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An Improved Multifractal Formalism and Self-Similar Measures

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Abstract

To characterize the geometry of a measure, its so-called generalized dimensions d_q have been introduced recently. The mathematically precise definition given by Falconer [F2] turns out to be unsatisfactory for reasons of convergence as well as of undesired sensitivity to the particular choice of coordinates in the negative q range. A new definition is introduced, which is based on box-counting too, but which carries relevant information about μ also for negative q. In particular, rigorous proofs are provided for the Legendre connection between generalized dimensions and the so-called multifractal spectrum and for the implicit formula giving the generalized dimensions of self-similar measures, which was until now known only for positive q.

1 Introduction

Given a compact set K in Euclidean space $\mathbb{R}^d$, such as the attractor of dynamical systems, the notion of Hausdorff dimension $d_{\mathrm{HD}}(K)$ [F2] has been used successfully to characterize K [FM]. But one single number such as the dimension is usually too crude and can only describe a global aspect of the geometry of K. More subtle structures may be detected when considering an appropriate measure with support K. Moreover, fractal sets are often insufficient in order to model nature. In a dynamical system, e.g. many essential features such as the long time behaviour of orbits can not be represented by a set, but rather by a measure. To give a second example, fractal sets may approximate porous media

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but not their content of some liquid. So, measures have become of increasing interest, in particular their local properties.

Throughout the paper μ will denote a Borel measure in $\mathbb{R}^d$ with bounded support K. To get an intuition of the kind of geometrical structure of μ studied in this paper, think of K as the union of infinitely many interwoven subsets K_α, usually fractals, with homogeneous concentration of μ. Based on this motivation μ has been termed *multifractal* [EvM, MEH, HJKPS], with the *multifractal decomposition* K_α.

To be more precise let $U(x,\delta)$ denote the closed ball of radius δ centered in x. The quantities

$$\overline{d}_\mu(x) := \limsup_{\delta\downarrow 0} \frac{\log \mu(U(x,\delta))}{\log \delta}, \qquad \underline{d}_\mu(x) := \liminf_{\delta\downarrow 0} \frac{\log \mu(U(x,\delta))}{\log \delta} \tag{1}$$

are called *upper (lower) pointwise dimension at x*. When they coincide, the common value is denoted by $d_\mu(x)$. In multifractal theory, one is interested in the Hausdorff dimension of sets like

$$\begin{aligned} K_\alpha &:= \{x : \underline{d}_\mu(x) = \overline{d}_\mu(x) = \alpha\} \\ \text{and } C_\alpha &:= \{x : \underline{d}_\mu(x) \le \alpha\}. \end{aligned} \tag{2}$$

Thereby, the Legendre transform has turned out to be a useful tool linking $f(\alpha) = d_{\mathrm{HD}}(K_\alpha)$ as a function of α, called the *multifractal spectrum of μ*, with the *singularity exponents* $\tau(q)$, which are given by [F2]

$$s_\delta(q) := \sum_{\mu(B)\neq 0} \mu(B)^q \qquad \tau(q) := \limsup_{\delta\downarrow 0} \frac{\log s_\delta(q)}{-\log \delta}. \tag{3}$$

(Here, the sum runs over a partition of $\mathbb{R}^d$ into cubes B of side δ.) The *generalized dimensions* $d_q := \tau(q)/(1-q)$ are interesting of their own: in the case of a dynamical system they are directly observable from the longtime behaviour of orbits [G, HP]. Moreover, they depend more regularly on the data $\mu(B)$ [JKL, HJKPS] and are therefore more easy to handle analytically and numerically.

According to the particular interests different notions of singularity exponents and 'dimension distributions' have been developed in various fields such as measure theory, dynamical systems and applied mathematics, i.e. with emphasis on box-counting methods. Here comes a short review on some of them.

The self-similar measures (SMF) are probably the best known multifractals. The multifractal spectrum for a large class of such measures has been calculated in [CM].

Thereby, the codespace is an invaluable tool (subsection 3.3). As an interesting corollary (31) one has

$$\underline{d}_\mu(x) = \overline{d}_\mu(x) = \alpha_1 \qquad \text{for } \mu\text{-almost every } x, \tag{4}$$

where α_1 does not depend on x. But note, that the range of $d_\mu(x)$ is a whole interval $[\alpha_\infty, \alpha_{-\infty}]$. The multifractal spectrum of broader classes of invariant measures have been found by Falconer, Edgar & Mauldin, Schmeling & Siegmund-Schultze, Collet et al. and Brown et al. [F3, EdM, S, CLP, BMP].

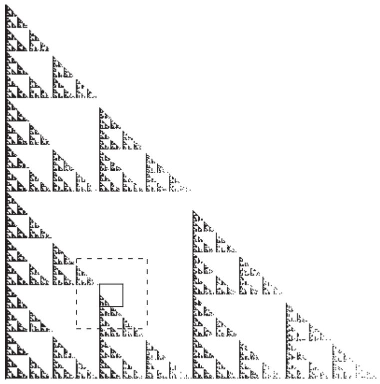
Of great interest are the ergodic invariant measures in the theory of dynamical systems. Here, naturally involved structures such as Markov partitions and Gibbs measures allow one to work with similar methods as with SMF. In particular, making essential assumptions on the structure of the invariant measure, formulae have been found for the Hausdorff dimension of its support [Ru, F1] as well as for the multifractal spectrum and related characteristics [BMP, CLP, BPTV, Ra, BR, L]. Thereby, some authors develop own notions of singularity exponents which serve as a powerful tool, but which only apply to the special situations under consideration. Relations between dimensionlike quantities (such as generalized dimensions and Hausdorff dimension) and characteristics of dynamical systems (such as Lyapunov exponents, entropy and pressure) are given in [BPTV, Y, P1]. Pesin [P3] gives a survey of different notions of 'generalized spectra for dimensions' which apply to arbitrary Borel measures μ. Two of them, denoted by γ_q and α_q, are reviewed in subsection 4.2. Roughly speaking, γ_q is sensitive to the geometry of the support of μ while α_q emphasizes on the set where μ is concentrated. As one might suggest, $\alpha_q \equiv \alpha_1$ in the situation of (4). On the other hand, γ_q coincides with our notion and is therefore different from α_q in general.

The approach by Cutler [C] is tailored to measures theory and applies to finite Borel measures μ, providing a 'dimension distribution' $\hat{\mu}$ of a random variable $\hat{\alpha}(x)$, which is related to $\underline{d}_\mu(x)$ through $\hat{\mu}([0,\alpha]) = \mu(C_\alpha)$. In the situation of (4), $\hat{\mu}$ reduces to the Dirac measure concentrated in α_1.

Finally, Falconer [F2] developed a multifractal formalism which is based on box-counting methods (see (3) and also [EvM, HP]). The advantage of such an approach is its relevance in numerical simulations. Unfortunately, (3) turns out to be unsatisfactory for reasons of convergence as well as for an undesired dependence on coordinates. The difficulties (as with the notions of Pesin) are imperceptibly hidden in the negative q domain (Ex. 1).

It is the aim of this paper to present an improvement of (3): a simple but effective change in the way of measuring the concentrations of μ is enough to make the singularity exponents a useful tool for negative q also. The notion presented here serves to detect

Figure 1: The picture shows the support K of a probability measure and a box B (solid) which intersects K in a point, say x. It becomes apparent that the enlarged and concentric box $(B)_1$ (dashed) constitutes a better approximation of a ball centered in x.



some structure of arbitrary Borel measures by box-counting methods. Since it coincides with γ_q introduced in [P3] it is also relevant for dynamical systems and measure theory. Finally, as a satisfying result, the well known formula [CM] for the spectrum of SMF is shown to hold also in our formalism. So, the whole range of $d_\mu(x)$ can be observed numerically through our notion F.

Here is the organization of the paper. Section two introduces the new notion and proves that spectrum and exponents are related through the Legendre transform. In section three the self-similar measures are treated. Section four gives examples and relations to the work of Pesin [P3].

2 An improved formalism

An improvement of (3) is proposed. Our idea is as simple as effective: we use the measure of boxes blown up by a factor three. The essential geometrical argument in the proofs below will be the following: whenever a box intersecting K is considered, the enlarged concentric box meets K in its 'middle' and is a better approximation of a ball centered in K than the original box (see figure 1). Thus, we feel that this method is more accurate to measure local behaviours such as the pointwise dimension.

This does not mean, however, that every multifractal can be described entirely by its spectrum. In particular, the newly defined singularity exponents may be infinite and so-called left-sided spectra may occur. For examples see [MEH, ME, R]. But it is important to notice that with the new concept infinite singularity exponents imply arbitrarily small balls with center in K and arbitrarily small measure, while in the former formalism $\tau(q) = \infty$ may equally well arise from inappropriate measurement (Ex. 1).

2.1 Basic Properties

Let G_δ be the family of all so-called *δ-boxes*, or simply *boxes* $B = \prod_{k=1}^{d} [l_k\delta, (l_k+1)\delta[$ with integral l_k and with nonvanishing measure. For $\kappa > 0$ let

$$(B)_\kappa := \prod_{k=1}^{d} [(l_k - \kappa)\delta, (l_k + 1 + \kappa)\delta[. \tag{5}$$

As will be shown, the particular choice of κ is of no importance, as long as it is kept fixed through the process. For numerical simulations it might be most convenient to choose $\kappa = 1$. We remind the reader that μ denotes a Borel measure in $\mathbb{R}^d$ with bounded support.

Definition 1 (Singularity Exponents and Generalized Dimensions) *For $q \in \mathbb{R}$ let*

$$S_\delta(q) := \sum_{B\in G_\delta} \Big(\mu((B)_1)\Big)^q \quad \textit{and} \quad T(q) = \limsup_{\delta\downarrow 0} \frac{\log S_\delta(q)}{-\log\delta}.$$

The value ∞ is allowed. If the $\lim_{\delta\to 0}$ exists for a particular q, then $T(q)$ will be called grid-regular. *The* generalized dimensions *[G, HP, HJKPS] are then given by*

$$D_q := \frac{1}{1-q} T(q) \qquad (q \neq 1) \qquad D_1 := \limsup_{\delta\downarrow 0} \frac{1}{\log(\delta)} \frac{\sum_{B\in G_\delta} \mu((B)_1)\log\mu((B)_1)}{\sum_{B\in G_\delta} \mu((B)_1)}.$$

Note that the condition '$\mu(B) \neq 0$' chooses the boxes, not '$\mu((B)_1) \neq 0$'. This is the central idea of the new formalism (see also figure 1).

The same argument that gives the independence of T from the choice of κ also proves its invariance under a considerable class of coordinate transformations and justifies the restriction of the δ to an admissible sequence. The sequence δ_n is called *admissible* if there is a $\upsilon > 0$ such that $\delta_n \geq \delta_{n+1} \geq \upsilon\delta_n$ for all n.

Proposition 2 *Let Φ be a bi-lipschitz coordinate transformation, let $\mu' := \mu(\Phi^{-1}(\cdot))$, $\kappa' > 0, \kappa > 0$ and let $(\delta_n)_{n\in\mathbb{N}}$ be an admissible sequence. Then for all $q \in \mathbb{R}$*

$$T(q) = \limsup_{n\to\infty} \frac{\log\left(\sum_{B\in G_{\delta_n}} \mu((B)_\kappa)^q\right)}{-\log\delta_n} = \limsup_{\delta\downarrow 0} \frac{\log\left(\sum_{B\in G'_\delta} \mu'((B)_{\kappa'})^q\right)}{-\log\delta}.$$

Remarks As a corollary of theorem 19 the former notion (3) and ours coincide for positive q, i.e. one has $\tau(q) = T(q)$ $(q \geq 0)$. Moreover, the same independence as above holds also for τ, but only when $q \geq 0$ [R], as the two admissible sequences in example 1 show.

Proof Let $G'_{\delta'}$ be the set of δ'-boxes B with $\mu'(B) \neq 0$, furthermore set $S'_{\delta'}(q, \kappa') = \sum_{B \in G'_{\delta'}} \mu'((B)_{\kappa'})^q$ and $S_\delta(q, \kappa) = \sum_{B \in G_\delta} \mu((B)_\kappa)^q$. One idea for the proof is to use the fact, that $S'_{\delta'}$ and S_δ are 'averages', i.e. sums. More precisely, to every $B' \in G'_{\delta'}$ we will assign a box $B_{B'} \in G_\delta$ such that $\mu'((B')_{\kappa'})^q \leq \mu((B_{B'})_\kappa)^q$. To meat the fact that different boxes B' can have the same counterpart $B_{B'}$ the terms in S_δ will be repeated sufficiently many times.

i) Take $\delta' > 0$, $B' \in G'_{\delta'}$. Writing $C := \Phi^{-1}(B')$ and $D := \Phi^{-1}((B')_{\kappa'})$ for short, $\mu(C) = \mu'(B') \neq 0$ and $\mathrm{diam}(D) \leq L \cdot \mathrm{diam}((B')_{\kappa'}) = L\sqrt{d}(1 + 2\kappa')\delta'$. For every $\delta > 0$, the choice of which is postponed at the moment, the δ-boxes constitute a covering of $\mathbb{R}^d$. Hence there must be one of them which meets C and is not a μ-nullset. Denote this box from G_δ by $B_{B'}$.

ii) Assume first that $q \geq 0$. The constructed box $B_{B'}$ should be large. Set $\eta_1 := \kappa^{-1} L\sqrt{d}(1 + 2\kappa')$ and take any δ from $[\eta_1\delta', \upsilon^{-1}\eta_1\delta']$. Since $\kappa\delta \geq \mathrm{diam}(D)$, D is contained in $(B_{B'})_\kappa$ and

$$\mu((B_{B'})_\kappa) \geq \mu(D) = \mu'((B')_{\kappa'}) \neq 0.$$

The same estimate holds for the q–th powers. The given relation $B' \to B_{B'}$ is not one-to-one, but the number of all δ'-boxes B' for which the same fixed box B^* has been assigned as $B_{B'}$ is bounded by the constant $b_1 := (L\sqrt{d}\upsilon^{-1}\eta_1 + 2)^d$. To see this note, that every B' with $B_{B'} = B^*$ must intersect $\Phi(B^*)$ and that $\mathrm{diam}(\Phi(B^*)) \leq L\sqrt{d}\delta \leq L\sqrt{d}\upsilon^{-1}\eta_1\delta'$. So, repeating each term in $S_\delta(q, \kappa)$ b_1 times produces a counterpart for every term in $S'_{\delta'}(q, \kappa')$, i.e.

$$S'_{\delta'}(q, \kappa') = \sum_{B' \in G'_{\delta'}} \mu'((B')_{\kappa'})^q \leq b_1 \sum_{B \in G_\delta} \mu((B)_\kappa)^q = b_1 S_\delta(q, \kappa). \tag{6}$$

Interchanging K with K' and κ with κ' yields corresponding constants η_2 and b_2 with

$$S_\delta(q, \kappa) \leq b_2 S'_{\delta'}(q, \kappa') \qquad \forall\, \delta' \in [\eta_2\delta, \upsilon^{-1}\eta_2\delta]. \tag{7}$$

iii) Given δ', n can be chosen such that $\delta_n \geq \eta_1\delta' > \delta_{n+1}$. Then $\delta_n \leq \upsilon^{-1}\delta_{n+1} \leq \eta_1\upsilon^{-1}\delta'$. Applying (7) with $\delta' = \eta_2\delta$ and (6) with $\delta = \delta_n$ implies

$$\limsup_{\delta\downarrow 0} \frac{\log S_\delta(q,\kappa)}{-\log\delta} \leq \limsup_{\delta'\downarrow 0} \frac{\log S'_{\delta'}(q,\kappa')}{-\log\delta'} \leq \limsup_{n\to\infty} \frac{\log S_{\delta_n}(q,\kappa)}{-\log\delta_n}.$$

Since the last term is smaller than the first, the desired equalities follow.

iv) Consider now the case $q < 0$. This time the constructed box $B_{B'}$ should be small. Set $\eta_3 := ((1+\kappa)\sqrt{d}L)^{-1}$. For any $\delta \leq \eta_3 \cdot \delta'$ the set $(B_{B'})_\kappa$ is contained in D and so $0 \neq \mu(B_{B'}) \leq \mu((B_{B'})_\kappa) \leq \mu(D) = \mu'((B')_{\kappa'})$. Raising the inequality to the q–th power reverses the sign. The rest of the argument is the same as above. $\diamondsuit$

Falconer's definition of the spectrum will now be modified in the same way as the one for the singularity exponents.

Definition 3 (Spectrum) *Let* $N_\delta(\alpha) := \#\{B \in G_\delta \,:\, \mu((B)_1) \geq \delta^\alpha\}$ *and define*

$$F(\alpha) := \lim_{\varepsilon\downarrow 0}\limsup_{\delta\downarrow 0} \frac{\log(N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon))}{-\log\delta},$$

where $\log 0 := -\infty$. *We will call this function the* box-counting spectrum *or just* spectrum *of* μ. *It will be shown to be related to the multifractal spectrum* $f(\alpha) = d_{\mathrm{HD}}(K_\alpha)$. *When the limit* $\delta \to 0$ *exists for a particular* α, *we call* $F(\alpha)$ *grid-regular.*

It should be emphasized that this notion uses the measures $\mu((B)_1)$, where the boxes B have been selected by the condition $\mu(B) \neq 0$. By definition F is positive when real-valued.

2.2 The Legendre Transform

An important tool in multifractal theory is the Legendre transform. First, we state a result which will not be used later but which supports the saying, that T is more regular than F [HJKPS, JKL]. In particular, T is always convex (lemma 5) while F need not be concave everywhere (Ex. 2).

Proposition 4

$$T(q) = \sup_{\alpha\in\mathbb{R}}(F(\alpha) - q\alpha) \qquad \forall q \neq 0.$$

Moreover, if there exists $\beta < \infty$ such that $\mu((B)_1) \geq \delta^\beta$ for all $B \in G_\delta$ and all sufficiently small $\delta > 0$, then

$$T(0) = \sup_{\alpha \in \mathbb{R}} F(\alpha).$$

Proof The essential idea is given in the proof of proposition 17.2 in [F2]. However, some refinement is needed for $q \leq 0$ which can be found in [R]. $\diamondsuit$

Lemma 5 (Infinite values of T) a) *If $q \geq 0$ then $-dq \leq T(q) \leq d$.*

b) *Either T is infinite for all $q < 0$ or T is real-valued for all $q \in \mathbb{R}$.*

c) *The function $T(q)$ is continuous, convex and nonincreasing where it is real-valued.*

Proof a) For any $q \geq 0$ one has $S_\delta(q) \leq \# G_\delta$. As a consequence of the boundedness of the support of μ there is a constant c with $\# G_\delta \leq c \cdot \delta^{-d}$. Hence, $T(q) \leq d$. Furthermore, there is a box $B \in G_\delta$ with $\mu(B) \geq 1/\# G_\delta$. Hence, $S_\delta(q) \geq \delta^{dq}/c^q$ and $T(q) \geq -dq$.
b) Fix $q < 0$ and let α be any positive number. If there is no β as in proposition 4, then there are arbitrarily small $\delta > 0$ with $S_\delta(q) \geq \delta^{q\alpha}$. Hence, $T(q) \geq -q\alpha$. Since α is arbitrary $T(q) = \infty$ for all $q < 0$. (This was not recognized in [F2].) If there exists a β as in proposition 4, then $1 \leq S_\delta(q) \leq \# G_\delta \cdot \delta^{q\beta}$ for sufficiently small $\delta > 0$. Hence, $0 \leq T(q) \leq d - q\beta < \infty$ for all $q < 0$.
c) Continuity and convexity as stated are properties of the Legendre transform. Obviously, $S_\delta(q)$ is nonincreasing in q for every $\delta > 0$ which carries over to T. $\diamondsuit$

By means of proposition 4 it is easy to calculate the singularity exponents once the spectrum is known. In typical applications however one will meet the converse situation: one would like to be able to deduce the spectrum from the singularity exponents. This would be straightforward if differentiability and concavity of the spectrum would be known *in advance*. Such properties can be established a priori only for a multifractal formalism distinct from ours [BMP, CM, CLP, EdM, BR], and may not hold in our situation (Ex. 2). Therefore, we prefer a different approach which does not make use of proposition 4.

We need a result of Ellis' on large deviations [E, page 3, theorem II.2]. Due to lemma 5 we do not need it in its full strength and restate a simplified version. Let $(\Omega_n, \mathcal{B}_n, P_n)$ be a sequence of probability spaces. For each n, let Y_n be a $\mathcal{B}_n-$ Borel-measurable map of Ω_n into $\mathbb{R}^N$. Given $t \in \mathbb{R}^N$ let

$$c_n := (1/a_n) \log E_n[\exp\langle t, Y_n \rangle],$$

where the $\{a_n\}$ are a fixed sequence of positive numbers tending to infinity, E_n denotes expectation with respect to P_n, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^N$.

Theorem 6 (Ellis) *Assume, that $c(t) := \lim_{n\to\infty} c_n(t)$ exists, is real, convex and differentiable for all $t \in \mathbb{R}^N$. Let*

$$I(z) := \sup_{t\in\mathbb{R}^N} \Big\{\langle t, z\rangle - c(t)\Big\}, \qquad z \in \mathbb{R}^N.$$

Given a subset A of $\mathbb{R}^N$, define $I(A) := \inf\{I(z) : z \in A\}$. Then
a) *For any closed subset H of $\mathbb{R}^N$,*

$$\limsup_{n\to\infty} (1/a_n) \log P_n[a_n^{-1} Y_n \in H] \le -I(H).$$

b) *For any open subset G of $\mathbb{R}^N$,*

$$\liminf_{n\to\infty} (1/a_n) \log P_n[a_n^{-1} Y_n \in G] \ge -I(G).$$

Our result on the Legendre transform in the direction opposite to proposition 4 is:

Theorem 7 *If T is grid-regular, differentiable and convex on $\mathbb{R}$, then*

$$F(\alpha) = \lim_{\varepsilon\downarrow 0} \lim_{\delta\downarrow 0} \frac{\log[N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon)]}{-\log\delta} = \inf_{q\in\mathbb{R}} (T(q) + q\alpha)$$

for all α. In particular, $F(\alpha) = T(q) - qT'(q)$ at $\alpha = -T'(q)$, F is continuous in the closure of the range of $-T'(q)$ and takes the value $-\infty$ elsewhere.

Proof Write $l(\alpha) = \inf_{q\in\mathbb{R}}(T(q) + q\alpha)$. The notation of theorem 6 is kept in use.

i) Fix any sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers which tends to zero. Let $\Omega_n := G_{\delta_n}$, associated with its powerset $\mathcal{B}_n$ and with the uniform probability distribution P_n. Choose the random variables $Y_n(B) := \log(\mu((B)_1)$ on G_{δ_n} and calculate their moment generating functions:

$$E_n[e^{qY_n}] = \frac{1}{\# G_{\delta_n}} \sum_{B\in G_{\delta_n}} \mu((B)_1)^q = \frac{1}{S_{\delta_n}(0)} S_{\delta_n}(q).$$

Choosing $a_n := -\log \delta_n$ leads to

$$c_n(q) = \frac{1}{a_n} \log E_n[e^{qY_n}] = \frac{\log S_{\delta_n}(q)}{-\log\delta_n} - \frac{\log S_{\delta_n}(0)}{-\log\delta_n}.$$

Thus

$$c(q) = T(q) - T(0) \qquad\qquad I(z) = T(0) - l(-z),$$

and the hypotheses of Ellis' theorem are satisfied.

ii) Fix any α and set $H := [-\alpha, \infty[\,, G :=]-\alpha, \infty[$. First, $I(H)$ and $I(G)$ will be computed using well known results about the Legendre transform. Let $[\alpha_\infty, \alpha_{-\infty}]$ denote the closure of the range of $-T'$. Obviously, $l(\alpha) = T(q) - qT'(q)$ at $\alpha = -T'(q)$ while $l(\alpha) = -\infty$ for $\alpha \notin [\alpha_\infty, \alpha_{-\infty}]$. There is a unique maximum of l at $\alpha_0 = -T'(0)$. With the continuity of l in $[\alpha_\infty, \alpha_{-\infty}]$ one obtains

$$I(H) = I(G) = \begin{cases} \infty = -l(\alpha) & \text{if } \alpha < \alpha_\infty, \\ T(0) - l(\alpha) & \text{if } \alpha_\infty < \alpha \le \alpha_0, \\ T(0) - l(\alpha_0) & \text{if } \alpha_0 < \alpha. \end{cases}$$

The value $\alpha = \alpha_\infty$ has to be omitted to guarantee the first equality. With

$$P_n[a_n^{-1} Y_n \in G] \le P_n[a_n^{-1} Y_n \in H] = P_n[Y_n \ge \log {\delta_n}^\alpha] = \frac{1}{S_{\delta_n}(0)} N_{\delta_n}(\alpha)$$

theorem 6 gives

$$\limsup_{n\to\infty} \frac{\log N_{\delta_n}(\alpha)}{-\log \delta_n} - T(0) = \limsup_{n\to\infty} (1/a_n) \log P_n[a_n^{-1} Y_n \in H] \le -I(H),$$

$$\liminf_{n\to\infty} \frac{\log N_{\delta_n}(\alpha)}{-\log \delta_n} - T(0) \ge \liminf_{n\to\infty} (1/a_n) \log P_n[a_n^{-1} Y_n \in G] \ge -I(G).$$

iii) Since the sequence δ_n was arbitrary, ii) yields

$$F^+(\alpha) := \lim_{\delta \downarrow 0} \frac{\log N_\delta(\alpha)}{-\log \delta} = \begin{cases} l(\alpha) & \text{if } \alpha \le \alpha_0,\ \alpha \ne \alpha_\infty \\ l(\alpha_0) & \text{if } \alpha > \alpha_0. \end{cases}$$

This function $F^+(\alpha)$ is strictly monotonous increasing in $]\alpha_\infty, \alpha_0[$. Thus, $F^+(\alpha - \varepsilon) < F^+(\alpha + \varepsilon)$ for any $\alpha \in]-\infty, \alpha_0]$ and any $\varepsilon > 0$, where $-\infty < -\infty$ is accepted. In the case $F^+(\alpha - \varepsilon) \ne -\infty$ this means that $N_\delta(\alpha + \varepsilon)$ grows essentially faster than $N_\delta(\alpha - \varepsilon)$. More precisely: choose $\eta > 0$ such that $F^+(\alpha - \varepsilon) + 3\eta \le F^+(\alpha + \varepsilon)$ and take $\delta_0(\eta) > 0$ such that

$$N_\delta(\alpha + \varepsilon) \ge \delta^{-F^+(\alpha+\varepsilon)+\eta}, \quad N_\delta(\alpha - \varepsilon) \le \delta^{-F^+(\alpha-\varepsilon)-\eta}, \quad \delta^\eta \le \frac{1}{2}$$

for all $\delta < \delta_0$. Then

$$\frac{1}{2}N_\delta(\alpha+\varepsilon) \leq N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon) \leq N_\delta(\alpha+\varepsilon) \qquad \forall\delta < \delta_0.$$

This is trivial when $F^+(\alpha-\varepsilon) = -\infty$. So, $F(\alpha)$ is grid-regular, and $F(\alpha) = F^+(\alpha+) = l(\alpha)$ for all $\alpha \in]-\infty, \alpha_0]$.

iv) To obtain $F(\alpha) = l(\alpha)$ for $\alpha \geq \alpha_0$ change the definitions of H and G to $H :=]-\infty, -\alpha]$ and $G :=]-\infty, -\alpha[$ and work with $M_\delta(\alpha) := \#\{B \in G_\delta : \mu((B)_1) < \delta^\alpha\}$ instead of $N_\delta(\alpha)$, using that $N_\delta(\alpha+\varepsilon) - N_\delta(\alpha-\varepsilon) = M_\delta(\alpha-\varepsilon) - M_\delta(\alpha+\varepsilon)$. $\diamondsuit$

For completeness we remark that [R] gives a 'local version' of the theorem above, the proof of which is based on proposition 4.

Theorem 8 *If T is differentiable at $q \neq 0$ and if $\alpha = -T'(q)$, then*

$$F(\alpha) = T(q) - qT'(q). \tag{8}$$

3 Self-Similar Measures

The self-similar measures (see below) are probably the simplest measures with nontrivial multifractal spectrum. Quite some time ago heuristic arguments have been given which suggest a simple formula for the singularity exponents $\tau(q)$ [HP]. But example 1 shows that a careful treatment is needed. So far we are not aware of a rigorous calculation of singularity exponents or spectrum based on box-counting. Therefore, this section is considered an important contribution in the multifractal theory.

Note, that the multifractal spectrum $f(\alpha) = d_{\text{HD}}(K_\alpha)$, has been calculated in [CM]. As one might hope, this f and our spectrum F coincide.

This section is divided into three parts. First, the usual formalism for Cantor sets and symbolic dynamics are introduced. For a deeper treatment of the statements made in this subsection, [Hut] is a good reference. Next, the singularity exponents T and the spectrum F of self-similar measures are computed. Finally, a short argument is given which leads directly to the multifractal decomposition of SMF.

3.1 Multiplicative Cascades

Fix a natural number r. To define a so-called *r-adic Cantor set K* take a compact subset V of $\mathbb{R}^d$ and choose r closed subsets $V_1, \ldots, V_r$ of V, not necessarily disjoint. Go on like this, inductively choosing r closed subsets $V_{\underline{i}*k}$ $(k = 1, \dots, r)$ of $V_{\underline{i}}$, where

$$\underline{i} := i_1 \dots i_n \ \in I_n := \{1, \dots, r\}^n \quad \text{and} \quad \underline{i} * k := i_1 \dots i_n k.$$

Call $\underline{i}$ a word of *length* $|\underline{i}| = n$. Finally define a cascade by

$$K_n := \bigcup_{\underline{i} \in I_n} V_{\underline{i}} \qquad K := \bigcap_{n \in \mathbb{N}} K_n. \tag{9}$$

The same labels $\underline{i}$, $\underline{j}$ etc. will be used for finite and infinite sequences. However, it will always be made explicit when a particular sequence is meant to be infinite. For any infinite or sufficiently long sequence $\underline{i}$ set $(\underline{i}|n) := i_1 \dots i_n$. If $\operatorname{diam}(V_{(\underline{i}|n)}) \to 0$ as $n \to \infty$ for any infinite sequence $\underline{i} \in I_\infty := \{1, \dots, r\}^{\mathbb{N}}$, then $\cap V_{(\underline{i}|n)}$ is a singleton, say $\{x_{\underline{i}}\}$, and the *coordinate map*

$$\pi \ : I_\infty \to K \quad \underline{i} \mapsto x_{\underline{i}}$$

is continuous and surjective. Now let $(p_1, \dots, p_r)$ be a *probability vector*, i.e. $p_i > 0$ and $p_1 + \dots + p_r = 1$, and let P be the product measure on I_∞ induced by the measure $\{j\} \mapsto p_j$ on the factors $\{1 \dots r\}$, i.e.

$$P[\{\underline{i} \in I_\infty \ : \ i_{k_m} = j_m, \ m = 1 \dots n\}] = p_{j_1} \cdot \dots \cdot p_{j_n} =: p_{\underline{j}} \tag{10}$$

for all $n \in \mathbb{N}$, all words $\underline{j}$ of length n and all integers $k_1 < \dots < k_n$.

Definition 9 *The measure $\mu := P(\pi^{-1}(\cdot))$ will be called a* Cantor multifractal *(*CMF*).*

Remark The construction of a CMF as described above has been termed *multiplicative cascade* [EvM].

The support of μ is $K = \pi(I_\infty)$, its total mass is $\mu(\mathbb{R}^d) = 1$. From its definition

$$\mu(V_{\underline{i}}) = P[\{\underline{j} \in I_\infty \ : \ \pi(\underline{j}) \in V_{\underline{i}}\}] \geq P[\{\underline{j} \in I_\infty \ : \ (\underline{j}|\,|\underline{i}|) = \underline{i}\}] = p_{\underline{i}} \tag{11}$$

with equality holding in particular if $V_{\underline{i}}$ does not intersect any $V_{\underline{k}}$ with $|\underline{k}| = |\underline{i}|$. The following lemma might be important in numerical simulations.

Lemma 10 *Given any μ, substituting the condition '$\mu(B) \neq 0$' in the definition of G_δ (see definition 1) by the condition '$B \cap K \neq \emptyset$' will not affect the value $T(q)$.*

In particular for $q = 0$:

Corollary 11 *The box dimension [F2] of the support of any μ equals $D_0 = T(0)$.*

Proof Let B denote a δ-box. Then, since $B \cap K \neq \emptyset$ is a stronger requirement than $\mu(B) \neq 0$,

$$S_\delta(q) \leq \sum_{B \cap K \neq \emptyset} \mu((B)_1)^q.$$

On the other hand if $B \cap K \neq \emptyset$ holds, then we must have

$$\mu((B)_{1/2}) \neq 0.$$

Thus there exists $C_B \in G_{\delta/2}$ and $D_B \in G_{2\delta}$, i.e. with nonvanishing measure, both meeting $(B)_{1/2}$. Hence $(C_B)_1 \subset (B)_1$, $(D_B)_1 \supset (B)_1$, and

$$\mu((B)_1)^q \leq \mu((C_B)_1)^q \ (q < 0) \qquad \mu((B)_1)^q \leq \mu((D_B)_1)^q \ (q \geq 0).$$

Moreover at most 2^d (resp. 5^d) δ-boxes B can share the same fixed C from $G_{\delta/2}$ as C_B (resp. the same fixed D from $G_{2\delta}$ as D_B). The estimate

$$\sum_{B \cap K \neq \emptyset} \mu((B)_1)^q \leq \sum_{B \in G_\delta} \mu((C_B)_1)^q \leq 2^d \sum_{C \in G_{\delta/2}} \mu((C)_1)^q \qquad (q < 0)$$

results and a similar one for $q \geq 0$. This proves the lemma. $\diamondsuit$

3.2 Singularity Exponents and Spectrum

Now let $(w_1, \ldots, w_r)$ be a set of contracting *similarities of* $\mathbb{R}^d$ *with ratios* $\lambda_1, \ldots, \lambda_r$, i.e. $\lambda_i \in]0,1[$ and $|w_i(x) - w_i(y)| = \lambda_i \cdot |x - y| \ \forall x \forall y \in \mathbb{R}^d$ $(i = 1, \ldots, r)$. Assume further the existence of a nonempty open bounded set O such that

$$w_i(O) \subset O \quad (i = 1, \ldots, r) \qquad \text{and} \qquad w_i(O) \cap w_j(O) = \emptyset \quad (i \neq j). \tag{12}$$

This property was termed *open set condition*, or *OSC* for short, and O is called a *basic open set*. Now, letting

$$V := \overline{O} \qquad \text{and} \quad V_{\underline{i}} := w_{\underline{i}}(\overline{O}) := w_{i_1} \circ \ldots \circ w_{i_n}(\overline{O}) \tag{13}$$

establishes an r-adic Cantor set, for which the coordinate map is well-defined.

Definition 12 *Assume (12). A CMF constructed by (13) is called a* Self-similar Multifractal (SMF) with ratios $\lambda_1, \ldots, \lambda_r$ and probability vector $(p_1, \ldots, p_r)$. *It is the unique probability measure with bounded support satisfying the invariance [Hut]*

$$\mu = \sum_{i=1}^{r} p_i \cdot \mu({w_i}^{-1}(\cdot)) \tag{14}$$

In order to compute the singularity exponents of a SMF one could deduce a recursive law for $S_\delta(q)$ from the invariance of μ. In [HP] a heuristic argument is given, which uses this idea. However, we prefer a different approach: we compare the covering of K by boxes B from G_δ with the covering by cylindrical sets $V_{\underline{i}}$ with $\underline{i}$ from J_δ, where

$$J_\delta := \{\underline{i} = i_1 \dots i_n \in I \, : \, \lambda_{\underline{i}} \le \delta < \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_{n-1}}\}. \tag{15}$$

Thereby we have the approximation

$$S_\delta(q)\delta^\gamma = \sum_{B \in G_\delta} \mu((B)_1)^q \delta^\gamma \simeq \sum_{\underline{i} \in J_\delta} \mu(V_{\underline{i}})^q \delta^\gamma \simeq \sum_{\underline{i} \in J_\delta} {p_{\underline{i}}}^q {\lambda_{\underline{i}}}^\gamma$$

in mind. When γ is chosen properly, the last sum equals exactly 1 for all δ, and $T(q)$ must equal γ. This procedure has the advantage of not using the maps w_i. Thus the result obtained is valid for multifractals arising from a more general construction than SMFs. We will need a lemma similar to lemma 9.2 in [F2].

Lemma 13 *Let (V_i) be a collection of subsets of $\mathbb{R}^d$ such that each V_i has diameter at most $\sigma_1\delta$ and contains a ball U_i of radius $\sigma_2\delta$. Assume that the U_i are disjoint. Then, any set W of diameter less or equal to $\sigma_3\delta$ intersects at the most $b = (\sigma_1 + \sigma_3)^d/\sigma_2^d$ of the closures $\overline{V_i}$.*

Proof One may assume $W \neq \emptyset$ and choose $x \in W$. The sets $\overline{V_i}$ intersecting W lie in the ball $U(x, (\sigma_1 + \sigma_3)\delta)$, and so do the corresponding interior balls U_i. Comparing volumes gives b. $\diamondsuit$

Proposition 14 *Let μ be a CMF, let $\rho_2 > \rho_1 > 0$ and let $\lambda_1, \ldots, \lambda_r$ be numbers from $]0,1[$ such that for every word $\underline{i} \in I$ there is a point $x_{\underline{i}}$ in $V_{\underline{i}}$ with*

$$U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}}) \subset V_{\underline{i}} \subset U(x_{\underline{i}}, \rho_2\lambda_{\underline{i}}), \tag{16}$$

$$U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}}) \cap V_{\underline{j}} = \emptyset \quad \textit{for all } \underline{j} \neq \underline{i} \textit{ with } |\underline{i}| = |\underline{j}|, \tag{17}$$

$$\mu(U(x_{\underline{i}}, \rho_1\lambda_{\underline{i}})) \neq 0. \tag{18}$$

Then $T(q)$ is grid-regular and satisfies

$$\sum_{i=1}^{r} p_i^q \lambda_i^{T(q)} = 1. \tag{19}$$

Proof By the strict monotonicity of the functions $x \mapsto p_i^q \lambda_i^x$ there is a unique solution of $\sum_{i=1}^r {p_i}^q {\lambda_i}^x = 1$ which is denoted by $\gamma(q)$.

o) Note first that J_δ can be obtained inductively from $L_1 := \{1, \dots, r\}$ in finitely many steps by letting $L_{n+1} := \{\underline{i} \in L_n : \lambda_{\underline{i}} \leq \delta\} \cup \{\underline{i} * k : \underline{i} \in L_n, 1 \leq k \leq r, \lambda_{\underline{i}} > \delta\}$. Three facts follow easily (compare also [Hut]):

$$\sum_{\underline{i} \in J_\delta} {p_{\underline{i}}}^q {\lambda_{\underline{i}}}^\gamma = 1, \tag{20}$$

$$K \subset \bigcup_{J_\delta} V_{\underline{i}} \tag{21}$$

and

$$U(x_{\underline{i}}, 2\rho_1 \lambda_{\underline{i}}) \cap U(x_{\underline{j}}, 2\rho_1 \lambda_{\underline{j}}) = \emptyset \tag{22}$$

for all $\underline{i} \neq \underline{j}$ from J_δ. (To prove (22) one may assume $n = |\underline{i}| \leq |\underline{j}|$. Then $\underline{i} \neq (\underline{j}|n)$ due to (15). Thus, (17) and $V_{\underline{j}} \subset V_{(\underline{j}|n)}$ prove the claim.) For convenience set

$$\underline{\lambda} := \min\{\lambda_1, \dots, \lambda_r\} \qquad \overline{\lambda} := \max\{\lambda_1, \dots, \lambda_r\}. \tag{23}$$

i) First let $q \geq 0$. Take $B \in G_\delta$. For the sake of brevity write $J_\delta(E) := \{\underline{i} \in J_\delta : V_{\underline{i}} \cap E \neq \emptyset\}$. By (22), (15) and lemma 13 there is a number b_1 independent of δ and B such that

$$\# J_\delta((B)_1) \leq b_1.$$

This allows the estimate

$$\mu((B)_1)^q \leq \Big(\sum_{J_\delta((B)_1)} \mu(V_{\underline{i}}) \Big)^q \leq \Big(b_1 \cdot \max_{J_\delta((B)_1)} \mu(V_{\underline{i}}) \Big)^q \leq {b_1}^q \cdot \sum_{J_\delta((B)_1)} \mu(V_{\underline{i}})^q.$$

Taking the sum over all $B \in G_\delta$ will yield an inequality where the right hand sum runs more than once over certain words of J_δ. But fixing $\underline{i} \in J_\delta$ and applying lemma 13 to $W = V_{\underline{i}}$ gives a constant b_2 such that

$$\#\{B \in G_\delta : V_{\underline{i}} \cap (B)_1 \neq \emptyset\} = \#\{B \in G_\delta : \underline{i} \in J_\delta((B)_1)\} \leq b_2.$$

Thus,

$$S_\delta(q) = \sum_{B \in G_\delta} \mu((B)_1)^q \leq {b_1}^q \sum_{B \in G_\delta} \sum_{\underline{i} \in J_\delta((B)_1)} \mu(V_{\underline{i}})^q \leq {b_1}^q b_2 \sum_{\underline{i} \in J_\delta} \mu(V_{\underline{i}})^q. \tag{24}$$

Finally $\mu(V_{\underline{j}})$ must be compared with $p_{\underline{j}}$. This is trivial for SMFs, but in general these two numbers are not equal. Again, the fact is used, that an average, i.e. a sum, has to be estimated. Take $\underline{j} \in J_\delta$. First $\pi^{-1}(V_{\underline{j}})$ is estimated: assume $x = \pi(\underline{k}) \in V_{\underline{j}}$. Due to (15) there is an integer n such that $\underline{i} := (\underline{k}|n) \in J_\delta$. Hence $x \in V_{\underline{i}} \cap V_{\underline{j}}$,

$$\pi^{-1}(V_{\underline{j}}) \subset \{\underline{k} \in I_\infty : \exists n \in \mathbb{N} \text{ with } (\underline{k}_\infty|n) \in J_\delta(V_{\underline{j}})\}$$

and by (11)

$$\mu(V_{\underline{j}}) = P[\pi^{-1}(V_{\underline{j}})] \leq \sum_{\underline{i} \in J_\delta(V_{\underline{j}})} p_{\underline{i}}.$$

Applying lemma 13 to $W = V_{\underline{j}}$ provides a constant b_3 with

$$\# J_\delta(V_{\underline{j}}) \leq b_3.$$

Consequently

$$\mu(V_{\underline{j}})^q \leq (b_3 \cdot \max_{J_\delta(V_{\underline{j}})} p_{\underline{i}})^q \leq b_3{}^q \sum_{\underline{i} \in J_\delta(V_{\underline{j}})} p_{\underline{i}}{}^q.$$

With similar ideas as above one obtains

$$\sum_{\underline{j} \in J_\delta} \mu(V_{\underline{j}})^q \leq b_3{}^q \sum_{\underline{j} \in J_\delta} \sum_{\underline{i} \in J_\delta(V_{\underline{j}})} p_{\underline{i}}{}^q \leq b_3{}^q b_3 \sum_{\underline{i} \in J_\delta} p_{\underline{i}}{}^q. \tag{25}$$

Set $c_1 := \max\{1, \underline{\lambda}^{-\gamma}\}$. Then, $\delta^\gamma \leq c_1 \lambda_{\underline{i}}{}^\gamma$ for any $\underline{i} \in J_\delta$ and all γ. The combination of (24), (25) and (20) reads

$$S_\delta(q)\delta^\gamma \leq b_1{}^q b_2 b_3{}^{q+1} \sum_{\underline{i} \in J_\delta} p_{\underline{i}}{}^q \delta^\gamma \leq b_1{}^q b_2 b_3{}^{q+1} c_1 \sum_{\underline{i} \in J_\delta} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = b_1{}^q b_2 b_3{}^{q+1} c_1.$$

This implies immediately $T(q) \leq \gamma(q)$ $(q \geq 0)$.

ii) Now $S_\delta(q)$ will be estimated from below. Take $\underline{i} \in J_{\delta'}$ where $\delta' = (3\rho_2)^{-1}\delta$. From $0 \neq p_{\underline{i}} \leq \mu(V_{\underline{i}})$ follows the existence of a box $B_{\underline{i}} \in G_\delta$ which meets $V_{\underline{i}}$. Since $\mathrm{diam}(V_{\underline{i}}) \leq 2\rho_2\lambda_{\underline{i}} < \delta$, the parallel body $(B_{\underline{i}})_1$ contains $V_{\underline{i}}$. Thus $p_{\underline{i}} \leq \mu((B_{\underline{i}})_1)$. Fix a δ-box B^*. Applying lemma 13 to $W = B^*$ shows, that B^* can meet at the most b_4 sets $V_{\underline{i}}$ with $\underline{i} \in J_{\delta'}$, where b_4 depends neither on δ nor on B^*. Consequently, $\#\{\underline{i} \in J_{\delta'} : B^* = B_{\underline{i}}\} \leq b_4$. Hence

$$\sum_{\underline{i} \in J_{\delta'}} p_{\underline{i}}{}^q \leq \sum_{\underline{i} \in J_{\delta'}} \mu((B_{\underline{i}})_1)^q \leq b_4 \sum_{B \in G_\delta} \mu((B)_1)^q.$$

With $c_2 = (3\rho_2)^\gamma \cdot \min\{1, \underline{\lambda}^{-\gamma}\}$ one has $\delta^\gamma \geq c_2 \lambda_{\underline{i}}{}^\gamma$ for all $\underline{i} \in J_{\delta'}$ due to (15). Thus, with (20)

$$S_\delta(q)\delta^\gamma \geq b_4^{-1} \sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q \delta^\gamma \geq b_4^{-1} c_2 \sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = \frac{c_2}{b_4}.$$

Since $\delta > 0$ is arbitrary, this implies with i) that $T(q)$ is grid-regular and equals γ.

iii) Now let $q < 0$. Take $B \in G_\delta$. Since $\mu(B) \neq 0$, (21) guarantees a word $\underline{i} \in J_{\delta'}$ such that $V_{\underline{i}}$ meets B. As in ii) $V_{\underline{i}}$ is a subset of $(B)_1$ and thus $0 \neq p_{\underline{i}} \leq \mu((B)_1)$. As in i) and ii), with a constant b_5 provided by lemma 13 and $c_3 = (3\rho_2)^\gamma \cdot \max\{1, \underline{\lambda}^{-\gamma}\}$,

$$S_\delta(q)\delta^\gamma = \delta^\gamma \sum_{B\in G_\delta} \mu((B)_1)^q \leq b_5 \delta^\gamma \sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q \leq b_5 c_3 \sum_{\underline{i}\in J_{\delta'}} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = b_5 c_3.$$

iv) Take $\underline{i} \in J_{\delta''}$ with $\delta'' = 3\sqrt{d}(\rho_1 \underline{\lambda})^{-1}\delta$. Here the precondition (18) is used, which implies the existence of a box $B(\underline{i}) \in G_\delta$ which meets $U(x_{\underline{i}}, \rho_1 \lambda_{\underline{i}})$. Applying (15) yields $\mathrm{diam}(B(\underline{i}))_1 = \rho_1 \underline{\lambda} \delta'' \leq \rho_1 \lambda_{\underline{i}}$ and thus

$$(B(\underline{i}))_1 \subset U(x_{\underline{i}}, 2\rho_1 \lambda_{\underline{i}}) \subset V_{\underline{i}}.$$

By (17)

$$\pi^{-1}(U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}})) \subset \{\underline{k} \in I_\infty \,:\, (\underline{k}|\,|\underline{i}|) = \underline{i}\}$$

which leads to

$$0 \neq \mu((B(\underline{i}))_1) \leq \mu(U(x_{\underline{i}}, 2\rho_1\lambda_{\underline{i}})) \leq p_{\underline{i}}.$$

It follows from (22) that distinct words $\underline{i}$ from $J_{\delta''}$ have distinct $B(\underline{i})$. Thus

$$S_\delta(q)\delta^\gamma = \delta^\gamma \sum_{B\in G_\delta} \mu((B)_1)^q \geq \delta^\gamma \sum_{\underline{i}\in J_{\delta''}} p_{\underline{i}}{}^q \geq c_4 \sum_{\underline{i}\in J_{\delta''}} p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^\gamma = c_4,$$

where $c_4 = (\rho_1\underline{\lambda})^\gamma (3\sqrt{d})^{-\gamma} \cdot \min\{1, \underline{\lambda}^{-\gamma}\}$. $\diamondsuit$

The above proposition enables us to give the singularity exponents of two types of measures: certain multiplicative cascades on $\mathbb{R}$ and self-similar measures (SMF).

Theorem 15 *Let μ be a CMF on $\mathbb{R}$. Assume the existence of positive numbers $\lambda_1, \ldots, \lambda_r$ and s, t such that for all sufficiently large $n \in \mathbb{N}$ the interiors of the sets $V_{\underline{i}}$ ($\underline{i} \in I_n$) are mutually disjoint intervals of length* $\mathrm{diam}V_{\underline{i}} \in [s\lambda_{\underline{i}}, t\lambda_{\underline{i}}]$. *Then, $T(q)$ is grid-regular for all $q \in \mathbb{R}$ and satisfies*

$$\sum_{i=1}^{r} p_i^q \lambda_i^{T(q)} = 1.$$

Proof Take a sufficiently long word $\underline{i}$. Among $V_{\underline{i}*1*1}, \ldots, V_{\underline{i}*r*r}$ there is at least one—say $V_{\underline{i}*j_1 j_2}$—with distance at least $s\underline{\lambda}^2\lambda_{\underline{i}}$ from the boundary of $V_{\underline{i}}$ (23). Choose m large enough to ensure $2t \cdot \overline{\lambda}^m < s \cdot \underline{\lambda}^2$ and set $\underline{j} := j_1 j_2 * 1 \ldots 1 \in I_m$. Then $V_{\underline{i}*\underline{j}} \subset V_{\underline{i}*j_1 j_2}$ and $\mathrm{diam}(V_{\underline{i}*\underline{j}}) \leq t\overline{\lambda}^m \lambda_{\underline{i}} < s/2 \cdot \underline{\lambda}^2 \lambda_{\underline{i}}$. Now choose $x_{\underline{i}} \in V_{\underline{i}*\underline{j}}$, $\rho_2 = t$ and $\rho_1 = s/2 \cdot \underline{\lambda}^2$. Since $V_{\underline{i}*\underline{j}} \subset \overline{U}(x_{\underline{i}}, \rho_1 \lambda_{\underline{i}})$, $p_{\underline{i}*\underline{j}} \neq 0$ and $\mathrm{dist}(x_{\underline{i}}, \partial V_{\underline{i}}) \geq 2\rho_1 \lambda_{\underline{i}}$, proposition 14 gives the desired result. $\diamondsuit$

Besides the OSC another condition is of interest in the context of self-similarity, the so-called *strong open set condition* (SOSC). The SOSC is said to hold for a set of contracting similarities $w_1, \ldots, w_r$, if there is a basic open set O which intersects the invariant set K. Of course SOSC implies OSC and seems to be more restrictive. The two conditions given in [BG], equivalent to SOSC and OSC respectively, support this view of things. Surprisingly, SOSC and OSC are equivalent, as was recently shown by Schief [Sch]. This satisfying and powerful result enables us to calculate the singularity exponents T of self-similar measures.

Theorem 16 (Singularity Exponents of Self-Similar Measures) *Let μ be a SMF with ratios $\lambda_1, \ldots, \lambda_r$ and probabilities $p_1, \ldots, p_r$. Then $T(q)$ is grid-regular for all $q \in \mathbb{R}$ and satisfies*

$$\sum_{i=1}^{r} p_i^q \lambda_i^{T(q)} = 1.$$

Proof Proposition 14 will be applied. Note first, that due to [Sch] there is a basic open set O which intersects K. Choose $x = \pi(\underline{i}) \in O \cap K$. Since O is open and bounded there is $\rho_2 > \rho_1 > 0$ such that $U(x, 2\rho_1) \subset O \subset \overline{O} \subset U(x, \rho_2)$. Letting $x_{\underline{k}} := w_{\underline{k}}(x)$ for all finite words $\underline{k}$ one finds

$$U(x_{\underline{k}}, 2\rho_1 \lambda_{\underline{k}}) \cap V_{\underline{j}} \subset w_{\underline{k}}(O) \cap w_{\underline{j}}(\overline{O}) = \emptyset$$

for all $\underline{k} \neq \underline{j}$ with $|\underline{k}| = |\underline{j}|$. This gives (17); (16) is evident. Finally, take an integer n such that $\overline{\lambda}^n \cdot \mathrm{diam}(O) \leq \rho_1$ and set $\underline{j} = i_1 \ldots i_n$. Then, the set $V_{\underline{j}}$ contains x, has diameter $\lambda_{\underline{j}} \cdot \mathrm{diam}(O)$ and is thus a subset of $U(x, \rho_1)$. From this

$$V_{\underline{k}*\underline{j}} = w_{\underline{k}}(V_{\underline{j}}) \subset w_{\underline{k}}(U(x, \rho_1)) = U(x_{\underline{k}}, \rho_1 \lambda_{\underline{k}}),$$

and $\mu(U(x_{\underline{k}}, \rho_1 \lambda_{\underline{k}})) \geq p_{\underline{k}*\underline{j}} \neq 0$. So, (18) is established as well. $\diamondsuit$

From theorem 7 and 16 the spectrum F of a self-similar measure follows immediately. To state some interesting features of it we set:

$$\alpha_\infty := \min_{i=1,\ldots,r} \frac{\log p_i}{\log \lambda_i} \quad \alpha_1 := \frac{\sum_{i=1}^r p_i \log p_i}{\sum_{i=1}^r p_i \log \lambda_i} \quad \alpha_0 := \frac{\sum_{i=1}^r \lambda_i^{D_0} \log p_i}{\sum_{i=1}^r \lambda_i^{D_0} \log \lambda_i} \quad \alpha_{-\infty} := \max_{i=1,\ldots,r} \frac{\log p_i}{\log \lambda_i},$$

where $D_0 = T(0)$ is the box dimension of $K = \mathrm{supp}(\mu)$. Thereby the various values of α have interpretations as particular pointwise dimensions or 'coarse Hölder exponents'. The latter notion is more accurate here and can be defined as [EvM]

$$\limsup_{\delta \downarrow 0} \frac{\log \mu(B(x,\delta))}{-\log \delta},$$

where $B(x,\delta)$ is the unique δ-box containing x. Then, α_∞ and $\alpha_{-\infty}$ are interpretated as the coarse Hölder exponents of the most probable and the most rarefied points respectively, and α_1 and a_0 as the coarse Hölder exponents which occur almost surely with respect to the underlying measure μ and the (restricted and normalized) D-dimensional Hausdorff measure respectively. For certain measures μ this can be made precise.

Corollary 17 (Spectrum of Self-Similar Measures) *Let μ be any SMF. Then, F is grid-regularand determined by theorem 7 and 16. More concretely, F attains its maximum D_0 at α_0, touches the internal bisector at $\alpha_1 = F(\alpha_1)$ and takes the value $-\infty$ for α outside $[\alpha_\infty, \alpha_{-\infty}]$. Moreover, the 2×2 equation system*

$$\begin{array}{rcll} \sum_{i=1}^r \left(\frac{p_i}{\lambda_i^\alpha}\right)^q \cdot \lambda_i^\gamma & = & 1 & \text{(a)} \\ \sum_{i=1}^r \log\left(\frac{p_i}{\lambda_i^\alpha}\right)\left(\frac{p_i}{\lambda_i^\alpha}\right)^q \cdot \lambda_i^\gamma & = & 0 & \text{(b)} \end{array} \tag{26}$$

is for every $\alpha \in]\alpha_\infty, \alpha_{-\infty}[$ uniquely solved by $\gamma = F(\alpha)$, $q = F'(\alpha)$, and

$$\sum_{p_i = \lambda_i^{\alpha_\infty}} \lambda_i^{F(\alpha_\infty)} = 1 \quad \textit{and} \quad \sum_{p_i = \lambda_i^{\alpha_{-\infty}}} \lambda_i^{F(\alpha_{-\infty})} = 1. \tag{27}$$

Remarks The assertion of the corollary is valid for any CMF, for which (19) holds with grid-regular T for all real q. In the case $r = 2$ (26) is explicitly solvable by introducing the variables $x_i = {p_i}^q {\lambda_i}^{\gamma - q\alpha}$. Setting $c_i = \log p_i - \alpha \log \lambda_i$ one finds

$$F(\alpha) = \frac{c_2 \log(-c_2) + (c_1 - c_2)\log(c_1 - c_2) - c_1 \log(c_1)}{\log \lambda_1 \log p_2 - \log \lambda_2 \log p_1}$$

for $\alpha \in]\alpha_\infty, \alpha_{-\infty}[$. Thereby $\alpha_\infty = \log p_1 / \log \lambda_1 < \log p_2 / \log \lambda_2 = \alpha_{-\infty}$ without loss of generality. Formulas free from the parameter q have been presented until now only for special cases [EvM, TV].

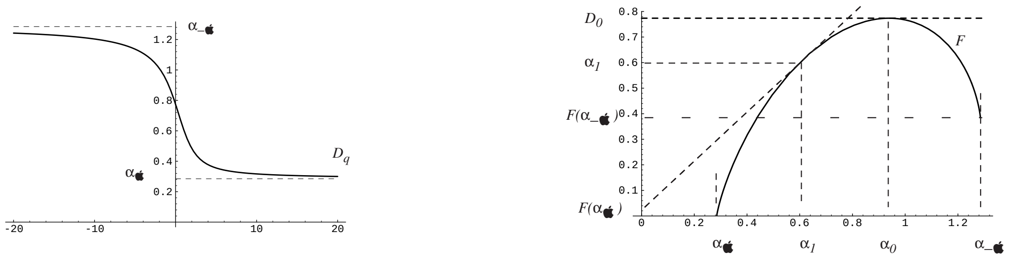


Figure 2: The generalized dimensions $D_q = T(q)/(1-q)$ and the spectrum $F(\alpha)$ of a self-similar measure with $r = 4$, ratios $\lambda_1 = \ldots = \lambda_4 = 1/6$ and probability vector $(.1, .1, .2, .6)$.

Proof The case $p_i = \lambda_i^D$ $(i = 1, \ldots, r)$ is trivial. Thus, $\alpha_\infty < \alpha_{-\infty}$ will be assumed. First, the range of $-T'$ is $]\alpha_\infty, \alpha_{-\infty}[$ by direct calculus as in [CM] or [R]. Now, it will be shown that (26) is solvable exactly if α lies in the range of $-T'$, and that it determines F and F'. Assume first that (γ_0, q_0) solves the system for some fixed α, and rewrite the equations as:

$$\begin{aligned} \sum_{i=1}^r p_i^{q_0} \lambda_i^{\gamma_0 - \alpha q_0} &= 1 \\ \sum_{i=1}^r p_i^{q_0} \lambda_i^{\gamma_0 - \alpha q_0} \log p_i &= \alpha \sum_{i=1}^r p_i^{q_0} \lambda_i^{\gamma_0 - \alpha q_0} \log \lambda_i \end{aligned}$$

So, necessarily $T(q_0) = \gamma_0 - \alpha q_0$ and $\alpha = -T'(q_0)$. On the other hand it is now easy to see that, if $\alpha = -T'(q_0)$, then $(T(q_0) + \alpha q_0, q_0)$ provides a solution of (26). By theorem 7 one has $F(\alpha) = T(q_0) - q_0 T'(q_0)$ which is γ_0. Finally, $F'(\alpha) = q_0$ since T is strictly convex and C^2, i.e. $T''(q) \neq 0$. As a consequence, the solution is unique.
It only remains to compute the values $F(a_{\pm\infty})$. For simplicity assume

$$\alpha_\infty = \frac{\log p_i}{\log \lambda_i} \qquad \Leftrightarrow \qquad i \in \{1, \ldots, t\}$$

for some $t < r$. The functions $c_i(\alpha) = \log(p_i) - \alpha \log(\lambda_i)$ $(i = 1, \ldots, r)$ are strictly increasing with zeros $\log p_i / \log \lambda_i$. There are numbers $\varepsilon > 0$ and $c'' > c' > 0$ such that $c' \leq -c_i(\alpha) \leq c''$ $(i = t+1, \ldots, r)$ and $c_k(\alpha) \geq 0$ $(k = 1, \ldots, t)$ for all $\alpha \in]\alpha_\infty, \alpha_\infty + \varepsilon[$.

Denote the solution of (26) by $(F(\alpha), q(\alpha))$. In the following limit $\alpha \downarrow \alpha_\infty$ is considered. Note first that $q(\alpha) = F'(\alpha) \to \infty$. Since $F(\alpha) \geq 0$, (26.b) implies

$$\sum_{k=1}^{t} \lambda_k^{F(\alpha)} c_k(\alpha) e^{c_k(\alpha)q(\alpha)} = \sum_{i=t+1}^{r} \lambda_i^{F(\alpha)} (-c_i(\alpha)) e^{c_i(\alpha)q(\alpha)} \leq rc'' \cdot e^{-c' \cdot q(\alpha)}.$$

The terms in the first sum are all positive. Thus, with $F(\alpha) \leq D_0$ one obtains

$$0 \leq c_k(\alpha)q(\alpha) \leq \text{const} \cdot q(\alpha) e^{-c' \cdot q(\alpha)} e^{-c_k(\alpha)q(\alpha)} \to 0 \quad (k = 1, \ldots, t).$$

On the other hand, $c_i(\alpha)q(\alpha) \to -\infty$ for $i = t+1, \ldots, r$ is trivial. Finally $F(\alpha) \to F(\alpha_\infty)$ for continuity reasons (theorem 7). Applying (26.a) gives $F(\alpha_\infty)$. A similar argument for $\alpha_{-\infty}$ completes the proof. $\diamondsuit$

3.3 Multifractal Decomposition

The multifractal decomposition of certain SMF on $\mathbb{R}$ has been given in [CLP] and, under less restricting conditions in [BMP]. In both papers the authors developed notions of singularity exponents which are carefully tailored to the particular situation and are not useful in a different context. However, they allow the computation of the multifractal spectrum $f(\alpha) = d_{\text{HD}}(K_\alpha)$ (compare (2)).

Cawley & Mauldin [CM] considered SMF in $\mathbb{R}^d$. Working on the codespace $I_\infty := \{1, \ldots, r\}^{\mathbb{N}}$ (subsection 3.1) they found the dimension of the sets corresponding to K_α:

For an SMF with ratios $\lambda_1, \ldots, \lambda_r$ and probability vector $(p_1, \ldots, p_r)$ let

$$\hat{K_\alpha} := \{\underline{i} \in I_\infty \; : \; \lim_{n\to\infty} \frac{\log p_{(\underline{i}|n)}}{\log \lambda_{(\underline{i}|n)}} = \alpha\}$$

Then $d_{\text{HD}}(\hat{K_\alpha}) = F(\alpha)$, where $F(\alpha)$ is given by corollary 17.

Provided the compact sets $w_i(K)$ ($K = \text{supp}(\mu)$) are mutually disjoint, the coordinate map π is bi-lipschitz and $K_\alpha = \pi(\hat{K_\alpha})$. Therefore, K_α has the same Hausdorff dimension as $\hat{K_\alpha}$. Abusing the notion of Hausdorff dimension ($d_{\text{HD}}(A) = -\infty$ iff $A = \emptyset$) this reads

Theorem 18 (Cawley & Mauldin) *Let μ be an SMF with ratios $\lambda_1, \ldots, \lambda_r$ and probability vector $(p_1, \ldots, p_r)$. If the sets $w_i(K)$ are mutually disjoint, then*

$$d_{\text{HD}}(K_\alpha) = F(\alpha).$$

Similar methods were used to provide multifractal decompositions in more general situations [F3, EdM, S]. Here, we present a short argument which leads directly to theorem 18.

Proof i) By assumption there is an $\varepsilon > 0$ with the following property: If O denotes the open ε-parallel body of K, i.e. $O = \{x : \exists y \in K \text{ such that } |x-y| < \varepsilon\}$, then the sets $w_i(\overline{O})$ are mutually disjoint. Consequently, O is a basic open set for the maps $w_1, \ldots, w_r$.

ii) Fix q for the moment. Let $\alpha(q) = -T'(q)$. Define the Borel measures $\mu_q := \hat{\mu}_q(\pi^{-1}(\cdot))$, where $\hat{\mu}_q$ is the product measure (10) on the codespace I_∞ induced by the probability vector $(\overline{p}_1, \ldots, \overline{p}_r)$ with $\overline{p}_i := p_i^q \lambda_i^{T(q)}$. Applying the Law of Large Numbers to the random variables $X_n := \log p_{i_n}$ and $Y_n := \log \lambda_{i_n}$ on I_∞ one obtains:

$$\frac{\log p_{(\underline{i}|n)}}{\log \lambda_{(\underline{i}|n)}} = \frac{(1/n)\log(X_1 + \ldots + X_n)}{(1/n)\log(Y_1 + \ldots + Y_n)} \to \frac{E[X_n]}{E[Y_n]} = \frac{\sum\limits_{i=1}^{r} p_i^q \lambda_i^{T(q)} \log p_i}{\sum\limits_{i=1}^{r} p_i^q \lambda_i^{T(q)} \log \lambda_i} = \alpha(q) \tag{28}$$

for $\hat{\mu}_q$ almost every $\underline{i} \in I_\infty$. In particular, $\mu_q(\hat{K}_{\alpha(q)}) = 1$. Furthermore, for any $\underline{i} \in \hat{K}_{\alpha(q)}$

$$\frac{\log \overline{p}_{(\underline{i}|n)}}{\log \lambda_{(\underline{i}|n)}} = q\,\frac{\log p_{(\underline{i}|n)}}{\log \lambda_{(\underline{i}|n)}} + T(q) \to T(q) - qT'(q) = F(\alpha(q)). \tag{29}$$

iii) As usual let $V_{\underline{i}} := w_{\underline{i}}(\overline{O})$. Let $\rho := \operatorname{diam}(\overline{O})$. Note, that $V_{\underline{i}}$ has diameter $\rho\lambda_{\underline{i}}$ and measure $\mu_q(V_{\underline{i}}) = \overline{p}_{\underline{i}} = p_{\underline{i}}{}^q \lambda_{\underline{i}}{}^{T(q)}$. Let $x \in K$. Due to i) there is a unique $\underline{j} \in I_\infty$ with $x = \pi(\underline{j})$. Let $r > 0$. Then, there exist unique integers n and m such that

$$\lambda_{(\underline{j}|n)} \le r/\rho < \lambda_{(\underline{j}|n-1)} \quad \text{and} \quad \lambda_{(\underline{j}|m+1)} \le r/\varepsilon < \lambda_{(\underline{j}|m)}.$$

Consequently,

$$x \in V_{(\underline{j}|n)} \subset U(x,r) \subset U(x, \varepsilon\lambda_{(\underline{j}|m)}) \subset V_{(\underline{j}|m)}$$

and

$$\frac{\log \overline{p}_{(\underline{j}|m)}}{\log \lambda_{(\underline{j}|m)} + \log(\varepsilon\underline{\lambda})} \le \frac{\log \mu_q(U(x,r))}{\log r} \le \frac{\log \overline{p}_{(\underline{j}|n)}}{\log \lambda_{(\underline{j}|n)} + \log(\rho\underline{\lambda})}.$$

Thus, for any $\underline{j} \in I_\infty$

$$\liminf_{n\to\infty} \frac{\log \overline{p}_{(\underline{j}|n)}}{\log \lambda_{(\underline{j}|n)}} = \underline{d}_{\mu_q}(\pi(\underline{j})) \le \overline{d}_{\mu_q}(\pi(\underline{j})) = \limsup_{n\to\infty} \frac{\log \overline{p}_{(\underline{j}|n)}}{\log \lambda_{(\underline{j}|n)}}. \tag{30}$$

Inserting the particular value $q = 1$ in (30) and observing $\mu_1 = \mu$ yields $K_\alpha = \pi(\hat{K_\alpha})$ for all α by definition (2). As a consequence, K_α is empty for $\alpha \notin [\alpha_\infty, \alpha_{-\infty}]$.

iv) Combining (29) and (30) shows that the pointwise dimension of μ_q equals

$$d_{\mu_q}(x) = T(q) - qT'(q) = F(\alpha(q)) \tag{31}$$

at all points x of $K_{\alpha(q)} = \pi(\hat{K_{\alpha(q)}})$. Moreover, $\mu_q(K_{\alpha(q)}) = 1$ by (28). By a famous theorem of Young [Y], also referred to as the Frostman lemma, the Hausdorff dimension of $K_{\alpha(q)}$ must equal $F(\alpha(q))$.

iv) Finally, the sets K_{α_∞} and $K_{\alpha_{-\infty}}$ are self-similar sets due to (30), i.e. invariant under the family of maps $\{w_i : \log p_i / \log \lambda_i = \alpha_{\pm\infty}\}$, respectively. The dimensions of self-similar sets are well-known [Hut]. Here they are given by (27). $\diamondsuit$

4 Further remarks

Here, we present examples which support the necessity of an improvement of (3) and which show that our spectrum need not be concave. Finally, we compare the notion of Pesin [P3] with ours.

4.1 Examples

Example 1 (Binomial Measure) A binomial measure [EvM] is simply an SMF on $\mathbb{R}$ with $r = 2$. As an example choose an arbitrary probability vector (p_1, p_2) and set $w_1(x) = x/3$ and $w_2(x) = (2 + x)/3$. Then, the invariant measure

$$\mu = p_1 \cdot \mu(w_1^{-1}(\cdot)) + p_2 \cdot \mu(w_2^{-1}(\cdot))$$

is supported by the well-known middle third Cantor set and is a binomial measure. When the definition (3) of $\tau(q)$ is rigorously applied one obtains

$$\tau(q) = \infty \qquad \text{whenever } q < 0.$$

Proof To every $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ with ${p_2}^{k_n} \leq (1/2 \cdot 3^{-n})^n$, because $p_2 < 1$. Without loss of generality $k_n \geq n+1$. Then $\delta_n := (1 - 3^{-k_n})3^{-n}$ lies in $[(1 - 3^{-n})3^{-n}, 3^{-n}]$.

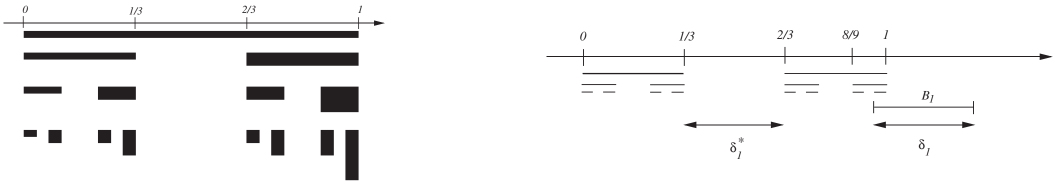


Figure 3: On the left: The construction of the Cantor binomial measure for $p_1 = 1/3$ and $p_2 = 2/3$. On the right an illustration concerning the exceptional behaviour of some δ_n-boxes ($n = 1$).

Since $(3^n+1)\delta_n \geq 1$ the box $B_n := [3^n\delta_n, (3^n+1)\delta_n[$ has very small measure: $B_n \cap [0,1] = [1 - 3^{-k_n}, 1]$, thus $\mu(B_n) = {p_2}^{k_n} \leq (\delta_n)^n$. For $q < 0$ it follows that

$$s_{\delta_n}(q) \geq \mu(B_n)^q \geq (\delta_n)^{nq},$$

which proves the claim. $\diamondsuit$

This examples makes clear, that $\tau(q)$ is not an appropriate notion of singularity exponents since the information of half of the q-domain is lost. One may hold against that the restriction of δ to the sequence $\delta_n^* = 3^{-n}$ allows to observe the expected exponents: $s_{\delta_n^*}(q) = ({p_1}^q + p_2^q)^n$ and $\tau^*(q) = T(q)$. However, working with such a notion would mean, that the structure of a measure to be investigated had to be known *in advance*. Even worse, since δ_n and δ_n^* are very close, numerical methods can not give reliable estimates of $\tau(q)$ for measures with less exact self-similar structures than binomial measures. In contrary to this, proposition 2 and theorem 14 assure that $T(q)$ is not sensible to small 'disturbance'. $\bigcirc$

Example 2 (Nonconcave spectrum) Consider the two families of maps

$$w_1(x) = \frac{x}{3} \quad w_2(x) = \frac{2+x}{3} \quad \text{and} \quad t_1(x) = \frac{4+x}{3} \quad t_2(x) = \frac{6+x}{3}$$

and the invariant measures

$$\mu_1 = 2/3 \cdot \mu_1(w_1^{-1}(\cdot)) + 1/3 \cdot \mu_1(w_2^{-1}(\cdot)) \quad \text{and} \quad \mu_2 = 8/9 \cdot \mu_2(t_1^{-1}(\cdot)) + 1/9 \cdot \mu_2(t_2^{-1}(\cdot)).$$

Let $\mu := (\mu_1 + \mu_2)/2$. A straightforward counting argument using the disjointness of the supports of μ_1 and μ_2 shows that $F(\alpha) = \max(F_1(\alpha), F_2(\alpha))$. This spectrum is not concave (figure 4). So, in general, $F(\alpha)$ need not equal the Legendre transform of $T(q)$ everywhere and proposition 4 can not be used blindly to obtain F from T. $\bigcirc$

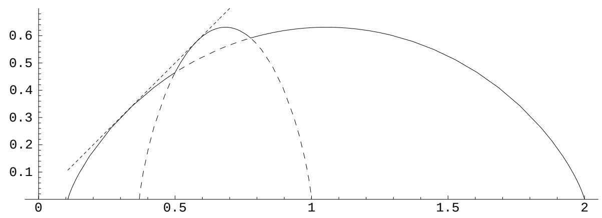


Figure 4: The spectrum F of $\mu = 1/2(\mu_1+\mu_2)$ as given in example 2. It is not concave and not everywhere differentiable. Thus, the inverse Legendre transform can not be applied blindly to obtain F from T. The dashed parts show the internal bisector of the axes and the spectra of μ_1 and μ_2.

4.2 The Notion Introduced by Pesin

In [P3] Pesin introduces different ways to define and study generalized dimensions. This is not the place to review them in full length. Therefore, we state only what is needed for the comparison with our formalism.

Following Hentschel & Procaccia [HP] Pesin defines the 'generalized spectrum for dimensions' for $q > 0$ as follows:

$$\overline{\gamma}_q(\mu) := \frac{1}{q} \limsup_{\delta\downarrow 0} \frac{1}{\log\delta} \log \int \mu(U(x,\delta))^q d\mu(x),$$
$$\underline{\gamma}_q(\mu) := \frac{1}{q} \liminf_{\delta\downarrow 0} \frac{1}{\log\delta} \log \int \mu(U(x,\delta))^q d\mu(x).$$

When $\overline{\gamma}_q(\mu) = \underline{\gamma}_q(\mu)$, then the common value is denoted by $\gamma_q(\mu)$. A modification of the above definition are the 'limit generalized spectrum for dimensions' $\overline{\alpha}_q(\mu)$ and $\underline{\alpha}_q(\mu)$, which are defined for $-1 \le q \le 0$, $q > 0$ through

$$\overline{\alpha}_q(\mu) := \frac{1}{q} \lim_{\varepsilon\downarrow 0} \sup_{\mu(Z)\ge 1-\varepsilon} \limsup_{\delta\downarrow 0} \frac{1}{\log\delta} \log \int_Z \mu(U(x,\delta))^q d\mu(x),$$

etc. For the reader familiar with the notion of capacity (box-dimension) of a measure [Y], the difference between $\overline{\gamma}_q(\mu)$ and $\overline{\alpha}_q(\mu)$ becomes apparent through the following two facts:

First, $\overline{\gamma}_q(\mu)$ coincides with the q-dimensional capacity of $\mathbb{R}^d$, while $\overline{\alpha}_q(\mu)$ coincides with q-dimensional capacity of μ [P3].

Secondly: When $d_\mu(x) = \delta$ for μ-almost every x, where δ does ot depend on x, then

$$\overline{\alpha}_q(\mu) = \underline{\alpha}_q(\mu) = \delta$$

for $-1 \leq q \leq 0$, $q > 0$ [P3, theorem 7]. This is the case in the situation of theorem 18 with $\delta = \alpha_1$ (see (31)). In contrary to this $\underline{\gamma}_q(\mu)$ is in general not constant then:

Proposition 19 *Let μ be an arbitrary Borel measure. For any $q > 0$ one has*

$$\underline{\gamma}_q(\mu) = -\frac{1}{q} T(q+1) = D_{q+1} = d_{q+1}.$$

Moreover, $\gamma_q(\mu)$ exists iff $T(q+1)$ is grid-regular.

Proof Let $q > 0$. Given δ let $B(x)$ denote the unique δ-box which contains x and write $B_1(x)$ for $(B(x))_1$.

i) Let $\delta > 0$. Then, $U(x,\delta)$ is a subset of $B_1(x)$ for every x. The measurable function $\mu(B_1(x))^q$ is constant on every δ-box B. Since the δ-boxes give a partition of the space

$$\int \mu(U(x,\delta))^q d\mu(x) = \sum_B \int_B \mu(B_1(x))^q d\mu(x) = \sum_{B \in G_\delta} \mu((B)_1)^q \mu(B) \leq S_\delta(q+1) \leq 1.$$

Thus, $-q\underline{\gamma}_q(\mu) \leq T(q+1)$.

ii) The ball $U(x, \sqrt{d} \cdot \delta)$ contains $B(x)$ for every x. Consequently,

$$\int \mu(U(x, \sqrt{d} \cdot \delta))^q d\mu(x) \geq \sum_{B \in G_\delta} \int_B \mu(B(x))^q d\mu(x) = \sum_{B \in G_\delta} \mu(B)^{q+1} = s_\delta(q+1)$$

by (3). Hence, $-q\underline{\gamma}_q(\mu) \geq \tau(q+1)$.

iii) Finally let $q \geq 0$. For any δ-box B

$$\mu((B)_1)^q \leq \Big(\sum_{C \in G_\delta,\, C \subset (B)_1} \mu(C) \Big)^q \leq \Big(3^d \max_{C \in G_\delta,\, C \subset (B)_1} \mu(C) \Big)^q \leq 3^{dq} \sum_{C \in G_\delta,\, C \subset (B)_1} \mu(C)^q.$$

Thus,

$$s_\delta(q) \leq S_\delta(q) = \sum_{B \in G_\delta} \mu((B)_1)^q \leq 3^{dq} \sum_{B \in G_\delta} \sum_{C \in G_\delta,\, C \subset (B)_1} \mu(C)^q = 3^{d(q+1)} s_\delta(q).$$

From this,

$$\tau(q) = T(q) \qquad \text{for } q \geq 0,$$

and the desired formula follows. Moreover, the given estimates imply that $\gamma_q(\mu)$ exists iff $T(q)$ is grid-regular. $\diamondsuit$

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