GENERALIZED JOINT SIGNAL REPRESENTATIONS AND OPTIMUM DETECTION

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ABSTRACT

Generalized joint signal representations (JSRs) extend the scope of joint time-frequency representations (TFRs) to a richer class of nonstationary signals, but their use, just as in the case of TFRs, has been primarily limited to qualitative, exploratory data analysis. To exploit their potential more fully, JSR-based statistical signal processing techniques need to be developed that can be successfully applied in real-world problems. In this paper, we present an optimal detection framework based on arbitrary generalized quadratic JSRs, thereby making it applicable in a wide variety of detection scenarios involving nonstationary stochastic signals, noise and interference. For any given class of generalized JSRs, we characterize the corresponding class of detection scenarios for which such JSRs constitute canonical detectors, and derive the corresponding JSR-based detectors. Our formulation also yields a very useful subspace-based interpretation in terms of corresponding linear JSRs that we exploit to design optimal detectors based on only partial signal information.

1. INTRODUCTION

Nonstationarity is a salient characteristic in many important signal processing scenarios. Examples include radar, sonar, speech, biological, geophysical and machine fault signals. Classical methods based on stationary spectral analysis are inadequate in such applications; new tools are needed to successfully tackle such problems.

Joint time-frequency representations (TFRs) facilitate a time-varying spectral analysis and are widely used in nonstationary signal processing [1]. However, new TFRs have been developed primarily for exploratory data analysis: that is, by using TFRs to get a visual display of the time-varying spectral energy in the signal and using this qualitative information as a starting point for further analysis/processing. Recently though, the need for detection in nonstationary scenarios has spurred interest in the use of TFRs as detectors [2, 3, 4]. However, most of the proposed techniques are *ad hoc* and/or do not justify the use of TFRs as detectors. Some very recent results have put TFR-based detection theory on a firm footing [5], thereby extending the use of TFRs to more quantitative application.

Fundamental to the structure of TFRs is the concept of time-frequency shifts, and this very property limits their scope to a relatively small class of nonstationary signals; qualitatively, TFRs are "well-matched" to a special, narrow class of signals. In an attempt to develop nonstationary tools well-matched to a broader class of signals, recent research in time-frequency analysis has yielded rapid progress in the development of new joint signal representations (JSRs) that analyze signals in terms of variables other than time and frequency. An important example is the affine class of joint time-scale representations (TSRs). The development is finally culminating in a fairly comprehensive theory for generalized JSRs with respect to arbitrary variables [1, 6, 7, 8].

Despite the proliferation of generalized JSRs, their development has been limited primarily to qualitative, exploratory data analysis tools. To exploit their potential more fully, JSR-based statistical signal processing techniques need to be developed that can be successfully applied to real-world problems involving random signals, noise and interference. One promising framework is that of nonstationary signal detection in the presence of noise and interference, which has received substantial attention lately; for example, in applications such as radar/sonar, engine knock detection [4] and machine health monitoring [3, 2]. Owing to the rich variety of signals involved in such applications, techniques beyond TFR-based detection are desirable.

In this paper, we present a framework for optimum detection based on any arbitrary class of generalized quadratic JSRs, thereby making it applicable in a broad class of nonstationary detection scenarios. Since such JSR-based detectors are canonical for detecting signals that have undergone certain unitary transformations, a covariance-based theory[1] for generalized JSRs is the appropriate vehicle for characterizing such detectors. We adopt the development of [8] because of its simplicity[2] and provide a brief review in the next section. In Section 3, we describe the JSR-based detection theory: for any arbitrary class of covariance-based JSRs, we characterize the corresponding class of detection scenarios for which such JSRs form canonical detectors, and also explicitly characterize the JSR-based detectors. A useful

*This work was supported by the Office of Naval Research under Grant No. N00014-95-1-0674, and the Schlumberger Foundation.

[1] Based on the covariance of JSRs to certain unitary transformations.

[2] We note that a covariance-based theory is also developed in [7] but it involves a complicated remapping of coordinates that is unnecessary from a detection viewpoint.

subspace-based interpretation of the detectors in described in Section 4, and concluding remarks are provided in Section 5.

2. COVARIANCE-BASED JSRS

Consider a parameter set $G \subset \mathbb{R}^2$ and a family of unitary operators $\{\mathbf{U}_{(a,b)}\}$, $(a,b) \in \mathbb{R}^2$, defined on $L^2(\mathbb{R})$; that is, $\langle \mathbf{U}_{(a,b)}s, \mathbf{U}_{(a,b)}s\rangle = \langle s, s\rangle$ for all $s \in L^2(\mathbb{R})$ and for all $(a,b) \in \mathbb{R}^2$. The two "coordinates" a and b represent the variables of interest, and, in a covariance-based approach, we are interested in all bilinear JSRs, with respect to the variables a and b, that are "covariant" to the unitary operator $\mathbf{U}_{(a,b)}$. For example, TFRs from Cohen's class are covariant to the time-frequency shift operator $\mathbf{T}_\mu \mathbf{F}_\nu$, where $(\mathbf{T}_\mu s)(t) = s(t-\mu)$ and $(\mathbf{F}_\nu s)(t) = e^{-j2\pi\nu t}s(t)$, and the TSRs from the affine class are covariant to the time-scale shift operator $\mathbf{T}_\mu \mathbf{D}_c$, where $(\mathbf{D}_c s)(t) = \frac{1}{\sqrt{c}}s(t/c)$. Under fairly natural assumptions, the family $\mathbf{U}_{(a,b)}$ must form a group under composition [8]; that is, the parameter set G is a group with the group operation $\bullet$ defined by [8]

$$\mathbf{U}_{(a,b)}\mathbf{U}_{(a',b')} = \mathbf{U}_{(a,b)\bullet(a',b')} \quad \text{(within a phase factor)}. \quad (1)$$

The appropriate covariance relation, then, is the group operation; that is, the JSRs satisfy [8]

$$(\mathbf{P}\mathbf{U}_{(a',b')}s)(a,b) = (\mathbf{P}s)((a',b')^{-1} \bullet (a,b)) \ , \quad (2)$$

for all (a,b), $(a',b') \in G$, where the JSR is denoted by the operator $\mathbf{P}$. All quadratic JSRs that satisfy (2) can be simply characterized as [8]

$$(\mathbf{P}s)(a,b;\Phi) = \langle \Phi \mathbf{U}_{(a,b)^{-1}}s, \mathbf{U}_{(a,b)^{-1}}s\rangle, \ (a,b) \in G, \quad (3)$$

where $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a linear operator characterizing the JSRs from the class.

3. GENERALIZED JSR-BASED DETECTION

As we mentioned in the Introduction, a covariance-based approach to generalized JSRs is the appropriate vehicle for a JSR-based detection framework. With the above characterization of covariance-based JSRs, we are now in a position to describe the JSR-based detection theory.

Consider an arbitrary family $\{\mathbf{U}_{(a,b)}\}$ of unitary operators satisfying (1) for some group G, and the corresponding class C of covariant JSRs as characterized by (3). Such a family of unitary operators could possibly model the effect of a transmission medium that produces a nondissipative signal distortion. Our objective is to characterize the class of detection scenarios for which the JSRs from C constitute canonical detectors, and also to characterize the corresponding JSR-based detectors.

Signal detection is simply a binary hypothesis testing problem:

$$\begin{aligned} H_0 &: \ x(t) = n(t) \\ H_1 &: \ x(t) = as(t) + n(t) \end{aligned} \quad (4)$$

where $t \in T$, the observation interval, x is the observed signal, s is the signal to be detected, n is additive noise and a is a positive parameter that may be unknown. Based on the observed signal x, it has to be decided whether the signal is present (H_1) or not (H_0). The optimal decision is made by comparing a real-valued function of data, $L(x)$, called the test statistic, to a threshold.

A key observation regarding JSR-based detectors is that they can realize a different quadratic function of the observed signal, x, at each (a,b) location. Thus, for optimal JSR-based detection, we focus on scenarios in which the optimal detector is a quadratic function of the observations. And, in order to exploit the degrees of freedom in a JSR (a different detector at each (a,b) location) we resort to composite hypothesis testing scenarios, analogous to the approach in [5].

3.1. Quadratic Detection Scenarios

It is well-known that for detecting a Gaussian signal in Gaussian noise, the optimal test statistic is quadratic. Thus, we focus attention on Gaussian detection[3] and assume that both n and s are zero-mean complex Gaussian processes, independent of each other, and characterized by their correlation functions $R_n(t_1,t_2) = E[n(t_1)n^*(t_2)]$ and $R_s(t_1,t_2)$. To exploit the degrees of freedom in JSR-based detectors, we consider composite hypothesis testing scenarios in which, under H_1, the signal, s, has two parameters (α,β) that may be unknown or random:

$$H_1 : x(t) = as(t;\alpha,\beta) + n(t) \ . \quad (5)$$

The idea is to associate the parameters (α,β) with the variables (a,b) of the JSR. The dependence of $s(t;\alpha,\beta)$ on (α,β) is completely characterized by the signal correlation function, which we denote by $R_s^{(\alpha,\beta)}$. We consider three important detection scenarios:

1. Arbitrary Gaussian signal in white Gaussian noise.
2. Arbitrary Gaussian signal in arbitrary Gaussian noise (deflection optimal detector).
3. Small amplitude arbitrary Gaussian signal in arbitrary Gaussian noise (locally optimal detector).

The optimal detector in all cases has the form [5, 9][4]

$$L(x) = \max_{(\alpha,\beta)} L^{(\alpha,\beta)}(x) = \max_{(\alpha,\beta)} [\langle \mathbf{Q}^{(\alpha,\beta)}x, x\rangle + F(\alpha,\beta)] \ , \quad (6)$$

with $\mathbf{Q}^{(\alpha,\beta)}$ given by [9]

$$\mathbf{Q}^{(\alpha,\beta)} = \begin{cases} \frac{1}{2N_0}\mathbf{R}_s^{(\alpha,\beta)}(\mathbf{R}_s^{(\alpha,\beta)} + N_0\mathbf{I})^{-1} & \text{Case 1} \\ \frac{1}{2}\mathbf{R}_n^{-1}\mathbf{R}_s^{(\alpha,\beta)}\mathbf{R}_n^{-1} & \text{Cases 2 and 3} \end{cases} \quad (7)$$

[3] Although some of the detectors are also optimal, with respect to the deflection criterion, for nonGaussian signals in Gaussian noise [5].

[4] For a fixed value of the parameters (α,β), the optimal detector is of the form $\langle \mathbf{Q}^{(\alpha,\beta)}x, x\rangle$, derived from the likelihood ratio (LR). For unknown or random parameters, a generalized LR test (GLRT) is used. For unknown parameters, maximum likelihood (ML) estimates of the parameters are used in the LR. For random parameters, the optimal detector does not admit a closed-form expression in general. In such cases, we propose MAP (maximum *a posteriori* probability) GLRT detectors in which MAP estimates of the parameters are used (see [5, 9] for details).

and the function $F(\alpha,\beta)$ depends on the joint pdf of (α,β) in the case of random parameters.[5]

3.2. Signal Dependence on Parameters

Since the first term in (6) is a quadratic form, if the parameters (α,β) could be identified as the variables (a,b) of a class of JSRs, the various detectors could be easily and efficiently realized using those JSRs. How should the signal $s(t;\alpha,\beta)$ depend on the parameters (α,β) so that the optimal detector (6) can be naturally realized using JSRs from C? Not surprisingly, the answer is intimately related to the family of unitary operators $\{\mathbf{U}_{(a,b)}\}$ defining the class C. By equating the quadratic form in (6) with (3), we find that for a given class of JSRs, characterized by the unitary operator $\mathbf{U}_{(a,b)}$ in (3), the test statistics can be naturally realized by the JSRs if and only if the operator $\mathbf{Q}^{(\alpha,\beta)}$ in (6) is of the form $((\alpha,\beta)\leftrightarrow(a,b))$.

$$\mathbf{Q}^{(a,b)}=\mathbf{U}_{(a,b)}\phi\mathbf{U}_{(a,b)}^{-1} \tag{8}$$

for some nonnegative definite linear operator ϕ.[6] Using (7), it can be verified that the form for $\mathbf{Q}^{(a,b)}$ in (8) translates into the following signal model:

$$\mathbf{R}_s^{(a,b)}=\mathbf{U}_{(a,b)}\mathbf{R}_0\mathbf{U}_{(a,b)}^{-1}\Leftrightarrow s(t;a,b)=\mathbf{U}_{(a,b)}s^{(a,b)}(t) \tag{9}$$

for some correlation function operator $\mathbf{R}_0$, where, for each (a,b), $s^{(a,b)}$ is any zero-mean Gaussian signal with correlation function R_0.[7] In particular, $s(t;a,b)$ could be one particular Gaussian signal (with correlation function R_0), say s_o, unitarily transformed by $\mathbf{U}_{(a,b)}$; that is, $s(t;a,b)=(\mathbf{U}_{(a,b)}s_o)(t)$.

3.3. JSR-based Optimal Detectors

Substituting (9) in (7) and comparing (6) with (3), we note that the optimal detectors in all the cases, corresponding to the signal model (9) in (5), can be naturally realized using JSRs from C as

$$L(x)=\max_{(a,b)}\left[(\mathbf{P}y)(a,b;\Phi)+F(a,b)\right]\ , \tag{10}$$

$$\text{where}\ \ y=\begin{cases} x & \text{Case 1}\\ \mathbf{R}_n^{-1}x & \text{Cases 2 and 3}\end{cases}\ , \tag{11}$$

the operator Φ characterizing the JSR is given by

$$\Phi=\begin{cases}\mathbf{R}_0(\mathbf{R}_0+N_0\mathbf{I})^{-1} & \text{Case 1}\\ \frac{1}{2}\mathbf{R}_0 & \text{Cases 2 and 3}\end{cases}\ , \tag{12}$$

N_0 is the noise power in Case 1, and the function $F(a,b)$ is given by

$$F(a,b)=\begin{cases}0 & \text{ML : Cases 1 and 2}\\ \frac{1}{2}\text{Trace}(\mathbf{R}_n^{-1}\mathbf{R}_s^{(a,b)}) & \text{ML : Case 3}\\ \log p(a,b) & \text{MAP : Cases 1 and 2}\\ \log p(a,b)+ & \\ \frac{1}{2}\text{Trace}(\mathbf{R}_n^{-1}\mathbf{R}_s^{(a,b)}) & \text{MAP : Case 3}\end{cases} \tag{13}$$

[5]Trace($\cdot$) denotes the trace of an operator (sum of eigenvalues).

[6]Since the test statistics are nonnegative for all (a,b).

[7]The operator $\mathbf{R}$ corresponding to a correlation function R is defined as $(\mathbf{R}s)(t)\equiv\int R(t,\tau)s(\tau)d\tau$.

Thus, an arbitrary family of unitary operators $\{\mathbf{U}_{(a,b)}\}$ satisfying (1) defines, on one hand, a class of JSRs covariant to $\mathbf{U}_{(a,b)}$ and, on the other, a corresponding class of signal detection problems for which such JSRs form canonical detectors. Essentially, any arbitrary second-order random signal that has been transformed by $\mathbf{U}_{(a,b)}$, with the parameters unknown or random, can be optimally detected via (10) by means of JSRs from the class C. In addition to the simple structure of the JSR-based detectors, we note that the (a,b) location at which the maximum occurs in (10) is actually the ML or the MAP estimate of the signal parameters (unknown or random parameters, respectively).

3.4. Example

Detection framework based on the hyperbolic class [10]. The hyperbolic class is covariant to "hyperbolic time-shifts" and scale changes: $\mathbf{U}_{(a,c)}=\mathbf{H}_a\mathbf{D}_c$ where the hyperbolic time-shift operator, $\mathbf{H}_a$, is defined in the frequency domain as $(\mathbf{H}_a\mathbf{F}s)(f)=e^{-j2\pi a\ln(f)}(\mathbf{F}s)(f)$. Thus, the TFRs from hyperbolic class form canonical detectors for random signals transformed by $\mathbf{U}_{(a,c)}$; that is, the corresponding family of *spectral* correlation functions is characterized as

$$\begin{aligned}&\mathbf{S}_s^{(a,c)}=\mathbf{U}_{(a,c)}\mathbf{S}_0\mathbf{U}_{(a,c)}^{-1}\ \Leftrightarrow\\ &S_s^{(a,c)}(f_1,f_2)\equiv E\{(\mathbf{F}s(a,c))(f_1)(\mathbf{F}s(a,c))^*(f_2)\}\\ &=\sum_k c\mu_k e^{-j2\pi a\ln(f_1)}V_k(cf_1)e^{j2\pi a\ln(f_2)}V_k^*(cf_2)\end{aligned} \tag{14}$$

where the μ_k's are the eigenvalues and the V_k's are the eigenfunctions (in the spectral domain) of the spectral correlation function S_0. This framework should be contrasted with that presented in [11] where the detection of *fixed* Gaussian signals (with no unknown parameters) is equivalently formulated using the hyperbolic class, and optimal detection of hyperbolic chirps with unknown parameters is discussed. Our formulation provides optimal detection of arbitrary nonstationary random signals with unknown (or random) scale and hyperbolic time-shift parameters.

4. SUBSPACE-BASED INTERPRETATION

For a given "window" function g, the unitary operator $\mathbf{U}_{(a,b)}$ defines a linear JSR $\mathbf{T}$ via

$$(\mathbf{T}s)(a,b;g)\equiv\langle\mathbf{U}_{(a,b)}g,s\rangle \tag{15}$$

which is covariant to a-b shifts (the operator $\mathbf{U}_{(a,b)}$) and analyzes the joint a-b content in the signal.[8] Moreover, $\mathbf{T}$ is the optimal matched filter for detecting the unitarily transformed deterministic signal, $g_{(a,b)}=\mathbf{U}_{(a,b)}g$, in additive Gaussian noise. Expressing the JSRs in terms of the eigenexpansion of Φ yields a structure in terms of such linear transforms:

$$(\mathbf{P}y)(a,b;\Phi)=\sum_k\mu_k\left|(\mathbf{T}s)(a,b;u_k)\right|^2\ , \tag{16}$$

[8]Generalization of the short-time Fourier transform and the wavelet transform.

where the eigenfunctions (u_k's) of Φ are the same as those of $\mathbf{R}_0$ and the eigenvalues (μ_k's) are given by

$$\mu_k = \begin{cases} \frac{\lambda_k}{\lambda_k+N_0} & \text{Case 1} \\ \lambda_k & \text{Cases 2 and 3} \end{cases} , \quad (17)$$

where the λ_k's are the eigenvalues of $\mathbf{R}_0$. The quadratic JSR detector in (16) is thus simply a weighted sum of the magnitude-squared outputs of a bank of linear a-b transforms corresponding to the eigenfunctions. This yields a subspace-based interpretation: detection of any signal is accomplished by taking a weighted (nonlinear) projection onto the subspace spanned by the eigenfunctions of the signal correlation function. If the signal correlation function is rank-1,[9] then the quadratic detector is effectively reduced to the magnitude-squared output of a linear detector (matched filter). The effect of the unknown or random parameters (a, b) is taken into account by taking their ML or MAP estimates, and then using the optimal detector corresponding to the estimated parameters.

The nonstationary structure of the signal in (9) is effectively determined by the eigenfunctions of $\mathbf{R}_0$. Given a set of eigenfunctions, different sets of eigenvalues generate a whole class of random signals with a similar underlying nonstationary structure.[10] Given knowledge of the eigenfunctions,[11] we can exploit the subspace-based interpretation to design optimal detectors for a class of random signals with different (unknown) distribution of energy in the different eigenmodes. More precisely, ML estimates of the signal eigenvalues can be incorporated into the detector structure, which, for Case 1, are given by [5]

$$\hat{\lambda}_k^{(a,b)}(x) = \max\{0, |(\mathbf{T}x)(a, b; u_k)|^2\} . \quad (18)$$

For Cases 2 and 3, an energy constraint must be imposed [5]

$$\mathrm{Trace}(\mathbf{R}_s^{(a,b)}) = \mathrm{Trace}(\mathbf{R}_0) = \sum_k \lambda_k \leq d , \quad (19)$$

for some bound $d > 0$, under which the ML estimates for the eigenvalues for Case 2 are given by [5]

$$\tilde{\lambda}_k = \begin{cases} d & \text{if } k = \arg\max_i \left\{ |(\mathbf{T}\mathbf{R}_n^{-1}x)(a, b; u_i)| \right\} \\ 0 & \text{else} \end{cases} \quad (20)$$

and those for Case 3 are [5]

$$\breve{\lambda}_k = \begin{cases} d & \text{if } k = \arg\max_i \left\{ A_i = |(\mathbf{T}\mathbf{R}_n^{-1}x)(a, b; u_i)|^2 \right. \\ & \quad \left. - \langle \mathbf{U}_{(a,b)}u_i, \mathbf{R}_n^{-1}\mathbf{U}_{(a,b)}u_i \rangle : A_i > 0 \right\} \\ 0 & \text{else} \end{cases} \quad (21)$$

These eigenvalue estimates can then be used as the combining weights for the matched-filter banks in (16) (via (17)) to realize the various optimal detectors as described by (10).

[9] Only one term in the eigenexpansion.

[10] The eigenvalues correspond to the energy in the different natural (eigen) modes of the random signal.

[11] Which may, for example, be available from an estimate of the signal correlation function, or, from a model for the signal eigenfunctions (such as a wavelet basis [3]).

5. CONCLUSIONS

By developing an optimal detection framework based on arbitrary covariance-based JSRs, the results of this paper facilitate the design of optimal JSR-based detection procedures for a rich variety of nonstationary scenarios. Such JSR-based detectors can be used to detect (in the presence of noise/interference) arbitrary second-order random signals that have been transformed by certain unitary transforms that may model a wide variety of nondissipative (energy preserving) signal distortions. The structure of the detectors yields a very useful subspace-based formulation that can be exploited to design optimal detectors even in situations in which partial signal information about the eigenfunctions only may be available.

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