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Robust Steady-State Tracking

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Abstract

The thesis solves the problem of finding an LTI controller that minimizes the steady-state tracking error of uncertain discrete-time systems. If a system is LTI, use of the "internal model principle" will assure that the error signal converges to zero. But this principle no longer applies when the physical plant is time-varying. This leads to the problem of how the steady-state value of the tracking error can be minimized in the time-varying case. The solution is provided in the following mathematical setting. The nominal model of the plant is SISO and LTI, and plant uncertainty is modeled by an arbitrary fading memory operator that is SISO, linear time-varying, and norm-bounded. A special case of the more general n-perturbation problem is also solved.

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To my beloved parents:

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and

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Chapter 1

Introduction

1.1 Background and Motivation

Research in linear control theory in the past two decades has been characterized by the recognition of the necessity to account for uncertainty in the plant. There is always a discrepancy between a physical system and its model, and therefore a controller that is designed for the model may not perform well with the physical plant. This problem may be addressed by designing the controller after the uncertainty itself has been modeled.

The dominant paradigm for the modeling of plant uncertainty has been the incorporation of norm-bounded perturbations. For example, if P is a linear time-invariant operator representing a nominal model, the physical system may be assumed to lie in the class of admissible systems:

$$\{P_\Delta : P_\Delta = P + \Delta, \quad \|\Delta\| \leq 1\}$$

where the Δ is an operator belonging to a pre-specified class, and $\|\Delta\|$ represents its induced norm. Uncertainty is thus accounted for in the arbitrariness of Δ, and is restricted by the the maximum allowable induced norm of the same. One then seeks to design a controller that performs well for every plant in the above set. More generally, uncertainty may be introduced at more than one point within the system, so that there are several Δ's. One may lump these uncertainty perturbations in a single *diagonal* perturbation operator, thus representing structured uncertainty.

The two primary goals in controller synthesis are maintaining stability and meeting performance specifications. Stability means that injection of bounded signals should produce only bounded signals within the system. Attaining stability in the face of all admissible perturbations is *robust stability.* An easily computable necessary and sufficient condition is available [KhPrsn90,KhPrsn91,KhPrsn93,DahBob95] for the robust stability of discrete-time systems when the signals are in ℓ_∞ and the perturbations are linear time-varying and bounded in the ℓ_∞-induced norm, namely the ℓ_1 norm. The same condition applies when the perturbations are in addition "fading horizon," meaning that their output eventually vanishes for vanishing inputs [Kham95].

Robust performance is defined for systems that are robustly stable. It means keeping some signals in the system "small" in the face of arbitrary disturbance signals which are all bounded above in the ℓ_∞ norm, the choice of this norm indicating that one is contending with disturbances that are persistent in time. When the signals are in ℓ_∞, the robust performance problem reduces to a robust stability problem. This holds for both of the perturbation classes that were mentioned in the last paragraph [KhPrsn91, KhPrsn93, Kham95].

A special performance criterion is *robust tracking.* It arises when the goal is for the system to track a fixed signal. The tracking is formulated as keeping an error signal small. The smallness of this signal can be specified in terms of different measures. One may choose the ℓ_∞ norm of the error signal. This would correspond to keeping the maximum deviation below a certain level over all time. This thesis, however, is concerned with the maximum deviation at the steady-state. The associated problem is called "robust steady-state tracking."

Recently [Kham95] has established analysis results that constitute necessary and sufficient conditions for a SISO system with multiple uncertain perturbations to achieve steady-state tracking. Furthermore, given a plant and a controller, these results enable one to easily compute the maximum steady-state error over the set of admissible perturbations.

What has not been hitherto addressed in the literature is the synthesis aspect of the problem, and that is the subject of this thesis. We will provide a solution to the problem of minimizing the worst-case steady-state tracking error (SSTE) over the set of stabilizing controllers.

The thesis is organized as follows. Chapter 2 provides the necessary background information. Chapter 3 poses the optimization problem in the case of a single perturbation, provides a solution, and proves that the optimal closed loop map is "FIR," i.e. has a "finite impulse response." It also shows how the method can be used to minimize the worst-case steady-state error subject to an upper bound on the norm of worst-case sensitivity. Chapter 4 provides two examples. The first example is of a plant for which the optimal SSTE is smaller than what can be obtained by inclusion of an internal model. The second example demonstrates that plants with optimally small SSTE's may have arbitrarily large worst-case sensitivity norms. Chapter 5 treats a special case of the more general n-perturbation SSTE minimization problem. Chapter 6 serves as a conclusion and points to issues that require further research.

1.2 Notation, Terminology, and Abbreviations

$\mathbf{Z}^+$ Set of nonnegative integers: $\{0, 1, 2, \cdots\}$.

$\mathbf{N}$ Set of natural numbers: $\{1, 2, 3, \cdots\}$.

$\mathbb{R}$ Set of real numbers.

$\mathbb{R}^+$ Set of nonnegative real numbers.

ℓ_∞	Space of all bounded sequences of real numbers, i.e. $x = \{x(k)\}_{k=0}^{\infty} \in \ell_\infty$ if and only if $\sup_k |x(k)| < \infty$. If $x \in \ell_\infty$ then $\|x\|_\infty = \sup_k |x(k)|$.
ℓ_1	Space of absolutely summable sequences. If $x \in \ell_1$ then $\sum_{k=0}^{\infty} |x(k)| < \infty$.
c_0	Subspace of ℓ_∞ of vanishing signals, i.e. sequences $x \in \ell_\infty$ which satisfy $x(k) \to 0$ as $k \to \infty$.
$<, \leq$	All vector and matrix inequalities are element-wise. Thus if $x, y \in \mathbb{R}^n$, $x \leq y$ means that $x_i \leq y_i$ for $i = 1, \cdots, n$.
$|.|$	Absolute value, applied element-wise when the argument is a vector.
$\|.\|_\infty$	See ℓ_∞.
$\|.\|$	Same as $\|.\|_1$.
$\|.\|_1$	ℓ_1 norm of a sequence. Given a sequence x, $\|x\|_1 = \sum_{k=0}^{\infty} |x(k)|$. Also, given a linear operator $T : \ell_\infty \to \ell_\infty$, $\|T\|_1$ denotes the ℓ_1 norm of the impulse response of T, and is also the same as its ℓ_∞ induced norm.
$\|.\|_{ss}$	Steady-state semi-norm. If $e \in \ell_\infty$, $\|e\|_{ss} := \lim_{k\to\infty} \sup_{i \geq k} |e(i)|$.
$\underline{\Delta}$	Set of all linear, causal operators mapping ℓ_∞ into itself, with induced ℓ_∞ norm less than or equal to one. Hence, $$\underline{\Delta} := \left\{ \Delta : \ell_\infty \to \ell_\infty : \sup_{x \neq 0} \frac{\|\Delta x\|_\infty}{\|x\|_\infty} \leq 1 \right\}.$$
$\underline{\Delta}_F$	The restriction of $\underline{\Delta}$ to fading memory operators. Thus $$\underline{\Delta}_F := \{\Delta \in \underline{\Delta};\ \Delta : c_0 \to c_0\}.$$

$\mathcal{D}(n)$ Set of all diagonal operators of the form $D = \mathrm{diag}(\Delta_1, \ldots, \Delta_n)$ where $\Delta_i \in \underline{\Delta}$.

$\mathcal{D}_F(n)$ Set of all diagonal operators of the form $D = \mathrm{diag}(\Delta_1, \ldots, \Delta_n)$ where $\Delta_i \in \underline{\Delta}_F$.

$\rho(.)$ Spectral radius.

λ-transform The complex variable of the λ-transform. See next item.

$\hat{}$ If Φ represents the impulse response of an LTI operator $T : \ell_\infty \longrightarrow \ell_\infty$, then $\hat{\Phi}$ is the λ-transform, $\hat{\Phi} := \sum_{i=0}^{\infty} \Phi_i \lambda^i$. Note that λ-transform is defined differently from what is normally called the Z-transform, in that $\lambda = z^{-1}$. Thus the unstable region is the unit disk, $|\lambda| \leq 1$.

$\widehat{}$ If M is a non-scalar stable operator with elements mapping l_∞ into itself, then $\hat{M}$ is obtained by replacing each element of M by the ℓ_1 norm of its impulse response.

$*$ Convolution operator.

FIR Finite impulse response.

LTI Linear time-invariant.

SSTE Worst-case steady-state tracking error semi-norm.

SISO Single-input, single-output.

causal Given a sequence x, let P_k, $k \in Z^+$, be the truncation operator, i.e. $P_k(x(0), x(1), \cdots) = (x(0), x(1), \cdots, x(k), 0, 0, \cdots)$. Then we say that an operator T is causal if $P_t T = P_t T P_t$ for all t.

fading memory An operator that maps c_0 into c_0.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we will give a summary of the results corresponding to the problem of finding the maximum error of a system at the steady-state, where the maximum is taken over the set of admissible perturbations. This quantity is properly referred to as the "worst-case steady-state tracking error." We will assume that we have a nominal plant with a stabilizing controller, that the admissible perturbations are SISO, linear, causal, and fading memory operators, and that the goal is to track a fixed signal.

This chapter treats an "analysis" problem, as it assumes that a nominal plant, a controller, and a set of perturbations are given, and that the goal is to assess the performance of the system at the steady-state. This problem was solved in [Kham95] in the case of multiple SISO perturbations ("structured uncertainty").

Before dealing with robust steady-state tracking, however, it is necessary to define robust stability and robust performance, and to state the corresponding analysis theorems and formulas.

2.2 Robust Stability

The robust stability problem is posed in terms of Figure 2.1. Here $\mathbf{M}$ represents a stable, linear time-invariant nominal system, including any nominal plants and controllers. Structured uncertainty is incorporated into Δ, whose diagonal elements

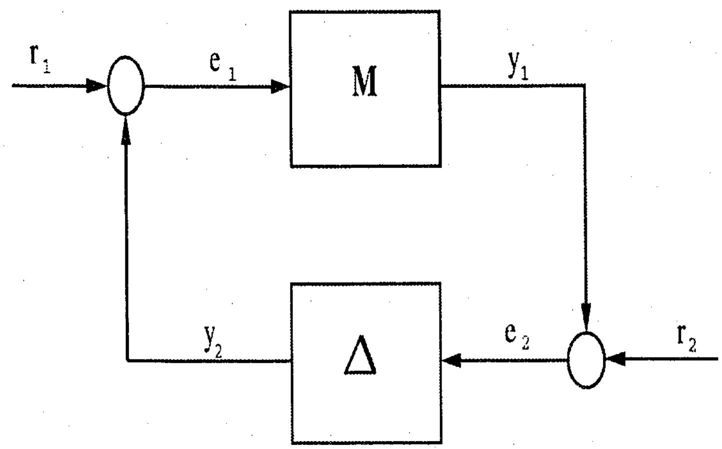

Figure 2.1: Robust Stability

represent the individual SISO perturbations within the system. Thus Δ is an arbitrary member of the set:

$$\mathcal{D}_F(n) := \{\text{diag}(\Delta_1, \cdots, \Delta_n) : \Delta_i \in \underline{\Delta}_F\}$$

where $\underline{\Delta}_F$; which represents the class to which the individual norm-bounded perturbations, Δ_i , belong; is defined to be the restriction of

$$\underline{\Delta} := \{\Delta : \ell_\infty \to \ell_\infty;\ \Delta \text{ is linear, causal and } \|\Delta\|_1 \leq 1\}$$

to fading memory operators, i.e. those operators that map the set of vanishing ℓ_∞ signals, c_0, into itself. In other words,

$$\underline{\Delta}_F := \{\Delta : c_0 \to c_0;\ \Delta \text{ is linear, causal and } \|\Delta\|_1 \leq 1\}$$

In summary, uncertainty is represented in the form of linear perturbations Δ_i within the system which are causal, SISO, norm-bounded, and fading memory. Note that they may be time-varying. These perturbations are selected from wherever they appear in the system and lumped together as the diagonal elements of a single operator called Δ. For this class of perturbations, robust stability is defined as follows.

Definition 2.1 *(Robust Stability: Fading Memory Perturbations)* The system in Figure 2.1 is robustly stable if for every $\Delta \in \mathcal{D}_F(n)$, (e_1, e_2, y_1, y_2) is bounded whenever (r_1, r_2) is bounded.

Thus, robust stability means that for all possible perturbations, all signals within the system remain bounded upon the injection of bounded signals into the system. The following theorem gives a simple necessary and sufficient condition for robust stability in terms of the ℓ_1 norms of the entries of the stable matrix $\mathbf{M}$.

Theorem 2.1 *(Robust Stability: Fading Memory Perturbations [Kham95])* The system in Figure 2.1 is robustly stable if and only if $\rho(\widehat{\mathbf{M}}) < 1$, where $\rho(.)$ is the spectral radius, and

$$\widehat{\mathbf{M}} := \begin{pmatrix} \|M_{11}\|_1 & \|M_{12}\|_1 & \cdots & \|M_{1,n}\|_1 \\ \|M_{21}\|_1 & \|M_{22}\|_1 & \cdots & \|M_{2,n}\|_1 \\ \vdots & \vdots & \vdots & \vdots \\ \|M_{n,1}\|_1 & \|M_{n,2}\|_1 & \cdots & \|M_{n,n}\|_1 \end{pmatrix}$$

Robust stability is defined similarly when Δ belongs to the following class of perturbations:

$$\mathcal{D}(n) := \{\mathrm{diag}(\Delta_1, \cdots, \Delta_n) : \Delta_i \in \underline{\Delta}\}$$

In fact, historically this class was treated before $\mathcal{D}_F(n)$. It turns out [Kham95] that $\rho\ (\hat{\mathbf{M}}) < 1$ is also a necessary and sufficient condition for robust stability against perturbations in $\mathcal{D}(n)$. Therefore, we have:

Corollary 2.1 *([Kham95])* A system is robustly stable against perturbations in $\mathcal{D}_F(n)$ if and only if it is robustly stable against perturbations in $\mathcal{D}(n)$.

2.3 Robust Performance

Besides stability, an important concern is achieving good performance, which is usually stated in terms of keeping some signals in the system small. We will define two performance criteria: robust ℓ_1-performance in this section (henceforth, simply "robust performance"), and robust steady-state tracking with a fixed input in the next section.

The robust performance problem is formulated in terms of Figure 2.2. **M** is stable, linear, and time-invariant, representing the nominal part of the system, including any nominal plants and controllers. The uncertainty is represented by Δ. It belongs to $\mathcal{D}(n)$ or $\mathcal{D}_F(n)$ as the case may be. Thus, its off-diagonal elements are zero, and its diagonal elements represent the individual norm-bounded SISO perturbations within the system. The output is e, while $r \in \ell_\infty$ represents the disturbance.

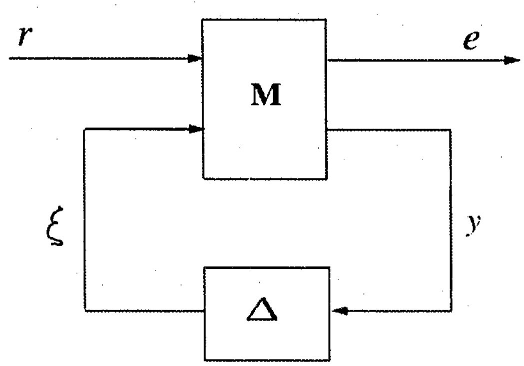

Figure 2.2: Robust Performance

The goal is to keep the effect of disturbance, r, on the output, e, small. Therefore, if T_{er} is the map that takes r to e, we would like the peak gain

$$\|T_{er}\| := \sup_{r \neq 0} \frac{\|e\|_\infty}{\|r\|_\infty}$$

to be small. If, for example, we achieve $\|T_{er}\| \leq \epsilon$, that would mean that as long as the persistent disturbance, $|r(k)|$, is bounded above by 1 over all time, the error magnitude $|e(k)|$, will be bounded above by ϵ over all time.

Moreover, we would like $\|T_{er}\|$ to be small in spite of the existing plant uncertainty. This gives rise to the notion of robust performance:

Definition 2.2 *(Robust Performance: Fading Memory Perturbations)* Suppose the system in Figure 2.2 is robustly stable against fading memory perturbations. Let T_{er} be as defined above. Let $\Delta \in \mathcal{D}_F(n)$. Then system is said to achieve robust performance if

$$\sup_{\Delta \in \mathcal{D}_F(n)} \|T_{er}\| < 1$$

Robust performance against perturbations in $\mathcal{D}(n)$ is defined similarly, except that $\mathcal{D}_F(n)$ is replaced throughout with $\mathcal{D}(n)$.

The following significant result [KhPrsn91, KhPrsn93] states that when $\Delta \in \mathcal{D}$, the robust performance problem is equivalent to a certain robust stability problem.

Theorem 2.2 *([DahBob95])* If $\Delta \in \mathcal{D}$, then the system in Figure 2.2 achieves robust performance if and only if the one in Figure 2.3 is robustly stable with $\Delta_p \in \underline{\Delta}$.

An important question is whether a similar equivalence holds when the class of perturbations is $\mathcal{D}_F$. Enough background material has already been presented to enable us to give an affirmative answer.

Corollary 2.2 *([Kham95])* The system in Figure 2.2 achieves robust performance against fading memory perturbations if and only if that of Figure 2.3 is robustly stable against fading memory perturbations.

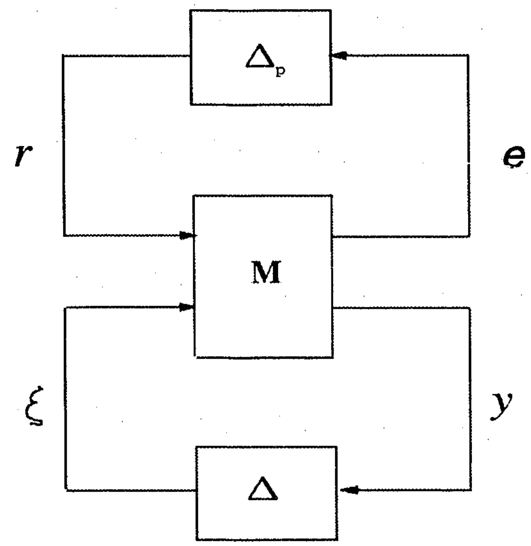

Figure 2.3: Robust Performance reduced to Robust Stability

Proof Suppose the system in Figure 2.2 achieves robust performance against fading memory perturbations ($\Delta \in \mathcal{D}_F$). Then, by the Small Gain Theorem it follows that the system in Figure 2.3 is robustly stable with respect to fading memory perturbations ($\Delta \in \mathcal{D}_F$ and $\Delta_p \in \underline{\Delta}_F$). Conversely, if the system in Figure 2.3 is robustly stable against fading memory perturbations, by Corollary 2.1 it is also robustly stable against $\Delta \in \mathcal{D}$ and $\Delta_p \in \underline{\Delta}$, which by Theorem 2.2 implies that the system in Figure 2.2 achieves robust performance against perturbations in $\mathcal{D}$. Since $\mathcal{D}_F \subset \mathcal{D}$, it also achieves robust performance against perturbations in $\mathcal{D}_F$. $\square$

During the rest of the thesis, we will be concerned only with fading memory perturbations. Thus, henceforth $\Delta \in \mathcal{D}_F$ and $\Delta_p \in \underline{\Delta}_F$.

The fact that robust performance has been reduced to robust stability along with Theorem 2.1 allows us to write down a simple necessary and sufficient condition for robust performance. Corresponding to the system in Figure 2.3 we have:

$$\mathbf{M} := \begin{pmatrix} M_{11} & \dots & M_{1,n+1} \\ \vdots & \ddots & \vdots \\ M_{n+1,1} & \dots & M_{n+1,n+1} \end{pmatrix}$$

By Theorem 2.1 the system achieves robust performance if and only if $\rho(\widehat{\mathbf{M}}) < 1$, where the "hat" on top indicates here and elsewhere in this chapter that each element of the LTI, stable matrix operator is replaced by its ℓ_1 norm. Next, we partition $\mathbf{M}$ as follows:

$$\begin{pmatrix} e \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{M_{11}} & \mathbf{M_{12}} \\ \mathbf{M_{21}} & \mathbf{M_{22}} \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} r \\ \xi \end{pmatrix}$$

Thus, $\mathbf{M_{11}}$ is 1×1, $\mathbf{M_{12}}$ is $1 \times n$, $\mathbf{M_{21}}$ is $n \times 1$, and $\mathbf{M_{22}}$ is $n \times n$.

Lemma 2.1 *([Kham94])* Let A be an $n+1 \times n+1$ nonnegative matrix where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and A_{11} is 1×1, A_{22} has dimension $n \times n$, and A_{12} and A_{21} have the appropriate dimensions. Then $\rho(A) < 1$ if and only if $\rho(A_{22}) < 1$ and $A_{11} + A_{12}(I - A_{22})^{-1}A_{21} < 1$.

Thus, the lemma implies that $\rho(\widehat{\mathbf{M}}) < 1$ if and only if

$$\rho(\widehat{\mathbf{M}}_{22}) < 1$$

$$\widehat{\mathbf{M}}_{11} + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1}\widehat{\mathbf{M}}_{21} < 1$$

The first condition is equivalent to robust stability. From these, we can obtain a formula for the worst-case sensitivity.

Corollary 2.3 *([Kham94])* If the system in Figure 2.2 is robustly stable, then its worst-case sensitivity norm is:

$$\sup_{\Delta \in \mathcal{D}_F} \|T_{er}\| = \widehat{\mathbf{M}}_{11} + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1}\widehat{\mathbf{M}}_{21}$$

Proof The system is robustly stable and $\sup_{\Delta \in \mathcal{D}_F} \|T_{er}\| < \gamma$ if and only if the new system obtained by scaling the signal e by $\frac{1}{\gamma}$ achieves robust performance. The latter is equivalent to:

$$\rho(\begin{pmatrix} \frac{1}{\gamma}\widehat{\mathbf{M}}_{11} & \widehat{\mathbf{M}}_{12} \\ \frac{1}{\gamma}\widehat{\mathbf{M}}_{21} & \widehat{\mathbf{M}}_{22} \end{pmatrix}) < 1$$

which by the preceding lemma and remarks holds if and only if

$$\widehat{\mathbf{M}}_{11} + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1}\widehat{\mathbf{M}}_{21} < \gamma$$

Therefore,

$$\sup_{\Delta \in \mathcal{D}_F} \|T_{er}\| = \widehat{\mathbf{M}}_{11} + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1}\widehat{\mathbf{M}}_{21}$$

□

2.4 Robust Steady-State Tracking

The tracking problem is formulated in terms of Figure 2.2. The situation is analogous to that of robust performance, but with two differences: First, the signal r is assumed to be fixed, i.e. known in advance. Secondly, the objective is to keep small not the peak magnitude of e, but its deviation at the steady-state. The signal r could, for example, be a step input we wish to track, and e could represent the tracking error.

To have a measure of the tracking error, we define the *steady-state semi-norm*, defined in the following fashion:

$$\|e\|_{ss} := \lim_{k\to\infty} \sup_{i\geq k} |e(i)|$$

Note that the semi-norm is well-defined for signals that do not have a limit as $k \to \infty$, such as sinusoids. Thus, this measure of the steady-state error can be thought of as "the maximum amplitude at infinity." The objective is to keep it small in the face of all possible perturbations. This gives rise to the following definition:

Definition 2.3 The system in Figure 2.2 is said to achieve robust steady-state tracking if it is robustly stable and if

$$\sup_{\Delta\in\mathcal{D}_F} \|e\|_{ss} < 1$$

In the case of the robust performance problem, as discussed in the last section, the analysis formula is in terms of the matrix $\widehat{\mathbf{M}}$, which is in turn composed of the ℓ_1 norms of the elements of $\mathbf{M}$. An analogous situation holds in the problem considered in this section. The difference, however, is that we will now have a combination of ℓ_1 norms and steady-state semi-norms of the elements of $\mathbf{M}$.

Theorem 2.3 *([Kham95])* The system achieves robust steady-state tracking if and only if $\rho(M_{SS}) < 1$, where $\rho(.)$ is the spectral radius, and

$$M_{SS} := \begin{pmatrix} \|M_{11}r\|_{ss} & \|M_{12}\|_1 & \cdots & \|M_{1,n+1}\|_1 \\ \|M_{21}r\|_{ss} & \|M_{22}\|_1 & \cdots & \|M_{2,n+1}\|_1 \\ \vdots & \vdots & \vdots & \vdots \\ \|M_{n+1,1}r\|_{ss} & \|M_{n+1,2}\|_1 & \cdots & \|M_{n+1,n+1}\|_1 \end{pmatrix}$$

Note that this matrix is identical to $\widehat{\mathbf{M}}$ except in its first column.

We next partition M_{SS} as follows:

$$M_{SS} = \begin{pmatrix} M_{SS_{11}} & \widehat{\mathbf{M}}_{12} \\ M_{SS_{21}} & \widehat{\mathbf{M}}_{22} \end{pmatrix}$$

where $M_{SS_{11}}$ is 1×1 and the other block elements have the appropriate dimensions. Applying Lemma 2.1 to Theorem 2.3 gives the following.

Corollary 2.4 *([Kham94])* The system achieves robust steady-state tracking if and only if:

$$\rho(\widehat{\mathbf{M}}_{22}) < 1$$

$$M_{SS_{11}} + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1} M_{SS_{21}} < 1$$

Again, the first condition is equivalent to robust stability. The corollary allows us to get a formula for the worst-case steady-state error.

Corollary 2.5 *([Kham94])* If the system is robustly stable, then its worst-case steady-state error is given by:

$$\sup_{\Delta \in \mathcal{D}_F} \|e\|_{ss} = M_{SS_{11}} + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1} M_{SS_{21}}$$

Proof The proof is similar to that of Corollary 2.3. □

Having the above formula for the tracking error paves the way for the latter's minimization.

This completes the exposition of the relevant results in the literature that pertain to robustness *analysis* with time-varying perturbations. We will henceforth focus on the primary objective of the thesis, namely the *synthesis* problem.

Chapter 3

Controller Synthesis with Single Perturbation

3.1 Introduction

The last chapter presented the analysis results in [Kham95] corresponding to the robust steady-state tracking problem with multiple SISO perturbations. In this chapter we specialize to a single perturbation and address the synthesis problem, i.e. the problem of finding a controller which optimizes the worst-case steady-state tracking error semi-norm (SSTE). We will present a method for obtaining the optimal closed loop transfer function.

3.2 Statement of the Problem

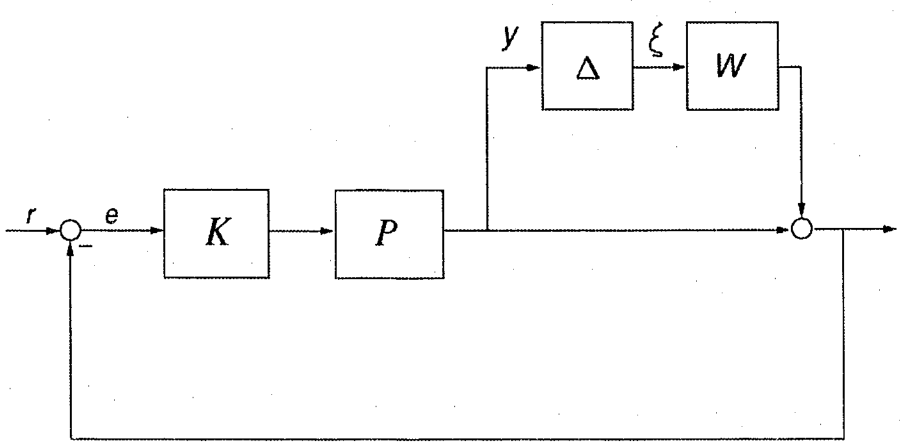

Figure 3.1: Single Perturbation: Multiplicative Uncertainty

The problem is posed in terms of Figure 3.1. P is the λ-transform of the SISO, LTI nominal plant; K of the LTI controller; and W of a stable weighting function. The fixed signal r is a unit step input. Uncertainty is represented by the SISO $\Delta \in \underline{\Delta}_F$. Thus, the physical plant is assumed to lie in the set parameterized by $P(1+W\Delta)$. This is called "multiplicative uncertainty." (Were we to choose "additive uncertainty" instead, i.e. $P+W\Delta$, we would obtain results similar to what will follow.) The system $\mathbf{M}$, as defined in Section 2.3 maps $(r \quad \xi)'$ to $(e \quad y)'$. It is straightforward to show that

$$\mathbf{M} = \begin{pmatrix} S & -SW \\ T & -TW \end{pmatrix}$$

where S and T are respectively the sensitivity and complimentary sensitivity functions:

$$S(\lambda) = \frac{1}{1+PK} \quad \text{and} \quad T(\lambda) = 1 - S = \frac{PK}{1+PK}$$

Therefore,

$$\widehat{\mathbf{M}} = \begin{pmatrix} \|S\| & \|SW\| \\ \|T\| & \|TW\| \end{pmatrix}.$$

For simplicity, we will assume that $W(\lambda)$ is a real number R, serving to scale the maximum radius of the perturbation Δ. By Corollary 2.3, this yields the following expression for the worst-case sensitivity norm:

$$\sup_{\|r\|_\infty < 1} \sup_{\Delta \in \underline{\Delta}_F} \|e\|_\infty = \frac{1}{R} \frac{\|S\|}{\frac{1}{R} - \|T\|}$$

Using the Final Value Theorem, we can write:

$$\|Sr\|_{ss} = |\lim_{k\to\infty} Sr(k)| = |\lim_{\lambda\to 1}(1-\lambda)S(\lambda)\frac{1}{(1-\lambda)}| = |S(1)|$$

So,

$$M_{SS} = \begin{pmatrix} |S(1)| & \|S\| \\ |T(1)| & \|T\| \end{pmatrix}$$

which, by Corollary 2.5, gives this expression for the SSTE with the unit step function as the input, r:

$$\sup_{\Delta \in \underline{\Delta}_F} \|e\|_{ss} = |S(1)| + |T(1)| \frac{\|S\|}{\frac{1}{R} - \|T\|}.$$

The goal is to find an optimal controller among the set of controllers which provide nominal stability. An optimal controller is one that provides robust stability and gives the smallest SSTE. Stated precisely, the problem is this:

Problem 1

Given that r is a unit step input, find the minimum SSTE,

$$\xi^* := \inf_{K stabilizing} \sup_{\Delta \in \underline{\Delta}_F} \|e\|_{ss}$$

where "K stabilizing" means that K stabilizes the nominal system.

3.3 Solution

Our approach to solving Problem 1 will be to perform the minimization over the set of feasible nominal closed-loop maps. Once an optimal closed-loop map is obtained, the controller can be easily derived. To that end, we will use the YJBK parameterization [DahBob95] of all stable closed loop maps S:

$$S(\lambda) = H(\lambda) - U(\lambda)Q(\lambda)$$

where H and U are fixed stable transfer functions which can be obtained through knowledge of the nominal plant, and Q is an arbitrary stable parameter. In other words, there is a one-to-one correspondence between stabilizing controllers and the set of all feasible closed loop maps, and the latter is parameterized as above. Let

$$\hat{\Phi}(\lambda) := S = \sum_{i=0}^{\infty} \Phi_i \lambda^i$$

Here, the "hat" is meant to signify that $\hat{\Phi}$ is a λ-transform. We will carry out the optimization over the coefficients of $\hat{\Phi}$, namely the Φ_i. To that end, we will make use of constraints on these coefficients which would be equivalent to the feasibility of $\hat{\Phi}$. These are known as the "interpolation constraints" and, as will presently become clear, are obtained from H and U in the YJBK parameterization.

Let $a_1, \cdots, a_n$ be the zeros of U in the unit circle. (For simplicity, assume that they are distinct). Then [DahBob95] $\hat{\Phi}$ is feasible if and only if the vector of its coefficients, Φ, satisfies the interpolation condition:

$$\overbrace{\begin{pmatrix} 1 & a_1 & a_1^2 \cdots \\ 1 & a_2 & a_2^2 \cdots \\ \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 \cdots \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{pmatrix}}^{\Phi} = \overbrace{\begin{pmatrix} H(a_1) \\ H(a_2) \\ \vdots \\ H(a_n) \end{pmatrix}}^{b}$$

Note how A, Φ, and b have been defined above. Therefore, we have:

$$S(1) = \hat{\Phi}(1) = \sum_{i=0}^{\infty} \Phi_i \quad \text{and} \quad T(1) = 1 - \sum_{i=0}^{\infty} \Phi_i$$

$$\|S\| = \|\Phi\| = \sum_{i=0}^{\infty} |\Phi_i| \quad \text{and} \quad \|T\| = \|1 - \Phi\| = |1 - \Phi_0| + \sum_{i=1}^{\infty} |\Phi_i|.$$

And we can further write

$$\|T\| < \frac{1}{R} \Leftrightarrow \text{Robust Stability} \Leftrightarrow |1 - \Phi_0| + \sum_{i=1}^{i=\infty} |\Phi_i| < \frac{1}{R}$$

$$K \text{ (nominally) stabilizing } \Leftrightarrow S \text{ feasible} \Leftrightarrow A\Phi = b$$

Since we shall perform the optimization over the coefficients of Φ, we will have to convert expressions involving S and T into ones involving Φ_i in accordance with the above relations. However, we will occasionally use expressions involving S, T, and $\hat{\Phi}$ as short hand for those in terms of Φ_i. The reason why is that the former are more compact and, besides, look more familiar. It will be clear from the context what is meant.

Define $h(\Phi)$ as follows:

$$h(\Phi) := \frac{\|S\|}{\frac{1}{R} - \|T\|} = \frac{\sum_{i=0}^{\infty} |\Phi_i|}{\frac{1}{R} - |1 - \Phi_0| - \sum_{i=1}^{\infty} |\Phi_i|}$$

Note that $\frac{h(\Phi)}{R}$ would be the expression for the worst-case sensitivity norm.

From the preceding discussion it follows that we can rewrite Problem 1 as follows:

Problem 2

$\xi^* = \inf \xi(\Phi) := \overbrace{|\hat{\Phi}(1)|}^{|S(1)|} + \overbrace{|1 - \hat{\Phi}(1)|}^{|T(1)|} h(\Phi)$

Over: $\Phi_i \in \mathbb{R}$

Subject to:

$\|T\| \leq 1/R$

$A\Phi = b$

We have replaced the constraint $\|T\| < 1/R$ with $\|T\| \leq 1/R$. If the optimal solution to this modified problem happens to satisfy $\|T\| = 1/R$, then we will simply redo the problem with the new constraint $\|T\| \leq 1/R - \epsilon$, with an appropriately chosen $\epsilon > 0$. In this manner, we will always keep the constraint set closed. Note also that ξ is, in general, not convex.

The goal is to reduce the above problem into one involving linear programs. If $h(\Phi)$ in Problem 2 were replaced with a constant, then the problem would immediately reduce to a linear program. This observation motivates the formulation of a new problem.

Problem 3

$\eta^* = \inf\ \eta(\Phi;\gamma) := |\hat{\Phi}(1)| + |1 - \hat{\Phi}(1)|\gamma$

Over: $\Phi_i \in \mathbb{R},\ \gamma \in \mathbb{R}$

Subject to:

$\|T\| \leq 1/R$

$A\Phi = b$

$h(\Phi) \leq \gamma$

We will show that this problem is equivalent to Problem 2.

Proposition 1 Let $h : \ell_\infty \longrightarrow \mathbb{R}^{+^n}$, $f : \ell_\infty \rightarrow \mathbb{R}^{+^n}$, and $g : \ell_\infty \longrightarrow \mathbb{R}^{+^n}$ be continuous. Let P be a polytope in ℓ_∞, i.e. a closed bounded set described by linear inequality constraints. Then the minimization

$$\xi^* = \inf_{x \in P} \xi(x) := h(x) + f(x)'g(x)$$

is equivalent to

$$\eta^* = \inf_{x \in P, \gamma \in \mathbb{R}^{+^n} :\ g(x) \leq \gamma} \eta(x;\gamma) := h(x) + f(x)'\gamma$$

Proof We will show that $\xi^* = \eta^*$. Let x^* be a minimizer of the first optimization problem and let $\gamma^* = g(x^*)$. Then:

$$\eta^* \leq \eta(x^*;\gamma^*) = h(x^*) + f(x^*)'\gamma^* = h(x^*) + f(x^*)'g(x^*) = \xi^*$$

This establishes that $\eta^* \leq \xi^*$. To show that $\xi^* \leq \eta^*$, now let (x^*, γ^*) be a minimizer of the second optimization problem. Then:

$$\xi^* \leq \xi(x^*) = h(x^*) + f(x^*)'g(x^*) \leq h(x^*) + f(x^*)'\gamma^* = \eta^*$$

Thus, the two objective functions achieve the same infimum value over their respective feasible sets. It is easy to show that every solution of the first problem is a solution of the second one, and vice-versa. □

The following corollary follows immediately from proposition 3.1 by letting $n = 1$.

Corollary 3.1 Problem 2 is equivalent to Problem 3.

We now present an algorithm for solving Problem 3.

Main Algorithm.

i) Perform the minimization:

$$\gamma_{min} = \inf_{\Phi_i \in \mathbb{R}:\ A\phi = b \text{ and } \|T\| < 1/R} h(\Phi)$$

This is nothing but R times the minimum achievable worst-case sensitivity for the system. An algorithm for obtaining it is given in [KhPrsn90].

ii) Let the parameter γ range over $[\gamma_{min}, \infty]$. (Should we wish to impose, as a new constraint in the problem, an upper bound $\frac{\gamma_{max}}{R}$ on the worst-case sensitivity, then we instead let γ range over $[\gamma_{min}, \gamma_{max}]$).

iii) For each fixed value of $\gamma \in [\gamma_{min}, \infty]$ perform the minimization:

$$\eta^*(\gamma) := \inf_{\phi_i \in \mathbb{R}\ S.T.:\ \|T\| \leq 1/R,\ A\Phi = b,\ h(\phi) \leq \gamma} \eta(\Phi; \gamma)$$

$$= \inf_{\Phi_i \in \mathbb{R}\ S.T.:\ \|T\| \leq 1/R,\ A\Phi = b,\ h(\Phi) \leq \gamma} |\hat{\Phi}(1)| + |1 - \hat{\Phi}(1)|\gamma$$

iv) Let η^* be the minimum value of $\eta^*(\gamma)$ over $\gamma \in [\gamma_{min}, \infty]$, and let Φ^* be a corresponding minimizer. Then (Φ^*, γ^*) is a solution to Problem 3.

In other words, after plotting $\eta^*(\gamma)$ versus γ, optimal solutions to Problem 3 occur at the global minima of the plot. Thus, the question is how to obtain $\eta^*(\gamma)$ for a given γ. Therefore, the key step in this algorithm is the minimization in step (iii), which is an infinite dimensional problem for each γ. The rest of this section is devoted to showing how this minimization can be performed. We will first show that (iii) may be formulated as an infinite dimensional linear program, and will then prove that its solution is finite dimensional, so that the optimization in step (iii) can be performed with only finitely many Φ_i as variables.

Reduction of step (iii) to a linear program.

In a series of manipulations, we transform the problem in (iii) into a linear program.

We introduce two new variables e_1 and e_2, replace the objective function $|\hat{\Phi}(1)| + |1 - \hat{\Phi}(1)|\gamma$ with $e_1 + e_2\gamma$, and add two new constraints:

$$|\hat{\Phi}(1)| \leq e_1 \quad \text{and} \quad |1 - \hat{\Phi}(1)| \leq e_2$$

We replace the left hand side constraint with two new ones:

$$\hat{\Phi}(1) \leq e_1 \quad \text{and} \quad -\hat{\Phi}(1) \leq e_1$$

Similarly, the right hand side constraint is replaced with the following:

$$1 - \hat{\Phi}(1) \leq e_2 \quad \text{and} \quad -1 + \hat{\Phi}(1) \leq e_2$$

Keep in mind throughout that $\hat{\Phi}(1) = \sum_{i=0}^{\infty} \Phi_i$. None of these transformations change the problem.

We replace each variable Φ_i with two nonnegative variables as defined below:

$$\Phi_i^+ := \begin{cases} \Phi_i & \text{if } \Phi_i > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \Phi_i^- := \begin{cases} -\Phi_i & \text{if } \Phi_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\Phi_i = \Phi_i^+ - \Phi_i^-$ and $|\Phi_i| = \Phi_i^+ + \Phi_i^-$. We make these substitutions in the constraints wherever Φ_i and $|\Phi_i|$ appear.

With these substitutions, the four constraints given above become:

$$-e_1 + \sum_{i=0}^{\infty} \Phi_i^+ - \Phi_i^- \leq 0 \qquad -e_1 + \sum_{i=0}^{\infty} -\Phi_i^+ + \Phi_i^- \leq 0$$

$$-e_2 + \sum_{i=0}^{\infty} -\Phi_i^+ + \Phi_i^- \leq -1 \qquad -e_2 + \sum_{i=0}^{\infty} \Phi_i^+ - \Phi_i^- \leq 1$$

or in matrix notation:

$$\begin{pmatrix} -1 & 0 & \underline{1} & -\underline{1} \\ -1 & 0 & -\underline{1} & \underline{1} \\ 0 & -1 & -\underline{1} & \underline{1} \\ 0 & -1 & \underline{1} & -\underline{1} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \Phi^+ \\ \Phi^- \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

where the inequality is element-wise and

$$\underline{1} := (1 \quad 1 \quad 1 \quad \ldots)$$

$$\Phi^+ := (\Phi_0^+ \quad \Phi_1^+ \quad \ldots)' \,, \qquad \Phi^- := (\Phi_0^- \quad \Phi_1^- \quad \ldots)'$$

We next manipulate the following pair of constraints:

$$h(\Phi) = \frac{\|S\|}{\frac{1}{R} - \|T\|} \leq \gamma \quad \text{and} \quad \|T\| < \frac{1}{R}$$

$$\Updownarrow$$

$$\|S\| \leq \gamma(\frac{1}{R} - \|T\|)$$

$$\Updownarrow$$

$$\sum_{i=0}^{\infty} |\Phi_i| \leq \frac{\gamma}{R} - \gamma|1 - \Phi_0| - \gamma \sum_{i=1}^{\infty} |\Phi_i|$$

$$\Updownarrow$$

$$\gamma|1 - \Phi_0| + |\Phi_0| + (1+\gamma)\sum_{i=1}^{\infty} |\Phi_i| \leq \frac{\gamma}{R}$$

$$\Updownarrow$$

$$\gamma - \gamma\Phi_0^+ + \gamma\Phi_0^- + \Phi_0^+ + \Phi_0^- + (1+\gamma)\sum_{i=1}^{\infty}(\Phi_i^+ + \Phi_i^-) \leq \frac{\gamma}{R}$$

$$-\gamma + \gamma\Phi_0^+ - \gamma\Phi_0^- + \Phi_0^+ + \Phi_0^- + (1+\gamma)\sum_{i=1}^{\infty}(\Phi_i^+ + \Phi_i^-) \leq \frac{\gamma}{R}$$

$$\Updownarrow$$

$$(1-\gamma)\Phi_0^+ + (1+\gamma)\sum_{i=1}^{\infty}\Phi_i^+ + (1+\gamma)\sum_{i=0}^{\infty}\Phi_i^- \leq \gamma(\frac{1}{R} - 1)$$

$$(1+\gamma)\sum_{i=0}^{\infty}\Phi_i^+ + (1-\gamma)\Phi_0^- + (1+\gamma)\sum_{i=1}^{\infty}\Phi_i^- \leq \gamma(\frac{1}{R} + 1)$$

$$\Updownarrow$$

$$\begin{pmatrix} \underline{g} & (1+\gamma)\underline{1} \\ (1+\gamma)\underline{1} & \underline{g} \end{pmatrix} \begin{pmatrix} \Phi^+ \\ \Phi^- \end{pmatrix} \leq \begin{pmatrix} \gamma(\frac{1}{R} - 1) \\ \gamma(\frac{1}{R} + 1) \end{pmatrix}$$

where

$$\underline{g} := (\, 1-\gamma \quad 1+\gamma \quad 1+\gamma \quad \ldots\,).$$

We can now write down the final formulation of (iii) as a linear program, LP:

$$\mu := \inf_x \underbrace{(\,1 \quad \gamma \quad \underline{0} \quad \underline{0}\,)}_{C} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ \Phi^+ \\ \Phi^- \end{pmatrix}}_{x}$$

Subject to the inequality constraints:

$$\begin{pmatrix} 0 & 0 & \phi & -I \\ 0 & 0 & -I & \phi \\ -1 & 0 & \underline{1} & -\underline{1} \\ -1 & 0 & -\underline{1} & \underline{1} \\ 0 & -1 & -\underline{1} & \underline{1} \\ 0 & -1 & \underline{1} & -\underline{1} \\ 0 & 0 & \underline{g} & (1+\gamma)\underline{1} \\ 0 & 0 & (1+\gamma)\underline{1} & \underline{g} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \Phi^+ \\ \Phi^- \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ \gamma(\frac{1}{R}-1) \\ \gamma(\frac{1}{R}+1) \end{pmatrix}$$

And equality constraints:

$$\begin{pmatrix} 0 & 0 & A & -A \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \Phi^+ \\ \Phi^- \end{pmatrix} = b$$

where I is an infinite dimensional identity matrix and ϕ is an infinite dimensional "square" zero matrix, and $\underline{1}$ and $\underline{g}$ are as defined before.

For computational purposes, it is useful to know that there is a finite dimensional optimal solution to the above minimization. That is what we will prove next. The result is that programs in step (iii) of the main algorithm are finite dimensional; and so is the minimization corresponding to the optimal solution. Hence, the optimal closed loop response is FIR.

Theorem 3.1 The solution to the above linear program, LP, is finite dimensional.

Proof We put the above linear program in the following form:

$\mu = \inf_x < x, \mathcal{C} >$

$\mathcal{A}x \leq \mathcal{B}$

$x \geq 0$

We have:

$$\mathcal{A} = \begin{pmatrix} -1 & 0 & \underline{1} & -\underline{1} \\ -1 & 0 & -\underline{1} & \underline{1} \\ 0 & -1 & -\underline{1} & \underline{1} \\ 0 & -1 & \underline{1} & -\underline{1} \\ 0 & 0 & \underline{g} & (1+\gamma)\underline{1} \\ 0 & 0 & (1+\gamma)\underline{1} & \underline{g} \\ 0 & 0 & A & -A \\ 0 & 0 & -A & A \end{pmatrix} \quad x = \begin{pmatrix} e_1 \\ e_2 \\ \Phi^+ \\ \Phi^- \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ \gamma(\frac{1}{R} - 1) \\ \gamma(\frac{1}{R} + 1) \\ b \\ -b \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 1 \\ \gamma \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

There are two steps to the proof. First we show that the dual of the above infinite dimensional linear program has a finite dimensional solution. Next, we show that there is no "duality gap," so that the dual formulation is indeed equivalent to the primal one. The dual problem is:

$\mu^d = \sup_y < \mathcal{B}, y >$

$\mathcal{A}^* y \leq \mathcal{C}$

$y \leq 0$

We rewrite $\mathcal{A}^* y \leq \mathcal{C}$ as follows:

$$\overbrace{\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ \underline{1}' & -\underline{1}' & -\underline{1}' & \underline{1}' & \underline{g}' & (1+\gamma)\underline{1}' & A^* & -A^* \\ -\underline{1}' & \underline{1}' & \underline{1}' & -\underline{1}' & (1+\gamma)\underline{1}' & \underline{g}' & -A^* & A^* \end{pmatrix}}^{\mathcal{A}^*} \overbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ z^+ \\ z^- \end{pmatrix}}^{y} \leq \overbrace{\begin{pmatrix} 1 \\ \gamma \\ 0 \\ 0 \\ \vdots \end{pmatrix}}^{c}$$

where $y_i \in \mathbb{R}$ for $i = 1, 2, \cdots, 6$; $z^+ \in \mathbb{R}^n$, and $z^- \in \mathbb{R}^n$.

From the first two rows of the matrix equation, we obtain:

$$-y_1 - y_2 \leq 1 \qquad - y_3 - y_4 \leq \gamma.$$

And from the last two rows of the matrix equation, we obtain that for $k \in \mathbf{N}$,

$$A_k^* z \leq -y_1 + y_2 + y_3 - y_4 - (1+\gamma)(y_5 + y_6)$$

$$-A_k^* z \leq \ y_1 - y_2 - y_3 + y_4 - (1+\gamma)(y_5 + y_6).$$

where A_k^* is the k'th row of A^*. By definition, the latter consists of the k'th powers of the zeros of U *within the unit circle.* Since $\|A_k^*\| \to 0$ as $k \to \infty$, there exists an integer N such that if $y_1, y_2, \cdots, y_6$, z^+, z^- satisfy the last two equations for $k = 1, \cdots, N$, then they also necessarily satisfy them for $k = N+1, N+2, \cdots$.

Therefore, the dual problem is equivalent to the following finite dimensional problem:

$\mu^d = {\mu_n}^d := \sup_y < \mathcal{B}, y >$

${\mathcal{A}_F}^* \; y \leq \mathcal{C}_F$

$y \leq 0$

where the subscripted quantities are appropriately truncated finite dimensional versions of the unsubscripted ones. There is no duality gap for finite dimensional programs. Therefore, the above optimization is equivalent to its primal formulation:

${\mu_n}^d = \mu_n := \inf_{x_F} < x_F, \mathcal{C}_F >$

$\mathcal{A}_F \; x_F \leq \mathcal{B}$

$x_F \geq 0$

This is the same as the original infinite dimensional primal problem, except for a reduction in the constraints. So, we can write:

$$\lim_{n \to \infty} \mu_n = \mu$$

Into the above equation, we now substitute μ^d for μ_n, and conclude that:

$$\mu = \mu^d$$

Therefore, there is no duality gap. Moreover, both the primal and dual problems are finite dimensional. $\square$

Chapter 4

Examples

4.1 Introduction

The results of Chapter 3 make it possible to optimize the tracking error using software developed in MATLAB. In this chapter, we apply these results to two concrete examples, both of which shed some light on the nature of the problem at hand. The examples are with reference to Figure 3.1, and the notation is the same as that in Chapter 3.

4.2 Example 1

In this example, the nominal plant is

$$P = \frac{\lambda + 0.5}{(\lambda + 0.7)(\lambda - 0.3)}$$

and R = 0.1.

Figure 4.1 shows the plot of $\eta^*(\gamma)$ versus $\frac{\gamma}{R}$. The horizontal axis corresponds to the upper bound on the tolerable worst-case sensitivity norm. Note that a transition occurs at $\frac{\gamma}{R} = 184.65$. To the left of 184.65, the plot has a negative slope. This means that the least achievable worst-case steady-state tracking error semi-norm (SSTE) decreases linearly as we ease the restriction on the maximum worst-case sensitivity norm that can be tolerated. For example, if we impose the constraint

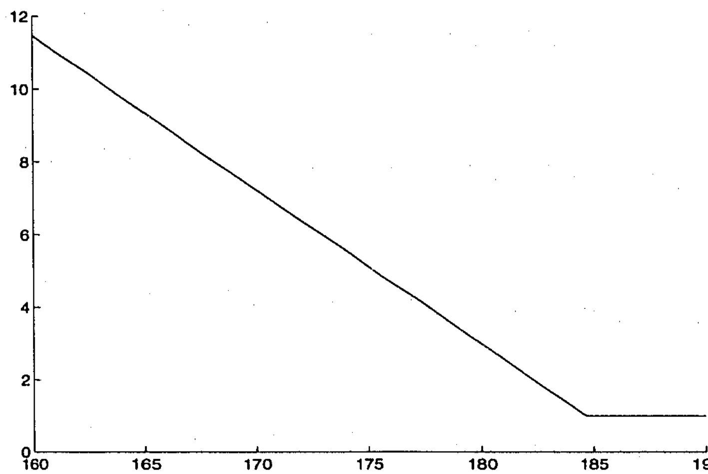

Figure 4.1: Example 1. Plot of $\eta^*(\gamma)$ versus $\frac{\gamma}{R}$.

$$\frac{1}{R}\frac{\|S\|}{\frac{1}{R}-\|T\|} \leq 170$$

then we can read off the graph that the minimum achievable worst-case tracking error will be 7.25. But if we ease the restriction to:

$$\frac{1}{R}\frac{\|S\|}{\frac{1}{R}-\|T\|} \leq 180$$

then the minimum SSTE reduces to 2.99.

To the right of 184.65, the graph is flat. Thus, beyond that point no improvement on the tracking error is possible. The optimal SSTE equals 1.

What now if we restrict ourselves to controllers which include an "internal model"? Since r is a unit step input, that would require K to include the factor $\frac{1}{\lambda-1}$. We lump this factor with the nominal plant instead, define a new nominal plant

$$P = \frac{\lambda+0.5}{(\lambda+0.7)(\lambda-0.3)(\lambda-1)},$$

and design an optimal controller.

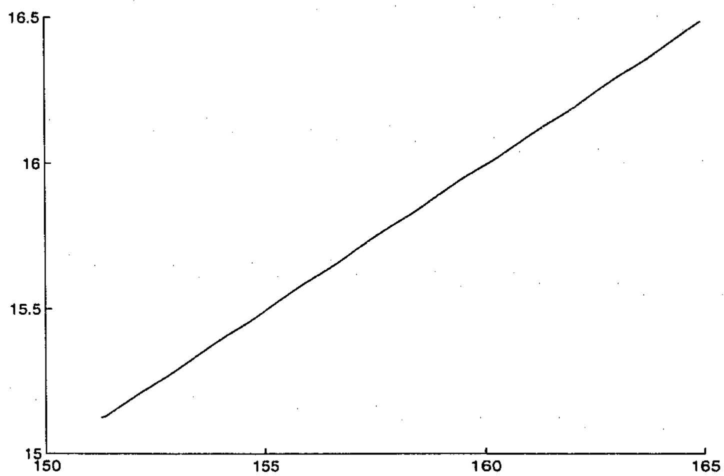

Figure 4.2: Example 1. With internal model. Plot of $\eta^*(\gamma)$ versus $\frac{\gamma}{R}$.

Figure 4.2 shows the plot of $\eta^*(\gamma)$ versus $\frac{\gamma}{R}$. The optimal SSTE (η^*) can be read off at $\frac{\gamma_{min}}{R}$, where the global minimum occurs, and equals 15.1. The fact that the global minimum occurs at the leftmost point means that the optimal SSTE is yielded in step (i) of the algorithm (Chapter 3). In this case, therefore, a solution minimizing the worst-case sensitivity also minimizes the SSTE.

The minimum SSTE value of 15.1 in this case can be compared to the optimal value of 1 in the absence of an internal model. Clearly, system performance severely deteriorates with inclusion of an internal model.

4.3 Example 2

In this example, the nominal plant is

$$P = \frac{\lambda - 0.5}{\lambda - 0.2}.$$

The goal is to explore the properties of the optimal solution as the maximum allowable perturbation radius, R, varies. The values obtained for the optimal SSTE for different R's are listed in the second column of Table 4.1 and plotted in Figure 4.3. As would be expected, the optimal SSTE increases monotonically with increasing plant uncertainty.

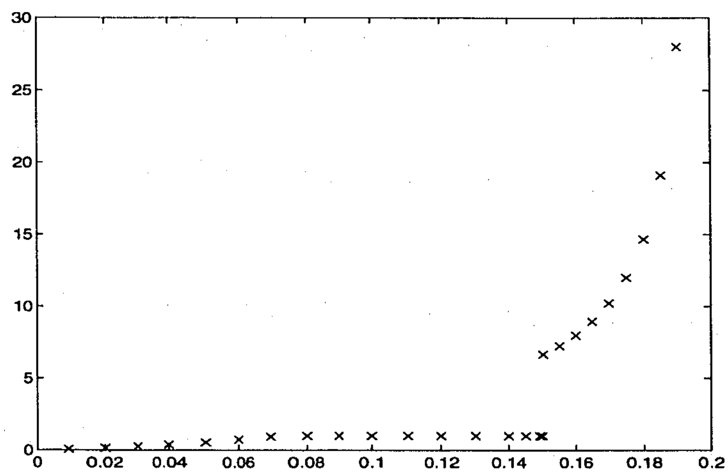

Figure 4.3: Example 2. Minimum SSTE (η^*) versus radius (R).

In general, there is no unique solution to the problem of minimizing the SSTE, and $\eta^*(\gamma)$ may have more than one global minimum. By choosing the leftmost minimum, however, we are assured to obtain an optimal solution of the smallest worst-case sensitivity norm. This procedure was used to obtain values for the worst-case sensitivity norms of the optimal solutions for different R's. These are listed in the third column of Table 4.1 and plotted in Figure 4.4.

Perturb. Radius (R)	Minimum SSTE	Minimum $\frac{1}{R}\frac{\|S\|}{\frac{1}{R}-\|T\|}$ for opt. sol.
0.01	0.0723	7.23
0.02	0.1576	7.88
0.03	0.2600	8.666
0.04	0.3850	9.6241
0.05	0.5411	10.8220
0.06	0.74158	12.3596
0.07	0.93357	12.3950
0.08	0.99944	12.3260
0.09	1	14.1821
0.1	1	17.0220
0.11	1	21.2840
0.12	1	28.40
0.13	1	42.63
0.14	1	85.5
0.145	1	172.1
0.149	1	905.3
0.1497	1	3557
0.150	6.6666	16.0009
0.155	7.2593	17.7803
0.160	8.0002	20.0033
0.165	8.9526	22.8595
0.170	10.2226	26.6700
0.175	12.0004	32.0024
0.180	14.6670	40.0020
0.185	19.1116	53.3354
0.190	28.0001	80.0003

Table 4.1: Example 2.

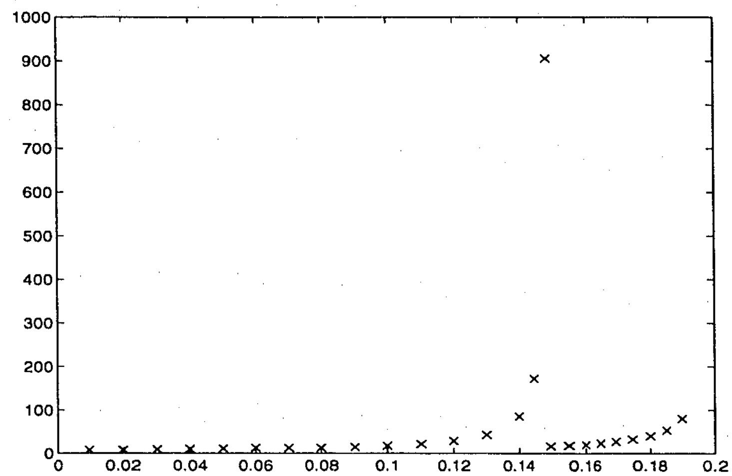

Figure 4.4: Example 2. Smallest worst-case sensitivity for optimal solution versus radius (R).

As R approaches $R^* \approx 0.14975564634839$ from the left, the worst-case sensitivity norm goes to ∞.[1] This shows that optimal solutions which yield the minimum SSTE may have arbitrarily large worst-case sensitivities.

It is instructive to look at the plot of $\eta^*(\gamma)$ for $R = 0.149$ in Figure 4.5. The optimum SSTE, $\eta^* = 1$. But this minimum may be attained only at the expense of a worst-case sensitivity norm of about 905. From the plot, it is clear that there is a trade-off between tracking error and sensitivity. For example, we can tell from the plot that if we constrain the sensitivity norm to be less than 400, then the least achievable SSTE will be 4.16.

As we have seen, optimal solutions may have very large worst-case sensitivity norms. This fact has important ramifications that will be discussed in the final chapter.

[1] It happens that $\frac{1}{R^*}$ equals the minimum of $\|T\|$ subject to $T(1) = 0$ and the interpolation conditions.

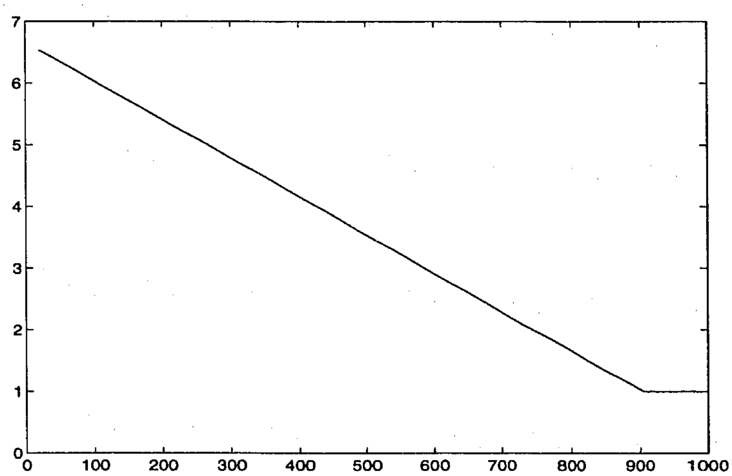

Figure 4.5: Example 2. Plot of $\eta^*(\gamma)$ versus $\frac{\gamma}{R}$, $R = 0.149$.

Chapter 5

Multiple Perturbations: Special Case

5.1 Introduction

Chapter 3 solved the SSTE minimization problem in the case of a single perturbation. This chapter gives a procedure for the minimization in the case of n perturbations, for $n \in \mathbf{N}$. The special case where the individual elements of $\mathbf{M}$ are affine in the YJBK parameter is treated only. The algorithm presented in Chapter 3 carries over, though with two key modifications.

First, there will now be n real parameters instead of just one. So, one will solve a linear program for each fixed value of a parameter $\gamma' \in \mathbb{R}^{+n}$.

Second, in general it will no longer be possible to carry out the optimization over the coefficients of the feasible closed-loop maps. Such a minimization problem would defy reduction to linear programs, for the elements of the $\mathbf{M}$ matrix will no longer be affine in the variables. This problem will be circumvented by minimizing instead over the coefficients $\{q_i\} \in \ell_1$ of the YJBK parameter, $Q = \sum_{i=0}^{\infty} q_i \lambda^i$.

5.2 Extension to Multiple Perturbations

The problem in this section is in terms of Figure 2.2, with the input r a unit step signal. Uncertainty is represented by $\Delta \in \mathcal{D}_F(n)$; there are n SISO perturbations.

As before, the operator $\mathbf{M} : (r \;\; \xi)' \to (e \;\; y)'$ is partitioned in this way:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix},$$

where $\mathbf{M}_{11}$ is 1×1, $\mathbf{M}_{12}$ is $1 \times n$, $\mathbf{M}_{21}$ is $n \times 1$, and $\mathbf{M}_{22}$ is $n \times n$. Accordingly, we have:

$$M_{ss} = \begin{pmatrix} |\mathbf{M}_{11}(1)| & \widehat{\mathbf{M}}_{12} \\ |\mathbf{M}_{21}(1)| & \widehat{\mathbf{M}}_{22} \end{pmatrix},$$

where the absolute values are applied element-wise, and the expressions $\mathbf{M}_{11}(1)$ and $\mathbf{M}_{21}(1)$ are meant to indicate that the λ-transforms of the individual elements are evaluated at $\lambda = 1$.

By Corollary 2.5, the SSTE is given by:

$$\sup_{\Delta \in \mathcal{D}_F} \|e\|_{ss} = |\mathbf{M}_{11}(1)| + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1}|\mathbf{M}_{21}(1)|.$$

The object is to minimize the above quantity subject to robust stability, $\rho(\widehat{\mathbf{M}}_{22}) < 1$. It will be shown that this minimization can be reduced to a linear program that depends on a parameter $\gamma' \in \mathbb{R}^{+^n}$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, $\gamma_i \in \mathbb{R}^+$. As γ varies, the linear program yields different optimal values. Global minima of these values will be optimal SSTE's. The solutions yielding such minima will be optimal solutions.

We assume that each individual element of $\mathbf{M}$ has a λ-transform of the form:

$$M_{ij} = H_{ij} - U_{ij}Q,$$

The assumption is not true in general. But it holds, for example, in the case of a plant with both multiplicative and additive perturbations [KhPrsn91]:
$P_\Delta = P + W_1 P \Delta_1 + W_2 \Delta_2$, where W_1 and W_2 are stable weighting functions, and $\Delta_1, \Delta_1 \in \underline{\Delta}_F$.

For convenience, the subscripts of H and U will be dropped. H and U are stable λ-transforms that are readily obtained from knowledge only of the nominal plant. We will thus consider them as given. $Q(\lambda)$ is the YJBK parameter. All feasible M_{ij} are obtained by letting Q range over the set of all stable rational transfer functions. The coefficients $\{q_i\} \in \ell_1$ of $Q(\lambda) = \sum_{i=0}^{\infty} q_i \lambda^i$ will be variables in the optimization problem. Minimizing over the $\{q_i\}$ is tantamount to minimizing over Q, and hence over the set of the feasible $\mathbf{M}$.

From the formula for the SSTE, it is clear that we need to obtain expressions for $|M_{ij}(1)|$ and $\|M_{ij}\|$ which are in terms of $\{q_i\}$. Ignoring the subscripts again, we let $\{h_k\}$ and $\{u_k\}$ be the FIR impulse responses of H and U. The impulse response of Q is, of course, $\{q_i\}_{i=0}^{\infty} \in \ell_1$. Then,

$$|M_{ij}(1)| = |H(1) - U(1)Q(1)| = |\sum_{k=0}^{\infty} h_k - (\sum_{k=0}^{\infty} u_k)(\sum_{k=0}^{\infty} q_k)|,$$

and

$$\begin{aligned} \|M_{ij}\| &= \|h_k - u_k * q_k\|_1 \\ &= \|h_k - \sum_{l=0}^{\infty} u_{k-l} q_l\|_1 \\ &= \sum_{k=0}^{\infty} |h_k - \sum_{l=0}^{\infty} u_{k-l} q_l| \end{aligned}$$

Note that considerable simplification will result when H and U are FIR. The expressions for $|M_{ij}(1)|$ and $\|M_{ij}\|$ are affine in the variables q_i. As it will turn out, this is what is needed in order to convert the optimization problem into a (parametric) linear program. In order for a mathematical programming problem to reduce to a linear program, it suffices for its objective and constraint functions to be summations of absolute values of terms that are affine in the variables. Such expressions are therefore what we have aimed at.

We are only one lemma away from the main result of this chapter.

Lemma 5.1 Let A be an $n \times n$ matrix with nonnegative elements. Let $x \in \mathbb{R}^{+^n}$ satisfy $x > (0 \ \cdots \ 0)'$. If $Ax < x$, then $\rho(A) < 1$. All inequalities are element-wise.

Proof The proof will be by mathematical induction on n, and will be based on Lemma 2.1. For $n = 1$, the statement of the lemma holds trivially. As the inductive hypothesis, assume the statement of the lemma holds for some $n \in \mathbf{N}$. Now, assume that $x \in \mathbb{R}^{n+1}$ and that A is an $n+1 \times n+1$ nonnegative matrix satisfying $Ax < x$. It suffices to prove that then $\rho(A) < 1$. To that end, we rewrite $Ax < x$ as follows:

$$\overbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ y \end{pmatrix}}^{x} < \overbrace{\begin{pmatrix} x_1 \\ y \end{pmatrix}}^{x}$$

Note how A and x have been partitioned; A_{11} is $n \times n$ and $y \in \mathbb{R}^n$. Carrying out the multiplication on the left hand side yields two inequalities:

$$A_{11}x_1 + A_{12}y < x_1,$$

$$A_{21}x_1 + A_{22}y < y.$$

From the second inequality, we have $A_{22}y < y$. By the inductive hypothesis, this implies that $\rho(A_{22}) < 1$. Hence, $(I - A_{22})$ is nonnegative and invertible. So, the second inequality can be rewritten as $(I - A_{22})^{-1}A_{21}x_1 < y$. Substituting this into the first inequality yields:

$$A_{11} + A_{12}(I - A_{22})^{-1}A_{21} < 1$$

By Lemma 2.1 this result, along with $\rho(A_{22}) < 1$, implies that $\rho(A) < 1$. □

Proposition 2

The SSTE minimization problem, $P1$:

$\inf |\mathbf{M}_{11}(1)| + \widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1}|\mathbf{M}_{21}(1)|$

Over $\{q_i\} \in \ell_1$

Subject to $\rho(\widehat{\mathbf{M}}_{22}) < 1$

is equivalent to $P2$:

$\inf |\mathbf{M}_{11}(1)| + \gamma|\mathbf{M}_{21}(1)|$

Over: $\{q_i\} \in \ell_1, \gamma' \in I\!R^{+^n}$

Subject to:

$\widehat{\mathbf{M}}_{12} + \gamma\widehat{\mathbf{M}}_{22} < \gamma.$

Proof By Proposition 1 from Chapter 3, $P1$ is equivalent to $P1'$:

$\inf |\mathbf{M}_{11}(1)| + \gamma|\mathbf{M}_{21}(1)|$

Over: $\{q_i\} \in \ell_1, \gamma' \in I\!R^{+^n}$

Subject to:

$\rho(\widehat{\mathbf{M}}_{22}) < 1$

$\widehat{\mathbf{M}}_{12}(I - \widehat{\mathbf{M}}_{22})^{-1} < \gamma$

The second constraint can be rewritten as

$$\widehat{\mathbf{M}}_{12} + \gamma\widehat{\mathbf{M}}_{22} < \gamma,$$

which means that $\gamma\widehat{\mathbf{M}}_{22} < \gamma$. This implies, by Lemma 5.1, that $\rho(\widehat{\mathbf{M}}_{22}) < 1$. Therefore, the first constraint is redundant as it is implied by the second one, and $P1'$ reduces to $P2$. $\square$

For each fixed γ, $P2$ is reduced in a straightforward fashion to a linear program. This can be seen from the fact that the expressions in both the objective function and the constraints are sums of absolute values of terms that are affine in the variables, q_i.

The problem has thus been reduced to a parametric linear program, the number of the parameters equaling the number of the perturbations Δ_i in the system.

In $P2$, there are infinitely many variables. The stable Q and the maps generated by it can be approximated arbitrarily closely by FIR maps. Therefore one may instead perform the minimization over FIR Q of impulse responses of length N, and let $N \to \infty$ in order to converge to the optimal SSTE. In other words, the minimum SSTE can be arbitrarily closely approximated by solving programs ($P3$) of the following form:

$\inf |\mathbf{M}_{11}(1)| + \gamma|\mathbf{M}_{21}(1)|$

Over: $q_i \in I\!R\ (i = 1, \cdots, N); \gamma' \in I\!R^{+n}$

Subject to:

$|\mathbf{M}_{12}(1)| + \gamma\widehat{\mathbf{M}}_{22} < \gamma.$

5.3 Remark

The results in this chapter will still hold if M_{ss} is replaced with $\widehat{\mathbf{M}}$, and the SSTE with the worst-case ℓ_1 norm of the system. As a result, the above procedure can be used to solve the robust ℓ_1 problem as well. This provides an alternative method to those discussed in [DahBob95]. In particular, the parametric programming method described in this chapter is comparable to the grid method discussed in [DahBob95, p. 356-7]. In both, a linear program is evaluated at each point of a grid.

Chapter 6

Conclusion

In control system design, often the objective is to track a fixed input signal which is known in advance, such as a unit step signal. If the physical plant is LTI, use of the "internal model principle" will assure that the error signal converges to zero [DahBob95, p. 103]. If the physical plant is time-varying, however, it may no longer be possible to achieve that. Nor is a controller with an internal model guaranteed to be in any sense optimal. This poses the problem of how the tracking error can be minimized. The principal contribution of the dissertation is to provide a solution to the minimization problem on the assumption that the nominal plant is LTI, and that plant uncertainty is represented in terms of a norm-bounded perturbation which may be time-varying.

In having solved the problem, however, we have gained insights which expose the potential *inadequacy* of the standard formulation of the problem.

In many cases, the standard formulation of the tracking problem and our solution to it will be satisfactory. Moreover, the engineer will be able to tell when that is the case: An optimal solution which has an acceptable worst-case sensitivity is a good solution. However, the case of the simple, first order nominal plant of Example 2 demonstrates that sometimes optimal solutions to the tracking error minimization problem may have arbitrarily large sensitivities. This means that the magnitude of the error signal e may become arbitrarily large before it eventually attains its advertised, "low," steady-state value. When that happens, the graphical technique presented in this dissertation can be used to impose an upper bound on the worst-case sensitivity. But this solution may not be entirely satisfactory. It is too conservative, since the quantity "worst-case sensitivity" is

$$\max_{\|r\|_\infty \leq 1, \Delta} \|T_{er} r\|_\infty,$$

i.e. the maximum over a rather large set of conceivable inputs r. But the input r is in fact a single fixed signal which we know in advance. Therefore, instead of imposing an upper bound γ_{max} on the worst-case sensitivity, a more appropriate formulation of the tracking problem would be as follows. Minimize the steady-state error subject to $\max_\Delta \|T_{er} r_0\|_\infty \leq \gamma_{max}$, where r_0 is a known, fixed signal. This problem requires investigation.

Chapter 5 solved the SSTE minimization in a special case of the more general n-perturbation problem, ($n \in \mathbf{N}$). As discussed in Section 5.3, the problem is structurally quite similar to the robust ℓ_1 problem, for which optimal solutions are FIR. Moreover, in the $n = 1$ case we proved that the solution is indeed FIR. Therefore, it is worthwhile to determine whether the FIR property is preserved when $n > 1$.

The method in Chapter 5 is equally applicable to the robust ℓ_1 problem. It remains to see how it fares compared to the techniques discussed in [DahBob95], particularly with the grid approach described therein on pages 356-57. Most importantly, one must investigate whether methods along the lines of [KunKon91, KonYaj92] can be used to obtain better algorithms for solving the parametric linear programs in the robust tracking and robust ℓ_1 problems. Efficient algorithms become indispensable when uncertainty is represented by many perturbations.

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