A multiscale data representation for distributed sensor networks: Proofs of basic characteristics and error bounds

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1. Characterization of the scaling and wavelet functions

Consider the vector $\underline{m}$ of N sensor measurements given by

$$\underline{m} = [m_1, m_2, ..., m_N]$$

The sensors are irregularly spaced on a $2D$ grid. We assume that the sensors sample a slowly varying field. To reconstruct the original field, we interpolate the sensor values over R^2 in the following manner. The approximate value of the field at any arbitrary location is equal to the value recorded by the sensor closest to it. Thus, the function space F spanned by the reconstructed field is the space of piecewise constants whose discontinuities lie on voronoi cell boundaries.

Our goal is to first find a suitable set of functions that span the function space F. Motivated by the Haar wavelet transform, we pick one function that captures the average value of the sensor field, and N funtions that capture the deviation of each sensor value from the average. Since we use $(N+1)$ functions that span the N dimensional function space F, we refer to the functions as frame functions. Example of such frame functions in $1-D$ are shown in Figure 1. The scaling function is constant over the entire region. The wavelet functions have discontinuities only at the boundaries of one voronoi cell. The wavelet functions have zero mean. Thus, the dot product of a wavelet function and the scaling function is zero.

We need to pick the parameters of the frame functions (namely the heights of the wavelet functions) appropriately. We choose these parameters such that the set of frame functions form a Parseval tight frame [1]. For Parseval tight frames, we can establish bounds on the energy of the error when wavelet coefficients are thresholded.

Let the support of the voronoi regions be Δ_i, for $i = 1, 2, ..., N$. Let $\Delta_{tot} = \sum_{i=1}^{N} \Delta_i$. Define the scaling function as

The i^{th} wavelet function, W_i, takes the value k_i for all the voronoi cells except at the i^{th} cell, where it takes value $k_i^{'}$. Since the average value of the wavelet functions are zero, we have $k_i^{'}\Delta_i + k_i(\Delta_{tot} - \Delta_i) = 0$. Let $\mathbf{\Delta}$ be the $N \times N$ diagonal matrix of the areas Δ_i, $i = 1...N$.

To satisfy the Parseval tight frame conditions, we look at the energy of the signal in the signal and wavelet domains. The energy of the signal is given by

$$E = \sum_{i=1}^{N} (m_i)^2 \Delta_i = \underline{m}^T \mathbf{\Delta} \underline{m}$$

The scaling coefficient s and the wavelet coefficient w are given by $s = \sum_{i=1}^{N} k_0 \Delta_i m_i$ and $w_j = k_j^{'} m_j \Delta_j + \sum_{i=1, i \neq j}^{N} k_j m_j \Delta_j$. For a Parseval tight frame,

$$\underline{m}^T \mathbf{\Delta} \underline{m} = s^2 + \sum_{i=1}^{N} w_i^2$$

The conditions for a Parseval tight frame can be expressed in a matrix form. Let $\mathbf{K}$ be the $(N+1) \times N$ matrix given by

$$\mathbf{K} = \begin{pmatrix} k_0 & k_0 & k_0 & \cdots & k_0 \\ k_1^{'} & k_1 & k_1 & \cdots & k_1 \\ k_2 & k_2^{'} & k_2 & \cdots & k_2 \\ k_3 & k_3 & k_3^{'} & \cdots & k_3 \\ \vdots & \vdots & \vdots & & \vdots \\ k_N & k_N & k_N & \cdots & k_N^{'} \end{pmatrix}$$

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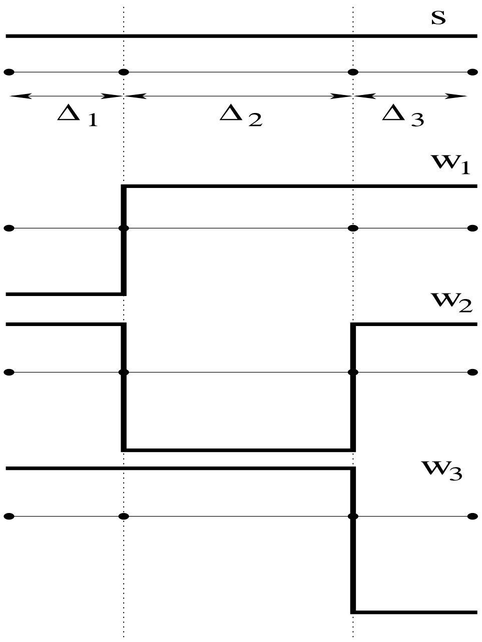

Fig. 1. *Computing the Wavelet coefficients in idea-1. We pass the polynomial-blind filter through every successive set of m points. In the above example $m = 2$ (blind upto quadratic polynomials) and the length of the blind-filter is $m + 2 = 4$. We have $N = 8$ above, and hance the number of wavelet coefficients is* 5.

The element $k_{i,j}$ in the matrix $\mathbf{K}$ represents the value of the frame function i at the voronoi cell surrounding sensor j. The condition for Parseval tight frame can be written as

$$\underline{m}^T \mathbf{\Delta} \underline{m} = \|\mathbf{K}\mathbf{\Delta}\underline{m}\|_2^2$$

Thus we have $\underline{m}^T \mathbf{\Delta} \underline{m} = (\mathbf{K}\mathbf{\Delta}\underline{m})^T(\mathbf{K}\mathbf{\Delta}\underline{m}) = \underline{m}^T \mathbf{\Delta}\mathbf{K}^T\mathbf{K}\mathbf{\Delta}\underline{m}$. For this condition to be satisfied, we require $\mathbf{\Delta} = \mathbf{\Delta}\mathbf{K}^T\mathbf{K}\mathbf{\Delta}$, or

$$\mathbf{K}^T\mathbf{K} = \mathbf{\Delta}^{-1}$$

We have $\mathbf{K}^T\mathbf{K}$ as a diagonal matrix. This motivates us to define a new matrix $\mathbf{T}$, also of dimension $(N+1) \times N$, as $\mathbf{T} = \mathbf{K}\mathbf{\Delta}^{\frac{1}{2}}$.

$$\mathbf{T} = \begin{pmatrix} k_0\sqrt{\Delta_1} & k_0\sqrt{\Delta_2} & k_0\sqrt{\Delta_3} & \cdots & k_0\sqrt{\Delta_N} \\ k_1^{'}\sqrt{\Delta_1} & k_1\sqrt{\Delta_2} & k_1\sqrt{\Delta_3} & \cdots & k_1\sqrt{\Delta_N} \\ k_2\sqrt{\Delta_1} & k_2^{'}\sqrt{\Delta_2} & k_2\sqrt{\Delta_3} & \cdots & k_2\sqrt{\Delta_3} \\ k_3\sqrt{\Delta_1} & k_3\sqrt{\Delta_2} & k_3^{'}\sqrt{\Delta_3} & \cdots & k_3\sqrt{\Delta_N} \\ \vdots & \vdots & \vdots & & \vdots \\ k_N\sqrt{\Delta_1} & k_N\sqrt{\Delta_2} & k_N\sqrt{\Delta_3} & \cdots & k_N^{'}\sqrt{\Delta_N} \end{pmatrix}$$

The matrix $\mathbf{T}$ satisfies the identity $\mathbf{T}^T\mathbf{T} = \mathbf{I}_{N\times N}$, which implies that the columns of $\mathbf{T}$ are orthonormal vectors in R^{N+1}. Since the number of columns in $\mathbf{T}$ is N, we can find one more vector $\underline{l}$ that is orthonormal to all the columns of $\mathbf{T}$. The columns of $\mathbf{T}$ together with the vector $\underline{l}$ form an orthonormal basis for R^{N+1}. We introduce the $(N+1) \times (N+1)$ matrix

$\mathbf{T}_{aug}$ that is obtained by augmenting the matrix $\mathbf{T}$ with the vector $\underline{l}$:

$$\mathbf{T}_{aug} = \begin{pmatrix} k_0\sqrt{\Delta_1} & k_0\sqrt{\Delta_2} & k_0\sqrt{\Delta_3} & \cdots & k_0\sqrt{\Delta_N} & l_0 \\ k_1^{'}\sqrt{\Delta_1} & k_1\sqrt{\Delta_2} & k_1\sqrt{\Delta_3} & \cdots & k_1\sqrt{\Delta_N} & l_1 \\ k_2\sqrt{\Delta_1} & k_2^{'}\sqrt{\Delta_2} & k_2\sqrt{\Delta_3} & \cdots & k_2\sqrt{\Delta_3} & l_2 \\ k_3\sqrt{\Delta_1} & k_3\sqrt{\Delta_2} & k_3^{'}\sqrt{\Delta_3} & \cdots & k_3\sqrt{\Delta_N} & l_3 \\ \vdots & \vdots & \vdots & & \vdots & \\ k_N\sqrt{\Delta_1} & k_N\sqrt{\Delta_2} & k_N\sqrt{\Delta_3} & \cdots & k_N^{'}\sqrt{\Delta_N} & l_N \end{pmatrix}$$

Thus $\mathbf{T}_{aug}^T\mathbf{T}_{aug} = \mathbf{I}_{(N+1)\times(N+1)}$. We can now solve for the frame parameters $k_{i,j}$ and the vector $\underline{l}$ by noting that the *rows* of the matrix $\mathbf{T}_{aug}$ are also orthonormal.

The first row is orthogonal to the other rows, and thus $\Delta_1 k_0 k_i + \Delta_2 k_0 k_i + ...\Delta_N k_0 k_i - \Delta_i k_0 k_i + \Delta_i k_0 k_i^{'} + l_0 l_i = 0$. The value of $\Delta_1 k_0 k_i + \Delta_2 k_0 k_i + ...\Delta_N k_0 k_i - \Delta_i k_0 k_i + \Delta_i k_0 k_i^{'}$ is zero because the wavelet functions have zero mean. Hence, the orthogonality condition requires $l_0 l_i = 0$, which implies

$$l_0 = 0$$

The requirement that the first row is of unit norm gives us $\Delta_1 k_0^2 + \Delta_2 k_0^2 + ...\Delta_N k_0^2 + l_0^2 = 1$. Since $l_0 = 0$, we have

$$k_0 = \frac{1}{\sqrt{\Delta_1 + \Delta_2 + ... + \Delta_N}} = \frac{1}{\sqrt{\Delta_{tot}}} \tag{1}$$

Next, we consider the requirement that rows i and j are orthogonal, for $i, j \neq 1$. We have $\Delta_1 k_i k_j + \Delta_2 k_i k_j + ...\Delta_N k_i k_j + l_i l_j - \Delta_i k_i k_j - \Delta_j k_i k_j - \Delta_i k_i k_j \frac{(\Delta_{tot} - \Delta_i)}{\Delta_i} - \Delta_j k_i k_j \frac{(\Delta_{tot} - \Delta_j)}{\Delta_j} = 0$. This requirement reduces to

$$l_i = \frac{k_i}{k_0}$$

Finally, the row i of matrix $\mathbf{T}_{aug}$ is of unit norm. Hence we have $\Delta_1 k_i^2 + \Delta_2 k_i^2 + ...\Delta_N k_1^2 + l_i^2 - \Delta_i k_i^2 + \Delta_i k_i^2 (\frac{\Delta_{tot} - \Delta_i}{\Delta_i})^2 = 1$, which reduces to

$$k_i = \frac{\sqrt{\Delta_i}}{\Delta_{tot}} \tag{2}$$

From equations (1) and (2), we have

$$l_i = \frac{k_i}{k_0} = \sqrt{\frac{\Delta_i}{\Delta_{tot}}} \tag{3}$$

Equations (1) and (2) specify the parameters of the frame functions in order to satisfy the requirements of a Parseval tight frame.

2. Derivation of analysis and synthesis matrices

To compute the transform coefficitents $\underline{w}$, where w_0 is the scaling coefficient and w_i, $i = 1...N$ are the wavelet coefficients, we use the analysis matrix, $\mathbf{T}_A$, to multiply the vector of sensor values $\underline{m}$. In other words, $\underline{w} = \mathbf{T}_A\underline{m}$. The analysis matrix, which is of dimension $(N+1)\times N$ is given by

$$\mathbf{T}_A = \mathbf{K\Delta} \tag{4}$$

Note that $\mathbf{T}_A^T\mathbf{T}_A = \mathbf{\Delta K^T K \Delta} = \sqrt{\mathbf{\Delta}}(\sqrt{\mathbf{\Delta}}\mathbf{K}^T)(\mathbf{K}\sqrt{\mathbf{\Delta}})\sqrt{\mathbf{\Delta}} = \sqrt{\mathbf{\Delta}}(\mathbf{T}^T)(\mathbf{T})\sqrt{\mathbf{\Delta}} = \mathbf{\Delta}$ The synthesis matrix, $\mathbf{T}_S$, is therefore

$$\mathbf{T}_S = \mathbf{\Delta}^{-1}\mathbf{T}_A^T \tag{5}$$

to ensure that we satisfy $\mathbf{T}_S\mathbf{T}_A = \mathbf{I}_{N\times N}$ The synthesis matrix is of dimension $N\times(N+1)$, and is used to compute the inverse wavelet transform: $\underline{m} = \mathbf{T}_S\underline{w}$.

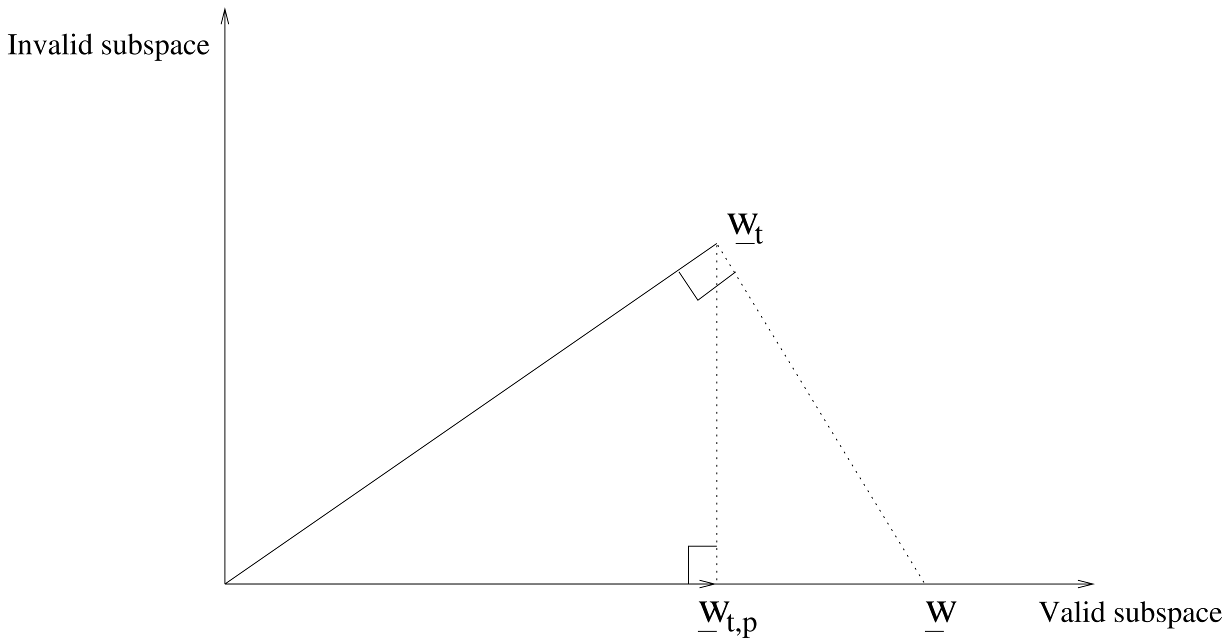

Fig. 2. *Computing the Wavelet coefficients in idea-1. We pass the polynomial-blind filter through every successive set of m points. In the above example* $m = 2$ *(blind upto quadratic polynomials) and the length of the blind-filter is* $m + 2 = 4$*. We have* $N = 8$ *above, and hance the number of wavelet coefficients is* 5.

3. Thresholding and error bounds

To facilitate the derivation of error bounds due to wavelet coefficient thresholding, we investigate the role of the vector $\underline{l}$. Note that $\underline{m}$ is in the space of R^N, whereas $\underline{w}$ exists in R^{N+1}. Clearly, the degrees of freedom of $\underline{w}$ that is obtained by computing the wavelet transform of signal $\underline{m}$ is N. Thus, the space of all valid wavelet coefficients lie in a subspace of dimension N. The valid subspace of $\underline{w}$ is spanned by the columns of $\mathbf{T}_A$, or equivalently, by the columns of $\mathbf{T}$. Clearly, the vector $\underline{l}$ which was used to augment the matrix $\mathbf{T}$ spans the 'invalid' subspace (of dimension one) of the wavelet coefficients.

In order to compute the bounds due to thresholding error, we need to calculate the strength of an arbitrary $\underline{w}$ in the invalid subspace. Since $\underline{l}$ is of unit norm, the projection of the vector $\underline{w}$ in the invalid subspace is given by $\underline{l}\underline{l}^T\underline{w}$. The of $\underline{w}$ on the valid subspace is given by $\underline{w}_p = \underline{w} - \underline{l}\underline{l}^T\underline{w}$. The vectors $\underline{w}$ and $\underline{w}_p$ are related by the analysis and synthesis matrices by: $\underline{w}_p = \mathbf{T}_A\mathbf{T}_S\underline{w}$ (note that $\mathbf{T}_A\mathbf{T}_S$ is the projection matrix into the valid subspace). Equivalently, $\mathbf{T}_S\underline{w}_p = \mathbf{T}_S\underline{w}$.

The vector of wavelet coefficients $\underline{w}$ lies in the valid subspace. Let $\underline{w}_t$ be the thresholed wavelet coefficients. Without loss of generality, let us assume that the wavelet coefficients $w_1, w_2, ..., w_q$, (where $q < N$) are thresholded to zero. We are interested in finding the energy of the signal that is obtained by computing the inverse wavelet transform of $\underline{w}_t$. Note that $\underline{w}_t$ need not be in the valid subspace, and hence the energy of the reconstructed signal will be the energy of the projection of $\underline{w}_t$ on the valid subspace. Let $\underline{w}_{t,p}$ be the projection of $\underline{w}_t$ on the valid subspace. Figure 2 captures the relationship between $\underline{w}$, $\underline{w}_t$ and $\underline{w}_{t,p}$.

Let E be the energy of the original signal. We have $E = \sum_{i=0}^{N} w_i^2$, because the frame functions form a Parseval tight frame. Let $\hat{E}$ be the energy of the reconstructed signal, which is equal to the energy of $\underline{w}_{t,p}$, which in turn is equal to the energy of $\underline{w}_t$ minus the energy of the projection of $\underline{w}_t$ along the invalid subspace. Thus,

$$\hat{E} = \|\underline{w}_t\|^2 - (\underline{l}^T\underline{w}_t)^2 = w_0^2 + \sum_{i=q+1}^{N} w_i^2 - (\sum_{i=q+1}^{N}\sqrt{\frac{\Delta_i}{\Delta_{tot}}}w_i)^2 = E - \sum_{i=1}^{q} w_i^2 - (\sum_{i=q+1}^{N}\sqrt{\frac{\Delta_i}{\Delta_{tot}}}w_i)^2$$

Since $\underline{w}$ lies in the valid subspace, we have $\sum_{i=1}^{N}\sqrt{\frac{\Delta_i}{\Delta_{tot}}}w_i = 0$, and hence $\sum_{i=1}^{q}\sqrt{\frac{\Delta_i}{\Delta_{tot}}}w_i = \sum_{i=q+1}^{N}\sqrt{\frac{\Delta_i}{\Delta_{tot}}}w_i$ The expression for $\hat{E}$ thus becomes

$$\hat{E} = E - \sum_{i=1}^{q} w_i^2 - (\sum_{i=1}^{q} \sqrt{\frac{\Delta_i}{\Delta_{tot}}} w_i)^2$$

or

$$E - \hat{E} = \sum_{i=1}^{q} w_i^2 + (\sum_{i=1}^{q} \sqrt{\frac{\Delta_i}{\Delta_{tot}}} w_i)^2 \tag{6}$$

Using equation (6), one can compute the *exact* reduction in signal energy in terms of the thresholded wavelet coefficients. The first term in equation (6) represents the loss in energy due to removing the wavelet coefficients, and the second term represents the loss in energy due to projection. Clearly, the reduction in energy $E - \hat{E} \geq \sum_{i=1}^{q} w_i^2$ has a lower bound given by

$$E - \hat{E} \geq \sum_{i=1}^{q} w_i^2$$

The more interesting question is whether we can find an upper bound for $E - \hat{E} \geq \sum_{i=1}^{q} w_i^2$. In order to compute the upper bound, we use Cauchy-Schwartz inequality for the second term in Equation (6):

$$E - \hat{E} \leq \sum_{i=1}^{q} w_i^2 + (\sum_{i=1}^{q} \frac{\Delta_i}{\Delta_{tot}})(\sum_{i=1}^{q} w_i^2)$$

$$E - \hat{E} \leq (\sum_{i=1}^{q} w_i^2)(1 + \sum_{j=1}^{q} \frac{\Delta_j}{\Delta_{tot}}) \tag{7}$$

Since $\sum_{j=1}^{N} \Delta_j = \Delta_{tot}$, we have $\sum_{j=1}^{q} \frac{\Delta_j}{\Delta_{tot}} \leq 1$. Hence the inequality (7) becomes $E - \hat{E} \leq 2(\sum_{i=1}^{q} w_i^2)$. This inequality gives an upper bound on the error in reconstruction in terms of the thresholded wavelet coefficients. Summarizing the upper and lower bound results, we have:

$$\sum_{i=1}^{q} w_i^2 \leq (E - \hat{E}) \leq 2(\sum_{i=1}^{q} w_i^2) \tag{8}$$

Inequality (8) guaratees that the reconstruction error is bounded. This is important in signal processing applications because thresholding a small wavelet coefficient ensures that the reconstruction error is not large. Hence the thresholding operation is stable.

4. References

[1] P. G. Casazza, M. Fickus, J. Kovacevic, M. Leon, and J. Tremain "A physical interpretation for finite tight frames," *Journal of Appl. and Comput. Harmonic Analysis*, 2004. Invited paper.