation with respect to random timing difference and data bits. We use the fact that the data is uniformly random to say that the four possible interferences are equally likely.

$$\begin{aligned} E(I_{i,j}) &= \tfrac{1}{4}E(I_{k,i}^{++}) + \tfrac{1}{4}E(I_{k,i}^{+-}) + \tfrac{1}{4}E(I_{k,i}^{-+}) + \tfrac{1}{4}E(I_{k,i}^{--}) \\ &= \tfrac{1}{4}E(I_{k,i}^{++} + I_{k,i}^{+-} + I_{k,i}^{-+} + I_{k,i}^{--}). \end{aligned} \tag{3.26}$$

Referring to Equations 2.21-24, we see that the only terms which don't sum to zero will be the constants, $\frac{1}{2}W_i$, giving $\frac{1}{4}E(\frac{4}{2}W_i)$ or just $\frac{1}{2}W_i$ as the mean.

The total N-user mean interference, then, will be

$$E(Z_I) = PT_c \sum_{\substack{k=1 \\ k \neq i}}^{N} E(I_{k,i}) = \frac{1}{2}PT_c(N-1)W_i. \tag{3.27}$$

The mean interference depends on the number of active users on the channel for nonzero W_i. This obviously poses a problem which, as discussed in [2], can be solved in one of two ways. The first, not at all trivial, way is to actively detect the number of users on the channel somehow, and subtract the appropriate offset to eliminate the mean. The second way would be to simply to use code families for which the weight is zero. While no commonly-used bipolar code families have this property, bipolar-equivalent supercodes do have this property as we will see in the next chapter.

3.5 Signal-to-Noise Ratio

We refer to Pursley [6] for the general expression for the signal-to-noise ratio of a matched-filter

$$\text{SNR} = \sqrt{\frac{(E[Z_s + Z_I] - E[Z_I])^2}{\text{Var}[\text{Z}_\text{I}]}}. \tag{3.28}$$

We must be a little careful with the numerator of this equation due to our varying mean situation. For now, we will take the approach of [2] and occasionally disregard terms involving W_i, since we assume we have eliminated them by other means. To calculate the variance, we start with

$$\text{Var}(Z_I) = E\left[P^2T_c^2\left(\sum_{\substack{k=1\\k\neq i}}^{N} I_{k,i}\right)^2\right] - E^2\left[PT_c\sum_{\substack{k=1\\k\neq i}}^{N} I_{k,i}\right], \tag{3.29}$$

where the expectation is taken over all possible time delays and data bits. We expand both terms and separate them into squared terms and cross terms to get

$$\text{Var}(Z_I) = P^2T_c^2\left(\sum_k E[I_{k,i}^2] + \sum_k\sum_{l\neq k} E[I_{k,i}I_{l,i}] - \sum_k E^2[I_{k,i}] - \sum_k\sum_{l\neq k} E[I_{k,i}]E[I_{l,i}]\right). \tag{3.30}$$

Since the data is uncorrelated ($E[b_ib_j] = \delta_{i,j}$), the second and fourth terms sum to zero and, substituting $\frac{1}{2}W_i$ for $E[I]$, get

$$\text{Var}(Z_I) = P^2T_c^2\sum_k\left(E[I_{k,i}^2] - \left(\frac{1}{2}W_i\right)^2\right). \tag{3.31}$$

We use Equation 3.12 to get

$$\mathrm{Var}(Z_I) = P^2T_c^2\sum_k \left(E\left[\left(\frac{1}{2T_c}\{b_k^{(0)}R_{a_k,a_i}(\tau_k) + b_k^{(-1)}\hat{R}_{a_k,a_i}(\tau_k) + T_cW_i\}\right)^2\right] - \frac{1}{4}W_i^2\right). \tag{3.32}$$

Since the data is zero mean and uncorrelated, the cross terms will average to zero. Also, the W_i terms will cancel, leaving

$$\mathrm{Var}(Z_I) = \frac{P^2}{4}\sum_k E\left[R^2_{a_k,a_i}(\tau_k) + \hat{R}^2_{a_k,a_i}(\tau_k)\right]. \tag{3.33}$$

Equation 3.33 is, within some constants, the same as the well-known variance of the bipolar interference. We have used a different normalization than [4] and others, so the only nontrivial difference is the factor of one fourth. A highly detailed derivation similar to what follows here, using perhaps more traditional normalization factors, can be found in [2].

Making use of the uniform distribution of τ_k and Equations 3.16 and 3.17, we replace the expectation with an integral and write

$$\begin{aligned} \mathrm{Var}(Z_I) &= \frac{P^2}{4}\sum_k \frac{1}{T}\int_0^T d\tau_k \qquad (3.34)\\ &\quad ([C_{k,i}(l_k - L)T_c + (C_{k,i}(l_k+1-L) - C_{k,i}(l_k-L))(\tau_k - l_kT_c)]^2 \\ &\quad +[C_{k,i}(l_k)T_c + (C_{k,i}(l_k+1) - C_{k,i}(l_k))(\tau_k - l_kT_c)]^2). \end{aligned}$$

But l_k, the discrete timing difference (in number of chips), is related to τ_k, the continuous timing difference. We express this as $\tau_k = l_kT_c + \lambda_k$ where λ_k is the chip-level timing difference which varies between zero and T_c. Then, we substitute this into Equation 3.34 and replace $\int_0^T d\tau_k$ with $\sum_{l=0}^{L-1}\int_{lT_c}^{(l+1)T_c} d\lambda_k$ to get

$$\begin{aligned} \mathrm{Var}(Z_I) &= \frac{P^2}{4T}\sum_{\substack{k=1\\k\neq i}}^{N}\sum_{l=0}^{L-1}\int_{lT_c}^{(l+1)T_c} d\lambda_k \qquad (3.35)\\ &\quad ([C_{k,i}(l_k - L)T_c + (C_{k,i}(l_k+1-L) - C_{k,i}(l_k-L))\lambda_k]^2 \\ &\quad +[C_{k,i}(l_k)T_c + (C_{k,i}(l_k+1) - C_{k,i}(l_k))\lambda_k]^2). \end{aligned}$$

3.5.1 Worst-Case SNR

We start with the simple case where all the users are synchronized at the chip level. This is called the worst case for reasons we will see in the next section. Chip-level synchronization means that λ_k is zero in Equation 3.35 so the integral reduces to multiplication by the constant T_c leaving

$$\mathrm{Var}(Z_I) = \frac{P^2T_c^2}{4L}\sum_k \sum_{l_k=0}^{L-1} \left(C_{k,i}^2(l_k - L) + C_{k,i}^2(l_k)\right). \tag{3.36}$$

We have also made use of the fact that $T_cL = T$. We can combine the two terms into one by extending the summation range to get

$$\mathrm{Var}(Z_I) = \frac{P^2T_c^2}{4L}\sum_k \mu_{k,i}(0), \tag{3.37}$$

where we have introduced the correlation parameter, $\mu_{k,i}(0)$, which comes from [4]

$$\mu_{k,i}(n) = \sum_{l=1-L}^{L-1} C_{k,i}(l)C_{k,i}(l+n). \tag{3.38}$$

Note that $\mu_{k,i}(0)$ is the variance of the discrete cross-correlation between a_i and a_k.

Referring to Equation 3.28, we get for the worst-case SNR,

$$\mathrm{SNR} = \sqrt{\frac{(\frac{PT}{2})^2}{\frac{P^2T_c^2}{4L}\sum_k \mu_{k,i}(0)}}. \tag{3.39}$$

And all the normalization constants cancel, anyway, to leave

$$\mathrm{SNR} = \left(\frac{1}{L^3}\sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{2}\mu_{k,i}(0)\right)^{-\frac{1}{2}}. \tag{3.40}$$

Here we disregarded the mean terms in the numerator of the SNR as we mentioned earlier.

Using a little foresight, we define the mean-squared interference parameter, $r_{x,y}$, also known as the average interference parameter (AIP), from

$$\mathrm{SNR} = \left(\frac{1}{6L^3}\sum_{\substack{k=1\\k\neq i}}^{N} r_{x,y}\right)^{-\frac{1}{2}}; \tag{3.41}$$

for arbitrary codes x and y. For the worst-case situation, then,

$$r_{k,i}^{WC} = 3\mu_{k,i}(0). \tag{3.42}$$

We have put the SNR in the same form as Equation 1.1, and get the same result for the AIP, as we would expect since we treat the two systems as scaled versions of each other.

3.5.2 Average-Case SNR

To compute the average-case SNR, we go back to Equation 3.35, and expand the terms to get

$$\begin{aligned} \mathrm{Var}(Z_I) \;=\; & \frac{P^2}{4T}\sum_k \sum_{l=1-L}^{L-1} \int_0^{T_c} d\lambda_k \{C_{k,i}^2(l_k)T_c^2 + (C_{k,i}(l_k+1) \\ & -C_{k,i}(l_k))^2\lambda_k^2 + 2C_{k,i}(l_k)(C_{k,i}(l_k+1) - C_{k,i}(l_k))T_c\lambda_k\}. \end{aligned} \tag{3.43}$$

Performing the integral and relabeling a few indices gives us

$$\mathrm{Var}(Z_I) = \frac{P^2}{4T}\sum_k \frac{T_c^3}{3}\{2\mu_{k,i}(0) + \mu_{k,i}(1)\}, \tag{3.44}$$

which provides an average-case SNR of

$$\mathrm{SNR} = \left(\frac{1}{6L^3} \sum_{\substack{k=1 \\ k\neq i}}^{N} \{2\mu_{k,i}(0) + \mu_{k,i}(1)\} \right)^{-\frac{1}{2}}, \tag{3.45}$$

which gives us

$$r_{k,i}^{AC} = 2\mu_{k,i}(0) + \mu_{k,i}(1), \tag{3.46}$$

again, equaling the bipolar result of Equation 1.3.

So the difference between the worst-case and the average-case is that one of the three discrete CCF variances, $\mu_{k,i}(0)$, is replaced by a $\mu_{k,i}(1)$ term. Referring back to Equation 3.38, we can see the reason for the designation of worst-case versus average-case. If the $C_{k,i}(l)$ are thought of as any string of numbers, then the sum of their squares will always be greater than or equal to the sum of $C_{k,i}(l)C_{k,i}(l+1)$, an equal number of cross terms, via the Cauchy-Schwarz inequality. In fact, as discussed in [2], for truly random codes a_i and a_j, $\mu_{i,j}(1)$ will be zero since all points on the correlation (except for at zero shift) will be equal to zero. We will look at this simplified system in the next chapter. We would hope that for codes with good randomness properties $\mu_{k,i}(1)$ will be very small, and we can get a significant reduction in the variance of the CCF's, and therefore, in the SNR. From now on, we will use simply $r_{k,i}$ to refer to the average-case AIP.

3.6 Summary of Our Method

The important equations we have derived, Equations 3.45 and 3.46, give us the signal-to-noise ratio of a unipolar transmission/bipolar reference system in terms of its codes' (bipolar) correlation parameters. For normal bipolar codes, these parameters have been investigated extensively (see, for example, [6]). It should be stressed that, while we use the unipolar versions of the codes instead of the bipolar, the only difference is a factor of four appearing in our equations. One could, for example, forego the supercodes and just use plain Gold codes in the system. The correlation parameters are provided in [6], and we would, as a result, see a signal-to-ratio equal to the bipolar result. We would still have the problem of a varying mean as in Section 3.4.2, of course, so this is not what we want. In the next chapter we will see how the supercodes, which don't have the varying-mean problem, will have correlation parameters which can be related to the ones from the bipolar case, which anyone can look up, allowing us to compare the performance of the supercode systems to the bipolar systems by comparing the signal-to-noise ratios.

Chapter 4

Specific System Analysis

We now want to relate the cross-correlations of our supercodes to the cross-correlations of the bipolar codes used to produce them. This will allow us to relate their average interference parameters and go on to compare their performance. This chapter contains the most important mathematical relations of the thesis, though, unlike the last chapter, no step performed in this chapter will require anything more complicated than simple algebra. The key is that wherever we write an equation for some term with a superscript such as X^{app} or Y^{int}, meaning for the appended or interleaved case, we will relate it to terms without superscripts, X and Y perhaps. These terms without superscripts will refer to the well-known bipolar case, so we are relating the supercode systems to the bipolar system.

4.1 Appended System

It is easy to see that we can describe U_i^{app}, the result of appending u_i and $\overline{u}_i$, as $U_{i,n}^{app} = u_{i,n} + \overline{u}_{i,n-L}$ and similarly, $A_{i,n}^{app} = a_{i,n} + (-a_{i,n-L})$ (recall that we assume $x_{i,j} = 0$ for $J < 0$ and $j > L - 1$). We will use this fact to eventually write the average interference parameter of appended codes in terms of correlation parameters from the original bipolar codes. The cross-correlation of the appended unipolar code with the appended bipolar reference code will therefore be

$$\begin{aligned}C_{U_k^{app},A_i^{app}}(l) &= C_{u_k,a_i}(l) + C_{u_k,-a_i}(l-L) + C_{\overline{u}_k,a_i}(l+L) + C_{\overline{u}_k,-a_i}(l) \\ &= C_{k,i}(l) - \tfrac{1}{2}\{C_{k,i}(l-L) + W_i + C_{k,i}(l+L) - W_i\}\end{aligned} \tag{4.1}$$

where the W_is cancel and we get

$$C_{k,i}^{app}(l) = C_{k,i}(l) - \frac{1}{2}\{C_{k,i}(l-L) + C_{k,i}(l+L)\}, \tag{4.2}$$

We see from Equation 4.2 that we have the original CCF plus two "cross terms" which arise since the codes are now $2L$ long. We can compute the average interference parameter with this from

$$\mu_{k,i}^{app}(0) = \sum_{l=1-2L}^{2L-1} \left(C_{k,i}^{app}(l)\right)^2 = \sum_{l=1-2L}^{2L-1} \left(C_{k,i}(l) - \frac{1}{2}\{C_{k,i}(l-L) + C_{k,i}(l+L)\}\right)^2 \tag{4.3}$$

which becomes

$$\begin{aligned}\mu_{k,i}^{app}(0) &= \sum_{l=1-2L}^{2L-1} (C_{k,i}^2(l) + \frac{1}{4}C_{k,i}^2(l-L) + \frac{1}{4}C_{k,i}^2(l+L) \\ &\quad -C_{k,i}(l)C_{k,i}(l-L) - C_{k,i}(l)C_{k,i}(l+L)).\end{aligned} \tag{4.4}$$

And since the summation indices go from $1-2L$ to $2L-1$, twice as far as any one term goes, we can make substitutions l' for $l-L$ and l'' for $l+L$ when appropriate, and using the fact that $\mu(-l) = \mu(l)$, we get

$$\mu_{k,i}^{app}(0) = \frac{3}{2}\mu_{k,i}(0) - 2\mu_{k,i}(L), \tag{4.5}$$

where the μ and r terms with no superscript refer to the original bipolar case as we have said. Also,

$$\mu_{i,j}^{app}(1) = \sum_{l=1-2L}^{2L-1} \{C_{k,i}(k)C_{k,i}(k+1) - \frac{1}{2}C_{k,i}(k)C_{k,i}(k-L+1) \tag{4.6}$$

$$-\frac{1}{2}C_{k,i}(k-L-1)C_{k,i}(k+1) + \frac{1}{4}C_{k,i}(k-L)C_{k,i}(k-L+1) \tag{4.7}$$

$$-\frac{1}{2}C_{k,i}(k+L)C_{k,i}(k+1) + \frac{1}{4}C_{k,i}(k+L)C_{k,i}(k+L+1)\},$$

which yields

$$\mu_{k,i}^{app}(1) = \frac{3}{2}\mu_{k,i}(1) - \mu_{k,i}(L-1) - \mu_{k,i}(L+1). \tag{4.8}$$

Since we assume an asynchronous system, we use Equation 3.46 for the AIP, and we combine these results to form

$$\begin{aligned} r_{k,i}^{app} &= 2\mu_{k,i}^{app}(0) + \mu_{k,i}^{app}(1) \\ &= 3\mu_{k,i}(0) + \frac{3}{2}\mu_{k,i}(1) - [\mu_{k,i}(L-1) + 4\mu_{k,i}(L) + \mu_{k,i}(L+1)] \\ &= \frac{3}{2}r_{k,i} - [\mu_{k,i}(L-1) + 4\mu_{k,i}(L) + \mu_{k,i}(L+1)], \end{aligned} \tag{4.9}$$

a rather complicated result which we look at empirically in Chapter 4.

What we can see here is that the average interference parameter for the appended system, which tells us the variance of the interference caused by a single interferer, is three halfs the bipolar system result with some additional terms. It makes sense that the variance of the appended cross-correlation will be fifty percent larger than the bipolar result if one considers Equation 4.2. At any given time, either $C_{k,i}(l-L)$ or $C_{k,i}(l+L)$ will be zero since the code length is L, so for codes with good randomness properties, we have one random number plus one half another independent random number. Random numbers would have the same variance, $r_{k,i}$, so the factor of one half gives a net variance of $\frac{3}{2}r_{k,i}$. The additional terms in Equation 4.8 appear because our two random numbers aren't independent, but based on what we have just considered, we expect these extra terms to be small for codes with good randomness properties. In the final analysis, then, the appended system doesn't look like it will give us the performance we want.

4.2 Interleaved System

Here we describe U_i^{int} as $U_{i,2n}^{int} = u_{i,n} + \overline{u}_{i,n+1}$. Then it is easy to see that

$$C_{U_k^{int},A_i^{int}}(l) = \begin{cases} C_{u_k,a_i}(\frac{1}{2}l) + C_{\overline{u}_k,-a_i}(\frac{1}{2}l) = C_{k,i}(\frac{1}{2}l) & , \text{ for } l \text{ even} \\ C_{u_k,-a_i}(\frac{1}{2}(l-1)) + C_{\overline{u}_k,a_i}(\frac{1}{2}(l+1)) & , \text{ for } l \text{ odd} \end{cases} \tag{4.10}$$

So this method reproduces the original cross-correlation of u_k and a_i at even shifts, but has cross terms at the odd shifts. Bounds on the cross terms have been discussed in [9]. We can do better than bounds on the terms, however. We write

$$C_{k,i}^{int}(l) = \begin{cases} C_{k,i}(\frac{1}{2}l) & , \text{ for } l \text{ even} \\ -\frac{1}{2}\{C_{k,i}(\frac{l-1}{2}) + W_i + C_{k,i}(\frac{l+1}{2}) - W_i\} & , \text{ for } l \text{ odd.} \end{cases} \tag{4.11}$$

So the weight terms cancel again leaving

$$C_{A_k,A_i}(l) = \begin{cases} C_{k,i}(\frac{1}{2}l) & , \text{ for } l \text{ even} \\ -\frac{1}{2}\{C_{k,i}(\frac{1}{2}l - \frac{1}{2}) + C_{k,i}(\frac{1}{2}l + \frac{1}{2})\} & , \text{ for } l \text{ odd,} \end{cases} \tag{4.12}$$

showing that the odd terms are the negatives of the linearly interpolated even terms: if the even terms take values A, B, C, ..., then the odd terms must be $-\frac{A+B}{2}$, $-\frac{B+C}{2}$, ... This, in itself, is a fairly interesting result.

The AIP, then, can be computed from

$$\mu_{k,i}^{int}(0) = \sum_{\substack{l=1-2L \\ l \text{ even}}}^{2L-1} \left(C_{k,i}(\frac{l}{2})\right)^2 + \sum_{\substack{l=1-2L \\ l \text{ odd}}}^{2L-1} (-\frac{1}{2})^2 \left(C_{k,i}(\frac{l-1}{2}) + C_{k,i}(\frac{l+1}{2})\right)^2. \tag{4.13}$$

We make the substitution $l' = \frac{l+1}{2}$ as needed to get

$$\mu_{k,i}^{int}(0) = \sum_{l=1-L}^{L-1} C_{k,i}^2(l) + \frac{1}{4}\sum_{l=1-L}^{L-1} \{C_{k,i}^2(l) + C_{k,i}^2(l+1) + C_{k,i}(l)C_{k,i}(l+1)\}, \tag{4.14}$$

which gives us

$$\mu_{k,i}^{int}(0) = \frac{3}{2}\mu_{k,i}(0) + \frac{1}{2}\mu_{k,i}(1), \tag{4.15}$$

and, similar computations yield

$$\mu_{k,i}^{int}(1) = -\mu_{k,i}(0) - \mu_{k,i}(1), \tag{4.16}$$

giving us the average-case AIP,

$$r_{k,i}^{int} = 2\mu_{k,i}(0) = r_{k,i} - \mu_{k,i}(1). \tag{4.17}$$

The above AIP is actually very close to the average-case bipolar result, differing only by the $\mu_{i,j}(1)$ term which we discussed earlier.

That this system seems to work so well, having a low variance for its cross-correlation, is understandable based on what we have seen. Our cross-correlation is equal to the bipolar cross correlation at even shifts and, at odd shifts, takes numbers interpolating the neighboring even shift numbers. We would expect the variance for this new system to be as good as the bipolar result, or perhaps slightly better, and this is what we see in Equation 4.16.

We now know everything we need to compare the interleaved system to the appended system, and both to the bipolar systems. We have written everything in terms of the well-known bipolar system correlation parameters, and all that is left is to see what these terms will be for specific real cases. Therefore, in the next chapter, we will numerically compute what these terms will be for specific code sets.

Chapter 5

Performance Results

We will now numerically compare the different coding methods using the parameters we have derived. We wish to compare our two unipolar systems to the bipolar system. We already have a fairly unified analysis, making use of the same equation for the signal-to-noise ratio, but there is still one key difference. In our bipolar-equivalent system, we formed supercodes of twice the length which were the basis of our analysis in Chapter 3. We must, therefore, consider all the systems using double the length to have an even comparison between the unipolar and the bipolar systems. We are basically accepting the factor of two increase in bandwidth as the cost of using bipolar-equivalence.

We have enumerated the different possible average interference parameters in Table 5.1.

Table 5.1 Average interference parameters

Method	AIP
Average-case bipolar	$2\mu_{k,i}(0) + \mu_{k,i}(1)$
Interleaved	$2\mu_{k,i}(0)$
Worst-case bipolar	$3\mu_{k,i}(0)$
Appended	$3\mu_{k,i}(0) + \frac{3}{2}\mu_{k,i}(1) - [\mu_{k,i}(L-1) + 4\mu_{k,i}(L) + \mu_{k,i}(L+1)]$

For the unipolar systems, we only consider the average case unless we explicitly state otherwise.

First, we will examine a theoretical case where we assume completely random codes in order to better recognize the dominant terms in the AIPs.

5.1 Random Codes

Some insight into the behavior of the correlation parameters can be gained by considering random codes, codes with perfect randomness, and therefore correlation, properties. This should give us a theoretical result which real codes will approach depending on how good their randomness properties are. The key to this analysis is to recognize that with a little reorganization of indices, we can write Equation 3.38 as [4]

$$\mu_{k,i}(n) = \sum_{l=1-L}^{L-1} C_k(l)C_i(l+n), \tag{5.1}$$

where we use the auto-correlation functions, defined as usual, instead of cross-correlations. Now, for random codes, the auto-correlation will be zero except at $C_k(0)$ where it will take value L. Because of this, Equation 5.1 will be zero except at $\mu_{k,i}(0)$ where it will take value L^2. Table 5.1, then, becomes rather trivial.

Table 5.2 Average interference parameters with perfectly random codes

Method	AIP
Average-case bipolar	$2L^2$
Interleaved	$2L^2$
Worst-case bipolar	$3L^2$
Appended	$3L^2$

We will see that these values form the greater part of the actual numerical results. As such, the results here tell us that the average interference parameters for the average-case bipolar and the average-case interleaved are very similar. The average interference parameters for the worst-case bipolar and the average-case appended are also very similar, but $\frac{3}{2}$ as large. Since the average interference parameter is in the denominator of the SNR, we would expect the SNR for the appended case to be about $\sqrt{\frac{2}{3}}$ as large, or about 18 percent lower, than the interleaved case.

5.2 Length-127 m-sequences

First, in Figure 5.1, we look at what the different AIPs from Table 5.1 are for various combinations of the 18 length-127 m-sequences. In this Figure, the x-axis gives the number k of a code, and on the y-axis is plotted $r_{k,1}$ for the four different systems. For length-127 codes, the random code result, $2L^2$, is approximately equal to 3.2×10^4, and $3L^2$ is approximately equal to 4.8×10^4. The data in Figure 5.1 can be seen to hover about these points. We can also see from this figure that the interleaved and average-case bipolar systems are indeed very similar, as are the appended and the worst-case bipolar.

In Figure 5.2, we show $\mu_{k,i}(1)$, the difference between the interleaved case and the bipolar case. Note that the scale is much smaller than the previous plot. Most importantly, we see the values going positive and negative rather randomly. Therefore, we cannot generally say whether $2\mu_{k,i}(0) + \mu_{k,i}(1)$ or $2\mu_{k,i}(0)$ will be greater, but we expect them not to differ by much.

In the next figure, Figure 5.3, we show $-\frac{3}{2}\mu_{k,i}(1)+\mu_{k,i}(L-1)+4\mu_{k,i}(L)+\mu_{k,i}(L+1)$, which is the difference between the asynchronous appended system and the worst-case bipolar system. We note again that the $\mu(n)$ terms vary randomly about zero for $n \neq 0$. These systems are interestingly similar, but neither are practical since their AIPs—and therefore, their variances—are roughly fifty percent larger than the

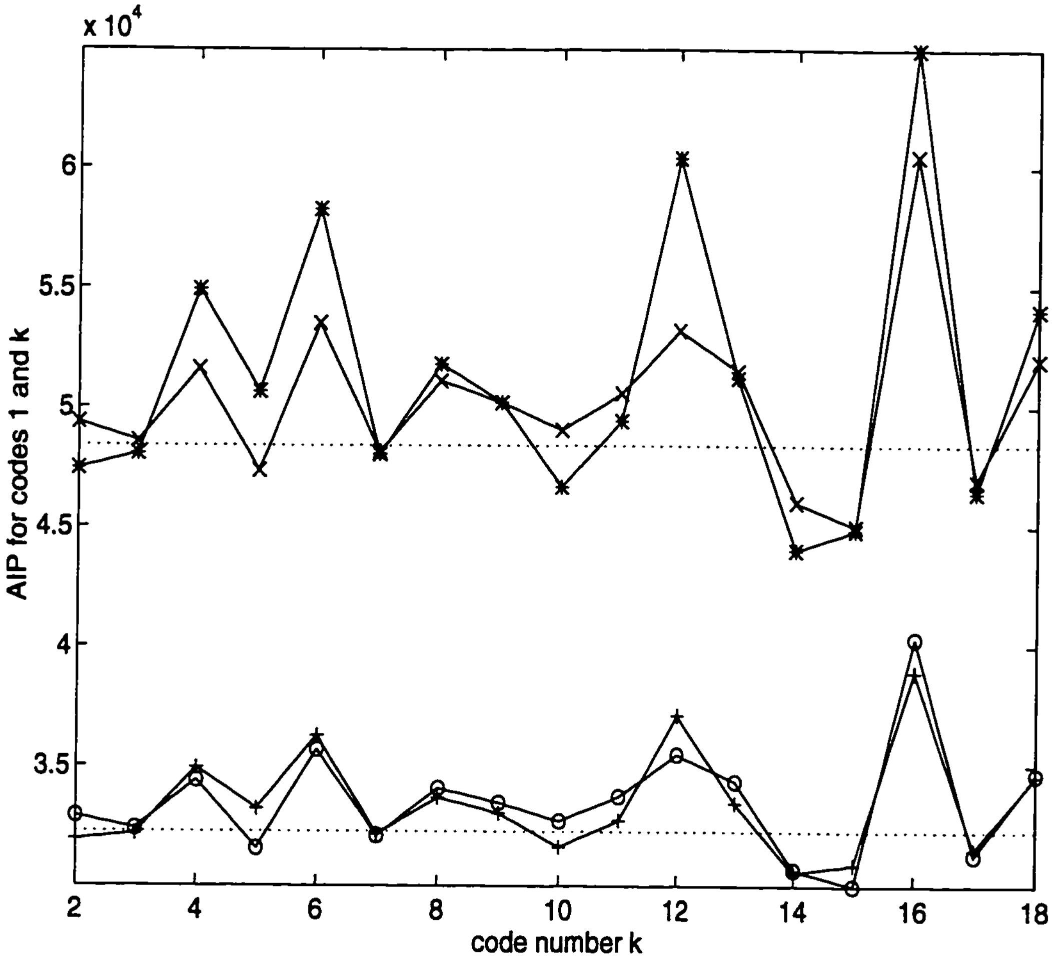


Figure 5.1 AIPs for Length 127 m-sequences for: asynchronous bipolar system (+), asynchronous interleaved code system (o), chip-synchronous bipolar system (×), and asynchronous appended case (∗). The dotted lines are at 3.2×10^4 and 4.8×10^4, corresponding to the random-code results.

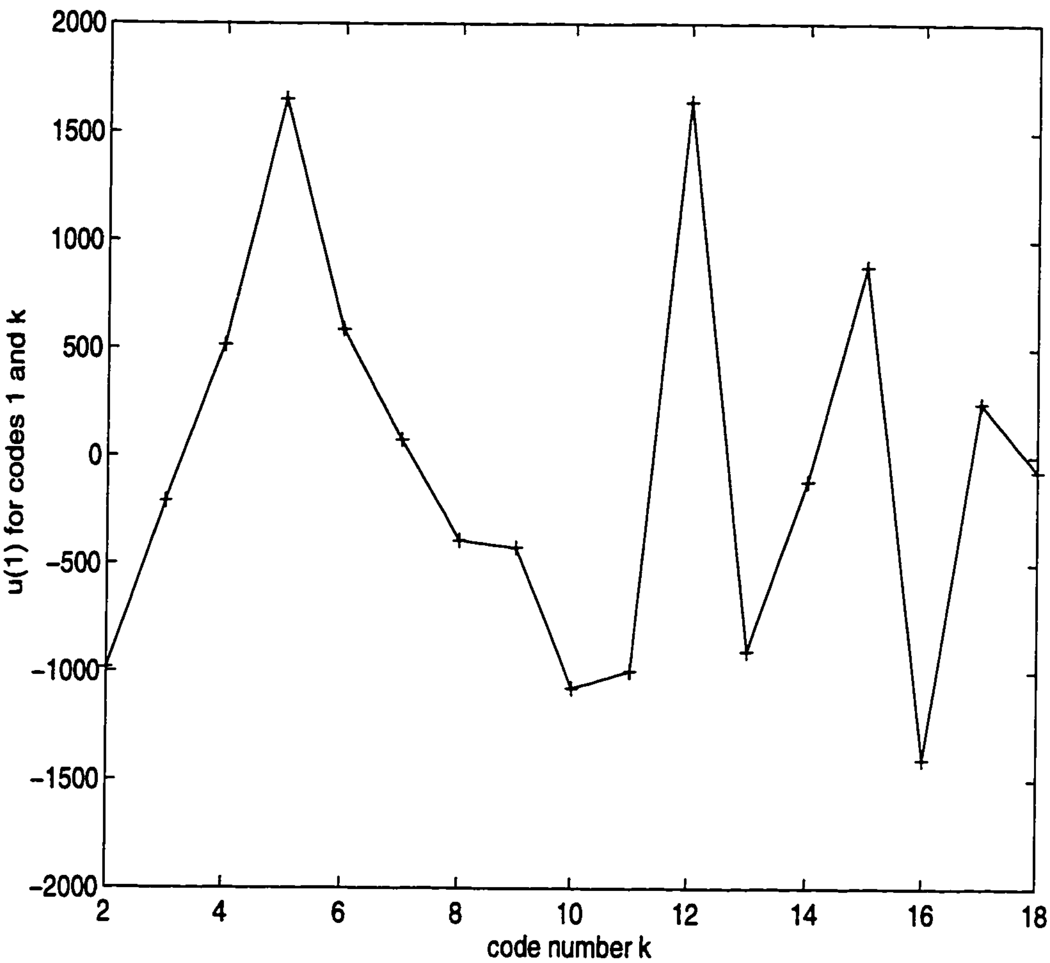


Figure 5.2 From the previous plot, $\mu_{k,i}(1)$, the difference between the bipolar AIP and the interleaved code AIP.

alternatives. The basic point we can see from all of these plots is how much the $\mu_{k,i}(0)$ term dominates the AIP.

5.3 Length-127 Gold Codes

We will now consider a set of codes that has more of a practical value in a CDMA system (since there are 129 of them), Gold codes. In Figure 5.4, we plot the AIPs for the bipolar system and the interleaved system. As before, $2L^2$ is 3.2×10^4, and we can see that the data stays within about twenty percent of this value.

In the next plot, Figure 5.5, we show $\mu_{k,i}(1)$, the difference between the bipolar average-case system and the interleaved system. Again, we see that difference hovers around zero, an order of magnitude less than the dominant term, and it depends greatly on the particular user's code. This suggests that the difference in performance between these two systems is a code-dependent parameter that is very small regardless.

With our respective AIPs in hand, we are equipped to compute the SNR, as

$$\mathrm{SNR} = \left(\frac{1}{6L^3} \sum_{\substack{k=1 \\ k \neq i}}^{N} r_{x,y} \right)^{-\frac{1}{2}}, \tag{5.2}$$

where now we decide which $r_{x,y}$ from Table 5.1 to use depending on our coding method. SNRs for increasing numbers of interferers are plotted in Figure 5.6. The difference between the two similar pairs of systems is barely discernible here. Note that this is for one particular choice of the i-th user.

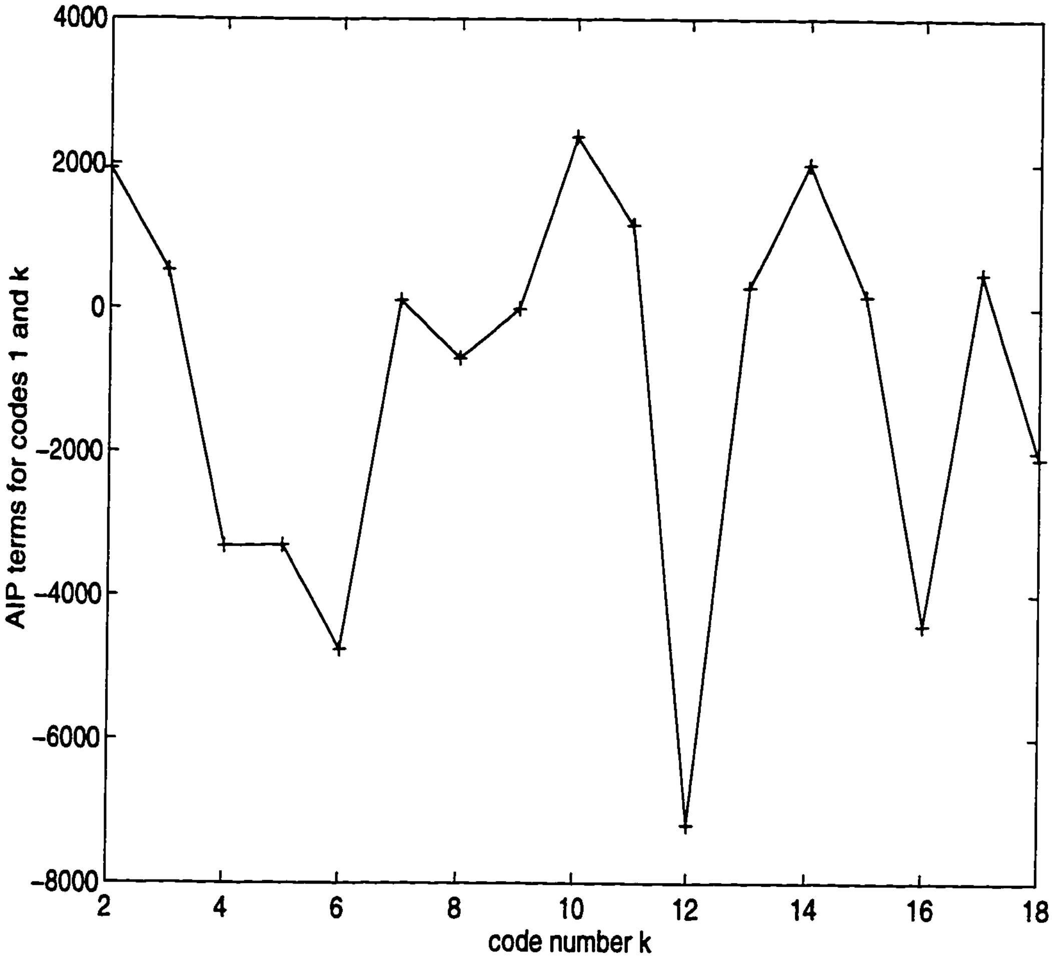


Figure 5.3 $-\frac{3}{2}\mu(1) + \mu_{k,i}(L-1) + 4\mu_{k,i}(L) + \mu_{k,i}(L+1)$, the difference between the bipolar chip-synchronous AIP and the asynchronous appended code AIP.

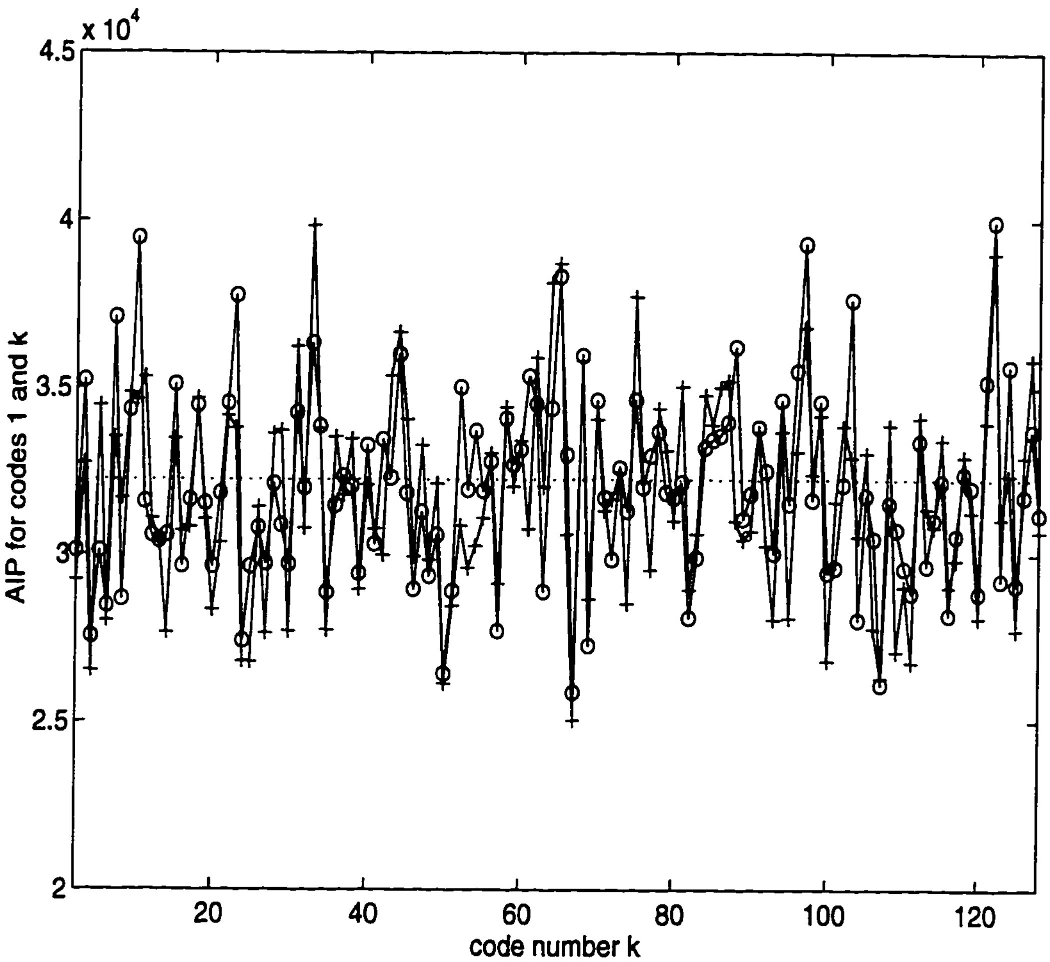


Figure 5.4 AIPs for Length 127 Gold codes for: original bipolar system (+), interleaved code system (o).

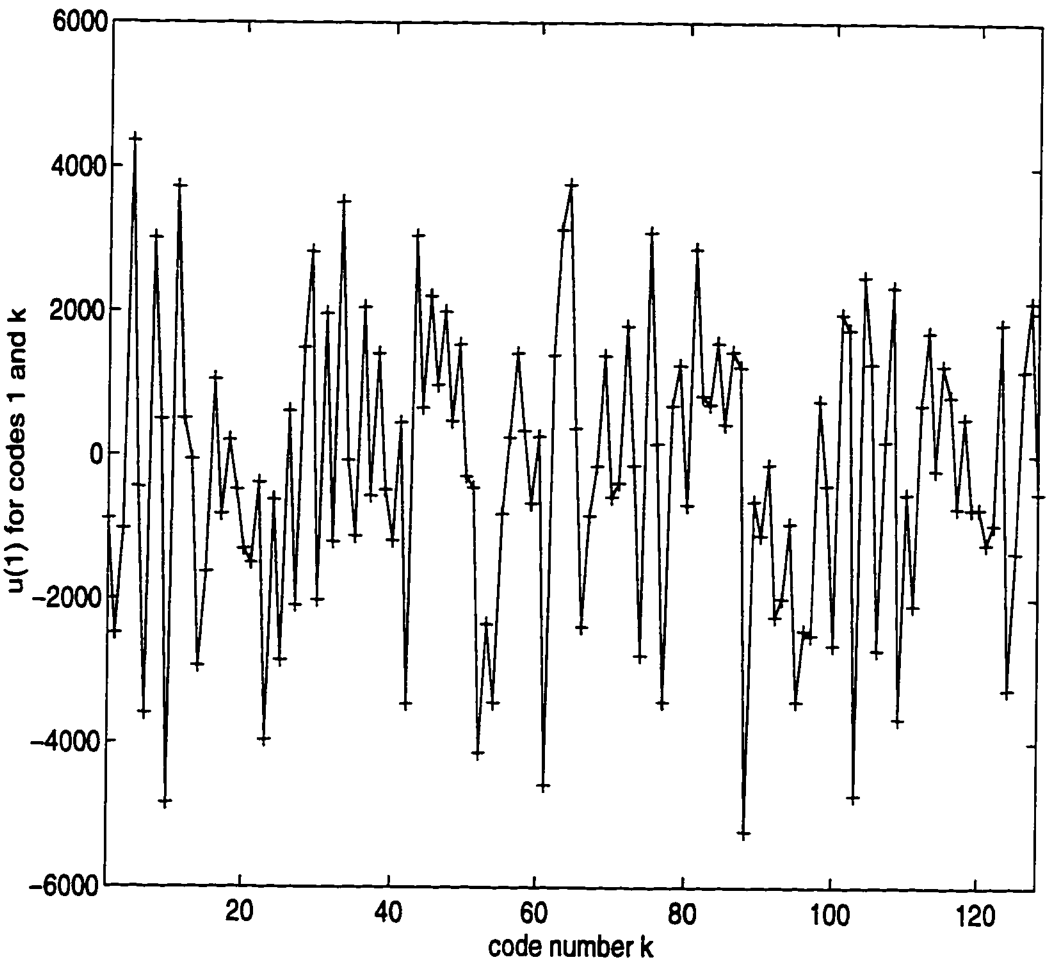


Figure 5.5 From the previous plot, $\mu_{k,i}(1)$, the difference between the bipolar AIP and the interleaved code AIP.

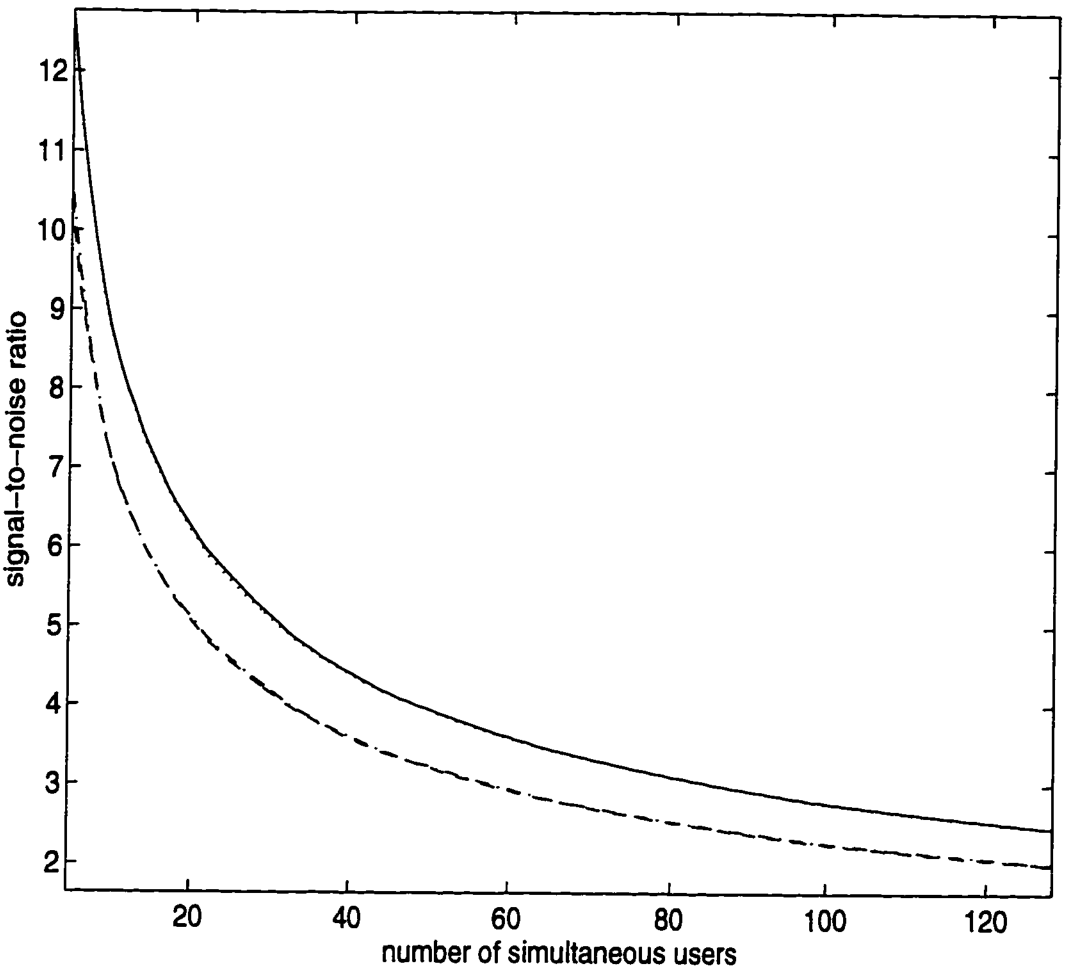


Figure 5.6 SNR vs. number of users for: original bipolar system (solid trace), interleaved code system (dotted), chip-synchronous bipolar system (dashed), and appended case (dash-dot).

5.3.1 Gaussian Approximation of the Probability of Error

Next, using the SNR, we make the somewhat controversial Gaussian approximation to get the probability of error from

$$P_E \approx Q(\text{SNR}). \tag{5.3}$$

where we have used the zero mean, unit variance error function Q.

$$Q(\text{SNR}) = \frac{1}{\sqrt{2\pi}} \int_{\text{SNR}}^{\infty} e^{\frac{-x^2}{2}} dx. \tag{5.4}$$

There has been a great deal said about the accuracy (or inaccuracy) of this approximation (see, for example, [4]). We performed a calculation of the exact probabilities of error in a worst-case situation to see for ourselves how well the approximation works. We did worst-case calculations, of course, because there are only a finite number of possible timing differences in this case, leading to much quicker results. Specifically, we computed the probability of error of the interleaved system where the only noise came from the multiple-access interference, which was a sum of terms from the discrete shifted cross-correlation. The shifts were varied uniformly over the possible discrete shifts. We counted the number of times an error was produced and compared this result to the worst-case probability of error from $Q(\text{SNR})$ where the average interference parameter in the SNR is $3\mu_{k,i}^{int}(0)$, as in Equation 3.44. Because it is not likely to be a useful system, we didn't consider the worst-case bipolar equivalent system in Chapter 4, but we can see by applying Equation 3.42 to Equation 4.15 that the average interference parameter will be

$$r_{k,i}^{int,WC} = \frac{9}{2}\mu_{k,i}(0) + \frac{3}{2}\mu_{k,i}(1). \tag{5.5}$$

Substituting this into Equation 5.2, we get the Gaussian-approximated result. Both results are compared in Figure 5.7. We can see from this data that, while there is a

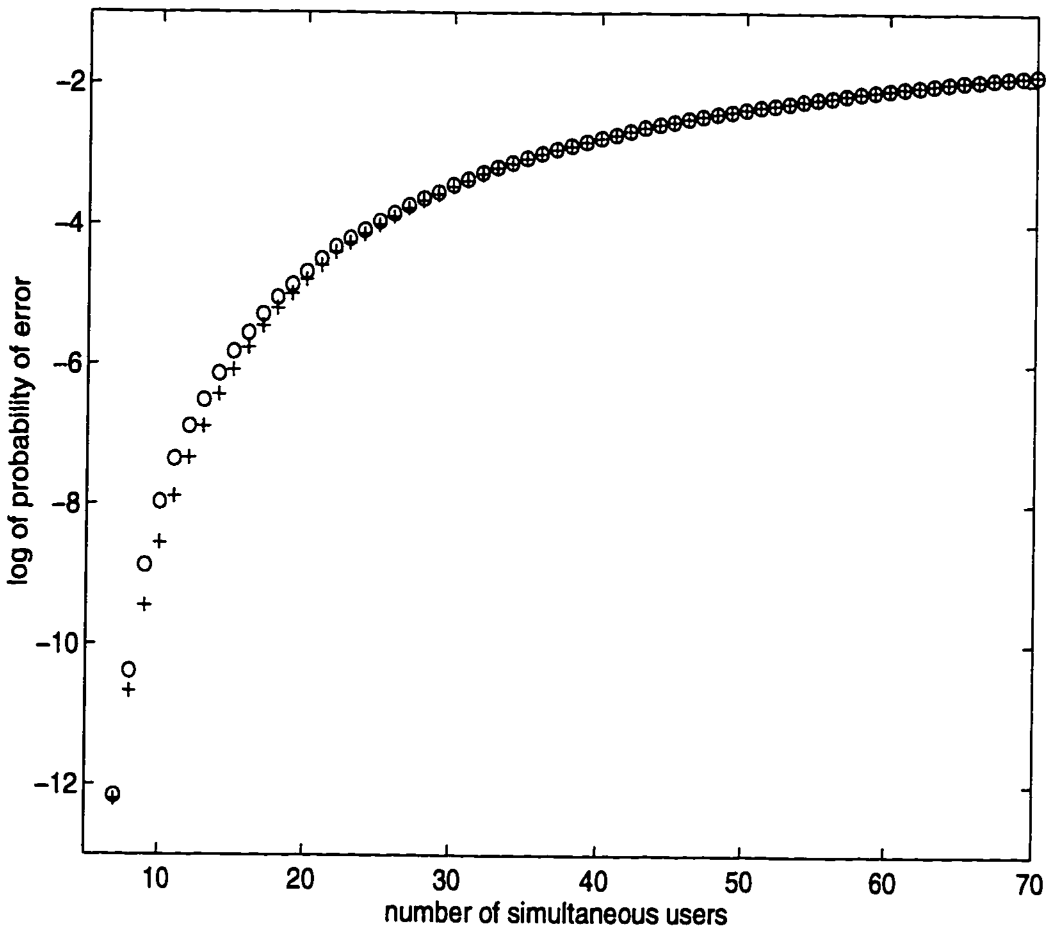


Figure 5.7 Comparison of exact probability of error (o) and Gaussian approximation (+) for the worst-case interleaved system.

noticeable difference between the exact and approximate results for low numbers of users, it is not an unreasonable approximation. Further, if we wish practically exact comparisons of systems, we need simply to look at the trends for 30 users and up, where the approximation becomes nearly perfect.

Using the Gaussian approximation now, for all four systems the probability of error versus number of simultaneous users is plotted in Figure 5.8. The plot shows that the interleaved code system actually performs very nearly the same as the original bipolar case, as we have come to expect.

Next, we try changing the i-th user's code to see what effect the specific code has on the performance. In Figure 5.9 we tried thirty different choices of the i-th code, and computed the probability for the four systems with two different numbers of interfering users. In other words, we have systems with thirty-one (or seventy-one) users transmitting simultaneously. We look at the performance of the systems based on their interference for thirty different possible choices of who to label our i-th user, and so consider a receiver with his code as reference. The bottom four traces show the results for the systems with thirty interferers, while the top four are the four systems with seventy interferers. We can see from this that the choice whether the interleaved or the bipolar performs better depends greatly on the particular user's code, and the same is true for the appended and worst-case systems. Another thing we can see is that the interleaved system's performance is actually more uniform over different codes, probably due to the lack of that $\mu(1)$ term. Also, these results are evident in the seventy-interferers plots as well, where we know the Gaussian approximation works well.

Of course, the Gaussian approximation isn't the only approximation we have made. We also chose to neglect stochastic noise sources such as source noise and detector shot noise. In the next section we will consider the latter.

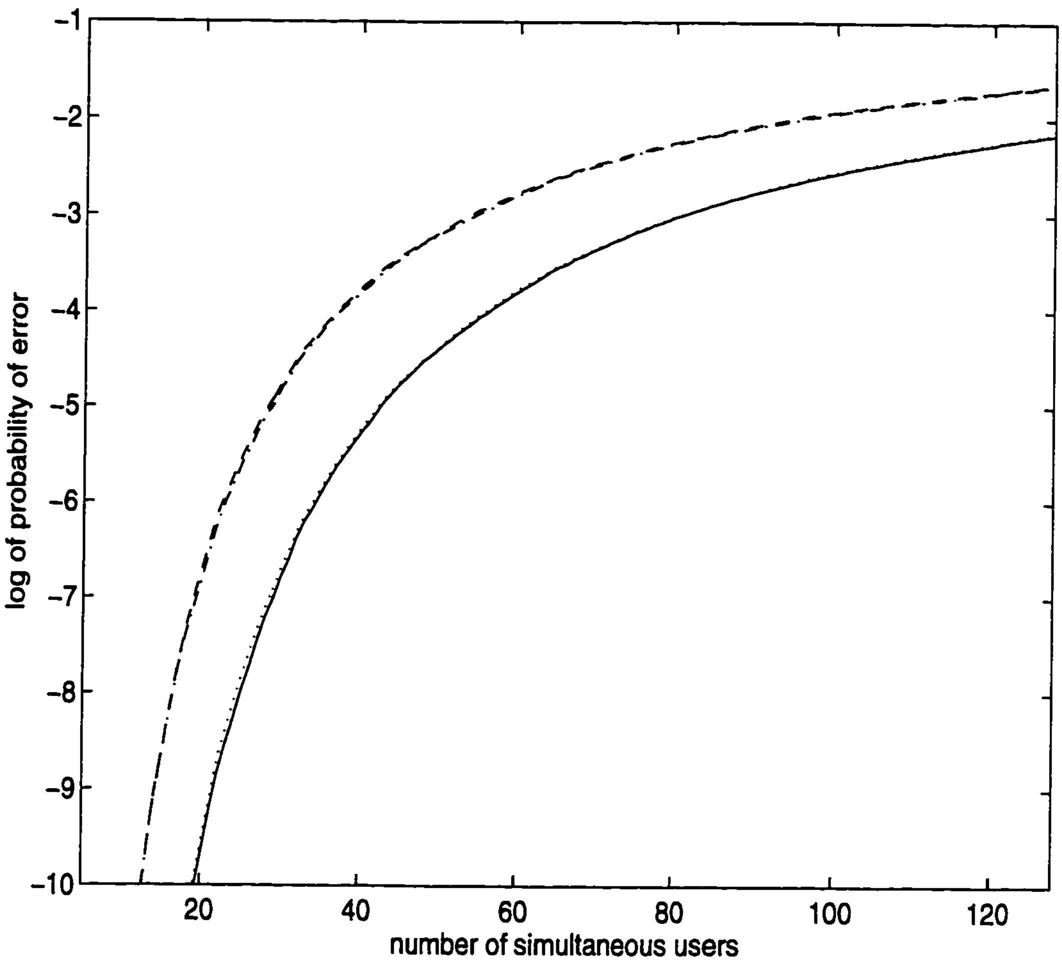


Figure 5.8 $\log_{10}(P_E)$ vs. number of users for: original bipolar system (solid trace), interleaved code system (dotted), chip-synchronous bipolar system (dashed), and appended case (dash-dot).

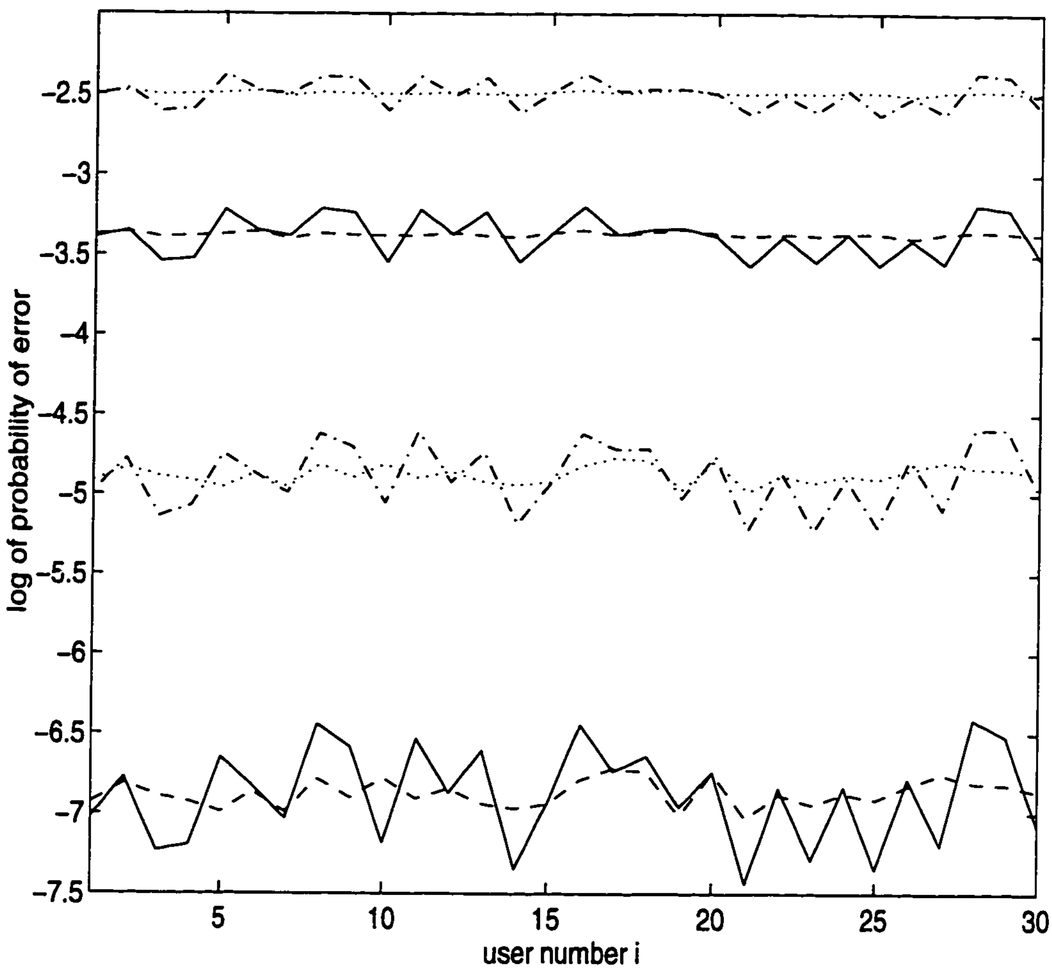


Figure 5.9 $\log_{10}(P_E)$ vs. i (as in the code a_i) with seventy users (top four traces) and thirty users (bottom four traces) all for: original bipolar system (solid trace), interleaved code system (dashed), chip-synchronous bipolar system (dash-dot), and appended case (dotted).

5.4 Implementation Considerations

In this section, we will consider a possible implementation of the asynchronous bipolar-equivalent system using a tapped-delay line correlator, and we will consider the system's performance in the presence of detector shot noise. The receiver for the system is shown in Figure 5.10. Assuming the i-th user, to whom this receiver belongs, is transmitting constant ones, or U_i's, we would expect the output of the top tapped-delay line to be something like

$$Z_{top}(T) = \frac{K}{2}(127 + \sum_k C_{U_k,U_i}(l_k)), \tag{5.6}$$

where the 127 comes from the i-th user signal, and there is interference from the other users. The K is the number of photons per chip, so this equation tells how many photons hit the detector. We have assumed a completely ideal system so far. We can write U_k and U_i in terms of bipolar codes A_k and A_i using Equation 3.2 and substitute this into Equation 5.6. Using Equation 3.19 and the fact that for the interleaved system, W_i is zero, we get

$$Z_{top}(T) = \frac{K}{2}(127 + 63.5(N-1) + \frac{1}{4}\sum_k C_{A_k,A_i}(l_k)), \tag{5.7}$$

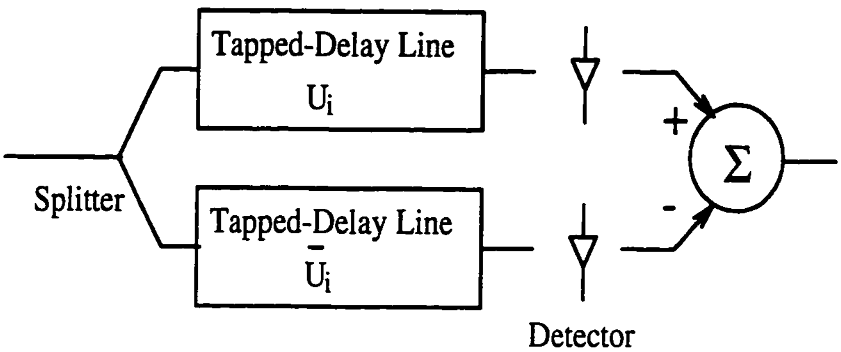


Figure 5.10 Fiber optic tapped-delay line based receiver.

where N is the total number of users. Modeling the detector noise as a Poisson process, we find the signal from our top detector to be

$$Z_{top}^{d}(T) = X_1, \tag{5.8}$$

where X_1 is a Poisson random variable with parameter λ_1.

$$\lambda_1 = \frac{K}{2}(127 + 63.5(N-1) + \frac{1}{4}\sum_k C_{A_k,A_i}(l_k)). \tag{5.9}$$

Along similar lines, the bottom detector yields

$$Z_{bot}^{d}(T) = X_2, \tag{5.10}$$

where X_2 is a Poisson random variable with parameter λ_2,

$$\lambda_1 = \frac{K}{2}(63.5(N-1) - \frac{1}{4}\sum_k C_{A_k,A_i}(l_k)). \tag{5.11}$$

The net received signal, therefore, will be

$$Z(T) = Z_{top}^{d}(T) - Z_{bot}^{d}(T) = X_1 - X_2, \tag{5.12}$$

in terms of number of detected photons.

We can approximate $X_1 - X_2$ with a Gaussian random variable with mean $\lambda_1 - \lambda_2$ and variance $\lambda_1 + \lambda_2$. This random variable simply adds a correction to the variance of our total interference in the amount $\frac{2(\lambda_1 - \lambda_2)}{K}$. So the correction decreases as expected with increasing transmission power.

In Figure 5.11 we provide the results of a simulation where the probability of error versus photon number was computed exactly in this system with 90 interfering users and compared to the Gaussian approximation just described. We see from this plot that the interleaved system continues to be superior to the appended system in the presence of increasing shot noise. We can also see that the Gaussian approximation works very well.

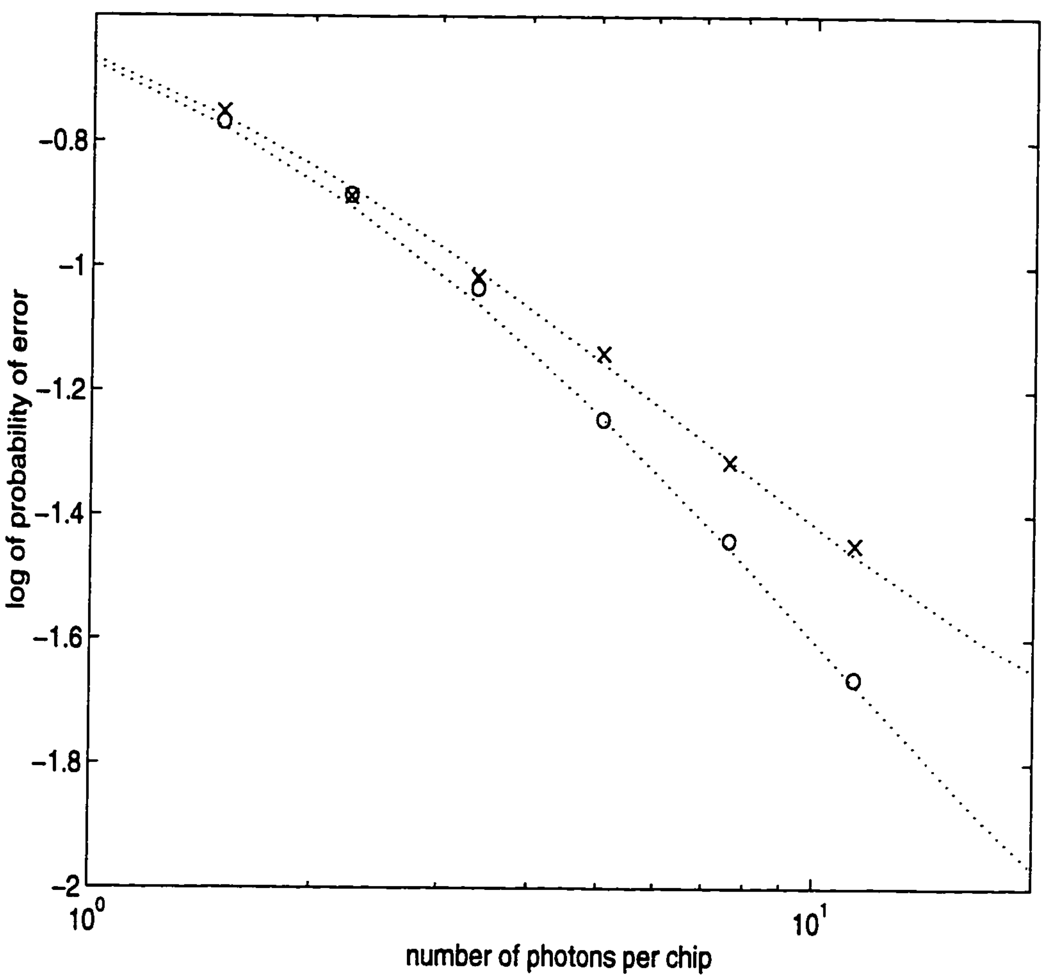


Figure 5.11 $\log_{10}(P_E)$ vs. number of photons per chip for: Appended system ($\times$), Interleaved system (o). The top and bottom dotted lines give the Gaussian approximated result for the two respective cases.

The probability of error in the interleaved system as transmission power increases for four different numbers of interfering users is plotted in Figure 5.12. The powers here range from 1 photon per chip to 10,000, which corresponds roughly to one microwatt and one milliwatt total power per bit respectively in a $1.55\mu m$ wavelength system transmitting chips at 10 GHz. We can see from this plot that the correction effect is insignificant above 100 photons, or a few hundred microwatts suggesting that the shot noise effect is completely negligible in a normal system.

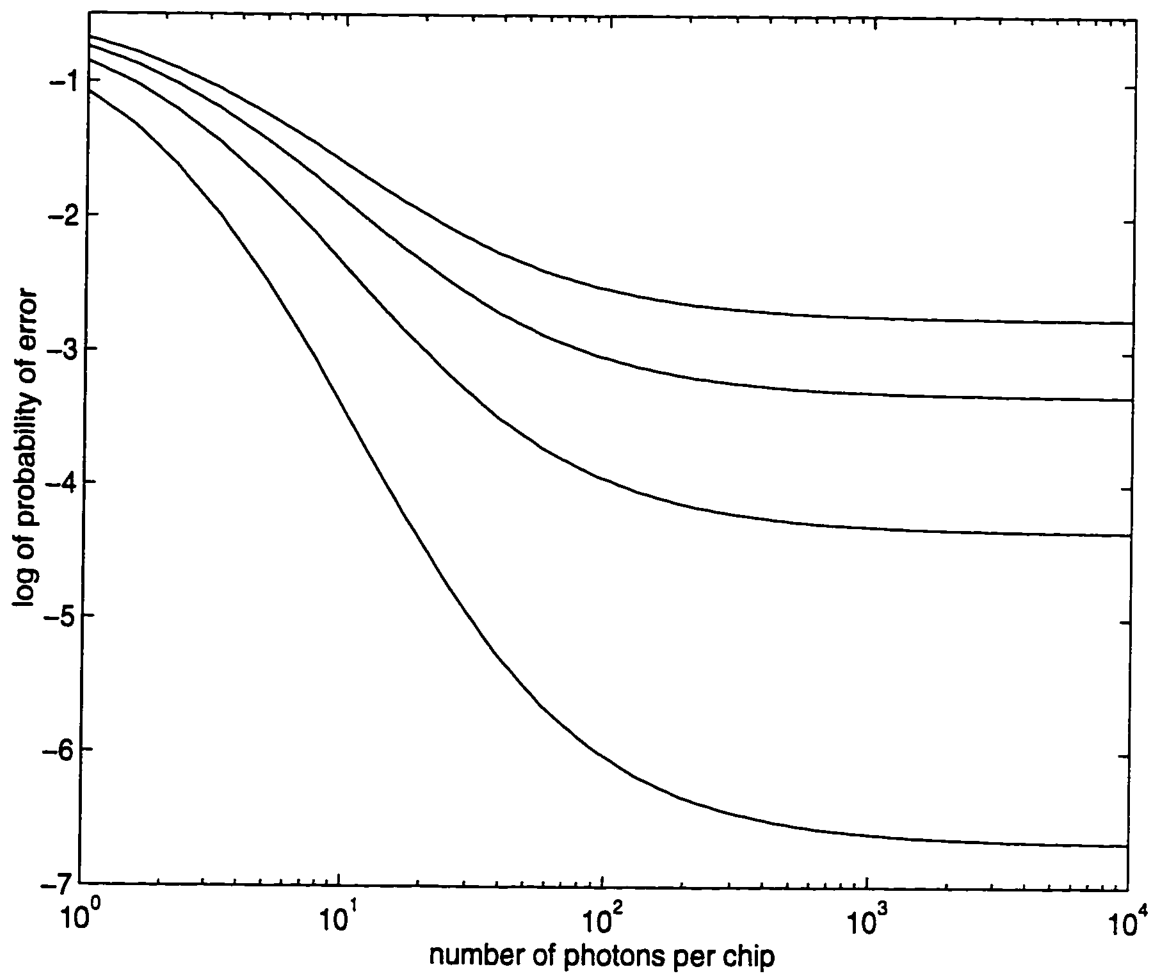


Figure 5.12 $\log_{10}(P_E)$ vs. number of photons per chip for the interleaved system with, from top to bottom: 90 users, 70 users, 50 users, 30 users.

Chapter 6

Conclusion

The goal of this work was to produce a time-encoded bipolar-equivalent system which gave us performance equivalent to the bipolar system, allowing, of course, for a factor of two increase in bandwidth requirement. We showed that, not only could this be achieved, but that it could be achieved in a system that is asynchronous when using the interleaved method. In this thesis, on the whole, we performed a complete analysis of the behavior of the two different asynchronous bipolar-equivalent CDMA methods we (and others) have found. We found that the cross-correlation functions for the interleaved codes could be represented very simply in terms of the CCF's of the original codes. This led directly to several results such as description of the exact values of the odd terms instead of just bounds as had been presented in [9]. Knowing this, we were further able to compute the average interference parameter for interleaved codes and compute the signal-to-noise ratio and approximate the probability of error, and compare these with other methods of achieving bipolar-equivalent CDMA.

We also discussed the difference between bipolar and unipolar systems, the $\mathbf{1}_k$ offset term in the codes which can pose problems depending on W_i, the weight of the code. All of this comes from the use of the matched-filter detector which outputs a correlation between the input and reference code. W_i is, after all, just the CCF between $\mathbf{1}_k$ and the reference code a_i. We found that while the W_i term may pose problems in the generic unipolar transmission/bipolar reference system, it becomes zero for the bipolar-equivalent systems.

6.1 Ramifications of our Results

Using the interleaving method, we are now able to build a time-encoded unipolar CDMA system which provides performance almost identical to that of the well-known bipolar asynchronous system. The only price we have paid for this is the increase in bandwidth requirement by a factor of two. In addition, as we showed in Chapter 2, our system can be implemented in a very simple, completely asynchronous, unipolar transmission/bipolar reference system.

6.2 Future Work

One might well question the reasons behind the choice of particular asynchronous bipolar equivalent coding methods analyzed in this thesis. We basically looked at the interleaved case, the method several other researchers are considering, and the appended case, the method used in spectrally-encoded systems. While the interleaved method is worth investigating for the reason just given, and in hindsight, it performs probably as well as can be done, might there be some other strange combination method which works better? We doubt it. Our result is, on average, as good as the best possible result. Of course, since our codes are doubly redundant, there could also be some kind of compression done to reduce the total code length, but it would probably require complicating our receiver, and may not be practical to implement.

Another interesting idea comes from the result concerning cross-correlations [11] where we replace codes $a_{i,n}$ and $a_{k,n}$ with $(-1)^n a_{i,n}$ and $(-1)^n a_{k,n}$ respectively. The CCF of these two will then be $(-1)^l C_{i,k}(l)$. Recall that the interleaved system produced a CCF that was almost a linearly interpolated version of the bipolar CCF except that the interpolated points were the negative of the actual linear interpolation. If we were to modulate the interleaved system's output with $(-1)^l$, we could get back the original linearly interpolated bipolar result. This therefore, would be equivalent to modulating the incoming data from the channel with $(-1)^n$, and doing the same to the reference code. Since linear interpolation can be considered a crude form of lowpass filtering (a filter with spectral response $\mathrm{sinc}^2(\omega)$) this idea has relevance to bandwidth reduction of the receiver.

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